DEPARTAMENTO DE ANÁLISIS MATEMÁTICO

SOBRE ESPACIOS Y ÁLGEBRAS DE FUNCIONES HOLOMORFAS.

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TESIS DOCTORAL
presentada por

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Valencia, 2001

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Chapter 1

Introducción

En esta memoria se tratan diversos problemas sobre holomorfía en infinitas dimensiones. La memoria se divide en tres capítulos, en cada uno de los cuales se aborda un problema distinto. En esta introducción se pretende dar aquellas definiciones y notaciones que son utilizados a lo largo de toda la memoria, así como un resumen amplio de los resultados obtenidos. De cada capítulo damos a continuación una breve introducción al problema tratado.

1.1 Definiciones básicas y notación

A lo largo de toda la memoria vamos a trabajar con espacios vectoriales localmente convexos, que a veces serán espacios de Banach. En cada momento se establecerá explícitamente de qué clase de espacios se está tratando y la notación utilizada. Vamos a fijar aquí los conceptos y la notación que serán utilizados a lo largo de toda memoria. Las nociones que se utilicen en cada capítulo serán establecidas cada una en su momento.

Si E y F son dos espacios vectoriales, denotaremos por $\mathcal{L}_a({}^nE;F)$ al espacio de aplicaciones $L : E^n \longrightarrow F$ n-lineales. Éste es un concepto púramente algebraico. Cuando E y F sean espacios localmente convexos, denotaremos por $\mathcal{L}({}^nE;F)$ al espacio de aplicaciones n-lineales y continuas. Cuando $n = 1$ escribiremos símplemente $\mathcal{L}(E;F)$. Si $F = \mathbb{C}$ escribimos $\mathcal{L}({}^nE)$ y si, además, $n = 1$, se denota E'. Una aplicación $L \in \mathcal{L}({}^nE;F)$ se llama *simétrica* si para cualesquiera $x_1, \ldots, x_n \in E$ y cualquier permutación $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ se cumple que

$$L(x_1, \ldots, x_n) = L(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Al subespacio de $\mathcal{L}_a({}^nE;F)$ de aplicaciones simétricas se le denota por $\mathcal{L}_a^s({}^nE;F)$.

Una aplicación $P : E \longrightarrow F$ es un *polinomio n-homogéneo* si existe alguna $L \in \mathcal{L}_a({}^nE;F)$ tal que, para todo $x \in E$,

$$P(x) = L(x, \ldots, x). \tag{1.1}$$

El espacio de polinomios n-homogéneos entre E y F se denota por $\mathcal{P}_a({}^nE;F)$. Las aplicaciones constantes se consideran como polinomios 0-homogéneos. Se llama *polinomio de grado* n entre E y F a toda aplicación P de modo que existen $P_0,\ldots,P_n$ con $P_k \in \mathcal{P}_a({}^kE;F)$ para cada k tales que

$$P = \sum_{k=1}^{n} P_k.$$

Se denota por $\mathcal{P}_a(E;F)$ al espacio de todos los polinomios entre E y F. Para un estudio detallado de los espacios de polinomios, su relación con las aplicaciones lineales y su representación como productos tensoriales puede consultarse [17], Capítulo 1.
Dado un polinomio $P \in \mathcal{P}_a({}^nE;F)$, por la Fórmula de Polarización, existe una única $\check{P} \in \mathcal{L}_a^s({}^nE;F)$ cumpliendo (1.1). De este modo se define un isomorfismo $\mathcal{P}_a({}^nE;F) \longrightarrow \mathcal{L}_a^s({}^nE;F)$ (ver [17], Corolario 1.7).
Un polinomio n-homogéneo es continuo si y sólo si la aplicación n-lineal asociada es continua. Del mismo modo, un polinomio es continuo si y sólo si los polinomios homogéneos que lo definen lo son. En este caso $\mathcal{P}({}^nE;F)$ y $\mathcal{P}(E;F)$ denotan respectivamente los espacios de polinomios n-homogéneos continuos y polinomios continuos. Cuando $F=\mathbb{C}$ denotaremos símplemente $\mathcal{P}({}^nE)$ y $\mathcal{P}(E)$.
En el caso en que E y F sean espacios de Banach se define una norma en $\mathcal{P}(E;F)$ haciendo

$$\|P\| = \sup_{\|x\|\leq 1} \|P(x)\|.$$

Con esta norma, tanto $\mathcal{P}({}^nE;F)$ como $\mathcal{P}(E;F)$ son, a su vez, espacios de Banach.

Con todo esto, dado dos espacios de Banach E y F y un abierto $U \subseteq E$, se dice que $f: U \longrightarrow F$ es *holomorfa* si para cada $x_0 \in U$ existe una bola $B(x_0;r) \subseteq U$ y una sucesión de polinomios $(P_k)_k \subseteq \mathcal{P}(E;F)$ de modo que la serie $\sum_{k=0}^{\infty} P_k(x-x_0)$ converge uniformemente a $f(x)$ para todo $x \in B(x_0;r)$. El espacio de funciones holomorfas de U a F se denota por $\mathcal{H}(U;F)$. Puede darse otra definición equivalente. Dados $x_0 \in U$ e $y_0 \in E$, se considera la función de una variable compleja $\lambda \mapsto f(x_0+\lambda y_0)$; entonces se dice que f es *G-holomorfa* si para cada x_0 e y_0, la tal función es holomorfa. Con esto, se prueba que f es holomorfa si y sólo si es G-holomorfa y continua. Esto equivale a su vez a que f sea continua y $f|_{U\cap M}$ sea holomorfa para todo subespacio finito dimensional M de E. Para un estudio más detallado de todas estas materias, véanse [17] y [58].

1.2 Operadores de composición

El primer capítulo se dedica al estudio de los operadores de composición. La idea original es bastante sencilla y natural. Tomamos el disco unidad complejo, que denotamos $\mathbb{D}$, y una función holomorfa $\phi : \mathbb{D} \longrightarrow \mathbb{D}$. Con esto se define un operador $f \mapsto f \circ \phi$ donde $f : \mathbb{D} \longrightarrow \mathbb{C}$ es una función holomorfa. Este tipo de operadores puede definirse, obviamente, con origen y rango en el espacio de todas las funciones holomorfas definidas en $\mathbb{D}$, un estudio pormenorizado de esta situación se encuentra en [69]. Otro tipo de espacios interesantes de funciones holomorfas son los espacios ponderados. Si $v : \mathbb{D} \longrightarrow [0, \infty[$ es una función acotada, continua y positiva (se llama peso a una tal función), se define el siguiente espacio de funciones holomorfas,

$$H_v^\infty(\mathbb{D}) = \{f \in H(\mathbb{D}) : \|f\|_v = \sup_{x \in B} v(x)|f(x)| < \infty\}.$$

Estos espacios, llamados ponderados, han sido estudiados en, por ejemplo, [4], [27], [67]. Pues bien, puede definirse el operador de composición entre dos de estos espacios y estudiar su propiedades dependiendo de la función que lo define o los pesos que definen los espacios, como se hace en [5] ó [73]. Nuestro objetivo en el capítulo es definir operadores de composición en espacios de funciones holomorfas en la bola unidad de un espacio de Banach y generalizar algunos de los resultados de [5].

Consideramos X un espacio de Banach y B su bola unidad abierta. En primer lugar, dado un peso $v : B \longrightarrow]0, \infty[$, definimos el espacio ponderado asociado de la forma natural

$$H_v^\infty(B) = \{f \in \mathcal{H}_b(B) : \|f\|_v = \sup_{x \in B} v(x)|f(x)| < \infty\}.$$

Por otro lado, de forma análoga al caso finito-dimensional, se define una condición de crecimiento asociada $u(x) = \frac{1}{v(x)}$ y a partir de ésta se define $\tilde{u} : B \longrightarrow]0, +\infty[$ por $\tilde{u}(x) = \sup_{f \in B_v} |f(x)|$ y un nuevo peso asociado $\tilde{v} = 1/\tilde{u}$. Se prueban una serie de propiedades de estas funciones. Por ejemplo, $\|f\|_v \leq 1$ si y sólo si $\|f\|_{\tilde{v}} \leq 1$, de dónde se deduce que $H_v^\infty(B) = H_{\tilde{v}}^\infty(B)$ isométricamente. También, $x \in B$ existe $f \in H_v^\infty(B)$ con $\|f\|_v \leq 1$ tal que $\tilde{u}(x) = |f_x(x)|$ (ver Proposición 2.2.4).

Ahora, dada $\phi : B \longrightarrow B$ y dos pesos v y w se define el operador de composición $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ como $C_\phi(f) = f \circ \phi$. Nuestro objetivo a lo largo del capítulo es encontrar condiciones sobre los pesos y sobre la función ϕ que hagan que el operador esté bien definido, sea continuo o que sea compacto. Empezamos por el estudio de cuándo está C_ϕ bien definido. A este respecto obtenemos los siguientes resultados.

Proposición 1.2.1
Si existe $0 < r < 1$ tal que $\phi(B) \subseteq rB$, entonces $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ está bien definido para dos pesos v, w cualesquiera.

Proposición 1.2.2
Sean v, w y ϕ tales que $\lim_{r \to 1^-} \sup_{\|\phi(x)\| > r} \dfrac{w(x)}{\tilde{v}(\phi(x))} < \infty$.
Entonces $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ está bien definido.

Para el caso de cuándo el operador es continuo damos condiciones en un caso general y, después, otra más sencilla para un caso algo más restrictivo.

Proposición 1.2.3
Sean v, w dos pesos y $\phi : B \longrightarrow B$ holomorfa. Entonces las siguientes afirmaciones son equivalentes,
(i) $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ *es continuo.*
(ii) $\sup_{x \in B} \frac{w(x)}{\tilde{v}(\phi(x))} = M < \infty$.
(iii) $\sup_{x \in B} \frac{\tilde{w}(x)}{\tilde{v}(\phi(x))} = M < \infty$.

En el caso en que v sea esencial (ver Definición 2.2.1) tenemos que $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ es continuo si y sólo si

$$\sup_{x \in B} \frac{w(x)}{v(\phi(x))} < \infty.$$

Consideramos ahora la situación en que H sea un espacio de Hilbert; en este caso se prueba.

Teorema 1.2.4
Sea B la bola unidad abierta de un espacio de Hilbert H y $v : B \longrightarrow]0, +\infty[$ un peso radial y decreciente respecto a $\|x\|$. Entonces las siguiente afirmaciones son equivalentes,
(i) $C_\phi : H_v^\infty(B) \longrightarrow H_v^\infty(B)$ *es continuo para toda* ϕ.
(ii) Para cada $(x_n)_{n \in \mathbb{N}} \subseteq B$ *tal que* $\|x_n\| = 1 - 2^{-n}$ *se cumple:*

$$\inf_{n \in \mathbb{N}} \frac{\tilde{v}(x_{n+1})}{\tilde{v}(x_n)} > 0.$$

Con ayuda de una versión generalizada del Lema de Schwarz este último teorema se prueba también para el caso más general (que incluye a los espacios de Hilbert) de los dominios simétricos acotados, esto es, la bola unidad abierta de un $JB^\star$-triple (véase Definición 2.4.6).

También buscamos condiciones que hagan que el operador de composición sea compacto. En esta línea, las pruebas se basan en el siguiente lema, cuya prueba sigue las mismas ideas que la de [69], Sección 2.4 y [5], Lemma 3.1.

Lema 1.2.5
Sea $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ *continuo Entonces las siguientes afirmaciones son equivalentes,*
(i) C_ϕ *es compacto.*
(ii) Para cada sucesión acotada $(f_n)_n \subseteq H_v^\infty(B)$ *tal que* $f_n \xrightarrow{\tau_0} 0$ *se cumple que* $\|C_\phi f_n\|_w \longrightarrow 0$.

Con este lema se prueban los siguientes resultados. En primer lugar tenemos una condición sobre la aplicación ϕ que hace que el operador sea compacto independientemente de los pesos.

Proposición 1.2.6
Sea $\phi : B \longrightarrow B$ *tal que* $\phi(B)$ *es relativamente compacto y* $\overline{\phi(B)} \subseteq B$. *Entonces,* $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ *es compacto para cualesquiera dos pesos* v, w.

Imponiendo algunas condiciones sobre los pesos se tiene la siguiente caracterización

Teorema 1.2.7
Sean v, w *dos pesos y* $\phi : B \longrightarrow B$ *con* $\phi(B)$ *relativamente compacto. Entonces,* $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ *es compacto si y sólo si*

$$\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}(\phi(x))} = 0.$$

De hecho, en un sentido podemos obtener incluso un límite más cómodo de manejar.

Teorema 1.2.8
Sean v, w *dos pesos y* $\phi : B \longrightarrow B$ *con* $\phi(B)$ *relativamente compacto tales que*

$$\lim_{\|x\|\to 1^-} \frac{w(x)}{\tilde{v}(\phi(x))} = 0.$$

Entonces $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ *es compacto.*

Así, imponiendo unas condiciones ligeramente más restrictivas sobre los pesos podemos conseguir una nueva caracterización.

Proposición 1.2.9
Sean v, w *dos pesos tales que* $\lim_{\|x\|\to 1^-} w(x) = 0$ *y* $\phi : B \longrightarrow B$ *con* $\phi(B)$ *relativamente compacto. Entonces,* $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ *es compacto si y sólo si*

$$\lim_{\|x\|\to 1^-} \frac{w(x)}{\tilde{v}(\phi(x))} = 0.$$

La última parte del capítulo se dedica al caso en que el operador está definido entre espacios ponderados definidos no ya por un peso sino por una familia de ellos (véase [27]). Así, se tienen condiciones relacionando la continuidad del operador cuando se considera la familia de pesos y cuando se considera sólo algunos de ellos de forma individual. Consideramos familias numerables de pesos V y W tales que $v(x) > 0$ y $w(x) > 0$ para todo $x \in B$, $v \in V$ y $w \in W$.

Proposición 1.2.10
Sea $\phi : B \longrightarrow B$ holomorfa y V, W dos familias de pesos tales que para cada $w \in W$ existe un $v \in V$ de manera que $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ es continuo.
Entonces, $C_\phi : \mathcal{H}V(B) \longrightarrow \mathcal{H}W(B)$ es continuo.

Proposición 1.2.11
Sean V, W dos familias de pesos y $\phi : B \longrightarrow B$ tales que el operador de composición $C_\phi : \mathcal{H}V(B) \longrightarrow \mathcal{H}W(B)$ es continuo.
Entonces, para cada $w \in W$ existen $v_{i_1}, \ldots, v_{i_m} \in V$ y $v = \sup_{j=1,\ldots,m} v_{i_j}$ tales que $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ es continuo.

A partir de esta proposición se tiene de forma inmediata lo siguiente.

Corolario 1.2.12
Sean V, W dos familias de pesos, V creciente, y $\phi : B \longrightarrow B$ tales que el operator de composición $C_\phi : \mathcal{H}V(B) \longrightarrow \mathcal{H}W(B)$ es continuo.
Entonces, para cada $w \in W$ existe $v \in V$ tal que $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ es continuo.

Se prueba entonces que, de hecho, V puede tomarse siempre creciente; es decir, que dada cualquier V, podemos definir V_1 creciente tal que $\mathcal{H}V(B) = \mathcal{H}V_1(B)$. Automáticamente tenemos una caracterización.

Proposición 1.2.13
Sean V, W dos familias de pesos y $\phi : B \longrightarrow B$ tales que el operator de composición. Entonces, $C_\phi : \mathcal{H}V(B) \longrightarrow \mathcal{H}W(B)$ es continuo si y solo si para cada $w \in W$ existe $v \in V$ tal que $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ es continuo.

1.3 Espectros en productos tensoriales de álgebras lmc

Si $\mathcal{A}$ es un álgebra unitaria cualquiera, en la teoría espectral clásica se define el espectro de $a \in \mathcal{A}$, denotado $\sigma(a)$, como aquellos $\lambda \in \mathbb{C}$ tales que $a - \lambda 1_{\mathcal{A}}$ no es invertible. Esta teoría clásica ha sido ampliamente estudiada y desarrollada. Durante la década de los 1930 Gelfand desarrolló un trabajo en el que relacionaba la teoría espectral en álgebras de Banach conmutativas con los homomorfismos de álgebras continuos $h : \mathcal{A} \longrightarrow \mathbb{C}$, cuyo espacio se denota por $\mathfrak{M}(\mathcal{A})$ (es un subespacio del dual de $\mathcal{A}$). Para cada $a \in \mathcal{A}$ defnió una aplicación $\hat{a} : \mathfrak{M}(\mathcal{A}) \longrightarrow \mathbb{C}$ por $\hat{a}(h) = h(a)$ y demostró que cada una de estas aplicaciones es continua (con la topología débil*). Además, la aplicación $\hat{\ } : \mathcal{A} \longrightarrow \mathcal{C}(\mathfrak{M}(\mathcal{A}))$ es un homomorfismo de álgebras continuo (véase p.e. [70], Capítulos 12 y 13). Con esto se tiene que

$$\sigma(a) = \{\hat{a}(h) : h \in \mathfrak{M}(\mathcal{A})\}.$$

Nuestro objetivo es, haciendo uso del espectro definido por Harte en los años 1970 (véase [33]) para familias de elementos de un álgebra, definir un espectro vectorial para elementos de un producto tensorial. Tomamos si $\mathcal{A}$ es un álgebra con unas ciertas propiedades y E un espacio localmente convexo. Consideramos el producto tensorial con una cierta topología $\mathcal{T}$, $\mathcal{A} \otimes_{\mathcal{T}} E$. Para cada $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ queremos definir un espectro $\sigma(\mathsf{T}) \subseteq E$ que generalice el clásico y el definido por Waelbroeck en la década de 1970 para elementos de $\mathcal{A}\hat{\otimes}_{\pi} X$ ($\mathcal{A}$ álgebra de Banach y X espacio de Banach). Los primeros pasos en esta dirección fueron dados en [18], [19], [20] y [73] para espacios y álgebras de Banach.

En primer lugar, se dice que un álgebra $\mathcal{A}$ es álgebra topológica si tiene una topología $\mathcal{T}$ de manera que las operaciones algebraicas son continuas. Un álgebra topológica $\mathcal{A}$ es localmente multiplicativamente convexa (lmc) si es un espacio localmente convexo cuya topología está definida por una familia de seminormas $(p_i)_{i\in I}$ tales que, para todo $x, y \in \mathcal{A}$ y todo $i \in I$,

$$p_i(xy) \leq p_i(x)p_i(y).$$

A las seminormas con esta propiedad se les llama multiplicativas. Por otro lado, en un álgebra $\mathcal{A}$ se define una operación por $a \circ b = a + b - ab$; esto da una operación asociativa con identidad 0, así, se dice que un elemento $a \in \mathcal{A}$ es quasi-invertible si existe algún $b \in \mathcal{A}$ de modo que $a \circ b = 0 = b \circ a$. Entonces un álgebra topológica $\mathcal{A}$ es Q-álgebra si el conjunto de elementos quasi-invertibles es abierto en $\mathcal{A}$. Si $\mathcal{A}$ es unitaria, entonces es Q-álgebra si y sólo si el conjunto de elementos invertibles es abierto. Durante todo el capítulo se trabaja con álgebras lmc y Q-álgebras.

Trabajamos también con productos tensoriales en los que consideramos una topología $\mathcal{T}$. Utilizamos topologías tensoriales uniformes (Definición 3.2.9), concepto que generaliza el de topologías compatibles definido por Grothendieck en [30]. Cuando tenemos dos álgebras lmc $\mathcal{A}$ y $\mathcal{B}$, el producto $\mathcal{A} \otimes \mathcal{B}$ es a su vez álgebra (se da una prueba). Estudiamos el caso de cuándo $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ es a su vez álgebra lmc. En esta línea obtenemos que tanto la topología π como la ε son uniformes. Además $\mathcal{A}\hat{\otimes}_{\pi}\mathcal{B}$ es álgebra lmc para cualesquiera $\mathcal{A}$ y $\mathcal{B}$ y $\mathcal{A}\hat{\otimes}_{\varepsilon}\mathcal{B}$ lo es cuando $\mathcal{A}$ sea álgebra uniforme. Consideramos a su vez topologías localmente convexas definidas a partir de las llamadas "normas de Lapresté" (véase [11], Sección 12.5), denotadas $\alpha_{r,s}$. Obtenemos que $\alpha_{r,s}$ es siempre uniforme y damos una clasificación de cuándo $\mathcal{A}\hat{\otimes}_{\alpha_{r,s}}\mathcal{B}$ es álgebra lmc.

Se empieza por definir una aplicación de Gelfand vectorial. Si E es un espacio localmente convexo completo y $\mathcal{T}$ es una topología uniforme podemos hacer la identificación $\mathbb{C}\hat{\otimes}_{\mathcal{T}} E \cong E$. Ahora tomamos un álgebra lmc $\mathcal{A}$ (no necesariamente conmutativa) y para cada $h \in \mathfrak{M}(\mathcal{A})$ podemos considerar la aplicación $h \otimes I_E :\longrightarrow E$. De este modo, para cada $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ se define su transformada de Gelfand como

$$\hat{\mathsf{T}} : \mathfrak{M}(\mathcal{A}) \longrightarrow E \quad , \quad \hat{\mathsf{T}}(h) = [h \otimes I_E](\mathsf{T}).$$

Aunque no podemos probar que sea siempre continua, sí tenemos lo siguiente.

Proposición 1.3.1
Sean $\mathcal{A}$ una Q-álgebra lmc, E un espacio localmente convexo completo y $\mathcal{T}$ una topología tensorial uniforme. Entonces, para cada $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$, la aplicación $\hat{\mathsf{T}} : \mathfrak{M}(\mathcal{A}) \longrightarrow E$ es continua.

Así pues, si $\mathcal{A}$ es Q-álgebra lmc tenemos definida la aplicación de Gelfand vectorial $\hat{} :$ $\mathcal{A}\hat{\otimes}_{\mathcal{T}} E \longrightarrow \mathcal{C}(\mathfrak{M}(\mathcal{A}); E)$ dada por $\hat{\mathsf{T}}(h) = [h \otimes I_E](\mathsf{T})$. Entonces se prueba que

Proposición 1.3.2
Sean $\mathcal{A}$ una Q-álgebra lmc, E un espacio localmente convexo completo y $\mathcal{T}$ una topología tensorial uniforme. Entonces, la aplicación de Gelfand, $\hat{} : \mathcal{A}\hat{\otimes}_{\mathcal{T}} E \longrightarrow \mathcal{C}(\mathfrak{M}(\mathcal{A}); E)$ es una aplicación lineal y continua.
Si, además $\mathcal{B}$ es un álgebra lmc completa y $\mathcal{T}$ una topología uniforme que satisface que $\mathcal{A} \otimes_{\mathcal{T}} \mathcal{B}$ es álgebra lmc, entonces la aplicación de Gelfand $\hat{} : \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B} \longrightarrow \mathcal{C}(\mathfrak{M}(\mathcal{A}), \mathcal{B})$ es un homomorfismo de álgebras continuo.

Con la transformada de Gelfand definida de esta manera, siguiendo el resultado del caso escalar y la definición dada en [18] y [73], para cada $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ se define el espectro de Waelbroeck como el siguiente conjunto,

$$\sigma_W(\mathsf{T}) = \{\hat{\mathsf{T}}(h) : h \in \mathfrak{M}(\mathcal{A})\} = \{[h \otimes I_E](\mathsf{T}) : h \in \mathfrak{M}(\mathcal{A})\} \subseteq E,$$

siendo $\mathcal{A}$ un álgebra lmc conmutativa y unitaria y E un espacio locamente convexo completo. Es bien sabido que en el caso clásico el espectro es compacto y que lo mismo ocurre en el caso de álgebras y espacios de Banach; en este caso se tiene el resultado análogo.

Proposición 1.3.3
Sea $\mathcal{A}$ una Q-álgebra lmc unitaria y conmutativa, E espacio localmente convexo completo y $\mathcal{T}$ una topología tensorial uniforme. Entonces, para cada $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$, el espectro $\sigma_W(\mathsf{T})$ es compacto.

Por otro lado, si $\mathcal{A}$ es álgebra lmc completa puede ponerse como un límite proyectivo reducido (ver Secciones 3.2.2 y 3.4.2) $\mathcal{A} = \varprojlim \mathcal{A}_i$ donde cada $\mathcal{A}_i$ es álgebra de Banach con proyecciones $\pi_i : \mathcal{A} \longrightarrow \mathcal{A}_i$. Lo que intentamos entonces es relacionar el espectro de Waelbroeck en $\mathcal{A}$ con los definidos en cada $\mathcal{A}_i$. En primer lugar, para cada $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$, denotamos $\mathsf{T}_i = (\pi_i \otimes I_E)(\mathsf{T}) \in \mathcal{A}_i\hat{\otimes}_{\mathcal{T}} E$. Imponiendo sobre la topología una condición no excesivamente restrictiva, que llamamos condición de límite proyectivo (Definición 3.4.3), obtenemos

Proposición 1.3.4
Sean $\mathcal{A}$ álgebra lmc unitaria, conmutativa y completa, E espacio localmente convexo completo y $\mathcal{T}$ topología tensorial uniforme que satisface la condición del límite proyectivo. Dado $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ se cumple,

(i) $\sigma_W(\mathsf{T}_i) \subseteq \sigma_W(\mathsf{T}_j)$ *para cualesquiera* $j > i$.
(ii) $\sigma_W(\mathsf{T}) = \bigcup_{i\in I} \sigma_W(\mathsf{T}_i)$.

Para el caso no conmutativo se definen espectros a izquierda, derivados de considerar invertibilidad por la izquierda. El tratamiento por la derecha es obviamente absolutamente análogo. Se comienza por considerar el espectro de Harte por la izquierda para familias de elementos de un álgebra (ver Definición 3.5.1). A partir de éste se define el espectro de Harte por la izquierda de $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$ como el conjunto $\sigma_H^{left}(\mathsf{T})$ de aquéllos $x'' \in E''$ tales que

$$1_{\mathcal{A}} \notin \{ \sum_{\substack{i\in F \\ F \text{ finito}}} a_i \left([I_{\mathcal{A}} \otimes x'](\mathsf{T}) \;-\; x''(x_i')1_{\mathcal{A}}\right) \;:\; a_i \in \mathcal{A},\; x_i' \in E'\}.$$

Tenemos el espectro definido en el bidual de E. La primera pregunta natural es, pues, ver en qué condiciones tenemos que está en E. Por medio de la aplicación $J_E : E \longrightarrow E''$ definida como $J_E(x)(x') = x'(x)$ podemos considerar a E como un subespacio de E''. Con esto tenemos la siguiente respuesta positiva.

