DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA

STUDY OF HOMOGENEOUS D'ATRI SPACES, OF THE JACOBI OPERATOR ON G.O. SPACES AND THE LOCALLY HOMOGENEOUS CONNECTIONS ON 2-DIMENSIONAL MANIFOLDS WITH THE HELP OF MATHEMATICA

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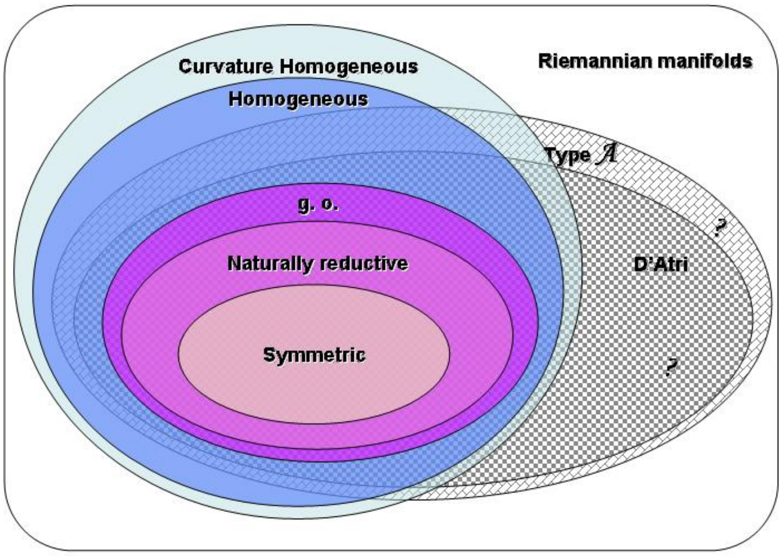
Study of homogeneous D'Atri spaces, of the Jacobi operator on g.o. spaces and the locally homogeneous connections on 2-dimensional manifolds with the help of MATHEMATICA©

(Estudio de los espacios de D'Atri homogéneos, del operador de Jacobi sobre g.o. espacios y de las conexiones localmente homogéneas sobre variedades 2-dimensionales con la ayuda de MATHEMATICA©)



Ph. D. Thesis presented by

Teresa Arias Marco

Departamento de Geometría y Topología
VNIVERSITAT DE VALÈNCIA
Valencia, 2007

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Teresa Arias Marco

Tesis Doctoral

Memoria realizada durante el disfrute de una beca de investigación F.P.U. del Ministerio de Educación y Cultura en el Departamento de Geometría y Topología de la Universidad de Valencia, bajo la dirección del Prof. Dr. Antonio Martínez Naveira y del Prof. Dr. Oldřich Kowalski, para la obtención del grado de Doctor y la mención de Doctor Europeo en Ciencias Matemáticas por la Universidad de Valencia.

Departamento de Geometría y Topología
VNIVERSITAT DE VALÈNCIA
Valencia, 2007

Dr. OLDŘICH KOWALSKI, Professor of Mathematics of the Charles University (Prague, Czech Republic) and Dr. ANTONIO MARTÍNEZ NAVEIRA, Professor of the Department of Geometry and Topology of the University of Valencia (Valencia, Spain)

CERTIFY: that the work

Study of homogeneous D'Atri spaces, of the Jacobi operator on g.o. spaces and the locally homogeneous connections on 2-dimensional manifolds with the help of MATHEMATICA©,

constitutes the Ph. D. Thesis of *Teresa Arias Marco*, carried out under the direction of Professors Oldřich Kowalski and Antonio Martínez Naveira. We, the undersigned, allow its submission to the Department of Geometry and Topology of the University of Valencia to be considered for the degree of Doctor and the mention of European Doctor of Mathematics Sciences at the University of Valencia.

Valencia, January 30th, 2007

Fdo. Oldřich Kowalski   Fdo. Antonio Martínez Naveira

Fdo. Teresa Arias Marco

A José Vicente, mi marido, y a mis padres.....

Agradecimientos (Acknowledgments)

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Contents

Introduction

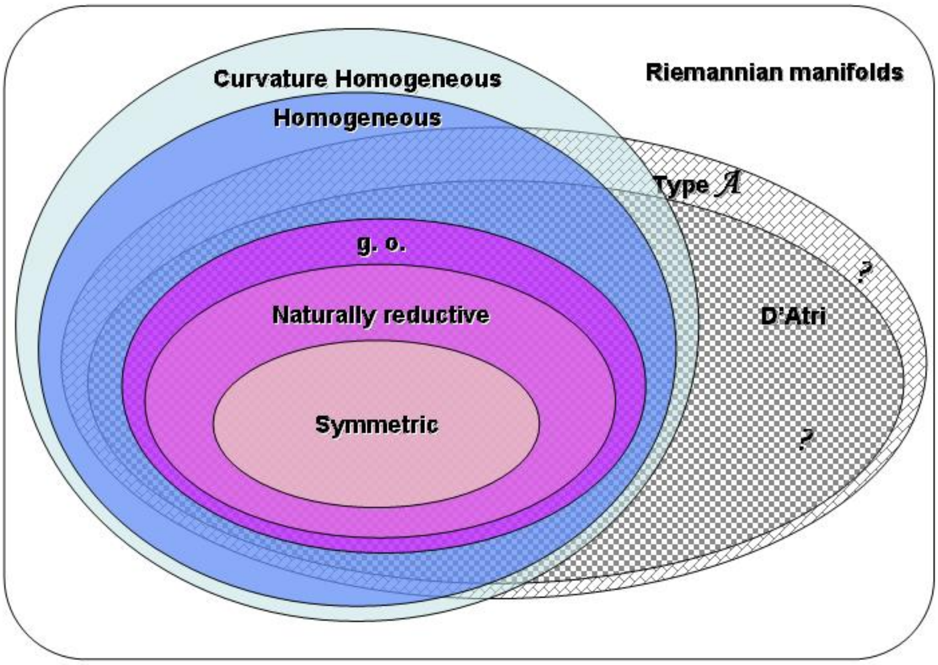
Nowadays, the concept of *homogeneity* is one of the fundamental notions in geometry although its meaning must be always specified for the concrete situations.

In this thesis, we consider the homogeneity of Riemannian manifolds and the homogeneity of manifolds equipped with affine connections. The first kind of homogeneity means that, for every smooth Riemannian manifold (M, g), its group of isometries $I(M)$ is acting transitively on M. Part I of this thesis fits into this philosophy. Afterwards in Part II, we treat the homogeneity concept of affine connections. This homogeneity means that, for every two points of a manifold, there is an affine diffeomorphism which sends one point into another. In particular, we consider a local version of the homogeneity, that is, we accept that the affine diffeomorphisms are given only locally, i.e., from a neighborhood onto a neighborhood.

More specifically, we devote Part I to solve the problem of checking if the three-parameter families of six and twelve-dimensional flag manifolds, $(M^6 = SU(3)/SU(1) \times SU(1) \times SU(1), g_{(c_1,c_2,c_3)})$, $(M^{12} = Sp(3)/SU(2) \times SU(2) \times SU(2), g_{(c_1,c_2,c_3)})$, constructed by N. R. Wallach in [W] are D'Atri spaces. Thus, we improve the results given in [AM-N]. Moreover, we obtain the complete 4-dimensional classification of homogeneous spaces of type $\mathcal{A}$. This allows us to prove correctly that every 4-dimensional homogeneous D'Atri space is naturally reductive. Furthermore, we prove that the curvature operator has constant osculating rank over g.o. spaces. As a consequence, we also present a method valid on every g.o. space to solve the Jacobi equation. This method extend the method given by A. M. Naveira and A. Tarrío in [N-T] for naturally reductive spaces. In Part II, we classify (locally) all locally homogeneous affine connections with arbitrary torsion on two-dimensional manifolds. Therefore, we generalize the result given by B. Opozda for torsion-less case in [Op.3]. Moreover, we obtain interesting consequences about flat connections with torsion.

In general, the study of these problems sometimes require a big number of straightforward symbolic computations. In such cases, it is a quite difficult task and a lot of time consuming, try to make correctly this kind of computations by hand. Thus, we try to organize our computations in (possibly) most systematic way so that the whole procedure is not excessively long. Also, because these topics are an ideal subject for a computer-aided research, we are using the software Mathematica©, throughout this work. But we put stress on the full transparency of this procedure.

More exactly, we devote Chapter 1 to make a brief overview of some special kinds of homogeneous Riemannian manifolds which will be of special relevance in Part I of this thesis. Furthermore, we recall the relevant material without proofs, to understand the following diagram:



Section 1.4 is the last of this chapter and it is intended to show how the software Mathematica 5.2 becomes useful. For that, we finish the study made by J. E. D'Atri and H. K. Nickerson in [D'A-N.2] of the three-parameter family of six-dimensional flag manifolds in the complex plane $(M^6, g_{(c_1,c_2,c_3)})$. (Now, it can be done in a quite short period of time). Also, we study a three-parameter family of twelve-dimensional flag manifolds in the quaternionic plane, $(M^{12}, g_{(c_1,c_2,c_3)})$. In particular, we conclude

"*the three-parameter families of flag manifolds* $(M^6, g_{(c_1,c_2,c_3)})$ *and* $(M^{12}, g_{(c_1,c_2,c_3)})$ *are D'Atri spaces if and only if they are naturally reductive spaces*".

Recall that the property of being a D'Atri space (i.e., a space with volume-preserving symmetries) is equivalent to the infinite number of curvature identities called the odd Ledger conditions. Moreover, a Riemannian manifold (M, g) satisfying the first odd Ledger condition is said to be of type $\mathcal{A}$. In addition, the classification of all 3-dimensional D'Atri spaces is well-known (see [K]). All of them are locally naturally reductive.

However, the first attempts to classify all 4-dimensional *homogeneous* D'Atri spaces were done by F. Podestà, A. Spiro, P. Bueken and L. Vanhecke in the papers [Po-Sp] and [Bu-V] (which are mutually complementary). The previous authors started with the corresponding classification of all spaces of type $\mathcal{A}$, but the classification in [Po-Sp] was incomplete, as we claim in [AM]. Now, in Chapter 2, we accomplish the complete classification of all 4-dimensional homogeneous spaces of type $\mathcal{A}$ in a simple and explicit form (see Theorem 2.1.2) and, as a consequence, we prove correctly that

"all 4-dimensional homogeneous D'Atri spaces are locally naturally reductive."

Moreover, Section 2.3 is devoted to correct Podesta and Spiro's Classification Theorem given in [Po-Sp].

On the other hand, note that in this first studies of Part I, we obtain geometric properties of a manifold from the curvature operator and its derivatives. However in the last chapter of Part I, we shall study just the opposite situation, i.e., we shall obtain properties of the curvature operator and its derivatives from geometric properties of a manifold.

A Riemannian g.o. manifold is a homogeneous Riemannian manifold (M, g) on which every geodesic is an orbit of a one-parameter group of isometries. It is known that every simply connected Riemannian g.o. manifold of dimension≤ 5 is naturally reductive. The first counter-example of a Riemannian g.o. manifold which is not naturally reductive is Kaplan's six-dimensional example that we shall denote it by N. On the other hand, A. M. Naveira and A. Tarrío in [N-T] have developed a method for solving the Jacobi equation in the manifold $Sp(2)/SU(2)$. This method is based on the fact that the Jacobi operator has constant osculating rank r over naturally reductive spaces, i.e., under some assumptions, its higher order derivatives from 1 to r are linear independent and from 1 to $r+1$ are linear dependent. In Chapter 3, we prove that

"the Jacobi operator has constant osculating rank over g.o. spaces"

and, as a consequence, we solve the Jacobi equation in the Kaplan example N. For that, we calculate that the constant osculating rank of the Jacobi operator $\mathcal{J}$ on N is 4. More concretely, we have obtained that the basic relation satisfied between the n^{th} covariant derivatives for $n = 1, ..., 5$ of the $(0, 4)$ - Jacobi operator along the arbitrary geodesic γ with initial vector x at the origin $p = \gamma_0$ of N is

$$\tfrac{1}{4}|\dot{\gamma}_0|^4 \mathcal{J}_0^{1)} + \tfrac{5}{4}|\dot{\gamma}_0|^2 \mathcal{J}_0^{3)} + \mathcal{J}_0^{5)} = 0$$

and the Jacobi operator can be written as

$$\mathcal{J}_t = c_0 + c_1 \cos(t) + c_2 \sin(t) + c_3 \cos(t/2) + c_4 \sin(t/2),$$

where

$$c_0 = \mathcal{J}_0 + 5\mathcal{J}_0^{2)} + 4\mathcal{J}_0^{4)}, \quad c_1 = \tfrac{1}{3}(\mathcal{J}_0^{2)} + 4\mathcal{J}_0^{4)}), \quad c_2 = \tfrac{-1}{3}(\mathcal{J}_0^{1)} + 4\mathcal{J}_0^{3)}),$$
$$c_3 = \tfrac{-16}{3}(\mathcal{J}_0^{2)} + \mathcal{J}_0^{4)}), \quad c_4 = \tfrac{8}{3}(\mathcal{J}_0^{1)} + \mathcal{J}_0^{3)}).$$

Moreover, we compare and present the main differences between this new method and the well-known method used by I. Chavel in [Ch.1], [Ch.2], W. Ziller in [Z] and J. Berndt, F. Tricerri, L. Vanhecke in [B-Tr-V, p. 51] to solve the Jacobi equation along a geodesic.

Finally, we devote Part II of this thesis to the area of affine differential geometry which is well-established and still in rapid development (see e.g. [N-S]). Also, many basic facts about affine transformation groups and affine Killing vector fields are known from the literature (see [Ko-N, vol.I] and [Ko]). Yet, it is remarkable that a seemingly easy problem to classify all locally homogeneous *torsion-less* connections in the plane domains was solved only recently by B. Opozda in [Op.3] (direct method) and by O. Kowalski, B. Opozda, Z. Vlášek in [K-Op-Vl.4] (group-theoretical method). Unfortunately, no relation between both classifications was given. See also the previous partial results in [K-Op-Vl.1] and [K-Op-Vl.2]. For dimension three, to make a classification seems to be a hard problem.

In Chapter 4 we classify all locally homogeneous affine connections with arbitrary torsion in the plane domains from the group-theoretical point of view and we obtain interesting consequences about flat connections with torsion. The stronger Classification Theorem for connections with torsion is the following:

Classification Theorem. *Let ∇ be a locally homogeneous affine connection with arbitrary torsion on a 2-dimensional manifold $\mathcal{M}$. Then, either ∇ is locally a Levi-Civita connection of the unit sphere or, in a neighborhood $\mathcal{U}$ of each point $m \in \mathcal{M}$, there is a system (u, v) of local coordinates and constants p, q, c, d, e, f, r, s such that ∇ is expressed in $\mathcal{U}$ by one of the following formulas:*

Type A

$$\nabla_{\partial_u}\partial_u = p\partial_u + q\partial_v, \quad \nabla_{\partial_u}\partial_v = c\partial_u + d\partial_v,$$
$$\nabla_{\partial_v}\partial_u = r\partial_u + s\partial_v, \quad \nabla_{\partial_v}\partial_v = e\partial_u + f\partial_v.$$

Type B

$$\nabla_{\partial_u}\partial_u = \tfrac{p\partial_u+q\partial_v}{u}, \quad \nabla_{\partial_u}\partial_v = \tfrac{c\partial_u+d\partial_v}{u},$$
$$\nabla_{\partial_v}\partial_u = \tfrac{r\partial_u+s\partial_v}{u}, \quad \nabla_{\partial_v}\partial_v = \tfrac{e\partial_u+f\partial_v}{u},$$

where not all p, q, c, d, e, f, r, s are zero.

Moreover, based on our computations, we illustrate the essential relationship between the classifications given in [K-Op-Vl.4] and [Op.3]. In addition, we prove that, for some Lie algebras $\mathfrak{g}$, all connections corresponding to such a $\mathfrak{g}$ are simultaneously of type A and of type B. These facts can be easily checked in the table of Section 4.1 that we use to summarize our results.

Finally, it is worthwhile to mention that our Classification Theorem has been just used by O. Kowalski and Z. Vlášek to solve the main existence problems concerning affine homogeneous geodesics in dimension two. See [K-Vl].

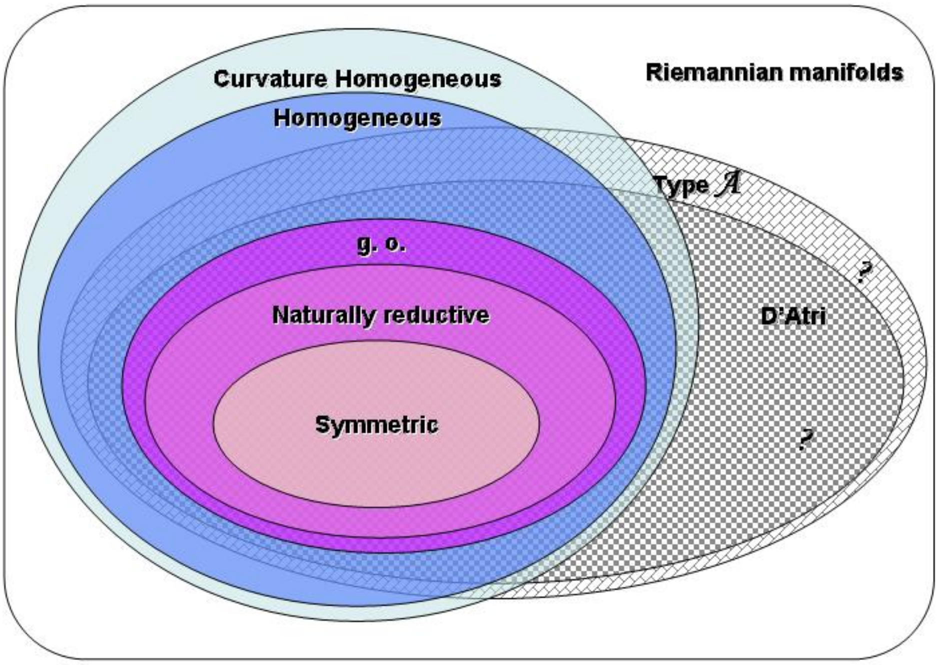
Part I

Homogeneous Riemannian manifolds

Chapter 1

Preliminaries

In this chapter, we shall make a brief overview of some special kinds of Riemannian manifolds. In particular, we shall recall the relevant material without proofs, to understand the following diagram:



This material is basic to facilitate access to the individual topics of the second and third Chapter. Besides, we shall show how the software MATHEMATICA 5.2 becomes useful. For that, we shall finish the study made by J. E. D'Atri and H. K. Nickerson in [D'A-N.2] of a three-parameter family of six-dimensional flag manifolds in the complex plane. (Now, it can be done in a quite short period of time). Also, we shall study a three-parameter family of twelve-dimensional flag manifolds in the quaternionic plane. In particular, for both families we shall prove that every flag manifold is a D'Atri space if and only if it is naturally reductive. These results will be published in [AM.2].

1.1 The concept of homogeneity on Riemannian manifolds

Let (M, g) be a smooth Riemannian manifold. (M, g) is said to be *homogeneous* if its group of isometries $I(M)$ is acting transitively on M.

Moreover, let $G \subset I(M)$ be a connected Lie group which acts transitively on a Riemannian manifold M and let $p \in M$ be a fixed point. If we denote by H the isotropy group at p, then M can be identified with the homogeneous manifold G/H. In general, there may be more than one such group $G \subset I(M)$. If, for example, we take a connected Lie group G' such that $G \neq G' \subset I(M)$ and G' also acts transitively on M, then there is another expression of M as G'/H' (where H' is the new isotropy group).

For any fixed choice $M = G/H$, G acts effectively on G/H from the left. The Riemannian metric g on M can be considered as a G-invariant metric on G/H. The pair $(G/H, g)$ is then called a *Riemannian homogeneous space*.

In [Be], L. Bérard Bergery published the 4-dimensional classification of Riemannian homogeneous spaces. In particular, he obtained the following

Proposition 1.1.1. *In dimension 4, each simply connected homogeneous Riemannian manifold M is either symmetric or isometric to a Lie group with a left-invariant metric. In the second case, either M is a solvable group or it is one of the groups $SU(2) \times \mathbb{R}$, $\widetilde{Sl(2, \mathbb{R})} \times \mathbb{R}$.*

On the other hand, I. M. Singer in [Si] introduced the following condition and proved the following Theorem which provides a new characterization for the concept of locally homogeneity on Riemannian manifolds.

Definition 1.1.2. *(M, g) satisfies the condition $P(k)$ if, for any $p, q \in M$, there is a linear isometry $F : T_pM \to T_qM$ such that $F^*(\nabla^i \mathcal{R}_q) = \nabla^i \mathcal{R}_p$ for $i = 0, 1, ..., k$. Here $\mathcal{R}$ denotes the curvature Riemannian tensor and $\nabla^0 \mathcal{R} = \mathcal{R}$.*

It is obvious that a homogeneous Riemannian manifold satisfies the condition $P(k)$ for all $k \geq 0$.

Theorem 1.1.3. *Let (M, g) be a connected, simply connected and complete Riemannian manifold. If (M, g) satisfies the condition $P(k)$ for a sufficiently large k (depending only on the dimension of M), then it is a homogeneous Riemannian manifold.*

To obtain more details about the Singer number $k_M = k - 1$ see [Bo-K-V], [Si] and [Gro, p.165].

Some years later, K. Sekigawa in [Se] gave the first examples of 3-dimensional Riemannian manifolds which satisfy the condition $P(0)$ but they are not (locally) homogeneous. This result gave sense to the concept of curvature homogeneous space introduced in [Si].

Definition 1.1.4. *A smooth Riemannian manifold M is called* curvature homogeneous *if, for any two points $p, q \in M$, there exists a linear isometry $F : T_pM \to T_qM$ such that $F^*\mathcal{R}_q = \mathcal{R}_p$, i.e. the condition $P(0)$ is satisfied.*

This is also equivalent to saying that, locally, there always exists a smooth field of orthonormal frames with respect to which the components of the curvature tensor $\mathcal{R}$ are constant functions (see, for instance, I. M. Singer [Si], or the monograph [Bo-K-V]). Hence it is obvious that *all principal Ricci curvatures are constant.* Clearly any homogeneous manifold is curvature homogeneous. On the other hand, the locally homogeneous Riemannian manifolds in dimensions ≥ 3 form a "negligible" subclass of all curvature homogeneous spaces (see a survey in [Bo-K-V]).

1.2 Reductive spaces, naturally reductive spaces and g.o. spaces

A *Riemannian homogeneous space* $(G/H, g)$ with its origin $p = \{H\}$ is always a *reductive homogeneous space* in the following sense (cf. [Ko-N, vol.II, p.190]): we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad} : H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There is a direct sum decomposition (*reductive decomposition*) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi : G \to G/H = M$. Using this natural identification and the scalar product g_p on T_pM, we obtain a scalar product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ which is obviously $\mathrm{Ad}(H)$-invariant.

Furthermore, note that if $(M = G/H, g)$ is a homogeneous space, for each $Z \in \mathfrak{g}$, the mapping $(t, q) \in \mathbb{R} \times M \mapsto \tau(\exp(tZ))(q)$ is a one-parameter group of isometries. Consequently, it induces a Killing vector field Z^*, called the *fundamental vector field* on M, given by

$$Z^*_q = \frac{d}{dt}_{|0}(\tau(\exp(tZ))(q)), \quad q \in M.$$

The following definition is well known from [Ko-N, chapter X, sections 2, 3]:

Definition 1.2.1. *A Riemannian homogeneous space $(G/H, g)$ is said to be* naturally reductive *if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ of $\mathfrak{g}$ satisfying the condition*

$$\langle [X, Z]_{\mathfrak{m}}, Y\rangle + \langle X, [Z, Y]_{\mathfrak{m}}\rangle = 0 \qquad \text{for all} \quad X, Y, Z \in \mathfrak{m}. \tag{1.1}$$

Here the subscript $\mathfrak{m}$ indicates the projection of an element of $\mathfrak{g}$ into $\mathfrak{m}$.

It is also well-known that the condition (1.1) is equivalent to the following more geometrical property:

$$\begin{gathered}\text{For any vector } X \in \mathfrak{m}\backslash\{0\}, \text{ the curve } \gamma(t) = \tau(\exp tX)(p) \\ \text{is a geodesic with respect to the Riemannian connection.}\end{gathered} \tag{1.2}$$

Here exp and $\tau(h)$ denote the Lie exponential map of G and the left transformation of G/H induced by $h \in G$ respectively. Thus, for a naturally reductive homogeneous space every geodesic on $(G/H, g)$ is an orbit of a one-parameter subgroup of the group of isometries.

The following concept was introduced for the first time in [Go.1] and nicely defined in [Du-K-Ni]. This concept became necessary after the special study made by A. Kaplan in [Ka.2].

Definition 1.2.2. *Let (M, g) be a homogeneous Riemannian manifold. Then (M, g) is said to be* naturally reductive *if there is a transitive group G of isometries for which the corresponding Riemannian homogeneous space $(G/H, g)$ is naturally reductive in the sense of Definition 1.2.1*

Let us denote by $I_0(M)$ the maximal connected group of isometries on (M, g). Examples are known such that $M = G/H$ is not naturally reductive for some small group $G \subset I_0(M)$ but it becomes naturally reductive if we write $M = G'/H'$ for a bigger group of isometries $G' \subset I_0(M)$.

Moreover, the naturally reductive manifolds have been studied by a number of authors as a natural generalization of *Riemannian symmetric spaces*, which may be characterized by the property that all geodesic symmetries are globally defined and they are isometries. See [D'A-Z], [Tr-V], [Tr-V.2], [Go-Z], [K-V.1], [K-V.3] and [K-P-V] for more advanced results on naturally reductive spaces.

Now, natural reductivity is still a special case of a more general property, which follows easily from (1.2):

Definition 1.2.3. *A Riemannian homogeneous space $(G/H, g)$ is called a* g.o. space *if each geodesic of $(G/H, g)$ (with respect to the Riemannian connection) is an orbit of a one-parameter subgroup $\{exp(tZ)\}$, $Z \in \mathfrak{g}$, of the group of isometries G.*

A homogeneous Riemannian manifold (M, g) is called a Riemannian g.o. manifold *if each geodesic of (M, g) is an orbit of a one-parameter group of isometries.*

The extensive study of g.o. spaces only started with A. Kaplan's paper [Ka.2] in 1983, because he gave the first counter-example of a Riemannian g.o. manifold which is not naturally reductive. This is a six-dimensional Riemannian nilmanifold with a two-dimensional center, one of the so-called "generalized Heisenberg groups" or "H-type groups". Subsequently, the class of generalized Heisenberg groups has provided a large number of further counter-examples. (See [R], [B-Tr-V]). A classification of all g.o. spaces in dimension less or equal to six is given by O. Kowalski and L. Vanhecke in [K-V.5]. All Riemannian g.o. manifolds of dimension ≤ 5 are proved to be naturally reductive. In dimension 6, new examples of g.o. spaces are given which are in no way naturally reductive. Moreover, in 2004, the first known 7-dimensional compact examples of Riemannian g.o. manifolds which are not naturally reductive were found by Z. Dušek, O. Kowalski, and S.Ž. Nikčević in [Du-K-Ni].

For more information on the relation between naturally reductive spaces and g.o. spaces, and also for the references to related topics, see [Al-Ar.1], [Al-Ar.2], [Du-K-Ni], [Go.2], [K-P-V] and [K-V.5].

1.3 D'Atri spaces

D'Atri spaces have been a topic of interest in Riemannian geometry since they were introduced by J. E. D'Atri and H. K. Nickerson [D'A-N.1], [D'A-N.2] and studied extensively by J. E. D'Atri in [D'A]. We may call them D'Atri spaces, following L. Vanhecke and T. J. Willmore in [V-Wi.1] and [V-Wi.2], where further interesting properties can be found. Some years later, they were reviewed by O. Kowalski, F. Prüfer, L. Vanhecke in [K-P-V] and also discussed in the book of T. J. Willmore [Wi].

Let (M, g) be a Riemannian manifold. (M, g) is called a *D'Atri space* (cf. [V-Wi.2]) if the local geodesic symmetries are volume-preserving. The local geodesic symmetry s_p at $p \in M$ is defined as follows: for all X in a normal neighbourhood of the origin in the tangent space T_pM we put $s_p(\exp_p(X)) = \exp_p(-X)$. (An equivalent definition in [D'A-N.1] says that each local geodesic symmetry should be divergence-preserving).

If all the geodesic symmetries are isometries, then M is said to be locally symmetric and this is equivalent to the curvature condition $\nabla\mathcal{R} = 0$. Thus it is clear that all *Riemannian symmetric spaces*, i.e. spaces with metric-preserving geodesic symmetries, *are D'Atri spaces.*

In [D'A-N.1], [D'A-N.2] it was proved that *all naturally reductive spaces are D'Atri spaces*, and another more simple proof was provided in [D'A]. The first proof of the fact that *all g.o. spaces are D'Atri spaces* was given in [K-V.2]. See also [K-V.4]. It is worthwhile to mention that an example of a non-homogeneous D'Atri space is not known. See [K-P-V] for a survey about the whole topic.

Let us note that the Jacobi vector fields play an important role in the work on D'Atri spaces. Recall (see [Ko-N, vol. 2, p. 63]) that a vector field Y along a geodesic γ of a manifold M is said to be a *Jacobi vector field* if and only if it satisfies the following second order differential equation, called *Jacobi equation*: $Y'' + \mathcal{R}_\gamma(Y) = 0$, where the operator $\mathcal{R}_\gamma = \mathcal{R}(\cdot, \gamma')\gamma'$ is called *Jacobi operator*. Using the calculation with the Jacobi operator, one can derive so-called "Ledger's recurrence formula" (see [Ru-Wa-Wi, p.62] or [K-P-V]). This recursion formula yields an infinite series of curvature conditions, known as *Ledger's conditions* L_q, $q \geq 2$. It was proved by L. Vanhecke in [V] that the "odd" Ledger conditions are already consequences of the "even" ones. Anyway, these inductively defined conditions soon become very complicated. Thus, the explicit form of L_q is known only for small values of q. In particular, the two first non-trivial odd Ledger conditions are

$$L_3 : (\nabla_X \rho)(X, X) = 0, \tag{1.3}$$

$$L_5 : \sum_{a,b=1}^{n} \mathcal{R}_{XE_aXE_b}(\nabla_X \mathcal{R})_{XE_aXE_b} = 0, \tag{1.4}$$

where X is any tangent vector at any point $m \in M$ and $\{E_1, ..., E_n\}$ is any orthonormal basis of T_mM. Here $\mathcal{R}$ denotes the curvature tensor and ρ the Ricci tensor of (M, g), respectively, and $n = dimM$.

Finally, let us recall the following well-known result [D'A-N.1]:

Theorem 1.3.1. *An analytic Riemannian manifold is a D'Atri space if, and only if, it satisfies the series of all Ledger conditions of odd order L_{2k+1}, $k \geq 1$.*

On the other hand, if *all* Ledger's condition L_q are satisfied, we obtain a characterization of so-called *harmonic spaces*.

The first odd Ledger condition L_3 is very important. Firstly, because it implies that M is analytic [Sz]. Hence Theorem 1.3.1 holds also in the case that a Riemannian manifold is only smooth. Secondly, because it also implies that the scalar curvature is constant [D'A-N.1]. Finally, because it defines a new kind of Riemannian manifolds.

Definition 1.3.2. *Riemannian manifold M is said to belong to class $\mathcal{A}$, or to be of type $\mathcal{A}$, if its Ricci curvature tensor ρ is cyclic-parallel that is, if $(\nabla_X \rho)(X, X) = 0$ for all vector field X tangent to M or, equivalently, if*

$$(\nabla_X \rho)(Y, Z) + (\nabla_Y \rho)(Z, X) + (\nabla_Z \rho)(X, Y) = 0$$

for all vector fields X,Y,Z tangent to M.

These Riemannian manifolds were introduced by A. Gray in [G] as a special subclass of (connected) Riemannian manifolds (M, g), called Einstein-like spaces, all of which have constant scalar curvature. See [Bu-V] for more references and details.

1.4 An example where Mathematica© is useful

In [W], N. R. Wallach constructed a family of Riemannian flag manifolds in the complex plane, $(M^6, g_{(c_1,c_2,c_3)})$, in the quaternionic plane, $(M^{12}, g_{(c_1,c_2,c_3)})$, and also in the octonionic plane $(M^{24}, g_{(c_1,c_2,c_3)})$ as examples of reductive homogeneous spaces. Here, c_1, c_2 and c_3 are positive real constants.

As concerns the first one, M^6, D'Atri and Nickerson in [D'A-N.2] proved that if two of the parameters c_1, c_2, c_3 are equal, the corresponding Riemannian space is of type $\mathcal{A}$. Moreover, for the case $c_1 = c_2 = 1$, $c_3 = 2$ they affirmed (without explicit argument) that the second odd Ledger condition L_5 is not satisfied.

Now, we shall finish their study of the L_5 condition over the manifold M^6. Of course, with all the relevant arguments. Further, we shall extend the study of the two-first odd Ledger conditions L_3, L_5 to the other Wallach's flag manifold M^{12}. Moreover, we shall correct the result given in [AM-N] where this problem over the manifold M^{12} was studied for the first time. In both cases, we shall conclude that every member of both families of Riemannian flag manifolds is a D'Atri space if and only if it is naturally reductive.

Many symbolic computations are required to make this study. Thus, to organize them in the most systematic way, we use the software MATHEMATICA 5.2 throughout this work. We put stress on the full transparency of this procedure.

However, we shall not treat along this section the 24-dimensional family of flag manifolds $(F_4/Spin(8), g_{(c_1,c_2,c_3)})$.

1.4.1 Preliminaries

Let $(M = G/H, g)$ a reductive Riemannian homogeneous space. In agreement with the notation of section 1.2 let us recall, following [Ko-N, vol.2,p.201], that the Riemannian connection for g is given by

$$\nabla_X Y = \tfrac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y), \tag{1.5}$$

where $U(X,Y)$ is the symmetric bilinear mapping of $\mathfrak{m}\times\mathfrak{m}$ into $\mathfrak{m}$ defined by

$$2\langle U(X,Y),Z\rangle = \langle X,[Z,Y]_{\mathfrak{m}}\rangle + \langle [Z,X]_{\mathfrak{m}},Y\rangle, \tag{1.6}$$

for all X, Y, $Z\in\mathfrak{m}$.

Note that the space M becomes naturally reductive if and only if $U\equiv 0$.

Let $\mathcal{R}$ denote the curvature tensor of the Riemannian connection ∇. Following [D'A-N.2] we have

$$\begin{aligned}\mathcal{R}(X,Y)Z = &-[[X,Y]_{\mathfrak{h}},Z] - \frac{1}{2}[[X,Y]_{\mathfrak{m}},Z]_{\mathfrak{m}} - U([X,Y]_{\mathfrak{m}},Z)\\ &+\frac{1}{4}[X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} + \frac{1}{2}[X,U(Y,Z)]_{\mathfrak{m}} + U(X,U(Y,Z))\\ &+\frac{1}{2}U(X,[Y,Z]_{\mathfrak{m}}) - \frac{1}{4}[Y,[X,Z]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{2}[Y,U(X,Z)]_{\mathfrak{m}}\\ &-U(Y,U(X,Z)) - \frac{1}{2}U(Y,[X,Z]_{\mathfrak{m}}),\end{aligned} \tag{1.7}$$

for all X, Y, $Z\in\mathfrak{m}$.

In addition, in [D'A-N.2] the authors showed how the Ledger conditions can be reformulated on reductive homogeneous spaces without explicit use of covariant derivatives. Their theorem below covers only the two first non-trivial odd conditions (1.3) and (1.4), but it is useful for checking concrete examples as in the next section.

Theorem 1.4.1. *Let $M^n = G/H$ be a reductive Riemannian homogeneous space. Let $\{E_1,\dots,E_n\}$ be an orthonormal basis of $\mathfrak{m}$ and let ρ denote the Ricci curvature tensor of the Riemannian connection. Then, the first two odd Ledger's conditions can be reformulated in the following way:*

$$L_3 \quad\equiv\quad \rho(X,U(X,X)) = \sum_{a=1}^{n}\langle\mathcal{R}(E_a,X)U(X,X),E_a\rangle = 0, \tag{1.8}$$

$$L_5 \quad \equiv \quad \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, X)X, X)U(X, X), E_a \rangle = 0 \tag{1.9}$$

for all $X \in \mathfrak{m}$. Or, equivalently

$$L_3 \quad \equiv \quad \rho(X, U(Y, Z)) + \rho(Y, U(Z, X)) + \rho(Z, U(X, Y)) = 0, \tag{1.10}$$

$$\begin{aligned} L_5 \equiv & \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, X)Y, Z)U(V, W), E_a \rangle + \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, Y)Z, V)U(W, X), E_a \rangle \\ & + \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, Z)V, W)U(X, Y), E_a \rangle + \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, V)W, X)U(Y, Z), E_a \rangle \\ & + \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, W)X, Y)U(Z, V), E_a \rangle = 0 \end{aligned} \tag{1.11}$$

for all $X, Y, Z, V, W \in \mathfrak{m}$.

In order to obtain examples using Theorem 1.4.1, we compute U from (1.6) and the curvature tensor $\mathcal{R}$ at the point p from (1.7).

1.4.2 Two families of flag manifolds

Let $SU(n)$ be the special unitary group and $Sp(n)$ be the symplectic group.

In the natural way, both $M^6 = SU(3)/SU(1) \times SU(1) \times SU(1)$ and $M^{12} = Sp(3)/SU(2) \times SU(2) \times SU(2)$ admit a reductive homogeneous decomposition [Wo-G].

Moreover, N. R. Wallach constructs an infinite number of metrics with strictly positive sectional curvature over the previous spaces [W].

Let $G = SU(3)$ or $Sp(3)$, and let $H = (SU(1) \times SU(1) \times SU(1))$ or $(Sp(1) \times Sp(1) \times Sp(1) \equiv SU(2) \times SU(2) \times SU(2))$. In agreement with the notation before, the Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$ or $\mathfrak{sp}(3)$ and $\mathfrak{h}$ is the subalgebra of diagonal matrices. To simplify notation, we use the same letter $\mathcal{K}$ for the complex plane $\mathbb{C}$ and for the quaternionic plane $\mathbb{H}$. Let us define $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ by

$$\mathfrak{m} = V_1 \oplus V_2 \oplus V_3,$$

where

$$V_1 = \left\{ \begin{bmatrix} 0 & z & 0 \\ -\bar{z} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, z \in \mathcal{K} \right\}, \qquad V_2 = \left\{ \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ -\bar{z} & 0 & 0 \end{bmatrix}, z \in \mathcal{K} \right\}$$

and

$$V_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & -\bar{z} & 0 \end{bmatrix}, z \in \mathcal{K} \right\}.$$

Let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{m}$ given by

$$\langle X,Y\rangle = \begin{cases} 0 & \text{if } X\in V_i,\ Y\in V_j,\ i\neq j, \\ -c_i Trace XY & \text{if } X,Y\in V_i,\ i=1,2,3. \end{cases} \quad (1.12)$$

where c_1, c_2 and c_3 are positive real parameters.

These spaces were introduced by N. R. Wallach in [W] where he also calculated from the formulas (1.6) and (1.12) that

$$U(X,Y) = \begin{cases} 0 & \text{if } X,Y\in V_i,\ i=1,2,3, \\ -\frac{c_i-c_j}{2c_k}[X,Y] & \text{if } X\in V_i,\ Y\in V_j,\ i\neq j\neq k. \end{cases} \quad (1.13)$$

Obviously, *the decomposition is naturally reductive if and only if* $c_1=c_2=c_3$.

Case $\mathcal{K}=\mathbb{C}$

For this case, the corresponding flag manifold is $M^6 = SU(3)/SU(1)\times SU(1)\times SU(1)$. Further, we know that J. E. D'Atri and H. K. Nickerson in [D'A-N.2] proved that if at least two of the parameters c_1, c_2, c_3 are equal, the corresponding Riemannian space is of type $\mathcal{A}$. Moreover, for the case $c_1=c_2=1$, $c_3=2$ they affirmed (without giving any argument) that the second odd Ledger condition L_5 is not satisfied. Now, we shall finish the study of the L_5 condition over the manifold M^6. For the convenience of the reader we repeat the relevant material from [D'A-N.2], thus making our exposition self-contained.

First, we define a basis $\{E_1, JE_1, E_2, JE_2, E_3, JE_3\}$ for $\mathfrak{m}$ taking $z=1,i$ in V_1, $z=1,-i$ in V_2 and $z=-1,-i$ in V_3, respectively. Note that implicitly we have defined the invariant almost complex structure $J:\mathfrak{m}\to\mathfrak{m}$ by

$$J\begin{pmatrix} 0 & a_{12} & a_{13} \\ -\bar{a_{12}} & 0 & a_{23} \\ -\bar{a_{13}} & -\bar{a_{23}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\,a_{12} & -i\,a_{13} \\ i\,\bar{a_{12}} & 0 & i\,a_{23} \\ -i\,\bar{a_{13}} & i\,\bar{a_{23}} & 0 \end{pmatrix}$$

i.e. for all $X\in\mathfrak{m}$ and $Y\in\mathfrak{h}$, it satisfies

$$J^2X=-X,\quad J[Y,X]_{\mathfrak{m}}=[Y,JX]_{\mathfrak{m}}.$$

Afterwards, we define a basis $\{K_1,K_2,K_3\}$ for $\mathfrak{h}$ taking

$$K_1=\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix},\ K_2=\begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix},\ K_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then, we get that the multiplication table for $\mathfrak{m}$ is given by

$$[E_l, JE_l]=2K_l,\quad l=1,2,3.$$

$$[E_l,E_m]=-[JE_l,JE_m]=E_n,\quad [E_l,E_m]=-[JE_l,JE_m]=E_n,$$

where (l, m, n) is a cyclic permutation of $(1, 2, 3)$. Moreover, we get

$$[K_l, E_l] = 2JE_l, \quad [K_l, JE_l] = -2E_l, \quad l = 1, 2, 3,$$

$$[K_l, E_m] = -2JE_m, \quad [K_l, JE_m] = E_m, \quad l \neq m, \ l, m \in \{1, 2, 3\}.$$

The curvature tensor can be computed from (1.7) with respect to this basis. The non-trivial cases are the following formulas (1.14) and the formulas obtained from (1.14) by using the operator J:

$$\begin{aligned} &\mathcal{R}(E_l, JE_l)E_l = -4JE_l, \\ &\mathcal{R}(E_l, JE_l)E_m = 2\mathcal{R}(E_l, E_m)JE_l = -2\mathcal{R}(JE_l, E_m)E_l = \tfrac{4-(c_l-c_m-c_n)^2}{2c_m c_n} JE_m, \\ &\mathcal{R}(E_l, E_m)E_l = \mathcal{R}(JE_l, E_m)JE_l = \left(\tfrac{(c_n-c_l)}{c_m} - \tfrac{(c_l-c_m-c_n)^2}{4c_m c_n}\right) E_m, \end{aligned} \tag{1.14}$$

for l, m, n distinct and $l, m, n \in \{1, 2, 3\}$.

Further, we obtain easily from (1.14) that the only non-trivial terms of the Ricci tensor are

$$\rho(E_l, E_l) = \rho(JE_l, JE_l) = \tfrac{(6c_m c_n + c_l^2 - c_m^2 - c_n^2)}{c_m c_n} \tag{1.15}$$

for l, m, n distinct and $l, m, n \in \{1, 2, 3\}$.

Now, we shall use (1.13) and (1.15) to compute the Ledger condition L_3, (1.10). The equation (1.10) has a purely algebraic character because the family of metrics $g_{(c_1,c_2,c_3)}$ is left-invariant. Hence, we can substitute for X, Y, Z every triplet chosen from the basis of $\mathfrak{m}$ (with possible repetition). Thus, the condition (1.10) is equivalent to a system of algebraic equations. Finally, we have obtained, after a lengthy by routine calculation, that the only non-trivial equation appears when

$$(X, Y, Z) \in \{(E_l, E_m, E_n), (E_l, JE_m, JE_n) \mid l, m, n \in \{1, 2, 3\}, l \neq m, l \neq n, m \neq n\}.$$

To be precise, the L_3 condition is equivalent to

$$\frac{(c_1 - c_2)(c_1 - c_3)(c_2 - c_3)}{c_1 c_2 c_3} = 0. \tag{1.16}$$

We conclude that *every member of the family of Riemannian flag manifolds* $(M^6, g_{(c_1,c_2,c_3)})$ *is of type* $\mathcal{A}$ *if and only if at least two of the parameters* c_1, c_2, c_3, *are equal.*

To finish, we shall prove that *the Ledger condition* L_5 *is satisfied if and only if* $c_1 = c_2 = c_3$.

Case $c_1 = c_l, \ l = 2, 3$.

Let us put $X = E_2, Y = E_3, Z = V = W = E_1$ in (1.11). Thus, for $l = 2$ we obtain using (1.12), (1.13) and (1.14) that (1.11) can be written in the form

$$(x - 1)(9x^2 + 24x + 80) = 0, \quad \text{for} \quad x = \tfrac{c_3}{c_1}. \tag{1.17}$$

Analogously for $l = 3$, we obtain that (1.11) can be written in the form

$$(x-1)(3x^2+8x+96)=0, \quad \text{for} \quad x=\tfrac{c_2}{c_1}. \tag{1.18}$$

In both equations (1.17), (1.18), the second order equation has negative discriminant. Then, if $c_1 = c_l$, $l = 2, 3$, the only possible real solution is $c_1 = c_2 = c_3$.

Case $c_2 = c_3$.

Let us put in (1.11) first $X = E_2, Y = JE_3, Z = W = E_1, V = JE_1$ and later $X = E_2, Y = JE_3, Z = JE_1, V = W = E_1$. Thus, we obtain a system of equations of the form

$$\begin{aligned}&(x-1)(x-4)(x^2+2x+4)=0,\\&(x-1)(x^2-4x-2)=0,\end{aligned} \tag{1.19}$$

respectively, where $x = \frac{c_1}{c_2}$. Here, the only solution of the system is $x = 1$. Then, if $c_2 = c_3$, the only possible solution is $c_1 = c_2 = c_3$.

As a conclusion, *every member of the family of Riemannian flag manifolds* $(M^6, g_{(c_1,c_2,c_3)})$ *is a D'Atri space if and only if it is naturally reductive.*

Case $\mathcal{K} = \mathbb{H}$

In this case, we shall make the study of the two-first odd Ledger conditions L_3, L_5 on the other Wallach's flag manifold, i.e. the twelve dimensional manifold $M^{12} = Sp(3)/SU(2) \times SU(2) \times SU(2)$. Moreover, we correct the result given in [AM-N] where this problem was studied for the first time.

From now on, we will denote by j_l, $l = 1, 2, 3$ the three quaternionic imaginary units i, j, k, respectively.

First, we shall define a basis for $\mathfrak{m}$. Let us introduce three invariants almost-complex structures $J_l : \mathfrak{m} \to \mathfrak{m}$, $l = 1, 2, 3$, by

$$J_l\begin{pmatrix} 0 & a_{12} & a_{13}\\ -\bar{a_{12}} & 0 & a_{23}\\ -\bar{a_{13}} & -\bar{a_{23}} & 0\end{pmatrix} = \begin{pmatrix} 0 & j_l a_{12} & -j_l a_{13}\\ j_l \bar{a_{12}} & 0 & j_l a_{23}\\ -j_l\bar{a_{13}} & j_l\bar{a_{23}} & 0\end{pmatrix}$$

for $l = 1, 2$ and

$$J_3\begin{pmatrix} 0 & a_{12} & a_{13}\\ -\bar{a_{12}} & 0 & a_{23}\\ -\bar{a_{13}} & -\bar{a_{23}} & 0\end{pmatrix} = \begin{pmatrix} 0 & j_3 a_{12} & j_3 a_{13}\\ j_3 \bar{a_{12}} & 0 & j_3 a_{23}\\ j_3\bar{a_{13}} & j_3\bar{a_{23}} & 0\end{pmatrix},$$

i.e. for all $X \in \mathfrak{m}$ and $Y \in \mathfrak{h}$, they satisfy

$$J_l^2X = -X, \quad J_l[Y,X]_{\mathfrak{m}} = [Y, J_lX]_{\mathfrak{m}} \quad \text{for} \quad l = 1, 2, 3,$$

$$J_lJ_mX = -J_mJ_lX = J_nX \quad \text{where} \quad (l,m,n) \text{ is a cyclic permutation of } (1,2,3).$$

On the other hand, it is easy to prove that the structures J_l, $l=1,2$ are nearly-Kähler (i.e. they satisfy $(\nabla_X J_l)X = 0$ for $X \in \mathfrak{m}$) and the structure J_3 is Hermitian (i.e. $(\nabla_X J_3)Y - (\nabla_{J_3 X} J_3)J_3 Y = 0$ for $X, Y \in \mathfrak{m}$), [G-He].