Proposición 1.3.5
Sean $\mathcal{A}$ una Q-álgebra lmc unitaria, E un espacio localmente convexo completo y $\mathcal{T}$ una topología tensorial uniforme. Tomamos $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$. Entonces, para todo $x'' \in \sigma_H^{left}(\mathsf{T})$ existe un $x \in E$ tal que $J_E(x) = x''$. Consiguientemente podemos identificar:

$$\sigma_H^{left}(\mathsf{T}) = \{x \in E : 1_{\mathcal{A}} \notin \{ \sum_{\substack{i\in F \\ F\text{finito}}} a_i([I_{\mathcal{A}} \otimes x_i'](\mathsf{T}) - x'(x)1_{\mathcal{A}}) : a_i \in \mathcal{A}, x_i' \in E'\}\}.$$

En el caso en que $\mathcal{A}$ sea completa, se define una acción de $\mathcal{A} \otimes E'$ sobre $\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$ por medio de

$$(b \otimes x')(a \otimes x) = x'(x)ba$$

y extendiendo por linealidad. De este modo podemos incluso hacer una nueva identificación, con un aspecto muy similar a la del espectro clásico,

$$\sigma_H^{left}(\mathsf{T}) = \{x \in E : \nexists \mathsf{Z} \in \mathcal{A} \otimes E' \text{ t.q. } < \mathsf{Z}, \mathsf{T} - 1_{\mathcal{A}} \otimes x >= 1_{\mathcal{A}}\}.$$

Si $\mathcal{A}$ es conmutativa tenemos dos espectros definidos, el de Harte y el de Waelbroeck. Se prueba que son el mismo conjunto.

Proposición 1.3.6
Sea $\mathcal{A}$ una Q-álgebra lmc unitaria y conmutativa, E un espacio localmente convexo completo y $\mathcal{T}$ una topología tensorial uniforme. Entonces, para cada $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$,

$$J_E(\sigma_W(\mathsf{T})) = \sigma_H(\mathsf{T}).$$

De modo análogo a lo que hicimos con el espectro de Waelbroeck, nos preguntamos por la relación entre el espectro de Harte y el de las proyecciones cuando $\mathcal{A}$ es un álgebra lmc completa. Obtenemos el siguiente resultado, que aunque análogo al del espectro de Waelbroeck, utiliza técnicas totalmente diferentes.

Proposición 1.3.7
Sean $\mathcal{A}$ una Q-álgebra lmc unitaria y completa, E un espacio localmente convexo completo y $\mathcal{T}$ una topología tensorial uniforme que satisfaga la condición del límite proyectivo. Entonces, para cada $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$,
(i) $\sigma_H^{left}(\mathsf{T}_i) \subseteq \sigma_H^{left}(\mathsf{T}_j)$ *para cualesquiera $j > i$.*
(ii) $\sigma_H^{left}(\mathsf{T}) = \bigcup_{i \in I} \sigma_H^{left}(\mathsf{T}_i)$.

En el caso en que tengamos dos álgebras $\mathcal{A}$ y $\mathcal{B}$, entonces $\mathcal{A} \otimes \mathcal{B}$ es de nuevo álgebra y tenemos definido el espectro por la izquierda clásico. Nos preguntamos qué relación hay entre el espectro clásico y el de Harte vectorial y probamos lo siguiente.

Proposición 1.3.8
Sean $\mathcal{A}$ un álgebra lmc unitaria y conmutativa, $\mathcal{B}$ un álgebra lmc unitaria y $\mathcal{T}$ una topología uniforme tal que $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ es Q-álgebra lmc. Tomemos $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$; entonces

$$\sigma_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}^{left}(\mathsf{T}) = \bigcup_{h \in \mathfrak{M}(\mathcal{A})} \sigma_{\mathcal{B}}^{left}([h \otimes I_{\mathcal{B}}](\mathsf{T})).$$

En la teoría escalar hay una serie de resultados conectando la invertibilidad de un elemento con la de su transformada de Gelfand. Utilizando diferentes técnicas y los resultados obtenidos hasta ahora probamos los siguientes resultados análogos.

Teorema 1.3.9
Sea $\mathcal{A}$ un álgebra de Fréchet unitaria y conmutativa, $\mathcal{B}$ un álgebra de Fréchet unitaria y $\mathcal{T}$ una topología tensorial uniforme tal que $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ es un álgebra lmc y que satisface la condición del límite proyectivo. Sea $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$; entonces las siguientes afirmaciones son equivalentes.
(i) T *es invertible por la izquierda en $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$.*
(ii) $\hat{\mathsf{T}}(h)$ *es invertible por la izquierda en $\mathcal{B}$ para todo $h \in \mathfrak{M}(\mathcal{A})$.*

Con este teorema podemos probar que para ciertos espacios X, si $\mathcal{B}$ es una cierta álgebra no conmutativa denotamos por $\mathcal{C}(X, \mathcal{B})$ el espacio de aplicaciones continuas de X en $\mathcal{B}$. Aplicando el teorema tenemos que $\mathbf{f} \in \mathcal{C}(X, \mathcal{B})$ es invertible por la izquierda si y sólo si $\mathbf{f}(x)$ es invertible por la izquierda en $\mathcal{B}$ para todo $x \in X$.

Proposición 1.3.10
Sea $\mathcal{A}$ una Q-álgebra de Fréchet unitaria y conmutativa, $\mathcal{B}$ un álgebra de Fréchet unitaria y $\mathcal{T}$ una topología tensorial uniforme tal que $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ es un álgebra lmc y que satisface la

condición del límite proyectivo. Sea $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$*; entonces las siguientes afirmaciones son equivalentes.*
(i) T *es invertible por la izquierda en* $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$*.*
(ii) $\hat{\mathsf{T}}(h)$ *es invertible por la izquierda en* $\mathcal{B}$ *para todo* $h \in \mathfrak{M}(\mathcal{A})$*.*
(iii) $\hat{\mathsf{T}}$ *es invertible por la izquierda en* $\mathcal{C}(\mathfrak{M}(\mathcal{A}), \mathcal{B})$*.*

Teorema 1.3.11
Sea $\mathcal{A}$ *un álgebra lmc unitaria y conmutativa,* $\mathcal{B}$ *un álgebra lmc unitaria y* $\mathcal{T}$ *una topología tensorial uniforme tal que* $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ *es una Q-álgebra lmc. Tomamos* $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$*; entonces las siguientes afirmaciones son equivalentes.*
(i) T *es invertible por la izquierda en* $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$*.*
(ii) $\hat{\mathsf{T}}(h)$ *es invertible por la izquierda en* $\mathcal{B}$ *para todo* $h \in \mathfrak{M}(\mathcal{A})$*.*
(iii) $\hat{\mathsf{T}}$ *es invertible por la izquierda en* $\mathcal{C}(\mathfrak{M}(\mathcal{A}), \mathcal{B})$*.*

Con esto tenemos un resultado similar al anterior (ver Ejemplo 3.6.7). Si K es un compacto con unas ciertas propiedades y $\mathcal{B}$ una cierta álgebra no conmutativa, sea $\mathcal{H}(K, \mathcal{B})$ el espacio de gérmenes holomorfos en K con valores en $\mathcal{B}$. Entonces, aplicando el teorema se tiene que $F \in \mathcal{H}(K, \mathcal{B})$ es invertible por la izquierda si y sólo si $F(k)$ es invertible por la izquierda en $\mathcal{B}$ para todo $k \in K$. Podemos conseguir aún un poco más, aplicando el tercer enunciado del teorema se tiene que F es invertible por la izquierda en $\mathcal{H}(K, \mathcal{B})$ si y sólo si es invertible por la izquierda en $\mathcal{C}(K, \mathcal{B})$.
Como corolario de los teoremas anteriores tenemos el siguiente, interesante en sí mismo.

Teorema 1.3.12
Sea $\mathcal{A} = \varprojlim \mathcal{A}_i$ *una Q-álgebra completa; entonces,*
$a \in \mathcal{A}$ *es invertible por la izquierda* $\Leftrightarrow$ a_i *es invertible por la izquierda en* $\mathcal{A}_i$ *para todo* i*.*

La última parte del capítulo se dedica al estudio de polinomios. Si $P \in \mathcal{P}_a(E; F)$, puede definirse otr polinomio $P_{\mathcal{A}} \in \mathcal{P}_a(\mathcal{A} \otimes E; \mathcal{A} \otimes F)$ de tal modo que, si P es n-homogéneo, se cumpla que

$$P_{\mathcal{A}}(a \otimes x) = a^n \otimes P(x).$$

Éste es un proceso algebraico que fue llevado a cabo en [19]. El problema es, cuando P es continuo, extenderlo a un cierto $P_{\mathcal{A}} \in \mathcal{P}(\mathcal{A}\hat{\otimes}_{\mathcal{T}}E; \mathcal{A}\hat{\otimes}_{\mathcal{T}}F)$. Se estudia el problema para las topologías π y ε. Con esto podemos relacionar los polinomios con los espectros que hemos definido y obtenemos los siguientes teoremas espectrales.

Proposición 1.3.13
Sea $\mathcal{A}$ *una Q-álgebra lmc,* E, F *espacios localmente convexos completos y* $\mathcal{T}$ *una toplogía tensorial uniforme. Entonces, para cada* $P \in \mathcal{P}(E; F)$ *tal que* $P_{\mathcal{A}} \in \mathcal{P}(\mathcal{A}\hat{\otimes}_{\mathcal{T}}E; \mathcal{A}\hat{\otimes}_{\mathcal{T}}F)$ *y todo* $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$ *se tiene,*

$$P(\sigma_H^{left}(\mathsf{T})) \subseteq \sigma_H^{left}(P_{\mathcal{A}}(\mathsf{T})).$$

Teorema 1.3.14
Sea $\mathcal{A}$ una Q-álgebra lmc completa y E un espacio locamente convexo completo con la propiedad de aproximación acotada. Sea $\mathcal{T}$ una topología tensorial uniforme definida por una familia de seminormas $\{p \otimes q\}$. Si $P \in \mathcal{P}(E;F)$ es tal que $P_{\mathcal{A}} \in \mathcal{P}({}^n\mathcal{A}\hat{\otimes}_{\mathcal{T}}E; \mathcal{A}\hat{\otimes}_{\mathcal{T}}F)$, entonces

$$P(\sigma_H^{left}(\mathsf{T})) = \sigma_H^{left}(P_{\mathcal{A}}(\mathsf{T}))$$

para todo $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$.

El capítulo finaliza probando con ayuda de los teoremas relativos a polinomios que el espectro de Harte es compacto.

Teorema 1.3.15
Sean $\mathcal{A}$ una Q-álgebra lmc completa y unitaria, E un espacio localmente convexo completo y $\mathcal{T}$ una topología tensorial uniforme. Entonces, para todo $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, el espectro de Harte vectorial $\sigma_H^{left}(\mathsf{T})$ es compacto en la topología de E.

1.4 Cotipo 2 de espacios de polinomios en espacios de sucesiones

Se dice que un espacio de Banach E tiene cotipo 2 si existe una cierta constante $\kappa > 0$ de manera que, para cualesquiera $x_1, \ldots, x_n \in E$,

$$\left(\sum_{k=1}^{n} \|x_k\|^2\right)^{1/2} \leq \kappa \left(\int_0^1 \|\sum_{k=1}^{n} r_k(t)x_k\|^2 dt\right)^{1/2},$$

siendo r_k las funciones de Rademacher clásicas (Definición 4.2.1). Con esto se define la constante de cotipo 2 de E, denotada por $\mathbf{C}_2(E)$ como la mejor constante en esta desigualdad. Se sabe que si E es infinito dimensional, entonces $\mathcal{P}({}^mE)$ no tiene nunca cotipo 2 (véanse [16] y [17] Proposición 1.54).
Si X es un espacio de Banach de sucesiones (ver Sección 4.3.1) esto incluye por ejemplo a los espacios ℓ_p, de Orlicz o de Lorentz, consideramos X_n como el espacio generado por los vectores $e_1, \ldots, e_n$. Lo anteriormente observado implica que la sucesión $(\mathbf{C}_2(\mathcal{P}({}^mX_n)))_n$ debe tender a ∞ (ver Nota 4.5.2). El objetivo de este capítulo es estudiar de el comportamiento asintótico de esa sucesión, es decir, "¿de qué manera se va a ∞?". De hecho, se conjetura que para todo espacio de Banach de sucesiones simétrico se cumple que

$$\mathbf{C}_2(\mathcal{P}({}^mX_n)) \asymp (n^{1/2})^{m-1}\mathbf{C}_2(X_n'),$$

dónde $(a_n) \asymp (b_n)$ significa que puede encontrarse una constante $K > 0$ de modo que $a_n \leq Kb_n$ y $b_n \leq Ka_n$ para todo $n \in \mathbb{N}$. Aunque no se prueba la conjetura en el caso más general sí se consigue para los casos en que X sea 2-convexo o bien 2-cóncavo y tenga convexidad no trivial (ver Definiciones 4.3.1 y 4.3.2).

Se comienza situando el problema en un contexto algo más general. El cotipo 2 puede verse como un caso particular de una situación mucho más general. Si $(\mathfrak{A}, A)$ es un ideal de operadores de Banach, decimos que un espacio de Banach E cumple la $\mathfrak{A}$-propiedad si $id_E \in \mathfrak{A}$ y se define la $\mathfrak{A}$-constante de E como $A(id_E)$.
Para probar la conjetura se hace uso de la representación del espacio de polinomios m-homogéneos como un producto tensorial simétrico; puesto que X_n es de dimensión finita podemos escribir $\mathcal{P}(^m X_n) = \otimes_{\varepsilon_s}^{m,s} X_n$ ([21]). El primer paso es mostrar que podemos trabajar no sólo con el producto tensorial simétrico, sino con el producto tensorial completo.

Teorema 1.4.1
Sea $(\mathfrak{A}, A)$ un ideal de operadores de Banach, X un espacio de Banach de sucesiones simétrico y $m \in \mathbb{N}$. Sea $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ tal que $a_{mn} \prec a_n$ (resp. $a_n \prec a_{mn}$); entonces las siguiente afirmaciones son equivalentes.
(i) $A(\mathcal{P}(^m X_n)) \prec a_n$ *(resp.* $a_n \prec A(\mathcal{P}(^m X_n))$*).*
(ii) $A(\otimes_{\varepsilon_s}^{m,s} X_n') \prec a_n$ *(resp.* $a_n \prec A(\otimes_{\varepsilon_s}^{m,s} X_n')$*).*
(iii) $A(\otimes_{\varepsilon}^{m} X_n') \prec a_n$ *(resp.* $a_n \prec A(\otimes_{\varepsilon}^{m} X_n')$*).*

Aunque la prueba de la conjetura se hace explícitamente para el caso del cotipo 2, la reducción al producto completo se hace para cualquier $\mathfrak{A}$-propiedad. Con esto la conjetura puede reformularse en los siguiente términos,

$$\mathbf{C}_2(\otimes_{\varepsilon}^{m} X_n) \asymp (n^{1/2})^{m-1} \mathbf{M}_{(2)}(X_n).$$

Esta estimación es la que después efectivamente se prueba.

En primer lugar, antes de probar la conjetura en los casos particulares que se mencionaron anteriormente, se prueba una estimación general, cierta para cualquier espacio de Banach de sucesiones simétrico X.

Lema 1.4.2
Sea X cualquier espacio de Banach de sucesiones simétrico y $m \in \mathbb{N}$; entonces,

$$\frac{(n^{1/2})^{m-1}}{\sqrt{\log(n+1)}} \prec \mathbf{C}_2(\otimes_{\varepsilon}^{m} X_n) \prec (n^{1/2})^{m}.$$

A partir de aquí se tiene la siguiente estimación para espacios de polinomios.

Teorema 1.4.3
Sea X cualquier espacio de Banach de sucesiones simétrico y $m \in \mathbb{N}$; entonces,

$$\frac{(n^{1/2})^{m-1}}{\sqrt{\log(n+1)}} \prec \mathbf{C}_2(\mathcal{P}(^m X_n)) \prec (n^{1/2})^{m}.$$

Si además X tiene convexidad no trivial, se tiene que

$$(n^{1/2})^{m-1} \prec \mathbf{C}_2(\mathcal{P}(^m X_n)) \prec (n^{1/2})^m.$$

La cota superior de esta estimación general es inmediata. Para la prueba de la cota inferior se parte del siguiente resultado de Pisier (véase [64], Capítulo 10 y Definición 4.6.3 para los a_k y Definición 4.2.5 para la norma l de un operador),

Proposición 1.4.4
Sea X un espacio de Banach y $T : \ell_2^n \longrightarrow X$ un operador lineal; entonces para cualquier q,

$$\sup_{k\in\mathbb{N}} k^{1/q} a_k(T) \leq \mathbf{C}_q(X)\ l(T).$$

Aplicando esto a nuestro caso particular y estimando convenientemente los elemento de la desigualdad se llega a que, para cualquier espacio de Banach de sucesiones simétrico X y cualquier m fijo,

$$(n^m)^{1/2} \frac{\|id : \ell_2^n \to X_n\|}{l(id : \ell_2^n \to X_n)} \prec \mathbf{C}_2(\otimes_\varepsilon^m X_n).$$

En el desarrollo de la prueba de esta cota inferior se obtienen resultados interesantes en sí mismos.

Proposición 1.4.5
Sean X, Y dos espacios de Banach de sucesiones simétricos; α, β dos normas en $\otimes^m X_n, \otimes^m Y_n$ respectivamente tales que todo $T \in S(\otimes^m \mathbb{K}^n)$ y todo $R \in S(\otimes^m \mathbb{K}^n)$ son isometrías cuando en los espacios se dotan con α y β; entonces,

$$\pi_2(\otimes_\alpha^m X_n \to \otimes_\beta^m Y_n) = (n^m)^{1/2} \frac{\| \otimes_2^m \ell_2^n \to \otimes_\beta^m Y_n\|}{\| \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n\|}.$$

El grupo $S(\otimes^m \mathbb{K}^n)$ se define en el Lema 4.6.1; baste decir que la norma ε cumple la condición de la proposición. Con esto tenemos el siguiente resultado sobre los números de aproximación (a_k) y de Weyl (x_k, Definición 4.6.3) de la identidad.

Lema 1.4.6
Sea α una norma en $\otimes^m X_n$ como en el enunciado de la Proposición 1.4.5. Entonces para todo $1 \leq k \leq \left[\frac{n^m}{2}\right] = \max\{r \in \mathbb{N} : r \leq \frac{n^m}{2}\}$ se tiene

$$\begin{aligned} &\|id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n\| \geq a_k(id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n) \\ \geq\ & x_k(id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n) \geq \frac{1}{\sqrt{2}} \|id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n\|. \end{aligned}$$

Como ha quedado dicho, la versión de la conjetura que se prueba es la del producto tensorial completo, y ni aún ésta se prueba en el caso más general. El primer caso es el de espacios 2-cóncavos.

Proposición 1.4.7
Sea X un espacio de Banach de sucesiones simétrico 2-cóncavo y $m \in \mathbb{N}$. Entonces

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \asymp (n^{1/2})^{m-1}.$$

La cota inferior se prueba a partir de la estimación que ya obtuvimos para el caso general. Para la cota superior se utilizan y estudian sucesiones débilmente sumantes y los llamados operadores (Y, X)-sumantes (Definición 4.7.4), que generalizan el concepto clásico de operadores (p, q)-sumantes. Estos operadores (Y, X)-sumantes han sido utilizados en [52].
El segundo caso en el que se prueba la conjetura es aquél en el que X es 2-convexo y tiene concavidad no trivial. Concretamente se tiene lo siguiente.

Proposición 1.4.8
Sea X un espacio de Banach de sucesiones simétrico 2-convexo con concavidad no trivial y $m \in \mathbb{N}$. Entonces

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \asymp (n^{1/2})^{m-1}\mathbf{M}_{(2)}(X_n) \asymp \frac{(n^{1/2})^m}{\sum_{i=1}^n \|e_i\|_X}.$$

Con estos resultados, utilizando el dual de Köthe de X (ver Definición 4.3.7), tenemos los resultados sobre espacios de polinomios que estábamos buscando.

Teorema 1.4.9
Sea X un espacio de Banach de sucesiones simétrico 2-convexo y $m \in \mathbb{N}$; entonces

$$\mathbf{C}_2(\mathcal{P}(^m X_n)) \asymp (n^{1/2})^{m-1}.$$

Teorema 1.4.10
Sea X un espacio de Banach de sucesiones simétrico 2-cóncavo con convexidad no trivial y $m \in \mathbb{N}$; entonces

$$\mathbf{C}_2(\mathcal{P}(^m X_n)) \asymp n^{\frac{m}{2}-1} \sum_{i=1}^n \|e_i\|_X.$$

Aplicando estos dos teoremas, junto con un estudio particular de ℓ_1, cubrimos los todos los espacios ℓ_p con $1 \leq p \leq \infty$ y nos permite dar la siguiente estimación,

$$\mathbf{C}_2(\mathcal{P}(^m \ell_p^n)) \asymp \begin{cases} \frac{(n^{1/2})^m}{\sqrt{\log(n+1)}} & \text{si} \quad p = 1 \\ n^{\frac{m}{2}-1} n^{1/p} & \text{si} \quad 1 < p \leq 2 \\ (n^{1/2})^{m-1} & \text{si} \quad 2 < p \leq \infty \end{cases}$$

También se obtienen resultados análogos para espacios de Orlicz, de Lorentz y espacios $\ell_{p,q}$.

Todos los resultados obtenidos son válidos tanto para espacios de sucesiones reales o complejos. Cuando el espacio de sucesiones es real es un retículo de Banach; por lo que en este caso puede utilizarse la teoría de retículos para obtener resultados nuevos. Se prueba también que los resultado que se obtengan en esta línea pueden trasladarse al caso complejo. Si X es un espacio de Banach de sucesiones complejo, definimos el espacio real

$$X(\mathbb{R}) = \{y \in X \ : \ y_n \in \mathbb{R} \ \text{ para todo } \ n\}$$

y lo dotamos de la topología inducida por X. Con esta notación tenemos el siguiente resultado.

Proposición 1.4.11
Sea X un espacio de Banach de sucesiones simétrico complejo. Entonces, para cada m

$$\mathbf{C}_2(\mathcal{P}(^m X(\mathbb{R})_n)) \prec \mathbf{C}_2(\mathcal{P}(^m X_n)) \prec \mathbf{C}_2(\mathcal{P}(^m X(\mathbb{R})_{2n})).$$

En particular, si $(a_n) \asymp (a_{2n})$ y $(b_n) \asymp (b_{2n})$, entonces

$$(a_n) \prec \mathbf{C}_2(\mathcal{P}(^m X(\mathbb{R})_n)) \prec (b_n)$$

si y sólo si

$$(a_n) \prec \mathbf{C}_2(\mathcal{P}(^m X_n)) \prec (b_n).$$

El capítulo termina con una sección en la que se da un resultado que, si bien no es sobre polinomios, nuestro interés inicial, sí utiliza en su demostración las técnicas desarrollada en las secciones precedentes. Se trata de una mejora de un resultado previo para espacios ℓ_p^n que se encuentra en [7].

Proposición 1.4.12
Sea X bien 2-cóncavo o 2-convexo con concavidad no trivial e Y bien 2-cóncavo o 2-convexo con concavidad no trivial; entonces

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \asymp \min(\sqrt{n}\mathbf{M}_{(2)}(Y_m), \ \sqrt{m}\mathbf{M}_{(2)}(X_n)).$$

Como corolario inmediato se tiene

Corolario 1.4.13
Sea X bien 2-cóncavo o 2-convexo con concavidad no trivial; entonces

$$\mathbf{C}_2(\mathcal{L}(X_n; X_n)) \asymp \sqrt{n}.$$

Chapter 2

Composition operators

2.1 Introduction

The starting idea of composition operators is simple and a very natural question. Consider $\mathbb{D}$ the open unit disc of $\mathbb{C}$ and a holomorphic map $\phi : \mathbb{D} \longrightarrow \mathbb{D}$. If $f : \mathbb{D} \longrightarrow \mathbb{C}$ is a holomorphic function, we can compose $f \circ \phi$ and try to analyze what happens when we let the f vary; in other words we define an operator between spaces of holomorphic functions and we want to study what properties does this operator have (continuity, compactness, ...). This obviously depends on which are the spaces considered. First candidates are Hardy spaces and a full study of the situation in this case can be found in [69].
There are two possible ways of approaching the generalization of these results. First one is try to go to higher dimensions, that is consider B the open unit ball of a Banach space and do the same kind of study. Some results in this trend, defining the operator between the space of holomorphic mappings of bounded type, can be found in [2] and [26].
But another possible way is to stay with $\mathbb{D}$ but consider different types of spaces of holomorphic functions, namely weighted spaces of holomorphic functions, which were studied in [4]. This step of defining the operator between weighted spaces was taken in [5], where conditions for continuity, compactness or integral representation are given.
Analogous weighted spaces of holomorphic mappings in Banach spaces have been defined and studied in [27]. Our aim in this chapter is to generalize some of the results in [5] when we consider B instead of $\mathbb{D}$ and define the composition operator between two weighted spaces of holomorphic functions between Banach spaces.

2.2 Weights. Weighted spaces

All through this chapter X will always denote a complex Banach space and B its open unit ball. We follow the notation in [5] and in [73],

Definition 2.2.1 A *weight* is any continuous bounded mapping $v : B \longrightarrow]0, +\infty[$.
We say that a weight v is *essential* if there exists $C > 0$ such that $v(x) \leq \tilde{v}(x) \leq C\ v(x)$ for all $x \in B$.
A weight v is said to be *radial* if $v(x_1) = v(x_2)$ whenever $\|x_1\| = \|x_2\|$.

A set $A \subset B$ is said to be B-bounded if it is bounded and $d(A, X \setminus B) > 0$. The space of those holomorphic functions $f : B \longrightarrow \mathbb{C}$ bounded on the B-bounded sets is denoted by $\mathcal{H}_b(B)$.
A mapping $f : B \longrightarrow [0, \infty[$ is said to vanish at infinity outside the B-bounded sets if for every $\varepsilon > 0$ there is B-bounded set $A \subset B$ such that $f(x) < \varepsilon$ for every $x \in A$.
Following the idea in [5], we define the spaces

$$\begin{aligned} H_v^\infty(B) &= \{f \in \mathcal{H}_b(B) : \|f\|_v = \sup_{x \in B} v(x)|f(x)| < \infty\} \qquad (2.1) \\ H_{v_0}^\infty(B) &= \{f \in \mathcal{H}_b(B) : v|f| \text{ van. at } \infty \text{ out. the } B\text{-bdd sets}\} \qquad (2.2) \end{aligned}$$

Remark 2.2.2
An equivalent definition of $H_{v_0}^\infty(B)$ can be given. In fact, we have that $v|f|$ vanishes at infinity outside the B-bounded sets if and only if $\lim_{\|x\| \to 1^-} v(x)|f(x)| = 0$. Indeed, take $\varepsilon > 0$. If $v|f|$ vanishes at infinity outside the B-bounded sets, we can find $A \subset B$, B-bounded, such that $v(x)|f(x)| < \varepsilon$ for all $x \in B \setminus A$. There is $0 < \delta < 1$ such that $A \subseteq B(0, \delta)$. Then $v(x)|f(x)| < \varepsilon$ for all $x \in B \setminus B(0, \delta)$. Hence $\lim_{\|x\| \to 1^-} v(x)|f(x)| = 0$.
Conversely, if $\lim_{\|x\| \to 1^-} v(x)|f(x)| = 0$ given any $\varepsilon > 0$ there is $0 < \delta < 1$ satisfying $v(x)|f(x)| < \varepsilon$ for all $x \in B \setminus B(0, \delta)$. Obviously $B(0, \delta)$ is B-bounded and $v|f|$ vanishes outside the B-bounded sets.
With this, we can write

$$H_{v_0}^\infty(B) = \{f \in \mathcal{H}_b(B) : \lim_{\|x\| \to 1^-} v(x)|f(x)| = 0\}.$$

Both $H_v^\infty(B)$ and $H_{v_0}^\infty(B)$ are Banach spaces. We denote their open unit balls by

$$B_v = \{f \in H_v^\infty(B) : \|f\|_v \leq 1\} \ , \ B_{v_0} = \{f \in H_{v_0}^\infty(B) : \|f\|_v \leq 1\}.$$

In [67] it is proved that in $H_v^\infty(B)$ the τ_v (norm) topology is finer than the τ_0 (compact-open) topology (Proposition 2.1.2) and that B_v is τ_0-compact (Proposition 2.1.3). When the weight has certain properties we have the following relationship between the unit balls of the two spaces.

Proposition 2.2.3
Let v be such that $\lim_{\|x\| \to 1^-} v(x) = 0$. Then B_{v_0} is τ_0-dense in B_v.

Proof.

Given any $f \in B_v$ and $n \in \mathbb{N}$, consider $B_n = B(0, 1-1/n)$ and define $f_n : B \longrightarrow B$ by $f_n(x) = f((1-1/n)x)$. Obviously $f_n \in \mathcal{H}_b(B)$. Also

$$\|f_n\|_v = \sup_{x\in B} v(x)|f_n(x)| = \sup_{x\in B_n} v(x)|f(x)| \leq \|f\|_v \leq 1.$$

Hence $f_n \in B_v$. Moreover $f_n \in H^\infty_{v_0}(B)$. Indeed, since $f \in \mathcal{H}_b(B)$ and B_n is a B-bounded set, there is $M > 0$ such that $\sup_{x\in B_n} |f(x)| \leq M$. Therefore $\sup_{x\in B} |f_n(x)| \leq M$ and

$$\lim_{\|x\|\to 1^-} v(x)|f_n(x)| \leq M \lim_{\|x\|\to 1^-} v(x) = 0.$$

By Remark 2.2.2 $f_n \in B_{v_0}$ for all $n \in \mathbb{N}$.
We have $(f_n)_n \subseteq B_{v_0}$ and we want it to converge to f uniformly on the compact subsets of B. Take $K \subseteq B$ compact and $\varepsilon > 0$. Since f is continuous, for each $x \in K$, we can find $\delta_x > 0$ with $B(x, \delta_x) \subseteq B$ and such that for all y satisfying that $\|x - y\| < \delta_x$ we have $\|f(x) - f(y)\| < \varepsilon/2$. Then $\{B(x, \delta_x/2) : x \in K\}$ is an open cover of K and there are $x_1, \ldots, x_n$ so that $K \subseteq \bigcup_{j=1}^n B(x_j, \delta_{x_j}/2)$. Consider $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \min\{\delta_{x_1}/2, \ldots, \delta_{x_n}/2\}$ and let $x \in K$. There is some x_j such that $\|x - x_j\| < \delta_{x_j}/2$. Hence

$$\|f(x) - f(x_j)\| < \frac{\varepsilon}{2}.$$

On the other hand, for $n \geq n_0$,

$$\left\|\left(1 - \frac{1}{n}\right)x - x\right\| = \frac{1}{n}\|x\| \leq \frac{1}{n} < \frac{\delta_{x_j}}{2}.$$

Therefore, $\|(1-1/n)x - x_j\| < \delta_{x_j}$ and

$$\left\|f\left(\left(1 - \frac{1}{n}\right)\right)x - f(x_j)\right\| < \frac{\varepsilon}{2}.$$

Putting all this together we obtain

$$\left\|f\left(\left(1 - \frac{1}{n}\right)\right)x - f(x)\right\| = \|f_n(x) - f(x)\| < \varepsilon.$$

This is true for all $x \in K$ and n is independent of x. Thus $f_n \longrightarrow f$ in τ_0.
q.e.d.

Now, given any weight v we can define an associated growth condition $u : B \longrightarrow]0, +\infty[$ by $u(x) = \frac{1}{v(x)}$. With this new function we can rewrite

$$B_v = \{f \in H^\infty_v(B) : |f| \leq u\}. \tag{2.3}$$

From this we define $\tilde{u} : B \longrightarrow]0, +\infty[$ by

$$\tilde{u}(x) = \sup_{f\in B_v} |f(x)|$$

and a new associated weight $\tilde{v} = 1/\tilde{u}$. All these functions are related in the following way.

Proposition 2.2.4
Let u be any weight; then
(*i*) $0 < \tilde{u} \leq u$, $0 < v \leq \tilde{v}$.
(*ii*) *$\tilde{u}$ (resp. $\tilde{v}$) is radial, continuous, decreasing or increasing whenever u (resp. v) is so.*
(*iii*) $\|f\|_v \leq 1 \Leftrightarrow \|f\|_{\tilde{v}} \leq 1$.
(*iv*) *For each $x \in B$ there exists $f_x \in B_v$ such that $\tilde{u}(x) = |f_x(x)|$.*
(*v*) *If $\lim\limits_{\|x\|\to 1^-} v(x) = 0$, then $\tilde{u}(x) = \sup_{f\in B_{v_0}} |f(x)|$.*
Proof.
(*i*) By (2.3) $\tilde{u}(x) = \sup_{f\in B_v}|f(x)| \leq u(x)$ for all $x \in B$ and $0 < \tilde{u} \leq u$. From the definition, $0 < v \leq \tilde{v}$.

(*iii*) Suppose first that $\|f\| \leq 1$; then $f \in B_v$. Obviously for each $x \in B$, $|f(x)| \leq \sup_{g\in B_v} |g(x)| = \tilde{u}(x)$. Hence $\|f\|_{\tilde{v}} \leq 1$.
If $\|f\|_{\tilde{v}} \leq 1$, given any $x \in B$, using (*i*), we have $|f(x)| \leq \tilde{u}(x) \leq u(x)$ and $\|f\| \leq 1$.

(*iv*) The evaluation functional is τ_0-continuous and B_v is τ_0-compact. Thus, the supremum in the definition of $\tilde{u}$ is actually a maximum.

(*v*) Let $x \in B$. By (*iv*) we can find $f_x \in B_v$ such that $\tilde{u}(x) = |f_x(x)|$. Let us see that $f \in H^\infty_{v_0}(B)$. Since $\lim_{\|x\|\to 1^-} v(x) = 0$, there is $K > 0$ such that $v(y) \leq K$ for all $y \in B$. But $f_x \in B_v$, thus $|f_x(y)| \leq 1/K$ for all $y \in B$ and

$$\lim_{\|y\|\to 1^-} v(y)|f_x(y)| \leq \frac{1}{K} \lim_{\|y\|\to 1^-} v(y) = 0.$$

This implies $f \in B_{v_0}$.
q.e.d.

As an immediate consequence of (*iii*) we have

Corollary 2.2.5
Given any weight v, $H^\infty_v(B) = H^\infty_{\tilde{v}}(B)$ holds isometrically.

2.3 Composition Operators. Definition

From now on we will always consider two weights v, w and $\phi : B \longrightarrow B$ holomorphic. The composition operator associated to ϕ is defined by

$$C_\phi : H^\infty_v(B) \longrightarrow H^\infty_w(B) \quad , \quad C_\phi(f) = f \circ \phi.$$

Obviously C_ϕ is linear. We want to find conditions on v, w or ϕ that guarantee that C_ϕ is well defined, continuous or compact. We begin by studying when is C_ϕ well defined.

Proposition 2.3.1
If there is some $0 < r < 1$ such that $\phi(B) \subseteq rB$, then $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is well defined for any two weights v, w.
Proof.
Since $\phi(B) \subseteq rB$, $\phi(B)$ is a B-bounded set. Then for each $f \in H_v^\infty(B)$ there is $K > 0$ such that $\sup_{y\in\phi(B)} |f(y)| \leq K$. Hence

$$\begin{aligned} \sup_{x\in B} w(x)|f\circ\phi(x)| &= \sup_{x\in B} w(x)|f(\phi(x))| \\ &\leq \sup_{x\in B} w(x) \ \sup_{x\in B} |f(\phi(x))| \leq C\cdot K < \infty. \end{aligned}$$

And $C_\phi(f) \in H_w^\infty(B)$.
q.e.d.

We can weaken slightly the condition on ϕ at the expense of imposing some restriction on the weights.

Proposition 2.3.2
Let v, w and ϕ be such that $\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} \dfrac{w(x)}{\tilde{v}(\phi(x))} < \infty$.
Then, $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is well defined.
Proof.
To simplify notation we call L the limit in the statement. Since L is finite, there is $r_0 \in]0,1[$ such that for any $r_0 < r < 1$,

$$\left| \sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}(\phi(x))} - L \right| < \frac{1}{2}.$$

This implies that for all $x \in B$ such that $\|\phi(x)\| > r_0$

$$\left| \frac{w(x)}{\tilde{v}(\phi(x))} - L \right| < \frac{1}{2}.$$

Let $x \in B$ and $f \in H_v^\infty(B)$. Suppose that $\|\phi(x)\| > r_0$, then

$$\begin{aligned} w(x)|f(\phi(x))| &= \frac{w(x)}{\tilde{v}(\phi(x))}\tilde{v}(\phi(x))|f(\phi(x))| \\ &\leq \left(\left| \frac{w(x)}{\tilde{v}(\phi(x))} - L \right| + |L| \right) \tilde{v}(\phi(x))|f(\phi(x))| \\ &\leq \left(\frac{1}{2} + |L| \right) \|f\|_{\tilde{v}} < \infty. \end{aligned}$$

Suppose now that $\|\phi(x)\| \leq r_0$. Since f is bounded in $\overline{B(0,r_0)}$ we have $w(x)|f(\phi(x))| \leq C\cdot K$.

Joining both cases we have $\sup_{x\in B} w(x)|f(\phi(x))| < \infty$ and $C_\phi(f) \in H_w^\infty(B)$ for all $f \in H_v^\infty(B)$.
q.e.d.

2.4 Continuity

We give now some results about the continuity of C_ϕ. We obtain first a condition for the general case and after that an easier condition for a more restricted case.

Remark 2.4.1
Given any two weights v and w and $\phi : B \longrightarrow$ holomorphic, the composition operator $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is (τ_0, τ_0)-continuous. Indeed, consider a net $f_\alpha \overset{\tau_0}{\longrightarrow} 0$ and take $K \subseteq B$ compact and $\varepsilon > 0$. Since ϕ is continuous, $\phi(K)$ is compact and we can find α_0 such that $|f_\alpha(y)| < \varepsilon$ for all $y \in \phi(K)$ and $\alpha \geq \alpha_0$. Now, given any $x \in K$ and $\alpha \geq \alpha_0$ we have $|C_\phi f_\alpha(x)| = |f_\alpha(\phi(x))| < \varepsilon$. With this $C_\phi f_\alpha \longrightarrow 0$ in τ_0.

Proposition 2.4.2
Let v, w be two weights and $\phi : B \longrightarrow B$ holomorphic. Then the following are equivalent,
(i) $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ *is continuous.*
(ii) $\displaystyle\sup_{x\in B} \frac{w(x)}{\tilde{v}(\phi(x))} = M < \infty.$
(iii) $\displaystyle\sup_{x\in B} \frac{\tilde{w}(x)}{\tilde{v}(\phi(x))} = M < \infty.$
Proof.
The implication $(iii) \Rightarrow (ii)$ is trivial, since $w \leq \tilde{w}$.
Assume that (ii) holds and let us show that C_ϕ is continuous. It is enough to check that $C_\phi(B_v) \subseteq H_w^\infty(B)$ is bounded. Let $f \in B_v$. For any $x \in B$ we have

$$w(x)|f(\phi(x))| = \frac{w(x)}{\tilde{v}(\phi(x))}\tilde{v}(\phi(x))|f(\phi(x))| \leq M\|f\|_{\tilde{v}} \leq M.$$

Hence

$$\|C_\phi(f)\|_w = \|f \circ \phi\|_w = \sup_{x\in B} w(x)|f(\phi(x))| \leq M.$$

And C_ϕ is continuous.
Suppose now that C_ϕ is continuous. If (iii) does not hold there exists $(x_n)_{n\in\mathbb{N}} \subseteq B$ such that $\tilde{w}(x_n) > n\ \tilde{v}(\phi(x_n))$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ take $f_n \in B_v$ so that

$$|f_n(\phi(x_n))| = \tilde{u}(\phi(x_n)) > \frac{\tilde{u}(\phi(x_n))}{2}.$$

Since C_ϕ is linear and continuous $C_\phi(B_v)$ is bounded in $H_w^\infty(B) = H_{\tilde{w}}^\infty(B)$ (see Proposition 2.2.5). Thus, we can find $C > 0$ such that $\|C_\phi(f)\|_{\tilde{w}} \leq C$ for all $f \in B_v$. In particular we have, for all $n \in \mathbb{N}$ and $x \in B$,

$$|f_n(\phi(x))|\tilde{w}(x) \leq C.$$

On the other hand, for each $n \in \mathbb{N}$

$$|f_n(\phi(x_n))|\tilde{w}(x_n) = |f_n(\phi(x_n))|\tilde{v}(\phi(x_n))\frac{\tilde{w}(x_n)}{\tilde{v}(\phi(x_n))} > \frac{n}{2}.$$

This leads to a contradiction and completes the proof.
q.e.d.

From Definition 2.2.1 and the Proposition 2.4.2 we get immediately.

Corollary 2.4.3
Let v be essential. Then, the operator $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is continuous if and only if

$$\sup_{x\in B} \frac{w(x)}{v(\phi(x))} < \infty.$$

We restrict now our attention to a slightly more particular case, this allows us to find a condition that does not depend on ϕ. Let B be the open unit ball of a Hilbert space H with a scalar product $(\cdot|\cdot)$. For each $a \in B$ we define the linear mapping $\Gamma(a) : B \longrightarrow B$ by

$$\Gamma(a)(x) = \frac{1}{1+v(a)}\ a\ (x|a)\ +\ v(a)\ x\ ,$$

where $v(a) = \sqrt{1-\|a\|^2}$. Using this mapping we define an automorphism of B, $\alpha_a : B \longrightarrow B$, by

$$\alpha_a(x) = \Gamma(a)\frac{x-a}{1-(x|a)}.$$

These Möbius transforms for Hilbert spaces were defined by Renaud in [66], where a deeper study can be found. Each one of them is holomorphic, and satisfies $\alpha_a(0) = a$, $\alpha_a^{-1} = \alpha_{-a}$ and

$$\sup_{\|x\|=r} \|\alpha_a(x)\| = \frac{\|a\|+r}{1+r\|a\|}\ \ ,\ \ \inf_{\|x\|=r} \|\alpha_a(x)\| = \frac{|\|a\|-r|}{1-r\|a\|}.$$

Also in [66] there is the following analogue of the classical Schwarz's Lemma,

Proposition 2.4.4
Let H, F be two Hilbert spaces and B_H, B_F their open unit balls.
Let $f : B_H \longrightarrow B_F$ holomorphic such that $f(0) = 0$. Then for all $x \in B_H$

$$\|f(x)\|_F \leq \|x\|_H.$$

With this we can prove the following result.

Theorem 2.4.5
Let B the open unit ball of a Hilbert space H and $v : B \longrightarrow]0,+\infty[$ a radial weight, decreasing with respect to $\|x\|$. Then the following are equivalent,
(i) $C_\phi : H_v^\infty(B) \longrightarrow H_v^\infty(B)$ is bounded for all ϕ.
(ii) Each $(x_n)_{n\in\mathbb{N}} \subseteq B$ such that $\|x_n\| = 1 - 2^{-n}$ satisfies:

$$\inf_{n\in\mathbb{N}} \frac{\tilde{v}(x_{n+1})}{\tilde{v}(x_n)} > 0. \tag{2.4}$$

Proof.
First of all note that if $\phi(0) = 0$, then C_ϕ is automatically continuous. Indeed, since v is decreasing, Proposition 2.4.4 implies

$$\begin{aligned} \|C_\phi(f)\|_v &= \sup_{\|x\|\leq 1} v(x)|f(\phi(x))| \leq \sup_{\|x\|\leq 1} v(\phi(x))|f(\phi(x))| \\ &\leq \sup_{\|x\|\leq 1} v(x)|f(x)| = \|f\|_v \end{aligned}$$

and C_ϕ is continuous.

For each $a \in B$ we have $\alpha_a : B \longrightarrow B$. Suppose that every C_{α_a} is continuous. Given any ϕ, let $a = \phi(0)$ and define $\psi = \alpha_a \circ \phi$. Obviously $\psi(0) = 0$ and C_ψ is continuous. Then we have $\phi = \alpha_{-a} \circ \psi$ and $C_\phi = C_\psi \circ C_{\alpha_{-a}}$ is continuous. We would have, then, that every C_ϕ is continuous. Therefore it is enough to prove

$$C_{\alpha_a} : H_v^\infty(B) \longrightarrow H_v^\infty(B) \text{ is continuous for all } a \in B \Leftrightarrow (2.4)$$

Let us begin by assuming that C_{α_a} is continuous for every $a \in B$. By Proposition 2.4.2, for each $a \in B$ we can find $M_a > 0$ such that $\tilde{v}(x) \leq M_a \tilde{v}(\alpha_a(x))$ for all $x \in B$. We also know that $\sup_{\|x\|=r} \|\alpha_a(x)\| = \frac{\|a\|+r}{1+r\|a\|}$ and it is attained at $x_0 = \frac{-r}{\|a\|}a$ (see [66]). Since v is radial so also is $\tilde{v}$ and

$$\tilde{v}(x) = \tilde{v}\left(\frac{-r}{\|a\|}a\right) \leq M_a \tilde{v}\left(\alpha_a\left(\frac{-r}{\|a\|}a\right)\right)$$

for every $x \in B$ with $\|x\| = r$. Define a new function, $l(r) = \tilde{v}(x)$ with $\|x\| = 1 - r$. Since $\tilde{v}$ is radial l is independent from the choice of x and it is well defined. With this new function we can rewrite the previous inequality in the following way

$$\begin{aligned} l(1-r) &= \tilde{v}(x) \leq M_a\ \tilde{v}\left(\alpha_a\left(\frac{-r}{\|a\|}a\right)\right) = M_a\ l\left(1 - \left\|\alpha_a\left(\frac{-r}{\|a\|}a\right)\right\|\right) \\ &= M_a\ l\left(1 - \frac{\|a\| + r}{1 + r\|a\|}\right) \end{aligned}$$

for all $0 < r < 1$.
Let us see that l is increasing. Take $r_1 \geq r_2$ and $x_1, x_2 \in B$ with $\|x_i\| = r_i$ for $i = 1, 2$. Then $\|x_1\| \leq \|x_2\|$ and, since $\tilde{v}$ is decreasing, $\tilde{v}(x_1) \geq \tilde{v}(x_2)$. Hence $l(x_1) \leq l(x_2)$.
Let now $s = 1 - r$, then

$$\begin{aligned} l\left(1 - \frac{\|a\| + r}{1 + r\|a\|}\right) &= l\left(1 - \frac{\|a\| + (1-s)}{1 + (1-s)\|a\|}\right) \\ &= l\left(\frac{1 + \|a\| - \|a\|s - \|a\| - 1 + s}{1 + (1-s)\|a\|}\right) \\ &= l\left(\frac{s(1 - \|a\|)}{1 + (1-s)\|a\|}\right). \end{aligned}$$

If $s < 1/2$ then $1 + \frac{\|a\|}{2} \leq 1 + \|a\|(1-s) \leq 1 + \|a\|$ and, since l is increasing

$$l\left(s\ \frac{1 - \|a\|}{1 + \|a\|}\right) \leq l\left(1 - \frac{\|a\| + (1-s)}{1 + (1-s)\|a\|}\right) \leq l\left(s\ \frac{1 - \|a\|}{1 + \|a\|/2}\right). \tag{2.5}$$

Taking $\|a\| = 2/5$ we have

$$s\ \frac{1 - \|a\|}{1 + \|a\|/2} = s\ \frac{1 - 2/5}{1 + \frac{2/5}{2}} = s\ \frac{3/5}{6/5} = \frac{s}{2}.$$

From this,

$$l(s) \leq M_a\ l\left(1 - \frac{\|a\| + (1-s)}{1 + (1-s)\|a\|}\right) \leq M_a\ l\left(\frac{s}{2}\right)$$

for s small enough.
Consider now $(x_n)_{n\in\mathbb{N}} \subseteq B$ with $\|x_n\| = 1 - 2^{-n}$. For n big enough we have $\tilde{v}(x_n) = l(2^{-n}) \leq M_a\ l(2^{-n-1}) = M_a \tilde{v}(x_{n+1})$. Choose $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\frac{\tilde{v}(x_{n+1})}{\tilde{v}(x_n)} \geq \frac{1}{M_a} > 0.$$

This implies

$$\inf_{n\in\mathbb{N}} \frac{\tilde{v}(x_{n+1})}{\tilde{v}(x_n)} > 0.$$

Let us suppose now that (2.4) is true, that is, all $(x_n)_n \subseteq B$ with $\|x_n\| = 1 - 2^{-n}$ satisfy that

$$\inf_{n\in\mathbb{N}} \frac{\tilde{v}(x_{n+1})}{\tilde{v}(x_n)} > 0.$$

Define a function l exactly in the same way as we did before. We can write (2.4) like $\inf_{n\in\mathbb{N}} \frac{l(2^{-(n+1)})}{l(2^{-n})} > 0$. There are $K > 0$ and $n_0 \in \mathbb{N}$ so that for every $n \geq n_0$,

$$l(2^{-(n+1)}) \geq K\ l(2^{-n}).$$

Let $t_0 = 2^{-(n_0+1)}$ and take $t < t_0$. Consider $n \in \mathbb{N}$ such that $2^{-(n-1)} \geq t > 2^{-n} \geq t/2 > 2^{-(n+1)}$. Since l is increasing,

$$l(t/2) \geq l(2^{-(n+1)}) \geq K\ l(2^{-n}) \geq K^2\ l(2^{-(n-1)}) \geq K^2\ l(t).$$

Thus, there is $M > 0$ such that $l(t) \leq M\ l(t/2)$ for all $t < t_0$.
For each $c > 0$ we have $n \in \mathbb{N}$ with $c < 2^n$. If $t < t_0$,

$$l(t) \leq M\ l(t/2) \leq \ldots \leq M^n\ l(t/2^n) \leq M^n\ l(t/c).$$

Take $c = \dfrac{1 + \|a\|}{1 - \|a\|}$ and use the first inequality in (2.5) to get that for each $a \in B$ there exists $K_a > 0$ such that

$$l(t) \leq K_a\ l(t/c) = K_a\ l\left(t\frac{1 - \|a\|}{1 + \|a\|}\right) \leq K_a l\left(1 - \frac{\|a\| + (1-t)}{1 + (1-t)\|a\|}\right)$$

for $t < t_0 \leq 1/2$.
Let now $t > t_0$. Since v is strictly positive, so is $\tilde{v}$ and l is strictly positive too. Define a function $h : [t_0, 1] \longrightarrow \mathbb{R}$ by $h(t) = \dfrac{l(t)}{l\left(1 - \frac{\|a\|+(1-t)}{1+(1-t)\|a\|}\right)}$. This is continuous and attains its maximum in $[t_0, 1]$. Let $C_a > 0$ such that

$$C_a \geq \frac{l(t)}{l\left(1 - \frac{\|a\|+(1-t)}{1+(1-t)\|a\|}\right)} > 0$$

For all $t_0 \leq t \leq 1$.
Joining both cases we can find a constant $M_a > 0$ that gives, for all $0 < t < 1$,

$$l(t) \leq M_a\ l\left(1 - \frac{\|a\| + (1-t)}{1 + (1-t)\|a\|}\right).$$

Hence, if $0 < r < 1$ and $\|x\| = r$, then

$$\begin{aligned} \tilde{v}(x) &= l(1-r) \leq M_a\ l\left(1 - \frac{\|a\| + r}{1 + r\|a\|}\right) \\ &\leq M_a\ l(1 - \|\alpha_a(x)\|) \leq M_a\ \tilde{v}(\alpha_a(x)). \end{aligned}$$

Applying Proposition 2.4.2, C_{α_a} is continuous.
q.e.d.

This proof can be easily adapted to a more general setting than the unit ball of Hilbert spaces, that is the bounded symmetric domains in any Banach space. Given D a domain in a Banach space, a *symmetry* at $a \in D$ is a biholomorphic map $s_a : D \longrightarrow D$ such that $s_a^2 = id$ and $s_a(a) = a$ is an isolated fixed point; for example, if B is the open unit ball of a Banach space X, there is symmetry at 0 given by $s_0(x) = -x$. A bounded symmetric domain is a bounded domain with symmetry at every point.