Finally, we define the adapted basis

$$\{E_1, J_1E_1, J_2E_1, J_3E_1, E_2, J_1E_2, J_2E_2, J_3E_2, E_3, J_1E_3, J_2E_3, J_3E_3\}$$

for $\mathfrak{m} = V_1 \oplus V_2 \oplus V_3$. In particular, we take for generating V_1 the elements

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_lE_1 = \begin{pmatrix} 0 & j_l & 0 \\ j_l & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ l = 1,2,3,$$

for generating V_2 the elements

$$E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_lE_2 = \begin{pmatrix} 0 & 0 & -j_l \\ 0 & 0 & 0 \\ -j_l & 0 & 0 \end{pmatrix}, \ l = 1,2,$$

$$J_3E_2 = \begin{pmatrix} 0 & 0 & j_3 \\ 0 & 0 & 0 \\ j_3 & 0 & 0 \end{pmatrix},$$

and for generating V_3 the elements

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_lE_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j_l \\ 0 & j_l & 0 \end{pmatrix}, \ l = 1,2,3.$$

Thus, we get an adapted basis for $\mathfrak{m}$ such that

$$[E_l, E_m] = -[J_pE_l, J_pE_m] = -E_n, \quad [E_l, J_pE_m] = [J_pE_l, E_m] = J_pE_n,$$
$$\text{where } p = 1,2 \quad \text{and} \quad (l,m,n) \text{ is a cyclic permutation of } (1,2,3),$$

$$[J_3E_1, J_3E_2] = -E_3, \quad [J_3E_2, J_3E_3] = -E_1, \quad [J_3E_3, J_3E_1] = E_2,$$
$$[E_1, J_3E_2] = -[J_3E_1, E_2] = -J_3E_3, \quad [E_2, J_3E_3] = -[J_3E_2, E_3] = J_3E_1,$$
$$[E_3, J_3E_1] = [J_3E_3, E_1] = -J_3E_2,$$
$$[J_pE_3, J_qE_1] = -[J_qE_3, J_pE_1] = J_rE_2 \ \text{ for } \ (p,q,r) \in \{(1,2,3),(1,3,2),(3,2,1)\},$$
$$[J_1E_l, J_2E_m] = -[J_2E_l, J_1E_m] = -J_3E_n \ \text{ for } \ (l,m,n) \in \{(1,2,3),(2,3,1)\},$$
$$[J_1E_l, J_3E_m] = [J_3E_l, J_1E_m] = J_2E_n \ \text{ for } \ (l,m,n) \in \{(2,1,3),(2,3,1)\},$$
$$[J_2E_l, J_3E_m] = [J_3E_l, J_2E_m] = J_1E_n \ \text{ for } \ (l,m,n) \in \{(1,2,3),(3,2,1)\}.$$

Now we introduce a basis $\{K_{lp} : l,p = 1,2,3\}$ for $\mathfrak{h}$. More explicitly, we take

$$K_{1l} = \begin{pmatrix} j_l & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ K_{2l} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & j_l & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ K_{3l} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & j_l \end{pmatrix}, l = 1,2,3.$$

Then, we get

$$
\begin{aligned}
[E_1, J_pE_1] =& 2(K_{1p} - K_{2p}) \text{ for } p = 1,2,3,\\
[J_pE_1, J_qE_1] =& 2(K_{1r} - K_{2r}) \text{ for } (p,q,r) \in \{(1,2,3),(2,3,1),(3,1,2)\},
\end{aligned}
$$

$$
\begin{aligned}
[E_2, J_pE_2] =& 2(-K_{1p} + K_{3p}) \text{ for } p = 1,2,\\
[E_2, J_3E_2] =& 2(K_{13} - K_{33}),\\
[J_pE_2, J_qE_2] =& 2(K_{1r} + K_{3r}) \text{ for } (p,q,r) \in \{(1,2,3),(1,3,2),(3,2,1)\},
\end{aligned}
$$

$$
\begin{aligned}
[E_3, J_pE_3] =& 2(K_{2p} - K_{3p}) \text{ for } p = 1,2,3,\\
[J_pE_3, J_qE_3] =& 2(K_{2r} - K_{3r}) \text{ for } (p,q,r) \in \{(1,2,3),(2,3,1),(3,1,2)\},
\end{aligned}
$$

$$[E_1, K_{1p}] = -[E_1, K_{2p}] = -J_pE_1,\ [E_1, K_{3p}] = 0 \text{ for } p = 1,2,3,$$

$$
\begin{aligned}
&[E_2, K_{1p}] = -[E_2, K_{3p}] = J_pE_2,\ [E_2, K_{2p}] = 0 \text{ for } p = 1,2,\\
&[E_2, K_{13}] = -[E_2, K_{33}] = -\ J_3E_2,\ [E_2, K_{23}] = 0,
\end{aligned}
$$

$$[E_3, K_{2p}] = -[E_1, K_{3p}] = -J_pE_3,\ [E_3, K_{1p}] = 0 \text{ for } p = 1,2,3,$$

$$[J_pE_1, K_{1p}] = -[J_pE_1, K_{2p}] = E_1,\ [J_pE_1, K_{3p}] = 0 \text{ for } p = 1,2,3,$$

$$[J_pE_2, K_{1p}] = -[J_pE_2, K_{3p}] = -E_2,\ [J_pE_2, K_{2p}] = 0 \text{ for } p = 1,2,3,$$

$$[J_pE_3, K_{2p}] = -[J_pE_3, K_{3p}] = E_3,\ [J_pE_3, K_{1p}] = 0 \text{ for } p = 1,2,3,$$

$$
\begin{gathered}
[J_pE_1, K_{lq}] = -[J_qE_1, K_{lp}] = J_rE_1,\ l = 1,2,\ [J_pE_1, K_{3q}] = [J_qE_1, K_{3p}] = 0,\\
\text{for } (p,q,r) \in \{(1,2,3),(2,3,1),(3,1,2)\},
\end{gathered}
$$

$$
\begin{aligned}
&[J_pE_2, K_{2q}] = [J_qE_2, K_{2p}] = 0, \quad \text{for } (p,q) \in \{(1,2),(1,3),(2,3)\},\\
&[J_2E_2, K_{l1}] = -[J_1E_2, K_{l2}] = J_3E_2,\ l = 1,3,\\
&[J_2E_2, K_{l3}] = [J_3E_2, K_{l2}] = J_1E_2,\ l = 1,3,\\
&[J_1E_2, K_{l3}] = [J_3E_2, K_{l1}] = -J_2E_2,\ l = 1,3,
\end{aligned}
$$

$$
\begin{gathered}
[J_pE_3, K_{lq}] = -[J_qE_3, K_{lp}] = J_rE_3,\ l = 2,3,\ [J_pE_3, K_{1q}] = [J_qE_3, K_{1p}] = 0,\\
\text{for } (p,q,r) \in \{(1,2,3),(2,3,1),(3,1,2)\}.
\end{gathered}
$$

The curvature tensor can be computed from (1.7) with respect to this basis. Let us denote by J_0 the identity and let us put $A = c_1^2 + (c_2 - c_3)^2 - 2c_1(c_2 + c_3)$. The non-trivial cases are the following formulas

$$
\begin{aligned}
\mathcal{R}(J_qE_l, J_pE_l)J_pE_l =& 4J_qE_l,\ p \neq q,\\
\mathcal{R}(J_qE_l, J_pE_m)J_pE_m =& \tfrac{-3c_n^2 + (c_l - c_m)^2 + 2c_n(c_l + c_m)}{4c_lc_n} J_qE_l,\\
\text{for distinct } & l,m,n \in \{1,2,3\},\ p,q \in \{0,1,2,3\},
\end{aligned}
$$

$$\begin{aligned}\mathcal{R}(E_l,E_m)J_pE_m =& -\mathcal{R}(E_l,J_pE_m)E_m = -J_p(\mathcal{R}(J_pE_l,E_m)J_pE_m)\\ =&J_p(\mathcal{R}(J_pE_l,J_pE_m)E_m) = \tfrac{A}{4c_lc_n}J_pE_l,\ p=1,2,\\ \mathcal{R}(E_l,E_m)J_3E_m =& -\mathcal{R}(E_l,J_3E_m)E_m = -J_3(\mathcal{R}(J_3E_l,E_m)J_3E_m)\\ =&J_3(\mathcal{R}(J_3E_l,J_3E_m)E_m) = \tfrac{(-1)^{l+m}A}{4c_lc_n}J_3E_l,\\ &\text{for distinct } \ l,m,n \in\{1,2,3\},\end{aligned}$$

$$\begin{aligned}\mathcal{R}(E_l,J_pE_m)J_qE_m =& -\mathcal{R}(E_l,J_qE_m)J_pE_m = \tfrac{(-1)^{(p+r+n!+1)}A}{4c_lc_n}J_rE_l,\\ &(l,m,n)\in\{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\},\\ \mathcal{R}(E_l,J_pE_m)J_qE_m =& -\mathcal{R}(E_l,J_qE_m)J_pE_m = \tfrac{(-1)^{(l!+1)}A}{4c_lc_n}J_rE_l,\\ &(l,m,n)\in\{(1,3,2),(3,1,2)\},\\ &\text{for } \ (p,q,r)\in\{(1,2,3),(2,3,1),(3,1,2)\},\end{aligned} \tag{1.20}$$

$$\begin{aligned}\mathcal{R}(J_pE_l,E_m)J_qE_m =& -\mathcal{R}(J_pE_l,J_qE_m)E_m = \tfrac{(-1)^{(q+r+n!)}A}{4c_lc_n}J_rE_l,\\ \mathcal{R}(J_rE_l,E_m)J_qE_m =& -\mathcal{R}(J_rE_l,J_qE_m)E_m = \tfrac{(-1)^{(q+r+n!+1)}A}{4c_lc_n}J_pE_l,\\ &(l,m,n)\in\{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\},\\ \mathcal{R}(J_pE_l,E_m)J_qE_m =& -\mathcal{R}(J_pE_l,J_qE_m)E_m = \tfrac{(-1)^{(l!+1)}A}{4c_lc_n}J_rE_l,\\ \mathcal{R}(J_rE_l,E_m)J_qE_m =& -\mathcal{R}(J_rE_l,J_qE_m)E_m = \tfrac{(-1)^{(l!)}A}{4c_lc_n}J_pE_l,\\ &(l,m,n)\in\{(1,3,2),(3,1,2)\},\\ &\text{for } \ (p,q,r)\in\{(1,2,3),(2,3,1),(3,1,2)\},\end{aligned}$$

$$\begin{aligned}\mathcal{R}(J_pE_l,J_qE_m)J_pE_m =& -\mathcal{R}(J_pE_l,J_pE_m)J_qE_m = \tfrac{(-1)^{q}A}{4c_lc_n}J_qE_l,\\ &(l,m,n)\in\{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\},\\ \mathcal{R}(J_pE_l,J_qE_m)J_pE_m =& -\mathcal{R}(J_pE_l,J_pE_m)J_qE_m = \tfrac{A}{4c_lc_n}J_qE_l,\\ &(l,m,n)\in\{(1,3,2),(3,1,2)\},\\ &\text{for distinct } \ p,q\in\{1,2,3\},\end{aligned}$$

$$\begin{aligned}\mathcal{R}(J_pE_l,J_qE_m)J_rE_m =& -\mathcal{R}(J_pE_l,J_rE_m)J_qE_m = \tfrac{(-1)^{(r+n!)}A}{4c_lc_n}E_l,\\ &(l,m,n)\in\{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\},\\ \mathcal{R}(J_pE_l,J_qE_m)J_rE_m =& -\mathcal{R}(J_pE_l,J_rE_m)J_qE_m = \tfrac{(-1)^{l!}A}{4c_lc_n}E_l,\\ &(l,m,n)\in\{(1,3,2),(3,1,2)\},\\ &\text{for } \ (p,q,r)\in\{(1,2,3),(2,3,1),(3,1,2)\},\end{aligned}$$

$$\begin{aligned}
\mathcal{R}(E_l, J_pE_l)J_qE_m &= \tfrac{(-1)^{(r+n!)}A}{2c_mc_n} J_rE_m,\\
\mathcal{R}(E_l, J_pE_l)J_rE_m &= \tfrac{(-1)^{(r+n!+1)}A}{2c_mc_n} J_qE_m,\\
&\quad (l,m,n) \in \{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\},\\
\mathcal{R}(E_l, J_pE_l)J_qE_m &= \tfrac{(-1)^{l!}A}{2c_mc_n} J_rE_m,\\
\mathcal{R}(E_l, J_pE_l)J_rE_m &= \tfrac{(-1)^{(l!+1)}A}{2c_mc_n} J_qE_m,\\
&\quad (l,m,n) \in \{(1,3,2),(3,1,2)\},
\end{aligned}$$

for $(p,q,r) \in \{(1,2,3),(2,3,1),(3,1,2)\}$,

$$\begin{aligned}
\mathcal{R}(E_l, J_pE_l)E_m &= -\,J_p(\mathcal{R}(E_l, J_pE_l)J_pE_m) = \tfrac{-A}{2c_mc_n} J_pE_m, \ p = 1,2,\\
\mathcal{R}(E_l, J_3E_l)E_m &= -\,J_3(\mathcal{R}(E_l, J_3E_l)J_3E_m) = \tfrac{(-1)^{(l+m+1)}A}{2c_mc_n} J_3E_m,
\end{aligned}$$

for distinct $l,m,n \in \{1,2,3\}$,

$$\begin{aligned}
\mathcal{R}(J_pE_l, J_qE_l)E_m &= -\,J_r(\mathcal{R}(J_pE_l, J_qE_l)J_rE_m) = \tfrac{(-1)^{(q+n!)}A}{2c_mc_n} J_rE_m,\\
&\quad (l,m,n) \in \{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\},\\
\mathcal{R}(J_pE_l, J_qE_l)E_m &= -\,J_r(\mathcal{R}(J_pE_l, J_qE_l)J_rE_m) = \tfrac{(-1)^{(m!)}A}{2c_mc_n} J_rE_m,\\
&\quad (l,m,n) \in \{(1,3,2),(3,1,2)\},
\end{aligned}$$

for $(p,q,r) \in \{(1,2,3),(2,3,1),(3,1,2)\}$,

$$\begin{aligned}
\mathcal{R}(J_pE_l, J_qE_l)J_pE_m &= \tfrac{(-1)^{r}A}{2c_mc_n} J_qE_m, \ r = \max(\{p,q\}),\\
&\quad (l,m,n) \in \{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\},\\
\mathcal{R}(J_pE_l, J_qE_l)J_pE_m &= \tfrac{A}{2c_mc_n} J_qE_m,\\
&\quad (l,m,n) \in \{(1,3,2),(3,1,2)\},
\end{aligned}$$

for distinct $p,q \in \{1,2,3\}$,

Further, we obtain easily from (1.20) that the only non-trivial terms of the Ricci tensor are

$$\rho(E_l,E_l) = \rho(J_pE_l, J_pE_l) = \tfrac{2(8c_mc_n + c_l^2 - c_m^2 - c_n^2)}{c_mc_n} \tag{1.21}$$

for l,m,n distinct and $p,l,m,n \in \{1,2,3\}$.

Now, we shall use (1.13) and (1.21) to compute the Ledger condition L_3, (1.10). The equation (1.10) has a purely algebraic character because the family of metrics $g_{(c_1,c_2,c_3)}$ is left-invariant. Hence, we can substitute for X,Y,Z every triplet chosen from the basis of $\mathfrak{m}$ (with possible repetition). Thus, the condition (1.10) is equivalent to a system of algebraic equations. Finally, we have obtained after a lengthy by routine calculation, that the only non-trivial equation appears when

$$\begin{aligned}
(X,Y,Z) \in \{&(E_l,E_m,E_n),(E_l,J_pE_m,J_pE_n),(J_lE_l,J_mE_m,J_nE_n),(J_lE_l,J_nE_m,J_mE_n),\\
&(J_nE_l,J_lE_m,J_mE_n) \mid p,l,m,n \in \{1,2,3\}, l \neq m, l \neq n, m \neq n\}.
\end{aligned}$$

To be precise, the L_3 condition is equivalent to

$$\frac{(c_1-c_2)(c_1-c_3)(c_2-c_3)}{c_1c_2c_3}=0. \tag{1.22}$$

We conclude that *every member of the family of Riemannian flag manifolds* $(M^{12}, g_{(c_1,c_2,c_3)})$ *is of type* $\mathcal{A}$ *if and only if at least two of the parameters* c_1, c_2, c_3, *are equal.*

To finish, we shall prove that *the* L_5 *Ledger condition is satisfied if and only if* $c_1 = c_2 = c_3$.

Case $c_1 = c_l,\ \ l = 2, 3$.

Let us put $X = E_2, Y = E_3, Z = V = W = E_1$ in (1.11). Thus, for $l = 2$ we obtain using (1.12), (1.13) and (1.20) that (1.11) can be written in the form

$$(x-1)(9x^2+48x+112)=0, \quad \text{for} \quad x=\tfrac{c_3}{c_1}. \tag{1.23}$$

Analogously for $l = 3$, we obtain that (1.11) can be written in the form

$$(x-1)(x^2+3x+36)=0, \quad \text{for} \quad x=\tfrac{c_2}{c_1}. \tag{1.24}$$

In both equations (1.23), (1.24), the second order equation has negative discriminant. Then, if $c_1 = c_l$, $l = 2, 3$, the only possible real solution is $c_1 = c_2 = c_3$.

Case $c_2 = c_3$.

Let us put in (1.11) first $X = E_2, Y = J_1E_3, Z = W = E_1, V = J_1E_1$ and later $X = E_2, Y = E_3, Z = V = W = E_1$. Thus, we obtain a system of equations of the form

$$\begin{aligned}&(x-1)(x-4)(3x^2-6x+4)=0,\\&(x-1)(7x^2-46x+48)=0,\end{aligned} \tag{1.25}$$

respectively, where $x = \frac{c_1}{c_2}$. Here, the only solution of the system is $x = 1$. Then, if $c_2 = c_3$, the only possible solution is $c_1 = c_2 = c_3$.

As a conclusion, *every member of the family of Riemannian flag manifolds* $(M^{12}, g_{(c_1,c_2,c_3)})$ *is a D'Atri space if and only if it is naturally reductive.*

Chapter 2

Classification of 4-dimensional D'Atri spaces

The classification of all 3-dimensional D'Atri spaces is well-known (see [K]). All of them are locally naturally reductive. The first attempts to classify all 4-dimensional *homogeneous* D'Atri spaces were done in the papers [Po-Sp] and [Bu-V] (which are mutually complementary). The previous authors started with the corresponding classification of all spaces of type $\mathcal{A}$, but the classification in [Po-Sp] was incomplete, as we claim in [AM].

In this Chapter we shall present the complete classification of all 4-dimensional homogeneous spaces of type $\mathcal{A}$ in a simple and explicit form and, as a consequence, we prove correctly that all 4-dimensional homogeneous D'Atri spaces are locally naturally reductive. Moreover, Section 2.3 is devoted to correct Podesta and Spiro's Classification Theorem given in [Po-Sp].

To facilitate access to the individual topics, this chapter is rendered as self-contained as possible. Anyway, see Chapter 1 for more details and the basic references about the main topics.

The results of this chapter will be published in [AM-K].

2.1 Preliminaries and Classification Theorem

A D'Atri space is defined as a Riemannian manifold (M, g) whose local geodesic symmetries are volume-preserving. D'Atri and Nickerson proved that every naturally reductive Riemannian manifold has this property. However, it is still an open problem whether all D'Atri spaces are locally homogeneous, even in the four-dimensional case.

Let us recall that the property of being a D'Atri space is equivalent to the infinite number of curvature identities called the *odd Ledger conditions* L_{2k+1}, $k \geq 1$. In particular, the two first non-trivial odd Ledger conditions are

$$L_3 : (\nabla_X \rho)(X, X) = 0,$$

$$L_5 : \sum_{a,b=1}^{n} \mathcal{R}_{XE_aXE_b}(\nabla_X \mathcal{R})_{XE_aXE_b} = 0,$$

where X is any tangent vector at any point $m \in M$ and $\{E_1, ..., E_n\}$ is any orthonormal basis of T_mM. Here $\mathcal{R}$ denotes the curvature tensor and ρ the Ricci tensor of (M, g), respectively, and $n = dimM$.

Thus, it is natural to start with the investigation of all homogeneous Riemannian 4-manifolds satisfying the simplest Ledger condition L_3, which is the first approximation of the D'Atri property. This condition is called in [Po-Sp] the "class $\mathcal{A}$ condition".

Likewise, from the L_3 condition and the contracted Bianchi identity in [D'A-N.1] has been proved that D'Atri spaces have constant scalar curvature. Hence in dimension 2, L_3 alone forces M to be a space of constant curvature and therefore locally symmetric (and thus, in particular, a D'Atri space). In dimension 3, O. Kowalski [K] found explicitly all the 3-dimensional D'Atri spaces using both Ledger conditions L_3, L_5 and the real analyticity condition. Some years later, H. Pedersen and P. Tod ([P-T]) eliminated L_5 from the argument with the following result:

Theorem 2.1.1. *All three-dimensional smooth Riemannian manifolds belonging to class $\mathcal{A}$ are locally homogeneous, and they are either locally symmetric or locally isometric to a naturally reductive spaces.*

Sketch of the proof: First by differentiations and other manipulations with the condition L_3 it is showed that all scalar invariants of the Ricci tensor are constant. By a result of [Pr-Tr-V] this implies that (M, g) is locally homogeneous. Local homogeneity and L_3 imply that (M, g) is naturally reductive and therefore a D'Atri space by a result of [Ab-Ga-V].

Therefore, Theorem 2.1.1 shows that L_3 alone is enough to force both homogeneity and the D'Atri property in dimension 3. Moreover in [P-T], the authors have also proved that in dimension 5 and higher, the condition L_3 alone forces neither homogeneity nor the D'Atri property. For dimension 4, we refer to [T] where K. P. Tod has proved by the spinor calculus that the condition L_3 alone forces neither homogeneity nor the D'Atri property, except in some particular cases. We shall devote this Chapter to classification of 4-dimensional homogeneous D'Atri spaces (not using the spinor calculus). In fact, we shall rectify the incomplete procedure made in [Po-Sp] and [Bu-V].

As the first and most extensive step of our classification procedure, we shall look for all 4-dimensional homogeneous spaces of class $\mathcal{A}$. In this direction, F. Podesta and A. Spiro ([Po-Sp]) published a classification theorem assuming that at most three of the constant Ricci eigenvalues are distinct. In their paper, (M, g) was not necessarily homogeneous but only curvature homogeneous, which is a more general situation. (See Section 1.1). Yet, there was a gap in their main Theorem, which we explain later. (See Section 2.3). They also pose the question if there are, in dimension four, spaces of class $\mathcal{A}$ with four distinct Ricci eigenvalues. Some years later, P. Bueken and L. Vanhecke ([Bu-V]) found a two-parameter family of such spaces. Yet, their presentation of this family was not explicit and without geometrical interpretation (they refer only to the computer results, which were not accessible). They also concluded in [Bu-V] that all simply connected homogeneous D'Atri spaces in dimension 4 are naturally reductive. Yet this final result was also not completely satisfactory just because of the gap in [Po-Sp], and because the new family of spaces in [Bu-V] was not described explicitly.

In this chapter, we derive the correct and complete local classification of all 4-dimensional homogeneous spaces of type $\mathcal{A}$ in a simple and explicit form. Our method is based on the classification of Riemannian homogeneous 4-spaces by L. Bérard Bergery ([Be]) and on the computer support, using the program Mathematica 5.0.

We shall now formulate our fundamental result, which will be proved in the next section.

Theorem 2.1.2 (Classification Theorem)**.** *Let (M, g) be a four-dimensional homogeneous Riemannian manifold of type $\mathcal{A}$. Then just the following five cases occur:*

i) *M is locally symmetric;*

ii) *(M, g) is locally isometric to a Riemannian product $M^3 \times \mathbb{R}$, where M^3 is a 3-dimensional Riemannian naturally reductive space with two distinct Ricci curvatures $(\rho_1, \rho_2 = \rho_1, \rho_3)$, $\rho_3 \neq \rho_1$. Thus M is locally isometric to a naturally reductive homogeneous space.*

iii) *(M, g) is locally isometric to a simply connected Lie group (G, g_γ), whose Lie algebra $\mathfrak{g}$ is described by*

$$[e_2, e_1] = e_2, \ [e_1, e_3] = e_3, \ [e_2, e_3] = e_4,$$

$$[e_1, e_4] = [e_2, e_4] = [e_3, e_4] = 0,$$

and which is endowed with the left-invariant metric

$$g_\gamma = \frac{4}{\gamma^2} w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3 + \gamma^2 w^4 \otimes w^4,$$

where $\gamma \in \mathbb{R}^+$ and $\{w^i\}$ is the dual basis of $\{e_i\}$. The metrics g_γ have Ricci eigenvalues $\rho_1 = \rho_2 = \rho_3 = \frac{-\gamma^2}{2}$, $\rho_4 = \frac{\gamma^2}{2}$ and are not isometric to each other for different values of γ. Moreover, the Riemannian manifolds (G, g_γ) are irreducible and not locally symmetric. They are not D'Atri spaces.

iv) *(M, g) is locally isometric to a simply connected Lie group $(G, g_{(c,k)})$, whose Lie algebra $\mathfrak{g}$ is described by*

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = \tfrac{A_+}{4} e_2, \quad [e_2, e_3] = \tfrac{A_-}{4} e_1,$$

$$[e_1, e_4] = 0, \quad [e_2, e_4] = 0, \quad [e_3, e_4] = 0,$$

where $A_\pm = 3 - 3k^2 \pm \sqrt{1 + 2k^2 - 3k^4} \geq 0$, $k \in]0, 1[\setminus \{\sqrt{\frac{5}{21}}\}$, *and which is endowed with the left-invariant metric*

$$g_{(c,k)} = \frac{1}{c^2}(w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3 + kw^3 \otimes w^4 + w^4 \otimes w^4),$$

where $\{w^i\}$ *is the dual basis of* $\{e_i\}$ *and* $c \in \mathbb{R}^+$ *is another parameter. The metrics* $g_{(c,k)}$ *have four distinct Ricci eigenvalues*

$$\begin{aligned} \rho_1 &= \tfrac{c^2}{8}(2 - 6k^2 - \sqrt{1 + 2k^2 - 3k^4}), \quad & \rho_2 &= \tfrac{c^2}{8}(2 - 6k^2 + \sqrt{1 + 2k^2 - 3k^4}), \\ \rho_3 &= \tfrac{c^2}{16}(3 - 3k^2 - \sqrt{9 - 2k^2 + 57k^4}), \quad & \rho_4 &= \tfrac{c^2}{16}(3 - 3k^2 + \sqrt{9 - 2k^2 + 57k^4}). \end{aligned}$$

Moreover, the Riemannian manifolds $(G, g_{(c,k)})$ *are irreducible and not locally symmetric. They are not D'Atri spaces.*

v) (M, g) *is locally isometric to a simply connected Lie group* (G, g_c), *whose Lie algebra* $\mathfrak{g}$ *is described by*

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = \tfrac{6}{7}e_2, \quad [e_2, e_3] = \tfrac{2}{7}e_1,$$

$$[e_1, e_4] = 0, \quad [e_2, e_4] = 0, \quad [e_3, e_4] = 0,$$

and which is endowed with the left-invariant metric

$$g_c = \frac{1}{c^2}(w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3 + \sqrt{\tfrac{5}{21}}w^3 \otimes w^4 + w^4 \otimes w^4),$$

where $c \in \mathbb{R}^+$ *and* $\{w^i\}$ *is the dual basis of* $\{e_i\}$. *The metrics* g_c *have Ricci eigenvalues* $\rho_1 = \rho_3 = \frac{-c^2}{14}$, $\rho_2 = \frac{3c^2}{14}$, $\rho_4 = \frac{5c^2}{14}$ *and are not isometric to each other for different values of c. Moreover, the Riemannian manifolds* (G, g_c) *are irreducible and not locally symmetric. They are not D'Atri spaces.*

It is well-known that every locally symmetric space is a D'Atri space and that, moreover, it is locally isometric to a naturally reductive homogeneous space. In addition, the Riemannian product spaces $M^3 \times \mathbb{R}$ described in *ii*) of the Classification Theorem 2.1.2 are locally isometric to naturally reductive homogeneous spaces and hence they are D'Atri spaces. On the other hand, we show that the spaces described in *iii*), *iv*) and *v*) do not satisfy the Ledger condition L_5 and thus, they cannot be D'Atri spaces. Combining these results with our Classification Theorem 2.1.2, we conclude with the following

Theorem 2.1.3 (Main Theorem)**.** *In dimension 4, all simply connected homogeneous D'Atri spaces are naturally reductive spaces (including symmetric spaces as special cases).*

2.2 Proof of the Classification Theorem

In [Be], L. Bérard Bergery published the classification of Riemannian homogeneous 4-spaces. In particular, he obtained the following

Proposition 2.2.1. *In dimension 4, each simply connected Riemannian homogeneous space M is either symmetric or isometric to a Lie group with a left-invariant metric. In the second case, either M is a solvable group or it is one of the groups $SU(2)\times\mathbb{R}$, $\widetilde{Sl(2,\mathbb{R})}\times\mathbb{R}$.*

Now, the main part of our computations is to check which of these spaces are of type $\mathcal{A}$. We shall work at the Lie algebra level and use Mathematica 5.0 for the computation. Let us start with the non-solvable group case and later we shall continue with the solvable case.

2.2.1 Non-Solvable Case (Study of $SU(2)\times\mathbb{R}$ and $\widetilde{Sl(2,\mathbb{R})}\times\mathbb{R}$)

Let $\mathfrak{g}_3$ be a unimodular Lie algebra with a scalar product $\langle\ ,\ \rangle_3$. According to [M], p. 305, there is an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathfrak{g}_3$ such that

$$[f_2, f_3] = af_1,\ \ [f_3, f_1] = bf_2,\ \ [f_1, f_2] = cf_3, \tag{2.1}$$

where a, b, c are real numbers. In the following, we shall study the cases $\mathfrak{g}_3 = \mathfrak{su}(2)$ and $\mathfrak{g}_3 = \mathfrak{sl}(2,\mathbb{R})$, which are characterized by the inequality $abc \neq 0$.

Let now $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}$ be a direct sum, and $\langle\ ,\ \rangle$ a scalar product on $\mathfrak{g}$ defined as follows: we choose a basis $\{f_1, f_2, f_3, f_4\}$ of unit vectors such that $\{f_1, f_2, f_3\}$ is an orthonormal basis of $\mathfrak{g}_3$ satisfying (2.1) and f_4 spans $\mathbb{R}$. Here $\mathbb{R}$ need not be orthogonal to $\mathfrak{g}_3$. In particular, we assume

$$[f_i, f_4] = 0, \quad \langle f_i, f_4\rangle = k_i,\ i = 1, 2, 3. \tag{2.2}$$

Here a, b, c, k_1, k_2, k_3 are arbitrary parameters where $\sum\limits_{i=1}^{3} k_i^2 < 1$ due to the positivity of the scalar product. Choosing a convenient orientation of f_4, we can always assume that $k_3 \geq 0$.

Now we replace the basis $\{f_i\}$ by the new basis $\{e_i\}$, $(i = 1, 2, 3, 4)$, putting

$$e_i = f_i,\ i = 1, 2, 3, \quad e_4 = \frac{1}{R}(f_4 - \sum_{i=1}^{3} k_i f_i) \tag{2.3}$$

where $R = \sqrt{1 - \sum\limits_{i=1}^{3} k_i^2} > 0$. Then we get an orthonormal basis for which

$$[e_2, e_3] = ae_1, \quad [e_3, e_1] = be_2, \quad [e_1, e_2] = ce_3,$$

$$[e_1,e_4]=\tfrac{1}{R}\left(k_3be_2-k_2ce_3\right),\quad [e_2,e_4]=\tfrac{1}{R}\left(k_1ce_3-k_3ae_1\right),\qquad (2.4)$$
$$[e_3,e_4]=\tfrac{1}{R}\left(k_2ae_1-k_1be_2\right).$$

Next, we shall consider the simply connected Lie group G with a left invariant Riemannian metric g corresponding to the Lie algebra $\mathfrak{g}$ and the scalar product $\langle\ ,\ \rangle$ on it. Here the vectors e_i determine some left-invariant vector fields on G.

According to our construction, the underlying group G is the direct product of the group $SU(2)$ or $\widetilde{Sl(2,\mathbb{R})}$ and the multiplicative group $\mathbb{R}^+$.

Now we are going to calculate the expression for the Levi-Civita connection, the curvature tensor, the Ricci matrix, and the condition for the Ricci tensor to be cyclic parallel.

We know that

$$\begin{aligned}2g(\nabla_X Z,Y)=&Zg(X,Y)+Xg(Y,Z)-Yg(Z,X)\\ &-g([Z,X],Y)-g([X,Y],Z)+g([Y,Z],X)\end{aligned}\qquad (2.5)$$

for every triplet (X,Y,Z) of vectors fields. Then using this formula we obtain by easy calculation

Lemma 2.2.2.

$$\nabla_{e_i}e_i=0,\ i=1,2,3,4,$$
$$\begin{aligned}
\nabla_{e_1}e_2&=\tfrac{(c+b-a)}{2}e_3+\tfrac{(a-b)k_3}{2R}e_4,\ \nabla_{e_2}e_1=\tfrac{(b-a-c)}{2}e_3+\tfrac{(a-b)k_3}{2R}e_4,\\
\nabla_{e_1}e_3&=\tfrac{(a-b-c)}{2}e_2+\tfrac{(c-a)k_2}{2R}e_4,\ \nabla_{e_3}e_1=\tfrac{(a+b-c)}{2}e_2+\tfrac{(c-a)k_2}{2R}e_4,\\
\nabla_{e_1}e_4&=\tfrac{(b-a)k_3}{2R}e_2+\tfrac{(a-c)k_2}{2R}e_3,\ \nabla_{e_4}e_1=\tfrac{-(b+a)k_3}{2R}e_2+\tfrac{(a+c)k_2}{2R}e_3,\\
\nabla_{e_2}e_3&=\tfrac{(a+c-b)}{2}e_1+\tfrac{(b-c)k_1}{2R}e_4,\ \nabla_{e_3}e_2=\tfrac{(c-a-b)}{2}e_1+\tfrac{(b-c)k_1}{2R}e_4,\\
\nabla_{e_2}e_4&=\tfrac{(b-a)k_3}{2R}e_1+\tfrac{(c-b)k_1}{2R}e_3,\ \nabla_{e_4}e_2=\tfrac{(a+b)k_3}{2R}e_1-\tfrac{(b+c)k_1}{2R}e_3,\\
\nabla_{e_3}e_4&=\tfrac{(a-c)k_2}{2R}e_1+\tfrac{(c-b)k_1}{2R}e_2,\ \nabla_{e_4}e_3=\tfrac{-(a+c)k_2}{2R}e_1+\tfrac{(b+c)k_1}{2R}e_2.
\end{aligned}\qquad (2.6)$$

Now, we denote by A_{ij} the elementary skew-symmetric operators whose corresponding action is given by the formulas $A_{ij}(e_l)=\delta_{il}e_j-\delta_{jl}e_i$. Then, by a lengthy but elementary calculation we get

Lemma 2.2.3. *The components of the curvature operator are*

$$\begin{aligned}
\mathcal{R}(e_1,e_2)&=\alpha_{1212}A_{12}+\alpha_{1213}A_{13}+\alpha_{1214}A_{14}+\alpha_{1223}A_{23}+\alpha_{1224}A_{24},\\
\mathcal{R}(e_1,e_3)&=\alpha_{1312}A_{12}+\alpha_{1313}A_{13}+\alpha_{1314}A_{14}+\alpha_{1323}A_{23}+\alpha_{1334}A_{34},\\
\mathcal{R}(e_1,e_4)&=\alpha_{1412}A_{12}+\alpha_{1413}A_{13}+\alpha_{1414}A_{14}+\alpha_{1424}A_{24}+\alpha_{1434}A_{34},\\
\mathcal{R}(e_2,e_3)&=\alpha_{2312}A_{12}+\alpha_{2313}A_{13}+\alpha_{2323}A_{23}+\alpha_{2324}A_{24}+\alpha_{2334}A_{34},\\
\mathcal{R}(e_2,e_4)&=\alpha_{2412}A_{12}+\alpha_{2414}A_{14}+\alpha_{2423}A_{23}+\alpha_{2424}A_{24}+\alpha_{2434}A_{34},\\
\mathcal{R}(e_3,e_4)&=\alpha_{3413}A_{13}+\alpha_{3414}A_{14}+\alpha_{3423}A_{23}+\alpha_{3424}A_{24}+\alpha_{3434}A_{34},
\end{aligned}\qquad (2.7)$$

where the coeficients $\alpha_{ijlm} = g(\mathcal{R}(e_i, e_j)e_l, e_m)$ satisfy the standard symmetries with respect to their indices and

$$\begin{aligned}
\alpha_{1212} &= \tfrac{1}{4R^2}((3c^2 - (a-b)^2 - 2c(a+b))R^2 - (a-b)^2k_3^2),\\
\alpha_{1213} &= \tfrac{1}{4R^2}((a-b)(a-c)k_2k_3),\\
\alpha_{1214} &= \tfrac{1}{4R}((a-c)(a-b+3c)k_2),\\
\alpha_{1223} &= \tfrac{1}{4R^2}((a-b)(b-c)k_1k_3),\\
\alpha_{1224} &= \tfrac{1}{4R}((b-c)(a-b-3c)k_1),\\
\alpha_{1313} &= \tfrac{1}{4R^2}((3b^2 - (a-c)^2 - 2b(a+c))R^2 - (a-c)^2k_2^2),\\
\alpha_{1314} &= \tfrac{1}{4R}((a-b)(a-c+3b)k_3),\\
\alpha_{1323} &= \tfrac{1}{4R^2}((a-c)(b-c)k_1k_2),\\
\alpha_{1334} &= \tfrac{1}{4R}((c-b)(c-a+3b)k_1),\\
\alpha_{1414} &= \tfrac{1}{4R^2}((4c^2 - (a+c)^2)k_2^2 + (4b^2 - (a+b)^2)k_3^2),\\
\alpha_{1424} &= \tfrac{1}{4R^2}((c(a+b-3c) + ab)k_1k_2),\\
\alpha_{1434} &= \tfrac{1}{4R^2}((b(a+c-3b) + ac)k_1k_3),\\
\alpha_{2323} &= \tfrac{1}{4R^2}((3a^2 - (b-c)^2 - 2a(b+c))R^2 - (b-c)^2k_1^2),\\
\alpha_{2324} &= \tfrac{1}{4R}((b-a)(3a+b-c)k_3),\\
\alpha_{2334} &= \tfrac{1}{4R}((a-c)(3a-b+c)k_2),\\
\alpha_{2424} &= \tfrac{1}{4R^2}((4c^2 - (b+c)^2)k_1^2 + (4a^2 - (a+b)^2)k_3^2),\\
\alpha_{2434} &= \tfrac{1}{4R^2}((a(-3a+b+c) + bc)k_2k_3),\\
\alpha_{3434} &= \tfrac{1}{4R^2}((4b^2 - (b+c)^2)k_1^2 + (4a^2 - (a+c)^2)k_2^2).
\end{aligned} \tag{2.8}$$

Further, we obtain easily

Lemma 2.2.4. *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1, e_2, e_3, e_4\}$ *is of the form*

$$\begin{pmatrix}
\beta_{11} & \frac{(c^2-ab)k_1k_2}{2R^2} & \frac{(b^2-ac)k_1k_3}{2R^2} & \frac{(b-c)^2k_1}{2R}\\
\frac{(c^2-ab)k_1k_2}{2R^2} & \beta_{22} & \frac{(a^2-bc)k_2k_3}{2R^2} & \frac{(a-c)^2k_2}{2R}\\
\frac{(b^2-ac)k_1k_3}{2R^2} & \frac{(a^2-bc)k_2k_3}{2R^2} & \beta_{33} & \frac{(a-b)^2k_3}{2R}\\
\frac{(b-c)^2k_1}{2R} & \frac{(a-c)^2k_2}{2R} & \frac{(a-b)^2k_3}{2R} & \beta_{44}
\end{pmatrix} \tag{2.9}$$

where

$$\beta_{11} = \tfrac{a^2-(b-c)^2}{2} + \tfrac{(a^2-b^2)k_3^2+(a^2-c^2)k_2^2}{2R^2},\ \beta_{22} = \tfrac{b^2-(a-c)^2}{2} + \tfrac{(b^2-a^2)k_3^2+(b^2-c^2)k_1^2}{2R^2},$$

$$\beta_{33} = \tfrac{c^2-(a-b)^2}{2} + \tfrac{(c^2-a^2)k_2^2+(c^2-b^2)k_1^2}{2R^2},\ \beta_{44} = \tfrac{-(b-c)^2k_1^2-(a-c)^2k_2^2-(a-b)^2k_3^2}{2R^2}.$$

Next, the condition for the metric g on G to be cyclic parallel (i.e. of type $\mathcal{A}$) is

$$(\nabla_X\rho)(Y,Z) + (\nabla_Y\rho)(Z,X) + (\nabla_Z\rho)(X,Y) = 0 \tag{2.10}$$

for every triplet (X, Y, Z) of vectors fields, where ρ is the Ricci tensor of type $(0, 2)$. This equation has a purely algebraic character because the metric g is left-invariant. Hence, we can substitute for X, Y, Z every triplet chosen from the basis $\{e_1, e_2, e_3, e_4\}$ (with possible repetition).

Here we obtain, by a lengthy but routine calculation

Lemma 2.2.5. *The condition* (2.10) *for the Ricci tensor of type* $(0, 2)$ *is equivalent to the system of algebraic equations*

$$\begin{aligned}
&(1,1,2) \to k_1k_3(b-a)(a-2b+c) = 0,\\
&(1,1,3) \to k_1k_2(a-c)(a+b-2c) = 0,\\
&(2,2,1) \to k_2k_3(a-b)(2a-b-c) = 0,\\
&(2,2,3) \to k_1k_2(c-b)(a+b-2c) = 0,\\
&(3,3,1) \to k_2k_3(c-a)(2a-b-c) = 0,\\
&(3,3,2) \to k_1k_3(b-c)(2b-a-c) = 0,\\
&(4,4,1) \to k_2k_3(2a-b-c)(b-c) = 0,\\
&(4,4,2) \to k_1k_3(a-2b+c)(a-c) = 0,\\
&(4,4,3) \to k_1k_2(a+b-2c)(b-a) = 0,\\
&(1,2,3) \to 2R^2(a-b)(a-c)(b-c) + k_1^2(c-b)(a(b+c)-2bc)\\
&\qquad\qquad + k_2^2(a-c)(b(a+c)-2ac) + k_3^2(b-a)(c(a+b)-2ab) = 0,\\
&(1,2,4) \to k_3(R^2(b-a)(2a-c)(2b-c) + k_1^2 b(2ab-3ac+2bc-c^2)\\
&\qquad\qquad + k_2^2 a(3bc+c^2-2ab-2ac) + k_3^2 4ab(b-a)) = 0,\\
&(1,3,4) \to k_2(R^2(c-a)(2a-b)(b-2c) + k_1^2 c(3ab-2ac-2bc+b^2)\\
&\qquad\qquad + k_2^2 4ac(a-c) + k_3^2 a(2ab-b^2+2ac-3bc)) = 0,\\
&(2,3,4) \to k_1(R^2(b-c)(2b-a)(a-2c) + k_1^2 4bc(c-b)\\
&\qquad\qquad + k_2^2 c(2ac+2bc-a^2-3ab) + k_3^2 b(a^2-2ab+3ac-2bc)) = 0,\\
&(1,1,4) \to k_1k_2k_3(a+b+c)(c-b) = 0,\\
&(2,2,4) \to k_1k_2k_3(a+b+c)(a-c) = 0,\\
&(3,3,4) \to k_1k_2k_3(a+b+c)(b-a) = 0.
\end{aligned} \tag{2.11}$$

Here the symbol "$(\alpha, \beta, \gamma) \to$" marks the substitution of $(e_\alpha, e_\beta, e_\gamma)$ for (X, Y, Z) respectively.

Now, the goal is to find the values of a, b, c, k_1, k_2 and k_3 which satisfy the system of equations (2.11) and to study each of these cases.

Proposition 2.2.6. *The only possible solutions of the system of algebraic equations* (2.11) *are, up to a re-numeration of the triplet $\{e_1, e_2, e_3\}$ the following ones:*

1. $a = b = c \neq 0,\ k_1,\ k_2,\ k_3$ *arbitrary.*
   Here, three of the four Ricci eigenvalues are equal and $\nabla\mathcal{R} = 0$*. Hence, the corresponding spaces belong to the case* $i)$ *of the Classification Theorem 2.1.2.*

2. $a = b \neq 0,\ a \neq c \neq 0,\ k_1 = k_2 = 0,\ k_3$ *arbitrary.*
   In this situation, the corresponding spaces are Riemannian direct products $M^3 \times \mathbb{R}$*, not locally symmetric, with the Ricci eigenvalues* $\rho_1 = \rho_2 = \frac{1}{2}(2ac - c^2),\ \rho_3 = \frac{c^2}{2},\ \rho_4 = 0$*. Hence, they give the case* $ii)$ *of the Classification Theorem 2.1.2.*

3. $a = \frac{cA_-}{4},\ b = \frac{cA_+}{4},\ c \neq 0,\ k_1 = k_2 = 0,\ k_3^2 \in\ ]0,1[\setminus\left\{\frac{2}{3},\frac{5}{21}\right\}$*, and* $A_\pm = 3 - 3k_3^2 \pm \sqrt{1 + 2k_3^2 - 3k_3^4} > 0$*.*
   For this situation, $(\nabla_{e_4}\mathcal{R})(e_4, e_2)e_4 \neq 0$ *and all Ricci eigenvalues are distinct. The corresponding spaces belong to the case* $iv)$ *of the Classification Theorem 2.1.2. Moreover, the* L_5 *condition is not satisfied here.*

4. $a = \frac{2c}{7},\ b = \frac{6c}{7},\ c \neq 0,\ k_1 = k_2 = 0,\ k_3 = \sqrt{\frac{5}{21}}$*.*
   The corresponding spaces give the case $v)$ *of the Classification Theorem 2.1.2. Moreover, the* L_5 *condition is not satisfied here.*

Proof. Because we can re-numerate the basis $\{e_1, e_2, e_3\}$ in arbitrary way (which implies corresponding permutation of the symbols a, b, c and the corresponding re-numeration of the parameters k_1, k_2, k_3), the system (2.11) is *symmetric* with respect to all such permutations and re-numerations. Then, in order to solve this system of equations, we can just consider the following cases:

A. $k_1k_2k_3 \neq 0$.

B. $k_1 = k_2 = 0$, k_3 arbitrary.

C. $k_1 = 0$ and $k_2k_3 \neq 0$.

Case A. $k_1k_2k_3 \neq 0$.

We first divide the formulas $(1,1,2)$ and $(2,2,1)$ by their nonzero coefficients k_1k_3, k_2k_3 and then subtract them. We obtain the necessary condition $b - a = 0$. Because the system (2.11) is symmetric with respect to all permutations, we get also $b - c = 0$. Hence the only possible solution under the condition $k_1k_2k_3 \neq 0$ is $a = b = c \neq 0$.

Now, we can extend this solution also to the case of *arbitrary* k_1, k_2, k_3. The system (2.11) is still satisfied and we obtain the case 1 of Proposition 2.2.6.

In particular, in this case we have the Ricci eigenvalues $\rho_1 = \rho_2 = \rho_3 = \frac{a^2}{2}$, $\rho_4 = 0$ and the curvature tensor (2.7) takes on the form

$$\mathcal{R}(e_1, e_2) = -\tfrac{a^2}{4}A_{12},\quad \mathcal{R}(e_1, e_3) = -\tfrac{a^2}{4}A_{13},\quad \mathcal{R}(e_2, e_3) = -\tfrac{a^2}{4}A_{23},$$
$$\mathcal{R}(e_1, e_4) = \mathcal{R}(e_2, e_4) = \mathcal{R}(e_3, e_4) = 0.$$

Moreover, from (2.6) we get $\nabla_{e_i} e_4 = 0$ for $i = 1, ..., 4$ and e_4 is a (globally) parallel vector field.

Now, the following Lemma is an immediate consequence of the well-known Ambrose-Singer Theorem.

Lemma 2.2.7. *On a Riemannian manifold (M, g), the Lie algebra $\psi(x)$ of the holonomy group $\Psi(x)$ with the reference point $x \in M$ ("the holonomy algebra") contains the Lie algebra generated by all curvature operators $R(X, Y)$, where $X, Y \in T_x M$.*

Using this lemma we see that the holonomy algebra $\psi(e)$ contains span(A_{12}, A_{13}, A_{23}). On the other hand, the holonomy group $\Psi(e)$ acts trivially on span(e_4). By the de Rham decomposition Theorem (see sections 5, 6 of Chapter IV in [Ko-N]), the corresponding Riemannian manifolds are (locally) direct products of a 3-dimensional Lie group and a real line. They are locally symmetric because the 3-dimensional factor is a space of constant curvature.

In conclusion, the corresponding spaces belong to the case *i*) of our Classification Theorem 2.1.2.

Case B. $k_1 = k_2 = 0$, k_3 arbitrary.

In this case, we have the following system of independent equations:

$$\begin{aligned} (1,2,3) \to &(a-b)(2(a-c)(b-c) + c(a+b-2c)k^2) = 0, \\ (1,2,4) \to &k(a-b)((2a-c)(2b-c) + c(2a+2b-c)k^2) = 0, \end{aligned} \tag{2.12}$$

where we put $k = k_3$. We suppose first that $a - b = 0$. If $a = c$ also holds, we obtain a subcase of the case 1 of Proposition 2.2.6. Hence, we can assume $a = b \neq 0$, $a \neq c \neq 0$, $k_1 = k_2 = 0$ and we obtain the case 2 of Proposition 2.2.6.

We want to show the remaining properties. MATHEMATICA 5.0 shows that this solution has the Ricci eigenvalues $\rho_1 = \rho_2 = \frac{1}{2}(2ac - c^2)$, $\rho_3 = \frac{1}{2}c^2$, $\rho_4 = 0$. Moreover, the basic curvature operators have the following expression:

$$\begin{gathered} \mathcal{R}(e_1, e_2) = \tfrac{1}{4}(3c - 4a)cA_{12}, \quad \mathcal{R}(e_1, e_3) = -\tfrac{1}{4}c^2 A_{13}, \quad \mathcal{R}(e_1, e_4) = 0, \\ \mathcal{R}(e_2, e_3) = -\tfrac{1}{4}c^2 A_{23}, \quad \mathcal{R}(e_2, e_4) = 0, \quad \mathcal{R}(e_3, e_4) = 0. \end{gathered}$$

Then the *Lie algebra* generated by curvature operators is just span(A_{12}, A_{13}, A_{23}). Analogously to Case A, we conclude that our spaces are direct products of 3-dimensional Lie group of *nonconstant* curvature and a real line. Hence they are not locally symmetric. The cyclic parallel condition for the whole space implies the cyclic parallel condition for the 3-dimensional factor. According to Theorem 2.1.1, the corresponding spaces must be naturally reductive. We obviously obtain the family from the case *ii*) of our Classification Theorem 2.1.2.

Assume now that $a \neq b$. Then we are left with the equations

$$\begin{aligned} (1,2,3) \to &(2(a-c)(b-c) + c(a+b-2c)k^2) = 0, \\ (1,2,4) \to &k((2a-c)(2b-c) + c(2a+2b-c)k^2) = 0. \end{aligned} \tag{2.13}$$

Here we can suppose $k \neq 0$ because otherwise we get $a = c$ or $b = c$, which is, up to a permutation, the case 2 of Proposition 2.2.6.

Now, due to $c \neq 0$, MATHEMATICA 5.0 gives, up to a permutation of the basis, the unique solution depending on two parameters c and k

$$a = \frac{c}{4}(3 - 3k^2 - \sqrt{1 + 2k^2 - 3k^4}), \quad b = \frac{c}{4}(3 - 3k^2 + \sqrt{1 + 2k^2 - 3k^4}). \tag{2.14}$$

Here we have the standard inequality $k^2 < 1$ (see the line below the formula (2.2)) and, due to $k \neq 0$ and $ab \neq 0$, we get the range $k^2 \in]0, 1[\setminus \left\{\frac{2}{3}\right\}$. The corresponding Ricci eigenvalues are

$$\begin{aligned} \rho_1 =& \tfrac{c^2}{8}(2 - 6k^2 - \sqrt{1 + 2k^2 - 3k^4}), \qquad \rho_2 = \tfrac{c^2}{8}(2 - 6k^2 + \sqrt{1 + 2k^2 - 3k^4}), \\ \rho_3 =& \tfrac{c^2}{16}(3 - 3k^2 - \sqrt{9 - 2k^2 + 57k^4}), \quad \rho_4 = \tfrac{c^2}{16}(3 - 3k^2 + \sqrt{9 - 2k^2 + 57k^4}). \end{aligned} \tag{2.15}$$

It is clear that these are functions of two independent variables c, k. Now, MATHEMATICA 5.0 gives that, due to the assumption $k^2 \in]0, 1[\setminus \left\{\frac{2}{3}\right\}$, we always have $\rho_1 \neq \rho_2$, $\rho_2 \neq \rho_3$, $\rho_1 \neq \rho_4$, $\rho_2 \neq \rho_4$, $\rho_3 \neq \rho_4$. But $\rho_1 = \rho_3$ can still occur, namely in the case when $k^2 = \frac{5}{21}$. Then, we obtain the cases 3 and 4 of Proposition 2.2.6. Now, using (2.7) and (2.8) we obtain, for all values of k, that the space of the curvature operators is $\operatorname{span}(A_{12}, A_{13}, A_{14}, A_{23}, A_{24})$. Hence the Lie algebra generated by these operators is $\mathfrak{so}(4)$. Using Lemma 2.2.7 we see that the action of the holonomy algebra on the tangent space T_eG is irreducible and hence the corresponding manifolds are irreducible. Moreover we can see easily that $(\nabla_{e_4}\mathcal{R})(e_4, e_2)e_4 \neq 0$ (for all values of k) and hence *the spaces are not locally symmetric*. Further, if we put $X = e_1 + e_2 + ve_4$, where v is a *nonzero* parameter, MATHEMATICA 5.0 shows that the Ledger condition $L_5(X) = 0$ can be written in the form $\phi_1(c, k) + \phi_2(c, k)v^2 = 0$ and, because v is a free parameter, this implies

$$\phi_1(c, k) = 59 + 11c^2 - 6c^3 + (250 - 22c^2 + 12c^3)k^2 + (11c^2 - 6c^3)k^4 = 0, \tag{2.16}$$

$$\phi_2(c, k) = (-262 + 39c)k^2 - (260 + 39c)k^4 = 0. \tag{2.17}$$

If $260 + 39c = 0$, the formula (2.17) leads to a contradiction. Hence $260 + 39c \neq 0$ and k^2 can be expressed from (2.17) in the form $k^2 = \frac{-262+39c}{13(20+3c)}$. Substituting this into (2.16), we obtain a cubic equation

$$4347200 - 392340c - 1155771c^2 + 544968c^3 = 0. \tag{2.18}$$

MATHEMATICA 5.0 says that (2.18) has only one real solution, namely $c = -1.57074...$ But this gives a negative value for k^2, a contradiction. As conclusion, we always have $L_5(e_1 + e_2 + ve_4) \neq 0$ for *some* $v \neq 0$, and the corresponding spaces do not satisfy the Ledger condition L_5.

Note that the case 3 of Proposition 2.2.6 is a family with four distinct Ricci eigenvalues and this is *an explicit presentation* of the family of spaces described by

P. Bueken and L. Vanhecke only implicitly in [Bu-V]. We conclude that our spaces belong to the case *iv*) of the Classification Theorem 2.1.2 as a generic subfamily. (The exceptional case $k^2 = \frac{2}{3}$ will be added later).

The case 4 of Proposition 2.2.6 with two equal Ricci eigenvalues has been presented in [AM] as the only missing family in the classification Theorem of [Po-Sp]- see Section 2.3 for more details. In particular, from this solution we obtain the spaces which give the case *v*) of our Classification Theorem 2.1.2.

<u>Case C</u>. $k_1 = 0$ and $k_2 k_3 \neq 0$.

First, we suppose that $2a - b - c \neq 0$. Then from the simplified equations $(2,2,1)$ and $(3,3,1)$ of (2.11) we obtain that $a = b = c$. Hence, the corresponding solution is a particular subcase of the case 1 of Proposition 2.2.6. Supposing $2a - b - c = 0$ we obtain the following more simple system of equations:

$$
\begin{aligned}
(1,2,3) &\to (a-b)(-4(a-b) - 3bk_2^2 + 3(2a-b)k_3^2) = 0, \\
(1,2,4) &\to (a-b)(-4a + 3b - 3bk_2^2 + 3(2a-b)k_3^2) = 0, \\
(1,3,4) &\to (a-b)(-2a + 3b - 3bk_2^2 + 3(2a-b)k_3^2) = 0.
\end{aligned}
$$

If $a - b = 0$, we conclude immediately that $a = b = c$. Thus we assume that $a - b \neq 0$. Dividing the equations $(1,2,4)$ and $(1,3,4)$ by the factor $(a-b)$ and subtracting both remaining equations we obtain the necessary condition $a = 0$ which is a contradiction to $abc \neq 0$. This concludes the proof of Proposition 2.2.6.

2.2.2 Solvable Case

We are going to analyze this case using the following result given by L. Bérard Bergery in [Be]:

Theorem 2.2.8. *In dimension 4, the solvable and simply connected Lie groups are:*

a) *The non-trivial semi-direct products* $E(2) \rtimes \mathbb{R}$ *and* $E(1,1) \rtimes \mathbb{R}$.

b) *The non-nilpotent semi-direct products* $H \rtimes \mathbb{R}$*, where* H *is the Heisenberg group.*

c) *All semi-direct products* $\mathbb{R}^3 \rtimes \mathbb{R}$.

As concern the semidirect products of the form $G = G_3 \rtimes \mathbb{R}$ in the above Theorem and all possible left-invariant metrics on them, we can construct all of them on the level of Lie algebras as follows: we consider the Lie algebra $\mathfrak{g}_3$ and the vector space $\mathfrak{g} = \mathfrak{g}_3 + \mathbb{R}$. Let $\{f_1, ..., f_4\}$ be any basis of $\mathfrak{g}$ such that $\mathfrak{g}_3$ =span$\{f_1, f_2, f_3\}$, $\mathbb{R}$ =span$\{f_4\}$. Let D be an arbitrary derivation of the algebra $\mathfrak{g}_3$ and let us define

$$
[f_4, f_i] = Df_i \quad \text{for} \quad i = 1, 2, 3. \tag{2.19}
$$

(This completes the multiplication table of the algebra $\mathfrak{g}_3$ to the multiplication table of $\mathfrak{g}$). Then we choose any scalar product $\langle\ ,\rangle$ on $\mathfrak{g}$ for which $\{f_1, f_2, f_3\}$ forms an orthonormal triplet but f_4 is just a unit vector which need not be orthonormal to $\mathfrak{g}_3$. Thus we have, as in the formula (2.2), $\langle f_i, f_4\rangle = k_i,\ i = 1, 2, 3$. Now, all semi-direct products $G_3 \rtimes \mathbb{R}$ with left-invariant metrics correspond to various choices of the derivations D of $\mathfrak{g}_3$ and to all scalar products given by the above rule. The algebra of all derivations D of $\mathfrak{g}_3$ will be usually represented in the corresponding matrix form.

Now, we shall study each of the cases from Theorem 2.2.8 separately following the construction indicated above and preserving the style of the Section 2.2.1.

Non-trivial semi-direct products $E(2) \rtimes \mathbb{R}$

Let $\mathfrak{e}(2)$ be the Lie algebra of $E(2)$ with a scalar product $\langle\ ,\ \rangle_3$. Then, there is an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathfrak{e}(2)$ such that

$$[f_2, f_3] = \gamma f_1,\ \ [f_3, f_1] = -\gamma f_2,\ \ [f_1, f_2] = 0 \tag{2.20}$$

where $\gamma \neq 0$ is a real number. The algebra of all derivations D of $\mathfrak{e}(2)$ is

$$\left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ c & d & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\},$$

when represented in the matrix form.

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{e}(2) + \mathbb{R}$, where the multiplication table is given by (2.20) and, according to the general formula (2.19), also by

$$\begin{gathered}[f_4, f_1] = af_1 + bf_2,\ [f_4, f_2] = -bf_1 + af_2,\ [f_4, f_3] = cf_1 + df_2, \\ \langle f_i, f_4\rangle = k_i,\ i = 1, 2, 3.\end{gathered} \tag{2.21}$$

Here $\gamma \neq 0, a, b, c, d, k_1, k_2, k_3$ are arbitrary parameters where $\sum\limits_{i=1}^{3} k_i^2 < 1$ due to the positivity of the scalar product. We exclude the case $a = b = c = d = 0$, i.e. the direct product $E(2) \times \mathbb{R}$.