Definition 2.4.6 A $JB^\star$-triple is a Banach space X with a triple product $\{\ ,\ ,\ \} : X^3 \longrightarrow X$ that is linear and symmetric on the first and third variables (symmetric in the sense that $\{x, y, z\} = \{z, y, x\}$ for all x, z) and antilinear on the second variable satisfying,
(i) The mapping $x\Box x$, given by $x\Box x(z) = \{x, x, z\}$ is Hermitian, $\sigma(x\Box x) \geq 0$ and $\|x\Box x\| = \|x\|^2$.
(ii) For every $a, b, x, y, z \in X$, the equality

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$$

holds.

For each $x, y \in X$ a linear mapping $x\Box y$ is defined by $x\Box y(z) = \{x, y, z\}$. Also, for $x \in X$ an antilinear mapping Q_x is defined by $Q_x(z) = \{x, z, x\}$. With these two mappings, fixing x and y, another very important one is defined as follows

$$B(x, y) = id - 2x\Box y + Q_x \circ Q_y \in \mathcal{L}(X; X).$$

From this, taking $x = y$, we have a new mapping $B_x = B(x, x)^{1/2}$; here the square root is taken in the sense of functional calculus, that is $B_x \circ B_x = B(x, x)$. It was proved in [42] that

$$\|B_x^{-1}\| = \frac{1}{1 - \|x\|^2}.$$

For background on $JB^\star$-triples, see [17], [31], [32] and [53].

Example 2.4.7
There are the main examples of $JB^\star$-triples: $\mathbb{C}$, Hilbert spaces and C*-algebras. On $\mathbb{C}$, the triple product is defined as $\{x, y, z\} = x\overline{y}z$. With this, $B(x, y)(z) = (1 - x\overline{y})^2 z$; doing $x = y$ we get that $B_x(z) = (1 - |x|^2)z$.
If H is a Hilbert space, the triple product can be defined in terms of the scalar product

$$\{x, y, z\} = \frac{1}{2}((x|y)z + (z|y)x).$$

Then $B(x, y)(z) = (1 - (x|y))(z - (z|y)x)$.
In the case of C*-algebras, the situation is the following; the triple product is defined by

$$\{x, y, z\} = \frac{1}{2}(xy^\star z + zy^\star x),$$

which gives that $B(x, y)(z) = (I - xy^\star)z(I - y^\star x)$, where I denotes the unity of the algebra. In this case,

$$B_x(z) = (I - xx^\star)^{1/2} z (I - x^\star x)^{1/2},$$

here the square root should be understood in terms of the algebra product.

It is a well know fact that the open unit ball of a Banach space is symmetric if and only if the space is a $JB^\star$-triple. Also, a bounded domain D is symmetric if and only if it has a transitive group of biholomorphic mappings $\{g_a\}_{a\in D}$ and a symmetry at some point p. In this case the bounded symmetric domain is biholomorphically equivalent to the unit ball of a $JB^\star$-triple and all biholomorphic mappings on the unit ball can be explicitly described. They are of the form Kg_a where K is a surjective linear isometry and g_a are Möbius mappings that satisfy $g_a(0) = a$ and $g_a^{-1} = g_{-a}$ (see [42]). These mappings can be defined from the triple product as

$$g_a(x) = a + (B(a,a)^{-1/2} \circ B(x,a))(x - Q_x(a)).$$

If s_0 denotes the symmetry at 0, the symmetry at any other point of the unit ball a is given by $g_a \circ s_0 \circ g_{-a}$.
The two clue facts in the proof of Theorem 2.4.5 are first the Schwarz Lemma that we use in the beginning and, later, the fact that the supremum of the mappings behaves in a certain way. Therefore, this is what we need in order to prove the result for bounded symmetric domains. In fact, the Schwarz Lemma can be proved for general Banach spaces. This is a well known fact, but the proof is simple and short and we include it here.

Proposition 2.4.8
Let X, Y be two Banach spaces and B_X, B_Y their open unit balls.
Let $f : B_X \longrightarrow B_Y$ holomorphic such that $f(0) = 0$; then, for all $x \in B_X$,

$$\|f(x)\|_Y \leq \|x\|_X.$$

Proof.
Let $x \in B_X$. Consider $y' \in Y'$ with $\|y'\| \leq 1$ and define the function $h : \mathbb{C} \longrightarrow \mathbb{C}$ by $h(\lambda) = y'\left(f\left(\lambda\frac{x}{\|x\|}\right)\right)$. Clearly h is holomorphic, $h(0) = 0$ and if $|\lambda| < 1$ we have

$$|h(\lambda)| = \left|y'\left(f\left(\lambda\frac{x}{\|x\|}\right)\right)\right| \leq \|y'\| \cdot \left\|f\left(\lambda\frac{x}{\|x\|}\right)\right\| < 1.$$

Then, applying the classical Schwarz Lemma, $|h(\lambda)| \leq |\lambda|$ for all $|\lambda| < 1$. But this is true independently of the choice of y'; this implies that for all $|\lambda| < 1$

$$\sup_{\|y'\|\leq 1}\left|y'\left(f\left(\lambda\frac{x}{\|x\|}\right)\right)\right| \leq |\lambda|.$$

This means that $\left\|f\left(\lambda\frac{x}{\|x\|}\right)\right\| \leq |\lambda|$ for all $|\lambda| < 1$. Taking $\lambda = \|x\|$ we get

$$\|f(x)\|_Y \leq \|x\|_X.$$

Since $x \in B_X$ was arbitrary, this is true for all $x \in B_X$.
q.e.d.

Thus, we have now the version of the Schwarz Lemma that we need. Let us study the behaviour of the supremum over spheres of the mappings g_a.

Lemma 2.4.9
Let B be a bounded symmetric domain (i.e., the open unit ball of a $JB^\star$-triple) and $\{g_a\}_{a\in B}$ the transitive group of biholomorphic mappings that define the symmetries. Then, for each $0 < r < 1$

$$\sup_{\|x\|=r} \|g_a(x)\| = \frac{\|a\| + r}{1 + r\|a\|}$$

and this supremum is attained at some point.

Proof.
First, for any bounded symmetric domain we show that $\|g_a(x)\| \leq \frac{\|a\|+\|x\|}{1+\|a\|\cdot\|x\|}$. It is well known (see [53]) that

$$\frac{1}{1 - \|g_a(x)\|^2} = \|B_a^{-1} \circ B(a, x) \circ B_x^{-1}\|.$$

In particular,

$$\begin{aligned} \frac{1}{1 - \|g_a(x)\|^2} &\leq \|B_a^{-1}\| \cdot \|B(a, x)\| \cdot \|B_x^{-1}\| \\ &\leq \frac{1}{1 - \|a\|^2} \, (1 + \|a\| \cdot \|x\|)^2 \, \frac{1}{1 - \|x\|^2}. \end{aligned}$$

And so

$$\begin{aligned} \|g_a(x)\|^2 &\leq 1 - \frac{(1 - \|a\|^2)(1 - \|x\|^2)}{(1 + \|a\| \cdot \|x\|)^2} \\ &= \frac{1 + 2\|a\| \cdot \|x\| + \|a\|^2\|x\|^2 - [1 - \|a\|^2 - \|x\|^2 + \|a\|^2\|x\|^2]}{(1 + \|a\| \; \|x\|)^2} \\ &= \frac{\|a\|^2 + 2\|a\| \cdot \|x\| + \|x\|^2}{(1 + \|a\| \cdot \|x\|)^2} \end{aligned}$$

giving

$$\|g_a(x)\| \leq \frac{\|a\| + \|x\|}{1 + \|a\| \cdot \|x\|}.$$

Next thing to show is that the bound is attained, in the sense that there exists $x \in B$, $\|x\| = r$ with $\|g_a(x)\| = \frac{\|a\|+r}{1+r\ \|a\|}$. Let us consider X_a the $JB^\star$-subtriple of X generated by a, that is, the smallest $JB^\star$-triple contained in X with same triple structure that contains a. Obviously, if we find $x \in X_a$ attaining the bound, then our problem will be solved. A deep result of Kaup (see [41]) shows that for any $JB^\star$-triple and $a \in X$, X_a is isometrically (triple) isomorphic to $\mathcal{C}_0(\Omega)$, where $\Omega \subseteq \mathbb{R}$ satisfies that $\Omega \cup \{0\}$ is compact. The Möbius maps on the unit ball of X_a, once composed with this isomorphism, give the following ones $g_c(z) = \frac{c+z}{1+\bar{c}\ z}$, where c and z are in the open unit ball of $\mathcal{C}_0(\Omega)$. If we take $z = \frac{r}{\|c\|}c$ then $z \in \mathcal{C}_0(\Omega)$ and $\|z\| = r$. We get

$$g_c(z) = \frac{\left(1 + \frac{r}{\|c\|}\right)\ c}{1 + |c|^2\ \frac{r}{\|c\|}} = \frac{r + \|c\|}{\|c\| + r\ |c|^2}\ c.$$

Now, $\|g_c(z)\| = (r + \|c\|)\ \left\|\frac{c}{\|c\|+r\ |c|^2}\right\| = (r + \|c\|)\ \sup_{\omega\in\Omega} \frac{|c|}{\|c\|+r\ |c|^2}(\omega)$. But since $|c| \leq \|c\| \leq 1$ and $r < 1$, it turns out that $\frac{|c|}{\|c\|+r\ |c|^2}$ is an increasing function of $|c|$, that is

$$\left\|\frac{c}{\|c\| + r\ |c|^2}\right\| = \frac{\|c\|}{\|c\| + r\ \|c\|^2} = \frac{1}{1 + r\ \|c\|}$$

This gives

$$\|g_c(z)\| = \frac{\|c\| + \|z\|}{1 + \|c\| \cdot \|z\|}.$$

Since $\|g_u(v)\| = \|g_v(u)\|$ we have what we wanted.
q.e.d.

Using Proposition 2.4.8 and Lemma 2.4.9 and repeating the proof of Theorem 2.4.5 we have

Theorem 2.4.10
Let B the open unit ball of a $JB^\star$-triple X and $v : B \longrightarrow]0, +\infty[$ a radial, decreasing with respect to $\|x\|$ weight. Then, the following are equivalent,
(i) $C_\phi : H_v^\infty(B) \longrightarrow H_v^\infty(B)$ is bounded for all ϕ.
(ii) Each $(x_n)_{n\in\mathbb{N}} \subseteq B$ such that $\|x_n\| = 1 - 2^{-n}$ satisfies:

$$\inf_{n\in\mathbb{N}} \frac{\tilde{v}(x_{n+1})}{\tilde{v}(x_n)} > 0.$$

2.5 Compactness

We give now conditions to have that C_ϕ is compact. Recall that an operator $T \in \mathcal{L}(E, F)$ is compact if the image of the open unit ball of E is relatively compact.
The proof of the following lemma is very similar to that of Section 2.4 in [69] and Lemma 3.1 in [5].

Lemma 2.5.1
Let $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ continuous; then the following are equivalent,
(i) C_ϕ is compact.

(ii) Each bounded sequence $(f_n)_n \subseteq H_v^\infty(B)$ such that $f_n \xrightarrow{\tau_0} 0$ satisfies that $\|C_\phi f_n\|_w \longrightarrow 0$.

Proof.

Suppose C_ϕ is compact. Then $C_\phi(B_v)$ is relatively compact in $H_w^\infty(B)$. Take $(f_n)_{n\in\mathbb{N}} \subseteq H_v^\infty(B)$ bounded such that $f_n \longrightarrow 0$ in τ_0. By Remark 2.4.1, $C_\phi f_n \xrightarrow{\tau_0} 0$. Since convergence in $\|\cdot\|_w$ implies that of τ_0, each $\|\cdot\|_w$-convergent subsequence of $(C_\phi f_n)_{n\in\mathbb{N}}$ will converge to 0.

If $(\|C_\phi f_n\|_w)_{n\in\mathbb{N}}$ does not converge to 0, there exist a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and $c > 0$ such that $\|C_\phi f_{n_k}\|_w \geq c$ for all $k \in \mathbb{N}$. But $(f_{n_k})_{k\in\mathbb{N}}$ is bounded and C_ϕ is compact, therefore $(C_\phi f_{n_k})_{k\in\mathbb{N}}$ is relatively compact and has a convergent subsequence. This new subsequence is also a subsequence of $(C_\phi f_n)_{n\in\mathbb{N}}$ and it must converge to 0. This gives a contradiction. So, $\lim_{n\to\infty} \|C_\phi f_n\|_w = 0$.

Assume *(ii)* holds. Let $(f_n)_{n\in\mathbb{N}} \subseteq B_v$. By [67], Proposition 2.1.3, B_v is τ_0-compact, in particular it is τ_0-bounded. Then, $(f_n)_n$ is τ_0-bounded. By Montel's Theorem we can extract a subsequence $g_k = f_{n_k}$ converging in τ_0 to $g \in H(B)$. For each $x \in B$ and $k \in \mathbb{N}$ we have $v(x)|g_k(x)| \leq \|g_k\|_v \leq 1$. Hence

$$1 \geq \lim_k v(x)|g_k(x)| = v(x) \lim_k |g_k(x)| = v(x)|g(x)|.$$

This implies $\sup_{x\in B} v(x)|g(x)| < \infty$ and $g \in H_v^\infty(B)$.

Thus, $(g_k - g)_{k\in\mathbb{N}}$ is bounded in $H_v^\infty(B)$ and $(g_k - g) \longrightarrow 0$ in τ_0. By hypothesis $\lim_{k\to\infty} \|C_\phi(g_k - g)\|_w = 0$. This implies that $C_\phi(B_v)$ is relatively compact and C_ϕ is compact.

q.e.d.

We will use this lemma several times.

Proposition 2.5.2

Let v, w be two weights and $\phi : B \longrightarrow B$ such that $\phi(B)$ is relatively compact and $\overline{\phi(B)} \subseteq B$. Then $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is compact.

Proof.

Since $\overline{\phi(B)} \subseteq B$ is compact,

$$\sup_{x\in B} \frac{1}{\tilde{v}(\phi(x))} = \sup_{y\in\phi(B)} \frac{1}{\tilde{v}(y)} \leq \sup_{y\in\overline{\phi(B)}} \frac{1}{\tilde{v}(y)} < \infty.$$

By definition w is bounded, this implies $\sup_{x\in B} \frac{w(x)}{\tilde{v}(\phi(x))} < \infty$. By Proposition 2.4.2, C_ϕ is continuous.

Let $(f_n)_{n\in\mathbb{N}} \subseteq H_v^\infty(B)$ τ_0-convergent to 0 and $\varepsilon > 0$. Let us write $C' = \sup_{x\in B} w(x) < \infty$. Since $\overline{\phi(B)} \subseteq B$ is compact, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\sup_{y\in\overline{\phi(B)}} |f_n(y)| < \frac{\varepsilon}{C'}.$$

Then, if $n \geq n_0$

$$\|C_\phi f_n\|_w = \sup_{x \in B} w(x)|f_n(\phi(x))| \leq C' \sup_{y \in \phi(B)} |f_n(y)| \leq C' \sup_{y \in \overline{\phi(B)}} |f_n(y)| < \varepsilon.$$

Hence $\|C_\phi f_n\|_w \longrightarrow 0$. Proposition 2.5.1 implies that C_ϕ is compact.
q.e.d.

We have now a condition on ϕ that makes C_ϕ compact independently from the weights. We are going to weaken the condition on ϕ and get a characterization; to do this we will again need to impose some condition on the weights. Before that we need the following fairly elementary remark.

Remark 2.5.3
Given any sequence of real numbers $(a_n)_{n\in\mathbb{N}} \subseteq]1/2, 1[$ with $\lim_n a_n = 1$ we can always find another one $(\alpha(n))_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\lim_n \alpha(n) = \infty$ and $a_n^{\alpha(n)} > 1/2$ for all $n \in \mathbb{N}$. Indeed, since $(a_n)_n$ tends to 1, $\lim_n \log a_n = 0$ and $\lim_n -\frac{\log 2}{\log a_n} = +\infty$. It is enough then to consider any sequence $(\alpha(n))_n$ such that $\alpha(n) > -\frac{\log 2}{\log a_n}$. Any such sequence satisfies $\log a_n^{\alpha(n)} > -\log 2$ and $a_n^{\alpha(n)} > 1/2$.

With this we prove the following characterization.

Theorem 2.5.4
Let v, w be two weights and $\phi : B \longrightarrow B$ with $\phi(B)$ relatively compact.
Then, $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is compact if and only if

$$\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}(\phi(x))} = 0. \tag{2.6}$$

Proof.
Let C_ϕ be compact and suppose that

$$\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}(\phi(x))} \neq 0.$$

So, we can find $(r_n)_n \subseteq]0, 1[$ with $\lim_n r_n = 1$ and $c > 0$ so that, for all $n \in \mathbb{N}$,

$$\sup_{\|\phi(x)\|>r_n} \frac{w(x)}{\tilde{v}(\phi(x))} \geq c.$$

From this we get a sequence $(x_n)_{n\in\mathbb{N}} \subseteq B$ with $\|\phi(x_n)\| > r_n$ and $w(x_n) \geq c\ \tilde{v}(\phi(x_n))$ for all $n \in \mathbb{N}$. Applying Proposition 2.2.4, for each $n \in \mathbb{N}$ choose $f_n \in B_v$ satisfying $|f_n(\phi(x_n))| = \tilde{u}(\phi(x_n))$.
On the other hand, since $\phi(B)$ is relatively compact we can suppose, going to a subsequence

if necessary, that $(\phi(x_n))_{n\in\mathbb{N}}$ converges to $x_0 \in \overline{B}$. Now, $1 > \|\phi(x_n)\| > r_n$ and $r_n \longrightarrow 1$; hence $\|x_0\| = 1$.
Applying Hahn-Banach Theorem, take $x' \in X'$ with $\|x'\| = x'(x_0) = \|x_0\| = 1$. Since $\lim_n |x'(\phi(x_n))| = |x'(x_0)| = 1$, there exists $n_0 \in \mathbb{N}$ such that $|x'(\phi(x_n))| > 1/2$ for all $n \geq n_0$. We are only interested in the behaviour of the limit of the sequence, therefore we can assume that the whole sequence satisfies this condition. By Remark 2.5.3 we can find $(\alpha(n))_{n\in\mathbb{N}} \subseteq \mathbb{N}$ with $\lim_n \alpha(n) = \infty$ satisfying

$$|x'(\phi(x_n))|^{\alpha(n)} > \frac{1}{2}$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ define $g_n(x) = x'(x)^{\alpha(n)} f_n(x)$ holomorphic. We have

$$\begin{aligned}
\sup_{x\in B} v(x)\ |x'(x)|^{\alpha(n)}\ |f_n(x)| &\leq \sup_{x\in B} v(x)\ \|x\|^{\alpha(n)}\ |f_n(x)| \\
&\leq \sup_{x\in B} v(x)\ |f_n(x)| \leq 1.
\end{aligned}$$

Hence $(g_n)_{n\in\mathbb{N}} \subseteq H_v^\infty(B)$ and is bounded. Since B_v is τ_0-bounded ([67], Proposition 2.1.3), given any $K \subseteq B$ compact there exists a $M > 0$ such that $\sup_{x\in K} |f_n(x)| \leq M$ for all $n \in \mathbb{N}$. On the other hand, since K is compact, there is $1 > C > 0$ with $\|x\| \leq C$ for all $x \in K$; with this,

$$\sup_{x\in K} |g_n(x)| = \sup_{x\in K} |x'(x)|^{\alpha(n)}\ |f_n(x)| \leq M\ \sup_{x\in K} \|x\|^{\alpha(n)} \leq M\ C^{\alpha(n)}.$$

The last term tends to 0 as $n \to \infty$. Thus, $(g_n)_{n\in\mathbb{N}} \subseteq H_v^\infty(B)$ is bounded and $g_n \longrightarrow 0$ uniformly over the compact subsets of B. By Lemma 2.5.1, $\|C_\phi(g_n)\|_w \longrightarrow 0$. On the other hand,

$$\begin{aligned}
\|C_\phi(g_n)\|_w &= \sup_{x\in B} w(x)\ |g_n(\phi(x))| \\
&\geq w(x_n)\ |g_n(\phi(x_n))| \\
&= w(x_n)\ |x'(\phi(x_n))|^{\alpha(n)}\ |f_n(\phi(x_n))| \\
&= w(x_n)\ |x'(\phi(x_n))|^{\alpha(n)}\ \tilde{u}(\phi(x_n)) \\
&= \frac{w(x_n)}{\tilde{v}(\phi(x_n))}\ |x'(\phi(x_n))|^{\alpha(n)} > c\frac{1}{2}.
\end{aligned}$$

This contradicts the fact that it converges to 0. Therefore,

$$\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}(\phi(x))} = 0.$$

Assuming now that (2.6) holds let us show that C_ϕ is compact. Since the limit is 0 we can find $0 < r_0 < 1$ such that, denoting $G = \{x \in B : \|\phi(x)\| \leq r_0\}$,

$$\sup_{x\in B} \frac{w(x)}{\tilde{v}(\phi(x))} = \sup_{x\in G} \frac{w(x)}{\tilde{v}(\phi(x))}.$$

We have $\overline{\phi(G)} \subseteq \overline{\phi(B)} \cap \overline{B(0,r_0)}$. This implies that $\overline{\phi(G)}$ is compact. Therefore there are $M, N > 0$ such that $0 < M < \tilde{v}(\phi(x)) < N$ for all $x \in G$. Then

$$\sup_{x\in B} \frac{w(x)}{\tilde{v}(\phi(x))} \leq \frac{1}{M} \sup_{x\in B} w(x) < \infty.$$

By Proposition 2.4.2, C_ϕ is continuous.
Let us see that C_ϕ is compact. By (2.6), given $\varepsilon > 0$ there is $r_0 \in]0,1[$ such that, for all $r_0 < r < 1$,

$$\sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}(\phi(x))} < \varepsilon.$$

This implies $w(x) < \varepsilon\ \tilde{v}(\phi(x))$ for all $\|\phi(x)\| > r_0$. Suppose that C_ϕ is not compact. By Lemma 2.5.1 there exists $(f_n)_{n\in\mathbb{N}} \subseteq B_v$ τ_0-converging to 0 such that $(\|C_\phi f_n\|_w)_{n\in\mathbb{N}}$ does not converge to 0. Taking a subsequence if necessary, there is $\lambda > 0$ such that $\|C_\phi f_n\|_w \geq \lambda$ for all $n \in \mathbb{N}$. We can consider a sequence $(x_n)_{n\in\mathbb{N}} \subseteq B$ with $w(x_n)|f_n(x_n)| \geq \lambda$ for all $n \in \mathbb{N}$.
If $\|\phi(x_n)\| \to 1$, there exists $n_1 \in \mathbb{N}$ with $\|\phi(x_n)\| > r_0$ for all $n \geq n_1$. So, for $n \geq n_1$, $w(x_n) < \varepsilon\ \tilde{v}(\phi(x_n))$. Applying Proposition 2.2.4,

$$\begin{aligned} \lambda &\leq w(x_n)\ |f_n(\phi(x_n))| < \varepsilon\ \tilde{v}(\phi(x_n))\ |f_n(\phi(x_n))| \\ &\leq \varepsilon\ \|f_n\|_{\tilde{v}} \leq \varepsilon. \end{aligned}$$

Hence $\lambda \leq \varepsilon$ for every $\varepsilon > 0$. This leads to a contradiction.
Suppose now that $(\|\phi(x_n)\|)_{n\in\mathbb{N}}$ does not converge to 1. Going to a subsequence if necessary we can choose $\eta \in]0,1[$ satisfying $\|\phi(x_n))\| \leq \eta$ for all $n \in \mathbb{N}$. Then $\overline{(\phi(x_n))_n} \subseteq \overline{B(0,\eta)} \subseteq B$. Since $\phi(B)$ is relatively compact, $\overline{(\phi(x_n))_n} \subseteq \overline{\phi(B)}$ is a closed subset of a compact set. This implies that $\overline{(\phi(x_n))_n}$ is compact. Given any $\varepsilon > 0$ and taking $C' = \sup_{x\in B} w(x)$ there is $n_2 \in \mathbb{N}$ such that for all $k \geq n_2$

$$\sup_{y\in\overline{(\phi(x_n))_n}} |f_k(y)| < \frac{\varepsilon}{C'}.$$

Hence, if $k \geq n_2$

$$\sup_{n\in\mathbb{N}} |f_k(\phi(x_n))| < \frac{\varepsilon}{C'}.$$

In particular, $\sup_{n\geq n_2} |f_n(\phi(x_n))| < \frac{\varepsilon}{C'}$. So, if $n \geq n_2$ we have $|f_n(\phi(x_n))| < \frac{\varepsilon}{C'}$ and $\lambda \leq w(x_n)\ |f_n(\phi(x_n))| < \varepsilon$. Thus $\lambda \leq \varepsilon$ for all $\varepsilon > 0$ This leads to a contradiction, coming from supposing that C_ϕ is not compact. Hence C_ϕ is compact.
q.e.d.

We can even prove a condition with an easier to handle limit.