This gives rise to a simply connected group space $(G = E(2) \rtimes \mathbb{R}, g)$.

Now we replace the basis $\{f_i\}$ by the new basis $\{e_i\}$, as in the formula (2.3). Then we get an orthonormal basis for which

$$\begin{gathered}[e_2, e_3] = \gamma e_1,\ [e_3, e_1] = -\gamma e_2,\ [e_1, e_2] = 0, \\ [e_4, e_1] = \tfrac{1}{R}\left(ae_1 + (b + k_3\gamma)e_2\right),\ [e_4, e_2] = \tfrac{1}{R}\left(-(b + k_3\gamma)e_1 + ae_2\right), \\ [e_4, e_3] = \tfrac{1}{R}\left((c + k_2\gamma)e_1 + (d - k_1\gamma)e_2\right).\end{gathered} \tag{2.22}$$

Next we are going to calculate, in the new basis, the expression for the Levi-Civita connection, the curvature tensor, the Ricci matrix, and the condition for the Ricci tensor to be cyclic parallel.

By an easy calculation we get

Lemma 2.2.9.

$$
\begin{gathered}
\nabla_{e_i} e_i = \tfrac{a}{R} e_4,\ i = 1,2, \quad \nabla_{e_i} e_i = 0,\ i = 3,4, \quad \nabla_{e_1} e_2 = 0 = \nabla_{e_2} e_1, \\
\nabla_{e_1} e_3 = \tfrac{(c+\gamma k_2)}{2R} e_4, \quad \nabla_{e_3} e_1 = -\gamma e_2 + \tfrac{(c+\gamma k_2)}{2R} e_4, \\
\nabla_{e_1} e_4 = -\tfrac{a}{R} e_1 - \tfrac{(c+\gamma k_2)}{2R} e_3, \quad \nabla_{e_4} e_1 = -\tfrac{(c+\gamma k_2)}{2R} e_3 + \tfrac{(b+\gamma k_3)}{R} e_2, \\
\nabla_{e_2} e_3 = \tfrac{(d-\gamma k_1)}{2R} e_4, \quad \nabla_{e_3} e_2 = \gamma e_1 + \tfrac{(d-\gamma k_1)}{2R} e_4, \\
\nabla_{e_2} e_4 = -\tfrac{a}{R} e_2 - \tfrac{(d-\gamma k_1)}{2R} e_3, \quad \nabla_{e_4} e_2 = -\tfrac{(d-\gamma k_1)}{2R} e_3 - \tfrac{(b+\gamma k_3)}{R} e_1, \\
\nabla_{e_3} e_4 = -\tfrac{(d-\gamma k_1)}{2R} e_2 - \tfrac{(c+\gamma k_2)}{2R} e_1, \quad \nabla_{e_4} e_3 = \tfrac{(d-\gamma k_1)}{2R} e_2 + \tfrac{(c+\gamma k_2)}{2R} e_1.
\end{gathered} \tag{2.23}
$$

Similarly to Lemma 2.2.3 we can now derive

Lemma 2.2.10. *The components of the curvature operator are*

$$
\begin{aligned}
&\mathcal{R}(e_1, e_2) = \alpha_{1212} A_{12} + \alpha_{1213} A_{13} + \alpha_{1223} A_{23}, \\
&\mathcal{R}(e_1, e_3) = \alpha_{1312} A_{12} + \alpha_{1313} A_{13} + \alpha_{1323} A_{23} + \alpha_{1334} A_{34}, \\
&\mathcal{R}(e_1, e_4) = \alpha_{1414} A_{14} + \alpha_{1424} A_{24} + \alpha_{1434} A_{34}, \\
&\mathcal{R}(e_2, e_3) = \alpha_{2312} A_{23} + \alpha_{2313} A_{13} + \alpha_{2323} A_{23} + \alpha_{2334} A_{34}, \\
&\mathcal{R}(e_2, e_4) = \alpha_{2414} A_{14} + \alpha_{2424} A_{24} + \alpha_{2434} A_{34}, \\
&\mathcal{R}(e_3, e_4) = \alpha_{3413} A_{13} + \alpha_{3414} A_{14} + \alpha_{3423} A_{23} + \alpha_{3424} A_{24} + \alpha_{3434} A_{34},
\end{aligned} \tag{2.24}
$$

where the coefficients $\alpha_{ijlm} = g(\mathcal{R}(e_i, e_j)e_l, e_m)$ satisfy the standard symmetries with respect to their indices and

$$
\begin{gathered}
\alpha_{1212} = \tfrac{a^2}{R^2}, \quad \alpha_{1213} = \tfrac{a(d-\gamma k_1)}{2R^2}, \quad \alpha_{1223} = -\tfrac{a(c+\gamma k_2)}{2R^2}, \quad \alpha_{1313} = -\tfrac{(c+\gamma k_2)^2}{4R^2}, \\
\alpha_{1323} = -\tfrac{(d-\gamma k_1)(c+\gamma k_2)}{4R^2}, \quad \alpha_{1334} = \tfrac{\gamma(-d+\gamma k_1)}{2R}, \quad \alpha_{1414} = \tfrac{4a^2-(c+\gamma k_2)^2}{4R^2}, \\
\alpha_{1424} = -\tfrac{(d-\gamma k_1)(c+\gamma k_2)}{4R^2}, \quad \alpha_{1434} = \tfrac{2a(c+\gamma k_2)+(d-\gamma k_1)(b+\gamma k_3)}{2R^2}, \\
\alpha_{2323} = -\tfrac{(d-\gamma k_1)^2}{4R^2}, \quad \alpha_{2334} = \tfrac{\gamma(c+\gamma k_2)}{2R}, \quad \alpha_{2424} = \tfrac{4a^2-(d-\gamma k_1)^2}{4R^2}, \\
\alpha_{2434} = \tfrac{2a(d-\gamma k_1)-(c+\gamma k_2)(b+\gamma k_3)}{2R^2}, \quad \alpha_{3434} = \tfrac{3((d-\gamma k_1)^2+(c+\gamma k_2)^2)}{4R^2}.
\end{gathered} \tag{2.25}
$$

Further, we obtain easily

Lemma 2.2.11. *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1, e_2, e_3, e_4\}$ *is of the form*

$$
\begin{pmatrix}
\frac{-4a^2+(c+\gamma k_2)^2}{2R^2} & \frac{(d-\gamma k_1)(c+\gamma k_2)}{2R^2} & \beta_{13} & \frac{\gamma(-d+\gamma k_1)}{2R} \\
\frac{(d-\gamma k_1)(c+\gamma k_2)}{2R^2} & \frac{-4a^2+(d-\gamma k_1)^2}{2R^2} & \beta_{23} & \frac{\gamma(c+\gamma k_2)}{2R} \\
\beta_{13} & \beta_{23} & \frac{-(c+\gamma k_2)^2-(d-\gamma k_1)^2}{2R^2} & 0 \\
\frac{\gamma(-d+\gamma k_1)}{2R} & \frac{\gamma(c+\gamma k_2)}{2R} & 0 & \beta_{44}
\end{pmatrix} \tag{2.26}
$$

where

$$\beta_{13} = \tfrac{-3a(c+\gamma k_2)+(-d+\gamma k_1)(b+\gamma k_3)}{2R^2}, \quad \beta_{23} = \tfrac{3a(-d+\gamma k_1)+(c+\gamma k_2)(b+\gamma k_3)}{2R^2},$$

$$\beta_{44} = \tfrac{-4a^2-(d-\gamma k_1)^2-(c+\gamma k_2)^2}{2R^2}.$$

Now we obtain the following analogue of Lemma 2.2.5:

Lemma 2.2.12. *The condition* (2.10) *for the Ricci tensor of type* $(0,2)$ *is equivalent to the system of algebraic equations*

$$\begin{aligned}
&(1,1,1) \rightarrow a(d-\gamma k_1) = 0,\\
&(1,1,2) \rightarrow a(c+\gamma k_2) = 0,\\
&(1,1,3) \rightarrow (d-\gamma k_1)(c+\gamma k_2) = 0,\\
&(1,1,4) \rightarrow -a(c^2-d^2+\gamma^2(k_2^2-k_1^2)) - 2cd(b+\gamma k_3)\\
&\qquad\quad -2\gamma k_2(ac+d(b+\gamma k_3)) + 2\gamma k_1((c+\gamma k_2)(b+\gamma k_3)-ad) = 0,\\
&(3,3,1) \rightarrow -3a(d-\gamma k_1) + (c+\gamma k_2)(b+\gamma k_3) = 0,\\
&(3,3,2) \rightarrow 3a(c+\gamma k_2) + (d-\gamma k_1)(b+\gamma k_3) = 0,\\
&(4,4,1) \rightarrow -a(d-\gamma k_1) + (c+\gamma k_2)(b+\gamma k_3) = 0,\\
&(4,4,2) \rightarrow a(c+\gamma k_2) + (d-\gamma k_1)(b+\gamma k_3) = 0,\\
&(1,2,3) \rightarrow (d-\gamma k_1)^2 - (c+\gamma k_2)^2 = 0,\\
&(1,2,4) \rightarrow -2a(d-\gamma k_1)(c+\gamma k_2)\\
&\qquad\quad +(c+\gamma k_2+d-\gamma k_1)(c+\gamma k_2-d+\gamma k_1)(b+\gamma k_3) = 0,\\
&(1,3,4) \rightarrow 2a(d-\gamma k_1)(b+\gamma k_3)\\
&\qquad\quad +(c+\gamma k_2)(-(b+\gamma k_3)^2+(a^2+R^2\gamma^2)) = 0,\\
&(2,3,4) \rightarrow -2a(c+\gamma k_2)(b+\gamma k_3)\\
&\qquad\quad +(d-\gamma k_1)(-(b+\gamma k_3)^2+(a^2+R^2\gamma^2)) = 0.
\end{aligned} \tag{2.27}$$

Here the symbol "$(\alpha,\beta,\gamma)\rightarrow$*" marks the substitution of* $(e_\alpha, e_\beta, e_\gamma)$ *for* (X,Y,Z) *respectively.*

Now, the goal is to find the values of $a, b, c, d, k_1, k_2, k_3$ and $\gamma \neq 0$ which satisfy the system of equations (2.27).

Proposition 2.2.13. *The unique solution of the system of algebraic equations* (2.27) *is given by the formula*

$$d = \gamma k_1,\ c = -\gamma k_2,\ \gamma \neq 0,\ a,\ b,\ k_1,\ k_2,\ k_3,\ \textit{arbitrary}. \tag{2.28}$$

The corresponding spaces belong to the case i) of the Classification Theorem 2.1.2.

Proof. From the subsystem of (2.27) formed by the equations $(1,1,3)$ and $(1,2,3)$ we obtain $(d-\gamma k_1) = (c+\gamma k_2) = 0$. Then, the remaining equations (2.27) are automatically satisfied.

Moreover, according to (2.26), the corresponding spaces have the Ricci eigenvalues $\rho_1 = \rho_2 = \rho_4 = -\frac{2a^2}{R^2}$, $\rho_3 = 0$ and the curvature tensor (2.24) takes on the form

$$\mathcal{R}(e_1,e_2) = \tfrac{a^2}{R^2}A_{12}, \quad \mathcal{R}(e_1,e_4) = \tfrac{a^2}{R^2}A_{14}, \quad \mathcal{R}(e_2,e_4) = \tfrac{a^2}{R^2}A_{24},$$
$$\mathcal{R}(e_1,e_3) = \mathcal{R}(e_2,e_3) = \mathcal{R}(e_3,e_4) = 0.$$

Then, either each of the spaces is flat (for $a=0$) or the space of the curvature operators is $\mathrm{span}(A_{12},A_{14},A_{24})$. Moreover, from (2.23) we get $\nabla_{e_i}e_3 = 0$ for all $i=1,...,4$ and e_3 is a parallel vector field. Using a complete analogue of the proof in <u>Case A</u> of Proposition 2.2.6, we conclude that the corresponding spaces belong to the case i) of our Classification Theorem 2.1.2.

Non-trivial semi-direct products $E(1,1)\rtimes\mathbb{R}$

Let $\mathfrak{e}(1,1)$ be the Lie algebra of $E(1,1)$ with a scalar product $\langle\ ,\ \rangle_3$. Then, there is an orthonormal basis $\{f_1,f_2,f_3\}$ of $\mathfrak{e}(1,1)$ such that

$$[f_2,f_3] = \gamma f_2,\ \ [f_3,f_1] = \gamma f_1,\ \ [f_1,f_2] = 0 \tag{2.29}$$

where $\gamma \neq 0$ is a real number. The algebra of all derivations D of $\mathfrak{e}(1,1)$ is

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & 0 \end{pmatrix} : a,b,c\in\mathbb{R} \right\},$$

when represented in the matrix form.

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{e}(1,1)+\mathbb{R}$, where the multiplication table is given by (2.29) and, according to the general formula (2.19), also by

$$\begin{gathered}[f_4,f_1] = af_1,\ [f_4,f_2] = af_2,\ [f_4,f_3] = bf_1 + cf_2,\\ \langle f_i,f_4\rangle = k_i,\ i=1,2,3.\end{gathered} \tag{2.30}$$

Here $\gamma\neq 0, a,b,c,k_1,k_2,k_3$ are arbitrary parameters where $\sum\limits_{i=1}^{3} k_i^2 < 1$, and we exclude the case $a=b=c=0$.

This gives rise to a simply connected group space $(G = E(1,1)\rtimes\mathbb{R}, g)$.

Now we replace the basis $\{f_i\}$ by the new basis $\{e_i\}$, as in the formula (2.3). Then we get an orthonormal basis for which

$$\begin{gathered}[e_2,e_3] = \gamma e_2,\ [e_3,e_1] = \gamma e_1,\ [e_1,e_2] = 0,\\ [e_4,e_1] = \tfrac{1}{R}\left((a-k_3\gamma)e_1\right),\ [e_4,e_2] = \tfrac{1}{R}\left((a+k_3\gamma)e_2\right),\\ [e_4,e_3] = \tfrac{1}{R}\left((b+k_1\gamma)e_1 + (c-k_2\gamma)e_2\right).\end{gathered} \tag{2.31}$$

Now we are going to calculate, in the new basis, the expression for the Levi-Civita connection, the curvature tensor, the Ricci matrix, and the condition for the Ricci tensor to be cyclic parallel.

By an easy calculation we get

Lemma 2.2.14.

$$\begin{gathered}
\nabla_{e_1}e_1 = \gamma e_3 + \tfrac{(a-\gamma k_3)}{R}e_4,\ \nabla_{e_2}e_2 = -\gamma e_3 + \tfrac{(a+\gamma k_3)}{R}e_4,\\
\nabla_{e_i}e_i = 0, i = 3,4,\ \nabla_{e_1}e_2 = 0 = \nabla_{e_2}e_1,\\
\nabla_{e_1}e_3 = -\gamma e_1 + \tfrac{(b+\gamma k_1)}{2R}e_4,\ \nabla_{e_3}e_1 = \tfrac{(b+\gamma k_1)}{2R}e_4,\\
\nabla_{e_1}e_4 = -\tfrac{(b+\gamma k_1)}{2R}e_3 + \tfrac{(-a+\gamma k_3)}{R}e_1,\ \nabla_{e_4}e_1 = -\tfrac{(b+\gamma k_1)}{2R}e_3,\\
\nabla_{e_2}e_3 = \gamma e_2 + \tfrac{(c-\gamma k_2)}{2R}e_4,\ \nabla_{e_3}e_2 = \tfrac{(c-\gamma k_2)}{2R}e_4,\\
\nabla_{e_2}e_4 = -\tfrac{(c-\gamma k_2)}{2R}e_3 - \tfrac{(a+\gamma k_3)}{R}e_2,\ \nabla_{e_4}e_2 = -\tfrac{(c-\gamma k_2)}{2R}e_3,\\
\nabla_{e_3}e_4 = -\tfrac{(b+\gamma k_1)}{2R}e_1 - \tfrac{(c-\gamma k_2)}{2R}e_2,\ \nabla_{e_4}e_3 = \tfrac{(b+\gamma k_1)}{2R}e_1 + \tfrac{(c-\gamma k_2)}{2R}e_2.
\end{gathered} \tag{2.32}$$

Similarly to Lemma 2.2.3 we can now derive

Lemma 2.2.15. *The components of the curvature operator are*

$$\begin{aligned}
\mathcal{R}(e_1,e_2) &= \alpha_{1212}A_{12} + \alpha_{1213}A_{13} + \alpha_{1214}A_{14} + \alpha_{1223}A_{23} + \alpha_{1224}A_{24},\\
\mathcal{R}(e_1,e_3) &= \alpha_{1312}A_{12} + \alpha_{1313}A_{13} + \alpha_{1314}A_{14} + \alpha_{1323}A_{23} + \alpha_{1334}A_{34},\\
\mathcal{R}(e_1,e_4) &= \alpha_{1412}A_{12} + \alpha_{1413}A_{13} + \alpha_{1414}A_{14} + \alpha_{1424}A_{24} + \alpha_{1434}A_{34},\\
\mathcal{R}(e_2,e_3) &= \alpha_{2312}A_{23} + \alpha_{2313}A_{13} + \alpha_{2323}A_{23} + \alpha_{2324}A_{24} + \alpha_{2334}A_{34},\\
\mathcal{R}(e_2,e_4) &= \alpha_{2412}A_{12} + \alpha_{2414}A_{14} + \alpha_{2423}A_{23} + \alpha_{2424}A_{24} + \alpha_{2434}A_{34},\\
\mathcal{R}(e_3,e_4) &= \alpha_{3413}A_{13} + \alpha_{3414}A_{14} + \alpha_{3423}A_{23} + \alpha_{3424}A_{24} + \alpha_{3434}A_{34},
\end{aligned} \tag{2.33}$$

where the coeficients $\alpha_{ijlm} = g(\mathcal{R}(e_i,e_j)e_l,e_m)$ satisfy the standard symmetries with respect to their indices and

$$\begin{gathered}
\alpha_{1212} = \tfrac{a^2+\gamma^2(-1+k_1^2+k_2^2)}{R^2},\quad \alpha_{1213} = \tfrac{(c-\gamma k_2)(a-\gamma k_3)}{2R^2},\quad \alpha_{1214} = \tfrac{-\gamma(c-\gamma k_2)}{2R},\\
\alpha_{1223} = \tfrac{-(b+\gamma k_1)(a+\gamma k_3)}{2R^2},\quad \alpha_{1224} = \tfrac{-\gamma(b+\gamma k_1)}{2R},\quad \alpha_{1313} = \tfrac{4R^2\gamma^2-(b+\gamma k_1)^2}{4R^2},\\
\alpha_{1314} = \tfrac{\gamma(a-\gamma k_3)}{R},\quad \alpha_{1323} = \tfrac{(b+\gamma k_1)(-c+\gamma k_2)}{4R^2},\quad \alpha_{1334} = \tfrac{\gamma(b+\gamma k_1)}{R},\\
\alpha_{1414} = \tfrac{4(a-\gamma k_3)^2-(b+\gamma k_1)^2}{4R^2},\quad \alpha_{1424} = \tfrac{(b+\gamma k_1)(-c+\gamma k_2)}{4R^2},\quad \alpha_{1434} = \tfrac{(b+\gamma k_1)(a-\gamma k_3)}{R^2},\\
\alpha_{2323} = \tfrac{4R^2\gamma^2-(c-\gamma k_2)^2}{4R^2},\quad \alpha_{2324} = \tfrac{-\gamma(a+\gamma k_3)}{R},\quad \alpha_{2334} = \tfrac{\gamma(-c+\gamma k_2)}{R},\\
\alpha_{2424} = \tfrac{4(a+\gamma k_3)^2-(c-\gamma k_2)^2}{4R^2},\quad \alpha_{2434} = \tfrac{(c-\gamma k_2)(a+\gamma k_3)}{R^2},\quad \alpha_{3434} = \tfrac{3((b+\gamma k_1)^2+(c-\gamma k_2)^2)}{4R^2}.
\end{gathered} \tag{2.34}$$

Further, we obtain easily

Lemma 2.2.16. *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1, e_2, e_3, e_4\}$ *is of the form*

$$\begin{pmatrix} \beta_{11} & \frac{(b+\gamma k_1)(c-\gamma k_2)}{2R^2} & \frac{(b+\gamma k_1)(-3a+\gamma k_3)}{2R^2} & \frac{\gamma(b+\gamma k_1)}{2R} \\ \frac{(b+\gamma k_1)(c-\gamma k_2)}{2R^2} & \beta_{22} & \frac{-(c-\gamma k_2)(3a+\gamma k_3)}{2R^2} & \frac{\gamma(-c+\gamma k_2)}{2R} \\ \frac{(b+\gamma k_1)(-3a+\gamma k_3)}{2R^2} & \frac{-(c-\gamma k_2)(3a+\gamma k_3)}{2R^2} & \beta_{33} & \frac{2\gamma^2 k_3}{R} \\ \frac{\gamma(b+\gamma k_1)}{2R} & \frac{\gamma(-c+\gamma k_2)}{2R} & \frac{2\gamma^2 k_3}{R} & \beta_{44} \end{pmatrix} \tag{2.35}$$

where

$$\beta_{11} = \frac{(b+\gamma k_1)^2 - 4a(a-\gamma k_3)}{2R^2}, \quad \beta_{22} = \frac{(c-\gamma k_2)^2 - 4a(a+\gamma k_3)}{2R^2},$$
$$\beta_{33} = -\frac{aR^2\gamma^2 + (b+\gamma k_1)^2 + (c-\gamma k_2)^2}{2R^2}, \quad \beta_{44} = -\frac{a(a^2+\gamma^2 k_3^2) + (b+\gamma k_1)^2 + (c-\gamma k_2)^2}{2R^2}.$$

Now we obtain the following analogue of Lemma 2.2.5:

Lemma 2.2.17. *The condition* (2.10) *for the Ricci tensor of type* $(0,2)$ *is equivalent to the system of algebraic equations*

$$\begin{aligned}
(1,1,1) &\to a(b+\gamma k_1) = 0, \\
(1,1,2) &\to a(c-\gamma k_2) = 0, \\
(1,1,3) &\to -4a^2 + 4\gamma^2(1-k_1^2-k_2^2) + (b+\gamma k_1)^2 + (c-\gamma k_2)^2 = 0, \\
(1,1,4) &\to -4\gamma k_3(a^2 - \gamma^2(1-k_1^2-k_2^2)) \\
&\qquad + (c-\gamma k_2)^2(-a+\gamma k_3) + (b+\gamma k_1)^2(a+\gamma k_3) = 0, \\
(1,2,4) &\to 2a(b+\gamma k_1)(c-\gamma k_2) = 0, \\
(1,3,4) &\to (b+\gamma k_1)(a(a+4\gamma k_3) + 3\gamma^2(k_3^2 - R^2)) = 0, \\
(2,3,4) &\to (c-\gamma k_2)(a(a-4\gamma k_3) + 3\gamma^2(k_3^2 - R^2)) = 0, \\
(3,3,1) &\to (b+\gamma k_1)(a+\gamma k_3) = 0,
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
(3,3,2) &\to (c-\gamma k_2)(a-\gamma k_3) = 0, \\
(4,4,1) &\to (b+\gamma k_1)(a+3\gamma k_3) = 0, \\
(4,4,2) &\to (c-\gamma k_2)(-a+3\gamma k_3) = 0.
\end{aligned}$$

Here the symbol "$(\alpha,\beta,\gamma) \to$*" marks the substitution of* $(e_\alpha, e_\beta, e_\gamma)$ *for* (X,Y,Z) *respectively.*

Now, we have

Proposition 2.2.18. *The unique solution of the system of algebraic equations* (2.36) *is, up to a re-numeration of the triplet* $\{e_1, e_2, e_3\}$,

$$a = \gamma\sqrt{1-k_1^2-k_2^2},\ b = -\gamma k_1,\ c = \gamma k_2,\ \gamma \neq 0,\ k_1,\ k_2,\ k_3 \text{ arbitrary.} \tag{2.37}$$

The corresponding spaces belong to the case i) of the Classification Theorem 2.1.2.

Proof. Suppose first $a \neq 0$. We obtain the formulas (2.37) from $(1,1,1)$, $(1,1,2)$ and $(1,1,3)$. Next we suppose $a = 0$. Then we obtain from $(1,1,3)$ that $1 - k_1^2 - k_2^2 \leq 0$, which is a contradiction. On the other hand, (2.36) is automatically satisfied by the solution (2.37).

Moreover, the corresponding spaces have the Ricci eigenvalues $\rho_1 = \frac{-2a^2 - k_3 2\gamma a}{R^2} = \rho_3$, $\rho_2 = \frac{-2a^2 + k_3 2\gamma a}{R^2} = \rho_4$. In addition, $\nabla\mathcal{R} = 0$, checking by MATHEMATICA 5.0. A routine computation shows that, in fact, every space is a direct product $M_2 \times M_2'$ of spaces of constant curvatures ρ_1 and ρ_2 (even for $k_3 = 0$ where $\rho_1 = \rho_2$). Hence, the corresponding spaces are locally symmetric and they belong to the case *i*) of the Classification Theorem 2.1.2.

Non-nilpotent semi-direct products $H \rtimes \mathbb{R}$

Let $\mathfrak{h}$ be the Lie algebra of H (the Heisenberg group) with a scalar product $\langle\ ,\ \rangle_3$. Then, there is an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathfrak{h}$ such that

$$[f_3, f_2] = 0, \ \ [f_3, f_1] = 0, \ \ [f_1, f_2] = \gamma f_3 \tag{2.38}$$

where $\gamma \neq 0$ is a real number. The algebra of all derivations D of $\mathfrak{h}$ is

$$\left\{ \begin{pmatrix} a & b & h \\ c & d & f \\ 0 & 0 & a+d \end{pmatrix} : a, b, c, d, h, f \in \mathbb{R} \right\},$$

when represented in the matrix form.

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{h} + \mathbb{R}$, where the multiplication table is given by (2.38) and, according to the general formula (2.19), also by

$$\begin{gathered} [f_4, f_1] = af_1 + bf_2 + hf_3, \quad [f_4, f_2] = cf_1 + df_2 + ff_3, \\ [f_4, f_3] = (a+d)f_3, \quad \langle f_i, f_4 \rangle = k_i, \ i = 1, 2, 3. \end{gathered} \tag{2.39}$$

Here $\gamma \neq 0, a, b, c, d, f, h, k_1, k_2, k_3$ are arbitrary parameters where $\sum_{i=1}^{3} k_i^2 < 1$. We exclude the nilpotent case $a = b = c = d = h = 0$. (See [Be]).

This gives rise to a simply connected group space $(G = H \rtimes \mathbb{R}, g)$.

Now we replace the basis $\{f_i\}$ by the new basis $\{e_i\}$, as in the formula (2.3). Then we get an orthonormal basis for which

$$\begin{gathered} [e_1, e_2] = \gamma e_3, \ [e_3, e_2] = [e_3, e_1] = 0, \ [e_4, e_1] = \tfrac{1}{R}\left(ae_1 + be_2 + (h + k_2\gamma)e_3\right), \\ [e_4, e_2] = \tfrac{1}{R}\left(ce_1 + de_2 + (f - k_1\gamma)e_3\right), \ [e_4, e_3] = \tfrac{1}{R}\left((a+d)e_3\right). \end{gathered} \tag{2.40}$$

Now we are going to calculate, in the new basis, the expression for the Levi-Civita connection, the curvature tensor, the Ricci matrix, and the condition for the Ricci tensor to be cyclic parallel.

By an easy calculation we get

Lemma 2.2.19.

$$
\begin{gathered}
\nabla_{e_1}e_1=\tfrac{a}{R}e_4,\ \nabla_{e_2}e_2=\tfrac{a}{R}e_4,\ \nabla_{e_3}e_3=\tfrac{(a+d)}{R}e_4,\ \nabla_{e_4}e_4=0,\\
\nabla_{e_1}e_2=\tfrac{\gamma}{2}e_3+\tfrac{(b+c)}{2R}e_4,\ \nabla_{e_2}e_1=-\tfrac{\gamma}{2}e_3+\tfrac{(b+c)}{2R}e_4,\\
\nabla_{e_1}e_3=-\tfrac{\gamma}{2}e_2+\tfrac{(h+\gamma k_2)}{2R}e_4=\nabla_{e_3}e_1,\ \nabla_{e_2}e_3=\tfrac{\gamma}{2}e_1+\tfrac{(f-\gamma k_1)}{2R}e_4=\nabla_{e_3}e_2,\\
\nabla_{e_1}e_4=-\tfrac{a}{R}e_1-\tfrac{(b+c)}{2R}e_2-\tfrac{(h+\gamma k_2)}{2R}e_3,\ \nabla_{e_4}e_1=\tfrac{(b-c)}{2R}e_2+\tfrac{(h+\gamma k_2)}{2R}e_3,\\
\nabla_{e_2}e_4=-\tfrac{(b+c)}{2R}e_1-\tfrac{d}{R}e_2-\tfrac{(f-\gamma k_1)}{2R}e_3,\ \nabla_{e_4}e_2=\tfrac{(-b+c)}{2R}e_1+\tfrac{(f-\gamma k_1)}{2R}e_3,\\
\nabla_{e_3}e_4=-\tfrac{(h+\gamma k_2)}{2R}e_1-\tfrac{(f-\gamma k_1)}{2R}e_2-\tfrac{(a+d)}{R}e_3,\ \nabla_{e_4}e_3=-\tfrac{(h+\gamma k_2)}{2R}e_1-\tfrac{(f-\gamma k_1)}{2R}e_2.
\end{gathered}
\tag{2.41}
$$

Similarly to Lemma 2.2.3 we can now derive

Lemma 2.2.20. *The components of the curvature operator are*

$$
\begin{aligned}
\mathcal{R}(e_1,e_2)&=\alpha_{1212}A_{12}+\alpha_{1213}A_{13}+\alpha_{1214}A_{14}+\alpha_{1223}A_{23}+\alpha_{1224}A_{24}+\alpha_{1234}A_{34},\\
\mathcal{R}(e_1,e_3)&=\alpha_{1312}A_{12}+\alpha_{1313}A_{13}+\alpha_{1314}A_{14}+\alpha_{1323}A_{23}+\alpha_{1324}A_{24}+\alpha_{1334}A_{34},\\
\mathcal{R}(e_1,e_4)&=\alpha_{1412}A_{12}+\alpha_{1413}A_{13}+\alpha_{1414}A_{14}+\alpha_{1423}A_{23}+\alpha_{1424}A_{24}+\alpha_{1434}A_{34},\\
\mathcal{R}(e_2,e_3)&=\alpha_{2312}A_{23}+\alpha_{2313}A_{13}+\alpha_{2314}A_{14}+\alpha_{2323}A_{23}+\alpha_{2324}A_{24}+\alpha_{2334}A_{34},\\
\mathcal{R}(e_2,e_4)&=\alpha_{2412}A_{12}+\alpha_{2413}A_{13}+\alpha_{2414}A_{14}+\alpha_{2423}A_{23}+\alpha_{2424}A_{24}+\alpha_{2434}A_{34},\\
\mathcal{R}(e_3,e_4)&=\alpha_{3412}A_{12}+\alpha_{3413}A_{13}+\alpha_{3414}A_{14}+\alpha_{3423}A_{23}+\alpha_{3424}A_{24}+\alpha_{3434}A_{34},
\end{aligned}
\tag{2.42}
$$

where the coeficients $\alpha_{ijlm}=g(\mathcal{R}(e_i,e_j)e_l,e_m)$ *satisfy the standard symmetries with respect to their indices and*

$$
\begin{gathered}
\alpha_{1212}=\tfrac{4ad+3\gamma^2R^2-(b+c)^2}{4R^2},\quad \alpha_{1213}=\tfrac{2a(f-\gamma k_1)-(b+c)(h+\gamma k_2)}{4R^2},\quad \alpha_{1214}=\tfrac{-3\gamma(h+\gamma k_2)}{4R},\\
\alpha_{1223}=\tfrac{(b+c)(f-\gamma k_1)-2d(h+\gamma k_2)}{4R^2},\quad \alpha_{1224}=\tfrac{3\gamma(-f+\gamma k_1)}{4R},\quad \alpha_{1234}=\tfrac{-(a+d)\gamma}{2R},\\
\alpha_{1313}=\tfrac{4a(a+d)-R^2\gamma^2-(h+\gamma k_2)^2}{4R^2},\quad \alpha_{1323}=\tfrac{2(a+d)(b+c)+(-f+\gamma k_1)(h+\gamma k_2)}{4R^2},\\
\alpha_{1314}=\tfrac{-(b+c)\gamma}{4R},\quad \alpha_{1324}=\tfrac{-d\gamma}{2R},\quad \alpha_{1334}=\tfrac{\gamma(f-\gamma k_1)}{4R},\quad \alpha_{1423}=\tfrac{a\gamma}{2R},\\
\alpha_{1414}=\tfrac{4a^2+(3b-c)(b+c)+3(h+\gamma k_2)^2}{4R^2},\quad \alpha_{1424}=\tfrac{4(ac+bd)+3(f-\gamma k_1)(h+\gamma k_2)}{4R^2},\\
\alpha_{1434}=\tfrac{(b-c)(f-\gamma k_1)+4(a+d)(h+\gamma k_2)}{4R^2},\quad \alpha_{2323}=\tfrac{4d(a+d)-R^2\gamma^2-(f-\gamma k_1)^2}{4R^2},\\
\alpha_{2324}=\tfrac{(b+c)\gamma}{4R},\quad \alpha_{2334}=\tfrac{-\gamma(h+\gamma k_2)}{4R},\quad \alpha_{2424}=\tfrac{-(b-3c)(b+c)+4d^2+3(f-\gamma k_1)^2}{4R^2},\\
\alpha_{2434}=\tfrac{4(a+d)(f-\gamma k_1)+(c-b)(h+\gamma k_2)}{4R^2},\quad \alpha_{3434}=\tfrac{4(a+d)^2-(f-\gamma k_1)^2-(h+\gamma k_2)^2}{4R^2}.
\end{gathered}
\tag{2.43}
$$

Further, we obtain easily

Lemma 2.2.21. *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1, e_2, e_3, e_4\}$ *is of the form*

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \frac{\gamma(-f+\gamma k_1)}{2R} \\ \beta_{12} & \beta_{22} & \beta_{23} & \frac{\gamma(h+\gamma k_2)}{2R} \\ \beta_{13} & \beta_{23} & \beta_{33} & 0 \\ \frac{\gamma(-f+\gamma k_1)}{2R} & \frac{\gamma(h+\gamma k_2)}{2R} & 0 & \beta_{44} \end{pmatrix} \tag{2.44}$$

where

$$\beta_{11} = \frac{-4a(a+d)-b^2+c^2-R^2\gamma^2-(h+\gamma k_2)^2}{2R^2}, \quad \beta_{12} = \frac{(-f+\gamma k_1)(h+\gamma k_2)-a(b+3c)-d(3b+c)}{2R^2},$$

$$\beta_{13} = \frac{c(f-\gamma k_1)-(2a+3d)(h+\gamma k_2)}{2R^2}, \quad \beta_{22} = \frac{b^2-c^2-4d(a+d)-R^2\gamma^2-(f-\gamma k_1)^2}{2R^2},$$

$$\beta_{23} = \frac{(3a+2d)(-f+\gamma k_1)+b(h+\gamma k_2)}{2R^2}, \quad \beta_{33} = \frac{-4(a+d)^2+R^2\gamma^2+(f-\gamma k_1)^2+(h+\gamma k_2)^2}{2R^2},$$

$$\beta_{44} = \frac{-(b+c)^2-4((a+d)^2-ad)-(f-\gamma k_1)^2-(h+\gamma k_2)^2}{2R^2}.$$

Now we obtain the following analogue of Lemma 2.2.5:

Lemma 2.2.22. *The condition* (2.10) *for the Ricci tensor of type* $(0,2)$ *is equivalent* *to the system of algebraic equations*

$$
\begin{aligned}
(1,1,1) \to & a(f-\gamma k_1)=0,\\
(1,1,2) \to & (b+c)(f-\gamma k_1)-a(h+\gamma k_2)=0,\\
(1,1,3) \to & a(b+3c)+(3b+c)d=0,\\
(1,1,4) \to & c(b(a-3d)-c(a+d)-(f-\gamma k_1)(h+\gamma k_2))\\
& +a(4d^2-R^2\gamma^2+(f-\gamma k_1)^2)=0,\\
(1,2,3) \to & 2a^2+b^2-c^2-2d^2=0,\\
(1,2,4) \to & -(b-c)(a-d)(a+d)+4(b+c)(bc-ad)-(b+c)R^2\gamma^2\\
& +b(f-\gamma k_1)^2+c(h+\gamma k_2)^2+(a+d)(-f+\gamma k_1)(h+\gamma k_2)=0,\\
(1,3,4) \to & (a(b+4c)+d(3b+2c))(-f+\gamma k_1)\\
& +(c(3b+2c)-4a(a-2d)+d^2)(h+\gamma k_2)=0,\\
(2,2,1) \to & d(f-\gamma k_1)-(b+c)(h+\gamma k_2)=0,\\
(2,2,2) \to & d(h+\gamma k_2)=0,\\
(2,2,4) \to & -b^2(a+d)+bc(d-3a)+d(4a^2-R^2\gamma^2)\\
& +(d(h+\gamma k_2)+b(-f+\gamma k_1))(h+\gamma k_2)=0,\\
(2,3,4) \to & (a^2+b(2b+3c)-4d(2a+d))(f-\gamma k_1)\\
& -(a(2b+3c)+d(4b+c))(h+\gamma k_2)=0,\\
(3,3,1) \to & (2a+d)(-f+\gamma k_1)+b(h+\gamma k_2)=0,\\
(3,3,2) \to & c(-f+\gamma k_1)+(a+2d)(h+\gamma k_2)=0,\\
(3,3,4) \to & a(b+c)^2+d((b+c)^2-4a(a+d)-h^2)+(a+d)R^2\gamma^2\\
& +h(b+c)(f-\gamma k_1)-a(f-\gamma k_1)^2\\
& +\gamma k_2((b+c)(f-\gamma k_1)-d(\gamma k_2+2h))=0,\\
(4,4,1) \to & a(-f+\gamma k_1)+c(h+\gamma k_2)=0,\\
(4,4,2) \to & b(-f+\gamma k_1)+d(h+\gamma k_2)=0.
\end{aligned}
\tag{2.45}
$$

Here the symbol "$(\alpha,\beta,\gamma)\to$" marks the substitution of $(e_\alpha,e_\beta,e_\gamma)$ for (X,Y,Z) respectively.

Now, the goal is to find the values of a,b,c,d,f,h,k_1,k_2,k_3 and $\gamma\neq 0$ which satisfy this system of equations and to study each of these cases.

Proposition 2.2.23. *The only possible solutions of the system of algebraic equations* (2.45) *are, up to a re-numeration of the triplet $\{e_1,e_2,e_3\}$, the following ones:*

1. $a=b=c=d=0,\ \gamma\neq 0,\ h\neq 0,\ f,k_1,k_2,k_3$ *arbitrary.*

2. $a=d=0,\ b=-c\neq 0,\ h=-\gamma k_2,\ f=\gamma k_1,\ \gamma\neq 0,\ k_1,k_2,k_3$ *arbitrary.*
 In these two first cases, the corresponding spaces are Riemannian direct products $M^3\times\mathbb{R}$, which are not locally symmetric. Hence, they give the case ii) of the Classification Theorem 2.1.2.

3. $a = d = \frac{\gamma R}{2},\ b = -c,\ h = -\gamma k_2,\ f = \gamma k_1,\ \gamma \neq 0,\ k_1, k_2, k_3$ *arbitrary.*
In this situation, the corresponding spaces are irreducible Riemannian manifolds with all Ricci eigenvalues equal to $-\frac{3\gamma^2}{2}$. *Hence, the corresponding spaces belong to the case* $i)$ *of the Classification Theorem 2.1.2.*

4. $a = -d,\ d^2 \leq \frac{\gamma^2}{4}R^2,\ b = c = \frac{1}{2}\sqrt{-4d^2 + \gamma^2 R^2},\ h = -\gamma k_2,\ f = \gamma k_1,\ \ \gamma \neq 0,$ $k_1,\ k_2,\ k_3$ *arbitrary.*
For this situation, the corresponding spaces are irreducible Riemannian manifolds, not locally symmetric, with the Ricci eigenvalues $\rho_1 = \rho_2 = \rho_4 = -\frac{\gamma^2}{2}$, $\rho_3 = \frac{\gamma^2}{2}$. *Moreover, they give the case* $iii)$ *of the Classification Theorem 2.1.2 and the* L_5 *condition is not satisfied here.*

Proof. Let first $b + 3c \neq 0$, then from $(1,1,3)$ we get $a = \frac{-(3b+c)d}{b+3c}$ and after substitution into $(1,2,3)$ we obtain $(b^2 - c^2)(16d^2 + (b+3c)^2) = 0$. Hence $b^2 = c^2$ and from $(1,2,3)$ $a^2 = d^2$. But, if $b = \pm c$, we get from $(1,1,3)$ $a = \mp d$.

If $b + 3c = 0$, we get from $(1,1,3)$ that $d(3b + c) = 0$, i.e. $8dc = 0$. If $c = 0$, then $b = 0$ and we get again $a^2 - d^2 = 0$. If $d = 0$, then from $(1,2,3)$ we obtain $a = 0$, $c = b = 0$. In conclusion, we only have to study the cases $a = \pm d$, $b = \mp c$.

Case A. $a = d,\ b = -c$.

In this case, the system (2.45) simplifies to

$$\begin{aligned}
&(1,1,1) \to d(f - \gamma k_1) = 0,\\
&(1,1,2) \to d(h + \gamma k_2) = 0,\\
&(1,1,4) \to d(4d^2 - \gamma^2(1 - k_1^2 - k_2^2 - k_3^2) = 0,\\
&(3,3,1) \to c(h + \gamma k_2) = 0,\\
&(1,1,2) \to c(f - \gamma k_1) = 0.
\end{aligned} \tag{2.46}$$

Now, firstly we suppose that $d = 0$. Then, if $c = 0$, we obtain the case 1 of the Proposition 2.2.23 (note that $h \neq 0$ because otherwise we would have the nilpotent semi-direct product) and, if $c \neq 0$, we obtain the case 2 of the Proposition 2.2.23.

In both cases 1 and 2 we obtain $(\nabla_{e_1}\mathcal{R})(e_1, e_2)e_1 = \frac{\gamma}{2R^2}(h^2 + 2\gamma k_2 h + \gamma^2(1 - k_1^2 - k_3^2))e_3 \neq 0$ and the corresponding spaces are not locally symmetric. Further, put $X = \frac{(-f+\gamma k_1)}{R\gamma}e_1 + \frac{(h+\gamma k_2)}{R\gamma}e_2 + e_4$. Then we check easily that $\nabla_{e_i}X = 0$ for $i = 1,2,3,4$ and X is (globally) parallel. Hence the action of the holonomy group $\Psi(e)$ is trivial on the 1-dimensional subspace $\mathrm{span}(X) \subset T_eG$. Consequently, according to the de Rham theorem, we have $(G, g) = M^3 \times \mathbb{R}$ when M^3 is not locally symmetric (and hence irreducible). According to Theorem 2.1.1, M^3 is naturally reductive and we obtain the case $ii)$ of our Classification Theorem 2.1.2.

On the other hand, if $d \neq 0$, it is clear from (2.46) that $f = \gamma k_1$, $h = -\gamma k_2$ and

$$(1,1,4) \to d(4d^2 - \gamma^2(1 - k_1^2 - k_2^2 - k_3^2)) = 0.$$

Hence, we obtain the case 3 of Proposition 2.2.23. From Lemma 2.2.21 we see that we have four equal Ricci eigenvalues $-\frac{3\gamma^2}{2}$. Then the corresponding spaces are Einstein and by a well-known theorem of G. R. Jensen (see [J]) they are locally symmetric. Hence, they belong to the case *i*) of our Classification Theorem 2.1.2.

Case B. $a = -d,\ b = c$.

In this case, the system (2.45) is reduced to

$$\begin{aligned}
&(1,1,1) \to d(f - \gamma k_1) = 0,\\
&(1,1,4) \to d(4c^2 + 4d^2 - \gamma^2(1 - k_1^2 - k_2^2 - k_3^2)) = 0,\\
&(1,2,4) \to c(4c^2 + 4d^2 - \gamma^2(1 - k_1^2 - k_2^2 - k_3^2)) = 0,\\
&(2,2,2) \to d(h + \gamma k_2) = 0,\\
&(3,3,1) \to c(h + \gamma k_2) = 0,\\
&(3,3,2) \to c(f - \gamma k_1) = 0.
\end{aligned} \tag{2.47}$$

Note that if we assume that $(-f + \gamma k_1) \neq 0$ or $(h + \gamma k_2) \neq 0$, we obtain a particular subcase of the case 1. Hence, we can assume that $f = \gamma k_1$ and $h = -\gamma k_2$. Thus, we obtain only two non-equivalent solutions: either $c = d = 0$, which is a particular subcase of the case 1, or the case 4 of Proposition 2.2.23. In the case 4 we have the Ricci eigenvalues $\rho_1 = \rho_2 = \rho_4 = \frac{-\gamma^2}{2}$, $\rho_3 = \frac{\gamma^2}{2}$, and the corresponding spaces are not locally symmetric due to $(\nabla_{e_1}\mathcal{R})(e_1, e_2)e_3 \neq 0$. Now, using (2.42) and (2.43), we obtain that the space of the curvature operators is spanned by the five operators $A_{12}, A_{13}, A_{14}, A_{23}, A_{24}$. Hence the Lie algebra generated by these operators is $\mathfrak{so}(4)$. We see that the action of the holonomy algebra on the tangent space T_eG is irreducible and hence the corresponding Riemannian manifolds are irreducible. Now, we make the following change of the basis:

$$e_1' = \tfrac{2}{\gamma}e_4,\ e_2' = e_1\cos(\alpha) + e_2\sin(\alpha),\ e_3' = -e_1\sin(\alpha) + e_2\cos(\alpha),\ e_4' = \gamma e_3, \tag{2.48}$$

where α is an angle satisfying $d\sin(2\alpha) + b\cos(2\alpha) = 0$. (Especially, we should put $\alpha = 0$ if $b = 0$ and $\alpha = \frac{3\pi}{4}$ if $d = 0$). Then the multiplication table for the new basis $\{e_1', e_2', e_3', e_4'\}$ (when using (2.40)) becomes exactly the same as in the case *iii*) of our Classification Theorem 2.1.2. We only have to change notation. The corresponding metric is also in accordance with the case *iii*). What remains is to prove that the condition L_5 is not satisfied.

Further, if we put $X = e_2 + ve_4$ where v is a *nonzero* parameter, Mathematica 5.0 shows that the Ledger condition $L_5(X) = 0$ can be written in the form

$$\phi_1(b,d) + \phi_2(b,d)v^2 + \phi_3(b,d)v^4 = 0$$

and, because v is a free parameter, this implies

$$\begin{aligned}
\phi_1(b,d) =& -1020 + 364b + 468d - 252bd + (20 - 13b + 30d + 8bd)4d^2\\
&- (61 + 15b - 21d - 14bd)4b^2 = 0,
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
\phi_2(b,d) =& 3564 - 1208b + 604d - 396bd - (140 + 17b + 16d - 8bd)4d^2\\
&- (51 + 3b + 7d - 13bd)4b^2 = 0,
\end{aligned} \tag{2.50}$$

$$\begin{aligned}\phi_3(b,d) = &-16 - 14b - 36d - 18bd - (4+b-d)2d^2 \\ &+ (3+b+d)4b^2 = 0.\end{aligned} \tag{2.51}$$

Mathematica 5.0 affirms that these equations have no common solution. Then the corresponding spaces do not satisfy the Ledger condition L_5 for *some* value $v \neq 0$ and thus, they cannot be D'Atri spaces. This concludes the proof of Proposition 2.2.23.

Semi-direct products $\mathbb{R}^3 \rtimes \mathbb{R}$

Let $\mathfrak{r}^3$ be the Lie algebra of $\mathbb{R}^3$ with a scalar product $\langle\ ,\ \rangle_3$. The algebra of all derivations D of $\mathfrak{r}^3$ is $\mathfrak{gl}(3,\mathbb{R})$. This means that the matrix form of D depends on 9 arbitrary parameters with respect to any fixed orthonormal basis of $\mathfrak{r}^3$. Moreover, if D is fixed, then we can make three convenient rotations in the coordinate planes to obtain a particular orthonormal basis $\{f_1, f_2, f_3\}$ for which the matrix form of D is a sum of a diagonal matrix and a skew-symmetric matrix. In other words, we have the general matrix form

$$D : \left\{ \begin{pmatrix} a & b & c \\ -b & f & h \\ -c & -h & p \end{pmatrix} : a, b, c, f, h, p \in \mathbb{R} \right\}$$

depending just on 6 parameters. The last statement can be proved by the following algebraic lemma:

Lemma 2.2.24. *Let A, $n \times n$ real matrix. There is an orthogonal matrix Q such that $QAQ^{-1} = \mathcal{D} + S$ where $\mathcal{D}$ is a diagonal matrix and S is a skew-symmetric one.*

Proof. Let $A_+ = \frac{A+A^t}{2}$, $A_- = \frac{A-A^t}{2}$ and $A = A_+ + A_-$. Note that A_+ is a symmetric matrix and A_- is a skew-symmetric one. Then we know that exist an orthogonal matrix Q ($Q^t = Q^{-1}$) such that $QA_+Q^{-1} = \mathcal{D}$ and consequently $QAQ^{-1} = \mathcal{D} + QA_-Q^{-1} = \mathcal{D} + S$. Here S is a skew-symmetric matrix because to $S^t = (QA_-Q^{-1})^t = (Q^{-1})^t(A_-)^tQ^t = Q(-A_-)Q^{-1} = -QA_-Q^{-1} = -S$.

Moreover, we have

$$[f_1, f_2] = 0, \ \ [f_1, f_3] = 0, \ \ [f_2, f_3] = 0. \tag{2.52}$$

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{r}^3 + \mathbb{R}$, where the multiplication table is given by (2.52) and,

$$\begin{aligned}&[f_4, f_1] = af_1 + bf_2 + cf_3, \quad [f_4, f_2] = -bf_1 + ff_2 + hf_3, \\ &[f_4, f_3] = -cf_1 - hf_2 + pf_3, \quad \langle f_i, f_4 \rangle = k_i, \ i = 1, 2, 3.\end{aligned} \tag{2.53}$$

Here $a, b, c, f, h, p, k_1, k_2, k_3$ are arbitrary parameters where $\sum\limits_{i=1}^{3} k_i^2 < 1$.

This gives rise to a simply connected group space $(G = \mathbb{R}^3 \rtimes \mathbb{R}, g)$.

Now we replace the basis $\{f_i\}$ by the new basis $\{e_i\}$, as in the formula (2.3). Then we get an orthonormal basis for which

$$\begin{aligned}&[e_1,e_2]=0,\ [e_1,e_3]=0,\ [e_2,e_3]=0,\ [e_4,e_1]=\tfrac{1}{R}\left(ae_1+be_2+ce_3\right),\\ &[e_4,e_2]=\tfrac{1}{R}\left(-be_1+fe_2+he_3\right),\ [e_4,e_3]=\tfrac{1}{R}\left(-ce_1-he_2+pe_3\right).\end{aligned} \tag{2.54}$$

Now we are going to calculate, in the new basis, the expression for the Levi-Civita connection, the curvature tensor, the Ricci matrix, and the condition for the Ricci tensor to be cyclic parallel.

By an easy calculation we get

Lemma 2.2.25.

$$\begin{gathered}\nabla_{e_1}e_1=\tfrac{a}{R}e_4,\ \nabla_{e_2}e_2=\tfrac{f}{R}e_4,\ \nabla_{e_3}e_3=\tfrac{p}{R}e_4,\ \nabla_{e_4}e_4=0,\\ \nabla_{e_1}e_2=0=\nabla_{e_2}e_1,\ \nabla_{e_1}e_3=0=\nabla_{e_3}e_1,\ \nabla_{e_2}e_3=0=\nabla_{e_3}e_2,\\ \nabla_{e_1}e_4=-\tfrac{a}{R}e_1,\ \nabla_{e_4}e_1=\tfrac{b}{R}e_2+\tfrac{c}{R}e_3,\ \nabla_{e_2}e_4=-\tfrac{f}{R}e_2,\\ \nabla_{e_4}e_2=-\tfrac{b}{R}e_1+\tfrac{h}{R}e_3,\ \nabla_{e_3}e_4=-\tfrac{p}{R}e_3,\ \nabla_{e_4}e_3=-\tfrac{c}{R}e_1-\tfrac{h}{R}e_2.\end{gathered} \tag{2.55}$$

Similarly to Lemma 2.2.3 we can now derive

Lemma 2.2.26. *The components of the curvature operator are*

$$\begin{aligned}\mathcal{R}(e_1,e_2)&=\tfrac{af}{R^2}A_{12}, & \mathcal{R}(e_1,e_4)&=\tfrac{a^2}{R^2}A_{14}+\tfrac{b(f-a)}{R^2}A_{24}+\tfrac{c(p-a)}{R^2}A_{34},\\ \mathcal{R}(e_1,e_3)&=\tfrac{ap}{R^2}A_{13}, & \mathcal{R}(e_2,e_4)&=\tfrac{b(f-a)}{R^2}A_{14}+\tfrac{f^2}{R^2}A_{24}+\tfrac{h(p-f)}{R^2}A_{34},\\ \mathcal{R}(e_2,e_3)&=\tfrac{fp}{R^2}A_{23}, & \mathcal{R}(e_3,e_4)&=\tfrac{c(p-a)}{R^2}A_{14}+\tfrac{h(p-f)}{R^2}A_{24}+\tfrac{p^2}{R^2}A_{34}.\end{aligned} \tag{2.56}$$

Further, we obtain easily

Lemma 2.2.27. *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1,e_2,e_3,e_4\}$ *is of the form*

$$\begin{pmatrix} -\frac{a(a+f+p)}{R^2} & \frac{b(a-f)}{R^2} & \frac{c(a-p)}{R^2} & 0\\ \frac{b(a-f)}{R^2} & -\frac{f(a+f+p)}{R^2} & \frac{h(f-p)}{R^2} & 0\\ \frac{c(a-p)}{R^2} & \frac{h(f-p)}{R^2} & -\frac{p(a+f+p)}{R^2} & 0\\ 0 & 0 & 0 & -\frac{a^2+f^2+p^2}{R^2}\end{pmatrix}. \tag{2.57}$$

Now we obtain the following analogue of Lemma 2.2.5

Lemma 2.2.28. *The condition* (2.10) *for the Ricci tensor of type* $(0,2)$ *is equivalent to the system of algebraic equations*

$$\begin{aligned}
&(1,1,4) \to -(f+p)a^2 + (f-a)b^2 + (p-a)c^2 + a(f^2+p^2) = 0,\\
&(1,2,4) \to (a+f-2p)ch + (a-f)bp = 0,\\
&(1,3,4) \to (p-a)cf + (a-2f+p)bh = 0,\\
&(2,2,4) \to (a-f)b^2 - (a+p)f^2 + (p-f)h^2 + f(a^2+p^2) = 0,\\
&(2,3,4) \to (f+p-2a)bc + (f-p)ah = 0.
\end{aligned} \tag{2.58}$$

Here the symbol "$(\alpha,\beta,\gamma) \to$" marks the substitution of $(e_\alpha, e_\beta, e_\gamma)$ for (X,Y,Z) respectively.

Now, the goal is to find the values of $a, b, c, f, h, p, k_1, k_2, k_3$ which satisfy this system of equations and to study each of these cases. Here MATHEMATICA 5.0 offers just 21 formally different solutions. But, using various numerations *and* various signs of the vectors e_1, e_2, e_3, we see easily that most of the solutions are mutually equivalent, and we can reduce the number of essentially different solutions to five. Then we get

Proposition 2.2.29. *The only possible solutions of the system of algebraic equations* (2.58) *are, up to a re-numeration of the triplet $\{e_1, e_2, e_3\}$, the following ones:*

1) $p = f = a,\ a, b, c, h, k_1, k_2, k_3$ *arbitrary.*
2) $b = c = f = p = 0,\ a, h, k_1, k_2, k_3$ *arbitrary.*
3) $a = b = c = 0,\ p = f,\ f, h, k_1, k_2, k_3$ *arbitrary.*
4) $c = h = p = 0,\ b = f = -a,\ a \neq 0, k_1, k_2, k_3$ *arbitrary.*
5) $b = \frac{a}{3},\ c = -h = \frac{4a}{3},\ f = -a,\ p = 0,\ a \neq 0, k_1, k_2, k_3$ *arbitrary.*

For the solution 1) we obtain from (2.57) that all four Ricci eigenvalues are equal to $-\frac{3a^2}{R^2}$. Then the corresponding spaces are Einstein and by [J] they are locally symmetric. Hence, they belong to the case *i*) of the Classification Theorem 2.1.2.

For the solution 2) we obtain from (2.57) that the corresponding spaces have the Ricci eigenvalues $\rho_1 = \rho_4 = -\frac{a^2}{R^2}$, $\rho_2 = \rho_3 = 0$. From (2.55) we see that the distribution $\mathrm{span}(e_2, e_3)$ is parallel and hence the holonomy group $\Psi(e)$ acts trivially on it. From the de Rham theorem we see that each space is a direct product $M^2 \times \mathbb{R}^2$, where M^2 is of constant curvature ρ_1. Hence, the corresponding spaces are locally symmetric and they belong to the case *i*) of our Classification Theorem 2.1.2.

For the solution 3) we obtain from (2.57) that the Ricci eigenvalues are $\rho_1 = 0$, $\rho_2 = \rho_3 = \rho_4 = -\frac{2f^2}{R^2}$ and from (2.56) that the curvature tensor takes on the form

$$\begin{gathered}
\mathcal{R}(e_2, e_3) = \tfrac{f^2}{R^2} A_{23}, \quad \mathcal{R}(e_2, e_4) = \tfrac{f^2}{R^2} A_{24}, \quad \mathcal{R}(e_3, e_4) = \tfrac{f^2}{R^2} A_{34},\\
\mathcal{R}(e_1, e_2) = \mathcal{R}(e_1, e_3) = \mathcal{R}(e_1, e_4) = 0.
\end{gathered}$$

We see that each of the spaces is either flat (for $f=0$) or it is a direct product $M^3\times\mathbb{R}$ where M^3 is a space of constant curvature. In the second case, the argument is exactly the same as in Case A of Proposition 2.2.6. The corresponding spaces belong to the case $i)$ of our Classification Theorem 2.1.2.

Under the hypothesis of the solution 4) we obtain from (2.57) that the Ricci eigenvalues are $\rho_1=\rho_4=-\frac{2a^2}{R^2},\ \rho_2=\frac{2a^2}{R^2},\ \rho_3=0$. Besides, it is easy to check that $(\nabla_{e_1}\mathcal{R})(e_1,e_2)e_1\neq 0$ and the curvature tensor (2.56) takes on the form

$$\begin{gathered}\mathcal{R}(e_1,e_2)=-\tfrac{a^2}{R^2}A_{12},\quad \mathcal{R}(e_1,e_3)=0,\quad \mathcal{R}(e_1,e_4)=\tfrac{a^2}{R^2}(A_{14}+2A_{24}),\\ \mathcal{R}(e_2,e_3)=0,\quad \mathcal{R}(e_2,e_4)=\tfrac{a^2}{R^2}(2A_{14}+A_{24}),\quad \mathcal{R}(e_3,e_4)=0.\end{gathered}$$

Then, the space of the curvature operators is obviously spanned by the three operators A_{12},A_{14},A_{24}. In addition, $\nabla_{e_i}e_3=0$ for all $i=1,...,4$. Consequently, according to Lemma 2.2.7 and the de Rham theorem the corresponding manifolds are (not locally symmetric) Riemannian direct products $M^3\times\mathbb{R}$. Moreover, according to Theorem 2.1.1, M^3 is naturally reductive and we obtain the case $ii)$ of our Classification Theorem 2.1.2.

Finally, we shall study the solution 5). We obtain from (2.57) that we have here four distinct Ricci eigenvalues

$$\rho_1=\frac{-2a^2}{3R^2},\ \rho_2=\frac{a^2(1-\sqrt{33})}{3R^2},\ \rho_3=\frac{a^2(1+\sqrt{33})}{3R^2},\ \rho_4=\frac{-2a^2}{R^2}.$$

Now, let us introduce a new basis $\{e'_1,e'_2,e'_3,e'_4\}$ by

$$e'_1=-\tfrac{R\sqrt{3}}{4a}e_4,\ e'_2=-\tfrac{R\sqrt{3}}{4a\sqrt{2}}(e_2-e_1),$$

$$e'_3=-\tfrac{R}{4a\sqrt{2}}(e_1+e_2+2e_3),\ e'_4=-\tfrac{R}{4a\sqrt{3}}(2e_1+2e_2+e_3).$$

Here $\langle e'_i,e'_i\rangle=\frac{3R^2}{16a^2}$, for $i=1,2,3,4$, the triplet $\{e'_1,e'_2,e'_3\}$ is orthogonal, $\langle e'_3,e'_4\rangle=\frac{3R^2}{16a^2}\sqrt{\frac{2}{3}}$ and $\langle e'_i,e'_4\rangle=0$, for $i=1,2$. Using the multiplication table (2.54) and the assumptions of the case 5 of Proposition 2.2.29, we obtain a new multiplication table

$$[e'_1,e'_2]=e'_3,\quad [e'_1,e'_3]=\tfrac{1}{2}e'_2,\quad [e'_2,e'_3]=[e'_4,e'_1]=[e'_4,e'_2]=[e'_4,e'_3]=0. \tag{2.59}$$

Now, if we compare this multiplication table and the scalar products $\langle e'_i,e'_j\rangle$ with the multiplication table and the family of metrics, $g_{(c,k)}$, in the case $iv)$ of the Classification Theorem 2.1.2, we see that this is exactly the subcase where $k^2=\frac{2}{3}$ and the parameter c in the metric is equal to $\frac{-4a}{R\sqrt{3}}$. Notice that it is the particular subcase which was omitted in the case 3 of Proposition 2.2.6 for rather formal reason that it was *not* generated on a non-solvable group $G_3\times\mathbb{R}$.

2.3 Podesta and Spiro's Classification Theorem

In [Po-Sp], F. Podestà and A. Spiro published the following classification theorem.

Theorem 2.3.1. *Let (M, g) be a 4-dimensional curvature homogeneous Riemannian manifold of type $\mathcal{A}$, not Einstein, with at most three distinct Ricci principal curvatures. Then just one of the following cases occurs:*

a) *M is locally symmetric;*

b) *(M, g) is locally isometric to a Riemannian product $M^3 \times \mathbb{R}$, where M^3 is a 3-dimensional Riemannian space with two distinct Ricci curvatures $(\rho_1,\ \rho_2 = \rho_1, \rho_3)$, $\rho_3 \neq \rho_1$: M^3 is the total space of a Riemannian submersion over a surface N of constant curvature $\rho_1 + \rho_3$; the fibres of this submersion are geodesics and the integrability tensor A of the submersion is given by $\sqrt{2\rho_3}w$, where w is the area form of N;*

c) *(M, g) is locally isometric to the simply connected Lie group (G, g_a), whose Lie algebra $\mathfrak{g}$ is described by*

$$[e_1, e_2] = -e_2,\ \ [e_1, e_3] = e_3,\ \ [e_2, e_3] = e_4,$$

$$[e_1, e_4] = [e_2, e_4] = [e_3, e_4] = 0,$$

endowed with the left-invariant metric g_a $(a \in \mathbb{R}^+)$

$$g_a = \frac{1}{a^2} w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3 + 4a^2 w^4 \otimes w^4,$$

$\{w^i\}$ being the dual basis of $\{e_i\}$. The metrics g_a have Ricci eigenvalues $\rho_1 = \rho_2 = \rho_3 = -2a^2$, $\rho_4 = -\rho_1 = 2a^2$ and are not isometric to each other for different values of a. Moreover, the Riemannian manifolds (G, g_a) are irreducible and not locally symmetric.

We are going to compare this Theorem with our Classification Theorem 2.1.2. The case c) of Theorem 2.3.1 is exactly the case iii) of our Classification Theorem 2.1.2. It suffices to put $a = \gamma/2$. Also, applying Theorem 2.1.1 to a direct product $M^3 \times \mathbb{R}$ we can see that the case b) of Theorem 2.3.1 coincides with the case ii) of our Classification Theorem 2.1.2 (and it is simplified herewith). (See also [Bu-V]). On the other hand, as we have claimed in [AM], Theorem 2.3.1 is incomplete because the case v) of our Classification Theorem 2.1.2 is missing there.

When the new family of examples was found, we contacted A. Spiro and F. Podestà who confirmed us that there was really a gap in the paper [Po-Sp] and they kindly asked us to publish the following *Erratum*: the formula in page 236, line 11, should read correctly

$$d_{34}^2(d_{12}^3 - d_{21}^3) - d_{24}^3(d_{13}^2 - d_{31}^2) = d_{34}^2 \left(d_{12}^3 \left(1 - \frac{\rho_2 - \rho_4}{\rho_3 - \rho_4} \right) - 2d_{31}^2 \frac{\rho_2 - \rho_4}{\rho_3 - \rho_4} \right) = 0.$$

Several weeks later they sent us a detailed and complete correction of the paper [Po-Sp] where they recovered the case v) of our Classification Theorem - in a bit different but still equivalent form. Also, they concluded that it was the only missing family. We reproduce here (with some cosmetic changes) the detailed erratum done by F. Podestà and A. Spiro, with their kind consent.

Erratum (February 26, 2005)**.** Let (M, g) be a 4-dimensional Riemannian manifold with constant Ricci principal curvatures ρ_i, $i = 1, ..., 4$ such that $\rho_1 = \rho_2$ and ρ_2, ρ_3, ρ_4 are all distinct. Let $\{e_i\}_{i=1,...,4}$ be a fixed set of vector fields, which gives an orthonormal frame at any point of M such that the Ricci tensor S is diagonal in such a frame, i.e. $S(e_i, e_j) = \rho_i \delta_{ij}$. Finally, we denote by d_{ij}^k the Christoffel symbols of the Levi-Civita connection with respect to the frame field $\{e_i\}_{i=1,...,4}$, i.e. the smooth functions $d_{ij}^k = g(\nabla_{e_i} e_j, e_k)$. Notice that $d_{ij}^k = -d_{ik}^j$ by orthonormality of the frame field $\{e_i\}_{i=1,...,4}$. The gap in the proof of Theorem 2.3.1 concerns the analysis of Subcase 1.1 of class $\mathcal{A}$ (see p. 234 of [Po-Sp]).

Under the hypothesis of Subcase 1.1 of [Po-Sp], there exists an orthonormal frame field $\{e_i\}_{i=1,...,4}$ in a neighborhood of any point $p \in M$ such that d_{ij}^k are all vanishing except for the following functions:

$$d_{32}^1 = -d_{31}^2 = -A, \quad d_{21}^3 = -d_{23}^1 = \frac{\rho_4 - \rho_2}{\rho_4 - \rho_3} A, \quad d_{12}^3 = -d_{13}^2 = -\frac{\rho_4 - \rho_2}{\rho_4 - \rho_3} A,$$

$$d_{32}^4 = -d_{34}^2 = f, \quad d_{23}^4 = -d_{24}^3 = -\frac{\rho_2 - \rho_4}{\rho_3 - \rho_4} f, \tag{2.60}$$

where A, f are smooth functions, $A > 0$ and f nonzero.

Now, we consider the Jacobi identity

$$[e_1, [e_3, e_4]] + [e_3, [e_4, e_1]] + [e_4, [e_1, e_3]] = 0$$

and the inner product of both sides with the vector field e_3. After that, we write each Lie bracket by means of the identities

$$[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i = (d_{ij}^k - d_{ji}^k) e_k,$$

obtaining from the previous claim that

$$(d_{12}^3 - d_{21}^3) d_{34}^2 + (-d_{24}^3)(d_{13}^2 - d_{31}^2) = fA \frac{\rho_4 - \rho_2}{\rho_4 - \rho_3} \left(3 - \frac{\rho_4 - \rho_2}{\rho_4 - \rho_3} \right) = 0.$$

Since $\rho_i - \rho_j \neq 0$ for any $i, j = 2, 3, 4$, we immediately get the following necessary relation between the Ricci eigenvalues ρ_i:

$$\frac{\rho_4 - \rho_2}{\rho_4 - \rho_3} = 3 \quad \text{or, equivalently,} \quad 2\rho_4 + \rho_2 - 3\rho_3 = 0. \tag{2.61}$$

On the other hand, A, ρ_i and f must satisfy the relations (3.2) of [Po-Sp], i.e. the expressions which give the components of the Ricci curvature tensor in terms of the Christoffel symbols d_{ij}^k. Substituting the expressions (2.60) and (2.61) in those relations, we get that A, ρ_i and f satisfy the following equations:

$$\rho_1 = \rho_2 = -2\frac{\rho_4 - \rho_2}{\rho_4 - \rho_3}A^2 = -6A^2, \quad \rho_3 = 2\left(\frac{\rho_4 - \rho_2}{\rho_4 - \rho_3}\right)^2 A^2 = 18A^2,$$

$$\rho_4 = 2\frac{\rho_4 - \rho_2}{\rho_4 - \rho_3}f^2 = 6f^2. \tag{2.62}$$

From (2.62) it follows immediately that A and $f = d_{32}^4$ are constants and, changing e_4 into $-e_4$, there is no loss of generality if we assume that $f > 0$. Moreover, substituting (2.62) into $(2.61)_2$, we obtain that

$$f = \sqrt{5}A \qquad \text{and hence that} \qquad \rho_4 = 30A^2. \tag{2.63}$$

According to (2.60), all Christoffel symbols are constant and the vector fields e_i, $i = 1, 2, 3, 4$, generate a 4-dimensional Lie algebra $\mathfrak{g}$ whose Lie brackets can be easily computed as follows:

$$\begin{aligned} [e_1, e_2] = -6Ae_3,\ [e_1, e_3] &= 2Ae_2,\ [e_1, e_4] = 0, \\ [e_2, e_3] = -2Ae_1 - 4A\sqrt{5}e_4,\ [e_2, e_4] &= 3A\sqrt{5}e_3,\ [e_3, e_4] = -A\sqrt{5}e_2. \end{aligned} \tag{2.64}$$

We constructed a new family of spaces of class $\mathcal{A}$ and, obviously, this is the only missing family in our Theorem 2 in [Po-Sp].

(*End of the Erratum*).

Now, let us introduce a new basis $\{e_1', e_2', e_3', e_4'\}$ by

$$e_1' = -\tfrac{1}{2\sqrt{21}A}e_2, \quad e_2' = -\tfrac{1}{2\sqrt{21}A}e_3,$$

$$e_3' = -\tfrac{1}{42A}(e_1 + 2\sqrt{5}e_4), \quad e_4' = -\tfrac{\sqrt{21}}{126A}(\sqrt{5}e_1 - 2e_4).$$

Here $\langle e_i', e_i' \rangle = \frac{1}{84A^2}$, for $i = 1, 2, 3, 4$, the triplet $\{e_1', e_2', e_3'\}$ is orthogonal, $\langle e_3', e_4' \rangle = \frac{1}{84A^2}\sqrt{\frac{5}{21}}$ and $\langle e_i', e_4' \rangle = 0$, for $i = 1, 2$. Using the multiplication table (2.64) we obtain a new multiplication table

$$[e_1', e_2'] = e_3',\ [e_3', e_1'] = \tfrac{6}{7}e_2',\ [e_2', e_3'] = \tfrac{2}{7}e_1',\ [e_4', e_1'] = [e_4', e_2'] = [e_4', e_3'] = 0. \tag{2.65}$$

Now, if we compare this multiplication table and the scalar products $\langle e_i', e_j' \rangle$ with the multiplication table and the family of metrics, g_c, in the case v) of the Classification Theorem 2.1.2, we see that we obtain exactly the same family of spaces via the substitution $c = -2\sqrt{21}A$.

Therefore, the classification by F. Podesta and A. Spiro should be now corrected as follows:

Theorem 2.3.2. *Let (M, g) be a 4-dimensional curvature homogeneous Riemannian manifold of type $\mathcal{A}$, not Einstein, with at most three distinct Ricci principal curvatures. Then just one of the following cases holds:* ***a)*** *(M, g) is locally symmetric, or one of the cases* ***b), c)*** *from Theorem 2.3.1 occurs, or the case* ***d)****, namely the family described in the case v) of the Classification Theorem 2.1.2 from Section 3.1 occurs.*

Note that, in the case of at most three distinct Ricci eigenvalues, the corrected result by Podesta and Spiro is stronger than our classification result because the homogeneity is replaced by the weaker assumption of curvature homogeneity.

Chapter 3

The Jacobi equation over g.o. spaces

A Riemannian g.o. manifold is a homogeneous Riemannian manifold (M, g) on which every geodesic is an orbit of a one-parameter group of isometries. It is known that every simply connected Riemannian g.o. manifold of dimension$\leq$5 is naturally reductive. The first counter-example of a Riemannian g.o. manifold which is not naturally reductive is Kaplan's six-dimensional example. On the other hand, A. M. Naveira and A. Tarrío have developed a method for solving the Jacobi equation in the manifold $Sp(2)/SU(2)$. This method is based on the fact that the Jacobi operator has constant osculating rank over naturally reductive spaces. In this chapter, we prove that the Jacobi operator has constant osculating rank over g.o. spaces and, as a consequence, we solve the Jacobi equation in the Kaplan example.

To facilitate access to the individual topics, this chapter is rendered as self-contained as possible. Anyway, we refer to Chapter 1 for more details and the basic references about the main topics.

3.1 Introduction

It is well-known that a Riemannian homogeneous space $(M, g) = G/H$ with its origin $p = \{H\}$ and with an ad(H)-invariant decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ is *naturally reductive* (with respect to this decomposition) if and only if the following holds:

$$\begin{array}{c}\text{For any vector } X \in \mathfrak{m}\backslash\{0\}, \text{ the curve } \gamma(t) = \tau(\exp tX)(p) \\ \text{is a geodesic with respect to the Riemannian connection}\end{array} \tag{3.1}$$

where exp and $\tau(h)$ denote the Lie exponential map of G and the left transformation of G/H induced by $h \in G$ respectively. Moreover, the naturally reductive spaces have been studied by a number of authors as a natural generalization of Riemannian symmetric spaces.

Now, natural reductivity is still a special case of a more general property, which follows easily from (3.1):

$$\begin{array}{c}\text{Each geodesic of } (M,g) = G/H, \text{ is an orbit of a} \\ \text{one-parameter group of isometries } \{\exp tZ\}, Z \in \mathfrak{g}.\end{array} \tag{3.2}$$

Riemannian homogeneous spaces $(M,g) = G/H$ with the property (3.2) will be called (Riemannian) g.o. spaces.

The extensive study of g.o. spaces only started with A. Kaplan's paper [Ka.2] in 1983, because he gave the first counter-example of a Riemannian g.o. manifold which is not naturally reductive. This is a six-dimensional Riemannian nilmanifold with a two-dimensional center, one of the so-called "generalized Heisenberg groups" or "H-type groups". It is worthwhile to mention that A. Kaplan also proved in [Ka.2] that any generalized Heisenberg group is a D'Atri space.

The resolution of the Jacobi equation on a Riemannian manifold can be quite a difficult task. In the Euclidean space the solution is trivial. For the symmetric spaces, the problem is reduced to a system of differential equations with constant coefficients. In [Ch.1] and [Ch.2], I. Chavel obtained a partial solution of this problem for the naturally reductive manifolds $V_1 = Sp(2)/SU(2)$ and $V_2 = SU(5)/(Sp(2) \times S^1)$. The method used by I. Chavel, which allows him to solve the Jacobi equation in some particular directions of the geodesic, is based on the use of the canonical connection. Nevertheless, his method does not seem to apply in a simple way to solve the Jacobi equation along a unit geodesic of an arbitrary direction. For naturally reductive compact homogeneous spaces, W. Ziller [Z] solves the Jacobi equation working with the canonical connection; but the solution can be considered of the qualitative type (it does not allow us to obtain in an easy way the explicit solutions of the Jacobi fields for any particular example nor for an arbitrary direction of the geodesic). The methods of solving the Jacobi equation used by I. Chavel and W. Ziller, are special cases of the following procedure, which is valid, in particular, on any g.o. space and any generalized Heisenberg group. (See [B-Tr-V, p. 51]).

Lemma 3.1.1. *Let M be a Riemannian manifold with Levi-Civita connection ∇ and $\gamma : I \to M$ a geodesic in M parametrized by arc length. We denote by $\dot\gamma$ the standard unit tangent field on I. Suppose there exists a $\nabla_{\dot\gamma}$-parallel skew-symmetric tensor field T_γ along γ such that the Jacobi operator $\mathcal{R}_\gamma$ along γ satisfies $\nabla_{\dot\gamma}\mathcal{R}_\gamma := \mathcal{R}'_\gamma = [\mathcal{R}_\gamma, T_\gamma]$. Then define a new covariant derivative*

$$\bar\nabla_{\dot\gamma} := \nabla_{\dot\gamma} + T_\gamma,$$

and put

$$\bar{\mathcal{R}}_\gamma := \mathcal{R}_\gamma + T_\gamma^2.$$

Then $\mathcal{R}_\gamma$, $\bar{\mathcal{R}}_\gamma$ and T_γ are $\bar\nabla_{\dot\gamma}$-parallel along γ and the Jacobi equation along γ is

$$\bar\nabla_{\dot\gamma}\bar\nabla_{\dot\gamma}B - 2T_\gamma\bar\nabla_{\dot\gamma}B + \bar{\mathcal{R}}_\gamma B = 0,$$

where B is a vector field along γ.

On the other hand, K. Tsukada in [Ts] gives a criterion for the existence of totally geodesic submanifolds of naturally reductive spaces. This criterion is based on the curvature tensor and on a finite number of its derivatives with respect to the Levi-Civita connection. In particular, to prove this result he used two basic formulae proved exclusively for naturally reductive spaces by K. Tojo in [To]. From these formulae he obtained that the curvature tensor can be considered as a curve in the space of curvature tensors on $\mathfrak{m}$. Later, using the general theory, he concluded that the curvature tensor has constant osculating rank, $r \in \mathbb{N}$, over naturally reductive spaces.

Some years later, K. Tsukada's result was applied by A. M. Naveira and A. Tarrío in [N-T] to give a method for solving the Jacobi equation $Y'' + \mathcal{R}_\gamma(Y) = 0$ on the naturally reductive manifold $Sp(2)/SU(2)$. Given the generality of the method, the authors conjectured that it could also be applied to solving the Jacobi equation in several other examples of naturally reductive homogeneous spaces. Indeed, they were not wrong because in [AM-Ba], we have successfully applied this method on the manifold $U(3)/(U(1) \times U(1) \times U(1))$.

In this chapter, we prove that the A. M. Naveira and A. Tarrío method can be also applied on g.o. spaces. In particular, we solve the Jacobi equation along a unit geodesic of arbitrary direction on Kaplan's example. Moreover, we check that our result on Kaplan's example coincide with the result obtained using Lemma 3.1.1 (see [B-Tr-V, Theorem of p. 52]).

We shall start, in Section 3.2, proving our general and main result on g.o. spaces:

> *"The Jacobi operator on a g.o. space has always constant osculating rank."*

From now on, we shall write $\mathcal{J}$ instead of $\mathcal{R}_\gamma$ for simplicity of notation.

We shall conclude, in Section 3.3, giving an application of the main result. More specifically, we shall start recalling some known definitions and results regarding "generalized Heisenberg groups". Later, we shall give the recursive expression for the n^{th} covariant derivative of the Jacobi operator at the origin of an H-type group. Finally, we shall calculate the constant osculating rank of the Jacobi operator on Kaplan's example obtaining that it is 4. More concretely, we have obtained that the basic relation satisfied between the n^{th} covariant derivatives for $n = 1, ..., 5$ of the $(0, 4)$ - Jacobi operator along the arbitrary geodesic γ with initial vector x at the origin $p = \gamma_0$ of N is

$$\tfrac{1}{4}|\dot\gamma_0|^4 \mathcal{J}_0^{1)} + \tfrac{5}{4}|\dot\gamma_0|^2 \mathcal{J}_0^{3)} + \mathcal{J}_0^{5)} = 0$$

and the Jacobi operator can be written as

$$\mathcal{J}_t = c_0 + c_1 \cos(t) + c_2 \sin(t) + c_3 \cos(t/2) + c_4 \sin(t/2),$$

where

$$c_0 = \mathcal{J}_0 + 5\mathcal{J}_0^{2)} + 4\mathcal{J}_0^{4)}, \quad c_1 = \tfrac{1}{3}(\mathcal{J}_0^{2)} + 4\mathcal{J}_0^{4)}), \quad c_2 = \tfrac{-1}{3}(\mathcal{J}_0^{1)} + 4\mathcal{J}_0^{3)}),$$
$$c_3 = \tfrac{-16}{3}(\mathcal{J}_0^{2)} + \mathcal{J}_0^{4)}), \quad c_4 = \tfrac{8}{3}(\mathcal{J}_0^{1)} + \mathcal{J}_0^{3)}).$$

Then, as a direct consequence, we shall solve the Jacobi equation using the new method given in [N-T]. In particular, we shall use the results of the previous sections and computer support, using the software MATHEMATICA 5.0. But we put stress on the full transparency of this procedure.

3.2 Main result

In this section, we follow the notation of Section 1.2.

Let $(G/H, g)$ be a g.o. space and let $Z \in \mathfrak{g}$. We denote by Z^* the corresponding fundamental vector field on M; that is

$$Z_q^* = \frac{d}{dt}_{|0}(\tau(\exp(tZ))(q))$$

for each $q \in M$. Moreover, a vector $Z \in \mathfrak{g}$ is called a *geodesic vector* if the curve $\tau(\exp(tZ))(p)$ is a geodesic.

The following property is direct from Definition 1.2.3 (see [K-V.5]).

Proposition 3.2.1. *G/H is a Riemannian g.o. space if and only if the projections of all geodesic vectors fill in the set $T_pM\backslash\{0\}$.*

In the remainder of this section, we shall generalize to Riemannian g.o. spaces some important results over naturally reductive spaces proved by K. Tojo and K. Tsukada in [To] and [Ts], respectively. Moreover, we shall develop their relation with the Jacobi operator which will be so useful for checking concrete examples.

Let (M, g) be a Riemannian g.o. space and $Z \in \mathfrak{g}$ be an arbitrary geodesic vector. We put

$$\mathrm{e}^{-\nabla_X} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \nabla_X^l, \quad X \in \mathfrak{m}.$$

Over g.o. spaces, ∇ denotes the Levi-Civita connection, so obviously the linear map ∇_X is skew-symmetric (i.e. $\nabla_X g$=0 for all $X \in \mathfrak{m}$). Therefore the mapping $\mathrm{e}^{-\nabla_X} : (\mathfrak{m}, \langle\ ,\ \rangle) \to (\mathfrak{m}, \langle\ ,\ \rangle)$ is an isometry and we can obtain the following lemmas.

Lemma 3.2.2. *The parallel translation along the geodesic*

$$\gamma_x(t) = \tau(exp(tZ))(p) \quad with \quad \gamma_x(0) = p,\ \gamma_x'(0) = x = Z_p^*$$

is given by

$$\tau(exp(tZ))_* e^{-\nabla_{tx}} : T_pM(= \mathfrak{m}) \longrightarrow T_{\gamma_x(t)}M.$$

Proof. Since (M, g) is a g.o. space, we can take

$$Y(t) = \tau(\exp(tZ))_*(\mathrm{e}^{-\nabla_{tx}}(y))$$

as a vector field along the geodesic $\gamma_x(t) = \tau(\exp(tZ))(p)$ such that $Y(0) = y \in \mathfrak{m}$.

From Proposition 1.2 of Chapter VI of [Ko-N], we have

$$\begin{aligned}\nabla_{\gamma'(t)}(\tau(\exp(tZ))_*(v)) &= \nabla_{\tau(\exp(tZ))_*(x)}(\tau(\exp(tZ))_*(v)) \\ &= \tau(\exp(tZ))_*(\nabla_x v)\end{aligned}$$

for all $v \in \mathfrak{m}$. Then we have

$$\begin{aligned}\nabla_{\gamma'(t)}Y(t) = {} & \nabla_{\gamma'(t)}(\tau(\exp(tZ))_*(\mathrm{e}^{-\nabla_{tx}}(y))) \\ = {} & \tau(\exp(tZ))_*(\nabla_x \circ (\mathrm{e}^{-\nabla_{tx}}(y))) + \tau(\exp(tZ))_* \left(\frac{d}{dt}(\mathrm{e}^{-\nabla_{tx}}(y))\right) \\ = {} & \tau(\exp(tZ))_* \left(\nabla_x \circ \sum_{l=0}^{\infty} \tfrac{(-1)^l}{l!} t^l \nabla_x^l(y) + \sum_{l=1}^{\infty} \tfrac{(-1)^l}{l-1!} t^{(l-1)} \nabla_x^l(y)\right) \\ = {} & 0.\end{aligned}$$

This proves the lemma.

Let $\mathcal{R}$ be the curvature tensor defined by

$$\mathcal{R}(U, V) = [\nabla_U, \nabla_V] - \nabla_{[U,V]},$$

where U and V are vector fields on M and let $P_{t_0 x}$ denote the parallel transport with respect to ∇ along the geodesic $\gamma_x(t) = \tau(\exp(tZ))(p)$ from p to $\gamma_x(t_0)$. We now define a $(1,3)-$tensor $\mathcal{R}_x(t)$ on T_pM and its n^{th} covariant derivative along $\gamma_x(t)$ as follows:

$$\mathcal{R}_x(t)(u, v)w = P_{tx}^{-1} \circ \mathcal{R}_{\gamma_x(t)}(P_{tx}u, P_{tx}v)P_{tx}w,$$

$$\begin{aligned}\mathcal{R}_x^{n)}(t)(u, v)w &= P_{tx}^{-1} \circ \mathcal{R}^{n)}_{\gamma_x(t)}(P_{tx}u, P_{tx}v)P_{tx}w \\ &= P_{tx}^{-1} \circ (\nabla^n_{\dot\gamma_x(t)}\mathcal{R})(P_{tx}u, P_{tx}v)P_{tx}w\end{aligned}$$

for u, v, $w \in T_pM$. Note that $\mathcal{R}_x^{0)}(t) = \mathcal{R}_x(t)$.

Lemma 3.2.3. *The (1,3)-tensor $\mathcal{R}_x^{n)}(t)$ on $\mathfrak{m}$ obtained by the parallel translation of the n^{th} covariant derivative of the curvature tensor along $\gamma_x(t)$ is given by*

$$\mathcal{R}_x^{n)}(t) = e^{\nabla_{tx}} \cdot \mathcal{R}_p^{n)} \tag{3.3}$$

where $x = Z_p^*$, $\mathcal{R}_p^{n)}$ denotes the n^{th} covariant derivative of the curvature tensor along $\gamma_x(t)$ at the origin p and $e^{\nabla_{tx}}\cdot$ denotes the action of $e^{\nabla_{tx}}$ on the space $\mathcal{R}(\mathfrak{m})$ of curvature tensors on $\mathfrak{m}$.

Proof. We denote briefly $dg_t = \tau(\exp(tZ))_*$. From Lemma 3.2.2 and due to $\mathrm{e}^{\nabla_{tx}}$ is an isometry, we have

$$\begin{aligned}
\left\langle \mathcal{R}_x^{n)}(t)(u,v)w,\xi \right\rangle &= \left\langle P_{tx}^{-1}\circ \mathcal{R}^{n)}_{\gamma_x(t)}(P_{tx}u,P_{tx}v)P_{tx}w,\xi \right\rangle \\
&= \left\langle \mathrm{e}^{\nabla_{tx}}\circ (dg_t)^{-1}\circ \mathcal{R}^{n}_{\gamma_x(t)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u),(dg_t)\mathrm{e}^{-\nabla_{tx}}(v))(dg_t)\mathrm{e}^{-\nabla_{tx}}(w),\xi \right\rangle \\
&= \left\langle (dg_t)^{-1}\circ \mathcal{R}^{n)}_{\gamma_x(t)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u),(dg_t)\mathrm{e}^{-\nabla_{tx}}(v))(dg_t)\mathrm{e}^{-\nabla_{tx}}(w),\mathrm{e}^{\nabla_{-tx}}(\xi) \right\rangle \\
&= \left\langle (dg_t)^{-1}\circ \mathcal{R}^{n)}_{\gamma_x(t)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u),(dg_t)\mathrm{e}^{-\nabla_{tx}}(v))(dg_t)\mathrm{e}^{-\nabla_{tx}}(w),(dg_t)^{-1}\circ(dg_t)\circ\mathrm{e}^{\nabla_{-tx}}(\xi) \right\rangle \\
&= g(\mathcal{R}^{n)}_{\gamma_x(t)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u),(dg_t)\mathrm{e}^{-\nabla_{tx}}(v))(dg_t)\mathrm{e}^{-\nabla_{tx}}(w),(dg_t)\mathrm{e}^{-\nabla_{tx}}(\xi)) \\
&= g((\nabla^n_{\dot\gamma_x(t)}\mathcal{R})((dg_t)\mathrm{e}^{-\nabla_{tx}}(u),(dg_t)\mathrm{e}^{-\nabla_{tx}}(v))(dg_t)\mathrm{e}^{-\nabla_{tx}}(w),(dg_t)\mathrm{e}^{-\nabla_{tx}}(\xi)) \\
&= g((\nabla^n_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}\mathcal{R})((dg_t)\mathrm{e}^{-\nabla_{tx}}(u),(dg_t)\mathrm{e}^{-\nabla_{tx}}(v))(dg_t)\mathrm{e}^{-\nabla_{tx}}(w),(dg_t)\mathrm{e}^{-\nabla_{tx}}(\xi)) \\
&= g((dg_t)(\nabla^n_{\mathrm{e}^{-\nabla_{tx}}(x)}\mathcal{R})(\mathrm{e}^{-\nabla_{tx}}(u),\mathrm{e}^{-\nabla_{tx}}(v))\mathrm{e}^{-\nabla_{tx}}(w),(dg_t)\mathrm{e}^{-\nabla_{tx}}(\xi)) \\
&= g((dg_t)\mathcal{R}_p^{n)}(\mathrm{e}^{-\nabla_{tx}}(u),\mathrm{e}^{-\nabla_{tx}}(v))\mathrm{e}^{-\nabla_{tx}}(w),(dg_t)\mathrm{e}^{-\nabla_{tx}}(\xi)) \\
&= \left\langle \mathcal{R}_p^{n)}(\mathrm{e}^{-\nabla_{tx}}(u),\mathrm{e}^{-\nabla_{tx}}(v))\mathrm{e}^{-\nabla_{tx}}(w),\mathrm{e}^{-\nabla_{tx}}(\xi) \right\rangle \\
&= \left\langle \mathrm{e}^{\nabla_{tx}}\mathcal{R}_p^{n)}(\mathrm{e}^{-\nabla_{tx}}(u),\mathrm{e}^{-\nabla_{tx}}(v))\mathrm{e}^{-\nabla_{tx}}(w),\xi \right\rangle .
\end{aligned}$$

We now recall the fundamental fact about a curve in the m - dimensional Euclidean space $\mathbb{R}^m$. Let $c : I \to \mathbb{R}^m$ be a curve defined on an open interval I of $\mathbb{R}$ into $\mathbb{R}^m$. We say that c has *constant osculating rank* r if, for all $t \in I$, its higher order derivatives $c^{1)}(t)$, ..., $c^{r)}(t)$ are linearly independent and $c^{1)}(t)$, ..., $c^{r+1)}(t)$ are linearly dependent in $\mathbb{R}^m$. It is a fundamental fact that if c has constant osculating rank r, there are smooth functions $a_1, \ldots, a_r : I \to \mathbb{R}$ such that

$$c(t) = c(0) + a_1(t)c^{1)}(0) + \cdots + a_r(t)c^{r)}(0) \quad \text{for all} \quad t \in I.$$

Let us return to a g.o. space M. Let $U(t)$, $V(t)$, $W(t)$ vector fields along the geodesic $\gamma_x(t)$ such that $U(0) = u$, $V(0) = v$, $W(0) = w$ are unit vectors in $\mathfrak{m}$. Since $\mathrm{e}^{\nabla_{tx}}$ is a 1-parameter subgroup of the group of linear isometries of $\mathcal{R}(\mathfrak{m})$, applying Proposition 4.9 of [D], it follows that $\mathcal{R}_x(t)$, or more explicitly

$$\mathcal{R}_x(t)(u,v)w = \mathrm{e}^{\nabla_{tx}}\cdot \mathcal{R}_p(\mathrm{e}^{\nabla_{-tx}}u,\mathrm{e}^{\nabla_{-tx}}v)\mathrm{e}^{\nabla_{-tx}}w,$$

is a curve in $\mathcal{R}(\mathfrak{m})$ with constant osculating rank r. Therefore we have

$$\mathcal{R}_x(t) = \mathcal{R}_x(0) + a_1(t)\mathcal{R}_x^{1)}(0) + \cdots + a_r(t)\mathcal{R}_x^{r)}(0) \quad \text{for all} \quad t \in I, \tag{3.4}$$

where $\mathcal{R}_x^{r)}(0)(u,v)w = (\nabla_x^r\mathcal{R})(u,v)w$. As a direct consequence, we obtain

Theorem 3.2.4. *Let (M,g) be a Riemannian g.o. space. Then there is always a finite real number r such that the curvature operator has constant osculating rank r.*

A useful technique to describe the curvature along a geodesic γ in a Riemannian manifold (M, g), with Riemannian curvature tensor $\mathcal{R}$, is the use of the Jacobi operator $\mathcal{J} = \mathcal{R}_\gamma = \mathcal{R}(\cdot, \dot{\gamma})\dot{\gamma}$. $\mathcal{J}$ determines a self-adjoint tensor field along γ.

In particular, from Lemma 3.2.3 the Jacobi operator on $\mathfrak{m}$ and its n^{th} covariant derivative along the geodesic $\gamma_x(t)$ is given by

$$\begin{aligned} \mathcal{J}_x^{n)}(t)(u) = \mathcal{R}_x^{n)}(t)(u,x)x &= P_{tx}^{-1} \circ \mathcal{R}^{n)}_{\gamma_x(t)}(P_{tx}u, P_{tx}x)P_{tx}x \\ &= P_{tx}^{-1} \circ (\nabla^n_{\dot{\gamma}_x(t)}\mathcal{R})(P_{tx}u, \dot{\gamma}_x(t))\dot{\gamma}_x(t) \\ &= P_{tx}^{-1} \circ \mathcal{J}^{n)}(P_{tx}u) \\ &= \mathrm{e}^{\nabla_{tx}}\mathcal{J}_p^{n)}(\mathrm{e}^{-\nabla_{tx}}(u)) \end{aligned} \tag{3.5}$$

for $u \in \mathfrak{m}$. Obviously, if $t = 0$ we have

$$\mathcal{J}_x^{n)}(0)(u) = \mathcal{J}_p^{n)}(u) = (\nabla_x^n\mathcal{R})(u,x)x. \tag{3.6}$$

Moreover, it is obvious that the property (3.3) can also be written as

$$\mathcal{R}_x^{n)}(t) = \mathrm{e}^{\nabla_{tx}} \cdot (\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}\mathcal{R}_p^{n-1)}). \tag{3.7}$$

Therefore, (3.5) becomes

$$\mathcal{J}_x^{n)}(t)(u) = \mathrm{e}^{\nabla_{tx}}(\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}\mathcal{J}_p^{n-1)})(\mathrm{e}^{-\nabla_{tx}}(u)) \tag{3.8}$$

for $u \in \mathfrak{m}$ and if $t = 0$ we have

$$\mathcal{J}_p^{n)}(u) = \mathcal{J}_x^{n)}(0)(u) = (\nabla_x\mathcal{J}_p^{n-1)})(u). \tag{3.9}$$

The following result is useful for checking concrete examples as in the next section.

Lemma 3.2.5. *The Jacobi operator $\mathcal{J}_x^{n)}(t)$ on $\mathfrak{m}$ obtained by the parallel translation of the n^{th} covariant derivative of the Jacobi operator along $\gamma_x(t)$ satisfies the following identity*

$$\mathcal{J}_x^{n)}(t)(u) = e^{\nabla_{tx}}\nabla_{e^{-\nabla_{tx}}(x)}(\mathcal{J}_p^{n-1)}(e^{-\nabla_{tx}}(u))) - e^{\nabla_{tx}}\mathcal{J}_p^{n-1)}(\nabla_{e^{-\nabla_{tx}}(x)}(e^{-\nabla_{tx}}(u)))$$

for $u \in \mathfrak{m}$ where $x = \dot{\gamma}_x(0) = Z_p^*$, $\mathcal{J}_p^{n)}$ denotes the n^{th} covariant derivative of the Jacobi operator along $\gamma_x(t)$ at the origin p and $e^{\nabla_{tx}} \cdot$ denotes the action of $e^{\nabla_{tx}}$ on the space $\mathcal{R}(\mathfrak{m})$ of curvature tensors on $\mathfrak{m}$. Moreover, in the particular case $t = 0$, using (3.6) *the identity becomes*

$$\begin{aligned} \mathcal{J}_x^{n)}(0)(u) &= \nabla_x(\mathcal{J}_x^{n-1)}(0)(u)) - \mathcal{J}_x^{n-1)}(0)(\nabla_x u) \\ &= \nabla_x((\nabla_x^{n-1}\mathcal{R})(u,x)x) - (\nabla_x^{n-1}\mathcal{R})(\nabla_x u, x)x. \end{aligned} \tag{3.10}$$

Proof. We denote briefly $dg_t = \tau(\exp(tZ))_*$. From Lemma 3.2.2, (3.5), the condition $\nabla_{\dot\gamma_x(t)}\dot\gamma_x(t) = 0$ and due to $\mathrm{e}^{\nabla_{tx}}$ being an isometry, we have

$$
\begin{aligned}
\left\langle \mathcal{J}_x^{n)}(t)(u), \xi \right\rangle &= \left\langle P_{tx}^{-1} \circ \mathcal{J}^{n)}(P_{tx}u), \xi \right\rangle = \left\langle P_{tx}^{-1} \circ (\nabla^{n)}_{\dot\gamma_x(t)}\mathcal{R})(P_{tx}u, \dot\gamma_x(t))\dot\gamma_x(t), \xi \right\rangle \\
&= \left\langle P_{tx}^{-1} \circ \nabla_{\dot\gamma_x(t)}((\nabla^{n-1)}_{\dot\gamma_x(t)}\mathcal{R})(P_{tx}u, \dot\gamma_x(t))\dot\gamma_x(t)), \xi \right\rangle \\
&\quad - \left\langle P_{tx}^{-1} \circ (\nabla^{n-1)}_{\dot\gamma_x(t)}\mathcal{R})(\nabla_{\dot\gamma_x(t)}(P_{tx}u), \dot\gamma_x(t))\dot\gamma_x(t)), \xi \right\rangle \\
&= \left\langle P_{tx}^{-1} \circ \nabla_{\dot\gamma_x(t)}(\mathcal{J}^{n-1)}(P_{tx}u)), \xi \right\rangle - \left\langle P_{tx}^{-1} \circ \mathcal{J}^{n-1)}(\nabla_{\dot\gamma_x(t)}(P_{tx}u)), \xi \right\rangle \\
&= \left\langle \mathrm{e}^{\nabla_{tx}} \circ (dg_t)^{-1} \circ \nabla_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}(\mathcal{J}^{n-1)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u))), \xi \right\rangle \\
&\quad - \left\langle \mathrm{e}^{\nabla_{tx}} \circ (dg_t)^{-1} \circ \mathcal{J}^{n-1)}(\nabla_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u))), \xi \right\rangle \\
&= \left\langle (dg_t)^{-1} \circ \nabla_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}(\mathcal{J}^{n-1)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u))), \mathrm{e}^{\nabla_{-tx}}\xi \right\rangle \\
&\quad - \left\langle (dg_t)^{-1} \circ \mathcal{J}^{n-1)}(\nabla_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u))), \mathrm{e}^{\nabla_{-tx}}\xi \right\rangle \\
&= g(\nabla_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}(\mathcal{J}^{n-1)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u))), (dg_t)\mathrm{e}^{\nabla_{-tx}}\xi) \\
&\quad - g(\mathcal{J}^{n-1)}(\nabla_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u))), (dg_t)\mathrm{e}^{\nabla_{-tx}}\xi) \\
&= g((dg_t)\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathcal{J}_p^{n-1)}(\mathrm{e}^{-\nabla_{tx}}(u))), (dg_t)\mathrm{e}^{\nabla_{-tx}}\xi) \\
&\quad - g((dg_t)\mathcal{J}_p^{n-1)}(\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathrm{e}^{-\nabla_{tx}}(u))), (dg_t)\mathrm{e}^{\nabla_{-tx}}\xi) \\
&= \left\langle \nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathcal{J}_p^{n-1)}(\mathrm{e}^{-\nabla_{tx}}(u))), \mathrm{e}^{\nabla_{-tx}}\xi \right\rangle - \left\langle \mathcal{J}_p^{n-1)}(\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathrm{e}^{-\nabla_{tx}}(u))), \mathrm{e}^{\nabla_{-tx}}\xi \right\rangle \\
&= \left\langle \mathrm{e}^{\nabla_{tx}}\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathcal{J}_p^{n-1)}(\mathrm{e}^{-\nabla_{tx}}(u))) - \mathrm{e}^{\nabla_{tx}}\mathcal{J}_p^{n-1)}(\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathrm{e}^{-\nabla_{tx}}(u))), \xi \right\rangle .
\end{aligned}
$$

3.3 Application of the main result

The aim of this section is to calculate the constant osculating rank of the Jacobi operator on Kaplan's example and, as a direct consequence, to solve the Jacobi equation using the new method given in [N-T]. Thus, we find the real value r of Theorem 3.2.4 and we show how the A. M. Naveira and A. Tarrío method can be applied in a concrete and well-known g.o. space.

More specifically, we solve the Jacobi equation along a unit geodesic of arbitrary direction on Kaplan's example. Moreover, we check that our result on Kaplan's example coincide with the result obtained using Lemma 3.1.1 by J. Berndt, F. Tricerri and L. Vanhecke in [B-Tr-V, p. 52].

Furthermore, due to Kaplan's example is a well-known "H-type group", we shall start recalling some known definitions and results regarding "generalized Heisenberg groups". In addition, we shall give the recursive expression for the n^{th} covariant derivative of the Jacobi operator at the origin of an H-type group.

This first section 3.3.1 and the computer support, using the software Mathematica 5.0 in a full transparency way, will be so useful in Section 3.3.2 to obtain our objective.

3.3.1 Preliminaries about H-type groups

Here, we shall recall some basic definitions and results regarding "generalized Heisenberg groups" and we shall prove some new results that will be very useful in Section 3.3.2.

Definition 3.3.1. *Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra with an inner product $\langle\ ,\ \rangle$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and let $\mathfrak{v}$ be its orthogonal complement. For each vector $A \in \mathfrak{z}$, the operator $j(A) : \mathfrak{v} \to \mathfrak{v}$ is defined by the relation*

$$\langle j(A)X, Y\rangle = \langle A, [X, Y]\rangle \quad \text{for all} \quad X, Y \in \mathfrak{v}. \tag{3.11}$$

The algebra $\mathfrak{n}$ is called a generalized Heisenberg algebra (H-type algebra) if, for each $A \in \mathfrak{z}$, the operator $j(A)$ satisfies the identity

$$j(A)^2 = -|A|^2 Id_{\mathfrak{v}} \tag{3.12}$$

where $|\ \ |^2$ denotes the quadratic form of the inner product $\langle\ ,\ \rangle$. A connected, simply connected Lie group whose Lie algebra is an H-type algebra is diffeomorphic to $\mathbb{R}^n$ and it is called an H-type group. It is endowed with a left-invariant metric.

In particular, the Lie algebra structure on $\mathfrak{n}$ is defined by extending the skew-symmetric bilinear map $[\ ,\ ] : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ to a bracket

$$[A + X, B + Y] = [X, Y]$$

where $A, B \in \mathfrak{z}$ and $X, Y \in \mathfrak{v}$.

The following Lemma collects some basic relations on H-type algebras. (See [Ka.1], [B-Tr-V, p. 24]).

Lemma 3.3.2. *Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be an H-type algebra. Then, the following relations are satisfied:*

$$|j(A)X| = |A||X|, \tag{3.13}$$

$$\langle j(A)X, j(B)X\rangle = |X|^2\langle A, B\rangle, \tag{3.14}$$

$$\langle j(A)X, j(A)Y\rangle = |A|^2\langle X, Y\rangle, \tag{3.15}$$

$$\langle j(A)X, Y\rangle + \langle X, j(A)Y\rangle = 0, \tag{3.16}$$

$$[X, j(A)X] = A|X|^2, \tag{3.17}$$

where $A, B \in \mathfrak{z}$ and $X, Y \in \mathfrak{v}$.

Proof. Putting $Y = j(A)X$ in (3.11) and using (3.12) we obtain (3.13). Polarizing (3.13) yields the relations (3.14) and (3.15). Considering that Y is $j(A)Y$ in (3.15) and using (3.12) we obtain (3.16). Finally, we shall obtain (3.17). For each $A \in \mathfrak{z}$ and $X \in \mathfrak{v}$ given, we have from (3.11) and (3.14) that $\langle B, [X, j(A)X]\rangle = \langle j(B)X, j(A)X\rangle = |X|^2\langle B, A\rangle = \langle B, A|X|^2\rangle$ for all $B \in \mathfrak{z}$.

H-type groups have been intrinsically described in [Ka.1] and [B-Tr-V, p. 28]. We shall recall some of these basic results with the aim of calculating what the expression of the Jacobi operator and its derivatives is over these kinds of spaces.

The Riemannian connection for any left-invariant metric on a Lie group satisfies

$$2\langle \nabla_X Y, Z\rangle = \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle$$

where X, Y, Z are arbitrary elements in the corresponding Lie algebra. In the case of an H-type group, (3.16) shows that the connection is given by

$$\begin{aligned} &\nabla_X Y = \tfrac{1}{2}[X,Y],\\ &\nabla_A X = \nabla_X A = -\tfrac{1}{2} j(A)X,\\ &\nabla_A B = 0, \end{aligned} \tag{3.18}$$

where $A, B \in \mathfrak{z}$ and $X, Y \in \mathfrak{v}$. A straightforward computation now shows that the Riemannian curvature tensor is given by

$$\begin{aligned} &\mathcal{R}(X,Y)Z = \tfrac{1}{4}(2j([X,Y])Z - j([Y,Z])X - j([Z,X])Y),\\ &\mathcal{R}(X,Y)A = \tfrac{1}{4}([Y, j(A)X] - [X, j(A)Y]),\\ &\mathcal{R}(X,A)Y = -\tfrac{1}{4}[X, j(A)Y],\\ &\mathcal{R}(X,A)B = -\tfrac{1}{4} j(A)j(B)X,\\ &\mathcal{R}(A,B)X = \tfrac{1}{4}(j(A)j(B)X - j(B)j(A)X),\\ &\mathcal{R}(A,B)C = 0, \end{aligned} \tag{3.19}$$

where $A, B, C \in \mathfrak{z}$ and $X, Y, Z \in \mathfrak{v}$.

With respect to the geodesics, $t \to \gamma(t)$, through the origin p of the H-type group N, we shall remember that they are described by means of the vector-valued functions $t \to X(t) \in \mathfrak{v}$, $t \to A(t) \in \mathfrak{z}$ as follows, $\gamma(t) = \exp(X(t) + A(t))$. Moreover, $X(0) = 0$, $A(0) = 0$ and the unit tangent vector of γ at the origin p is given by $\dot\gamma_0 = \dot X_0 + \dot A_0$ where $\dot f_0$ denotes $(df/dt)_{t=0}$ of any real or vector-valued function $f(t)$. (See [Ka.1], [Ka.2], [B-Tr-V, p. 30] and [Tr-V] for more detailed results on geodesics over H-type spaces).

Now we define the mappings $\zeta_{(n,A)} : \mathfrak{z} \to \mathfrak{z}$, $\nu_{(n,A)} : \mathfrak{z} \to \mathfrak{v}$, $\zeta_{(n,X)} : \mathfrak{v} \to \mathfrak{z}$ and $\nu_{(n,X)} : \mathfrak{v} \to \mathfrak{v}$ in a recurrent way for each $n \in \mathbb{N}$ by

$$\begin{aligned} \zeta_{(0,A)}(B) =& \tfrac{1}{4}|\dot X_0|^2 B,\\ \nu_{(0,A)}(B) =& \tfrac{1}{2} j(B)j(\dot A_0)\dot X_0 - \tfrac{1}{4} j(\dot A_0)j(B)\dot X_0,\\ \zeta_{(0,X)}(Y) =& \tfrac{1}{4}[\dot X_0, j(\dot A_0)Y] - \tfrac{1}{2}[Y, j(\dot A_0)\dot X_0],\\ \nu_{(0,X)}(Y) =& \tfrac{1}{4}|\dot A_0|^2 Y + \tfrac{3}{4} j([Y,\dot X_0])\dot X_0, \end{aligned} \tag{3.20}$$

$$
\begin{aligned}
\zeta_{(n,A)}(B) =& \tfrac{1}{2}([\dot{X}_0, \nu_{(n-1,A)}(B)] + \zeta_{(n-1,X)}(j(B)\dot{X}_0)),\\
\nu_{(n,A)}(B) =& \tfrac{1}{2}(\nu_{(n-1,X)}(j(B)\dot{X}_0) - j(\dot{A}_0)\nu_{(n-1,A)}(B) - j(\zeta_{(n-1,A)}(B))\dot{X}_0),\\
\zeta_{(n,X)}(Y) =& \tfrac{1}{2}([\dot{X}_0, \nu_{(n-1,X)}(Y)] + \zeta_{(n-1,X)}(j(\dot{A}_0)Y) - \zeta_{(n-1,A)}([\dot{X}_0, Y])),\\
\nu_{(n,X)}(Y) =& \tfrac{1}{2}(\nu_{(n-1,X)}(j(\dot{A}_0)Y) - j(\dot{A}_0)\nu_{(n-1,X)}(Y)\\
& - j(\zeta_{(n-1,X)}(Y))\dot{X}_0 - \nu_{(n-1,A)}([\dot{X}_0, Y])),
\end{aligned} \tag{3.21}
$$

where $B \in \mathfrak{z}$ and $Y \in \mathfrak{v}$.