Proposition 2.5.5
Let v, w be two weights and $\phi : B \longrightarrow B$ with $\phi(B)$ relatively compact such that

$$\lim_{\|x\| \to 1^-} \frac{w(x)}{\tilde{v}(\phi(x))} = 0.$$

Then $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is compact.
Proof.
We begin by showing that C_ϕ is continuous. Given $\varepsilon > 0$ there is some $0 < r_0 < 1$ such that, for $r_0 \leq \|x\| < 1$,

$$\left| \frac{w(x)}{\tilde{v}(\phi(x))} \right| < \varepsilon.$$

Then obviously,

$$\sup_{\|x\| > r_0} \frac{w(x)}{\tilde{v}(\phi(x))} \leq \varepsilon.$$

We study now the supremum of this expression on $\{x \in B : \|x\| \leq r_0\}$. We claim that $\sup_{\|x\| \leq r_0} \|\phi(x)\| \neq 1$. Suppose that there exists $(x_n)_n$ with $\|x_n\| \leq r_0$ such that $\|\phi(x_n)\| \longrightarrow 1$. Since $\phi(B)$ is relatively compact we can extract a subsequence $(\phi(x_{n_k}))_k$ converging to y_0 with $\|y_0\| = 1$.
By the Hahn-Banach Theorem we can choose $x' \in X'$ such that $\|x'\| = 1$ and $x'(y_0) = 1$. Define a mapping $\psi = x' \circ \phi : B \longrightarrow \mathbb{C}$. Clearly, ψ is bounded and $\sup_{\|x\| \leq r_0} |\psi(x)| = 1$. We denote by $\mathbb{D}$ the open unit disc of $\mathbb{C}$. Let $a = \psi(0)$. Take $g_a : \mathbb{D} \longrightarrow \mathbb{D}$ the Möbius transform such that $g_a(a) = 0$. The mapping $g_a \circ \psi : B \longrightarrow \mathbb{D}$ clearly satisfies $g_a \circ \psi(0) = 0$. By the Schwarz's Lemma (Proposition 2.4.8),

$$|g_a \circ \psi(x)| \leq \|x\|$$

for all $x \in B$. This implies $g_a \circ \psi(\overline{B(0, r_0)}) \subseteq \overline{D(0, r_0)}$. Hence

$$\psi(\overline{B(0, r_0)}) \subseteq g_a^{-1}(\overline{D(0, r_0)}).$$

The last set is compact in $\mathbb{D}$. Therefore there is $0 < s < 1$ such that $\psi(\overline{B(0, r_0)}) \subseteq \overline{D(0, s)}$. Then

$$\sup_{\|x\| \leq r_0} |\psi(x)| \leq s < 1.$$

This leads to a contradiction and proves our claim. Hence there exists $0 < t < 1$ such that

$$\sup_{\|x\| \leq r_0} |\phi(x)| \leq t < 1.$$

This means that $\phi(B(0,r_0)) \subseteq \phi(\overline{B(0,r_0)}) \subseteq \overline{B(0,t)}$ and

$$\overline{\phi(B(0,r_0))} \subseteq \overline{B(0,t)} \cap \overline{\phi(B)}.$$

Thus $\overline{\phi(B(0,r_0))}$ is compact and there are $M, N > 0$ such that $0 < M \leq \tilde{v}(\phi(x)) \leq N$ for all $\|x\| \leq r_0$. Therefore

$$\sup_{\|x\| \leq r_0} \frac{w(x)}{\tilde{v}(\phi(x))} \leq \frac{1}{M} \sup_{\|x\| \leq r_0} w(x) < \infty.$$

This gives

$$\sup_{x \in B} \frac{w(x)}{\tilde{v}(\phi(x))} < \infty$$

and C_ϕ is continuous.
Let us suppose that C_ϕ is not compact. From Lemma 2.5.1 this means that there is a τ_0-null sequence of functions $(f_n)_n \subseteq B_v$ such that $(\|C_\phi(f_n)\|_w)_n$ does not converge to 0 in $\mathbb{C}$. Going to a subsequence if necessary we can assume that there is $\lambda > 0$ such that

$$\sup_{x \in B} w(x)|f_n(\phi(x))| = \|C_\phi(f_n)\|_w \geq \lambda > 0$$

for all $n \in \mathbb{N}$. Choose $(x_n)_n \subseteq B$ with $w(x_n)|f_n(\phi(x_n))| \geq \lambda$ for all n and suppose that $\|x_n\| \to 1$. Given any $\varepsilon > 0$ there is $0 < r_0 < 1$ such that, for every $r_0 \leq \|x\| < 1$,

$$\frac{w(x)}{\tilde{v}(\phi(x))} < \varepsilon.$$

Take n_1 such that $\|x_n\| \geq r_0$ for all $n \geq n_1$. Then

$$w(x_n) \leq \varepsilon \tilde{v}(\phi(x_n))$$

for all $n \geq n_1$. Hence

$$\lambda \leq w(x_n)|f_n(\phi(x_n))| < \varepsilon \tilde{v}(\phi(x_n))|f_n(\phi(x_n))| \leq \varepsilon \|f_n\|_{\tilde{v}} < \varepsilon.$$

Therefore, $\lambda \leq \varepsilon$ for all $\varepsilon > 0$, but $\lambda > 0$. We have, then, to assume that $(\|x_n\|)_n$ does not converge to 1. Taking a subsequence if necessary we can choose $1 > \eta > 0$ such that $\|x_n\| \leq \eta$ for every n. From what we have already seen, $\overline{\phi(B(0,\eta))}$ is compact. This implies that $\overline{\phi((x_n)_n)}$ is also compact. Let $\varepsilon > 0$ and write $C' = \sup_{x \in B} w(x)$. Since $f_n \to 0$ in τ_0, there is n_2 such that, for $k \geq n_2$

$$\sup_{y \in \overline{\phi((x_n)_n)}} |f_k(y)| < \frac{\varepsilon}{C'}.$$

This gives $|f_n(\phi(x_n))| < \varepsilon / C'$ for all $n \geq n_2$. Hence

$$\lambda \leq w(x_n)|f_n(\phi(x_n))| < \varepsilon.$$

Since ε was arbitrary this again gives a contradiction and finally shows that C_ϕ is compact.
q.e.d.

Imposing some not very restrictive conditions on the weights we can get another characterization.

Proposition 2.5.6
Let v, w be two weights so that $\lim_{\|x\|\to 1^-} w(x) = 0$ and $\phi : B \longrightarrow B$ with $\phi(B)$ relatively compact.
Then, $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is compact if and only if

$$\lim_{\|x\|\to 1^-} \frac{w(x)}{\tilde{v}(\phi(x))} = 0.$$

Proof.
One implication has already been proved in Proposition 2.5.5. To prove the other one, assume

$$\lim_{\|x\|\to 1^-} \frac{w(x)}{\tilde{v}(\phi(x))} \neq 0.$$

Then we have a sequence $(x_n)_{n\in\mathbb{N}} \subseteq B$ with $\lim_n \|x_n\| = 1$ and $c > 0$ such that $\frac{w(x_n)}{\tilde{v}(\phi(x_n))} \geq c$ for all $n \in \mathbb{N}$. By Proposition 2.2.4, for each $n \in \mathbb{N}$ we get $f_n \in B_v$ such that $|f_n(\phi(x_n))| = \tilde{u}(\phi(x_n))$.
Since $\phi(B)$ is relatively compact we can assume, going to a subsequence if necessary, that $(\phi(x_n))_{n\in\mathbb{N}}$ converges to $x_0 \in \overline{B}$. If that $\|x_0\| \neq 1$ then

$$0 = \lim_n w(x_n) \geq c \ \lim_n \tilde{v}(\phi(x_n)) = c \ \tilde{v}(x_0) > 0.$$

Thus $\|x_0\| = 1$. From now on, applying the Hahn-Banach Theorem to get $x' \in X'$, defining g_n like in Theorem 2.5.4 and proceeding exactly in the same way we get the contradiction we are looking for.
q.e.d.

2.6 Spaces defined by families of weights

We consider now a countable family V of continuous non-negative weights $v : B \longrightarrow [0, +\infty[$ so that for each $x \in B$ there exists $v \in V$ such that $v(x) > 0$. In these conditions we define the spaces

$$\begin{aligned} \mathcal{H}V(B) &= \{f \ \text{ holom. } : \ p_v = \sup_{x\in B} v(x)|f(x)| < \infty \text{ for all } \ v \in V\} \\ \mathcal{H}V_0(B) &= \{f \in \mathcal{H}V(B) : \forall v \in V, \ v|f| \ \text{ van. out. the } B\text{-bdd sets}\}. \end{aligned}$$

Obviously the family of seminorms $(p_v)_{v\in V}$ is separating; so we endow both spaces with the locally convex topology τ_V generated by $(p_v)_{v\in V}$. For a more complete study of the properties of these spaces see [27] or [67].

Definition 2.6.1 ([27], Definition 1) A family of non-negative continuous weights V defined on B is said to satisfy the *Condition I* if for each B-bounded set $A \subseteq B$ there exists $v \in V$ such that $\inf\{v(x) : x \in A\} > 0$.

It is a well known fact that (see [27])

Proposition 2.6.2
Let V satisfy Condition I. Then $\mathcal{H}V(B) \subseteq \mathcal{H}_b(B)$ and τ_V is stronger than τ_b (uniform convergence over the B-bounded sets). Moreover, $\mathcal{H}V(B)$ and $\mathcal{H}V_0(B)$ are Fréchet spaces.

With this notation, given $\phi : B \longrightarrow B$ holomorphic and two families of weights V and W, we define a composition operator $C_\phi : \mathcal{H}V(B) \longrightarrow \mathcal{H}W(B)$. We want to study when is it continuous. Note that we are assuming that $v(x) > 0$ and $w(x) > 0$ for all $x \in X$ and all $v \in V$, $w \in W$. This obviously implies that both V and W satisfy Condition I.

Proposition 2.6.3
Let $\phi : B \longrightarrow B$ holomorphic and two families of weights V and W such that for each $w \in W$ there exists $v \in V$ such that $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is continuous.
Then $C_\phi : \mathcal{H}V(B) \longrightarrow \mathcal{H}W(B)$ is continuous.
Proof.
Let $\Omega \subseteq \mathcal{H}V(B)$ be bounded. Let $w \in W$. Choose v such that $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is continuous. We have $K = \sup_{f\in\Omega} p_v(f) < \infty$.
On the other hand Proposition 2.4.2 implies

$$M = \sup_{x\in B} \frac{w(x)}{\tilde{v}(\phi(x))} < \infty.$$

Then, for any $f \in \Omega$ and $x \in B$,

$$\begin{aligned} w(x)|f(\phi(x))| &= \frac{w(x)}{\tilde{v}(\phi(x))}\tilde{v}(\phi(x))|f(\phi(x))| \\ &\le M\ p_{\tilde{v}}(f) = M\ p_v(f) \le M \cdot K. \end{aligned}$$

Since $x \in B$ is arbitrary, $\sup_{x\in B} w(x)|f(\phi(x))| \le M \cdot K$ for all $f \in \Omega$. Therefore, for all $w \in W$,

$$\sup_{f\in\Omega} p_w(C_\phi f) \le M \cdot K < \infty.$$

Hence $C_\phi(\Omega) \subseteq \mathcal{H}W(B)$ is τ_W-bounded and C_ϕ is continuous.
q.e.d.

Lemma 2.6.4
Let $V = (v_n)_{n=1}^\infty$ be a family of weights and $i_1, \ldots, i_m \in \mathbb{N}$. Define $v(x) = \max_{j=1,\ldots,m} v_{i_j}(x)$, $x \in B$, which is a weight. Consider $V_1 = V \cup \{v\}$; then

$$\mathcal{H}V(B) = \mathcal{H}V_1(B).$$

Proof.
We obviously have $\mathcal{H}V_1(B) \subseteq \mathcal{H}V(B)$. For the converse inclusion, take $f \in \mathcal{H}V(B)$ and let $M = \max_{j=1,\dots,m} p_{v_{i_j}}(f)$. Given $x \in B$ there is $j_0 \in \mathbb{N}$ such that $v(x) = v_{i_{j_0}}(x)$. Then,

$$v(x)|f(x)| = v_{i_{j_0}}(x)|f(x)| \leq p_{v_{i_{j_0}}}(f) \leq M.$$

Since x was arbitrary

$$p_v(f) = \sup_{x \in B} v(x)|f(x)| \leq M < \infty$$

and $f \in \mathcal{H}V_1(B)$.
q.e.d.

With this we can go now from the continuity with the families to continuity for individual weights.

Proposition 2.6.5
Let V, W be two families of weights and $\phi : B \longrightarrow B$ such that the composition operator $C_\phi : \mathcal{H}V(B) \longrightarrow \mathcal{H}W(B)$ is continuous.
Then for each $w \in W$ there exist $v_{i_1}, \dots, v_{i_m} \in V$ and $v = \sup_{j=1,\dots,m} v_{i_j}$ so that $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is continuous.
Proof.
Let $w \in W$ and fix $\Omega = \{g \in H_w^\infty(B) : p_w(g) \leq 1/n\}$, neighbourhood of 0 in $H_w^\infty(B)$. Consider now $\tilde{\Omega} = \{g \in \mathcal{H}W(B) : p_w(g) \leq 1/n\}$. Obviously $\tilde{\Omega} \subseteq \Omega$. Since $C_\phi : \mathcal{H}V(B) \longrightarrow \mathcal{H}W(B)$ is continuous, there are $v_{i_1}, \dots, v_{i_m} \in V$ and $n_1, \dots, n_m \in \mathbb{N}$ so that

$$\tilde{\Lambda} = \cap_{j=1}^m \{f \in \mathcal{H}V(B) : p_{v_{i_j}}(f) \leq 1/n_j\}$$

satisfies $C_\phi(\tilde{\Lambda}) \subseteq \tilde{\Omega}$. Define $v = \sup_{j=1,\dots,m} v_{i_j}$. Let $n_0 = \max_{j=1,\dots,m} n_j$. For all $j = 1, \dots, m$

$$\Lambda = \{f \in \mathcal{H}V(B) : p_v(f) \leq 1/n_0\} \subseteq \{f \in \mathcal{H}V(B) : p_{v_{i_j}}(f) \leq 1/n_j\}.$$

Therefore $C_\phi(\Lambda) \subseteq C_\phi(\tilde{\Lambda}) \subseteq \tilde{\Omega} \subseteq \Omega$. Since $\Lambda \subseteq H_v^\infty(B)$ is a neighbourhood of 0, $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is continuous.
q.e.d.

From this we have immediately

Corollary 2.6.6
Let V, W be two families of weights, V increasing and $\phi : B \longrightarrow B$ such that $C_\phi : \mathcal{H}V(B) \longrightarrow \mathcal{H}W(B)$ is continuous.
Then for all $w \in W$ there exists $v \in V$ such that $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is continuous.

If $V = (v_n)_n$ is a family of weights, for each $m \in \mathbb{N}$ we define $\overline{v}_m(x) = \max_{j=1,\ldots,m} v_j(x)$ and write $V_1 = (\overline{v}_m)_m$. Following the same steps of the proof of Lemma 2.6.4 we can prove that

$$\mathcal{H}V(B) = \mathcal{H}V_1(B).$$

In other words, the family of weights can always be chosen to be increasing. Thus we have the following.

Proposition 2.6.7
Let V, W be two families of weights and $\phi : B \longrightarrow B$ holomorphic.
Then, $C_\phi : \mathcal{H}V(B) \longrightarrow \mathcal{H}W(B)$ is continuous if and only if for all $w \in W$ there exists $v \in V$ such that $C_\phi : H_v^\infty(B) \longrightarrow H_w^\infty(B)$ is continuous.

Chapter 3

Spectra in tensor products of lmc algebras

3.1 Introduction

Spectral theory was developed at the beginning of the 20th century and is now a classical subject. During the 1930's Gelfand developed his theory, in which he showed the relationship between spectra and the multiplicative linear functionals or the maximal closed ideals of the algebra (these two sets are essentially the same). A good study of all this classical theory is given in [70].
The step of going from spectra of a single element the spectra to a family of elements was given during the 1950's by Waelbroeck and others for the commutative case. Left, right and joint spectra for families of elements in a non-commutative algebra were defined by R.E. Harte during the 1970's (see [33]). Alternatively L. Waelbroeck focused his efforts on defining a vector spectrum, instead of the classical scalar spectrum. He defined the spectrum for elements of $\mathcal{A}\hat{\otimes}_\pi X$, where $\mathcal{A}$ was a commutative unital Banach algebra and X was a Banach space.
C.Taylor, jointly with S.Dineen and R.E.Harte (see [74] and [18], [19], [20]), recently developed a vector Gelfand theory for elements in $\mathcal{A}\hat{\otimes}_\gamma X$, where $\mathcal{A}$ is any Banach algebra, X any Banach space and γ is a uniform tensor norm and generalized the Waelbroeck spectrum. Using the spectrum defined by Harte, they defined a left spectrum when $\mathcal{A}$ is not commutative. In a series of three papers they presented their results.
Our aim in this chapter is to generalize some of their results for lmc algebras and Q-algebras and locally convex spaces. We obtain results concerning left invertibility of continuous mappings and of holomorphic germs with values in a non-commutative algebra.

3.2 Topological preliminaries

3.2.1 Topological algebras

Topological algebras have been long and widely studied. A detailed study can be found in [23], [28], [36], [50], [54], [57]. We only present here the basic definitions and properties that will be needed later. As its name suggests, a topological algebra is an object with two structures that are in principle different. First of all it is an algebra, and has a topological structure. These two structures are connected in the following way.

Definition 3.2.1 An algebra $\mathcal{A}$ is said to be a *topological algebra* if it has a topology $\mathcal{T}$ such that the algebra operations $+ : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, $. : \mathbb{C} \times \mathcal{A} \longrightarrow \mathcal{A}$, $\cdot : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ are continuous.

Clearly every topological algebra is a topological vector space.

lmc algebras

A topological algebra can, as a topological vector space, be locally convex. This could in principle give us a new structure, but we need some extra topological conditions on the inner multiplication, slightly more demanding than being just continuous.

Definition 3.2.2 A topological algebra $\mathcal{A}$ is *locally multiplicatively convex (lmc)* if it is a locally convex space whose topology is defined by a a family of seminorms $(p_i)_{i \in I}$ such that for all $x, y \in \mathcal{A}$ and all $i \in I$,

$$p_i(xy) \leq p_i(x)p_i(y).$$

Seminorms with this property are called *multiplicative*.

An equivalent definition can be given in terms of neighbourhoods of 0 in the following way. First, given any two sets $A, B \subseteq \mathcal{A}$ the product is defined to be $A \cdot B = \{xy : x \in A,\ y \in B\}$. With this we say that a topological algebra is locally multiplicatively convex if it is a locally convex space with a basis of neighbourhoods of $0, \mathcal{U}$, satisfying that $U \cdot U \subseteq U$ for all $U \in \mathcal{U}$.

Every Banach algebra is clearly a lmc algebra. We say that a lmc algebra is Fréchet if it is complete and the family of seminorms defining the topology is countable.

Q-algebras

An interesting class of topological algebras is the class of Q-algebras. Two different definitions of Q-algebras exist in the literature. They are simply two totally different and independent concepts that unfortunately have the same name. The ones that we are going to use were first used by Kaplansky during the 1940's when studying the radical of a ring

(see [38], [39]). This led him to define what he called Q-rings. This concept was immediately used for algebras ([54]).
We begin by defining the 'circle' operation $\circ$ in an algebra $\mathcal{A}$. Given any two $a, b \in \mathcal{A}$, let $a \circ b = a + b - ab$. This is an associative operation with identity 0. An element $a \in \mathcal{A}$ is said to be *quasi-invertible* if there exists $b \in \mathcal{A}$ so that $a \circ b = 0 = b \circ a$.

Definition 3.2.3 A topological algebra $\mathcal{A}$ is *Q-algebra* if the set of quasi-invertible elements is open in $\mathcal{A}$.

For a complete survey on the definitions and properties of lmc and Q-algebras, we refer to [23], [24], [28], [36],[57],[59], [68]. Clearly every Banach algebra is a Q-algebra.
For any locally convex space E, the topological dual will be denoted by E'. On it we will always, unless stated otherwise, use the weak* topology.
The set of non-zero continuous homomorphisms from $\mathcal{A}$ to $\mathbb{C}$ is denoted by $\mathfrak{M}(\mathcal{A})$. This is a subset of $\mathcal{A}'$. Q-algebras have very interesting properties. We state some of them in the following proposition.

Proposition 3.2.4
Let $\mathcal{A}$ be a Q-algebra; then
(i) *If $\mathcal{A}$ is unital, $\mathfrak{M}(\mathcal{A})$ is weak*-compact (see [78], Proposition 10).*
(ii) *$\mathfrak{M}(\mathcal{A})$ is equicontinuous (see [78], Theorem 6).*
(iii) *Every proper maximal ideal is closed ([36], Theorem 1.6; [59], Proposition 2.4).*
(iv) *Every complex homomorphism is continuous ([24]; [59], Corollary 2.5).*

A subset $M \subseteq \mathcal{L}(E; F)$ is equicontinuous if for each neighbourhood of 0, $V \subseteq F$, there is some neighbourhood U of 0 in E, such that $f(U) \subseteq V$ for every $f \in M$. Equivalently, for each continuous seminorm on F, say q, there exists p, continuous seminorm on E, such that $q(f(x)) \leq p(x)$ for all $f \in M$ and $x \in E$.

Remark 3.2.5
Let $\mathcal{A}$ be a Q-algebra and $\mathcal{B}$ a subalgebra of $\mathcal{A}$ that is a complemented subspace of $\mathcal{A}$ in such a way that the projection $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ is an algebra homomorphism. Then $\mathcal{B}$ is also a Q-algebra. Indeed, given any $x, y \in \mathcal{A}$,

$$\pi(x \circ y) = \pi(x) + \pi(y) - \pi(xy) = \pi(x) + \pi(y) - \pi(x)\pi(y) = \pi(x) \circ \pi(y).$$

Also, $\pi(0) = 0$, therefore the set of quasi-invertible elements in $\mathcal{B}$ is the projection of the set in $\mathcal{A}$. Since every projection is an open mapping, $\mathcal{B}$ is a Q-algebra.

Remark 3.2.6
The situation becomes nicer when $\mathcal{A}$ is unital; in this case it is easily shown that $x \in \mathcal{A}$ is quasi-invertible if and only if $1 - x$ is invertible. Thus, if $\mathcal{A}$ is unital, it is Q-algebra if and only in the set of invertible elements, which will be always denoted by $\mathcal{A}_{inv}$, is open. In fact this is the definition given in [57].

Example 3.2.7
If E is a locally convex space and $K \subseteq E$ compact, then $\mathcal{H}(K)$ is a Q-algebra. If, furthermore, E is metrizable, $\mathcal{H}(K)$ is a lmc-algebra; see [57], Prop. 28.1.
On the other hand, $\mathcal{H}(\mathbb{C})$ with the pointwise multiplication is an algebra but with the τ_0 topology of convergence over the compact sets it is not a Q-algebra. We know that a function f is invertible if and only if $f(z) \neq 0$ for every $z \in \mathbb{C}$. Then, if $f \in \mathcal{H}(\mathbb{C})$ is invertible, either it is constant or $f(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Take a non-zero null sequence $(\omega_n)_n$. Suppose that f is not constant and define $g_n = f - \omega_n$. For each $n \in \mathbb{N}$ there exists some $z_n \in \mathbb{C}$ such that $f(z_n) = \omega_n$. Then, $g_n(z_n) = f(z_n) - \omega_n = 0$ and g_n is non-invertible. Let us now show that $g_n \xrightarrow{\tau_0} f$. Take $K \subseteq \mathbb{C}$ compact; given $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ so that $|\omega_n| < \varepsilon$ for every $n \geq n_0$. For $n \geq n_0$ we have;

$$\sup_{z \in K} |f(z) - g_n(z)| = \sup_{z \in K} |f(z) - f(z) + \omega_n| = |\omega_n| < \varepsilon.$$

Therefore $g_n \xrightarrow{\tau_0} f$.
If $f \equiv c$ is constant and invertible then $c \neq 0$. Let $h_n(z) = c - \omega_n z$. Then h_n is a non-zero polynomial and hence has a zero and is non-invertible. Clearly $h_n \longrightarrow c$ as $n \to \infty$.
This shows that the set of invertible elements is not open and, since $\mathcal{H}(\mathbb{C})$ is unital, it is not a Q-algebra.

The next example is well known in the literature ([24] or [49], Sections VI and VII).

Example 3.2.8
Let X be a completely regular Hausdorff topological space and consider the algebra of continuous complex functions over X, $\mathcal{C}(X)$. We show that $\mathcal{C}(X)$ is a Q-algebra if and only if X is compact. If X is compact, then $\mathcal{C}(X)$ is a Banach algebra and, therefore, a Q-algebra.
Suppose now that it is not compact we are going to show in this case that $\mathfrak{M}(\mathcal{C}(X))$ is not equicontinuous. For each $x \in X$, we consider the evaluation map $\delta_x : \mathcal{C}(X) \longrightarrow \mathbb{C}$ given by $\delta_x(f) = f(x)$. Each $\delta_x \in \mathfrak{M}(\mathcal{C}(X))$. By [15], Theorem 1, $\delta : X \longrightarrow \mathfrak{M}(\mathcal{C}(X))$ is a homeomorphism. Thus, we can identify $X \cong \mathfrak{M}(\mathcal{C}(X))$.
The topology in $\mathcal{C}(X)$ is defined by the seminorms

$$p_K(f) = \sup_{x \in K} |f(x)|,$$

where K ranges over all the compact subsets of X. Then, $\mathfrak{M}(\mathcal{C}(X))$ is equicontinuous if and only if there exists some $K \subseteq X$ compact such that $|\delta_x(f)| \leq p_K(f)$ for all $x \in X$ and all $f \in \mathcal{C}(X)$. In order to see that this is false, consider any compact $K \subseteq X$. Since X is non-compact $X \setminus K \neq \emptyset$. Consider $x_0 \in X \setminus K$. Since X is completely regular, we can find a continuous mapping $f : X \longrightarrow [0,1]$ satisfying $f(x) = 0$ for all $x \in K$ and $f(x_0) = 1$. Then

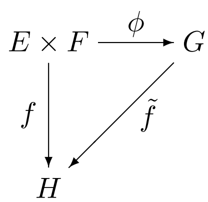
$$|\delta_{x_0}(f)| = |f(x_0)| = 1 > 0 = \sup_{x \in K} |f(x)| = p_K(f).$$

Thus, $\mathfrak{M}(\mathcal{C}(X))$ is not equicontinuous and $\mathcal{C}(X)$ is not a Q-algebra.

3.2.2 Tensor products

Definitions and properties

The tensor product of two vector spaces is in principle a purely algebraic concept. Given any two vector spaces E and F there exists a unique vector space G and a bilinear mapping $\phi : E \times F \longrightarrow G$ with the following universal property; for each vector space H and each bilinear mapping $f : E \times F \longrightarrow H$ there is a unique linear mapping $\tilde{f} : G \longrightarrow H$ such that $f = \tilde{f} \circ \phi$.