Proposition 3.3.3. *The n-derivative of the Jacobi operator at the origin p of the H-type group N is given by*

$$
\begin{aligned}
\mathcal{J}^{n)}_{\dot{\gamma}_0}(0)(B) = \nabla^n_{\dot{\gamma}_0}\mathcal{R}(B, \dot{\gamma}_0)\dot{\gamma}_0 = \zeta_{(n,A)}(B) + \nu_{(n,A)}(B),\\
\mathcal{J}^{n)}_{\dot{\gamma}_0}(0)(Y) = \nabla^n_{\dot{\gamma}_0}\mathcal{R}(Y, \dot{\gamma}_0)\dot{\gamma}_0 = \zeta_{(n,X)}(Y) + \nu_{(n,X)}(Y).
\end{aligned} \tag{3.22}
$$

where $B \in \mathfrak{z}$ and $Y \in \mathfrak{v}$.

Proof. For $n = 0$, using (3.19), (3.17), (3.12) and (3.20) we get that

$$
\begin{aligned}
\mathcal{J}_{\dot{\gamma}_0}(0)(B) =& \mathcal{R}(B, \dot{X}_0)\dot{X}_0 + \mathcal{R}(B, \dot{X}_0)\dot{A}_0 + \mathcal{R}(B, \dot{A}_0)\dot{X}_0 + \mathcal{R}(B, \dot{A}_0)\dot{A}_0\\
=& \tfrac{1}{4}|\dot{X}_0|^2 B + \tfrac{1}{2}j(B)j(\dot{A}_0)\dot{X}_0 - \tfrac{1}{4}j(\dot{A}_0)j(B)\dot{X}_0\\
=& \zeta_{(0,A)}(B) + \nu_{(0,A)}(B),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_{\dot{\gamma}_0}(0)(Y) =& \mathcal{R}(Y, \dot{X}_0)\dot{X}_0 + \mathcal{R}(Y, \dot{X}_0)\dot{A}_0 + \mathcal{R}(Y, \dot{A}_0)\dot{X}_0 + \mathcal{R}(Y, \dot{A}_0)\dot{A}_0\\
=& \tfrac{1}{4}[\dot{X}_0, j(\dot{A}_0)Y] - \tfrac{1}{2}[Y, j(\dot{A}_0)\dot{X}_0] + \tfrac{1}{4}|\dot{A}_0|^2 Y + \tfrac{3}{4}j([Y, \dot{X}_0])\dot{X}_0\\
=& \zeta_{(0,X)}(Y) + \nu_{(0,X)}(Y).
\end{aligned}
$$

Finally, assuming that (3.22) is true for $n-1$, we shall prove the result for n using (3.10), (3.18) and (3.21).

$$
\begin{aligned}
\mathcal{J}^{n)}_{\dot{\gamma}_0}(0)(B) =& \nabla_{\dot{\gamma}_0}((\nabla^{n-1}_{\dot{\gamma}_0}\mathcal{R})(B, \dot{\gamma}_0)\dot{\gamma}_0) - (\nabla^{n-1}_{\dot{\gamma}_0}\mathcal{R})(\nabla_{\dot{\gamma}_0}B, \dot{\gamma}_0)\dot{\gamma}_0\\
=& \nabla_{\dot{X}_0}((\nabla^{n-1}_{\dot{\gamma}_0}\mathcal{R})(B, \dot{\gamma}_0)\dot{\gamma}_0) + \nabla_{\dot{A}_0}((\nabla^{n-1}_{\dot{\gamma}_0}\mathcal{R})(B, \dot{\gamma}_0)\dot{\gamma}_0)\\
& + \tfrac{1}{2}(\nabla^{n-1}_{\dot{\gamma}_0}\mathcal{R})(j(B)\dot{X}_0, \dot{\gamma}_0)\dot{\gamma}_0\\
=& \nabla_{\dot{X}_0}(\zeta_{(n-1,A)}(B) + \nu_{(n-1,A)}(B)) + \nabla_{\dot{A}_0}(\zeta_{(n-1,A)}(B) + \nu_{(n-1,A)}(B))\\
& + \tfrac{1}{2}(\zeta_{(n-1,X)}(j(B)\dot{X}_0) + \nu_{(n-1,X)}(j(B)\dot{X}_0))\\
=& -\tfrac{1}{2}j(\zeta_{(n-1,A)}(B))\dot{X}_0 + \tfrac{1}{2}[\dot{X}_0, \nu_{(n-1,A)}(B)] - \tfrac{1}{2}j(\dot{A}_0)(\nu_{(n-1,A)}(B))\\
& + \tfrac{1}{2}\zeta_{(n-1,X)}(j(B)\dot{X}_0) + \tfrac{1}{2}\nu_{(n-1,X)}(j(B)\dot{X}_0)\\
=& \zeta_{(n,A)}(B) + \nu_{(n,A)}(B),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}^{n)}_{\dot\gamma_0}(0)(Y) =&\nabla_{\dot\gamma_0}((\nabla^{n-1}_{\dot\gamma_0}\mathcal{R})(Y,\dot\gamma_0)\dot\gamma_0) - (\nabla^{n-1}_{\dot\gamma_0}\mathcal{R})(\nabla_{\dot\gamma_0}Y,\dot\gamma_0)\dot\gamma_0 \\
=&\nabla_{\dot X_0}((\nabla^{n-1}_{\dot\gamma_0}\mathcal{R})(Y,\dot\gamma_0)\dot\gamma_0) + \nabla_{\dot A_0}((\nabla^{n-1}_{\dot\gamma_0}\mathcal{R})(Y,\dot\gamma_0)\dot\gamma_0) \\
&-\tfrac{1}{2}(\nabla^{n-1}_{\dot\gamma_0}\mathcal{R})([\dot X_0,Y],\dot\gamma_0)\dot\gamma_0 + \tfrac{1}{2}(\nabla^{n-1}_{\dot\gamma_0}\mathcal{R})(j(\dot A_0)Y,\dot\gamma_0)\dot\gamma_0 \\
=&\nabla_{\dot X_0}(\zeta_{(n-1,X)}(Y)+\nu_{(n-1,X)}(Y)) + \nabla_{\dot A_0}(\zeta_{(n-1,X)}(Y)+\nu_{(n-1,X)}(Y)) \\
&-\tfrac{1}{2}(\zeta_{(n-1,A)}([\dot X_0,Y]) + \nu_{(n-1,A)}([\dot X_0,Y])) \\
&+\tfrac{1}{2}(\zeta_{(n-1,X)}(j(\dot A_0)Y) + \nu_{(n-1,X)}(j(\dot A_0)Y)) \\
=&-\tfrac{1}{2}j(\zeta_{(n-1,X)}(Y))\dot X_0 + \tfrac{1}{2}[\dot X_0,\nu_{(n-1,X)}(Y)] - \tfrac{1}{2}j(\dot A_0)(\nu_{(n-1,X)}(Y)) \\
&-\tfrac{1}{2}\zeta_{(n-1,A)}([\dot X_0,Y]) - \tfrac{1}{2}\nu_{(n-1,A)}([\dot X_0,Y]) \\
&+\tfrac{1}{2}\zeta_{(n-1,X)}(j(\dot A_0)Y) + \tfrac{1}{2}\nu_{(n-1,X)}(j(\dot A_0)Y) \\
=&\zeta_{(n,X)}(Y) + \nu_{(n,X)}(Y).
\end{aligned}
$$

3.3.2 Kaplan's example

Let $\mathfrak{n}$ be a vector space of dimension 6 equipped with a scalar product and let $\{E_1,E_2,E_3,E_4,E_5,E_6\}$ form an orthonormal basis. The elements E_5 and E_6 span the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{n}$. The structure of a Lie algebra on $\mathfrak{n}$ is given by the following relations:

$$
\begin{aligned}
&[E_1,E_2]=0,\\
&[E_1,E_3]=E_5, \quad [E_2,E_3]=E_6,\\
&[E_1,E_4]=E_6, \quad [E_2,E_4]=-E_5, \quad [E_3,E_4]=0,\\
&[E_k,E_5]=0, \quad \text{for} \quad k=1,...,4,\\
&[E_k,E_6]=0, \quad \text{for} \quad k=1,...,4,\\
&[E_5,E_6]=0.
\end{aligned}
\tag{3.23}
$$

Moreover, from (3.11) we easily obtain that

$$
\begin{aligned}
&j(E_5)E_1=E_3, \quad j(E_5)E_2=-E_4, \quad j(E_5)E_3=-E_1, \quad j(E_5)E_4=E_2,\\
&j(E_6)E_1=E_4, \quad j(E_6)E_2=E_3, \quad j(E_6)E_3=-E_2, \quad j(E_6)E_4=-E_1.
\end{aligned}
\tag{3.24}
$$

The condition (3.12) for the operators $j(A)$ can be easily verified from (3.24). Thus, the relation (3.23) defines an H-type algebra.

Moreover, Z. Dušek in [Du] expresses the H-type group N corresponding to $\mathfrak{n}$ as a homogeneous space G/H where $H\cong SU(2)$ and $G=N\rtimes H$. Here the group G is not the full isometry group of N, but the group N is a g.o. space with respect to this group. As a consequence, we can apply Theorem 3.2.4 to conclude that

Proposition 3.3.4. *The osculating rank r of the Jacobi operator $\mathcal{J}$ is constant in the Kaplan example.*

Now we want to calculate what the value of $r \in \mathbb{N}$ is. We shall therefore start by calculating what relation is satisfied between the covariant derivatives of the Jacobi operator along an arbitrary geodesic γ parameterized by arclength with initial vector x at the origin p of N. In the following we always suppose that $x \in \mathfrak{n}$ is an arbitrary unit vector, that is, $x = \sum_{i=1}^{6} x_i E_i$, $\sum_{i=1}^{6}(x_i)^2 = 1$. Thus, following the notation of Section 3.3.1, $\dot{X}_0 = \sum_{i=1}^{4} x_i E_i$ and $\dot{A}_0 = \sum_{\alpha=5}^{6} x_\alpha E_\alpha$. Furthermore, we denote by $\{Q_i\}_{i=1}^{6}$ the orthonormal frame field along γ obtained by parallel translation of the basis $\{E_i\}$ along γ.

Under this assumptions and using (3.20), (3.23) and (3.24), we obtain the following basic result:

Lemma 3.3.5. *The operator $[\dot{X}_0, Y]$ where $Y \in \mathfrak{v}$, on Kaplan's example is given by*

$$\begin{aligned} [\dot{X}_0, Q_1] &= -x_3 Q_5 - x_4 Q_6, \quad [\dot{X}_0, Q_2] = x_4 Q_5 - x_3 Q_6, \\ [\dot{X}_0, Q_3] &= x_1 Q_5 + x_2 Q_6, \quad [\dot{X}_0, Q_4] = -x_2 Q_5 + x_1 Q_6. \end{aligned} \tag{3.25}$$

The operator $j(\dot{A}_0)(Y)$ where $Y \in \mathfrak{v}$, on Kaplan's example is given by

$$\begin{aligned} j(\dot{A}_0)(Q_1) &= x_5 Q_3 + x_6 Q_4, \quad j(\dot{A}_0)(Q_2) = x_6 Q_3 - x_5 Q_4, \\ j(\dot{A}_0)(Q_3) &= -x_5 Q_1 - x_6 Q_2, \quad j(\dot{A}_0)(Q_4) = -x_6 Q_1 + x_5 Q_2. \end{aligned} \tag{3.26}$$

The operator $j(B)(\dot{X}_0)$ where $B \in \mathfrak{z}$, on Kaplan's example is given by

$$\begin{aligned} j(Q_5)(\dot{X}_0) &= -x_3 Q_1 + x_4 Q_2 + x_1 Q_3 - x_2 Q_4, \\ j(Q_6)(\dot{X}_0) &= -x_4 Q_1 - x_3 Q_2 + x_2 Q_3 + x_1 Q_4. \end{aligned} \tag{3.27}$$

The mappings $\zeta_{(0,X)}$, $\nu_{(0,X)}$, $\zeta_{(0,A)}$, $\nu_{(0,A)}$ on Kaplan's example are given by

$$\begin{aligned}
\zeta_{(0,X)}(Q_1) &= \tfrac{1}{4}((-x_1x_5 - 3x_2x_6)Q_5 + (3x_2x_5 - x_1x_6)Q_6), \\
\zeta_{(0,X)}(Q_2) &= \tfrac{1}{4}((-x_2x_5 + 3x_1x_6)Q_5 + (-3x_1x_5 - x_2x_6)Q_6), \\
\zeta_{(0,X)}(Q_3) &= \tfrac{1}{4}((-x_3x_5 - 3x_4x_6)Q_5 + (3x_4x_5 - x_3x_6)Q_6), \\
\zeta_{(0,X)}(Q_4) &= \tfrac{1}{4}((-x_4x_5 + 3x_3x_6)Q_5 + (-3x_3x_5 - x_4x_6)Q_6), \\
\nu_{(0,X)}(Q_1) &= \tfrac{1}{4}((-3(x_3^2 + x_4^2) + x_5^2 + x_6^2)Q_1 + 3(x_1x_3 + x_2x_4)Q_3 + 3(x_1x_4 - x_2x_3)Q_4), \\
\nu_{(0,X)}(Q_2) &= \tfrac{1}{4}((-3(x_3^2 + x_4^2) + x_5^2 + x_6^2)Q_2 + 3(x_2x_3 - x_1x_4)Q_3 + 3(x_1x_3 + x_2x_4)Q_4), \\
\nu_{(0,X)}(Q_3) &= \tfrac{1}{4}(3(x_1x_3 + x_2x_4)Q_1 + 3(x_2x_3 - x_1x_4)Q_2 + (-3(x_1^2 + x_2^2) + x_5^2 + x_6^2)Q_3), \\
\nu_{(0,X)}(Q_4) &= \tfrac{1}{4}(3(x_1x_4 - x_2x_3)Q_1 + 3(x_1x_3 + x_2x_4)Q_2 + (-3(x_1^2 + x_2^2) + x_5^2 + x_6^2)Q_4), \\
\zeta_{(0,A)}(Q_\alpha) &= \tfrac{1}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)Q_\alpha, \quad \alpha = 5, 6, \\
\nu_{(0,A)}(Q_5) &= \tfrac{1}{4}((-x_1x_5 - 3x_2x_6)Q_1 + (3x_1x_6 - x_2x_5)Q_2 \\
&\quad + (-x_3x_5 - 3x_4x_6)Q_3 + (-x_4x_5 + 3x_3x_6)Q_4), \\
\nu_{(0,A)}(Q_6) &= \tfrac{1}{4}((3x_2x_5 - x_1x_6)Q_1 + (-3x_1x_5 - x_2x_6)Q_2 \\
&\quad + (3x_4x_5 - x_3x_6)Q_3 + (-3x_3x_5 - x_4x_6)Q_4).
\end{aligned} \tag{3.28}$$

Moreover, by a lengthy but elementary calculation using Proposition 3.3.3, (3.21), (3.23), (3.24), Lemma 3.3.5 and the linearity of all involve operators, we get for each $n \in \{0,1,2,3,4,5\}$, the n^{th} covariant derivative of the $(0,4)$ - Jacobi operator at the origin p for Kaplan's example, N. This is given by the matrix

$$\mathcal{J}_0^{n)} = \left(\mathcal{J}_{ij}^{n}(0)\right) \quad \text{for} \quad i,j = 1,...,6,$$

where

$$\begin{aligned}\mathcal{J}_{ij}^{n}(0) =& \langle \mathcal{J}_p^{n)}(Q_i), Q_j\rangle(0) = \langle \mathcal{J}_{\dot\gamma_0}^{n)}(0)(Q_i), Q_j\rangle(0)\\ =& \begin{cases} \langle \zeta_{(n,X)}(Q_i) + \nu_{(n,X)}(Q_i), Q_j\rangle(0) & \text{for} \ \ i = 1,...,4. \\ \langle \zeta_{(n,A)}(Q_i) + \nu_{(n,A)}(Q_i), Q_j\rangle(0) & \text{for} \quad i = 5,6. \end{cases}\end{aligned}$$

From now on, we will write the expression without (0) at the end when no confusion can arise. More concretely,

$$\begin{aligned} \mathcal{J}_{\alpha\beta}^{n)}(0) &= \langle \zeta_{(n,A)}(Q_\alpha), Q_\beta\rangle, \quad \mathcal{J}_{\alpha j}^{n)}(0) = \langle \nu_{(n,A)}(Q_\alpha), Q_j\rangle, \\ \mathcal{J}_{i\beta}^{n)}(0) &= \langle \zeta_{(n,X)}(Q_i), Q_\beta\rangle, \quad \mathcal{J}_{ij}^{n)}(0) = \langle \nu_{(n,X)}(Q_i), Q_j\rangle, \end{aligned} \tag{3.29}$$

for $i,j = 1,...,4$ and $\alpha,\beta = 5,6$.

The explicit expressions of $\mathcal{J}_{ij}^{n}(0)$ for $n = 0,...,5$ can be seen in Section 3.3.3.

Now, we easily obtain from the explicit expressions of $\mathcal{J}_{ij}^{n}(0)$ for $n = 1,3,5$ the following result that is the most important intermediate one.

Lemma 3.3.6. *The relation satisfied between the n^{th} covariant derivatives for $n = 1,3,5$ of the $(0,4)$ - Jacobi operator along the arbitrary geodesic γ with initial vector x at the origin $p = \gamma_0$ of Kaplan's example, N, is*

$$\tfrac{1}{4}|\dot\gamma_0|^4 \mathcal{J}_0^{1)} + \tfrac{5}{4}|\dot\gamma_0|^2 \mathcal{J}_0^{3)} + \mathcal{J}_0^{5)} = 0. \tag{3.30}$$

Proof. It is a straightforward computation, using (3.46), (3.48) and (3.50), to check that

$$\tfrac{1}{4}|\dot\gamma_0|^4 \mathcal{J}_{ij}^{1}(0) + \tfrac{5}{4}|\dot\gamma_0|^2 \mathcal{J}_{ij}^{3}(0) + \mathcal{J}_{ij}^{5}(0) = 0, \ i,j = 1,...,6.$$

Thus, we obtain (3.30).

The relation (3.30) contains the most basic and the most important information of the Jacobi operator at the origin of N. Now, we shall use it to obtain two new intermediates results which will be so useful to find the general form of the Jacobi operator along a geodesic γ in N.

Lemma 3.3.7. *The relations satisfied between the n^{th} covariant derivatives of the $(0,4)$ - Jacobi operator along the arbitrary geodesic γ with initial vector x at the origin $p = \gamma_0$ of N are*

$$\tfrac{1}{4}|\dot\gamma_0|^4 \mathcal{J}_0^{k+1)} + \tfrac{5}{4}|\dot\gamma_0|^2 \mathcal{J}_0^{k+3)} + \mathcal{J}_0^{k+5)} = 0 \quad \textit{for} \quad k = 0,1,2,... \tag{3.31}$$

Proof. For $i = 0$, we directly get (3.30). Now, let us assume that (3.31) is true for $k = i$ and we shall prove the result for $k = i + 1$ using (3.9)

$$\begin{aligned}\mathcal{J}_0^{i+6)} = \nabla_x \mathcal{J}_0^{i+5)} &= -\nabla_x \left(\tfrac{1}{4}|\dot\gamma_0|^4 \mathcal{J}_0^{i+1)} + \tfrac{5}{4}|\dot\gamma_0|^2 \mathcal{J}_0^{i+3)}\right) \\ &= -\tfrac{1}{4}|\dot\gamma_0|^4 \nabla_x \mathcal{J}_0^{i+1)} - \tfrac{5}{4}|\dot\gamma_0|^2 \nabla_x \mathcal{J}_0^{i+3)} \\ &= -\tfrac{1}{4}|\dot\gamma_0|^4 \mathcal{J}_0^{i+2)} - \tfrac{5}{4}|\dot\gamma_0|^2 \mathcal{J}_0^{i+4)}.\end{aligned}$$

Proposition 3.3.8. *The relation satisfied between the n^{th} covariant derivatives for $n = 1, 3, 5$ of the $(0, 4)$ - Jacobi operator along the arbitrary geodesic γ with initial vector x of Kaplan's example, N, is*

$$\tfrac{1}{4}|\dot\gamma_0|^4 \mathcal{J}_t^{1)} + \tfrac{5}{4}|\dot\gamma_0|^2 \mathcal{J}_t^{3)} + \mathcal{J}_t^{5)} = 0. \tag{3.32}$$

Proof. Using the expansion in Taylor's series on the Jacobi operator $\mathcal{J}_t$, it is clear that

$$\mathcal{J}_t^{n)} = \sum_{i=n}^{\infty} \frac{t^{i-n}}{(i-n)!} \mathcal{J}_0^{i)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{J}_0^{k+n)}$$

for $n = 1, 3, 5$. Therefore, using the relation (3.31) we conclude

$$\tfrac{1}{4}|\dot\gamma_0|^4 \mathcal{J}_t^{1)} + \tfrac{5}{4}|\dot\gamma_0|^2 \mathcal{J}_t^{3)} + \mathcal{J}_t^{5)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\tfrac{1}{4}|\dot\gamma_0|^4 \mathcal{J}_0^{k+1)} + \tfrac{5}{4}|\dot\gamma_0|^2 \mathcal{J}_0^{k+3)} + \mathcal{J}_0^{k+5)}\right) = 0.$$

On the other hand, from Theorem 3.2.4 and (3.4) we know that a finite real number r and smooth functions $a_1, ..., a_r : I \to \mathbb{R}$ exist such that the Jacobi operator is given by the following relation

$$\mathcal{J}_t = \mathcal{J}_0 + a_1(t)\mathcal{J}_0^{1)} + \cdots + a_r(t)\mathcal{J}_0^{r)}. \tag{3.33}$$

More specifically, on Kaplan's example this relation is the following

Theorem 3.3.9. *Let N be Kaplan's example. Then the Jacobi operator along the geodesic γ has constant osculating rank 4 and it can be written as*

$$\mathcal{J}_t = \mathcal{J}_0 + a_1(t)\mathcal{J}_0^{1)} + a_2(t)\mathcal{J}_0^{2)} + a_3(t)\mathcal{J}_0^{3)} + a_4(t)\mathcal{J}_0^{4)} \tag{3.34}$$

where

$$\begin{aligned} a_1(t) &= \tfrac{1}{3}(8\sin(t/2) - \sin(t)), \quad & a_2(t) &= 5 + \tfrac{1}{3}(\cos(t) - 16\cos(t/2)), \\ a_3(t) &= \tfrac{1}{3}(8\sin(t/2) - 4\sin(t)), \quad & a_4(t) &= 4 + \tfrac{4}{3}(\cos(t) - 4\cos(t/2)). \end{aligned} \tag{3.35}$$

Proof. From Proposition 3.3.8 and due to $|\dot\gamma_0| = 1$, we only have to solve the following homogeneous lineal ordinary differential equation of order 5:

$$\tfrac{1}{4}\mathcal{J}_t^{1)} + \tfrac{5}{4}\mathcal{J}_t^{3)} + \mathcal{J}_t^{5)} = 0.$$

Following the general theory about ordinary differential equations we have that $r^5 + \frac{5}{4}r^3 + \frac{1}{4}r$ is its characteristic polinomy whose roots are $\{0, \pm\frac{\imath}{2}, \pm\imath\}$. Thus, the Jacobi operator is given by

$$\mathcal{J}_t = c_0 + c_1 \cos(t) + c_2 \sin(t) + c_3 \cos(t/2) + c_4 \sin(t/2), \tag{3.36}$$

where c_l, $l = 1, ..., 4$ are arbitrary parameters. Now, it remains only to find the value of c_l. From (3.36) it is easy to obtain the relations

$$\begin{aligned} \mathcal{J}_0 &= c_0 + c_1 + c_3, \\ \mathcal{J}_0^{2)} &= -\left(c_1 + \tfrac{c_3}{4}\right), \\ \mathcal{J}_0^{4)} &= c_1 + \tfrac{c_3}{16}, \end{aligned} \qquad \begin{aligned} \mathcal{J}_0^{1)} &= c_2 + \tfrac{c_4}{2}, \\ \mathcal{J}_0^{3)} &= -\left(c_2 + \tfrac{c_4}{8}\right). \end{aligned}$$

Then, we have

$$c_0 = \mathcal{J}_0 + 5\mathcal{J}_0^{2)} + 4\mathcal{J}_0^{4)}, \quad c_1 = \tfrac{1}{3}(\mathcal{J}_0^{2)} + 4\mathcal{J}_0^{4)}), \quad c_2 = \tfrac{-1}{3}(\mathcal{J}_0^{1)} + 4\mathcal{J}_0^{3)}),$$
$$c_3 = \tfrac{-16}{3}(\mathcal{J}_0^{2)} + \mathcal{J}_0^{4)}), \quad c_4 = \tfrac{8}{3}(\mathcal{J}_0^{1)} + \mathcal{J}_0^{3)}).$$

We conclude the proof substituting these values in (3.36).

Finally in this section, as a direct consequence of Theorem 3.3.9 we shall solve the Jacobi equation $Y_t'' + \mathcal{J}_t Y_t = 0$ along the geodesic $\gamma(t)$ and with respect to the Levi-Civita connection ∇ on the Kaplan example N. Let A_t be the Jacobi tensor field along the geodesic γ with initial values $A_0 = 0$, $A_0^{1)} = Id$ where we consider the covariant differentiation with respect to γ' and Id is the identity transformation of $T_{\gamma_0}M$. Remember that $\{Q_i\}_{i=1}^6$ denotes the orthonormal frame field along γ obtained by $\nabla_{\dot\gamma}$-parallel translation of the basis $\{E_i\}$ along γ. Thus $Y_t = A_t Q_t$ and Jacobi's equation is simplified to $A_t^{2)} + \mathcal{J}_t A_t = 0$. To conclude, we shall show what the Jacobi equation solution is in the following Proposition. To prove it we follow the same steps that A.M. Naveira and A. Tarrío took in [N-T]. Anyway, we shall give the details of the proof to make our exposition self-contained.

Proposition 3.3.10. *On the manifold N, the Jacobi tensor field A_t along the geodesic γ with initial values $A_0 = 0$, $A_0^{1)} = Id$ is given by*

$$A_t = \sum_{k=0}^{\infty} \beta_k(0) \tfrac{t^k}{k!} \tag{3.37}$$

where $\alpha_0(t) = \alpha_1(t) = \beta_0(t) = 0$, $\beta_1(t) = Id$ and

$$\beta_k(t) = \alpha_{k-1}(t) + \beta_{k-1}'(t), \quad \alpha_k(t) = \alpha_{k-1}'(t) - \mathcal{J}_t \beta_{k-1}(t) \quad \text{for} \quad k \geq 2.$$

Moreover, the coefficients $\beta_k(0)$ are functions only of $\mathcal{J}_0$, $\mathcal{J}_0^{1)}$, $\mathcal{J}_0^{2)}$, $\mathcal{J}_0^{3)}$ and $\mathcal{J}_0^{4)}$.

Proof. If we successively derive $A_t^{2)} = -A_t\mathcal{J}_t$, it has

$$A_t^{i)} = \alpha_i(t)A_t + \beta_i(t)A_t^{1)},$$

where

$$\beta_i(t) = \alpha_{i-1}(t) + \beta'_{i-1}(t), \quad \alpha_i(t) = \alpha'_{i-1}(t) - \mathcal{J}_t\beta_{i-1}(t) \quad \text{for} \quad i \geq 2.$$

Thus, if $t = 0$ it has $A_0^{0)} = \beta_0(0) = 0$, $A_0^{1)} = \beta_1(0) = Id$, $A_0^{2)} = \beta_2(0) = 0$, $A_3^{0)} = \beta_3(0) = -\mathcal{J}_0$, and, in general,

$$A_0^{i)} = \alpha_{i-1}(0) + \beta'_{i-1}(0) = \beta_i(0).$$

Now the result follows using the expansion in Taylor's series of A_t and Theorem 3.3.9.

In the remainder of the section, we shall assume that the unit vector $\dot\gamma_0 = \dot X_0 = \sum_{i=1}^4 x_iE_i$ (or, equivalently, that $x_5 = x_6 = 0$). Moreover, we shall see that the Jacobi field $Y_1(t)$ on Kaplan's example calculated using Proposition 3.3.10, coincides with the Jacobi field calculated using [B-Tr-V, Theorem of p. 52].

We obtain the following result from Proposition 3.3.10 and a straightforward computation using Theorem 3.3.9.

Lemma 3.3.11. *The general expression for each Jacobi vector field $Y_i(t)$ along the arbitrary geodesic γ is given by*

$$Y_i(t) = \sum_{j=1}^{i-1} F_{ij}Q_j + F_{ii}Q_i + \sum_{k=i+1}^{6} F_{ik}Q_k, \quad i = 1, ..., 6 \tag{3.38}$$

where, for $m \in \{1, ..., 6\}$

$$\begin{aligned} F_{im} = &- \mathcal{J}_{0_{im}}\tfrac{t^3}{3!} - 2\mathcal{J}_{0_{im}}^{1)}\tfrac{t^4}{4!} + ((\mathcal{J}_0\mathcal{J}_0)_{im} - 3\mathcal{J}_{0_{im}}^{2)})\tfrac{t^5}{5!} \\ &+ (2(\mathcal{J}_0^{1)}\mathcal{J}_0)_{im} + 4(\mathcal{J}_0\mathcal{J}_0^{1)})_{im} - 4\mathcal{J}_{0_{im}}^{3)})\tfrac{t^6}{6!} \\ &+ (-(\mathcal{J}_0\mathcal{J}_0\mathcal{J}_0)_{im} + 3(\mathcal{J}_0^{2)}\mathcal{J}_0)_{im} + 10(\mathcal{J}_0^{1)}\mathcal{J}_0^{1)})_{im} + 10(\mathcal{J}_0\mathcal{J}_0^{2)})_{im} - 5\mathcal{J}_{0_{im}}^{4)})\tfrac{t^7}{7!} \\ &+ (-2(\mathcal{J}_0^{1)}\mathcal{J}_0\mathcal{J}_0)_{im} - 4(\mathcal{J}_0\mathcal{J}_0^{1)}\mathcal{J}_0)_{im} + 4(\mathcal{J}_0^{3)}\mathcal{J}_0)_{im} - 6(\mathcal{J}_0\mathcal{J}_0\mathcal{J}_0^{1)})_{im} \\ &+ 18(\mathcal{J}_0^{2)}\mathcal{J}_0^{1)})_{im} + 30(\mathcal{J}_0^{1)}\mathcal{J}_0^{2)})_{im} + 20(\mathcal{J}_0\mathcal{J}_0^{3)})_{im} + \tfrac{6}{4}\mathcal{J}_{0_{im}}^{1)} + \tfrac{30}{4}\mathcal{J}_{0_{im}}^{3)})\tfrac{t^8}{8!} + O(t^9). \end{aligned}$$

Moreover, using (3.45), (3.46), (3.47), (3.48), (3.49) and (3.38) we obtain

Lemma 3.3.12. *The Jacobi field $Y_1(t)$ along the geodesic γ with $\dot\gamma_0 = \dot X_0$ is given by*

$$\begin{aligned} Y_1(t) = &tQ_1 + S(t)((x_3^2 + x_4^2)Q_1 - (x_1x_3 + x_2x_4)Q_3 + (x_2x_3 - x_1x_4)Q_4) \\ &+ T(t)(\tfrac{1}{2}t^2(x_3Q_5 + x_4Q_6)), \end{aligned} \tag{3.39}$$

where

$$\begin{aligned} S(t) &= \tfrac{1}{8}t^3 - \tfrac{1}{128}t^5 + \tfrac{1}{9216}t^7 + O(t^9), \\ T(t) &= -\tfrac{1}{12}t^2 + \tfrac{1}{480}t^4 - \tfrac{1}{53760}t^6 + O(t^7). \end{aligned} \tag{3.40}$$

Now, following the notation of Lemma 3.1.1, we denote by $\{P_i\}_{i=1}^6$ the orthonormal frame field obtained by $\bar{\nabla}_{\dot{\gamma}}$-parallel translation of the basis $\{E_i\}$ along γ. Thus, from Theorem of [B-Tr-V, p. 52] we have along the geodesic γ with $\dot{\gamma}_0 = \dot{X}_0$ that the Jacobi field

$$Y_1(t) = B_{P_1}(t) = tP_1 + \tfrac{1}{2}t^2[P_1, \dot{\gamma}_0] \overset{(3.23)}{=} tP_1 + \tfrac{1}{2}t^2(x_3P_5 + x_4P_6). \tag{3.41}$$

The following result is analogous to Lemma 5 of [Ch.2] and establishes the relation between the parallel orthonormal frame fields $\{Q_i\}_{i=1}^6$ and $\{P_i\}_{i=1}^6$ along the arbitrary geodesic γ.

Lemma 3.3.13. *Under the assumptions of Lemma 3.1.1, the relation*

$$Q_i(t) = \sum_{j=1}^{6} a_{ij}(t)P_j(t) \quad \textit{for} \quad i = 1, ..., 6$$

implies that the matrix $(a_{ij}(0)) = Id$ *and*

$$a'_{ij}(t) = \sum_{k=1}^{6} a_{ik}(t)(T_\gamma P_k(t))_j \quad \textit{for} \quad i, j = 1, ..., 6.$$

Proof. Note that $\nabla_{\dot{\gamma}} = \bar{\nabla}_{\dot{\gamma}} - T_\gamma$, implies that $\nabla_{\dot{\gamma}}X = \bar{\nabla}_{\dot{\gamma}}X - T_\gamma X$, where X is any vector field along γ. Then for each $i = 1, ..., 6$,

$$\begin{aligned}
0 = \nabla_{\dot{\gamma}}Q_i(t) &= \nabla_{\dot{\gamma}}\left(\sum_{j=1}^{6} a_{ij}(t)P_j(t)\right) \\
&= \sum_{j=1}^{6} \nabla_{\dot{\gamma}}a_{ij}(t)P_j(t) + \sum_{j=1}^{6} a_{ij}(t)\nabla_{\dot{\gamma}}P_j(t) \\
&= \sum_{j=1}^{6} a'_{ij}(t)P_j(t) + \sum_{j=1}^{6} a_{ij}(t)(\bar{\nabla}_{\dot{\gamma}}P_j(t) - T_\gamma P_j(t)) \\
&= \sum_{j=1}^{6} a'_{ij}(t)P_j(t) - \sum_{j=1}^{6} a_{ij}(t)\sum_{k=1}^{6}(T_\gamma P_j(t))_k P_k(t) \\
&= \sum_{j=1}^{6}\left(a'_{ij}(t) - \sum_{k=1}^{6} a_{ik}(t)(T_\gamma P_k(t))_j\right)P_j(t)
\end{aligned}$$

which implies the lemma by the linear independence of $\{P_j(t)\}_{j=1}^6$ for all t.

Finally, we shall use Lemma 3.3.13 to compare (3.39) with (3.41). Thus, we shall first calculate the matrix $(a_{ij}(t))$ assuming that $x_5 = x_6 = 0$. From now on, if there is no confusion we shall denote it by (a_{ij}).

Lemma 3.3.14. *The matrix* (a_{ij}) *such that* $Q_i = \sum_{j=1}^{6} a_{ij} P_j$, $i = 1, ..., 6$ *along the geodesic* γ *with* $\dot{\gamma}_0 = \dot{X}_0$ *is given by*

$$a_{11} = a_{22} = x_1^2 + x_2^2 + \cos(t/2)(x_3^2 + x_4^2), \quad a_{12} = a_{21} = a_{34} = a_{43} = a_{56} = a_{65} = 0,$$
$$a_{13} = a_{24} = a_{31} = a_{42} = (1 - \cos(t/2))(x_1x_3 + x_2x_4), \; a_{16} = -a_{25} = a_{52} = \sin(t/2)x_4,$$
$$a_{55} = a_{66} = \cos(t/2), \quad a_{14} = -a_{23} = -a_{32} = a_{41} = (-1 + \cos(t/2))(x_2x_3 - x_1x_4),$$
$$a_{33} = a_{44} = \cos(t/2)(x_1^2 + x_2^2) + x_3^2 + x_4^2, \quad a_{15} = a_{26} = -a_{51} = \sin(t/2)x_3,$$
$$a_{35} = a_{46} = -a_{53} = -\sin(t/2)x_1, \quad a_{36} = -a_{45} = a_{54} = -\sin(t/2)x_2.$$

Proof. The details of this proof can be seen in Section 3.3.4.

Second, we substitute Q_i by $\sum_{j=1}^{6} a_{ij} P_j$, $i = 1, ..., 6$ in (3.39) obtaining that

$$\begin{aligned} Y_1(t) =& (t(x_1^2 + x_2^2) + (t - f(t))(x_3^2 + x_4^2))P_1 + f(t)(x_1x_3 + x_2x_4)P_3 \\ &- f(t)(x_2x_3 - x_1x_4)P_4 + \tfrac{1}{2}t^2 g(t)(x_3P_5 + x_4P_6) \\ =& tP_1 - f(t)((x_3^2 + x_4^2)P_1 - (x_1x_3 + x_2x_4)P_3 + (x_2x_3 - x_1x_4)P_4) \\ &+ \tfrac{1}{2}t^2 g(t)(x_3P_5 + x_4P_6), \end{aligned} \tag{3.42}$$

where

$$\begin{aligned} f(t) =& t(1 - \cos(t/2)) - \cos(t/2)S(t) + \tfrac{t^2}{2}\sin(t/2)T(t), \\ g(t) =& \tfrac{2}{t}\sin(t/2) + \cos(t/2)T(t) + \tfrac{2}{t^2}\sin(t/2)S(t). \end{aligned}$$

Then, (3.41) is equal to (3.42) if and only if $f(t) = 0$, $g(t) = 1$ or, equivalently, if

$$S(t) = \tfrac{1}{2}t(-2 + 2\cos(t/2) + t\sin(t/2)), \quad T(t) = \cos(t/2) - \tfrac{2}{t}\sin(t/2). \tag{3.43}$$

Finally, it is easy to check that (3.43) is true because their Taylor's series development at $t = 0$ coincide with (3.40). Thus, the expressions for the Jacobi field $Y_1(t)$ along the geodesic γ with $\dot{\gamma}_0 = \dot{X}_0$ given in (3.39) and (3.41) are equivalent.

As a final conclusion, note that the method used by J. Berndt, F. Tricerri and L. Vanhecke in [B-Tr-V, p.52] is exact and the method that we used to obtain Proposition 3.3.10 gives approximations to the result as good as the number of terms of the series (3.37) that we were able to calculate. However, with the first method it is not always possible to obtain general explicit results as on H-type groups (recall the results gives by I. Chavel in [Ch.1], [Ch.2] and W. Ziller in [Z]), while the new method, propose in [N-T] on naturally reductive spaces and now extending on every g.o. space, can be always applied with easy and straightforward computations.

3.3.3 Appendix A

In this section, we present the explicit expressions of the n^{th} covariant derivative of the $(0, 4)$ - Jacobi operator along an arbitrary geodesic γ with initial vector x at the origin p of Kaplan's example N for $n = 0, ..., 5$. Here, we suppose that $x \in \mathfrak{n}$

is an arbitrary unit vector, that is, $x=\sum_{i=1}^{6} x^i E_i$, $\sum_{i=1}^{6}(x^i)^2=1$. Furthermore, we denote by $\{Q_i\}_{i=1}^{6}$ the orthonormal frame field along γ obtained by parallel translation of the basis $\{E_i\}$ along γ. Moreover, remember that

$$\begin{aligned}
\mathcal{J}^{n)}_{\alpha\beta}(0)&=\langle\zeta_{(n,A)}(Q_\alpha),Q_\beta\rangle, \quad \mathcal{J}^{n)}_{\alpha j}(0)=\langle\nu_{(n,A)}(Q_\alpha),Q_j\rangle,\\
\mathcal{J}^{n)}_{i\beta}(0)&=\langle\zeta_{(n,X)}(Q_i),Q_\beta\rangle, \quad \mathcal{J}^{n)}_{ij}(0)=\langle\nu_{(n,X)}(Q_i),Q_j\rangle,
\end{aligned} \tag{3.44}$$

for $i,j=1,...,4$ and $\alpha,\beta=5,6$.

Thus, using (3.44), Lemma 3.3.5 and (3.21), for $n=0$ we have $\mathcal{J}^{0)}_0=\left(\mathcal{J}^0_{ij}(0)\right)$, $i,j=1,...,6$, where

$$\begin{aligned}
&\mathcal{J}^0_{11}(0)=\mathcal{J}^0_{22}(0)=\tfrac14(-3x_3^2-3x_4^2+x_5^2+x_6^2), \quad \mathcal{J}^0_{12}(0)=\mathcal{J}^0_{34}(0)=\mathcal{J}^0_{56}(0)=0,\\
&\mathcal{J}^0_{13}(0)=\mathcal{J}^0_{24}(0)=\tfrac34(x_1x_3+x_2x_4), \quad \mathcal{J}^0_{14}(0)=-\mathcal{J}^0_{23}(0)=\tfrac34(-x_2x_3+x_1x_4),\\
&\mathcal{J}^0_{15}(0)=\tfrac14(-x_1x_5-3x_2x_6),\ \mathcal{J}^0_{16}(0)=\tfrac14(3x_2x_5-x_1x_6),\ \mathcal{J}^0_{25}(0)=\tfrac14(-x_2x_5+3x_1x_6),\\
&\mathcal{J}^0_{26}(0)=\tfrac14(-3x_1x_5-x_2x_6),\ \mathcal{J}^0_{35}(0)=\tfrac14(-x_3x_5-3x_4x_6),\ \mathcal{J}^0_{36}(0)=\tfrac14(3x_4x_5-x_3x_6),\\
&\mathcal{J}^0_{33}(0)=\mathcal{J}^0_{44}(0)=\tfrac14(-3x_1^2-3x_2^2+x_5^2+x_6^2), \quad \mathcal{J}^0_{45}(0)=\tfrac14(-x_4x_5+3x_3x_6),\\
&\mathcal{J}^0_{46}(0)=\tfrac14(-3x_3x_5-x_4x_6), \quad \mathcal{J}^0_{55}(0)=\mathcal{J}^0_{66}(0)=\tfrac14(x_1^2+x_2^2+x_3^2+x_4^2).
\end{aligned} \tag{3.45}$$

For $n=1$ we have $\mathcal{J}^{1)}_0=\left(\mathcal{J}^1_{ij}(0)\right)$, $i,j=1,...,6$, where

$$\begin{aligned}
\mathcal{J}^1_{11}(0)&\overset{(3.44)}{=}\langle\nu_{(1,X)}(Q_1),Q_1\rangle\\
&\overset{(3.21)}{=}\langle\tfrac12(\nu_{(0,X)}(j(\dot A_0)Q_1)-j(\dot A_0)\nu_{(0,X)}(Q_1)-j(\zeta_{(0,X)}(Q_1))\dot X_0-\nu_{(0,A)}([\dot X_0,Q_1])),Q_1\rangle\\
&\overset{Lem.3.3.5}{=}\tfrac12(\langle\nu_{(0,X)}(x_5Q_3+x_6Q_4),Q_1\rangle\\
&-\langle j(\dot A_0)(\tfrac14((-3(x_3^2+x_4^2)+x_5^2+x_6^2)Q_1+3(x_1x_3+x_2x_4)Q_3+3(x_1x_4-x_2x_3)Q_4)),Q_1\rangle\\
&-\langle j(\tfrac14((-x_1x_5-3x_2x_6)Q_5+(3x_2x_5-x_1x_6)Q_6))\dot X_0,Q_1\rangle\\
&-\langle\nu_{(0,A)}(-x_3Q_5-x_4Q_6),Q_1\rangle)\\
&=\tfrac12(x_5\langle\nu_{(0,X)}(Q_3),Q_1\rangle+x_6\langle\nu_{(0,X)}(Q_4),Q_1\rangle\\
&-\tfrac14((-3(x_3^2+x_4^2)+x_5^2+x_6^2)\langle j(\dot A_0)(Q_1),Q_1\rangle\\
&+3(x_1x_3+x_2x_4)\langle j(\dot A_0)(Q_3),Q_1\rangle+3(x_1x_4-x_2x_3)\langle j(\dot A_0)(Q_4),Q_1\rangle)\\
&-\tfrac14((-x_1x_5-3x_2x_6)\langle j(Q_5)\dot X_0,Q_1\rangle+(3x_2x_5-x_1x_6)\langle j(Q_6)\dot X_0,Q_1\rangle)\\
&+x_3\langle\nu_{(0,A)}(Q_5),Q_1\rangle+x_4\langle\nu_{(0,A)}(Q_6),Q_1\rangle)
\end{aligned} \tag{3.46}$$

$$
\begin{aligned}
&\overset{Lem.3.3.5}{=} \tfrac{1}{2}(x_5(\tfrac{3}{4}(x_1x_3+x_2x_4))+x_6(\tfrac{3}{4}(x_1x_4-x_2x_3))\\
&+\tfrac{-3}{4}(x_1x_3+x_2x_4)(-x_5)+\tfrac{-3}{4}(x_1x_4-x_2x_3)(-x_6)\\
&-\tfrac{1}{4}((-x_1x_5-3x_2x_6)(-x_3)+(3x_2x_5-x_1x_6)(-x_4))\\
&+x_3(\tfrac{1}{4}(-x_1x_5-3x_2x_6))+x_4(\tfrac{1}{4}(3x_2x_5-x_1x_6)))\\
&=\tfrac{1}{2}(x_1(x_3x_5+x_4x_6)+3x_2(x_4x_5-x_3x_6)),\\
\mathcal{J}^1_{12}(0)&=x_1(-x_4x_5+x_3x_6)+x_2(x_3x_5+x_4x_6),\\
\mathcal{J}^1_{13}(0)&=\tfrac{1}{4}(x_5(-x_1^2-3x_2^2+x_3^2+3x_4^2)+x_6(2x_1x_2-2x_3x_4)),\\
\mathcal{J}^1_{14}(0)&=\tfrac{1}{4}(x_6(-x_1^2-3x_2^2+3x_3^2+x_4^2)+x_5(-2x_1x_2-2x_3x_4)),\\
\mathcal{J}^1_{15}(0)&=\tfrac{1}{4}(x_3(2x_1^2+2x_2^2+2x_3^2+2x_4^2-x_5^2+x_6^2)-2x_4x_5x_6),\\
\mathcal{J}^1_{16}(0)&=\tfrac{1}{4}(x_4(2x_1^2+2x_2^2+2x_3^2+2x_4^2+x_5^2-x_6^2)-2x_3x_5x_6),\\
\mathcal{J}^1_{22}(0)&=\tfrac{1}{2}(-3x_1(x_3x_5+x_4x_6)+x_2(-x_4x_5+x_3x_6)),\\
\mathcal{J}^1_{23}(0)&=\tfrac{1}{4}(x_6(-3x_1^2-x_2^2+x_3^2+3x_4^2)+x_5(2x_1x_2+2x_3x_4)),\\
\mathcal{J}^1_{24}(0)&=\tfrac{1}{4}(x_5(3x_1^2+x_2^2-3x_3^2-x_4^2)+x_6(2x_1x_2-2x_3x_4)),\\
\mathcal{J}^1_{25}(0)&=\tfrac{1}{4}(x_4(-2x_1^2-2x_2^2-2x_3^2-2x_4^2+x_5^2-x_6^2)-2x_3x_5x_6),\\
\mathcal{J}^1_{26}(0)&=\tfrac{1}{4}(x_3(2x_1^2+2x_2^2+2x_3^2+2x_4^2+x_5^2-x_6^2)+2x_4x_5x_6),\\
\mathcal{J}^1_{33}(0)&=\tfrac{1}{2}(x_1(-x_3x_5+3x_4x_6)+x_2(-3x_4x_5-x_3x_6)),\\
\mathcal{J}^1_{34}(0)&=x_1(-x_4x_5-x_3x_6)+x_2(x_3x_5-x_4x_6),\\
\mathcal{J}^1_{35}(0)&=\tfrac{1}{4}(-x_1(2x_1^2+2x_2^2+2x_3^2+2x_4^2-x_5^2+x_6^2)+2x_2x_5x_6),\\
\mathcal{J}^1_{36}(0)&=\tfrac{1}{4}(-x_2(2x_1^2+2x_2^2+2x_3^2+2x_4^2+x_5^2-x_6^2)+2x_1x_5x_6),\\
\mathcal{J}^1_{44}(0)&=\tfrac{1}{2}(x_1(3x_3x_5-x_4x_6)+x_2(x_4x_5+3x_3x_6)),\\
\mathcal{J}^1_{45}(0)&=\tfrac{1}{4}(x_2(2x_1^2+2x_2^2+2x_3^2+2x_4^2-x_5^2+x_6^2)+2x_1x_5x_6),\\
\mathcal{J}^1_{46}(0)&=\tfrac{1}{4}(-x_1(2x_1^2+2x_2^2+2x_3^2+2x_4^2+x_5^2-x_6^2)-2x_2x_5x_6),\\
\mathcal{J}^1_{55}(0)&=\mathcal{J}^1_{56}(0)=\mathcal{J}^1_{66}(0)=0.
\end{aligned}
$$

For $n=2$ we have $\mathcal{J}^{2)}_0=\left(\mathcal{J}^2_{ij}(0)\right)$, $i,j=1,...,6$, where

$$
\begin{aligned}
\mathcal{J}^2_{11}(0)=&\tfrac{1}{4}(x_1^2(2x_3^2+2x_4^2-x_5^2-x_6^2)+x_2^2(2x_3^2+2x_4^2-3x_5^2-3x_6^2)\\
&+2((x_3^2+x_4^2)^2+2(x_4x_5-x_3x_6)^2)),\\
\mathcal{J}^2_{12}(0)=&\tfrac{1}{2}(x_5x_6(-2x_3^2+x_4^2)+2x_3x_4(x_5^2-x_6^2)+x_1x_2(x_5^2+x_6^2)),\\
\mathcal{J}^2_{13}(0)=&\tfrac{1}{4}(4x_5x_6(x_2x_3+x_1x_4)-x_1x_3(2x_1^2+2x_2^2+2x_3^2+2x_4^2+x_5^2+5x_6^2)\\
&-x_2x_4(2x_1^2+2x_2^2+2x_3^2+2x_4^2+7x_5^2+3x_6^2)),\\
\mathcal{J}^2_{14}(0)=&\tfrac{1}{4}(4x_5x_6(x_1x_3-x_2x_4)+x_2x_3(2x_1^2+2x_2^2+2x_3^2+2x_4^2+3x_5^2+7x_6^2)\\
&-x_1x_4(2x_1^2+2x_2^2+2x_3^2+2x_4^2+5x_5^2+x_6^2)),\\
\mathcal{J}^2_{15}(0)=&\tfrac{1}{8}(x_1x_5(-3x_1^2-3x_2^2-3x_3^2-3x_4^2+x_5^2+x_6^2)\\
&+x_2x_6(5x_1^2+5x_2^2+5x_3^2+5x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^2_{16}(0)=&\tfrac{1}{8}(x_1x_6(-3x_1^2-3x_2^2-3x_3^2-3x_4^2+x_5^2+x_6^2)\\
&-x_2x_5(5x_1^2+5x_2^2+5x_3^2+5x_4^2+x_5^2+x_6^2)),
\end{aligned}
\tag{3.47}
$$

$$
\begin{aligned}
\mathcal{J}^2_{22}(0) =&\tfrac{1}{4}(x_2^2(2x_3^2+2x_4^2-x_5^2-x_6^2)+x_1^2(2x_3^2+2x_4^2-3x_5^2-3x_6^2)\\
&+2((x_3^2+x_4^2)^2+2(x_4x_6-x_3x_5)^2)),\\
\mathcal{J}^2_{23}(0) =&\tfrac{1}{4}(4x_5x_6(x_1x_3-x_2x_4)+x_1x_4(2x_1^2+2x_2^2+2x_3^2+2x_4^2+3x_5^2+7x_6^2)\\
&-x_2x_3(2x_1^2+2x_2^2+2x_3^2+2x_4^2+5x_5^2+x_6^2)),\\
\mathcal{J}^2_{24}(0) =&\tfrac{1}{4}(-4x_5x_6(x_2x_3+x_1x_4)-x_1x_3(2x_1^2+2x_2^2+2x_3^2+2x_4^2+7x_5^2+3x_6^2)\\
&-x_2x_4(2x_1^2+2x_2^2+2x_3^2+2x_4^2+x_5^2+5x_6^2)),\\
\mathcal{J}^2_{25}(0) =&\tfrac{1}{8}(-x_1x_6(5x_1^2+5x_2^2+5x_3^2+5x_4^2+x_5^2+x_6^2)\\
&-x_2x_5(3x_1^2+3x_2^2+3x_3^2+3x_4^2-x_5^2-x_6^2)),\\
\mathcal{J}^2_{26}(0) =&\tfrac{1}{8}(x_1x_5(5x_1^2+5x_2^2+5x_3^2+5x_4^2+x_5^2+x_6^2)\\
&+x_2x_6(-3x_1^2-3x_2^2-3x_3^2-3x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^2_{33}(0) =&\tfrac{1}{4}(x_3^2(2x_1^2+2x_2^2-x_5^2-x_6^2)+x_4^2(2x_1^2+2x_2^2-3x_5^2-3x_6^2)\\
&+2((x_1^2+x_2^2)^2+2(x_2x_5-x_1x_6)^2)),\\
\mathcal{J}^2_{34}(0) =&\tfrac{1}{2}(2x_5x_6(-x_1^2+x_2^2)+x_3x_4(x_5^2+x_6^2)+2x_1x_2(x_5^2-x_6^2)),\\
\mathcal{J}^2_{35}(0) =&\tfrac{1}{8}(x_3x_5(-3x_1^2-3x_2^2-3x_3^2-3x_4^2+x_5^2+x_6^2)\\
&+x_4x_6(5x_1^2+5x_2^2+5x_3^2+5x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^2_{36}(0) =&\tfrac{1}{8}(x_3x_6(-3x_1^2-3x_2^2-3x_3^2-3x_4^2+x_5^2+x_6^2)\\
&-x_4x_5(5x_1^2+5x_2^2+5x_3^2+5x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^2_{44}(0) =&\tfrac{1}{4}(x_3^2(2x_1^2+2x_2^2-3x_5^2-3x_6^2)+x_4^2(2x_1^2+2x_2^2-x_5^2-x_6^2)\\
&+2((x_1^2+x_2^2)^2+2(x_2x_6-x_1x_5)^2)),\\
\mathcal{J}^2_{45}(0) =&\tfrac{1}{8}(-x_3x_6(5x_1^2+5x_2^2+5x_3^2+5x_4^2+x_5^2+x_6^2)\\
&+x_4x_5(-3x_1^2-3x_2^2-3x_3^2-3x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^2_{46}(0) =&\tfrac{1}{8}(x_3x_5(5x_1^2+5x_2^2+5x_3^2+5x_4^2+x_5^2+x_6^2)\\
&+x_4x_6(-3x_1^2-3x_2^2-3x_3^2-3x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^2_{55}(0) =&\tfrac{-1}{4}(x_1^2+x_2^2+x_3^2+x_4^2)(2x_1^2+2x_2^2+2x_3^2+2x_4^2-x_5^2+x_6^2),\\
\mathcal{J}^2_{56}(0) =&\tfrac{1}{2}(x_1^2+x_2^2+x_3^2+x_4^2)x_5x_6,\\
\mathcal{J}^2_{66}(0) =&\tfrac{-1}{4}(x_1^2+x_2^2+x_3^2+x_4^2)(2x_1^2+2x_2^2+2x_3^2+2x_4^2+x_5^2-x_6^2).
\end{aligned}
$$

For $n=3$ we have $\mathcal{J}_0^{3)}=\left(\mathcal{J}^3_{ij}(0)\right)$, $i,j=1,...,6$, where

$$
\begin{aligned}
\mathcal{J}^3_{11}(0) =&\tfrac{1}{8}(x_1(x_3x_5+x_4x_6)(-7x_1^2-7x_2^2-7x_3^2-7x_4^2-x_5^2-x_6^2)\\
&+x_2(x_3x_6-x_4x_5)(9x_1^2+9x_2^2+9x_3^2+9x_4^2+15x_5^2+15x_6^2)),\\
\mathcal{J}^3_{12}(0) =&(x_1(x_4x_5-x_3x_6)-x_2(x_3x_5+x_4x_6))(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2),\\
\mathcal{J}^3_{13}(0) =&\tfrac{1}{16}(x_5(7x_1^4+16x_1^2x_2^2+9x_2^4+2x_2^2x_3^2-7x_3^4-2x_1^2x_4^2-16x_3^2x_4^2-9x_4^4\\
&+x_1^2x_5^2+15x_2^2x_5^2-x_3^2x_5^2-15x_4^2x_5^2+x_1^2x_6^2+15x_2^2x_6^2-x_3^2x_6^2-15x_4^2x_6^2)\\
&+x_6(-2x_1^3x_2-2x_2^3x_1-2x_1x_2x_3^2+2x_1^2x_3x_4+2x_2^2x_3x_4+2x_3^3x_4\\
&-2x_1x_2x_4^2+2x_3x_4^3-14x_1x_2x_6^2+14x_3x_4x_6^2-14x_1x_2x_5^2+14x_3x_4x_5^2)),
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
\mathcal{J}^3_{14}(0) =&\tfrac{1}{16}(x_6(7x_1^4 + 16x_1^2x_2^2 + 9x_2^4 - 2x_1^2x_3^2 - 9x_3^4 + 2x_2^2x_4^2 - 16x_3^2x_4^2 - 7x_4^4\\
&+ x_1^2x_5^2 + 15x_2^2x_5^2 - 15x_3^2x_5^2 - x_4^2x_5^2 + x_1^2x_6^2 + 15x_2^2x_6^2 - 15x_3^2x_6^2 - x_4^2x_6^2)\\
&+ x_5(2x_1^3x_2 + 2x_1x_2^3 + 2x_1x_2x_3^2 + 2x_1^2x_3x_4 + 2x_2^2x_3x_4 + 2x_3^3x_4\\
&+ 2x_1x_2x_4^2 + 2x_3x_4^3 + 14x_1x_2x_5^2 + 14x_3x_4x_5^2 + 14x_1x_2x_6^2 + 14x_3x_4x_6^2)),\\
\mathcal{J}^3_{15}(0) =&\tfrac{1}{16}(x_3(-8x_1^4 - 16x_1^2x_2^2 - 8x_2^4 - 16x_1^2x_3^2 - 16x_2^2x_3^2 - 8x_3^4 - 16x_1^2x_4^2 - 16x_2^2x_4^2\\
&- 16x_3^2x_4^2 - 8x_4^4 - x_1^2x_5^2 - x_2^2x_5^2 - x_3^2x_5^2 - x_4^2x_5^2 + x_5^4 - 15x_1^2x_6^2 - 15x_2^2x_6^2\\
&- 15x_3^2x_6^2 - 15x_4^2x_6^2 - x_6^4) + 2x_4x_5x_6(7x_1^2 + 7x_2^2 + 7x_3^2 + 7x_4^2 + x_5^2 + x_6^2)),\\
\mathcal{J}^3_{16}(0) =&\tfrac{1}{16}(x_4(-8x_1^4 - 16x_1^2x_2^2 - 8x_2^4 - 16x_1^2x_3^2 - 16x_2^2x_3^2 - 8x_3^4 - 16x_1^2x_4^2 - 16x_2^2x_4^2\\
&- 16x_3^2x_4^2 - 8x_4^4 - 15x_1^2x_5^2 - 15x_2^2x_5^2 - 15x_3^2x_5^2 - 15x_4^2x_5^2 - x_5^4 - x_1^2x_6^2 - x_2^2x_6^2\\
&- x_3^2x_6^2 - x_4^2x_6^2 + x_6^4) + 2x_3x_5x_6(7x_1^2 + 7x_2^2 + 7x_3^2 + 7x_4^2 + x_5^2 + x_6^2)),\\
\mathcal{J}^3_{22}(0) =&\tfrac{1}{8}(x_1(x_3x_5 + x_4x_6)(9x_1^2 + 9x_2^2 + 9x_3^2 + 9x_4^2 + 15x_5^2 + 15x_6^2)\\
&+ x_2(x_3x_6 - x_4x_5)(-7x_1^2 - 7x_2^2 - 7x_3^2 - 7x_4^2 - x_5^2 - x_6^2)),\\
\mathcal{J}^3_{23}(0) =&\tfrac{1}{16}(x_6(9x_1^4 + 16x_1^2x_2^2 + 7x_2^4 + 2x_1^2x_3^2 - 7x_3^4 - 2x_2^2x_4^2 - 16x_3^2x_4^2 - 9x_4^4\\
&+ 15x_1^2x_5^2 + x_2^2x_5^2 - x_3^2x_5^2 - 15x_4^2x_5^2 + 15x_1^2x_6^3 + x_2^2x_6^2 - x_3^2x_6^2 - 15x_4^2x_6^2)\\
&+ x_5(-2x_1^3x_2 - 2x_1x_2^3 - 2x_1x_2x_3^2 - 2x_1^2x_3x_4 - 2x_2^2x_3x_4 + 2x_3^3x_4\\
&- 2x_1x_2x_4^2 - x_3x_4^3 - 14x_1x_2x_5^2 - 14x_3x_4x_5^2 - 14x_1x_2x_6^2 - 14x_3x_4x_6^2)),\\
\mathcal{J}^3_{24}(0) =&\tfrac{1}{16}(x_5(-9x_1^4 - 16x_1^2x_2^2 - 7x_2^4 + 2x_2^2x_3^2 + 9x_3^4 - 2x_1^2x_4^2 + 16x_3^2x_4^2 + 7x_4^4\\
&- 15x_1^2x_5^2 - x_2^2x_5^2 + 15x_3^2x_5^2 + x_4^2x_5^2 - 15x_1^2x_6^2 - x_2^2x_6^2 + 15x_3^2x_6^2 + x_4^2x_6^2)\\
&+ x_6(-2x_1^3x_2 - 2x_2^3x_1 - 2x_1x_2x_3^2 + 2x_1^2x_3x_4 + 2x_2^2x_3x_4 + 2x_3^3x_4\\
&- 2x_1x_2x_4^2 + 2x_3x_4^3 - 14x_1x_2x_5^2 + 14x_3x_4x_5^2 - 14x_1x_2x_6^2 + 14x_3x_4x_6^2)),\\
\mathcal{J}^3_{25}(0) =&\tfrac{1}{16}(x_4(8x_1^4 + 16x_1^2x_2^2 + 8x_2^4 + 16x_1^2x_3^2 + 16x_2^2x_3^2 + 8x_3^4 + 16x_1^2x_4^2 + 16x_2^2x_4^2\\
&+ 16x_3^2x_4^2 + 8x_4^4 + x_1^2x_5^2 + x_2^2x_5^2 + x_3^2x_5^2 + x_4^2x_5^2 - x_5^4 + 15x_1^2x_6^2 + 15x_2^2x_6^2\\
&+ 15x_3^2x_6^2 + 15x_4^2x_6^2 + x_6^4) + 2x_3x_5x_6(7x_1^2 + 7x_2^2 + 7x_3^2 + 7x_4^2 + x_5^2 + x_6^2)),\\
\mathcal{J}^3_{26}(0) =&\tfrac{1}{16}(x_3(-8x_1^4 - 16x_1^2x_2^2 - 8x_2^4 - 16x_1^2x_3^2 - 16x_2^2x_3^2 - 8x_3^4 - 16x_1^2x_4^2 - 16x_2^2x_4^2\\
&- 16x_3^2x_4^2 - 8x_4^4 - 15x_1^2x_5^2 - 15x_2^2x_5^2 - 15x_3^2x_5^2 - 15x_4^2x_5^2 - x_5^4 - x_1^2x_6^2 - x_2^2x_6^2\\
&- x_3^2x_6^2 - x_4^2x_6^2 + x_6^4) - 2x_4x_5x_6(7x_1^2 + 7x_2^2 + 7x_3^2 + 7x_4^2 + x_5^2 + x_6^2)),\\
\mathcal{J}^3_{33}(0) =&\tfrac{1}{8}(x_3(x_1x_5 + x_2x_6)(7x_1^2 + 7x_2^2 + 7x_3^2 + 7x_4^2 + x_5^2 + x_6^2)\\
&+ x_4(x_1x_6 - x_2x_5)(-9x_1^2 - 9x_2^2 - 9x_3^2 - 9x_4^2 - 15x_5^2 - 15x_6^2)),\\
\mathcal{J}^3_{34}(0) =&(x_1(x_4x_5 + x_3x_6) - x_2(x_3x_5 - x_4x_6))(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2),\\
\mathcal{J}^3_{35}(0) =&\tfrac{1}{16}(x_1(8x_1^4 + 16x_1^2x_2^2 + 8x_2^4 + 16x_1^2x_3^2 + 16x_2^2x_3^2 + 8x_3^4 + 16x_1^2x_4^2 + 16x_2^2x_4^2\\
&+ 16x_3^2x_4^2 + 8x_4^4 + x_1^2x_5^2 + x_2^2x_5^2 + x_3^2x_5^2 + x_4^2x_5^2 - x_5^4 + 15x_1^2x_6^2 + 15x_2^2x_6^2\\
&+ 15x_3^2x_6^2 + 15x_4^2x_6^2 + x_6^4) - 2x_2x_5x_6(7x_1^2 + 7x_2^2 + 7x_3^2 + 7x_4^2 + x_5^2 + x_6^2)),\\
\mathcal{J}^3_{36}(0) =&\tfrac{1}{16}(x_2(8x_1^4 + 16x_1^2x_2^2 + 8x_2^4 + 16x_1^2x_3^2 + 16x_2^2x_3^2 + 8x_3^4 + 16x_1^2x_4^2 + 16x_2^2x_4^2\\
&+ 16x_3^2x_4^2 + 8x_4^4 + 15x_1^2x_5^2 + 15x_2^2x_5^2 + 15x_3^2x_5^2 + 15x_4^2x_5^2 + x_5^4 + x_1^2x_6^2 + x_2^2x_6^2\\
&+ x_3^2x_6^2 + x_4^2x_6^2 - x_6^4) - 2x_1x_5x_6(7x_1^2 + 7x_2^2 + 7x_3^2 + 7x_4^2 + x_5^2 + x_6^2)),
\end{aligned}
$$

$$\begin{aligned}
\mathcal{J}^3_{44}(0) =&\tfrac{1}{8}(x_3(x_1x_5+x_2x_6)(-9x_1^2-9x_2^2-9x_3^2-9x_4^2-15x_5^2-15x_6^2)\\
&+x_4(x_1x_6-x_2x_5)(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^3_{45}(0) =&\tfrac{1}{16}(x_2(-8x_1^4-16x_1^2x_2^2-8x_2^4-16x_1^2x_3^2-16x_2^2x_3^2-8x_3^4-16x_1^2x_4^2-16x_2^2x_4^2\\
&-16x_3^2x_4^2-8x_4^4-x_1^2x_5^2-x_2^2x_5^2-x_3^2x_5^2-x_4^2x_5^2+x_5^4-15x_1^2x_6^2-15x_2^2x_6^2\\
&-15x_3^2x_6^2-15x_4^2x_6^2-x_6^4)-2x_1x_5x_6(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^3_{46}(0) =&\tfrac{1}{16}(x_1(8x_1^4+16x_1^2x_2^2+8x_2^4+16x_1^2x_3^2+16x_2^2x_3^2+8x_3^4+16x_1^2x_4^2+16x_2^2x_4^2\\
&+16x_3^2x_4^2+8x_4^4+15x_1^2x_5^2+15x_2^2x_5^2+15x_3^2x_5^2+15x_4^2x_5^2+x_5^4+x_1^2x_6^2+x_2^2x_6^2\\
&+x_3^2x_6^2+x_4^2x_6^2-x_6^4)+2x_2x_5x_6(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^3_{55}(0) =& \mathcal{J}^3_{56}(0)=\mathcal{J}^3_{66}(0)=0.
\end{aligned}$$

For $n=4$ we have $\mathcal{J}_0^{4)}=\left(\mathcal{J}^4_{ij}(0)\right)$, $i,j=1,...,6$, where

$$\begin{aligned}
\mathcal{J}^4_{11}(0) =&\tfrac{1}{16}(x_1^2(-8x_1^2x_3^2-16x_3^4-8x_1^2x_4^2-16x_4^4+7x_1^2x_5^2-x_3^2x_5^2-17x_4^2x_5^2+x_5^4+7x_1^2x_6^2\\
&-17x_3^2x_6^2-x_4^2x_6^2+2x_5^2x_6^2+x_6^4)+x_2^2(-8x_2^2x_3^2-16x_3^4-8x_2^2x_4^2-16x_4^4\\
&+9x_2^2x_5^2+x_3^2x_5^2-15x_4^2x_5^2+15x_5^4+9x_2^2x_6^2-15x_3^2x_6^2+x_4^2x_6^2+30x_5^2x_6^2+15x_6^4)\\
&+x_1^2x_2^2(-16x_3^2-16x_4^2+16x_5^5+16x_6^2)+x_3^2(-8x_3^4-8x_3^2x_5^2-24x_3^2x_6^2\\
&-16x_5^2x_6^2-16x_6^4)+x_4^2(-8x_4^4-24x_4^2x_5^2-16x_5^4-8x_4^2x_6^2-16x_5^2x_6^2)\\
&+x_3^2x_4^2(-32x_1^2-32x_2^2-24x_3^2-24x_4^2-32x_5^2-32x_6^2)\\
&+x_3x_4x_5x_6(32x_1^2+32x_2^2+32x_3^2+32x_4^2+32x_5^2+32x_6^2)),\\
\mathcal{J}^4_{12}(0) =&\tfrac{1}{8}(x_1x_2(x_5^2+x_6^2)(-x_1^2-x_2^2-x_3^2-x_4^2-7x_5^2-7x_6^2)\\
&+x_3x_4(-8x_5^4+8x_6^4+8(x_6^2-x_5^2)(x_1^2+x_2^2+x_3^2+x_4^2))\\
&+x_5x_6(8x_3^4-8x_4^4+8(x_3^2-x_4^2)(x_1^2+x_2^2+x_5^2+x_6^2))),\\
\mathcal{J}^4_{13}(0) =&\tfrac{1}{16}(x_1x_3(8(x_1^2+x_2^2)^2+(x_3^2+x_4^2)(16x_1^2+16x_2^2+8x_3^2+8x_4^2)\\
&+x_5^2(15x_1^2+15x_2^2+15x_3^2+15x_4^2+x_5^2+9x_6^2)\\
&+x_6^2(31x_1^2+31x_2^2+31x_3^2+31x_4^2+9x_5^2+17x_6^2))\\
&+x_2x_4(8(x_1^2+x_2^2)^2+(x_3^2+x_4^2)(16x_1^2+16x_2^2+8x_3^2+8x_4^2)\\
&+x_5^2(33x_1^2+33x_2^2+33x_3^2+33x_4^2+31x_5^2+23x_6^2)\\
&+x_6^2(17x_1^2+17x_2^2+17x_3^2+17x_4^2+23x_5^2+15x_6^2))\\
&-16x_5x_6((x_1x_4+x_2x_3)(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2))),\\
\mathcal{J}^4_{14}(0) =&\tfrac{1}{16}(x_2x_3(-8(x_1^2+x_2^2)^2+(x_3^2+x_4^2)(-16x_1^2-16x_2^2-8x_3^2-8x_4^2)\\
&+x_5^2(-17x_1^2-17x_2^2-17x_3^2-17x_4^2-15x_5^2-23x_6^2)\\
&+x_6^2(-33x_1^2-33x_2^2-33x_3^2-33x_4^2-23x_5^2-31x_6^2))\\
&+x_1x_4(8(x_1^2+x_2^2)^2+(x_3^2+x_4^2)(16x_1^2+16x_2^2+8x_3^2+8x_4^2)\\
&+x_5^2(31x_1^2+31x_2^2+31x_3^2+31x_4^2+17x_5^2+9x_6^2)
\end{aligned}\tag{3.49}$$

$$
\begin{aligned}
&+ x_6^2(15x_1^2 + 15x_2^2 + 15x_3^2 + 15x_4^2 + 9x_5^2 + x_6^2))\\
&+ 16x_5x_6((-x_1x_3 + x_2x_4)(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2))),\\
\mathcal{J}^4_{15}(0) =& \tfrac{1}{32}(x_1x_5((x_1^2 + x_2^2 + x_3^2 + x_4^2)(15x_1^2 + 15x_2^2 + 15x_3^2 + 15x_4^2 + x_5^2 + x_6^2)\\
&+ (x_5^2 + x_6^2)(x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2))\\
&- x_2x_6((x_1^2 + x_2^2 + x_3^2 + x_4^2)(17x_1^2 + 17x_2^2 + 17x_3^2 + 17x_4^2 + 15x_5^2 + 15x_6^2)\\
&+ (x_5^2 + x_6^2)(15x_1^2 + 15x_2^2 + 15x_3^2 + 15x_4^2 + x_5^2 + x_6^2)),\\
\mathcal{J}^4_{16}(0) =& \tfrac{1}{32}(x_1x_6((x_1^2 + x_2^2 + x_3^2 + x_4^2)(15x_1^2 + 15x_2^2 + 15x_3^2 + 15x_4^2 + x_5^2 + x_6^2)\\
&+ (x_5^2 + x_6^2)(x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2))\\
&+ x_2x_5((x_1^2 + x_2^2 + x_3^2 + x_4^2)(17x_1^2 + 17x_2^2 + 17x_3^2 + 17x_4^2 + 15x_5^2 + 15x_6^2)\\
&+ (x_5^2 + x_6^2)(15x_1^2 + 15x_2^2 + 15x_3^2 + 15x_4^2 + x_5^2 + x_6^2)),\\
\mathcal{J}^4_{22}(0) =& \tfrac{1}{16}(x_1^2(-8x_1^2x_3^2 - 16x_3^4 - 8x_1^2x_4^2 - 16x_4^4 + 9x_1^2x_5^2 - 15x_3^2x_5^5 + x_4^2x_5^2 + 15x_5^4\\
&+ 9x_1^2x_6^2 + x_3^2x_6^2 - 15x_4^2x_6^2 + 30x_5^2x_6^2 + 15x_6^4) + x_2^2(-8x_2^2x_3^2 - 16x_3^4 - 8x_2^2x_4^2\\
&- 16x_4^4 + 7x_2^2x_5^2 - 17x_3^2x_5^2 - x_4^2x_5^2 + x_5^4 + 7x_2^2x_6^2 - x_3^2x_6^2 - 17x_4^2x_6^2 + 2x_5^2x_6^2\\
&+ x_6^4) + x_1^2x_2^2(-16x_3^2 - 16x_4^2 + 16x_5^2 + 16x_6^2) + x_3^2(-8x_3^4 - 24x_3^2x_5^2 - 16x_5^4\\
&- 8x_3^2x_6^2 - 16x_5^2x_6^2) + x_4^2(-8x_4^4 - 8x_4^2x_5^2 - 24x_4^2x_6^2 - 16x_5^2x_6^2 - 16x_6^4)\\
&+ x_3^2x_4^2(-32x_1^2 - 32x_2^2 - 24x_3^2 - 24x_4^2 - 32x_5^2 - 32x_6^2)\\
&- x_3x_4x_5x_6(32x_1^2 + 32x_2^2 + 32x_3^2 + 32x_4^2 + 32x_5^2 + 32x_6^2)),\\
\mathcal{J}^4_{23}(0) =& \tfrac{1}{16}(x_2x_3(8(x_1^2 + x_2^2)^2 + (x_3^2 + x_4^2)(16x_1^2 + 16x_2^2 + 8x_3^2 + 8x_4^2)\\
&+ x_5^2(31x_1^2 + 31x_2^2 + 31x_3^2 + 31x_4^2 + 17x_5^2 + 9x_6^2)\\
&+ x_6^2(15x_1^2 + 15x_2^2 + 15x_3^2 + 15x_4^2 + 9x_5^2 + x_6^2))\\
&+ x_1x_4(-8(x_1^2 + x_2^2)^2 + (x_3^2 + x_4^2)(-16x_1^2 - 16x_2^2 - 8x_3^2 - 8x_4^2)\\
&+ x_5^2(-17x_1^2 - 17x_2^2 - 17x_3^2 - 17x_4^2 - 15x_5^2 - 23x_6^2)\\
&+ x_6^2(-33x_1^2 - 33x_2^2 - 33x_3^2 - 33x_4^2 - 23x_5^2 - 31x_6^2))\\
&+ 16x_5x_6((-x_1x_3 + x_2x_4)(x_1^2 + x_2^2 + x_3^2 + 16x_4^2 + 16x_5^2 + 16x_6^2))),\\
\mathcal{J}^4_{24}(0) =& \tfrac{1}{16}(x_1x_3(8(x_1^2 + x_2^2)^2 + (x_3^2 + x_4^2)(16x_1^2 + 16x_2^2 + 8x_3^2 + 8x_4^2)\\
&+ x_5^2(33x_1^2 + 33x_2^2 + 33x_3^2 + 33x_4^2 + 31x_5^2 + 23x_6^2)\\
&+ x_6^2(17x_1^2 + 17x_2^2 + 17x_3^2 + 17x_4^2 + 23x_5^2 + 15x_6^2))\\
&+ x_2x_4(8(x_1^2 + x_2^2)^2 + (x_3^2 + x_4^2)(16x_1^2 + 16x_2^2 + 8x_3^2 + 8x_4^2)\\
&+ x_5^2(15x_1^2 + 15x_2^2 + 15x_3^2 + 15x_4^2 + x_5^2 + 9x_6^2)\\
&+ x_6^2(31x_1^2 + 31x_2^2 + 31x_3^2 + 31x_4^2 + 9x_5^2 + 17x_6^2))\\
&16x_5x_6((x_1x_4 + x_2x_3)(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2))),\\
\mathcal{J}^4_{25}(0) =& \tfrac{1}{32}(x_2x_5((x_1^2 + x_2^2 + x_3^2 + x_4^2)(15x_1^2 + 15x_2^2 + 15x_3^2 + 15x_4^2 + x_5^2 + x_6^2)\\
&+ (x_5^2 + x_6^2)(x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2))\\
&+ x_1x_6((x_1^2 + x_2^2 + x_3^2 + x_4^2)(17x_1^2 + 17x_2^2 + 17x_3^2 + 17x_4^2 + 15x_5^2 + 15x_6^2)\\
&+ (x_5^2 + x_6^2)(15x_1^2 + 15x_2^2 + 15x_3^2 + 15x_4^2 + x_5^2 + x_6^2)),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}^4_{26}(0) =&\tfrac{1}{32}(x_2x_6((x_1^2+x_2^2+x_3^2+x_4^2)(15x_1^2+15x_2^2+15x_3^2+15x_4^2+x_5^2+x_6^2)\\
&+(x_5^2+x_6^2)(x_1^2+x_2^2+x_3^2+x_4^2-x_5^2-x_6^2))\\
&-x_1x_5((x_1^2+x_2^2+x_3^2+x_4^2)(17x_1^2+17x_2^2+17x_3^2+17x_4^2+15x_5^2+15x_6^2)\\
&+(x_5^2+x_6^2)(15x_1^2+15x_2^2+15x_3^2+15x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^4_{33}(0) =&\tfrac{1}{16}(x_3^2(-16x_1^4-16x_2^4-8x_1^2x_3^2-8x_2^2x_3^2-x_1^2x_5^2-17x_2^2x_5^2+7x_3^2x_5^2+x_5^4-17x_1^2x_6^2\\
&-x_2^2x_6^2+7x_3^2x_6^2+2x_5^2x_6^2+x_6^4)+x_4^2(-16x_1^4-16x_2^4-8x_1^2x_4^2-8x_2^2x_4^2\\
&+x_1^2x_5^2-15x_2^2x_5^2+9x_4^2x_5^2+15x_5^4-15x_1^2x_6^2+x_2^2x_6^2+9x_4^2x_6^2+30x_5^2x_6^2+15x_6^4)\\
&+x_3^2x_4^2(-16x_1^2-16x_2^2+16x_5^2+16x_6^2)+x_1^2(-8x_1^4-8x_1^2x_5^2-24x_1^2x_6^2\\
&-16x_5^2x_6^2-16x_6^4)+x_2^2(-8x_2^4-24x_2^2x_5^2-16x_5^4-8x_2^2x_6^2-16x_5^2x_6^2)\\
&+x_1^2x_2^2(-24x_1^2-24x_2^2-32x_3^2-x_4^2-32x_5^2-32x_6^2)\\
&+x_1x_2x_5x_6(32x_1^2+32x_2^2+32x_3^2+32x_4^2+32x_5^2+32x_6^2)),\\
\mathcal{J}^4_{34}(0) =&\tfrac{1}{8}(x_3x_4(x_5^2+x_6^2)(-x_1^2-x_2^2-x_3^2-x_4^2-7x_5^2-7x_6^2)\\
&+x_1x_2(-8x_5^4+8x_6^4+8(x_6^2-x_5^2)(x_1^2+x_2^2+x_3^2+x_4^2))\\
&+x_5x_6(8x_1^4-8x_2^4+8(x_1^2-x_2^2)(x_3^2+x_4^2+x_5^2+x_6^2))),\\
\mathcal{J}^4_{35}(0) =&\tfrac{1}{32}(x_3x_5((x_1^2+x_2^2+x_3^2+x_4^2)(15x_1^2+15x_2^2+15x_3^2+15x_4^2+x_5^2+x_6^2)\\
&+(x_5^2+x_6^2)(x_1^2+x_2^2+x_3^2+x_4^2-x_5^2-x_6^2))\\
&-x_4x_6((x_1^2+x_2^2+x_3^2+x_4^2)(17x_1^2+17x_2^2+17x_3^2+17x_4^2+15x_5^2+15x_6^2)\\
&+(x_5^2+x_6^2)(15x_1^2+15x_2^2+15x_3^2+15x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^4_{36}(0) =&\tfrac{1}{32}(x_3x_6((x_1^2+x_2^2+x_3^2+x_4^2)(15x_1^2+15x_2^2+15x_3^2+15x_4^2+x_5^2+x_6^2)\\
&+(x_5^2+x_6^2)(x_1^2+x_2^2+x_3^2+x_4^2-x_5^2-x_6^2))\\
&+x_4x_5((x_1^2+x_2^2+x_3^2+x_4^2)(17x_1^2+17x_2^2+17x_3^2+17x_4^2+15x_5^2+15x_6^2)\\
&+(x_5^2+x_6^2)(15x_1^2+15x_2^2+15x_3^2+15x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^4_{44}(0) =&\tfrac{1}{16}(x_3^2(-16x_1^4-16x_2^4-8x_1^2x_3^2-8x_2^2x_3^2-15x_1^2x_5^2+x_2^2x_5^2+9x_3^2x_5^2+15x_5^4+x_1^2x_6^2\\
&-15x_2^2x_6^2+9x_3^2x_6^2+30x_5^2x_6^2+15x_6^4)+x_4^2(-16x_1^4-16x_2^4-8x_1^2x_4^2-x_2^2x_4^2\\
&-17x_1^2x_5^2-x_2^2x_5^2+7x_4^2x_5^2+x_5^4-x_1^2x_6^2-17x_2^2x_6^2+7x_4^2x_6^2+2x_5^2x_6^2+x_6^4)\\
&+x_3^2x_4^2(-16x_1^2-16x_2^2+16x_5^2+16x_6^2)+x_1^2(-8x_1^4-24x_1^2x_5^2-16x_5^4\\
&-8x_1^2x_6^2-16x_5^2x_6^2)+x_2^2(-8x_2^4-8x_2^2x_5^2-24x_6^4-16x_5^2x_6^2-16x_6^4)\\
&+x_1^2x_2^2(-24x_1^2-24x_2^2-32x_3^2-32x_4^2-32x_5^2-32x_6^2)\\
&+x_1x_2x_5x_6(32x_1^2+32x_2^2+32x_3^2+32x_4^2+32x_5^2+32x_6^2)),\\
\mathcal{J}^4_{45}(0) =&\tfrac{1}{32}(x_4x_5((x_1^2+x_2^2+x_3^2+x_4^2)(15x_1^2+15x_2^2+15x_3^2+15x_4^2+x_5^2+x_6^2)\\
&+(x_5^2+x_6^2)(x_1^2+x_2^2+x_3^2+x_4^2-x_5^2-x_6^2))\\
&+x_3x_6((x_1^2+x_2^2+x_3^2+x_4^2)(17x_1^2+17x_2^2+17x_3^2+17x_4^2+15x_5^2+15x_6^2)\\
&+(x_5^2+x_6^2)(15x_1^2+15x_2^2+15x_3^2+15x_4^2+x_5^2+x_6^2)),\\
\mathcal{J}^4_{46}(0) =&\tfrac{1}{32}(x_4x_6((x_1^2+x_2^2+x_3^2+x_4^2)(15x_1^2+15x_2^2+15x_3^2+15x_4^2+x_5^2+x_6^2)\\
&+(x_5^2+x_6^2)(x_1^2+x_2^2+x_3^2+x_4^2-x_5^2-x_6^2))\\
&-x_3x_5((x_1^2+x_2^2+x_3^2+x_4^2)(17x_1^2+17x_2^2+17x_3^2+17x_4^2+15x_5^2+15x_6^2)\\
&+(x_5^2+x_6^2)(15x_1^2+15x_2^2+15x_3^2+15x_4^2+x_5^2+x_6^2)),
\end{aligned}
$$

$$\begin{aligned}
\mathcal{J}^4_{55}(0) =&\tfrac{1}{16}(x_1^2+x_2^2+x_3^2+x_4^2)(8x_1^4+8x_2^4+8x_3^4+8x_4^4-x_5^4+x_6^4+16x_3^2x_4^2+x_3^2x_5^2\\
&+x_4^2x_5^2+15x_3^2x_6^2+15x_4^2x_6^2+x_2^2(16x_3^2+16x_4^2+x_5^2+15x_6^2)\\
&+x_1^2(16x_2^2+16x_3^2+16x_4^2+x_5^2+15x_6^2)),\\
\mathcal{J}^4_{56}(0) =&\tfrac{-1}{8}(x_1^2+x_2^2+x_3^2+x_4^2)x_5x_6(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2),\\
\mathcal{J}^4_{66}(0) =&\tfrac{1}{16}(x_1^2+x_2^2+x_3^2+x_4^2)(8x_1^4+8x_2^4+8x_3^4+8x_4^4+x_5^4-x_6^4+16x_3^2x_4^2+15x_3^2x_5^2\\
&+15x_4^2x_5^2+x_3^2x_6^2+x_4^2x_6^2+x_2^2(16x_3^2+16x_4^2+15x_5^2+x_6^2)\\
&+x_1^2(16x_2^2+16x_3^2+16x_4^2+15x_5^2+x_6^2)).
\end{aligned}$$

For $n=5$ we have $\mathcal{J}_0^{5)} = \left(\mathcal{J}^5_{ij}(0)\right)$, $i,j=1,...,6$, where

$$\begin{aligned}
\mathcal{J}^5_{11}(0) =&\tfrac{-1}{8}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_1(x_3x_5\\
&+x_4x_6)+3x_2(x_4x_5-x_3x_6))+\tfrac{5}{4}(x_1(x_3x_5+x_4x_6)(-7x_1^2-7x_2^2-7x_3^2-7x_4^2\\
&-x_5^2-x_6^2)+x_2(x_3x_6-x_4x_5)(9x_1^2+9x_2^2+9x_3^2+9x_4^2+15x_5^2+15x_6^2))),\\
\mathcal{J}^5_{12}(0) =&\tfrac{-1}{4}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_1(-x_4x_5\\
&+x_3x_6)+x_2(x_3x_5+x_4x_6))+5((x_1(x_4x_5-x_3x_6)-x_2(x_3x_5+x_4x_6))(x_1^2\\
&+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2))),\\
\mathcal{J}^5_{13}(0) =&\tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_5(-x_1^2\\
&-3x_2^2+x_3^2+3x_4^2)+x_6(2x_1x_2-2x_3x_4)))+\tfrac{5}{4}(x_5(7x_1^4+16x_1^2x_2^2+9x_2^4\\
&+2x_2^2x_3^2-7x_3^4-2x_1^2x_4^2-16x_3^2x_4^2-9x_4^4+x_1^2x_5^2+15x_2^2x_5^2-x_3^2x_5^2-15x_4^2x_5^2\\
&+x_1^2x_6^2+15x_2^2x_6^2-x_3^2x_6^2-15x_4^2x_6^2)+x_6(-2x_1^3x_2-2x_2^3x_1-2x_1x_2x_3^2\\
&+2x_1^2x_3x_4+2x_2^2x_3x_4+2x_3^3x_4-2x_1x_2x_4^2+2x_3x_4^3-14x_1x_2x_6^2+14x_3x_4x_6^2\\
&-14x_1x_2x_5^2+14x_3x_4x_5^2)))),\\
\mathcal{J}^5_{14}(0) =&\tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_6(-x_1^2\\
&+-3x_2^2+3x_3^2+x_4^2)+x_5(-2x_1x_2-2x_3x_4))\tfrac{5}{4}(x_6(7x_1^4+16x_1^2x_2^2+9x_2^4\\
&-2x_1^2x_3^2-9x_3^4+2x_2^2x_4^2-16x_3^2x_4^2-7x_4^4+x_1^2x_5^2+15x_2^2x_5^2-15x_3^2x_5^2-x_4^2x_5^2\\
&+x_1^2x_6^2+15x_2^2x_6^2-15x_3^2x_6^2-x_4^2x_6^2)+x_5(2x_1^3x_2+2x_1x_2^3+2x_1x_2x_3^2\\
&+2x_1^2x_3x_4+2x_2^2x_3x_4+2x_3^3x_4+2x_1x_2x_4^2+2x_3x_4^3+14x_1x_2x_5^2+14x_3x_4x_5^2\\
&+14x_1x_2x_6^2+14x_3x_4x_6^2))),\\
\mathcal{J}^5_{15}(0) =&\tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_3(2x_1^2\\
&+2x_2^2+2x_3^2+2x_4^2-x_5^2+x_6^2)-2x_4x_5x_6)+\tfrac{5}{4}(x_3(-8x_1^4-16x_1^2x_2^2-8x_2^4\\
&-16x_1^2x_3^2-16x_2^2x_3^2-8x_3^4-16x_1^2x_4^2-16x_2^2x_4^2-16x_3^2x_4^2-8x_4^4-x_1^2x_5^2\\
&-x_2^2x_5^2-x_3^2x_5^2-x_4^2x_5^2+x_5^4-15x_1^2x_6^2-15x_2^2x_6^2-15x_3^2x_6^2-15x_4^2x_6^2-x_6^4)\\
&+2x_4x_5x_6(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2))),
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
\mathcal{J}^5_{16}(0) =&\tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_4(2x_1^2\\
&+2x_2^2+2x_3^2+2x_4^2+x_5^2-x_6^2)-2x_3x_5x_6)+\tfrac{5}{16}(x_4(-8x_1^4-16x_1^2x_2^2-8x_2^4\\
&-16x_1^2x_3^2-16x_2^2x_3^2-8x_3^4-16x_1^2x_4^2-16x_2^2x_4^2-16x_3^2x_4^2-8x_4^4-15x_1^2x_5^2\\
&-15x_2^2x_5^2-15x_3^2x_5^2-15x_4^2x_5^2-x_5^4-x_1^2x_6^2-x_2^2x_6^2-x_3^2x_6^2-x_4^2x_6^2+x_6^4)\\
&+2x_3x_5x_6(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2))),\\
\mathcal{J}^5_{22}(0) =&\tfrac{-1}{8}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(-3x_1(x_3x_5\\
&+x_4x_6)+x_2(-x_4x_5+x_3x_6))+\tfrac{4}{8}(x_1(x_3x_5+x_4x_6)(9x_1^2+9x_2^2+9x_3^2+9x_4^2\\
&+15x_5^2+15x_6^2)+x_2(x_3x_6-x_4x_5)(-7x_1^2-7x_2^2-7x_3^2-7x_4^2-x_5^2-x_6^2))),\\
\mathcal{J}^5_{23}(0) =&\tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_6(-3x_1^2\\
&-x_2^2+x_3^2+3x_4^2)+x_5(2x_1x_2+2x_3x_4))+\tfrac{5}{4}(x_6(9x_1^4+16x_1^2x_2^2+7x_2^4\\
&+2x_1^2x_3^2-7x_3^4-2x_2^2x_4^2-16x_3^2x_4^2-9x_4^4+15x_1^2x_5^2+x_2^2x_5^2-x_3^2x_5^2\\
&-15x_4^2x_5^2+15x_1^2x_6^3+x_2^2x_6^2-x_3^2x_6^2-15x_4^2x_6^2)+x_5(-2x_1^3x_2-2x_1x_2^3\\
&-2x_1x_2x_3^2-2x_1^2x_3x_4-2x_2^2x_3x_4+2x_3^3x_4-2x_1x_2x_4^2-x_3x_4^3-14x_1x_2x_5^2\\
&-14x_3x_4x_5^2-14x_1x_2x_6^2-14x_3x_4x_6^2))),\\
\mathcal{J}^5_{24}(0) =&\tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_5(3x_1^2+\\
&x_2^2-3x_3^2-x_4^2)+x_6(2x_1x_2-2x_3x_4))+\tfrac{5}{4}(x_5(-9x_1^4-16x_1^2x_2^2-7x_2^4+2x_2^2x_3^2\\
&+9x_3^4-2x_1^2x_4^2+16x_3^2x_4^2+7x_4^4-15x_1^2x_5^2-x_2^2x_5^2+15x_3^2x_5^2+x_4^2x_5^2-15x_1^2x_6^2\\
&-x_2^2x_6^2+15x_3^2x_6^2+x_4^2x_6^2)+x_6(-2x_1^3x_2-2x_2^3x_1-2x_1x_2x_3^2+2x_1^2x_3x_4\\
&+2x_2^2x_3x_4+2x_3^3x_4-2x_1x_2x_4^2+2x_3x_4^3-14x_1x_2x_5^2+14x_3x_4x_5^2-14x_1x_2x_6^2\\
&+14x_3x_4x_6^2))),\\
\mathcal{J}^5_{25}(0) =&\tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_4(-2x_1^2\\
&-2x_2^2-2x_3^2-2x_4^2+x_5^2-x_6^2)-2x_3x_5x_6)+\tfrac{5}{4}(x_4(8x_1^4+16x_1^2x_2^2+8x_2^4\\
&+16x_1^2x_3^2+16x_2^2x_3^2+8x_3^4+16x_1^2x_4^2+16x_2^2x_4^2+16x_3^2x_4^2+8x_4^4+x_1^2x_5^2\\
&+x_2^2x_5^2+x_3^2x_5^2+x_4^2x_5^2-x_5^4+15x_1^2x_6^2+15x_2^2x_6^2+15x_3^2x_6^2+15x_4^2x_6^2+x_6^4)\\
&+2x_3x_5x_6(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2))),\\
\mathcal{J}^5_{26}(0) =&\tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_3(2x_1^2\\
&+2x_2^2+2x_3^2+2x_4^2+x_5^2-x_6^2)+2x_4x_5x_6)+\tfrac{5}{4}(x_3(-8x_1^4-16x_1^2x_2^2-8x_2^4\\
&-16x_1^2x_3^2-16x_2^2x_3^2-8x_3^4-16x_1^2x_4^2-16x_2^2x_4^2-16x_3^2x_4^2-8x_4^4-15x_1^2x_5^2\\
&-15x_2^2x_5^2-15x_3^2x_5^2-15x_4^2x_5^2-x_5^4-x_1^2x_6^2-x_2^2x_6^2-x_3^2x_6^2-x_4^2x_6^2+x_6^4)\\
&-2x_4x_5x_6(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2))),\\
\mathcal{J}^5_{33}(0) =&\tfrac{-1}{8}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_1(-x_3x_5\\
&+3x_4x_6)+x_2(-3x_4x_5-x_3x_6))+\tfrac{5}{4}(x_3(x_1x_5+x_2x_6)(7x_1^2+7x_2^2+7x_3^2+7x_4^2\\
&+x_5^2+x_6^2)+x_4(x_1x_6-x_2x_5)(-9x_1^2-9x_2^2-9x_3^2-9x_4^2-15x_5^2-15x_6^2))),\\
\mathcal{J}^5_{34}(0) =&(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)^2(-x_1(x_4x_5+x_3x_6)+x_2(x_3x_5-x_4x_6)),
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}^5_{35}(0) =& \tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(-x_1(2x_1^2 \\
&+2x_2^2+2x_3^2+2x_4^2-x_5^2+x_6^2)+2x_2x_5x_6)+\tfrac{5}{4}(x_1(8x_1^4+16x_1^2x_2^2+8x_2^4 \\
&+16x_1^2x_3^2+16x_2^2x_3^2+8x_3^4+16x_1^2x_4^2+16x_2^2x_4^2+16x_3^2x_4^2+8x_4^4+x_1^2x_5^2 \\
&+x_2^2x_5^2+x_3^2x_5^2+x_4^2x_5^2-x_5^4+15x_1^2x_6^2+15x_2^2x_6^2+15x_3^2x_6^2+15x_4^2x_6^2+x_6^4) \\
&-2x_2x_5x_6(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2))), \\
\mathcal{J}^5_{36}(0) =& \tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(-x_2(2x_1^2 \\
&+2x_2^2+2x_3^2+2x_4^2+x_5^2-x_6^2)+2x_1x_5x_6)+\tfrac{5}{4}(x_2(8x_1^4+16x_1^2x_2^2+8x_2^4 \\
&+16x_1^2x_3^2+16x_2^2x_3^2+8x_3^4+16x_1^2x_4^2+16x_2^2x_4^2+16x_3^2x_4^2+8x_4^4+15x_1^2x_5^2 \\
&+15x_2^2x_5^2+15x_3^2x_5^2+15x_4^2x_5^2+x_5^4+x_1^2x_6^2+x_2^2x_6^2+x_3^2x_6^2+x_4^2x_6^2-x_6^4) \\
&-2x_1x_5x_6(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2))), \\
\mathcal{J}^5_{44}(0) =& \tfrac{-1}{8}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_1(3x_3x_5 \\
&-x_4x_6)+x_2(x_4x_5+3x_3x_6))+\tfrac{5}{4}(x_3(x_1x_5+x_2x_6)(-9x_1^2-9x_2^2-9x_3^2-9x_4^2 \\
&-15x_5^2-15x_6^2)+x_4(x_1x_6-x_2x_5)(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2))), \\
\mathcal{J}^5_{45}(0) =& \tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(x_2(2x_1^2 \\
&+2x_2^2+2x_3^2+2x_4^2-x_5^2+x_6^2)+2x_1x_5x_6)+\tfrac{5}{4}(x_2(-8x_1^4-16x_1^2x_2^2-8x_2^4 \\
&-16x_1^2x_3^2-16x_2^2x_3^2-8x_3^4-16x_1^2x_4^2-16x_2^2x_4^2-16x_3^2x_4^2-8x_4^4-x_1^2x_5^2 \\
&-x_2^2x_5^2-x_3^2x_5^2-x_4^2x_5^2+x_5^4-15x_1^2x_6^2-15x_2^2x_6^2-15x_3^2x_6^2-15x_4^2x_6^2-x_6^4) \\
&-2x_1x_5x_6(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2))), \\
\mathcal{J}^5_{46}(0) =& \tfrac{-1}{16}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)((x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)(-x_1(2x_1^2 \\
&+2x_2^2+2x_3^2+2x_4^2+x_5^2-x_6^2)-2x_2x_5x_6)+\tfrac{5}{4}(x_1(8x_1^4+16x_1^2x_2^2+8x_2^4 \\
&+16x_1^2x_3^2+16x_2^2x_3^2+8x_3^4+16x_1^2x_4^2+16x_2^2x_4^2+16x_3^2x_4^2+8x_4^4+15x_1^2x_5^2 \\
&+15x_2^2x_5^2+15x_3^2x_5^2+15x_4^2x_5^2+x_5^4+x_1^2x_6^2+x_2^2x_6^2+x_3^2x_6^2+x_4^2x_6^2-x_6^4) \\
&+2x_2x_5x_6(7x_1^2+7x_2^2+7x_3^2+7x_4^2+x_5^2+x_6^2))), \\
&\mathcal{J}^5_{55}(0) = \mathcal{J}^5_{56}(0) = \mathcal{J}^5_{66}(0) = 0.
\end{aligned}$$

3.3.4 Appendix B

In this appendix, we shall prove the Lemma 3.3.14.

Let γ be a geodesic of Kaplan's example N such that $\dot\gamma_0 = \sum_{i=1}^4 x_i E_i$ (or, equivalently, that $x_5 = x_6 = 0$). Moreover, following the notation of Lemma 3.1.1, we denote by $\{P_i\}_{i=1}^6$ the orthonormal frame field obtained by $\bar\nabla_{\dot\gamma}$-parallel translation of the basis $\{E_i\}$ along γ.

Thus, from [B-Tr-V, p. 45] we know that

$$T_\gamma P_i = \tfrac{1}{2}[P_i, \dot\gamma_0],\ i = 1,2,3,4, \quad T_\gamma P_\alpha = \tfrac{1}{2} j(P_i)(\dot\gamma_0),\ \alpha = 5,6.$$

More specifically, under our assumptions and using (3.23) and (3.24) we have

$$\begin{aligned} T_\gamma P_1 = \tfrac{1}{2}(x_3P_5 + x_4P_6), \quad T_\gamma P_2 = \tfrac{1}{2}(x_3P_6 - x_4P_5),\\ T_\gamma P_3 = \tfrac{-1}{2}(x_1P_5 + x_2P_6), \quad T_\gamma P_4 = \tfrac{1}{2}(x_2P_5 - x_1P_6),\\ T_\gamma P_5 = \tfrac{1}{2}(x_1P_3 - x_2P_4 - x_3P_1 + x_4P_2),\\ T_\gamma P_6 = \tfrac{1}{2}(x_1P_4 + x_2P_3 - x_3P_2 - x_4P_1). \end{aligned} \tag{3.51}$$

Now, from Lemma 3.3.13 and using (3.51) we obtain for each $i \in \{1,2,3,4,5,6\}$, the following homogeneous system of ordinary differential equations, $a_i' = M \cdot a_i$, with initial condition $a_{ii}(0) = 1$, $a_{ij}(0) = 0$, $j \neq i$:

$$\begin{pmatrix} a'_{i1} \\ a'_{i2} \\ a'_{i3} \\ a'_{i4} \\ a'_{i5} \\ a'_{i6} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2}x_3 & -\frac{1}{2}x_4 \\ 0 & 0 & 0 & 0 & \frac{1}{2}x_4 & -\frac{1}{2}x_3 \\ 0 & 0 & 0 & 0 & \frac{1}{2}x_1 & \frac{1}{2}x_2 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}x_2 & \frac{1}{2}x_1 \\ \frac{1}{2}x_3 & -\frac{1}{2}x_4 & -\frac{1}{2}x_1 & \frac{1}{2}x_2 & 0 & 0 \\ \frac{1}{2}x_4 & \frac{1}{2}x_3 & -\frac{1}{2}x_2 & -\frac{1}{2}x_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \\ a_{i6} \end{pmatrix}. \tag{3.52}$$

Finally, we shall solve (3.52) following the general theory of ordinary differential equations. The eigenvalues associated to matrix M are $\{0, -\frac{1}{2}\imath, \frac{1}{2}\imath\}$, all of them with multiplicity 2. A basis $\{v_1, v_2\}$ of $\ker M^2$ and a basis $\{w_1, w_2\}$ of $\ker(M - \frac{\imath}{2}Id)^2$ are given by

$$\begin{aligned} v_1 = (\tfrac{x_1x_3+x_2x_4}{x_3^2+x_4^2}, \tfrac{x_2x_3-x_1x_4}{x_3^2+x_4^2}, 1, 0, 0, 0), \quad v_2 = (\tfrac{x_1x_4-x_2x_3}{x_3^2+x_4^2}, \tfrac{x_1x_3+x_2x_4}{x_3^2+x_4^2}, 0, 1, 0, 0),\\ w_1 = (\tfrac{x_3}{x_2}, \tfrac{-x_4}{x_2}, \tfrac{-x_1}{x_2}, 1, \tfrac{-\imath}{x_2}, 0), \quad w_2 = (\tfrac{\imath(x_1x_3+x_2x_4)}{x_2}, \tfrac{\imath(x_2x_3-x_1x_4)}{x_2}, \tfrac{-\imath(x_1^2+x_2^2)}{x_2}, 0, \tfrac{x_1}{x_2}, 1). \end{aligned}$$

Moreover,

$$\begin{aligned} e^{tM}v_1 = v_1 + tM \cdot v_1 = v_1, \quad e^{tM}v_2 = v_2 + tM \cdot v_2 = v_2,\\ e^{tM}w_1 = e^{\frac{\imath t}{2}}(w_1 + t(M - \tfrac{\imath}{2}Id) \cdot w_1) = e^{\frac{\imath t}{2}}w_1,\\ e^{tM}w_2 = e^{\frac{\imath t}{2}}(w_2 + t(M - \tfrac{\imath}{2}Id) \cdot w_2) = e^{\frac{\imath t}{2}}w_2. \end{aligned}$$

Thus, the general solution of the system (3.52) is

$$a_i = C_{i1}e^{tM}v_1 + C_{i2}e^{tM}v_2 + C_{i3}\mathrm{Re}(e^{tM}w_1) + C_{i4}\mathrm{Im}(e^{tM}w_1) + C_{i5}\mathrm{Re}(e^{tM}w_2) + C_{i6}\mathrm{Im}(e^{tM}w_2)$$

or, more explicitly, it is

$$
\begin{aligned}
a_{i1}(t) =& \tfrac{x_1x_3+x_2x_4}{x_3^2+x_4^2}C_{i1} + \tfrac{x_1x_4-x_2x_3}{x_3^2+x_4^2}C_{i2} + \tfrac{x_3}{x_2}(\cos(t/2)C_{i3} + \sin(t/2)C_{i4}) \\
&+ \tfrac{x_1x_3+x_2x_4}{x_2}(\cos(t/2)C_{i6} - \sin(t/2)C_{i5}), \\
a_{i2}(t) =& \tfrac{x_2x_3-x_1x_4}{x_3^2+x_4^2}C_{i1} + \tfrac{x_1x_3+x_2x_4}{x_3^2+x_4^2}C_{i2} - \tfrac{x_4}{x_2}(\cos(t/2)C_{i3} + \sin(t/2)C_{i4}) \\
&+ \tfrac{x_2x_3-x_1x_4}{x_2}(\cos(t/2)C_{i6} - \sin(t/2)C_{i5}), \\
a_{i3}(t) =& C_{i1} - \tfrac{x_1}{x_2}(\cos(t/2)C_{i3} + \sin(t/2)C_{i4}) \\
&+ \tfrac{x_1^2-x_2^2}{x_2}(\sin(t/2)C_{i5} - \cos(t/2)C_{i6}), \\
a_{i4}(t) =& C_{i2} + \cos(t/2)C_{i3} + \sin(t/2)C_{i4}, \\
a_{i5}(t) =& \tfrac{1}{x_2}(\sin(t/2)C_{i3} - \cos(t/2)C_{i4}) + \tfrac{x_1}{x_2}(\cos(t/2)C_{i5} + \sin(t/2)C_{i6}), \\
a_{i6}(t) =& \cos(t/2)C_{i5} + \sin(t/2)C_{i6}.
\end{aligned}
\tag{3.53}
$$

where $C_{i1}, C_{i2}, C_{i3}, C_{i4}, C_{i5}, C_{i6}$ are arbitrary parameters.

Now, it only remains to know the value of the constants C_{ik}, $k = 1, ..., 6$, for each $i \in \{1, .., 6\}$. For that, we solve the system $a_{ii}(0) = 1$, $a_{ij}(0) = 0$, $j \neq 1$, for each $i \in \{1, .., 6\}$ and, easily, we obtain

$$
\begin{gathered}
C_{11} = x_1x_3 + x_2x_4, \quad C_{12} = x_1x_4 - x_2x_3, \quad C_{13} = x_2x_3 - x_1x_4, \\
C_{14} = C_{15} = 0, \quad C_{16} = x_4, \\
C_{21} = x_2x_3 - x_1x_4, \quad C_{22} = x_1x_3 + x_2x_4, \quad C_{23} = -x_1x_3 - x_2x_4, \\
C_{24} = C_{25} = 0, \quad C_{26} = x_3, \\
C_{31} = x_3^2 + x_4^2, \quad C_{32} = C_{33} = C_{34} = C_{35} = 0, \quad C_{36} = -x_2, \\
C_{41} = C_{44} = C_{45} = 0, \quad C_{42} = x_3^2 + x_4^2, \quad C_{43} = x_1^2 + x_2^2, \quad C_{46} = -x_1, \\
C_{51} = C_{52} = C_{53} = C_{55} = C_{56} = 0, \quad C_{54} = -x_2, \\
C_{61} = C_{62} = C_{63} = C_{66} = 0, \quad C_{64} = x_1, \quad C_{65} = 1.
\end{gathered}
\tag{3.54}
$$

Therefore, we obtain a_{ij}, $j = 1, ..., 6$, of Lemma 3.3.14 if we substitute the value of C_{ik}, $k = 1, ..., 6$, given in (3.54) in (3.53) for each $i \in \{1, ..., 6\}$.