The pair (G, ϕ) is called the *tensor product of E and F*. Usually the mapping is not explicitly mentioned and the vector space is denoted by $E \otimes F$. The vectors in the tensor product are called tensors. Let $x \otimes y = \phi(x, y)$.
By uniqueness of the construction $E \otimes F$ is the set of formal finite sums $\sum_{i=1}^{n} x_i \otimes y_i$, taking into account that tensors can be represented by different formal sums. With this representation of the tensor product, given any two linear mappings $f : E_1 \longrightarrow E_2$ and $g : F_1 \longrightarrow F_2$ we can define a new mapping $f \otimes g : E_1 \otimes F_1 \longrightarrow E_2 \otimes F_2$ by $(f \otimes g)\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} f(x_i) \otimes g(y_i)$. This mapping is well defined and linear. Moreover, if M and N are two equicontinuous sets of linear mappings, we can consider the set $M \otimes N = \{f \otimes g : f \in M,\ g \in N\}$. A deeper and more detailed study of tensor products of vector spaces con be found in [11], [17], [30], [37], [45].

Since $E \otimes F$ is a vector space, it may be endowed with some topologies that make it a topological vector space and these topologies may be generated from those of E and F. When $\mathcal{T}$ is a topology for $E \otimes F$, we write $E \otimes_{\mathcal{T}} F$ for $(E \otimes F, \mathcal{T})$ and $E \hat{\otimes}_{\mathcal{T}} F$ for the completion. The problem of how to generate these topologies was considered by Grothendieck in [30]. In his work the definition of compatible tensor topology is given (Chapter 3, Section 3). Inspired by it we give the following definition.

Definition 3.2.9 We say that $\mathcal{T}$ is a *uniform* tensor topology for locally convex spaces if for every pair E, F of locally convex spaces:
1) $E \otimes_{\mathcal{T}} F$ is a locally convex space.
2) The canonical bilinear mapping $E \times F \longrightarrow E \otimes_{\mathcal{T}} F$ is separately continuous.
3) If $f \in \mathcal{L}(E_1; E_2)$ and $g \in \mathcal{L}(F_1; F_2)$, then $f \otimes g \in \mathcal{L}(E_1 \otimes_{\mathcal{T}} F_1; E_2 \otimes_{\mathcal{T}} F_2)$.

4) If $M \subseteq \mathcal{L}(E_1; E_2)$ and $N \subseteq \mathcal{L}(F_1; F_2)$ are equicontinuous, then $M \otimes N \subseteq \mathcal{L}(E_1 \otimes_{\mathcal{T}} F_1; E_2 \otimes_{\mathcal{T}} F_2)$ is equicontinuous.

The last property is called the *mapping property* in [6] and in [11]. Clearly it implies condition 3. We keep both conditions because they appeared in this way in Grothendieck's definition of compatible topology.
This definition generalizes that of uniform tensor norm for Banach spaces (see [11], Section 12.1), in the sense that a uniform tensor norm defines a uniform tensor topology when restricted to normed spaces.

When $\mathcal{A}$, $\mathcal{B}$ are two algebras, a product can be defined on $\mathcal{A} \otimes \mathcal{B}$ by using the universal properties of the tensor product,

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$

and extending it by linearity. A proof of the fact that $\mathcal{A} \otimes \mathcal{B}$ is an algebra can be found in [18] and [50]. We give here a different proof of this fact. In order to see that it is well defined, fix first $(a, b) \in \mathcal{A} \times \mathcal{B}$ and suppose that $\sum_{i=1}^n x_i \otimes y_i = \sum_{j=1}^n \tilde{x}_j \otimes \tilde{y}_i$. It is a well known fact that $\mathcal{A} \otimes \mathcal{B} \cong (\mathrm{Bil}(\mathcal{A}, \mathcal{B}))^*$ ([37], Section 1.6). We have, for all $B \in \mathrm{Bil}(\mathcal{A}, \mathcal{B})$

$$\sum_{i=1}^n B(x_i, y_i) = \sum_{j=1}^n B(\tilde{x}_j, \tilde{y}_j) \tag{3.1}$$

Now, given any such B, we define $B_{(a,b)} \in \mathrm{Bil}(\mathcal{A}, \mathcal{B})$ by $B_{(a,b)}(x, y) = B(ax, by)$. Then, using (3.1),

$$\sum_{i=1}^n B_{(a,b)}(x_i, y_i) = \sum_{j=1}^n B_{(a,b)}(\tilde{x}_j, \tilde{y}_j).$$

From this we obtain

$$\sum_{i=1}^n B(ax_i, by_i) = \sum_{j=1}^n B(a\tilde{x}_j, b\tilde{y}_j).$$

This means that $(a \otimes b)\left(\sum_{i=1}^n x_i \otimes y_i\right) = (a \otimes b)\left(\sum_{j=1}^n \tilde{x}_j \otimes \tilde{y}_j\right)$. Proceeding in the same way we have $\left(\sum_{i=1}^n x_i \otimes y_i\right)(a \otimes b) = \left(\sum_{j=1}^n \tilde{x}_j \otimes \tilde{y}_j\right)(a \otimes b)$. Suppose now that $\sum_{i=1}^n x_i \otimes y_i = \sum_{j=1}^n \tilde{x}_j \otimes \tilde{y}_j$ and $\sum_{k=1}^m z_k \otimes w_k = \sum_{l=1}^m \tilde{z}_l \otimes \tilde{w}_l$; then:

$$\begin{aligned} &\left(\sum_{i=1}^n x_i \otimes y_i\right)\left(\sum_{k=1}^m z_k \otimes w_k\right) - \left(\sum_{j=1}^n \tilde{x}_j \otimes \tilde{y}_j\right)\left(\sum_{l=1}^m \tilde{z}_l \otimes \tilde{w}_l\right) \\ = &\left(\sum_{i=1}^n x_i \otimes y_i\right)\left(\sum_{k=1}^m z_k \otimes w_k\right) - \left(\sum_{i=1}^n x_i \otimes y_i\right)\left(\sum_{l=1}^m \tilde{z}_l \otimes \tilde{w}_l\right) \end{aligned}$$

$$
\begin{aligned}
&+\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\left(\sum_{l=1}^{m} \tilde{z}_l \otimes \tilde{w}_l\right)-\left(\sum_{j=1}^{n} \tilde{x}_j \otimes \tilde{y}_j\right)\left(\sum_{l=1}^{m} \tilde{z}_l \otimes \tilde{w}_l\right)\\
= &\sum_{i=1}^{n}\left((x_i \otimes y_i)\left(\sum_{k=1}^{m} z_k \otimes w_k\right)-(x_i \otimes y_i)\left(\sum_{l=1}^{m} \tilde{z}_l \otimes \tilde{w}_l\right)\right)\\
&+\sum_{l=1}^{m}\left(\left(\sum_{i=1}^{n} x_i \otimes y_i\right)(\tilde{z}_l \otimes \tilde{w}_l)-\left(\sum_{j=1}^{n} \tilde{x}_j \otimes \tilde{y}_j\right)(\tilde{z}_l \otimes \tilde{w}_l)\right)=0.
\end{aligned}
$$

Therefore the product is well defined on $\mathcal{A} \otimes \mathcal{B}$. The fact that it is an algebra is an easy exercise. Note that if both $\mathcal{A}$ and $\mathcal{B}$ are unital, then $\mathcal{A} \otimes \mathcal{B}$ is also unital and $1_{\mathcal{A}} \otimes 1_{\mathcal{B}}$ is the identity element.
It is also interesting to ask under which conditions $\mathcal{A} \otimes_{\mathcal{T}} \mathcal{B}$ is a lmc algebra.

Remark 3.2.10
Take $h : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ and $g : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ two algebra homomorphisms. Then $h \otimes g$ is linear. We show that it is multiplicative. Let $\mathsf{T} = \sum_{i=1}^{n} a_i \otimes b_i$, $\mathsf{S} = \sum_{j=1}^{m} c_j \otimes d_j$; we have

$$
\begin{aligned}
(h \otimes g)(\mathsf{TS}) &= (h \otimes g)\left(\sum_{i,j=1}^{n,m} a_i c_j \otimes b_i d_j\right)\\
&= \sum_{i,j=1}^{n,m} h(a_i c_j) \otimes g(b_i d_j)\\
&= \sum_{i,j=1}^{n,m} h(a_i)h(c_j) \otimes g(b_i)g(d_j)\\
&= \left(\sum_{i=1}^{n} h(a_i) \otimes g(b_i)\right)\left(\sum_{j=1}^{n} h(c_j) \otimes g(d_j)\right)\\
&= (h \otimes g)(\mathsf{T})\ (h \otimes g)(\mathsf{S}).
\end{aligned}
$$

Hence $h \otimes g$ is an algebra homomorphism. If furthermore $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2$ are lmc algebras and $\mathcal{A}_1 \otimes_{\mathcal{T}} \mathcal{B}_1$ and $\mathcal{A}_2 \otimes_{\mathcal{T}} \mathcal{B}_2$ are lmc algebras and both h and g are continuous, then

$$h \otimes g : \mathcal{A}_1 \otimes_{\mathcal{T}} \mathcal{B}_1 \longrightarrow \mathcal{A}_2 \otimes_{\mathcal{T}} \mathcal{B}_2$$

is a continuous algebra homomorphism.

Remark 3.2.11
Let $\mathcal{A}$ be a lmc-algebra whose topology is generated by a system of seminorms $\{p : p \in \mathcal{P}\}$. Each continuous seminorm is uniformly continuous and, therefore, admits a unique

extension, $\hat{p}$, to the completion $\hat{\mathcal{A}}$. The topology on $\hat{\mathcal{A}}$ is generated by the system of seminorms $\{\hat{p} : p \in \mathcal{P}\}$. Let us show that we can extend the product defined in $\mathcal{A}$ to $\hat{\mathcal{A}}$ so that $\hat{\mathcal{A}}$ is again a lmc algebra. What we need to show, then, is that $\hat{p}(ab) \leq \hat{p}(a)\hat{p}(b)$ for any two $a, b \in \hat{\mathcal{A}}$ and any p.

Suppose first that $a \in \hat{\mathcal{A}}$ and $b \in \mathcal{A}$. Take a net $(a_\alpha)_\alpha$ in $\mathcal{A}$ converging to a. Consider any $p \in \mathcal{P}$ and let $\varepsilon > 0$. Since $(a_\alpha)_\alpha$ converges it is Cauchy and there exists α_0 so that $p(a_\alpha - a_\beta) < \frac{\varepsilon}{p(b)}$ (w.l.o.g. we can assume that $p(b) \neq 0$) for all $\alpha, \beta \geq \alpha_0$. Then, if $\alpha, \beta \geq \alpha_0$

$$p(a_\alpha b - a_\beta b) = p((a_\alpha - a_\beta)b) \leq p(a_\alpha - a_\beta)p(b) < \varepsilon.$$

Therefore, $(a_\alpha b)_\alpha$ is a Cauchy net and let

$$ab = \lim_\alpha a_\alpha b.$$

This is well defined. Indeed, if $(c_\beta)_\beta$ is another net converging to a we have for any p, $p(a_\alpha b - c_\beta b) = p((a_\alpha - c_\beta)b) \leq p(a_\alpha - c_\beta)p(b) \longrightarrow 0$. Thus, $\lim_\alpha a_\alpha b = \lim_\beta c_\beta b$. In this way we define a product $\hat{\mathcal{A}} \times \mathcal{A} \longrightarrow \hat{\mathcal{A}}$ satisfying that

$$\hat{p}(ab) = \lim_\alpha p(a_\alpha b) \leq \lim_\alpha p(a_\alpha)p(b) = \hat{p}(a)p(b)$$

for all $a \in \hat{\mathcal{A}}$ and $b \in \mathcal{A}$ and any $p \in \mathcal{P}$.

Now, for $b \in \hat{\mathcal{A}}$, take a net $(b_\alpha)_\alpha$ converging to b and proceeding in the same way we define a product $\hat{\mathcal{A}} \times \hat{\mathcal{A}} \longrightarrow \hat{\mathcal{A}}$ satisfying

$$\hat{p}(ab) \leq \hat{p}(a)\hat{p}(b)$$

for every $a, b \in \hat{\mathcal{A}}$ and $p \in \mathcal{P}$. Therefore, $\hat{\mathcal{A}}$ is a lmc-algebra.

This means that, in order to check that $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is a lmc-algebra, it is enough to consider $\mathcal{A} \otimes \mathcal{B}$.

Examples

In order to check that the concept of uniform tensor topology is not void and therefore it makes sense to consider it, let us see some examples of such topologies.

Example 3.2.12
Let E, F be two locally convex spaces whose topologies are defined by families of seminorms $(p_\alpha)_\alpha$ and $(q_\beta)_\beta$, then let

$$(p_\alpha \otimes_\pi q_\beta)(\mathsf{T}) = \inf\{\sum_{i=1}^{n} p_\alpha(x_i)q_\beta(y_i) \ : \ \mathsf{T} = \sum_{i=1}^{n} x_i \otimes y_i \ , \ n \in \mathbb{N}\}.$$

This defines a seminorm in $E \otimes F$ and $(p_\alpha \otimes_\pi q_\beta)_{\alpha,\beta}$ is a system that defines a locally convex topology called π (projective). If $\mathcal{U}$ and $\mathcal{V}$ are fundamental systems of convex, balanced neighbourhoods of 0 for E and F, respectively, then

$$\{\Gamma(U \otimes V) \ : \ U \in \mathcal{U} \ , \ V \in \mathcal{V}\}$$

is a fundamental system of convex, balanced neighbourhoods of 0 for $E \otimes_\pi F$, where $\Gamma(A)$ denotes the convex hull of A. It is well known that π satisfies the two first conditions of Definition 3.2.9 to be a uniform norm (see [49]).
Take E_1, E_2, F_1, F_2 locally convex spaces and $M \subseteq \mathcal{L}(E_1; E_2)$ and $N \subseteq \mathcal{L}(F_1; F_2)$ equicontinuous. For E_1, E_2, F_1, F_2, consider fundamental systems of neighbourhoods, $(U^1_\alpha)_\alpha, (U^2_\beta)_\beta, (V^1_\mu)_\mu$, $(V^2_\eta)_\eta$ respectively. Take $\Gamma(U^2_\beta \otimes V^2_\eta)$. By equicontinuity, we can find U^1_α and V^1_μ such that $f(U^1_\alpha) \subseteq U^2_\beta$ and $g(V^1_\mu) \subseteq V^2_\eta$ for every $f \in M$ and $g \in N$. Take any $f \in M$ and $g \in N$, since both are linear we have

$$(f \otimes g)(\Gamma(U^1_\alpha \otimes V^1_\mu)) = \Gamma((f \otimes g)(U^1_\alpha \otimes V^1_\mu)) \subseteq \Gamma(f(U^1_\alpha) \otimes g(V^1_\mu)) \subseteq \Gamma(U^2_\beta \otimes V^2_\eta).$$

Thus, the π topology is uniform.
If $\mathcal{A}, \mathcal{B}$ are two lmc algebras, the seminorms that generate the π topology are multiplicative (see [49]) and, then, $\mathcal{A} \otimes_\pi \mathcal{B}$ is a lmc algebra for any two lmc algebras $\mathcal{A}, \mathcal{B}$.

Before giving the next example, let us introduce some notation. If E is a locally convex space and p is a continuous seminorm in E, we denote by B_p the set $\{x \in E : p(x) \leq 1\}$. For every $x \in E$ we have $p(x) = \sup_{x' \in B_p^\circ} |x'(x)|$. With this notation, a lmc-algebra $\mathcal{A}$ is said to be *uniform* if for every $x \in \mathcal{A}$

$$p(x) = \sup_{h \in B_p^\circ \cap \mathfrak{M}(\mathcal{A})} |h(x)|.$$

Example 3.2.13
Given any two locally convex spaces E, F whose topologies are generated by two families of continuous seminorms $(p_\alpha)_\alpha$ and $(q_\beta)_\beta$ respectively, for each pair of continuous seminorms define for each $\mathsf{T} \in E \otimes F$

$$(p \otimes_\varepsilon q)(\mathsf{T}) = \sup_{\substack{x' \in B_p^\circ \\ y' \in B_q^\circ}} |(x' \otimes y')(\mathsf{T})|.$$

This defines a seminorm and the family $(p_\alpha \otimes_\varepsilon q_\beta)_{\alpha,\beta}$ generates a locally convex topology in $E \otimes F$, called the ε (injective) topology. If $\mathcal{U}$ and $\mathcal{V}$ are respectively two fundamental systems of neighbourhoods of 0 in E and F then

$$\{(U^\circ \otimes V^\circ)^\circ : U \in \mathcal{U}, V \in \mathcal{V}\}$$

is a fundamental system of neighbourhoods of 0 in $E \otimes_\varepsilon F$. For a more detailed and deeper study of this topology [45] Section 44 can be consulted. The proof of the fact that ε satisfies the first two conditions of a uniform tensor topology can be found in [45], Section 44.2. We show that it also satisfies the fourth condition. Take E_1, E_2, F_1, F_2 locally convex spaces with basis of convex, balanced neighbourhoods $(U_{1\alpha})_\alpha, (U_{2\beta})_\beta, (V_{1\mu})_\mu, (V_{2\eta})_\eta$. Then, $((U^\circ_{1\alpha} \otimes V^\circ_{1\mu})^\circ)_{\alpha,\mu}$ is a basis of neighbourhoods of 0 in $E_1 \otimes_\varepsilon F_1$ and $((U^\circ_{2\beta} \otimes V^\circ_{2\eta})^\circ)_{\beta,\eta}$ is a similar basis for $E_2 \otimes_\varepsilon F_2$. Consider $M \subseteq \mathcal{L}(E_1; E_2)$ and $N \subseteq \mathcal{L}(F_1; F_2)$ equicontinuous. We want to show that $M \otimes N \subseteq \mathcal{L}(E_1 \otimes_\varepsilon F_1; E_2 \otimes_\varepsilon F_2)$ is also equicontinuous. Consider $(U^\circ_{2\beta} \otimes V^\circ_{2\eta})^\circ$. Since M and N are equicontinuous, we can find $U_{1\alpha}$ such that $f(U_{1\alpha}) \subseteq U_{2\beta}$ for all $f \in M$ and $V_{1\mu}$ such that $g(V_{1\mu}) \subseteq V_{2\eta}$. Take any $f \in M$, $g \in N$ and $\mathsf{T} \in (U^\circ_{1\alpha} \otimes V^\circ_{1\mu})^\circ$. We have $|(x'_1 \otimes y'_1)(\mathsf{T})| \leq 1$ for all $x'_1 \in U^\circ_{1\alpha}$ and all $y'_1 \in V^\circ_{1\mu}$. Consider $x'_2 \in U^\circ_{2\beta} \subseteq E'_2$ and $y'_2 \in V^\circ_{2\eta} \subseteq F'_2$. We have $x'_2 \circ f \in E'_1$ and, if $x_1 \in U_{1\alpha}$, then $f(x_1) \in U_{2\beta}$ and, from this, $|(x'_2 \circ f)(x_1)| = |x'_2(f(x_1))| \leq 1$. Therefore, $x'_2 \circ f \in U^\circ_{1\alpha}$. In the same way $y'_2 \circ g \in V^\circ_{1\mu}$ and hence $(x'_2 \circ f) \otimes (y'_2 \circ g) \in U^\circ_{1\alpha} \otimes V^\circ_{1\mu}$. From this we have

$$(x'_2 \otimes y'_2)((f \otimes g)(\mathsf{T})) = ((x'_2 \circ f) \otimes (y'_2 \circ g))(\mathsf{T}) \leq 1.$$

Therefore

$$(f \otimes g)(\mathsf{T}) \in (U^\circ_{2\beta} \otimes V^\circ_{2\eta})^\circ.$$

And this implies that $(f \otimes g)((U^\circ_{1\alpha} \otimes V^\circ_{1\mu})^\circ) \subseteq (U^\circ_{2\beta} \otimes V^\circ_{2\eta})^\circ$ for any $f \in M$ and $g \in N$. Hence $M \otimes N$ is equicontinuous and the ε topology is uniform.

Now suppose that $\mathcal{A}$ is a uniform algebra. Then for any lmc-algebra $\mathcal{B}$ we show that $\mathcal{A} \otimes_\varepsilon \mathcal{B}$ is lmc algebra. Let p, q be continuous seminorms on $\mathcal{A}, \mathcal{B}$ respectively and let $\mathsf{T} = \sum_{i=1}^n x_i \otimes y_i,\ \mathsf{S} = \sum_{j=1}^m z_j \otimes w_j \in \mathcal{A} \otimes \mathcal{B}$. Consider the sets B_p and B_q; then

$$\begin{aligned}
(p \otimes_\varepsilon q)(\mathsf{TS}) &= \sup_{\substack{x' \in B^\circ_p \\ y' \in B^\circ_q}} |(x' \otimes y')(\mathsf{TS})| \\
&= \sup_{\substack{x' \in B^\circ_p \\ y' \in B^\circ_q}} |(x' \otimes y')\left(\sum_{i,j=1}^{n,m} x_i z_j \otimes y_i w_j\right)| \\
&= \sup_{\substack{x' \in B^\circ_p \\ y' \in B^\circ_q}} |\sum_{i,j=1}^{n,m} x'(x_i z_j)\ y'(y_i w_j)| \\
&= \sup_{\substack{x' \in B^\circ_p \\ y' \in B^\circ_q}} |x'\left(\sum_{i,j=1}^{n,m} x_i z_j\ y'(y_i w_j)\right)| \\
&= \sup_{\substack{h \in B^\circ_p \cap \mathfrak{M}(\mathcal{A}) \\ y' \in B^\circ_q}} |h\left(\sum_{i,j=1}^{n,m} x_i z_j y'(y_i w_j)\right)|
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\substack{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})\\ y'\in B_q^\circ}} |\sum_{i,j=1}^{n,m} h(x_i)h(z_j)y'(y_iw_j)| \\
&= \sup_{\substack{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})\\ y'\in B_q^\circ}} |\ y'\left(\sum_{i,j=1}^{n,m} h(x_i)h(z_j)y_iw_j\right)| \\
&= \sup_{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})} |\ q\left(\sum_{i,j=1}^{n,m} h(x_i)h(z_j)y_iw_j\right)| \\
&\leq \sup_{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})} |\ q\left(\sum_{i=1}^{n} h(x_i)y_i\right)| \\
&\quad \sup_{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})} |\ q\left(\sum_{j=1}^{m} h(z_j)w_j\right)| \\
&= \sup_{\substack{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})\\ y'\in B_q^\circ}} |\sum_{i=1}^{n} h(x_i)\ y'(y_i)| \sup_{\substack{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})\\ y'\in B_q^\circ}} |\sum_{j=1}^{m} h(z_j)\ y'(w_j)| \\
&= \sup_{\substack{x'\in B_p^\circ\\ y'\in B_q^\circ}} |(x'\otimes y')(\mathsf{T})| \sup_{\substack{x'\in B_p^\circ\\ y'\in B_q^\circ}} |(x'\otimes y')(\mathsf{S})| \\
&= (p\otimes_\varepsilon q)(\mathsf{T})\ \ (p\otimes_\varepsilon q)(\mathsf{S}).
\end{aligned}
$$

Hence $\mathcal{A}\otimes_\varepsilon\mathcal{B}$ is a lmc algebra if $\mathcal{A}$ is a uniform lmc algebra and $\mathcal{B}$ is any lmc algebra.

Example 3.2.14
In [30], section 3, Grothendieck defines the inductive topology to be the only Hausdorff locally convex topology $\mathcal{T}_{ind}$ on $E\otimes F$ such that for any other locally convex space G, the natural algebraic isomorphism $\mathrm{Bil}(E\times F;G)\longrightarrow\mathcal{L}(E\otimes F;G)$ sends exactly the separately continuous bilinear mappings into the $\mathcal{T}_{ind}$-continuous linear mappings. Moreover, the separately equicontinuous sets of bilinear mappings correspond to the $\mathcal{T}_{ind}$-equicontinuous sets of linear mappings on $E\otimes F$. Hence, if $M\subseteq\mathcal{L}(E_1;E_2)$ and $N\subseteq\mathcal{L}(F_1;F_2)$ are equicontinuous, $M\times N$ is separately equicontinuous and its image by the isomorphism, $M\otimes N\subseteq\mathcal{L}(E_1\otimes F_1;E_2\otimes F_2)$ is $\mathcal{T}_{ind}$-equicontinuous and $\mathcal{T}_{ind}$ is a uniform tensor topology.

Topologies defined from norms

A complete locally convex space E can be described as a projective limit of Banach spaces, $E=\varprojlim E_i$, in the following way. Suppose the topology in E is given by a directed set of seminorms $(p_i)_{i\in I}$ such that $p_i\leq p_j$ whenever $i\leq j$. Consider the projection $\pi_i : E\longrightarrow E/\ker p_i$ and let $\|\pi_i(x)\|_i=p_i(x)$. This is a norm on $E/\ker p_i$. We denote its completion by $(E_i, \| \|_i)$ and $\pi_i : E \longrightarrow E_i$ is a continuous linear mapping (not necessarily onto). We then have mappings $\pi_{ij} : E_j \longrightarrow E_i$ for $i \leq j$ such that $\pi_{ij} \circ \pi_j = \pi_i$. Write $x_i = \pi_i(x)$. The map $E \longrightarrow \lim\limits_{\leftarrow}(E_i, \pi_i)$ given by $x \mapsto (x_i)_{i \in I}$ is a homeomorphism that provides the desired representation. Note that by construction $\pi_i(E)$ is dense in E_i. Projective systems satisfying this condition are called *reduced*.

A uniform tensor norm α (see [11], 12.1) assigns to each pair of normed spaces X, Y a norm on $X \otimes Y$ such that:

1) $\varepsilon \leq \alpha \leq \pi$.

2) If $T_i \in \mathcal{L}(E_i; Fi)$, $i = 1, 2$, then $T_1 \otimes T_2 \in \mathcal{L}(E_1 \otimes_\alpha E_2; F_1 \otimes_\alpha F2)$ and $\|T_1 \otimes T_2\| \leq \|T_1\| \cdot \|T_2\|$ (it satisfies the metric mapping property).

In fact, it can be proved that the converse inequality is always true.