This concludes the proof.

Part II

Homogeneous affine connections

Chapter 4

Locally homogeneous affine connections on 2-dimensional manifolds

Homogeneity is one of the fundamental notions in geometry although its meaning must be always specified for the concrete situations. In this chapter we consider the homogeneity of manifolds equipped with affine connections. This homogeneity means that, for every two points of a manifold, there is an affine diffeomorphism which sends one point into another. We shall treat a local version of the homogeneity, that is, we admit that the affine diffeomorphisms are given only locally, i. e., from a neighborhood onto a neighborhood.

Locally homogeneous Riemannian structures were first studied by I. M. Singer in [Si]. Many years later, B. Opozda worked out an affine version of Singer's theory in [Op.1] and [Op.2].

Two-dimensional locally homogeneous Riemannian manifolds are those with constant curvature. In contrast to this situation there are many locally homogeneous affine structures on 2-dimensional manifolds. In this chapter we classify (locally) all locally homogeneous affine connections with arbitrary torsion on two-dimensional manifolds. Therefore, we generalize the result given by B. Opozda for torsion-less case in [Op.3].

The results of this chapter will be published in [AM-K.2].

4.1 Introduction and main results about the classification problem

The field of affine differential geometry is well-established and still in quick development (see e.g. [N-S]). Also, many basic facts about affine transformation groups and affine Killing vector fields are known from the literature (see [Ko-N, vol.I] and [Ko]). Yet, it is remarkable that a seemingly easy problem to classify all locally homogeneous *torsion-less* connections in the plane domains was solved only recently in [Op.3] (direct method) and in [K-Op-Vl.4] (group-theoretical method). Unfortunately, no relation between both classifications was given. See also the previous partial results in [K-Op-Vl.1] and [K-Op-Vl.2]. For dimension three, to make a classification seems to be a hard problem.

The original result by B. Opozda [Op.3] was the following:

Theorem 4.1.1. *Let ∇ be a torsion-less locally homogeneous affine connection on a 2-dimensional manifold $\mathcal{M}$. Then, either ∇ is a Levi-Civita connection of constant curvature or, in a neighborhood $\mathcal{U}$ of each point $m \in \mathcal{M}$, there is a system (u, v) of local coordinates and constants p, q, c, d, e, f such that ∇ is expressed in $\mathcal{U}$ by one of the following formulas:*

Type A

$$\nabla_{\partial_u}\partial_u = p\partial_u + q\partial_v, \quad \nabla_{\partial_u}\partial_v = c\partial_u + d\partial_v, \quad \nabla_{\partial_v}\partial_v = e\partial_u + f\partial_v.$$

Type B

$$\nabla_{\partial_u}\partial_u = \tfrac{p\partial_u+q\partial_v}{u}, \quad \nabla_{\partial_u}\partial_v = \tfrac{c\partial_u+d\partial_v}{u}, \quad \nabla_{\partial_v}\partial_v = \tfrac{e\partial_u+f\partial_v}{u}.$$

(Here the "Levi-Civita connection" involves also the Lorentzian case. For an application of this result see [K-Op-Vl.3]).

In the next section we are going to classify all locally homogeneous affine connections with arbitrary torsion in the plane domains from the group-theoretical point of view. This means that we always start with a specific *transitive* Lie algebra $\mathfrak{g}$ of vector fields from the list of P. J. Olver [O] (see Section 4.4) and we are looking for all affine connections with arbitrary torsion for which, in the same domain and with respect to the same local coordinates, $\mathfrak{g}$ is the *full* algebra of affine Killing vector fields. Such connections are called *corresponding* to $\mathfrak{g}$. It happens quite often that the given Lie algebra of vector fields does not admit any invariant affine connection or it only admits torsion-less invariant affine connections.

Finally, we prove some simple algebraic lemmas which enable to decide very easily if a connection corresponding to a Lie algebra $\mathfrak{g}$ has, in some local coordinate system (u', v'), Christoffel symbols of type A, or of type B, respectively. In such a case we say shortly that such a connection is of type A, or of type B, respectively. Due to our lemmas, the whole procedure depends only on the structure of the algebra $\mathfrak{g}$.

We try to organize our computation in (possibly) most systematic way so that the whole procedure is not excessively long. Also, because this topic is an ideal subject for a computer-aided research, we are using the software MATHEMATICA 5.2, throughout this work. But we put stress on the full transparency of this procedure.

Now, we shall formulate the stronger Classification Theorem for connections with torsion. Based on our computations, we illustrate here the essential relationship between the classifications given in [K-Op-Vl.4] and [Op.3]. Moreover we prove that, for some Lie algebras $\mathfrak{g}$, all connections corresponding to such a $\mathfrak{g}$ are simultaneously of type A and of type B. These facts can be easily checked in the table that we use to summarize our results.

Theorem 4.1.2 (Classification Theorem)**.** *Let ∇ be a locally homogeneous affine connection with arbitrary torsion on a 2-dimensional manifold $\mathcal{M}$. Then, either ∇ is locally a Levi-Civita connection of the unit sphere or, in a neighborhood $\mathcal{U}$ of each point $m \in \mathcal{M}$, there is a system (u, v) of local coordinates and constants p, q, c, d, e, f, r, s such that ∇ is expressed in $\mathcal{U}$ by one of the following formulas:*

Type A

$$\nabla_{\partial_u}\partial_u = p\partial_u + q\partial_v, \quad \nabla_{\partial_u}\partial_v = c\partial_u + d\partial_v,$$

$$\nabla_{\partial_v}\partial_u = r\partial_u + s\partial_v, \quad \nabla_{\partial_v}\partial_v = e\partial_u + f\partial_v.$$

Type B

$$\nabla_{\partial_u}\partial_u = \tfrac{p\partial_u + q\partial_v}{u}, \quad \nabla_{\partial_u}\partial_v = \tfrac{c\partial_u + d\partial_v}{u},$$

$$\nabla_{\partial_v}\partial_u = \tfrac{r\partial_u + s\partial_v}{u}, \quad \nabla_{\partial_v}\partial_v = \tfrac{e\partial_u + f\partial_v}{u},$$

where not all p, q, c, d, e, f, r, s are zero.

Proof. Let us start with the presentation of the following table. This is a refinement of the tables 1 and 6 from [O] (see Section 4.4) completed by additional information. In each case, or subcase, we get a Lie algebra of vector fields given by its generators. We are looking for all locally homogeneous connections which (in the same local coordinates) are *corresponding* to a Lie algebra in question. Moreover, T denotes the torsion tensor, "VCS" means that all Christoffel symbols vanish with respect to the *given* coordinates (u, v) and the label "flat" means that the Ricci tensor vanishes. In the column $T \neq 0$, if we write "in general", we mean that the torsion tensor is different from zero except some special cases. Some additional properties of the Ricci tensor will be studied separately, later.

Properties of connections associated with the (refined) Olver list.

Case	***Generators***	***Remarks***	***Type A***	***Type B***	**$T \neq 0$**
1.1	∂_v, $v\partial_v - u\partial_u$, $v^2\partial_v - 2uv\partial_u$.		No	Yes	In general
1.2	∂_v, $v\partial_v - u\partial_u$, $v^2\partial_v - (2uv+1)\partial_u$.	∇ is the Levi-Civita connection of a Lorentzian metric with constant curvature.	No	Yes	Never

Case	***Generators***	***Remarks***	***Type A***	***Type B***	$T\neq 0$
1.3	∂_v, $v\partial_v$, $u\partial_u$, $v^2\partial_v - uv\partial_u$.		Yes (Flat)	Yes	Never
1.4	∂_v, $v\partial_v$, $v^2\partial_v$, ∂_u, $u\partial_u$, $u^2\partial_u$.	No corresponding invariant affine connection.			
1.5	∂_v, $\eta_1(v)\partial_u, ..., \eta_k(v)\partial_u$, $k \in \mathbb{Z}$, $k \geq 1$.	The functions $\eta_1(v)$,..., $\eta_k(v)$ satisfy a k^{th} order constant coefficient homogeneous linear ordinary differential equation $\mathcal{D}[u] = 0$.			
1.5 a)	∂_u, ∂_v.	$k = 1$	Yes	No	In general
1.5 b)	∂_v, $\mathrm{e}^v\partial_u$.	$k = 1$	No	Yes	In general
1.5 c)	∂_u, ∂_v, $\mathrm{e}^v\partial_u$.	$k = 2$	Yes	Yes	In general
1.5 d)	∂_u, ∂_v, $v\partial_u$.	$k = 2$	Yes	No	In general
1.5 e)	∂_v, $\mathrm{e}^{\alpha v}\partial_u$, $\mathrm{e}^{\beta v}\partial_u$, $\alpha \neq \beta$, $\alpha, \beta \neq 0$.	This case becomes equivalent to the case 1.6 e')			
1.5 f)	∂_v, $\mathrm{e}^{\alpha v}\partial_u$, $v\mathrm{e}^{\alpha v}\partial_u$, $\alpha \neq 0$.	This case becomes equivalent to the case 1.6 f')			
1.5 g)	$\mathrm{e}^{\alpha v}\cos(\beta v)\partial_u$, $\mathrm{e}^{\alpha v}\sin(\beta v)\partial_u$, ∂_v, $\beta \neq 0$.	This case becomes equivalent to the case 1.6 g')			
1.5 h)	$\eta_1(v)\partial_u, ...$, $\eta_k(v)\partial_u$, ∂_v, $k > 2$.	No corresponding invariant affine connection.			
1.6	∂_v, $u\partial_u$, $\eta_1(v)\partial_u, ..., \eta_k(v)\partial_u$, $k \in \mathbb{Z}$, $k \geq 1$.	The functions $\eta_1(v)$,..., $\eta_k(v)$ satisfy a k^{th} order constant coefficient homogeneous linear ordinary differential equation $\mathcal{D}[u] = 0$.			
1.6 a')	∂_u, ∂_v $u\partial_u$.	$k = 1$	Yes	No	In general
1.6 b')	∂_v, $\mathrm{e}^v\partial_u$ $u\partial_u$.	$k = 1$	Yes	Yes	In general

Case	*Generators*	*Remarks*	*Type A*	*Type B*	$T \neq 0$
1.6 c')	∂_u, ∂_v, $\mathrm{e}^v\partial_u$, $u\partial_u$.	$k=2$	Yes	Yes	In general
1.6 d')	∂_u, ∂_v, $v\partial_u$, $u\partial_u$.	$k=2$	Yes	No	In general
1.6 e')	∂_v, $\mathrm{e}^{\alpha v}\partial_u$, $\mathrm{e}^{\beta v}\partial_u$, $u\partial_u$, $\alpha \neq \beta$, $\alpha, \beta \neq 0$.	$k=2$	Yes	Yes	In general
1.6 f')	∂_v, $\mathrm{e}^{\alpha v}\partial_u$, $v\mathrm{e}^{\alpha v}\partial_u$, $u\partial_u$, $\alpha \neq 0$.	$k=2$	Yes	Yes	In general
1.6 g')	$\mathrm{e}^{\alpha v}\cos(\beta v)\partial_u$, $\mathrm{e}^{\alpha v}\sin(\beta v)\partial_u$, ∂_v, $u\partial_u$, $\beta \neq 0$.	$k=2$	Yes	No	In general
1.6 h')	$\eta_1(v)\partial_u$, ..., $\eta_k(v)\partial_u$, ∂_v, $u\partial_u$, $k>2$.	No corresponding invariant affine connection.			
1.7	∂_u, ∂_v, $v\partial_v + \alpha u\partial_u$, $v\partial_u$, ..., $v^{k-1}\partial_u$, $k \in \mathbb{Z}$, $k \geq 1$.	$k=1$, $\alpha=0$	Yes	Yes	In general
		$k=1$, $\alpha = 1/2, 2$ or $k=2$, $\alpha=2$	Yes (Flat)	Yes	Never
		$k=1$, $\alpha \neq 0, 1/2, 2$ or $k=2$, $\alpha \neq 2$	Yes (VCS)	Yes	Never
		$k>2$	No corresponding invariant affine connection.		
1.8	∂_u, ∂_v, $v\partial_v + (ku+v^k)\partial_u$, $v\partial_u$, ..., $v^{k-1}\partial_u$, $k \in \mathbb{Z}$, $k \geq 1$.	$k=1$	Yes (VCS)	Yes	Never
		$k \geq 2$	No corresponding invariant affine connection.		
1.9	∂_u, ∂_v, $v\partial_v$, $u\partial_u$, $v\partial_u$, ..., $v^{k-1}\partial_u$, $k \in \mathbb{Z}$, $k \geq 1$.	$k=1,2$	Yes (VCS)	Yes	Never
		$k>2$	No corresponding invariant affine connection.		

Case	***Generators***	***Remarks***	***Type A***	***Type B***	***T≠ 0***
1.10	∂_v, ∂_u, $v\partial_u$, ..., $v^{k-1}\partial_u$, $2v\partial_v + (k-1)u\partial_u$, $v^2\partial_v + (k-1)uv\partial_u$, $k \in \mathbb{Z}$, $k \geq 1$.	No corresponding invariant affine connection.			
1.11	∂_v, ∂_u, $v\partial_u$, ..., $v^{k-1}\partial_u$, $v\partial_v$, $u\partial_u$, $v^2\partial_v + (k-1)uv\partial_u$, $k \in \mathbb{Z}$, $k \geq 1$.	No corresponding invariant affine connection.			
2.1	∂_v, ∂_u, $\alpha(v\partial_v + u\partial_u)$ $+u\partial_v - v\partial_u$.		Yes (VCS)	Yes	Never
2.2	∂_v, $v\partial_v + u\partial_u$, $(v^2 - u^2)\partial_v + 2uv\partial_u$.	∇ is the Levi-Civita Connection of the hyperbolic plane.	No	Yes	Never
2.3	$u\partial_v - v\partial_u$, $(1 + v^2 - u^2)\partial_v + 2uv\partial_u$, $2uv\partial_v + (1 - v^2 + u^2)\partial_u$.	∇ is the Levi-Civita Connection of the sphere.	No	No	Never
2.4	∂_v, ∂_u, $v\partial_v + u\partial_u$, $u\partial_v - v\partial_u$.		Yes (VCS)	Yes	Never
2.5	∂_v, ∂_u, $v\partial_v - u\partial_u$, $u\partial_v$, $v\partial_u$.		Yes (VCS)	Yes	Never
2.6	∂_v, ∂_u, $v\partial_v$, $u\partial_u$, $u\partial_v$, $v\partial_u$.		Yes (VCS)	Yes	Never
2.7	∂_v, ∂_u, $v\partial_v + u\partial_u$, $u\partial_v - v\partial_u$, $(v^2 - u^2)\partial_v + 2uv\partial_u$, $2uv\partial_v + (u^2 - v^2)\partial_u$.	No corresponding invariant affine connection.			
2.8	∂_v, ∂_u, $v\partial_v$, $u\partial_u$, $u\partial_v$, $v\partial_u$, $v^2\partial_v + uv\partial_u$, $uv\partial_v + u^2\partial_u$.	No corresponding invariant affine connection.			

It is sufficient to check that this table is correct and this will prove our Theorem. Our check will be done step by step.

Remark 4.1.3. *As concerns the remark in Case 1.2, recall that, if a pseudo-Riemannian metric g has the constant positive curvature 1, then the reversed metric $-g$ has the constant negative curvature -1. See, for instance [O'N, p. 92].*

4.2 Checking the table step by step

4.2.1 Preliminaries

Let ∇ be an affine connection on a manifold M. It is locally homogeneous, if for each two points $x, y \in M$ there exists a neighborhood $\mathcal{U}$ of x, a neighborhood $\mathcal{V}$ of y and an affine transformation $\varphi : \mathcal{U} \to \mathcal{V}$ such that $\varphi(x) = y$. It means that φ is a (local) diffeomorphism such that

$$\nabla_{\varphi_* X} \varphi_* Y = \varphi_*(\nabla_X Y)$$

holds for every vector fields X, Y defined in $\mathcal{U}$.

Now, let us recall the following criterion describing affine Killing vector fields. An affine Killing vector field X is characterized by the equation:

$$[X, \nabla_Y Z] - \nabla_Y [X, Z] - \nabla_{[X,Y]} Z = 0 \tag{4.1}$$

which has to be satisfied for arbitrary vector fields Y, Z (see Proposition 2.2 in Chapter VI of [Ko-N]). The following assertion is standard:

Proposition 4.2.1. *A smooth connection ∇ on $\mathcal{M}$ is locally homogeneous if and only if it admits, in a neighborhood of each point $p \in \mathcal{M}$, at least two linearly independent affine Killing vector fields.*

From now on, we assume that M is 2-dimensional. We choose a fixed coordinate domain $\mathcal{U}(u, v) \subset \mathcal{M}$ and we express a vector field X in the form $X = a(u, v)\partial_u + b(u, v)\partial_v$. Then, for a connection ∇ with arbitrary torsion in $\mathcal{U}(u, v)$, we put

$$\begin{aligned} \nabla_{\partial_u} \partial_u &= A(u,v)\partial_u + B(u,v)\partial_v, & \nabla_{\partial_u} \partial_v &= C(u,v)\partial_u + D(u,v)\partial_v, \\ \nabla_{\partial_v} \partial_u &= E(u,v)\partial_u + F(u,v)\partial_v, & \nabla_{\partial_v} \partial_v &= G(u,v)\partial_u + H(u,v)\partial_v. \end{aligned} \tag{4.2}$$

In the following, we will often denote the functions $a(u, v)$, $b(u, v)$, $A(u, v)$, $B(u, v)$, $C(u, v)$, $D(u, v)$, $E(u, v)$, $F(u, v)$, $G(u, v)$, $H(u, v)$ by a, b, A, B, C, D, E, F, G, H respectively, if there is no risk of confusion.

Writing the formula (4.1) in local coordinates, we find that any affine Killing vector field X must satisfy eight basic equations. We shall write these equations in the simplified notation:

$$\begin{aligned}
&1) \quad a_{uu} + Aa_u - Ba_v + (C + E)b_u + A_u a + A_v b = 0, \\
&2) \quad b_{uu} + 2Ba_u + (F + D - A)b_u - Bb_v + B_u a + B_v b = 0, \\
&3) \quad a_{uv} + (A - D)a_v + Gb_u + Cb_v + C_u a + C_v b = 0, \\
&4) \quad b_{uv} + Da_u + Ba_v + (H - C)b_u + D_u a + D_v b = 0, \\
&5) \quad a_{uv} + (A - F)a_v + Gb_u + Eb_v + E_u a + E_v b = 0, \\
&6) \quad b_{uv} + Fa_u + Ba_v + (H - E)b_u + F_u a + F_v b = 0, \\
&7) \quad a_{vv} - Ga_u + (C + E - H)a_v + 2Gb_v + G_u a + G_v b = 0, \\
&8) \quad b_{vv} + (D + F)a_v - Gb_u + Hb_v + H_u a + H_v b = 0.
\end{aligned} \tag{4.3}$$

Moreover, after some direct calculations, we obtain the following formulas for the Ricci tensor:

$$\begin{aligned} Ric(\partial_u, \partial_u) &= B_v - F_u + B(H-E) + F(A-D), \\ Ric(\partial_u, \partial_v) &= D_v - H_u + CF - BG, \\ Ric(\partial_v, \partial_u) &= E_u - A_v + CF - BG, \\ Ric(\partial_v, \partial_v) &= G_u - C_v + C(H-E) + G(A-D). \end{aligned} \tag{4.4}$$

Now, we shall establish two lemmas as a basic tool.

We know that if a connection is of type A, then it has constant Christoffel symbols in some local coordinates (u, v) and hence, by (4.3), it admits the pair of Killing vector fields $\{\partial_v, \partial_u\}$. Thus we have two linearly independent Killing vector fields X, Y such that $[X, Y] = 0$.

Further, if a connection is of type B, all Christoffel symbols are of the form $\frac{constant}{u}$ in some local coordinates (u, v), and it follows easily from the formula (4.3) that the connection admits a pair of Killing vector fields $\{\partial_v, u\partial_u + v\partial_v\}$. Hence we have two linearly independent Killing vector fields X, Y such that $[X, Y] = X$.

Now, we want to prove also the converse statements, and we summarize all in our lemmas.

Lemma 4.2.2. *For any Lie algebra $\mathfrak{g}$ of vector fields (from the refined Olver list) defined on a simply connected domain $\mathcal{U}(u, v)$ of the plane, the following two assertions are equivalent:*

(i) *In $\mathfrak{g}$, there are two linearly independent vector fields X, Y such that $[X, Y] = 0$.*

(ii) *All connections corresponding to $\mathfrak{g}$ are of type A.*

Proof. For the implication $(ii) \to (i)$ see above. It remains to show the implication $(i) \to (ii)$. From a well-known result about commuting vector fields, which is valid in every dimension, we deduce that there is a local coordinate system (x, y) on the given domain such that $X = \partial_x$, $Y = \partial_y$. Formula (4.3) (written in the new coordinates) shows that all Christoffel symbols are constant.

Lemma 4.2.3. *For any Lie algebra $\mathfrak{g}$ of vector fields (from the refined Olver list) defined on a simply connected domain $\mathcal{U}(u, v)$ of the plane, the following two assertions are equivalent:*

(i) *In $\mathfrak{g}$, there are two linearly independent vector fields X, Y such that $[X, Y] = X$.*

(ii) *All connections corresponding to $\mathfrak{g}$ are of type B.*

Proof. For the implication $(ii) \to (i)$ see above. It remains to show the implication $(i) \to (ii)$. Let $\mathcal{U}(u,v) \subset \mathbb{R}^2$ be a local coordinate system and $X = p(u,v)\partial_u + q(u,v)\partial_v$, $Y = r(u,v)\partial_u + s(u,v)\partial_v$ be two linearly independent vector fields such that $[X,Y] = X$. From now on, we will denote the functions $p(u,v)$, $q(u,v)$, $r(u,v)$, $s(u,v)$ by p, q, r, s respectively. Then, if we put $A = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$, the matrix A is nonsingular and

$$A^{-1} = \frac{1}{(ps - qr)} \begin{pmatrix} s & -r \\ -q & p \end{pmatrix}.$$

Moreover, the condition $[X,Y] = X$ is satisfied if and only if

$$\begin{aligned} pr_u - rp_u - sp_v + qr_v &= p, \\ ps_u - rq_u - sq_v + qs_v &= q. \end{aligned} \tag{4.5}$$

Now, we are going to prove that there exists a convenient local coordinate system, (x,y), in which the vector fields X, Y, are of the form ∂_y, $x\partial_x + y\partial_y$, respectively. Put $x = f(u,v)$, $y = g(u,v)$. Then $\partial_u = f_u\partial_x + g_u\partial_y$, $\partial_v = f_v\partial_x + g_v\partial_y$. Consequently, in the new coordinate system

$$X = (pf_u + qf_v)\partial_x + (pg_u + qg_v)\partial_y,$$

$$Y = (rf_u + sf_v)\partial_x + (rg_u + sg_v)\partial_y.$$

Hence, $X = \partial_y$ and $Y = x\partial_x + y\partial_y$ if and only if the following system of partial differential equations for f and g is satisfied:

$$pf_u + qf_v = 0, \quad pg_u + qg_v = 1, \quad rf_u + sf_v = f, \quad rg_u + sg_v = g.$$

Now, using the Cramer's rule, we obtain the explicit expression of this system in the form

$$f_u = \frac{-qf}{ps - qr}, \quad f_v = \frac{pf}{ps - qr}, \quad g_u = \frac{s - qg}{ps - qr}, \quad g_v = \frac{-r + pg}{ps - qr}.$$

Finally, using both equations of (4.5) and the above relations $pf_u + qf_v = 0$, $pg_u + qg_v = 1$, is easy to prove that the integrability conditions, $(f_u)_v = (f_v)_u$ and $(g_u)_v = (g_v)_u$, are satisfied. Hence, in our simply connected domain, the above system of partial differential equations is completely integrable. Substituting $X = \partial_y$ and $Y = x\partial_x + y\partial_y$, respectively, into the system (4.3) (written in the new coordinates x, y), we find easily that all corresponding connections are of type B.

From now on, we shall generalize and improve the results given in [K-Op-Vl.4], using among others tools our lemmas 4.2.2 and 4.2.3.

4.2.2 Lie algebras containing all translations

We shall first study Case 1.4, the cases 1.7-1.11, Case 2.1 and the cases 2.4-2.8 from refined Olver's list. Here we generalize the results of Sections 2, 3 and the parts 1), 2) of Theorem 6.1 of [K-Op-Vl.4] where the torsion was supposed to be zero. In the following we assume that our connections have eight Christoffel symbols given by the formula (4.2). The calculations done by MATHEMATICA 5.2 are very similar to those made by Maple V Release 4 in [K-Op-Vl.4]. Anyway, we shall show the details of the proofs to make our exposition self-contained.

Firstly, we need some technical lemmas.

Lemma 4.2.4. *If the connection ∇ admits the Killing vector fields ∂_u, ∂_v, $(pu + qv)\partial_u + (ru + sv)\partial_v$, where p, q, r, s are constants such that*

$$ps - qr \neq 0, \quad 2p^2 + 2s^2 + 9qr - 5ps \neq 0, \tag{4.6}$$

then all Christoffel symbols $A, B, ..., H$ with respect to the coordinates u, v are zero, i.e., ∇ is locally flat.

Proof. First, due to the Killing vector fields ∂_u, ∂_v are admitted, it is obvious from (4.3) that all Christoffel symbols $A, B, ..., H$ are constants. Substituting now $pu+qv$ and $ru+sv$ for $a(u, v)$ and $b(u, v)$, respectively, into (4.3), we obtain the following system of 8 linear homogeneous algebraic equations for 8 unknown constants $A, B, ..., H$, namely

$$\begin{aligned}
&1)\quad pA - qB + rC + rE = 0,\\
&2)\quad -rA + (2p - s)B + rD + rF = 0,\\
&3)\quad qA + sC - qD + rG = 0,\\
&4)\quad qB - rC + pD + rH = 0,\\
&5)\quad qA + sE - qF + rG = 0,\\
&6)\quad qB - rE + pF + rH = 0,\\
&7)\quad qC + qE + (2s - p)G - qH = 0,\\
&8)\quad qD + qF - rG + sH = 0.
\end{aligned} \tag{4.7}$$

The determinant of this system can be expressed in the form

$$-(2p^2 + 2s^2 + 9qr - 5ps)(ps - qr)^3,$$

which concludes the proof.

Lemma 4.2.5. *If ∇ is locally flat and (u, v) is a coordinate system in which the Christoffel symbols of ∇ vanish, then the connection does not admit a Killing vector field of the form $Z = a(u, v)\partial_u + b(u, v)\partial_v$ where $a(u, v)$ or $b(u, v)$ is a proper quadratic polynomial of u, v.*

Proof. Because all coefficients $A, ..., H$ are zero, we see at once from (4.3) a contradiction.

Now we proves the first classification results.

Lemma 4.2.6. *The following Lie algebras from refined Olver's list characterize, as algebras of affine Killing vector fields, connections with vanishing Christoffel symbols:*
Case 1.7 for $\alpha \neq 0, 1/2, 2$ *and* $k = 1, 2$,
Case 1.7 for $\alpha = 0$, *or* $\alpha = 1/2$, *and* $k = 2$,
Case 1.8 for $k = 1$,
Case 1.9 for $k = 1, 2$,
the cases 2.1, 2.4, 2.5 and 2.6.

Proof. Since all these algebras contain the operators ∂_u, ∂_v, it follows easily from (4.3) that the algebras quoted above consist of affine Killing vector fields of the connection with constant Christoffel symbols. Thus, it remains to show that each of these algebras *enforces* the vanishing of all Christoffel symbols.

The case 1.8 for $k = 1$ involves the operator $(u + v)\partial_u + v\partial_v$, i.e., one with $p = q = s = 1$, $r = 0$, which satisfies the inequalities (4.6) from Lemma 4.2.4.

The case 1.9 always involves the operator $u\partial_u + v\partial_v$, which also satisfies the conditions (4.6). But if $k > 2$ from Lemma 4.2.5 we obtain a contradiction.

The case 2.1 involves the operator with $p = \alpha$, $q = -1$, $r = 1$, $s = \alpha$. We see again that (4.6) is satisfied. The cases 2.4, 2.5 and 2.6 are now obvious.

It remains to discuss the case 1.7, in which we have an additional operator with $p = \alpha$, $q = 0$, $r = 0$ and $s = 1$. Here the values p, q, r, s satisfy (4.6) except the cases $\alpha = 0$, $\alpha = 2$ and $\alpha = 1/2$. Moreover, if $\alpha \notin \{0, 1/2, 2\}$ and $k > 2$ from Lemma 4.2.5 we obtain a contradiction. Thus, for $\alpha \notin \{0, 1/2, 2\}$ we get our assertion. Let be now $\alpha = 0$, or $\alpha = 1/2$, and $k = 2$. Then we have the second additional operator $v\partial_u$, i.e. one with $p = r = s = 0$, $q = 1$. From (4.7) we obtain $A = B = D = F = 0$ and, because $\alpha \neq 2$, applying (4.7) for the first operator we get finally $C = E = G = H = 0$.

This completes the proof

Lemma 4.2.7. *The following Lie algebras of vector fields do not admit any invariant affine connection:*
Case 1.4,
Case 1.7 for $k > 2$,
Case 1.8 for $k \geq 2$,
Case 1.9 for $k > 2$,
the cases 1.10, 1.11 and the cases 2.7, 2.8.

Proof. If there is such a connection, its Christoffel symbols must be again constant. For the case 1.7 with $\alpha \neq 2$ the result follows from Lemma 4.2.6 and Lemma 4.2.5. If $\alpha = 2$ in the case 1.7, we substitute $p = 2$, $q = r = 0$ and $s = 1$ into the system

(4.7). We obtain $A = B = C = D = E = F = H = 0$. Now the operator $v^2\partial_u$ has to be added. We put $a(u,v) = v^2$, $b(u,v) = 0$ in (4.3) and the 7th equation gives a contradiction.

For the case 1.8 with $k \geq 2$ we put $a(u,v) = ku + v^k$, $b(u,v) = v$ in (4.3). We obtain that $A = B = C = D = E = F = H = 0$ and consequently, the following contradiction from the 7th equation of (4.3):

$$(k^2 - k)v^{k-2} + (2 - k)G = 0.$$

For the remaining cases, the result follows from Lemma 4.2.6, Lemma 4.2.4 and Lemma 4.2.5.

Proposition 4.2.8. *The Lie algebras from Case 1.7 produce just the following locally homogeneous connections:*

a) $\alpha = 0$ *and* $k = 1$. *Then the Christoffel symbols are constant parameters given by* $B = C = E = G = H = 0$, D, F *and* A *arbitrary. Here, the torsion tensor is not zero if and only if* $D \neq F$ *and the Ricci tensor, Ric, is zero if and only if* $Ric(\partial_u, \partial_u) = F(A - D)$ *is zero.*

b) $\alpha = 1/2$ *and* $k = 1$. *Then the Christoffel symbols are constant parameters given by* $A = C = D = E = F = G = H = 0$ *and* B *arbitrary. Here the torsion tensor and the Ricci tensor are always zero.*

c) $\alpha = 2$ *and* $k = 1, 2$. *Then the Christoffel symbols are constant parameters given by* $A = B = C = D = E = F = H = 0$ *and* G *arbitrary. Here the torsion tensor and the Ricci tensor are always zero.*

d) The VCS connections described in Lemma 4.2.6.

Proof. The locally homogeneous connections given in the cases a), b) and c) are obvious from the equation (4.7) where we put $p = \alpha$, $q = r = 0$, $s = 1$ in each case, and $p = r = s = 0$, $q = 1$ in the last subcase $\alpha = 2$, $k = 2$. The statements about the torsion tensor and the Ricci tensor follow directly after substituting the above Christoffel symbols in the definition and in the system (4.4), respectively.

Now, we shall investigate, in each case, if the corresponding connections are of type A, or of type B, respectively.

Proposition 4.2.9. *The connections with VCS (like those from Lemma 4.2.6) are of both types A and B.*

Proof. Obviously, every connection with vanishing Christoffel symbols is of type A. On the other hand, we can use the transformation of the coordinates $u' = e^u$, $v' = v$ in the formula (4.2) and we obtain the expression belonging to type B. That is

$$\nabla_{\partial_{u'}}\partial_{u'} = \tfrac{-1}{u'}\partial_{u'}, \quad \nabla_{\partial_{u'}}\partial_{v'} = \nabla_{\partial_{v'}}\partial_{u'} = \nabla_{\partial_{v'}}\partial_{v'} = 0. \tag{4.8}$$

Proposition 4.2.10. *The connections described in Proposition 4.2.8 are of both types A and B.*

Proof. First, all connections described in Proposition 4.2.8 are of type A. Further, it is easy to check that the assumption (i) of Lemma 4.2.3 is satisfied for each algebra described in Case 1.7 of the refined Olver list. Thus, all corresponding connections are of type B. For the explicit illustration, we give more details to the subcase b) of Proposition 4.2.8.

We know that the Lie algebra of vector fields given in the case 1.7 for $\alpha = 1/2$ and $k = 1$ it is generated by $\{\partial_v, v\partial_v + \frac{1}{2}u\partial_u, \partial_u\}$, and it admits an invariant affine connection whose constant Christoffel symbols are given by $A = C = D = E = F = G = H = 0$ and B arbitrary. Applying the change of the coordinates $u' = u^2$, $v' = v$ to the Lie algebra and also to the formula (4.2), and using for (u', v') the original notation (u, v), we obtain the Lie algebra $\{\partial_v, v\partial_v + u\partial_u, 2\sqrt{u}\partial_u\}$, and the new Christoffel symbols are

$$A(u) = \frac{-1}{2u},\ B(u) = \frac{B}{4u},\ C(u) = D(u) = E(u) = F(u) = G(u) = H(u) = 0.$$

where B is the old constant Christoffel symbol.

Moreover, for the subcase a), or c), of Proposition 4.2.8, the proper change of coordinates making the Christoffel symbols to be of type B is $u' = \mathrm{e}^u$, $v' = v$, or $u' = \sqrt{u}$, $v' = v$, respectively.

4.2.3 Lie algebras of the cases 1.5 and 1.6

In these cases we have the Killing vector field ∂_v. Thus it is obvious from (4.3) that all Christoffel symbols $A, ..., H$ are functions of the variable u only. Looking at the generators in the cases 1.5, 1.6, we are lead to the investigation of the situation when we have additional Killing vector fields of the form $f(v)\partial_u$, where $f(v)$ is a function. So, we substitute $a(u, v) = f(v)$ and $b(u, v) = 0$ into the system (4.3). After doing that, we obtain the system

$$\begin{aligned}
&1)\quad A'(u) - B(u)(f'(v)/f(v)) = 0,\\
&2)\quad B'(u) = 0,\\
&3)\quad C'(u) + (A(u) - D(u))(f'(v)/f(v)) = 0,\\
&4)\quad D'(u) + B(u)(f'(v)/f(v)) = 0,\\
&5)\quad E'(u) + (A(u) - F(u))(f'(v)/f(v)) = 0,\\
&6)\quad F'(u) + B(u)(f'(v)/f(v)) = 0,\\
&7)\quad G'(u) + (C(u) + E(u) - H(u))(f'(v)/f(v)) + f''(v)/f(v) = 0,\\
&8)\quad H'(u) + (D(u) + F(u))(f'(v)/f(v)) = 0.
\end{aligned} \tag{4.9}$$

Now we are going to study this system of linear ordinary differential equations extending the method used in Section 4 of [K-Op-Vl.4]. Moreover, we are going to correct the gap that occurred in the classification.

Let ∇ be a locally homogeneous connection with a fixed Killing vector field of the form $f(v)\partial_u$. Then we can just consider the following cases:

Case I. $f'(v) = 0$.

Then the corresponding Killing vector field is a constant multiple of ∂_u. Then all corresponding Christoffel symbols are constant. We obtain Case 1.5 a) of the refined Olver list. Assuming now that the Lie algebra of Killing vector fields also contains $u\partial_u$, we obtain Case 1.6 a'). Then, we substitute $a(u,v) = u$ and $b(u,v) = 0$ into the system (4.3). After a routine calculation we obtain $A = B = D = F = G = 0$.

Case II. $f'(v) \neq 0$ everywhere.

Even here, $B(u)$ must be constant, $B(u) = B$, according to the equation 2) of (4.9).

Case II.1. $B \neq 0$.

From the equation 1) we see that $f'(v)/f(v)$ is a non-zero constant (due to the separation of variables), i.e., $f(v) = \mathrm{e}^{lv}$ where $l \in \mathbb{R}^*$. Using the transformation of the coordinate $v : v' = lv$ (which does not change the form of the algebras of the cases 1.5 and 1.6) we can assume that $l = 1$ and $f(v) = \mathrm{e}^v$. Thus, we have Case 1.5 b) of the refined Olver list. Integrating now the system (4.9), we obtain the solution

$$\begin{gathered} A(u) = C_1u + C_2, \quad B(u) = C_1, \quad H(u) = C_1u^2 - (C_3 + C_5)u + C_7, \\ C(u) = -C_1u^2 + (C_3 - C_2)u + C_4, \quad D(u) = -C_1u + C_3, \\ E(u) = -C_1u^2 + (C_5 - C_2)u + C_6, \quad F(u) = -C_1u + C_5, \\ G(u) = C_1u^3 + (C_2 - C_3 - C_5)u^2 + (C_7 - C_4 - C_6 - 1)u + C_8, \end{gathered} \tag{4.10}$$

where $C_1, \ldots, C_8$ are constant parameters and $C_1 \neq 0$.

Assume now that the Lie algebra of Killing vector fields also contain $u\partial_u$, as required in Case 1.6 b'). Then we solve the system (4.3) assuming that the Christoffel symbols are given by (4.10), $a(u,v) = u$ and $b(u,v) = 0$. This gives necessarily $C_1 = 0$, which is a contradiction.

Case II.2. $B = 0$.

Then, by 1), 4) and 6) of (4.9), A, D and F are constants.

Case II.2.1. $f'(v)/f(v)$ is a nonzero constant.

Then, we can assume again that $f(v) = \mathrm{e}^v$ and we have Case 1.5 b) of the refined Olver list again. Moreover, we obtain the formula (4.10) with $C_1 = 0$.

Assume now that the Lie algebra of Killing vector fields also contains $u\partial_u$ as required in Case 1.6 b'). Then we solve the system (4.3) assuming that $a(u,v) = u$, $b(u,v) = 0$ and the Christoffel symbols are given by (4.10) with $C_1 = 0$. This gives

$$\begin{gathered} A = B = D = F = 0, \quad C(u) = C_4, \quad E(u) = C_6, \\ H(u) = C_7, \quad G(u) = (C_7 - C_6 - C_4 - 1)u, \end{gathered} \tag{4.11}$$

where C_4, C_6, C_7 are constant parameters.

Case II.2.2. $f'(v)/f(v)$ is not a constant.

Then, under our assumptions that $B = 0$ and A, D, F are constants, we get from (4.9)

$$\begin{gathered} A = B = D = F = 0, \quad C(u) = 2c_1, \quad E(u) = 2c_2, \\ H(u) = 2c_3, \quad G(u) = c_4 u + c_5, \end{gathered} \tag{4.12}$$

where c_1, c_2, c_3, c_4, c_5 are constants. Moreover the 7th equation of (4.9) takes on the form

$$f''(v) + 2(c_1 + c_2 - c_3)f'(v) + c_4 f(v) = 0. \tag{4.13}$$

We see that, for any choice of the parameters in (4.12), the corresponding equation (4.13) is uniquely determined, and the solutions $f(v)$ form a uniquely determined two-dimensional vector space. We are now in the case $k = 2$. The generators of this vector space are the following pairs of functions, which depend on the sign of the discriminant $c_0^2 - c_4$, where $c_0 = c_3 - c_1 - c_2$. If $c_0^2 - c_4 > 0$, then there is a constant $c_6 > 0$ such that $c_6^2 = c_0^2 - c_4$ and we get a pair of generators $\{\mathrm{e}^{(c_0+c_6)v}, \mathrm{e}^{(c_0-c_6)v}\}$. If $c_0^2 - c_4 = 0$, then we have a pair of generators $\{\mathrm{e}^{c_0 v}, v\mathrm{e}^{c_0 v}\}$. If $c_0^2 - c_4 < 0$, then there is a constant $c_6 > 0$ such that $c_6^2 = c_4 - c_0^2$ and the pair of generators is $\{\mathrm{e}^{c_0 v}\cos(c_5 v), \mathrm{e}^{c_0 v}\sin(c_5 v)\}$.

Case II.2.2.1. $c_4 = 0$.

In the special case $c_4 = 0$ there is a constant solution of (4.13), and we have the Killing vector field ∂_u. If, in (4.13), the coefficient $c_1 + c_2 - c_3$ is non-zero, then the additional basic solution of (4.13) is $f(v) = \mathrm{e}^v$, and from (4.9) we obtain $c_3 = \frac{1}{2} + c_1 + c_2$. We have Case 1.5 c). If $c_1 + c_2 - c_3 = 0$, then the additional solution is $f(v) = v$ and we have Case 1.5 d).

Assume now that the Lie algebra of Killing vector fields also contain $u\partial_u$ as required in the cases 1.6 c') and 1.6 d'). Then by solving the system (4.3), under our assumptions, we obtain easily that $c_5 = 0$ in both cases.

Case II.2.2.2. $c_4 \neq 0$.

Thus, we have the cases 1.5 e), 1.5 f) and 1.5 g) of the refined Olver list. Here, we change the coordinate system in the following way: $u' = u + \frac{c_5}{c_4}$, $v' = v$. Then, in the new coordinate system, which we denote again by (u, v), we have still the same generators ∂_v, $\eta_1(v)\partial_u$, $\eta_2(v)\partial_u$. Moreover, the Christoffel symbols (4.12) get the new form

$$\begin{gathered} A = B = D = F = 0, \quad C(u) = 2c_1, \quad E(u) = 2c_2, \\ H(u) = 2c_3, \quad G(u) = c_4 u, \end{gathered} \tag{4.14}$$

where c_1, c_2, c_3, $c_4 \neq 0$ are constants. Now, it is easy to check from (4.3) that the connection given by (4.14) has an additional Killing vector field $u\partial_u$. This means, the cases 1.5 e), f), g) and 1.6 e'), f'), g') from refined Olver's list *coincide*, respectively.

Now, we substitute the Christoffel symbols (4.14) and, separately, the vector fields $\mathrm{e}^{\alpha v}\partial_u$, $\mathrm{e}^{\beta v}\partial_u$, $\alpha \neq \beta$, $\alpha, \beta \neq 0$, in (4.9). We obtain by easy calculations that the corresponding Christoffel symbols to Case 1.6 e') are

$$\begin{gathered} A = B = D = F = 0, \quad C(u) = -\alpha - \beta + 2(c_3 - c_2), \\ E(u) = 2c_2, \quad H(u) = 2c_3, \quad G(u) = \alpha\beta u, \end{gathered} \tag{4.15}$$

where c_2, c_3, $\alpha \neq 0$, $\beta \neq 0$ are constants and $\alpha \neq \beta$.

Now, we proceed as before but with the vector fields $\mathrm{e}^{\alpha v}\partial_u$, $v\mathrm{e}^{\alpha v}\partial_u$, $\alpha \neq 0$. Here, the corresponding Christoffel symbols to Case 1.6 f') are

$$\begin{aligned} A = B = D = F = 0, \quad C(u) = 2(c_3 - c_2 - \alpha), \\ E(u) = 2c_2, \quad H(u) = 2c_3, \quad G(u) = \alpha^2 u, \end{aligned} \tag{4.16}$$

where c_2, c_3, $\alpha \neq 0$ are constants.

Finally, with the same method but with the vector fields $\cos(\beta v)\mathrm{e}^{\alpha v}\partial_u$, $\sin(\beta v)\mathrm{e}^{\alpha v}\partial_u$, $\beta \neq 0$, we obtain that the corresponding Christoffel symbols to Case 1.6 g') are

$$\begin{aligned} A = B = D = F = 0, \quad C(u) = 2(c_3 - c_2 - \alpha), \\ E(u) = 2c_2, \quad H(u) = 2c_3, \quad G(u) = (\alpha^2 + \beta^2)u, \end{aligned} \tag{4.17}$$

where c_2, c_3, α, $\beta \neq 0$ are constants.

Now we summarize the obtained results.

Theorem 4.2.11. *Let ∇ be a locally homogeneous connection whose corresponding algebra of Killing vector fields belongs to Case* 1.5 *(and not to Case 1.6). Then $k = 1$, or $k = 2$, and we have:*

(*i*) *For $k = 1$ there are two non-equivalent algebras of Killing vector fields, namely those spanned by the following pairs:*

- *a)* $\{\partial_v, \partial_u\}$. *Here all Christoffel symbols are arbitrary constants. Moreover, the torsion tensor, T, is not zero if and only if $C \neq E$ or $D \neq F$.*
- *b)* $\{\partial_v, \mathrm{e}^v\partial_u\}$. *Here the Christoffel symbols can be expressed by the formula* (4.10) *admitting also the case $C_1 = 0$. Moreover $T \neq 0$ if and only if $C_3 \neq C_5$ or $C_4 \neq C_6$.*

(*ii*) *For $k = 2$ there are two non-equivalent algebras of Killing vector fields, namely those spanned by the following triplets:*

- *c)* $\{\partial_v, \partial_u, \mathrm{e}^v\partial_u\}$. *Here, the corresponding Christoffel symbols are given by* (4.12) *with $c_4 = 0$ and $c_3 = \frac{1}{2} + c_1 + c_2$. Moreover $T \neq 0$ if and only if $c_1 \neq c_2$.*
- *d)* $\{\partial_v, \partial_u, v\partial_u\}$. *Here, the corresponding Christoffel symbols are given by* (4.12) *with $c_4 = 0$ and $c_3 = c_1 + c_2$. Moreover $T \neq 0$ if and only if $c_1 \neq c_2$.*

For $k > 2$, an invariant affine connection ∇ does not exist.

Theorem 4.2.12. *Let ∇ be a locally homogeneous connection whose corresponding algebra of Killing vector fields belong to Case* 1.6. *Then $k = 1$, or $k = 2$, and we have:*

(*i*) *For $k = 1$ there are two non-equivalent algebras of Killing vector fields, namely those spanned by the following triplets:*

a') $\{\partial_v, u\partial_u, \partial_u\}$. Here the Christoffel symbols are constant parameters given by $A = B = D = F = G = 0$, C, E and H arbitrary. Moreover, the torsion tensor, T, is not zero if and only if $C \neq E$.

b') $\{\partial_v, u\partial_u, \mathrm{e}^v\partial_u\}$. Here the Christoffel symbols are given by (4.11)*. Moreover $T \neq 0$ if and only if $C_4 \neq C_6$.*

(*ii*) *For $k = 2$ there are five non-equivalent algebras of Killing vector fields, namely those spanned by the following quadruplets:*

c') $\{\partial_v, u\partial_u, \partial_u, e^v\partial_u\}$. Here, the corresponding Christoffel symbols are given by (4.12) *with $c_4 = c_5 = 0$ and $c_3 = \frac{1}{2} + c_1 + c_2$. Moreover $T \neq 0$ if and only if $c_1 \neq c_2$.*

d') $\{\partial_v, u\partial_u, \partial_u, v\partial_u\}$. Here, the corresponding Christoffel symbols are given by (4.12) *with $c_4 = c_5 = 0$ and $c_3 = c_1 + c_2$. Moreover $T \neq 0$ if and only if $c_1 \neq c_2$.*

e') $\{\partial_v, u\partial_u, e^{\alpha v}\partial_u, e^{\beta v}\partial_u, \}$, $\alpha \neq \beta$, $\alpha, \beta \neq 0$. Here, the corresponding Christoffel symbols are given by (4.15)*. Moreover, $T \neq 0$ if and only if $2c_3 - 4c_2 - \alpha - \beta \neq 0$.*

f') $\{\partial_v, u\partial_u, e^{\alpha v}\partial_u, v e^{\alpha v}\partial_u, \}$, $\alpha \neq 0$. Here, the corresponding Christoffel symbols are given by (4.16)*. Moreover, $T \neq 0$ if and only if $c_3 - 2c_2 - \alpha \neq 0$.*

g') $\{\partial_v, u\partial_u, cos(\beta v) e^{\alpha v}\partial_u, sin(\beta v) e^{\alpha v}\partial_u, \}$, $\alpha, \beta \neq 0$. Here, the corresponding Christoffel symbols are given by (4.17)*. Moreover, $T \neq 0$ if and only if $c_3 - 2c_2 - \alpha \neq 0$.*

For $k > 2$, an invariant affine connection ∇ does not exist.

Now, we shall investigate, in each case, if the corresponding connections are of type A, or of type B, respectively.

Proposition 4.2.13. *The connections corresponding to the Lie algebras from the cases 1.5 a), 1.5 d), 1.6 a'), 1.6 d') and 1.6 g') are of type A but not of type B.*

Proof. Obviously, in the cases 1.5 a), 1.5 d), 1.6 a') and 1.6 d') we have constant Christoffel symbols.

For Case 1.6 g'), it is easy to check that the assumption (i) of Lemma 4.2.2 is satisfied. Thus the corresponding connections are of type A. For the explicit illustration, we only need to use the change of the coordinates $u' = \ln u$, $v' = v$. Then the Christoffel symbols become of type A.

On the other hand, a direct computation shows that none of the Lie algebras above admits a subalgebra $\mathrm{span}(X, Y)$ such that $[X, Y] = X$. Thus, by Lema 4.2.3, the corresponding connections are not of type B.

Proposition 4.2.14. *The connections corresponding to the Lie algebra from Case 1.5 b) are of type B but not of type A.*

Proof. It is easy to check that the assumption (i) of Lemma 4.2.3 is satisfied. Thus the corresponding connections are of type B. For the explicit illustration, we only need to use the change of the coordinates $u' = \mathrm{e}^{-v}$, $v' = u\mathrm{e}^{-v}$. Then the Christoffel symbols become of type B.

On the other hand, it is obvious from Lema 4.2.2 that the corresponding connections are not of type A.

Proposition 4.2.15. *The connections corresponding to the Lie algebras from the cases 1.5 c), 1.6 b'), 1.6 c'), 1.6 e') and 1.6 f') are of both types A and B.*

Proof. Obviously, the corresponding connections ∇ to the Lie algebras of the cases 1.5 c) and 1.6 c') have constant Christoffel symbols.

For the cases 1.6 b'), 1.6 e') and 1.6 f'), it is easy to check that the assumption (i) of Lemma 4.2.2 is satisfied. Thus the corresponding connections are of type A. For the explicit illustration, we only need to use, in these three cases, the change of the coordinates $u' = \ln u$, $v' = v$. Then the Christoffel symbols become of type A.

On the other hand, it is easy to check that the assumption (i) of Lemma 4.2.3 is satisfied for each Lie algebra. Thus the corresponding connections are also of type B. For the explicit illustration, in the cases 1.5 c), 1.6 b') and 1.6 c'), we only need to use the change of the coordinates $u' = \mathrm{e}^{-v}$, $v' = u\mathrm{e}^{-v}$, and in the cases 1.6 e') and 1.6 f'), the change of the coordinates $u' = \mathrm{e}^{-\alpha v}$, $v' = u\mathrm{e}^{-\alpha v}$, $\alpha \neq 0$. Then the Christoffel symbols become of type B.

4.2.4 Other cases

We are now left with the cases 1.1-1.3 and 2.2, 2.3 from refined Olver's list. First we generalize the results of Section 5, the part 6) of Theorem 6.1 and the parts 2), 3), 4) of Theorem 6.4 of [K-Op-Vl.4] where the torsion was supposed to be zero. Now we assume that our connections have eight Christoffel symbols given by the formula (4.2). The calculations are again very similar but, as before, we shall show the details of the proofs to make our exposition self-contained.

Proposition 4.2.16. *The Lie algebras from the cases 1.1, 1.2, 1.3, 2.2 and 2.3 of the refined Olver list produce just the following homogeneous connections:*

a) For Case 1.1, the Christoffel symbols are given by

$$\begin{aligned} A(u) = \frac{-1}{2u}, \quad B(u) = 0, \quad C(u) = cu, \quad D(u) = \frac{1}{2u}, \\ E(u) = eu, \quad F(u) = \frac{1}{2u}, \quad G(u) = gu^3, \quad H(u) = (c+e)u, \end{aligned} \tag{4.18}$$

with three arbitrary parameters c, e, g. Here the torsion tensor, T, is not zero if and only if $c \neq e$ and the Ricci tensor, Ric, is zero if and only if $g = -2e^2$, $c = -2e$.

b) *For Case 1.2, the Christoffel symbols are given by*

$$\begin{aligned} A(u) = B(u) = D(u) = F(u) = 0, \quad C(u) = E(u) = -2u, \\ G(u) = 4u^3, \quad H(u) = 2u. \end{aligned} \tag{4.19}$$

Here, $T = 0$ and $Ric \neq 0$. Moreover, this connection is the Levi-Civita connection of the Lorentzian metric $Ric = -4dudv + 4u^2dv^2$ of constant positive curvature 1, or, equivalently, of the Lorentzian metric $-Ric$ of constant negative curvature -1.

c) *For Case 1.3, the Christoffel symbols are given by*

$$\begin{aligned} A(u) = B(u) = C(u) = E(u) = G(u) = H(u) = 0, \\ D(u) = F(u) = \frac{1}{u}. \end{aligned} \tag{4.20}$$

Here, $T = 0$ and $Ric = 0$.

d) *For Case 2.2, the Christoffel symbols are given by*

$$\begin{aligned} A(u) = D(u) = F(u) = -G(u) = \frac{-1}{u}, \\ B(u) = C(u) = E(u) = H(u) = 0. \end{aligned} \tag{4.21}$$

Here $T = 0$, and $(-Ric) = (du^2 + dv^2)/u^2$ is a Riemannian metric of constant negative curvature -1. Then ∇ is locally the Levi-Civita connection of the standard hyperbolic plane.

e) *For Case 2.3, the Christoffel symbols are given by*

$$\begin{aligned} A = -\rho_u,\ B = \rho_v,\ C = -\rho_v,\ D = -\rho_u,\ E = -\rho_v,\ F = -\rho_u, \\ G = \rho_u,\ H = -\rho_v, \quad \text{where } \rho = log(1 + u^2 + v^2). \end{aligned} \tag{4.22}$$

Here, $T = 0$ and $Ric = 4(du^2 + dv^2)/(1 + u^2 + v^2)^2$ is a Riemannian metric of constant positive curvature 1. Then ∇ is locally the Levi-Civita connection of the unit sphere.

Proof. The cases 1.1-1.3 should involve the Killing vector fields ∂_v, $u\partial_u - v\partial_v$. Hence, the Christoffel symbols of ∇ depend only on u and they satisfy the system of equations:

$$\begin{aligned} A(u) + uA'(u) &= 0, \\ D(u) + uD'(u) &= 0, \\ F(u) + uF'(u) &= 0, \\ -C(u) + uC'(u) &= 0, \\ -E(u) + uE'(u) &= 0, \\ -H(u) + uH'(u) &= 0, \\ 3B(u) + uB'(u) &= 0, \\ -3G(u) + uG'(u) &= 0. \end{aligned} \tag{4.23}$$

The general solution is of the form

$$\begin{aligned} &A(u) = \frac{p}{u}, \quad B(u) = \frac{q}{u^3}, \quad C(u) = cu, \quad D(u) = \frac{d}{u}, \\ &E(u) = eu, \quad F(u) = \frac{f}{u}, \quad G(u) = gu^3, \quad H(u) = hu. \end{aligned} \tag{4.24}$$

Now, in Case 1.1, we have the third Killing vector field for which $a(u,v) = -2uv$, $b(u,v) = v^2$. Making corresponding substitutions in (4.3), we get the following relations among constant:

$$q = 0, \quad p = -1/2, \quad d = f = 1/2, \quad h = c + e. \tag{4.25}$$

Hence we get the final solution (4.18).

In Case 1.2 we have to put $a(u,v) = -(2uv+1)$, $b(u,v) = v^2$. An easy calculation shows that here $p = q = d = f = 0$, $c = e = -2$, $g = 4$, $h = 2$. We get (4.19).

In Case 1.3 we have two additional Killing vector fields, e.g. $u\partial_u$ and $-uv\partial_u + v^2\partial_v$. An easy calculation yields $d = f = 1$ and the other constants zero. We have (4.20).

The remaining cases are 2.2 and 2.3.

In Case 2.2, we have again the Killing vector field ∂_v and hence all Christoffel symbols are functions of the variable u only. Further, the occurrence of the Killing vector field $u\partial_u + v\partial_v$ enforces equations analogous to (4.23), namely

$$\Phi(u) + u\Phi'(u) = 0 \quad \text{for} \quad \Phi = A(u), B(u), ..., H(u). \tag{4.26}$$

Hence we get:

$$\begin{aligned} &A(u) = \frac{p}{u}, \quad B(u) = \frac{q}{u}, \quad C(u) = \frac{c}{u}, \quad D(u) = \frac{d}{u}, \\ &E(u) = \frac{e}{u}, \quad F(u) = \frac{f}{u}, \quad G(u) = \frac{g}{u}, \quad H(u) = \frac{h}{u}. \end{aligned} \tag{4.27}$$

The last Killing vector field is characterized by $a(u,v) = 2uv$, $b(u,v) = v^2 - u^2$. Substituting from this and (4.27) in (4.3), we get a system of linear equations for the constants p, q, c,..., h and hence

$$q = c = e = h = 0, \quad p = d = f = -1, \quad g = 1. \tag{4.28}$$

We get the final formula (4.21).

It remains the most complicated Case 2.3. Here we have three Killing vector fields and we write the equation (4.3) for each of these Killing vector fields separately. As a result, we obtain a system of 24 (proper) partial differential equations, namely

$$\begin{aligned}
uA_v - vA_u + B + C + E &= 0,\\
uB_v - vB_u - A + D + F &= 0,\\
uC_v - vC_u - A + D + G &= 0,\\
uD_v - vD_u - B - C + H &= 0,\\
uE_v - vE_u - A + F + G &= 0,\\
uF_v - vF_u - B - E + H &= 0,\\
uG_v - vG_u - C - E + H &= 0,\\
uH_v - vH_u - D - F - G &= 0,
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
&(1-u^2+v^2)A_v + 2uvA_u + 2vA - 2uB - 2uC - 2uE = 0,\\
&(1-u^2+v^2)B_v + 2uvB_u + 2uA + 2vB - 2uD - 2uF - 2 = 0,\\
&(1-u^2+v^2)C_v + 2uvC_u + 2uA + 2vC - 2uD - 2uG + 2 = 0,\\
&(1-u^2+v^2)D_v + 2uvD_u + 2uB + 2uC + 2vD - 2uH + 2 = 0,\\
&(1-u^2+v^2)E_v + 2uvE_u + 2uA + 2vE - 2uF - 2uG + 2 = 0,\\
&(1-u^2+v^2)F_v + 2uvF_u + 2uB + 2uE + 2vF - 2uH = 0,\\
&(1-u^2+v^2)G_v + 2uvG_u + 2uC + 2uE + 2vG - 2uH = 0,\\
&(1-u^2+v^2)H_v + 2uvH_u + 2uD + 2uF + 2uG + 2vH + 2 = 0,
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
&(1+u^2-v^2)A_u + 2uvA_v + 2uA + 2vB + 2vC + 2vE + 2 = 0,\\
&(1+u^2-v^2)B_u + 2uvB_v - 2vA + 2uB + 2vD + 2vF + 2 = 0,\\
&(1+u^2-v^2)C_u + 2uvC_v - 2vA + 2uC + 2vD + 2vG = 0,\\
&(1+u^2-v^2)D_u + 2uvD_v - 2vB - 2vC + 2uD + 2vH + 2 = 0,\\
&(1+u^2-v^2)E_u + 2uvE_v - 2vA + 2uE + 2vF + 2vG = 0,\\
&(1+u^2-v^2)F_u + 2uvF_v - 2vB - 2vE + 2uF + 2vH + 2 = 0,\\
&(1+u^2-v^2)G_u + 2uvG_v - 2vC - 2vE + 2uG + 2vH - 2 = 0,\\
&(1+u^2-v^2)H_u + 2uvH_v - 2vD - 2vF - 2vG + 2uH = 0.
\end{aligned} \tag{4.31}$$

We shall solve first the system (4.29)-(4.31) as a system of *linear algebraic equations* for the unknowns $A, B, ..., H$ and their first partial derivatives. We obtain a unique solution

$$\begin{aligned}
&A = \tfrac{-2u}{1+u^2+v^2},\ A_u = \tfrac{-2(1-u^2+v^2)}{(1+u^2+v^2)^2},\ A_v = \tfrac{4uv}{(1+u^2+v^2)^2},\\
&B = \tfrac{2v}{1+u^2+v^2},\ B_u = \tfrac{-4uv}{(1+u^2+v^2)^2},\ B_v = \tfrac{2(1+u^2-v^2)}{(1+u^2+v^2)^2},\\
&C = \tfrac{-2v}{1+u^2+v^2},\ C_u = \tfrac{4uv}{(1+u^2+v^2)^2},\ C_v = \tfrac{-2(1+u^2-v^2)}{(1+u^2+v^2)^2},\\
&D = \tfrac{-2u}{1+u^2+v^2},\ D_u = \tfrac{-2(1-u^2+v^2)}{(1+u^2+v^2)^2},\ D_v = \tfrac{4uv}{(1+u^2+v^2)^2},\\
&E = \tfrac{-2v}{1+u^2+v^2},\ E_u = \tfrac{4uv}{(1+u^2+v^2)^2},\ E_v = \tfrac{-2(1+u^2-v^2)}{(1+u^2+v^2)^2},
\end{aligned} \tag{4.32}$$

$$F = \tfrac{-2u}{1+u^2+v^2},\ F_u = \tfrac{-2(1-u^2+v^2)}{(1+u^2+v^2)^2},\ F_v = \tfrac{4uv}{(1+u^2+v^2)^2},$$
$$G = \tfrac{2u}{1+u^2+v^2},\ G_u = \tfrac{2(1-u^2+v^2)}{(1+u^2+v^2)^2},\ G_v = \tfrac{-4uv}{(1+u^2+v^2)^2},$$
$$H = \tfrac{-2v}{1+u^2+v^2},\ H_u = \tfrac{4uv}{(1+u^2+v^2)^2},\ H_v = \tfrac{-2(1+u^2-v^2)}{(1+u^2+v^2)^2},$$

But hence we easily see that the expressions for $A, B, ..., H$ are unique solutions of (4.29)-(4.31) as a system of PDE. We can write briefly (4.22).

Finally, the statements about the properties of each connection are straightforward computations using definitions and the system (4.4).

This finishes the proof.

Now, we shall investigate, in each case, if the corresponding connections are of type A or of type B, respectively.

Proposition 4.2.17. *The connections corresponding to the Lie algebras from the cases 1.1, 1.2 and 2.2 are of type B but not of type A.*

Proof. The Lie algebras described in the cases 1.1, 1.2 and 2.2 of the refined Olver list are isomorphic to $\mathfrak{sl}(2)$, which is a simple Lie algebra. It does not admit any subalgebra $\operatorname{span}(X, Y)$ such that $[X, Y] = 0$. Thus, by Lema 4.2.2, the corresponding connections are not of type A.

Obviously, the connection given by the formula (4.21) (Case 2.2) is of type B. Further it is easy to check that the assumption (i) of Lemma 4.2.3 is satisfied for the algebra $\mathfrak{sl}(2)$ occurring in the cases 1.1 and 1.2 of the refined Olver list. Thus the corresponding connections are of type B. In particular, for the explicit illustration, we only need to use in both cases 1.1 and 1.2 the change of the coordinates $u' = \frac{1}{u}$, $v' = v$. Then the Christoffel symbols become of type B.

Proposition 4.2.18. *The connection corresponding to the Lie algebra from Case 1.3 (and given by* (4.20)*) is of both types A and B.*

Proof. Obviously, the connection ∇ given by the formula (4.20) is of type B. Because $T = 0$ and $Ric = 0$, it is known that, in some local coordinates, all Christoffel symbols vanish.

Proposition 4.2.19. *The connection corresponding to the Lie algebra from Case 2.3 is not of type A and also not of type B.*

Proof. The Lie algebra described in Case 1.3 of the refined Olver list is isomorphic to $\mathfrak{so}(3)$. Hence it does not admit any subalgebra $\operatorname{span}(X, Y)$ such that $[X, Y] = 0$, or such that $[X, Y] = X$. By the lemmas 4.2.2 and 4.2.3, the corresponding connections are neither of type A, nor of type B.

4.3 Results about flat connections

Let (M^n, ∇) be an affine manifold. We say that the affine connection ∇ is *flat* if and only if the curvature tensor $\mathcal{R}$ vanishes on M. Moreover, the following result it is well-known. (See [Gol, p. 49]).

Theorem 4.3.1. *Let (M^n, ∇) be an affine manifold. Then, the curvature tensor and the torsion tensor vanish on M if and only if around each point there exists a local coordinate system such that all Christoffel symbols vanish.*

Thus, if $Ric = 0$ and $T = 0$ on (M^2, ∇), all Christoffel symbols can be made constant and the corresponding affine manifolds are of type A. Now, we shall give an example which shows that the last result can not be extended to the flat connections with torsion.

Example 4.3.2. In the particular case a) of Proposition 4.2.16 (Case 1.1) we see that the conditions $Ric = 0$ and $T \neq 0$ give a one-parameter family of connections given by

$$g = -2e^2, \quad c = -2e, \quad e \neq 0. \tag{4.33}$$

We are going to prove that the connections of this family (which are flat and with nonzero torsion) are *not* of type A. It is sufficient to show that these connections are still *corresponding* to the Lie algebra from Case 1.1, or, equivalently, that this algebra is the full Killing algebra for this family of connections.

Under the assumptions (4.33), the system of partial differential equations (4.3) is reduced to

$$\begin{array}{ll}
1) & a_{uu} - \frac{a_u}{2u} - eub_u + \frac{a}{2u^2} = 0, \\
2) & b_{uu} + \frac{3b_u}{2u} = 0, \\
3) & a_{uv} - \frac{a_v}{u} - 2e^2u^3b_u - 2eub_v - 2ea = 0, \\
4) & b_{uv} + \frac{a_u}{2u} + eub_u - \dfrac{a}{2u^2} = 0, \\
5) & a_{uv} - \frac{a_v}{u} - 2e^2u^3b_u + eub_v + ea = 0, \\
6) & b_{uv} + \frac{a_u}{2u} - 2eub_u - \dfrac{a}{2u^2} = 0, \\
7) & a_{vv} + 2e^2u^3a_u - 4e^2u^3b_v - 6e^2u^2a = 0, \\
8) & b_{vv} + \frac{a_v}{u} + 2e^2u^3b_u - eub_v - ea = 0.
\end{array} \tag{4.34}$$

Now we shall solve this system.

First, we subtract the equations 4) and 6) and we obtain that $b_u = 0$. Consequently, $b_{uv} = b_{uu} = 0$.

Secondly, we subtract the equations 3) and 5) and we obtain that $b_v = \frac{-a}{u}$. Hence $b_{vu} = \frac{a}{u^2} - \frac{a_u}{u}$ and $b_{vv} = \frac{-a_v}{u}$. Moreover, the integrability condition $b_{uv} = b_{vu} (= 0)$ gives that $a_u = \frac{a}{u} = -b_v$. As a consequence, the first derivatives of a_u are $a_{uu} = 0$ and $a_{uv} = \frac{a_v}{u}$. Then from the equation 7) we obtain $a_{vv} = 0$.

Hence we see easily that the system of partial differential equations (4.34) can be reduced to the following four conditions:

$$i)\ b_u = 0, \quad ii)\ b_v = \tfrac{-a}{u}, \quad iii)\ a_u = \tfrac{a}{u}, \quad iv)\ a_{vv} = 0. \tag{4.35}$$

Now, from the conditions iii) and i) we obtain that $a(u, v) = uf(v)$ and $b(u, v) = g(v)$, where $f(v)$ and $g(v)$ are arbitrary real functions. Thus the conditions ii) and

iv) become $g'(v)+f(v)=0$ and $uf''(v)=0$, respectively. Hence $f(v)=c_1v+c_2$ and $g(v)=-\frac{c_1}{2}v^2-c_2v+c_3$, where c_1, c_2, c_3 are arbitrary constant parameters. Using the original functions $a(u,v)$, $b(u,v)$, we see that (4.34) is satisfied.

As a conclusion, the locally homogeneous connection (4.18) for $T\neq 0$ and $Ric=0$ admits the following family of Killing vector fields:

$$X=a(u,v)\partial_u+b(u,v)\partial_v=(c_1vu+c_2u)\partial_u+(-\tfrac{c_1}{2}v^2-c_2v+c_3)\partial_v,$$

where c_1, c_2, c_3 are arbitrary constant parameters. Obviously X is a linear combination of the generators of Case 1.1. Then a new Killing vector field does not occur and our 1-parameter family of flat locally homogeneous connections with non-zero torsion is still corresponding to Case 1.1 and hence it is not of type A.

Finally, we shall study the case $T\neq 0$, $\nabla T=0$ and $Ric=0$ on 2-dimensional affine manifolds.

Using the notation of Section 4.2.1 and by a straightforward computation, we obtain the following formulas for vanishing of the covariant derivative of the torsion tensor:

$$\begin{aligned}&DE-FC=0,\\&(C-E)B+A(F-D)=0,\\&(C-E)H+G(F-D)=0.\end{aligned}\tag{4.36}$$

Theorem 4.3.3. *Let ∇ be a locally homogeneous flat affine connection on a 2-dimensional manifold $\mathcal{M}$ such that $T\neq 0$ and $\nabla T=0$. Then, ∇ is of type A.*

Proof. The only cases from the refined Olver list such that the torsion tensor T could be nonzero and the connections corresponding to the Lie algebra are not of type A, are the cases 1.1 and 1.5 b).

In the particular case a) of Theorem 4.2.16 (Case 1.1 of the refined Olver list), the conditions $T=0$ and $\nabla T=0$ are equivalent. This is a straightforward computation using (4.18) and (4.36). Thus, this case is in contradiction with the assumptions of Theorem 4.3.3 and we can omit it.

Now, it remains to start from Case 1.5 b) of the refined Olver list. According to Theorem 4.2.11, part (i) b), all such connections are described by the formula (4.10). Because we shall put additional geometric conditions on these connections, we must admit also occurrence of *non-corresponding* connections to Case 1.5 b) given by the formula (4.10). In particular, we must admit the possibility $C_1=0$. What we are going to show is that the connections satisfying assumptions of Theorem 4.3.3 are *corresponding* either to Case 1.5 c) or to Case 1.6 e'). In both cases, the corresponding Lie algebras give connections of type A.

First, we shall found the family of connections such that $T\neq 0$, $\nabla T=0$ and $Ric=0$. From Theorem 4.2.11 b) we know that $T\neq 0$ if and only if $C_3\neq C_5$ or

$C_4 \neq C_6$. From (4.10) and (4.36) we know that $\nabla T = 0$ if and only if

$$\begin{aligned} i)&C_3C_6 - C_4C_5 = 0, \\ ii)&C_2(-C_3 + C_5) + C_1(C_4 - C_6) = 0, \\ iii)&(C_3 - C_5 + 2(C_3C_6 - C_4C_5)) = 0, \\ iv)&(C_4 - C_6)C_7 - (C_3 + C_5)C_8 = 0. \end{aligned} \tag{4.37}$$

From (4.10) and (4.4) we know that $Ric = 0$ if and only if

$$\begin{aligned} v)&(C_2 - C_3)C_5 + C_1(1 - C_6 + C_7) = 0, \\ vi)&C_3 + C_5 + C_4C_5 - C_1C_8 = 0, \\ vii)&-C_2 + C_5 + C_4C_5 - C_1C_8 = 0, \\ viii)&-1 - C_4 - C_6 - C_4C_6 + C_7 + C_4C_7 + C_2C_8 - C_3C_8 = 0. \end{aligned} \tag{4.38}$$

Substituting $i)$ in $iii)$ we obtain $C_3 = C_5$. Then, the condition $\nabla T = 0$ and the equation $i)$ give $C_4 \neq C_6$ and $C_3 = C_5 = 0$. Consequently, from $ii)$, $iv)$ and $vii)$ we obtain $C_1 = C_2 = C_7 = 0$. Now, the equation $viii)$ gives $C_4 = -1$ (or, equivalently, $C_6 = -1$). Therefore, we obtain the following subfamily of connections:

$$\begin{aligned} A(u) = B(u) = D(u) = F(u) = H(u) = 0, \\ C(u) = -1, \quad E(u) = C_6, \quad G(u) = -C_6u + C_8, \end{aligned} \tag{4.39}$$

where C_6, C_8 are constant parameters and $C_6 \neq -1$.

If now $C_6 = 0$, it is clear that the Christoffel symbols are the same as in the formula (4.12), with the additional conditions $c_4 = c_3 = c_2 = 0$, $c_1 = -1/2$, which is a special subcase of the case (ii) c) in Theorem 4.2.11, i.e., a special subcase of Case 1.5 c) of the refined Olver list. These connections are of type A.

Finally, if $C_6 \neq 0$ we change the coordinate system in the following way: $u' = u - \frac{C_8}{C_6}$, $v' = v$. Then, in the new coordinate system, which we denote again by (u, v), we have still the same generators $\{\partial_v, \mathrm{e}^v\partial_u\}$. Moreover, the Christoffel symbols (4.39) get the new form

$$\begin{aligned} A(u) = B(u) = D(u) = F(u) = H(u) = 0, \\ C(u) = -1, \quad E(u) = C_6, \quad G(u) = -C_6u, \end{aligned} \tag{4.40}$$

where $C_6 \in \mathbb{R} \setminus \{0, -1\}$. Now, it is easy to check from (4.3) that the connection given by (4.40) has two additional Killing vector fields $u\partial_u$ and $\mathrm{e}^{-vC_6}\partial_u$. Thus we found the equivalence with Case 1.6 e') of the refined Olver list. This concludes the proof.

4.4 Olver's tables

According to [O], taking into account the comments in page 61, the classification of all *transitive* Lie algebras of vector fields in $\mathbb{R}^2$ is given by Table 1 and Table

6 ([O], pages 472 and 475, respectively). (This result belongs originally to Sophus Lie). We present the Olver's tables with a slight modification, denoting them as Table 1 and Table 2.

	Generators	Dim	Structure
1.1	$\partial_v,\ v\partial_v - u\partial_u,\ v^2\partial_v - 2uv\partial_u$	3	$\mathfrak{sl}(2)$
1.2	$\partial_v,\ v\partial_v - u\partial_u,\ v^2\partial_v - (2uv+1)\partial_u$	3	$\mathfrak{sl}(2)$
1.3	$\partial_v,\ v\partial_v,\ u\partial_u,\ v^2\partial_v - uv\partial_u$	4	$\mathfrak{gl}(2)$
1.4	$\partial_v,\ v\partial_v,\ v^2\partial_v,\ \partial_u,\ u\partial_u,\ u^2\partial_u$	6	$\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$
1.5	$\partial_v,\ \eta_1(v)\partial_u,\ \ldots,\ \eta_k(v)\partial_u$	$k+1$	$\mathbb{R}\ltimes\mathbb{R}^k$
1.6	$\partial_v,\ u\partial_u,\ \eta_1(v)\partial_u,\ \ldots,\ \eta_k(v)\partial_u$	$k+2$	$\mathbb{R}^2\ltimes\mathbb{R}^k$
1.7	$\partial_v,\ v\partial_v+\alpha u\partial_u,\ \partial_u,\ v\partial_u,\ldots,\ v^{k-1}\partial_u$	$k+2$	$\mathfrak{a}(1)\ltimes\mathbb{R}^k$
1.8	$\partial_v,\ v\partial_v+(ku+v^k)\partial_u,\ \partial_u,\ v\partial_u,\ldots,\ v^{k-1}\partial_u$	$k+2$	$\mathfrak{a}(1)\ltimes\mathbb{R}^k$
1.9	$\partial_v,\ v\partial_v,\ u\partial_u,\ \partial_u,\ v\partial_u,\ldots,\ v^{k-1}\partial_u$	$k+3$	$\mathfrak{c}(1)\ltimes\mathbb{R}^k$
1.10	$\partial_v,\ 2v\partial_v+(k-1)u\partial_u,\ v^2\partial_v+(k-1)uv\partial_u,$ $\partial_u,\ v\partial_u,\ldots,\ v^{k-1}\partial_u$	$k+3$	$\mathfrak{sl}(2)\ltimes\mathbb{R}^k$
1.11	$\partial_v,\ v\partial_v,\ v^2\partial_v+(k-1)uv\partial_u,\ u\partial_u,$ $\partial_u,\ v\partial_u,\ldots,\ v^{k-1}\partial_u$	$k+4$	$\mathfrak{gl}(2)\ltimes\mathbb{R}^k$

TABLE 1. Transitive, Imprimitive Lie Algebras of Vector Fields in $\mathbb{R}^2$.

Remark 4.4.1 (from p.472 of [O]). *Here* $\mathfrak{c}(1)=\mathfrak{a}(1)\oplus\mathbb{R}$.

In Cases 1.5 and 1.6, the functions $\eta_1(v),\ldots,\eta_k(v)$ *satisfy a* k^{th} *order constant coefficient homogeneous linear ordinary differential equation* $\mathcal{D}[u]=0$.

In Cases 1.5-1.11 we require $k\geq 1$. *Note, though, that if we set* $k=0$ *in Case 1.10, and replace u by* u^2*, we obtain Case 1.1. Similarly, if we set* $k=0$ *in Case 1.11, we obtain Case 1.3. Cases 1.7 and 1.8 for* $k=0$ *are equivalent to the Lie algebra span*$\{\partial_v, e^v\partial_u\}$ *of type 1.5. Case 1.9 for* $k=0$ *is equivalent to the Lie algebra span*$\{\partial_v,\partial_u,u\partial_u\}$ *of type 1.6.*

	Generators	Dim	Structure
2.1	$\partial_v,\ \partial_u,\ \alpha(v\partial_v+u\partial_u)+u\partial_v-v\partial_u$	3	$\mathbb{R}\ltimes\mathbb{R}^2$
2.2	$\partial_v,\ v\partial_v+u\partial_u,\ (v^2-u^2)\partial_v+2uv\partial_u$	3	$\mathfrak{sl}(2)$
2.3	$u\partial_v-v\partial_u,\ (1+v^2-u^2)\partial_v+2uv\partial_u,$ $2uv\partial_v+(1-v^2+u^2)\partial_u$	3	$\mathfrak{so}(3)$
2.4	$\partial_v,\ \partial_u,\ v\partial_v+u\partial_u,\ u\partial_v-v\partial_u$	4	$\mathbb{R}^2\ltimes\mathbb{R}^2$
2.5	$\partial_v,\ \partial_u,\ v\partial_v-u\partial_u,\ u\partial_v,\ v\partial_u$	5	$\mathfrak{sa}(2)$
2.6	$\partial_v,\ \partial_u,\ v\partial_v,\ u\partial_v,\ v\partial_u,\ u\partial_u$	6	$\mathfrak{a}(2)$
2.7	$\partial_v,\ \partial_u,\ v\partial_v+u\partial_u,\ u\partial_v-v\partial_u,$ $(v^2-u^2)\partial_v+2uv\partial_u,\ 2uv\partial_v+(u^2-v^2)\partial_u$	6	$\mathfrak{so}(3,1)$
2.8	$\partial_v,\ \partial_u,\ v\partial_v,\ u\partial_v,\ v\partial_u,\ u\partial_u,$ $v^2\partial_v+uv\partial_u,\ uv\partial_v+u^2\partial_u$	8	$\mathfrak{sl}(3)$

TABLE 2. Primitive Lie Algebras of Vector Fields in $\mathbb{R}^2$.

Resumen

Hoy en día, el concepto de *homogeneidad* es una noción fundamental en geometría aunque su significado debe ser especificado para cada situación en concreto.

A lo largo de esta memoria, se consideran dos tipos bien diferenciados de homogeneidad: la de las variedades riemannianas (M, g) y la de las variedades afines (M, ∇). El primer tipo de homogeneidad se define como aquel que tiene la propiedad de que el grupo de isometrías $I(M)$ actúa transitivamente sobre M. La Parte I, recoge todos los resultados que hemos obtenido en esta dirección. Sin embargo, en la Parte II se presentan los resultados obtenidos sobre conexiones afines homogéneas. Una conexión afín se dice homogénea si para cada par de puntos de la variedad M existe un difeomorfismo afín que envía un punto en otro. En este caso, también se considera una versión local de homogeneidad. Así, se admite que los difeomorfismos afines sean definidos sólo localmente, es decir; de un entorno en otro.

Más específicamente, la Parte I de esta tesis está dedicada a resolver el problema de comprobar si las familias de dimensión seis y doce de las variedades bandera 3-paramétricas, $(M^6 = SU(3)/SU(1) \times SU(1) \times SU(1), g_{(c_1,c_2,c_3)})$, $(M^{12} = Sp(3)/SU(2) \times SU(2) \times SU(2), g_{(c_1,c_2,c_3)})$, construidas por N. R. Wallach en [W] son espacios de D'Atri. Por tanto, se mejoran los resultados presentados en [AM-N]. Más aún, en el segundo Capítulo, se obtiene la clasificación completa de los espacios homogéneos de tipo $\mathcal{A}$ cuatro dimensionales que permite probar correctamente que todo espacio de D'Atri homogéneo de dimensión cuatro es naturalmente reductivo. Finalmente, en el tercer Capítulo se prueba que en cualquier g.o. espacio el operador de Jacobi a lo largo de una geodésica tiene rango osculador constante y, como consecuencia, se presenta un método para resolver la ecuación de Jacobi sobre cualquier g.o. espacio. Este método extiende de una manera natural el método propuesto, sólamente para espacios naturalmente reductivos, por A. M. Naveira y A. Tarrío en [N-T].

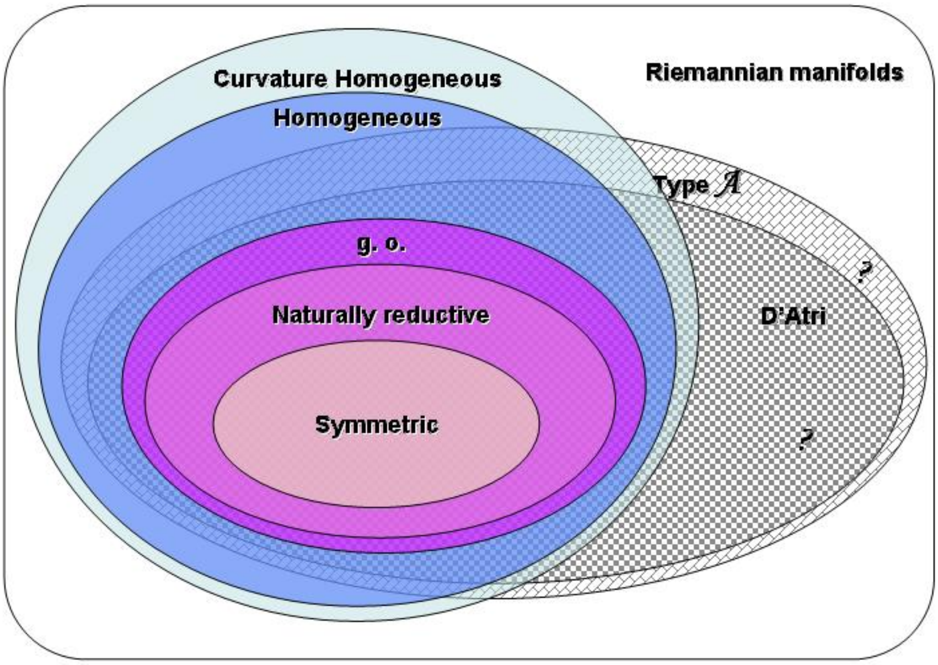
La Parte II se destina a clasificar (localmente) todas las conexiones afines localmente homogéneas con torsión arbitraria sobre variedades 2-dimensionales. Por tanto, se generaliza el resultado dado por B. Opozda para el caso sin torsión en [Op.3]. Para finalizar el cuarto Capítulo, se prueban algunos resultados interesantes sobre conexiones llanas con torsión.

En general, el estudio de estos problemas requiere a veces, un gran número de cálculos simbólicos aunque sencillos. En dichas ocasiones, realizarlos correctamente a mano es una tarea ardua que requiere mucho tiempo. Por ello, se intenta or-

ganizar todos estos cálculos de la manera más sistemática posible de forma que el procedimiento no resulte excesivamente largo. Este tipo de investigación es ideal para utilizar la ayuda del ordenador; así, cuando resulte conveniente, utilizaremos la ayuda del software MATHEMATICA© para desarrollar con total transparencia el método de resolución que más se adecua a cada uno de los problemas a resolver.

Parte I. Variedades riemannianas homogéneas

El Capítulo 1 de esta tesis denominado "Preliminares" está dedicado a realizar un breve revisión de algunos tipos especiales de variedades riemannianas homogéneas que son de especial relevancia a lo largo de todo este estudio. En particular, se recoge todo el material necesario, aunque sin demostraciones, para comprender el siguiente diagrama:



Además, el objetivo de la última sección de este Capítulo, la Sección 1.4, es mostrar como la utilización del software MATHEMATICA 5.2 puede resultar muy útil. Para ello, se termina (ya que ahora es posible realizarlo en un tiempo factible) el estudio comenzado por J. E. D'Atri y H. K. Nickerson en [D'A-N.2] relativo a la familia seis dimensional de variedades bandera 3-paramétrica en el espacio complejo, $(M^6, g_{(c_1,c_2,c_3)})$. Más concretamente, J. E. D'Atri y H. K. Nickerson probaron que, si al menos dos de los parámetros c_1, c_2, c_3 son iguales, la primera condición de Ledger L_3 era satisfecha en la correspondiente variedad bandera. Además, para el caso concreto $c_1 = c_2 = 1$, $c_3 = 2$ afirmaron (sin mostrar ningún argumento) que la segunda condición de Ledger L_5 no se satisfacía. Ahora, se rehace su estudio y se concluye éste, probando que la condición L_5 se satisface si y sólo si $c_1 = c_2 = c_3$. Lo que equivale a afirmar

"*cada miembro de la familia 3-paramétrica de variedades bandera* $(M^6, g_{(c_1,c_2,c_3)})$ *es un espacio de D'Atri si y sólo si, es un espacio naturalmente reductivo*".

Además, siguiendo el mismo método se amplia nuestro estudio a la familia 3-paramétrica de variedades bandera de dimensión doce en el espacio cuaterniónico, $(M^{12}, g_{(c_1,c_2,c_3)})$, concluyendo análogamente que

"*cada miembro de la familia 3-paramétrica de variedades bandera* $(M^{12}, g_{(c_1,c_2,c_3)})$ *es un espacio de D'Atri si y sólo si, es un espacio naturalmente reductivo*".

Es interesante recordar que la propiedad de ser un *espacio de D'Atri* (es decir; un espacio cuyas simetrías conservan el volumen) es equivalente a que se satisfagan un número infinito de identidades curvatura, denominadas condiciones de Ledger de orden impar, L_{2k+1}, $k \geq 1$. En particular, una variedad riemanniana (M, g) que satisface la primera condición de Ledger L_3 se denomina *de tipo* $\mathcal{A}$.

La clasificación de los espacios de D'Atri de dimensión tres es bien conocida; todos ellos son naturalmente reductivos, véase [K]. Sin embargo, el primer intento de clasificar los espacios de D'Atri *homogéneos* de dimensión cuatro fué realizado por F. Podestà, A. Spiro, P. Bueken y L. Vanhecke en [Po-Sp] y [Bu-V], los cuales son mutuamente complementarios. Estos autores comenzaron clasificando los espacios de tipo $\mathcal{A}$, pero la clasificación dada en [Po-Sp], asumiendo como mucho que tres de los valores propios del tensor de Ricci son distintos y el espacio es "curvature homogeneous", es incompleta, como se ha probado en [AM]. Además, la familia 2-paramétrica de espacios con cuatro valores propios del tensor de Ricci distintos dada en [Bu-V], que complementa la anterior clasificación sólo en el caso homogéneo, presenta los resultados de una manera no explícita y sin ninguna interpretación geométrica.

Por ello, el segundo Capítulo de esta tesis está dedicado a obtener de una forma simple y explícita la clasificación completa de los espacios homogéneos de tipo $\mathcal{A}$. En particular, se ha obtenido el resultado siguiente, denominado "Theorem 2.1.2" a lo largo de toda la memoria.

Teorema de Clasificación. *Sea* (M, g) *una variedad riemanniana homogénea de dimensión cuatro de tipo* $\mathcal{A}$. *Entonces, sólo pueden darse uno de los cinco casos siguientes:*

i) M *es localmente simétrica;*

ii) (M, g) *es localmente isométrica a un producto riemanniano* $M^3 \times \mathbb{R}$, *donde* M^3 *es un espacio riemanniano naturalmente reductivo tres dimensional con dos curvaturas de Ricci distintas* $(\rho_1,\ \rho_2 = \rho_1, \rho_3)$, $\rho_3 \neq \rho_1$. *Así,* M *es localmente isométrica a un espacio homogéneo naturalmente reductivo.*

iii) (M, g) *es localmente isométrica a un grupo de Lie simplemente conexo* (G, g_γ), *cuya álgebra de Lie* $\mathfrak{g}$ *se describe por*

$$[e_2, e_1] = e_2, \ [e_1, e_3] = e_3, \ [e_2, e_3] = e_4,$$

$$[e_1, e_4] = [e_2, e_4] = [e_3, e_4] = 0,$$

y que está equipada con la métrica invariante a izquierda dada por

$$g_\gamma = \frac{4}{\gamma^2} w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3 + \gamma^2 w^4 \otimes w^4,$$

donde $\gamma \in \mathbb{R}^+$ *y* $\{w^i\}$ *es la base dual de* $\{e_i\}$. *En particular, los valores propios del tensor de Ricci asociados a las métricas* g_γ *son* $\rho_1 = \rho_2 = \rho_3 = \frac{-\gamma^2}{2}$, $\rho_4 = \frac{\gamma^2}{2}$, *los cuales no son isométricos entre si para cualquier valor de* γ. *Además, las variedades riemannianas* (G, g_γ) *son irreducibles y no son localmente simétricas. Más aún, no son espacios de D'Atri.*

iv) (M, g) *es una variedad localmente isométrica a un grupo de Lie simplemente conexo* $(G, g_{(c,k)})$, *cuya álgebra de Lie* $\mathfrak{g}$ *se describe por*

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = \tfrac{A_+}{4} e_2, \quad [e_2, e_3] = \tfrac{A_-}{4} e_1,$$

$$[e_1, e_4] = 0, \quad [e_2, e_4] = 0, \quad [e_3, e_4] = 0,$$

donde $A_\pm = 3 - 3k^2 \pm \sqrt{1 + 2k^2 - 3k^4} \geq 0$, $k \in]0, 1[\setminus \{\sqrt{\frac{5}{21}}\}$, *y está equipada con la métrica invariante a izquierda*

$$g_{(c,k)} = \frac{1}{c^2}(w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3 + k w^3 \otimes w^4 + w^4 \otimes w^4),$$

donde $\{w^i\}$ *es la base dual de* $\{e_i\}$ *y* $c \in \mathbb{R}^+$ *es otro parámetro. Las métricas* $g_{(c,k)}$ *tienen los cuatro valores propios del tensor de Ricci distintos, que están dados por:*

$$\rho_1 = \tfrac{c^2}{8}(2 - 6k^2 - \sqrt{1 + 2k^2 - 3k^4}), \qquad \rho_2 = \tfrac{c^2}{8}(2 - 6k^2 + \sqrt{1 + 2k^2 - 3k^4}),$$
$$\rho_3 = \tfrac{c^2}{16}(3 - 3k^2 - \sqrt{9 - 2k^2 + 57k^4}), \quad \rho_4 = \tfrac{c^2}{16}(3 - 3k^2 + \sqrt{9 - 2k^2 + 57k^4}).$$

Además, las variedades riemannianas $(G, g_{(c,k)})$ *son irreducibles y no son localmente simétricas. Más aún, no son espacios de D'Atri.*

v) (M, g) *es localmente isométrica a un grupo de Lie simplemente conexo* (G, g_c), *cuya álgebra de Lie* $\mathfrak{g}$ *se describe por*

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = \tfrac{6}{7} e_2, \quad [e_2, e_3] = \tfrac{2}{7} e_1,$$

$$[e_1, e_4] = 0, \quad [e_2, e_4] = 0, \quad [e_3, e_4] = 0,$$

y está equipada con la métrica invariante a izquierda

$$g_c = \frac{1}{c^2}(w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3 + \sqrt{\tfrac{5}{21}} w^3 \otimes w^4 + w^4 \otimes w^4),$$

donde $c \in \mathbb{R}^+$ *y* $\{w^i\}$ *es la base dual de* $\{e_i\}$. *En particular, los valores propios del tensor de Ricci asociados a las métricas* g_c *son* $\rho_1 = \rho_3 = \frac{-c^2}{14}$, $\rho_2 = \frac{3c^2}{14}$, $\rho_4 = \frac{5c^2}{14}$, *los cuales no son isométricos entre si para cualquier valor de* c. *Además, las variedades riemannianas* (G, g_c) *son irreducibles y no son localmente simétricas. Más aún, no son espacios de D'Atri.*

Como consecuencia, también se prueba correctamente que

"todos los espacios de D'Atri homogéneos de dimensión cuatro son, localmente, espacios naturalmente reductivos."

El último apartado de este Capítulo (Sección 2.3) se concentra en presentar la corrección del teorema de clasificación de F. Podesta y A. Spiro dado en [Po-Sp]. Finalmente, cabe para ello resaltar que bajo la suposición de que como mucho tres de los valores propios del tensor de Ricci sean distintos, esta nueva versión del teorema de clasificación es más fuerte que nuestro "Theorem 2.1.2", ya que la condición de homogeneidad es reemplazada por la condición de "curvature homogeneous".

Hasta ahora, en estos primeros capítulos de la Parte I, se han obtenido propiedades geométricas de la variedad, utilizando el operador curvatura y sus derivadas a través de las denominadas condiciones de Ledger de orden impar. Sin embargo, en el tercer y último Capítulo de esta primera parte, se obtendrán propiedades del operador curvatura y sus derivadas, utilizando para ello algunas de las propiedades geométricas de la variedad.

Una *g.o. variedad riemanniana* es una variedad riemanniana homogénea (M, g) en la cual cada geodésica es una órbita de un grupo 1-paramétrico de isometrías. Además, es bien conocido que toda g.o. variedad riemanniana de dimensión$\leq$5 es naturalmente reductiva. El primer contraejemplo de g.o. variedad riemanniana que no es naturalmente reductiva fue dado por A. Kaplan in [Ka.2]. En concreto, este espacio es una nilvariedad riemanniana seis dimensional con centro de dimensión dos, uno de los denominados grupos de Heisenberg generalizados o grupos de tipo H.

Es bien sabido que resolver la ecuación de Jacobi sobre una variedad riemanniana puede resultar una tarea ardua. En el espacio euclideo, la solución es trivial. Para los espacios simétricos, el problema se reduce a un sistema de ecuaciones diferenciales ordinarias con coeficientes constantes. Sin embargo, en [Ch.1] y [Ch.2], I. Chavel obtuvo solamente una solución parcial de este problema al intentar abordarlo sobre las variedades naturalmente reductivas con métrica normal $V_1 = Sp(2)/SU(2)$ y $V_2 = SU(5)/(Sp(2) \times S^1)$. El método utilizado por I. Chavel que está basado en el uso de la conexión canónica, le permitió resolver la ecuación de Jacobi solamente en algunas direcciones particulares de la geodésica. Utilizando también un método basado en el uso de la conexión canónica, W. Ziller en [Z] resuelve la ecuación de Jacobi sobre espacios homogéneos naturalmente reductivos compactos. Pero la solución obtenida por W. Ziller puede ser considerada de tipo cualitativo ya que no permite obtener fácilmente soluciones explícitas de los campos de Jacobi para un ejemplo concreto o una dirección arbitraria de la geodésica.

El método utilizado por I. Chavel y W. Ziller para resolver la ecuación de Jacobi es un caso especial del procedimiento siguiente ("Lemma 3.1.1" del Capítulo 3 de la tesis), válido para cualquier g.o. espacio y cualquier grupo de Heisenberg generalizado (ver [B-Tr-V, pág. 51]):

Lema. *Sea M una variedad riemanniana, ∇ la conexión de Levi Civita de M y $\gamma : I \to M$ una geodésica en M parametrizada por la longitud de arco tal que $\dot\gamma$ denota el campo vectorial tangente estándar unitario sobre I. Se supone además, que a lo largo de γ existe un campo tensorial T_γ antisimétrico y $\nabla_{\dot\gamma}$-paralelo tal que el operador de Jacobi a lo largo de γ, $\mathcal{R}_\gamma$, satisface $\nabla_{\dot\gamma}\mathcal{R}_\gamma := \mathcal{R}'_\gamma = [\mathcal{R}_\gamma, T_\gamma]$. Por tanto, este define una nueva derivada covariante*

$$\bar\nabla_{\dot\gamma} := \nabla_{\dot\gamma} + T_\gamma,$$

y poniendo

$$\bar{\mathcal{R}}_\gamma := \mathcal{R}_\gamma + T_\gamma^2.$$

Entonces, $\mathcal{R}_\gamma$, $\bar{\mathcal{R}}_\gamma$ y T_γ son $\bar\nabla_{\dot\gamma}$-paralelos a lo largo de γ y la ecuación de Jacobi a lo largo de γ es

$$\bar\nabla_{\dot\gamma}\bar\nabla_{\dot\gamma}B - 2T_\gamma\bar\nabla_{\dot\gamma}B + \bar{\mathcal{R}}_\gamma B = 0,$$

donde B es un campo vectorial a lo largo de la geodésica γ.

Por otro lado, K. Tsukada en [Ts] propuso un nuevo criterio para comprobar la existencia de subvariedades totalmente geodésicas en espacios naturalmente reductivos. Este criterio está basado en el estudio del tensor curvatura y un número finito de sus derivadas (con respecto a la conexión de Levi-Civita). En particular, para probar este resultado utilizó dos fórmulas probadas, exclusivamente para espacios naturalmente reductivos, por K. Tojo en [To]. Con ellas, K. Tsukada concluyó que el tensor curvatura puede ser considerado como una curva sobre el espacio de los tensores curvatura sobre $\mathfrak{m}$. Más aún, como consecuencia pudo asegurar que en todo espacio naturalmente reductivo el tensor curvatura tiene *rango osculador constante* $r \in \mathbb{N}$. Es decir; que las derivadas covariantes del tensor curvatura de la primera a la r-ésima son linealmente independientes y, sin embargo, de la primera a la $(r+1)$-ésima son linealmente dependientes.

Este resultado ha sido utilizado por A. M. Naveira y A. Tarrío en [N-T] para obtener un nuevo método de resolución de la ecuación de Jacobi, $Y'' + \mathcal{R}_\gamma(Y) = 0$, sobre el espacio naturalmente reductivo $Sp(2)/SU(2)$. Dada la generalidad del método utilizado, los autores conjeturaron que éste método podría ser utilizado para resolver dicha ecuación en otros muchos ejemplos de espacios naturalmente reductivos. De hecho, su conjetura parece ser acertada ya que este método ha sido aplicado satisfactoriamente a la variedad bandera $U(3)/(U(1) \times U(1) \times U(1))$ en [AM-Ba].

En el Capítulo 3, se ha probado que el método desarrollado en [N-T] sobre espacios naturalmente reductivos también puede aplicarse sobre todo g.o. espacio. Además, se resuelve la ecuación de Jacobi a la largo de una geodésica unitaria de dirección arbitraria en el ejemplo de Kaplan N.

En particular, en la Sección 3.2 se prueba nuestro resultado principal ("Theorem 3.2.4" del Capítulo 3):

> *"El operador de Jacobi en un g.o. espacio siempre tiene rango osculador constante."*

Después, en la Sección 3.3, se presenta una aplicación de este resultado: la resolución de la ecuación de Jacobi a la largo de una geodésica unitaria de dirección arbitraria en el ejemplo de Kaplan N. Más específicamente, se comienza recordando conceptos y resultados básicos sobre "grupos de Heisenberg generalizados" ó "grupos de tipo H". Además, se obtiene una fórmula recursiva para el cálculo de la n-ésima derivada covariante del operador de Jacobi en el origen para este tipo de espacios. Con ella, se calcula el rango osculador constante del operador de Jacobi $\mathcal{J}$ ("Lemma 3.3.6" y "Theorem 3.3.9"). En particular, se obtiene que es 4 ya que la relación básica satisfecha entre las cinco primeras derivadas covariantes del operador de Jacobi de tipo $(0,4)$ a lo largo de una geodésica arbitraria γ con vector inicial x en el origen $p = \gamma_0$ de N es

$$\tfrac{1}{4}|\dot\gamma_0|^4 \mathcal{J}_0^{1)} + \tfrac{5}{4}|\dot\gamma_0|^2 \mathcal{J}_0^{3)} + \mathcal{J}_0^{5)} = 0$$

y el operador de Jacobi puede ser escrito de la forma siguiente:

$$\mathcal{J}_t = c_0 + c_1 \cos(t) + c_2 \sin(t) + c_3 \cos(t/2) + c_4 \sin(t/2),$$

donde,

$$c_0 = \mathcal{J}_0 + 5\mathcal{J}_0^{2)} + 4\mathcal{J}_0^{4)}, \quad c_1 = \tfrac{1}{3}(\mathcal{J}_0^{2)} + 4\mathcal{J}_0^{4)}), \quad c_2 = \tfrac{-1}{3}(\mathcal{J}_0^{1)} + 4\mathcal{J}_0^{3)}),$$
$$c_3 = \tfrac{-16}{3}(\mathcal{J}_0^{2)} + \mathcal{J}_0^{4)}), \quad c_4 = \tfrac{8}{3}(\mathcal{J}_0^{1)} + \mathcal{J}_0^{3)}).$$

Finalmente y como una consecuencia directa, resolvemos la ecuación de Jacobi en el ejemplo de Kaplan utilizando el método propuesto en [N-T]. (Véase "Proposition 3.3.10"). Además, se comprueba (en un caso especial) que nuestro resultado sobre el ejemplo de Kaplan coincide con el resultado obtenido por J. Berndt, F. Tricerri y L. Vanhecke, utilizando el lema anterior ([B-Tr-V, Teorema de la pág. 52]).

Es importante observar que el método utilizado por J. Berndt, F. Tricerri y L. Vanhecke proporciona resultados exactos mientras que este nuevo, utilizado para obtener la "Proposition 3.3.10", da aproximaciones del resultado tanto mejores como mayor sea el número de términos de la serie (3.37) calculados. Sin embargo, con el primer método no siempre es posible obtener resultados tan exactos y precisos como los obtenidos para los grupos de tipo H (recordar los estudios de I. Chavel en [Ch.1], [Ch.2] y W. Ziller en [Z]) mientras que el nuevo método propuesto sobre espacios naturalmente reductivos en [N-T], cuya aplicación ha sido ahora extendida sobre todo g.o. espacio, puede ser siempre aplicado realizando unos sencillos aunque laboriosos cálculos metódicos y además, su solución es válida para una dirección arbitraria.

Parte II. Conexiones afines homogéneas

El campo de la geometría diferencial afín está bien establecido pero a su vez en continuo desarrollo (ver, por ejemplo, [N-S]). Además, son muchos los resultados conocidos sobre grupos de transformaciones afines y campos vectoriales de Killing afines ([Ko-N, vol.I] y [Ko]). Pero es importante remarcar que el problema de clasificar todas las conexiones localmente homogéneas *sin torsión* en el plano ha sido resuelto recientemente por B. Opozda en [Op.3] (utilizando un método directo) y por O. Kowalski, B. Opozda, Z. Vlášek en [K-Op-Vl.4] (utilizando la teoría de grupos), pero sin ninguna relación entre ambas clasificaciones. Resultados parciales previos a ambas fueron publicados en [K-Op-Vl.1] y [K-Op-Vl.2].

Las variedades riemannianas localmente homogéneas de dimensión dos son aquellas que tienen curvatura constante. Sin embargo, hay muchas estructuras afines localmente homogéneas sobre variedades de dimensión dos. Por ello, clasificar las conexiones afines homogéneas sobre el plano no es una tarea sencilla y resolver este problema en el caso tres dimensional parece ser un duro problema a resolver, todavía abierto.

Por todo ello, el cuarto Capítulo de esta tesis está dedicado a clasificar las conexiones afines localmente homogéneas *con torsión arbitraria* en el plano. El resultado obtenido, denominado “Theorem 4.1.2”, es el siguiente:

Teorema de Clasificación. *Sea ∇ una conexión afín localmente homogénea con torsión arbitraria sobre una variedad de dimensión dos $\mathcal{M}$. Entonces, ó bien ∇ es, localmente, la conexión de Levi-Civita sobre la esfera unidad ó bien, en un entorno $\mathcal{U}$ de cada punto $m \in \mathcal{M}$, hay un sistema de coordenadas locales (u, v) y constantes p, q, c, d, e, f, r, s tales que ∇ se expresa en $\mathcal{U}$ mediante una de las fórmulas siguientes:*

Tipo A

$$\nabla_{\partial_u}\partial_u = p\partial_u + q\partial_v, \quad \nabla_{\partial_u}\partial_v = c\partial_u + d\partial_v,$$
$$\nabla_{\partial_v}\partial_u = r\partial_u + s\partial_v, \quad \nabla_{\partial_v}\partial_v = e\partial_u + f\partial_v.$$

Tipo B

$$\nabla_{\partial_u}\partial_u = \tfrac{p\partial_u+q\partial_v}{u}, \quad \nabla_{\partial_u}\partial_v = \tfrac{c\partial_u+d\partial_v}{u},$$
$$\nabla_{\partial_v}\partial_u = \tfrac{r\partial_u+s\partial_v}{u}, \quad \nabla_{\partial_v}\partial_v = \tfrac{e\partial_u+f\partial_v}{u},$$

donde p, q, c, d, e, f, r, s no se anulan simultáneamente.

Este teorema ha sido obtenido desde el punto de vista de la teoría de grupos. Ello quiere decir que se comienza fijando una álgebra de Lie transitiva $\mathfrak{g}$ de campos vectoriales de la lista dada por P. J. Olver [O] (Sección 4.4) y se buscan todas las conexiones afines con torsión arbitraria para las cuales, en el mismo dominio y con respecto al mismo sistema de coordenadas, $\mathfrak{g}$ es una álgebra de campos vectoriales de Killing *completa*. En dicho caso, también se dice que dichas conexiones son *correspondientes* a $\mathfrak{g}$. Además, ocurre bastante a menudo, que el álgebra de Lie de campos vectoriales fijada no admite ninguna conexión afín ó solamente admite conexiones afines invariantes sin torsión.

Por otra parte, se prueban unos sencillos lemas, ("Lemma 4.2.2" y "Lemma 4.2.3"), que nos permiten decidir fácilmente si una conexión correspondiente al álgebra de Lie $\mathfrak{g}$ tiene, en algún sistema de coordenadas locales (u', v'), símbolos de Christoffel de tipo A, o de tipo B, respectivamente. En dicho caso, se dice, para abreviar, que dicha conexión es de tipo A, o de tipo B, respectivamente. Además, gracias a estos lemas, todo el procedimiento de la clasificación depende solamente de la estructura del álgebra $\mathfrak{g}$.

Más aún, los cálculos realizados a lo largo de la Sección 4.2 ilustran la relación esencial existente entre las clasificaciones dadas en [K-Op-Vl.4] y [Op.3], y prueban que todas las conexiones correspondientes a *algunas* álgebras de Lie $\mathfrak{g}$ son simultáneamente de tipo A y de tipo B. Todos estos hechos pueden ser fácilmente comprobados en la tabla de la sección 4.1, donde se recogen de una forma organizada todos los resultados obtenidos.

En la Sección 4.3 se obtienen consecuencias interesantes sobre conexiones llanas con torsión. En particular, se da un ejemplo ("Example 4.3.2") que muestra que el conocido resultado *"En una variedad afín, el tensor curvatura R y el tensor torsión T se anulan si y sólo si en un entorno de cada punto de la variedad existe un sistema de coordenadas local tal que todos los símbolos de Christoffel se anulan"*, no es cierto en el caso $T \neq 0$. Además, se prueba el resultado siguiente ("Theorem 4.3.3"):

"*toda conexión afín localmente homogénea y llana ($R = 0$) sobre una variedad de dimensión dos tal que $T \neq 0$ y $\nabla T = 0$, es de tipo A.*"

Para finalizar, parece interesante mencionar que nuestro teorema de clasificación ya ha sido utilizado en [K-Vl], dónde O. Kowalski y Z. Vlášek resuelven los principales problemas de existencia en dimensión dos, relativos a geodésicas afines homogéneas.

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