From a uniform tensor norm we can generate a tensor topology for complete locally convex spaces in the following way. Take any two complete locally convex spaces with basis of continuous seminorms $(p_i)_{i \in I}$ and $(q_j)_{j \in J}$ respectively. Following the previous notation we have projections $\pi_i : E \longrightarrow E_i$ and $\zeta_j : F \longrightarrow F_j$. Now, if $\mathsf{T} \in E \otimes F$, for each i and j we have $(\pi_i \otimes \zeta_j)(\mathsf{T}) \in E_i \otimes F_j$, but this is a tensor product of two normed spaces, on which α can act. We define

$$p_i \otimes_\alpha q_j(\mathsf{T}) = \|(\pi_i \otimes \zeta_j)(\mathsf{T})\|_\alpha.$$

The system $(p_i \otimes_\alpha q_j)_{i \in I,\ j \in J}$ defines a locally convex topology on $E \otimes F$, that we call $\mathcal{T}_\alpha$. We have, for basic tensors,

$$\begin{aligned}(p_i \otimes_\alpha q_j)(x \otimes y) &= \|(\pi_i \otimes \zeta_j)(x \otimes y)\|_\alpha = \|\pi_i(x) \otimes \zeta_j(y)\|_\alpha \\ &= \|\pi_i(x)\| \cdot \|\zeta_j(y)\| = p_i(x) \cdot q_j(y).\end{aligned}$$

This obviously implies that the canonical embedding $E \times F \longrightarrow E \otimes_\alpha F$ is separately continuous. All this construction can be found in [11], Section 35.2. There it is also proved that $\mathcal{T}_\alpha$ satisfies the mapping property, i.e., given any two equicontinuous sets $M \subseteq \mathcal{L}(E; G)$ and $N \subseteq \mathcal{L}(F; L)$, then $M \otimes N \subseteq \mathcal{L}(E \otimes_\alpha F; G \otimes_\alpha L)$ is again equicontinuous. Since the proof is not long, nor difficult, we reproduce it here. Take any seminorm in $G \otimes_\alpha L$, $p_2 \otimes_\alpha q_2$. There exist p_1 and q_1, seminorms in E and F respectively, such that $p_2(f(x)) \leq p_1(x)$ for all $f \in M$ and $x \in E$ and that $q_2(g(y)) \leq q_1(y)$ for all $g \in N$ and $y \in F$. Now, given any $f \in M$ we can define a mapping $f_{p_1 p_2} : E_{p_1} \longrightarrow G_{p_2}$ by $f_{p_1 p_2}(\pi_{p_1}(x)) = \pi_{p_2}(f(x))$. This is a mapping between normed spaces, whose norm can be calculated:

$$\|f_{p_1 p_2}(\pi_{p_1}(x))\|_{p_2} = \|\pi_{p_2}(f(x))\|_{p_2} = p_2(f(x)) \leq p_1 = \|x\|_{p_1}.$$

This implies that $\|f_{p_1 p_2}\| \leq 1$. In the same way, for every $g \in N$ we have $g_{q_1 q_2} : F_{q_1} \longrightarrow L_{q_2}$ with $\|g_{q_1 q_2}\| \leq 1$. Then, for all $f \otimes g \in M \otimes N$, since α satisfies the metric mapping property,

$$\begin{aligned}(p_2 \otimes_\alpha q_2)((f \otimes g)(\mathsf{T})) &= \|(\pi_{p_2} \otimes \zeta_{p_2})((f \otimes g)(\mathsf{T}))\|_\alpha \\ &= \|(f_{p_1 p_2} \otimes g_{q_1 q_2})(\pi_{p_1} \otimes \zeta_{p_1})(\mathsf{T}))\|_\alpha\end{aligned}$$

$$\begin{aligned} &\leq \|f_{p_1p_2} \otimes g_{q_1q_2}\| \, \|(\pi_{p_1} \otimes \zeta_{p_1})(\mathsf{T}))\|_\alpha \\ &= \|f_{p_1p_2}\| \, \|g_{q_1q_2}\| \, \|(\pi_{p_1} \otimes \zeta_{p_1})(\mathsf{T}))\|_\alpha \\ &\leq (p_1 \otimes_\alpha q_1)(\mathsf{T}). \end{aligned}$$

Hence $M \otimes N$ is equicontinuous and every uniform tensor norm α generates a uniform the locally convex tensor topology $\mathcal{T}_\alpha$.
The requirement that E and F are complete is not a big restriction since if α is a finitely generated norm, that is, the norm of an element $\mathsf{T} \in X \otimes Y$ can be obtained as follows

$$\|\mathsf{T}\|_\alpha = \inf\{\alpha(\mathsf{T}; Z, W) : \mathsf{T} \in Z \otimes W\}$$

and Z ranges over all the finite-dimensional subspaces of X and W over all the finite-dimensional subspaces of Y, then the topology $\mathcal{T}_\alpha$ also satisfies $E\hat{\otimes}_{\mathcal{T}_\alpha}F = \hat{E}\hat{\otimes}_{\mathcal{T}_\alpha}\hat{F}$ ([11], Section 35.2).
Obviously both the π and the ε topologies can be obtained in this way. Nevertheless, since they are the two most important topologies we preferred to study them independently.
Many examples of uniform tensor norms that generate uniform tensor topologies can be found in [11], Sections 12.5 and 12.7. Take for example the d_∞ norm, defined by

$$\|\mathsf{T}\|_{d_\infty} = \inf_{\mathsf{T}=\sum_{i=1}^n x_i \otimes y_i} \left(\sup_{\|x'\| \leq 1} \sum_{i=1}^n |x'(x_i)| \quad \sup_{i=1,\ldots,n} \|y_i\| \right).$$

The tensor topology for locally convex spaces generated by d_∞ coincides with the λ topology used in [34]. To see this we first note that

$$\|\mathsf{T}\|_{d_\infty} \geq \inf_{\mathsf{T}=\sum_{i=1}^n x_i \otimes y_i} \left(\sup_{\|x'\| \leq 1} \sum_{i=1}^n |x'(x_i)| \, \|y_i\| \right)$$

for all $\mathsf{T} \in X \otimes Y$. The right hand side generates the λ topology. Take a tensor T and a representation $\mathsf{T} = \sum_{i=1}^n x_i \otimes y_i$. Transform this into the representation $\mathsf{T} = \sum_{i=1}^n x_i \|y_i\| \otimes \frac{y_i}{\|y_i\|}$. From this,

$$\|\mathsf{T}\|_{d_\infty} \leq \sup_{\|x'\| \leq 1} \sum_{i=1}^n |x'(x_i \|y_i\|)| \sup_{i=1,\ldots,n} \left\| \frac{y_i}{\|y_i\|} \right\| = \sup_{\|x'\| \leq 1} \sum_{i=1}^n |x'(x_i)| \, \|y_i\|.$$

This gives the equality that we were looking for. We have then, that the λ topology introduced by H.P. Lotz and used in [34] is a particular case of a tensor topology defined from a uniform tensor norm and, hence, is uniform.
This provides us with a large family of uniform tensor topologies for locally convex spaces.
A very important class of uniform tensor norms is the Lapresté's norms ([11], Section 12.5),

of which the preceding d_∞ is a particular case. Before defining them we need some other concepts. Given any sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ and $1 \le r \le \infty$ we have the classical ℓ_r norms,

$$\ell_r(\lambda_n) = \left(\sum_{n=1}^{\infty} |\lambda_n|^r\right)^{1/r}$$

for $1 \le r < \infty$ and $\ell_\infty(\lambda_n) = \sup_{n\in\mathbb{N}} |\lambda_n|$. Also, for sequences in a normed space $(x_n)_{n\in\mathbb{N}} \subseteq X$, we can define the weak r-norm in the following two ways, recalling the duality between ℓ_r and $\ell_{r'}$, where $\frac{1}{r} + \frac{1}{r'} = 1$,

$$w_r(x_n) = \sup_{x' \in B_{X'}} \left(\sum_{n=1}^{\infty} |x'(x_n)|^r\right)^{1/r} = \sup\{\| \sum_{n=1}^{N} \lambda_n x_n \| : N \in \mathbb{N}, \ell_{r'}(\lambda_n) \le 1\}.$$

for $1 \le r < \infty$ and $w_\infty(x_n) = \sup_{n\in\mathbb{N}} \|x_n\|$. With this, for $1 \le r, s \le \infty$ with $\frac{1}{r} + \frac{1}{s} \ge 1$, take the unique t, $1 \le t \le \infty$ such that $\frac{1}{t} + \frac{1}{r'} + \frac{1}{s'} = 1$ and we define for X and Y normed spaces the Lapresté tensor norm,

$$\alpha_{r,s}(\mathsf{T}) = \inf\{\ell_t(\lambda_i) w_{s'}(x_i) w_{r'}(y_i) : \mathsf{T} = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i\}.$$

If E is a locally convex space and p a continuous seminorm, let

$$w_r^p(x_n) = \sup_{x' \in B_p^\circ} \left(\sum_{n=1}^{\infty} |x'(x_n)|^r\right)^{1/r} = \sup\{p\left(\sum_{n=1}^{N} \lambda_n x_n\right) : N \in \mathbb{N}, \ell_{r'}(\lambda_n) \le 1\}$$

for $1 \le r < \infty$ and $w_\infty^p(x_n) = \sup_{n\in\mathbb{N}} p(x_n)$. So, if E and F are two locally convex spaces whose topologies are defined by seminorms $(p_i)_{i\in I}$ and $(q_j)_{j\in J}$ respectively, then the tensor topology defined by the $\alpha_{r,s}$ norm is generated by the seminorms

$$(p \otimes_{\alpha_{r,s}} q)(\mathsf{T}) = \inf\{\ell_t(\lambda_i) w_{s'}^p(x_i) w_{r'}^q(y_i) : \mathsf{T} = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i\}.$$

All these topologies are uniform.

Lemma 3.2.15

Let $\mathcal{A}$ be a uniform lmc algebra.
Given $(x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, consider $(x_n z_m)_{n,m}$. Then, for any continuous seminorm p, and any r,

$$w_r^p(x_n z_m) \le w_r^p(x_n)\ w_r^p(z_m).$$

Moreover, if $r = \infty$, the result is also true for any $\mathcal{A}$, not necessarily uniform.
Proof.
If $1 < r < \infty$ we have, using the duality between ℓ_r and $\ell_{r'}$,

$$\begin{aligned}
w_r^p(x_n z_m) &= \sup_{x' \in B_p^\circ} \left(\sum_{n,m} |x'(x_n z_m)|^r \right)^{1/r} \\
&= \sup_{\substack{x' \in B_p^\circ \\ \sum_{n,m} |\lambda_{nm}|^{r'} \le 1}} \left| \sum_{n,m} \lambda_{nm} x'(x_n z_m) \right| \\
&= \sup_{\substack{x' \in B_p^\circ \\ \sum_{n,m} |\lambda_{nm}|^{r'} \le 1}} \left| x' \left(\sum_{n,m} \lambda_{nm} x_n z_m \right) \right| \\
&= \sup_{\sum_{n,m} |\lambda_{nm}|^{r'} \le 1} \; \sup_{h \in B_p^\circ \cap \mathfrak{M}(\mathcal{A})} \left| \sum_{n,m} \lambda_{nm} h(x_n) h(z_m) \right| \\
&= \sup_{h \in B_p^\circ \cap \mathfrak{M}(\mathcal{A})} \left(\sum_{n,m} |h(x_n) h(z_m)|^r \right)^{1/r} \\
&\le \sup_{h \in B_p^\circ \cap \mathfrak{M}(\mathcal{A})} \left(\sum_{n} |h(x_n)|^r \right)^{1/r} \sup_{h \in B_p^\circ \cap \mathfrak{M}(\mathcal{A})} \left(\sum_{m} |h(z_m)|^r \right)^{1/r} \\
&= \sup_{x' \in B_p^\circ} \left(\sum_{n} |x'(x_n)|^r \right)^{1/r} \sup_{x' \in B_p^\circ} \left(\sum_{m} |x'(z_m)|^r \right)^{1/r} \\
&= w_r^p(x_n)\; w_r^p(z_m).
\end{aligned}$$

When $r = 1$, the proof is even simpler, since we do not need to use the duality. For the case $r = \infty$ we have, for any lmc algebra $\mathcal{A}$,

$$\begin{aligned}
w_\infty^p(x_n z_m) &= \sup_{n,m} p(x_n z_m) \le \sup_{n,m} p(x_n) p(z_m) \\
&= \sup_{n} p(x_n) \sup_{m} p(z_m) = w_\infty^p(x_n) w_\infty^p(z_m).
\end{aligned}$$

q.e.d.

Proposition 3.2.16
Let $\mathcal{A}, \mathcal{B}$ be two lmc algebras. Then,
(1) *If both $\mathcal{A}$ and $\mathcal{B}$ are uniform algebras, then $\mathcal{A} \otimes_{\alpha_{r,s}} \mathcal{B}$ is a lmc algebra for all r, s.*
(2) *If $\mathcal{A}$ is uniform, then $\mathcal{A} \otimes_{\alpha_{1,s}} \mathcal{B}$ is a lmc algebra for any lmc algebra $\mathcal{B}$ and for all s.*
Proof.

Take p, q two continuous seminorms on $\mathcal{A}, \mathcal{B}$ respectively and $\mathsf{T}, \mathsf{S} \in \mathcal{A} \otimes \mathcal{B}$. Consider two representations $\mathsf{T} = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$ and $\mathsf{S} = \sum_{j=1}^{m} \mu_j z_j \otimes w_j$. We then have a representation for the product $\mathsf{TS} = \sum_{i,j=1}^{n,m} \lambda_i \mu_j x_i z_j \otimes y_i w_j$. Clearly

$$\ell_t(\lambda_i \mu_j) = \left(\sum_{i,j=1}^{n,m} |\lambda_i \mu_j|^t \right)^{1/t} = \left(\sum_{i=1}^{n} |\lambda_i|^t \right)^{1/t} \left(\sum_{j=1}^{m} |\mu_j|^t \right)^{1/t} = \ell_t(\lambda_i) \ell_t(\mu_j).$$

From this,

$$\begin{aligned} (p \otimes_{\alpha_{r,s}} q)(\mathsf{TS}) &\leq \ell_t(\lambda_i \mu_j) w^p_{s'}(x_i z_j) w^q_{r'}(y_i w_j) \\ &\leq \ell_t(\lambda_i) \ell_t(\mu_j) w^p_{s'}(x_i) w^p_{s'}(z_j) w^q_{r'}(y_i) w^q_{r'}(w_j) \\ &= \left(\ell_t(\lambda_i) w^p_{s'}(x_i) w^q_{r'}(y_i) \right) \left(\ell_t(\mu_j) w^p_{s'}(z_j) w^q_{r'}(w_j) \right). \end{aligned}$$

Taking the infimum over all possible representations we get

$$(p \otimes_{\alpha_{r,s}} q)(\mathsf{TS}) \leq (p \otimes_{\alpha_{r,s}} q)(\mathsf{T})(p \otimes_{\alpha_{r,s}} q)(\mathsf{S}).$$

This shows that $\mathcal{A} \otimes_{\alpha_{r,s}} \mathcal{B}$ is a lmc algebra.
Note that if $s = 1$, then $s' = \infty$ and, as we saw in Lemma 3.2.15, we do not need $\mathcal{A}$ uniform.
q.e.d.

This implies that $\mathcal{A} \otimes_\lambda \mathcal{B}$ (see [34]) is a lmc algebra when $\mathcal{A}$ is uniform (for any $\mathcal{B}$), since $\lambda = d_\infty = \alpha_{1,\infty}$ ([11], Section 12.7). We can also recover the result in [49] on the π topology, since $\pi = \alpha_{1,1}$.

3.3 A vector Gelfand mapping

The classical Gelfand theory has widely studied since its first appearance during the 1930's. It is a scalar theory; our goal is to give an analogous vector theory in the context of tensor products. What is done in the classical theory is, given a commutative Banach algebra and $\mathfrak{M}(\mathcal{A})$ its space of linear multiplicative functionals, consider for each $x \in \mathcal{A}$ the mapping $\hat{x} : \mathfrak{M}(\mathcal{A}) \longrightarrow \mathbb{C}$, called the *Gelfand transform* of x, defined by $\hat{x}(h) = h(x)$. This mapping is continuous and from it the *Gelfand mapping*, $\hat{} : \mathcal{A} \longrightarrow \mathcal{C}(\mathfrak{M}(\mathcal{A}))$, is defined by $x \mapsto \hat{x}$. This is a continuous algebra homomorphism with interesting properties For a more detailed study, see [70], Chapters 12 and 13. In [18] and [74] the authors define a vector Gelfand mapping on tensor products of Banach algebras and Banach spaces. Following the same ideas we define a Gelfand mapping in our new context.

3.3.1 Definition

Let $\mathcal{A}$ be a lmc-algebra and E a locally convex space. Given $h \in \mathfrak{M}(\mathcal{A})$ and the identity mapping $I_E : E \longrightarrow E$ we can consider $h \otimes I_E : \mathcal{A}\hat{\otimes}_{\mathcal{T}} E \longrightarrow \mathbb{C}\hat{\otimes}_{\mathcal{T}} E$. It is a well known fact that if E is complete $\mathbb{C}\hat{\otimes}_{\mathcal{T}} E \cong E$; from now on, in order to simplify notation, we are going to assume that E is complete and, keeping in mind this identification, write E instead of $\mathbb{C}\hat{\otimes}_{\mathcal{T}} E$.

Definition 3.3.1 Let $\mathcal{A}$ be a lmc algebra, E a complete locally convex space and $\mathcal{T}$ a uniform tensor topology. For each $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ we define its *Gelfand transform*,

$$\hat{\mathsf{T}} : \mathfrak{M}(\mathcal{A}) \longrightarrow E \ \ , \ \ \hat{\mathsf{T}}(h) = [h \otimes I_E](\mathsf{T}).$$

We first check if this mapping is continuous. Unfortunately this is not always true, although we can prove it for Q-algebras. In $\mathfrak{M}(\mathcal{A})$ we consider the weak* topology induced from $(\mathcal{A}', \sigma(\mathcal{A}', \mathcal{A}))$.

Proposition 3.3.2
Let $\mathcal{A}$ be a lmc Q-algebra, E a complete locally convex space and $\mathcal{T}$ a uniform tensor topology. Then, for each $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$, the mapping $\hat{\mathsf{T}} : \mathfrak{M}(\mathcal{A}) \longrightarrow E$ is continuous.
Proof.
Fix $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ and $h \in \mathfrak{M}(\mathcal{A})$ and consider a net $(h_\alpha)_\alpha \subseteq \mathfrak{M}(\mathcal{A})$ weakly*-converging to h. Let q be any continuous seminorm on E and let $\varepsilon > 0$. Since $\mathcal{A}$ is a Q-algebra, $\mathfrak{M}(\mathcal{A})$ is equicontinuous and, therefore, so is $\mathfrak{M}(\mathcal{A}) \otimes \{I_E\}$. Then we can find some seminorm ρ_1 on $\mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ such that $q([g \otimes I_E](\mathsf{Z})) \leq \rho_1(\mathsf{Z})$ for all $\mathsf{Z} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ and $g \in \mathfrak{M}(\mathcal{A})$. On the other hand, h is continuous and there is some other seminorm on $\mathcal{A}\hat{\otimes}_{\mathcal{T}} E$, say ρ_2, such that $q([h \otimes I_E](\mathsf{Z})) \leq \rho_2(\mathsf{Z})$ for all Z. Choose $\mathsf{S} = \sum_{i=1}^n a_i \otimes x_i$ such that

$$\max\{\rho_1(\mathsf{T} - \mathsf{S}), \rho_2(\mathsf{T} - \mathsf{S})\} < \frac{\varepsilon}{4}.$$

Since $h_\alpha \to h$ in the weak* topology, we have $h_\alpha(a_i) \to h(a)$ for all $a \in \mathcal{A}$. Let $K = \max_{i=1,\ldots,n} q(x_i)$ and for each $i = 1, \ldots, n$ we can find α_i such that, for $\alpha \geq \alpha_i$, satisfies $|h_\alpha(a_i) - h(a_i)| < \frac{\varepsilon}{2\ n\ K}$. Let $\alpha_0 = \max(\alpha_1, \ldots, \alpha_n)$. If $\alpha \geq \alpha_0$, we have

$$\begin{aligned}
& q([h_\alpha \otimes I_E](\mathsf{T}) - [h \otimes I_E](\mathsf{T})) = q([(h_\alpha - h) \otimes I_E](\mathsf{T} - \mathsf{S} + \mathsf{S}) \\
\leq\ & q([h_\alpha \otimes I_E](\mathsf{T} - \mathsf{S})) + q([h \otimes I_E](\mathsf{T} - \mathsf{S})) + q([(h_\alpha - h) \otimes I_E](\mathsf{S})) \\
\leq\ & \rho_1(\mathsf{T} - \mathsf{S}) + \rho_2(\mathsf{T} - \mathsf{S}) + q\left(\sum_{i=1}^n (h_\alpha(a_i) - h(a_i))x_i\right) \\
\leq\ & \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \sum_{i=1}^n |h_\alpha(a_i) - h(a_i)| q(x_i) \\
\leq\ & \frac{\varepsilon}{2} + nK\frac{\varepsilon}{2\ n\ K} = \varepsilon.
\end{aligned}$$

Thus, $\lim_\alpha q([h_\alpha \otimes I_E](\mathsf{T}) - [h \otimes I_E](\mathsf{T})) = 0$ and $\hat{\mathsf{T}} \in \mathcal{C}(\mathfrak{M}(\mathcal{A}); E)$.
q.e.d.

It is well known that $\mathcal{H}(\mathbb{C})\hat{\otimes}_\varepsilon E = \mathcal{H}(\mathbb{C}; E)$ and $\mathfrak{M}(\mathcal{H}(\mathbb{C})) \cong \mathbb{C}$ topologically (by means of the evaluation functionals). Let $\mathsf{F} = \sum_{i=1}^n f_i \otimes x_i \in \mathcal{H}(\mathbb{C}) \otimes E$. Given any $z \in \mathbb{C}$ we have

$$\hat{\mathsf{F}}(\delta_z) = [\delta_z \otimes I_E](\mathsf{F}) = \sum_{i=1}^n f_i(z)x_i = \mathsf{F}(z).$$

Continuity and density show that for all $\mathsf{F} \in \mathcal{H}(\mathbb{C})\hat{\otimes}_\varepsilon E$ and all $z \in \mathbb{C}$ the equality $\hat{\mathsf{F}}(\delta_z) = \mathsf{F}(z)$ (as a function in $\mathcal{H}(\mathbb{C}; E)$) holds. Then obviously $\hat{\mathsf{F}}$ is continuous. This provides us with an example that shows that Q-algebra is not a necessary condition in Proposition 3.3.2.
With Proposition 3.3.2 we can define the vector Gelfand mapping.

Definition 3.3.3 Let $\mathcal{A}$ be a lmc Q-algebra, E a complete locally convex space and $\mathcal{T}$ a uniform tensor topology; we define the *vector Gelfand mapping* to be the mapping $\hat{}$: $\mathcal{A}\hat{\otimes}_\mathcal{T} E \longrightarrow \mathcal{C}(\mathfrak{M}(\mathcal{A}); E)$ given by $\hat{\mathsf{T}}(h) = [h \otimes I_E](\mathsf{T})$.

3.3.2 The Gelfand mapping as an algebra homomorphism

If $\mathcal{A}$ is a unital lmc Q-algebra, $\mathfrak{M}(\mathcal{A})$ is $\omega^\star$-compact. Given a locally convex space E and a continuous seminorm q we consider the seminorm in $\mathcal{C}(\mathfrak{M}(\mathcal{A}), E)$ given by $\tilde{q}(F) = \sup_{h \in \mathfrak{M}(\mathcal{A})} q(F(h))$ for $F \in \mathcal{C}(\mathfrak{M}(\mathcal{A}), E)$. If the topology of E is defined by a family of seminorms $(q_\beta)_\beta$, the family $(\tilde{q}_\beta)_\beta$ defines a locally convex topology in $\mathcal{C}(\mathfrak{M}(\mathcal{A}), E)$. With this notation we have the following result.

Proposition 3.3.4
Let $\mathcal{A}$ be a unital lmc Q-algebra, E any complete locally convex space, $\mathcal{T}$ a uniform topology for the tensor product. Then, the Gelfand mapping $\widehat{\ }$: $\mathcal{A}\hat{\otimes}_\mathcal{T} E \longrightarrow \mathcal{C}(\mathfrak{M}(\mathcal{A}), E)$ is a continuous linear mapping.
If, furthermore, $\mathcal{B}$ is a complete lmc algebra and $\mathcal{A}\hat{\otimes}_\mathcal{T}\mathcal{B}$ is a lmc algebra, then the Gelfand mapping $\widehat{\ }$: $\mathcal{A}\hat{\otimes}_\mathcal{T}\mathcal{B} \longrightarrow \mathcal{C}(\mathfrak{M}(\mathcal{A}), \mathcal{B})$ is a continuous algebra homomorphism.
Proof.
The Gelfand mapping is easily seen to be linear. To see that it is continuous we have to show that, for every $\tilde{q}$ there is some p, a continuous seminorm on $\mathcal{A}\hat{\otimes}_\mathcal{T} E$, such that $\tilde{q}(\hat{\mathsf{T}}) \le p(\mathsf{T})$ for all $\mathsf{T} \in \mathcal{A}\hat{\otimes}_\mathcal{T} E$. Take, then, $\tilde{q}$ any continuous seminorm on $\mathcal{C}(\mathfrak{M}(\mathcal{A}), E)$.
Since $\mathfrak{M}(\mathcal{A}) \otimes \{I_E\}$ is equicontinuous, there exists a continuous seminorm on $\mathcal{A}\hat{\otimes}_\mathcal{T} E$, say p, such that $q([h \otimes I_E](\mathsf{T})) \le p(\mathsf{T})$ for every $\mathsf{T} \in \mathcal{A}\hat{\otimes}_\mathcal{T} E$ and $h \in \mathfrak{M}(\mathcal{A})$. Hence

$$\sup_{h \in \mathfrak{M}(\mathcal{A})} q([h \otimes I_E](\mathsf{T})) \le p(\mathsf{T})$$

and the Gelfand mapping is continuous.
Suppose $\mathcal{B}$ is a complete lmc algebra. Let $\mathsf{T} := \sum_{i=1}^{k} a_i \otimes c_i$ and $\mathsf{S} := \sum_{j=1}^{n} b_j \otimes d_j$ belong to $\mathcal{A} \otimes \mathcal{B}$ and $h \in \mathfrak{M}(\mathcal{A})$. Then

$$\begin{aligned}
\widehat{\mathsf{TS}}(h) &= [h \otimes I_{\mathcal{B}}](\mathsf{TS}) \\
&= [h \otimes I_{\mathcal{B}}]\left(\sum_{i,j=1}^{k,n} a_i b_j \otimes c_i d_j\right) \\
&= \sum_{i,j=1}^{k,n} h(a_i b_j) c_i d_j \\
&= \sum_{i,j=1}^{k,n} h(a_i) h(b_j) c_i d_j \\
&= \sum_{i=1}^{k} h(a_i) c_i \sum_{j=1}^{n} h(b_j) d_j \\
&= [h \otimes I_{\mathcal{B}}](\mathsf{T})\,[h \otimes I_{\mathcal{B}}](\mathsf{S}) \\
&= \hat{\mathsf{T}}(h)\,\hat{\mathsf{S}}(h).
\end{aligned}$$

Hence $\widehat{\mathsf{TS}} = \hat{\mathsf{T}}\hat{\mathsf{S}}$ in $\mathcal{A} \otimes \mathcal{B}$. By a density argument, together with properties of $\mathcal{T}$, this extends to all of $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ and the Gelfand mapping is an algebra homomorphism.
q.e.d.

3.4 Commutative case

We are going to consider tensor products $\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, where $\mathcal{A}$ is a lmc algebra, which in many cases will be a Q-algebra, $\mathcal{T}$ a uniform tensor topology and E a locally convex space. The results obtained and the techniques differ depending on whether the algebra is commutative or not; for this reason we study both cases separately.

3.4.1 The Waelbroeck spectrum

In a unital algebra $\mathcal{A}$ the spectrum of an element $a \in \mathcal{A}$ is defined as follows

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1_{\mathcal{A}} \text{ not invertible.}\} \subseteq \mathbb{C}. \tag{3.2}$$

This is a classical concept that we call the scalar spectrum. In the same way, using left or right invertibility, left and right spectra may be defined. Our goal now is to define analogous concepts for elements in $\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$ that we call vector spectra.

If $\mathcal{A}$ is a commutative, unital Banach algebra, then the scalar spectrum of each $a \in \mathcal{A}$ can be represented in the following way (see e.g. [70], Section 70, Theorem B),

$$\sigma(a) = \{\hat{a}(h) : h \in \mathfrak{M}(\mathcal{A})\}.$$

We have just defined a vector valued Gelfand mapping, using it we can define a vector spectrum when the algebra is commutative.

Definition 3.4.1 Let $\mathcal{A}$ be a commutative, unital lmc algebra, E a complete locally convex space and $\mathcal{T}$ a uniform topology on $\mathcal{A} \otimes E$. Given $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$, the *Waelbroeck spectrum* of T is defined to be the following set:

$$\sigma_W(\mathsf{T}) = \{[h \otimes I_E](\mathsf{T}) \, : \, h \in \mathfrak{M}(\mathcal{A})\}.$$

L. Waelbroeck defined in [77] his spectrum during the 1970's for elements of $\mathcal{A}\hat{\otimes}_\pi X$, where $\mathcal{A}$ is a unital commutative Banach algebra and X is a Banach space. In this case each tensor has a representation $\mathsf{T} = \sum_{i=1}^{\infty} a_i \otimes x_i$ with $a_i \in \mathcal{A}$, $x_i \in X$ and $\sum_{i=1}^{\infty} \|a_i\| \cdot \|x_i\| < \infty$. Then he defined the spectrum,

$$\sigma_W(\mathsf{T}) = \{\sum_{i=1}^{\infty} h(a_i) x_i : h \in \mathfrak{M}(\mathcal{A})\} \subseteq X.$$

Years later C. Taylor realized in his Ph.D. Dissertation ([74]) that the above sum is the form $[h \otimes I_E](\mathsf{T})$ takes when T has such a representation. With this observation he defined the Waelbroeck spectrum for $\mathsf{T} \in \mathcal{A}\hat{\otimes}_\gamma X$, when $\mathcal{A}$ is a commutative unital Banach algebra, X a Banach space and γ a uniform tensor norm. This definition inspired the one we give here for more general algebras and spaces.

We have already seen that when $\mathcal{A}$ is also a Q-algebra the Gelfand mapping has particularly good properties. This suggests that the Waelbroeck spectrum has also interesting properties. For instance, Waelbroeck proved that his spectrum is compact ([77]), Taylor proved the analogous result in [74], Section 2.2. When $\mathcal{A}$ is a Q-algebra, we also obtain this result.

Proposition 3.4.2
Let $\mathcal{A}$ be a commutative, unital lmc Q-algebra, E a complete locally convex space and $\mathcal{T}$ a uniform topology on $\mathcal{A} \otimes E$. Then, for each $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$, the spectrum $\sigma_W(\mathsf{T})$ is compact in E.
Proof.
We have, using the Gelfand transform, $\sigma_W(\mathsf{T}) = \{\hat{\mathsf{T}}(h) : h \in \mathfrak{M}(\mathcal{A})\}$. Since $\mathcal{A}$ is a Q-algebra, $\mathfrak{M}(\mathcal{A})$ is $\omega^\star$-compact on $\mathcal{A}'$. Furthermore, the mapping $\hat{\mathsf{T}} : \mathfrak{M}(\mathcal{A}) \longrightarrow E$ is continuous when we consider the $\omega^\star$ topology on $\mathfrak{M}(\mathcal{A})$ and the locally convex topology in E. Then, $\sigma_W(\mathsf{T})$ is the continuous image of a compact set and, hence, compact.
q.e.d.

3.4.2 Projective limits

Let $\mathcal{A}$ be a complete lmc algebra whose topology is given by a directed family of multiplicative seminorms $(p_i)_{i\in I}$ so that $p_i \leq p_j$ for all $i \leq j$. In particular, $\mathcal{A}$ is a complete locally convex space and we have already seen that then $\mathcal{A}$ can be realized as a projective limit $\mathcal{A} = \varprojlim \mathcal{A}_i$ where each $\mathcal{A}_i$ is a Banach space. By the submultiplicativity of the seminorms, $\ker p_i$ is a two-sided ideal of $\mathcal{A}$ and $\mathcal{A}/\ker p_i$ is well defined as an algebra. Therefore, $(\mathcal{A}_i, \|\ \|_i)$ is a Banach algebra and $\pi_i : \mathcal{A} \longrightarrow \mathcal{A}_i$ is for each $i \in I$ a continuous algebra homomorphism (not necessarily surjective). A natural question now is to try to relate the Waelbroeck spectrum of an element with those of its projections. This was studied in [8] when $\mathcal{A}$ is a Banach algebra, E a Banach space and the π topology is used. Following the same steps given there we are going to consider our more general case. We need to impose a new condition on the tensor topology.

Definition 3.4.3 We say that a uniform tensor topology satisfies the *projective limit condition* if for every pair of projective systems $(E_\alpha)_\alpha, (F_\beta)_\beta$ with projective limits $E = \varprojlim E_\alpha$ and $F = \varprojlim F_\beta$, we have that $E\hat{\otimes}_{\mathcal{T}}F = \varprojlim E_\alpha\hat{\otimes}_{\mathcal{T}}F_\beta$.

This condition is obviously satisfied by any uniform tensor topology coming from a uniform tensor norm.

Lemma 3.4.4
Let $\mathcal{A} = \varprojlim \mathcal{A}_i$ be a complete lmc algebra and $f : \mathcal{A} \longrightarrow \mathbb{C}$ a continuous linear mapping. Then there exist some i and $f_i : \mathcal{A}_i \longrightarrow \mathbb{C}$ linear, continuous such that $f = f_i \circ \pi_i$ (i.e. f factorizes through some of the Banach algebras).
If f is a homomorphism, then so also is f_i.
Proof.
Since f is continuous, we can find some seminorm p_i satisfying that $|f(a)| \leq p_i(a)$ for all $a \in \mathcal{A}$. This means that $\ker p_i \subseteq \ker f$ and we can find some continuous $f_i : \mathcal{A}/\ker p_i \longrightarrow \mathbb{C}$, linear, such that $f_i \circ \pi_i = f$.
q.e.d.

Remark 3.4.5
For $i, j, k \in I$ with $k > j > i$ we have, in completed tensor products with a uniform tensor topology satisfying the projective limit condition,

$$\begin{aligned}(\pi_{ij} \otimes I_E) \circ (\pi_{jk} \otimes I_E) &= (\pi_{ik} \otimes I_E)\\ (\pi_{ij} \otimes I_E) \circ (\pi_j \otimes I_E) &= (\pi_i \otimes I_E)\end{aligned}$$

In the first case,

$$\pi_{ij} \otimes I_E : \mathcal{A}_j\hat{\otimes}_{\mathcal{T}}E \longrightarrow \mathcal{A}_i\hat{\otimes}_{\mathcal{T}}E, \qquad \pi_{jk} \otimes I_E : \mathcal{A}_k\hat{\otimes}_{\mathcal{T}}E \longrightarrow \mathcal{A}_j\hat{\otimes}_{\mathcal{T}}E,$$
$$\pi_{ik} \otimes I_E : \mathcal{A}_k\hat{\otimes}_{\mathcal{T}}E \longrightarrow \mathcal{A}_i\hat{\otimes}_{\mathcal{T}}E.$$

Take $\mathsf{T} = \sum_{p=1}^{n} a_p \otimes x_p \in \mathcal{A}_k \otimes E$. Then

$$\begin{aligned}
(\pi_{ij} \otimes I_E) \circ (\pi_{jk} \otimes I_E)(\mathsf{T}) &= (\pi_{ij} \otimes I_E)\left(\sum_{p=1}^{n} \pi_{jk}(a_p) \otimes x_p\right) \\
&= \sum_{p=1}^{n} \pi_{ij}(\pi_{jk}(a_p)) \otimes x_p = \sum_{p=1}^{n} \pi_{ik}(a_p) \otimes x_p \\
&= (\pi_{ik} \otimes I_E)(\mathsf{T}) \in \mathcal{A}_i \otimes E.
\end{aligned}$$

Since the mappings are continuous, we have the equality in the completed tensor products. The second equality is proved in the same way.

Now, for $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ we denote, following [8], $\mathsf{T}_i = (\pi_i \otimes I_E)(\mathsf{T}) \in \mathcal{A}_i\hat{\otimes}_{\mathcal{T}} E$. If $\mathsf{T} = \sum_{k=1}^{n} a_k \otimes x_k$, then, clearly

$$\mathsf{T}_i = (\pi_i \otimes I_E)\left(\sum_{k=1}^{n} a_k \otimes x_k\right) = \sum_{k=1}^{n} \pi_i(a_k) \otimes x_k.$$

It is also clear, from Remark 3.4.5, that $(\pi_{ij} \otimes I_E)(\mathsf{T}_j) = \mathsf{T}_i$ for $i < j$.

Lemma 3.4.6

Let $\mathcal{A}$ be a complete unital lmc algebra, E a complete locally convex space and $\mathcal{T}$ a uniform tensor topology that satisfies the projective limit condition. If $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$, $j > i$ and $h \in \mathfrak{M}(\mathcal{A}_i)$ then,

(i) $h \circ \pi_{ij} \in \mathfrak{M}(\mathcal{A}_j)$ and $(h \circ \pi_{ij}) \otimes I_E(\mathsf{T}_j) = h \otimes I_E(\mathsf{T}_i)$.

(ii) $h \circ \pi_i \in \mathfrak{M}(\mathcal{A})$ and $(h \circ \pi_i) \otimes I_E(\mathsf{T}) = h \otimes I_E(\mathsf{T}_i)$.

Proof.

We prove (i) and (ii) is proved in the same way. First of all, $h \circ \pi_{ij} \in \mathfrak{M}(\mathcal{A}_j)$, since $h : \mathcal{A}_i \longrightarrow \mathbb{C}$, $\pi_{ij} : \mathcal{A}_j \longrightarrow \mathcal{A}_i$ and both are algebra homomorphisms.

For the second part, take first $\mathsf{T} = \sum_{k=1}^{n} a_k \otimes x_k$. Then

$$\begin{aligned}
(h \circ \pi_{ij}) \otimes I_E(\mathsf{T}_j) &= (h \circ \pi_{ij}) \otimes I_E\left(\sum_{k=1}^{n} \pi_j(a_k) \otimes x_k\right) \\
&= \sum_{k=1}^{n} (h \circ \pi_{ij})(\pi_i(a_k)) \otimes x_k \\
&= (h \otimes I_E)\left(\sum_{k=1}^{n} \pi_i(a_k) \otimes x_k\right) \\
&= (h \otimes I_E)(\mathsf{T}_i).
\end{aligned}$$

Using a density argument we complete the proof.

q.e.d.

With this we can finally prove the relationship between the spectrum of an element and those of its projections.

Proposition 3.4.7
Let $\mathcal{A}$ be a complete, commutative, unital lmc algebra, E a complete locally convex space and $\mathcal{T}$ a uniform tensor topology satisfying the projective limit condition. Then for all $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$,
(i) $\sigma_W(\mathsf{T}_i) \subseteq \sigma_W(\mathsf{T}_j)$ *whenever* $j > i$.
(ii) $\sigma_W(\mathsf{T}) = \bigcup_{i\in I} \sigma_W(\mathsf{T}_i)$.

Proof.
Let $h \in \mathfrak{M}(\mathcal{A}_i)$. By Lemma 3.4.6,

$$[h \otimes I_E](\mathsf{T}_i) = [(h \circ \pi_{ij}) \otimes I_E](\mathsf{T}_j) = [(h \circ \pi_i) \otimes I_E](\mathsf{T})$$

and $h \circ \pi_{ij} \in \mathfrak{M}(\mathcal{A}_j)$, $h \circ \pi_i \in \mathfrak{M}(\mathcal{A})$ for all $i < j$. From this,

$$\sigma_W(\mathsf{T}_i) \subseteq \sigma_W(\mathsf{T}_j) \subseteq \sigma_W(\mathsf{T}).$$

So, we have the first statement and the first inclusion in (ii).
Let $h \in \mathfrak{M}(\mathcal{A})$. By Lemma 3.4.4, we can find $h_i : \mathcal{A}/\ker p_i \longrightarrow \mathbb{C}$ such that $h_i \circ \pi_i = h$; then,

$$[h \otimes I_E](\mathsf{T}) = [(h_i \circ \pi_i) \otimes I_E](\mathsf{T}) = [h_i \otimes I_E](\mathsf{T}_i).$$

Therefore $\sigma_W(\mathsf{T}) \subseteq \bigcup_{i\in I} \sigma_W(\mathsf{T}_i)$.

q.e.d.

This result generalizes Proposition 2.3 in [8].

3.5 Non-commutative case

3.5.1 The algebraic Harte spectrum

Definition

We now have a vector spectrum when $\mathcal{A}$ is commutative. This was inspired by the relationship between the scalar spectrum and the Gelfand transform. In the non-commutative case we cannot use this property. To proceed we return to the definition of the scalar spectrum recalled in (3.2). If the algebra is not commutative left and right invertible elements can be considered. This leads us to consider left and right spectra. We always work from the left. Everything done in this setting can be immediately translated to the right invertible setting.
Given $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we have $\lambda \in \sigma^{left}(a)$ if and only if $a - \lambda 1_{\mathcal{A}}$ is not left invertible. With this idea R.E. Harte defined during the 1970's a left spectrum for an arbitrary family of elements of a Banach algebra.

Definition 3.5.1 Given a family $A = (a_i)_{i\in I} \subseteq \mathcal{A}$, its *left joint Harte spectrum*, $\sigma_H^{left}(A) \subseteq \mathbb{C}^I$, is defined to be the following set,

$$\lambda = (\lambda_i)_{i\in I} \in \sigma_H^{left}(A) \Leftrightarrow 1_{\mathcal{A}} \notin \{ \sum_{\substack{i\in F\subseteq I \\ F \text{finite}}} b_i(a_i - \lambda_i 1_{\mathcal{A}}) \ : \ b_i \in \mathcal{A}\}.$$

Note that the set of finite sums on the right hand side is nothing else but the left ideal generated by $(a_i)_{i\in I}$. Also, when the family consists of just one element, the joint spectrum is the scalar left spectrum.

Remark 3.5.2
Since we are only considering finite sums, clearly $(\lambda_i)_{i\in I} \in \sigma_H^{left}((a_i)_{i\in I})$ if and only if $(\lambda_j)_{j\in J} \in \sigma_H^{left}((a_j)_{j\in J})$ for each finite set $J \subseteq I$.

Properties

In [18] properties of the Harte spectrum are proved for the case when $\mathcal{A}$ is a Banach algebra. We give here some analogous properties in the case of lmc algebras that will be very useful later. We begin with a technical lemma. Topological divisors of 0 for lmc algebras were introduced by Michael in [54] and proved a result (Proposition 11.6) to which our next result is very close. Nevertheless, the proof we give here is different from that and is inspired by Theorem B of Section 66 in [70].

Lemma 3.5.3
Let $\mathcal{A}$ be a unital lmc Q-algebra and denote by $S_{\mathcal{A}}$ the set of non-invertible elements of $\mathcal{A}$. Then, for $z \in \partial S_{\mathcal{A}}$, there exist a continuous multiplicative seminorm p and a net $(z_\alpha)_\alpha \subseteq \mathcal{A}$ such that $p(z_\alpha) = 1$ for all α and $\lim_\alpha p(z_\alpha z) = 0 = \lim_\alpha p(z z_\alpha)$.
Proof.
Since $z \in \partial S_{\mathcal{A}}$, there exist a net $(r_\alpha)_\alpha \subseteq \mathcal{A}$ of invertibles converging to z. On the other hand, since $\mathcal{A}$ is a Q-algebra the set of invertible elements is open and therefore we can find some $\varepsilon > 0$ and p a continuous multiplicative seminorm so that $\{a \in \mathcal{A} \ : \ p(a-1) < \varepsilon\} \subseteq \mathcal{A}_{inv}$. Suppose that $(p(r_\alpha^{-1}))_\alpha$ is bounded. Then, since $(r_\alpha)_\alpha$ converges to z,

$$p(r_\alpha^{-1}(z - r_\alpha)) \leq p(r_\alpha^{-1})p(z - r_\alpha) \longrightarrow 0.$$

But $r_\alpha^{-1}(z - r_\alpha) = r_\alpha^{-1}z - 1_{\mathcal{A}}$, and for some α, $r_\alpha^{-1}z$ is invertible. From this $z = r_\alpha(r_\alpha^{-1}z)$ is invertible, contradicting the fact that $z \in \partial S_{\mathcal{A}} \subseteq S_{\mathcal{A}}$. Thus, $(p(r_\alpha^{-1}))_\alpha$ is not bounded and we can assume that $\lim_\alpha p(r_\alpha^{-1}) = \infty$.
Let $z_\alpha = \frac{r_\alpha^{-1}}{p(r_\alpha^{-1})}$. Obviously $p(z_\alpha) = 1$ for all α. Also,

$$\begin{aligned} z z_\alpha &= z\frac{r_\alpha^{-1}}{p(r_\alpha^{-1})} = \frac{1_{\mathcal{A}} + z r_\alpha^{-1} - 1_{\mathcal{A}}}{p(r_\alpha^{-1})} = \frac{1_{\mathcal{A}} + z r_\alpha^{-1} - r_\alpha r_\alpha^{-1}}{p(r_\alpha^{-1})} \\ &= \frac{1_{\mathcal{A}} + (z - r_\alpha) r_\alpha^{-1}}{p(r_\alpha^{-1})} = \frac{1_{\mathcal{A}}}{p(r_\alpha^{-1})} + (z - r_\alpha) z_\alpha. \end{aligned}$$

From this, $p(zz_\alpha) \leq \frac{1}{p(r_\alpha^{-1})} + p(z - r_\alpha)p(z_\alpha) \longrightarrow 0$. In the same way we prove that $p(z_\alpha z)$ also tends to 0.
q.e.d.

If $\mathcal{J} \subset \mathcal{I}$ are any two sets, a canonical projection $\pi_{\mathcal{J}}$ is defined from $\mathbb{C}^{\mathcal{I}}$ to $\mathbb{C}^{\mathcal{J}}$ by $\pi_{\mathcal{J}}((\lambda_i)_{i\in\mathcal{I}}) = (\lambda_j)_{j\in\mathcal{J}}$. This mapping is used in the next two results.

Proposition 3.5.4
Let $\mathcal{A}$ a unital lmc Q-algebra, $(a_i)_{i\in I}$, $(b_j)_{j\in J} \subseteq \mathcal{A}$. Then:

$$\sigma_H^{left}((a_i)_{i\in I}, (b_j)_{j\in J}) \subseteq \sigma_H^{left}((a_i)_{i\in I}) \times \sigma_H^{left}((b_j)_{j\in J}).$$

If furthermore the following commutativity conditions are satisfied for every $i, k \in I$ and $j \in J$.

$$a_i a_k = a_k a_i \quad , \quad a_i b_j = b_j a_i,$$

then $\pi_J(\sigma_H^{left}((a_i)_{i\in I}, (b_j)_{j\in J})) = \sigma_H^{left}((b_j)_{j\in J})$.
Proof.
The first statement is obvious from Remark 3.5.2. Let us prove the second one by transfinite induction over $|I|$. Suppose first that $|I| = 1$. In this case we can assume $(a_i)_{i\in I} = a \in \mathcal{A}$. Take $(\mu_j)_{j\in J} \in \sigma_H^{left}((b_j)_{j\in J}) \in \mathbb{C}^J$.
Denote by N the closed left ideal generated in $\mathcal{A}$ by $(b_j - \mu_j 1_{\mathcal{A}})_{j\in J}$. Obviously, $1_{\mathcal{A}} \notin N$. Define now

$$M = \{y \in \mathcal{A} \ : \ Ny \subseteq N\}.$$

Clearly $1_{\mathcal{A}} \in M$ and it is easily seen that M is an algebra. Then it is a unital subalgebra of $\mathcal{A}$. Let $n \in N$. We have $Nn \subseteq NN \subseteq \mathcal{A}N \subseteq N$ and $N \subseteq M$. By construction N is a closed two-sided ideal in M. Consider the unital algebra M/N.
For any $x \in \mathcal{A}$ and $j \in J$ we have, applying our commutativity conditions,

$$x(b_j - \mu_j 1_{\mathcal{A}})a = x(b_j a - \mu_j 1_{\mathcal{A}} a) = x(ab_j - a\mu_j 1_{\mathcal{A}}) = xa(b_j - \mu_j 1_{\mathcal{A}}) \in N.$$

Then, for a finite sum, $\left(\sum_{j\in F} x_j(b_j - \mu_j 1_{\mathcal{A}})\right) a \in N$. This implies $Na \subseteq N$ and $a \in M$. Take $\lambda \in \sigma_{M/N}(a + N) \subseteq \mathbb{C}$. In the notation of Lemma 3.5.3, $(a - \lambda 1_{\mathcal{A}}) + N \in \partial S_{M/N}$. Suppose $(\lambda, (\mu_j)_{j\in J}) \notin \sigma_H^{left}(a, (b_j)_{j\in J}) \subseteq \mathbb{C}^{I\cup J}$. Then we can find $a' \in \mathcal{A}$ and $(b'_j)_{j\in J} \subseteq \mathcal{A}$, all zero but a finite number of j, such that

$$1_{\mathcal{A}} = a'(a - \lambda 1_{\mathcal{A}}) + \sum_{j\in J} b'_j(b_j - \mu_j 1_{\mathcal{A}}).$$

If $y \in M$, then

$$y - a'(a - \lambda 1_{\mathcal{A}})y = \sum_{j\in J} b'_j(b_j - \mu_j 1_{\mathcal{A}})y \in N.$$

This means that $y + N = (a - \lambda 1_{\mathcal{A}})y + N$. If $z \in N$ then $a'z \in N$ and, from the previous equality,

$$y + w = a'(a - \lambda 1_{\mathcal{A}})y + a'z = a'[(a - \lambda 1_{\mathcal{A}})y + z]. \tag{3.3}$$

The topology on M/N is generated by the seminorms given by $\tilde{p}(y+N) := \inf_{z\in N} p(y+z)$, where p range over all the continuous seminorms on M. Since the z was arbitrary, (3.3) implies

$$\tilde{p}(y + N) \leq p(a')\tilde{p}((a - \lambda 1_{\mathcal{A}})y + N) \tag{3.4}$$

for all $y \in M$ and all continuous seminorms p.
On the other hand, since $(a - \lambda 1_{\mathcal{A}}) + N \in \partial S_{M/N}$, Lemma 3.5.3 implies that there exist a net $(y_\alpha)_\alpha \subseteq M$ and a seminorm p such that $\tilde{p}(y_\alpha + N) = 1$ for all α and

$$\tilde{p}((a - \lambda 1_{\mathcal{A}})y_\alpha + N) \longrightarrow 0.$$

This obviously contradicts (3.4) and, from this, we have $(\lambda, (\mu_j)_{j\in J}) \in \sigma_H^{left}(a, (b_j)_{j\in J})$.
Since $\pi_j(\lambda, (\mu_j)_{j\in J}) = (\mu_j)_{j\in J}$, we have our result for the case $|I| = 1$. The general case is proved using transfinite induction exactly in the same way as in Proposition 11 in [18].
q.e.d.

Definition 3.5.5 A family of elements in an algebra, $(a_i)_{i\in I} \subseteq \mathcal{A}$ is called *commutative* if $a_i a_j = a_j a_i$ for all $i, j \in I$

Corollary 3.5.6
Let $\mathcal{A}$ be a unital lmc Q-algebra and $(a_i)_{i\in I} \subseteq \mathcal{A}$ a commutative system. Then, for any $J \subseteq I$,

$$\pi_J(\sigma_H^{left}((a_i)_{i\in I})) = \sigma_H^{left}((a_j)_{j\in J}).$$

3.5.2 The vector valued Harte spectrum

If the indexing set of the family is a locally convex space, $I = E$, we can interpret $A = (a_x)_{x\in E}$ as a mapping $A : E \longrightarrow \mathcal{A}$ given by $A(x) = a_x$. Analogously, $\lambda = (\lambda_x)_{x\in E}$ defines a mapping $\lambda : E \longrightarrow \mathbb{C}$ by $\lambda(x) = \lambda_x$. Under certain conditions some properties of A as a mapping are inherited by the elements of the spectrum.

Lemma 3.5.7
Let $\mathcal{A}$ be a Q-algebra; then, if $A \in \mathcal{L}(E; \mathcal{A})$ and $\lambda \in \sigma_H^{left}(A)$, then $\lambda \in E'$.
If, furthermore, $E = \mathcal{B}$ is a lmc algebra and A is an algebra homomorphism, then so also is λ.
Proof.
We have

$$A(\beta\, x) - \lambda(\beta\, x)1_{\mathcal{A}} - \beta 1_{\mathcal{A}}(A(x) - \lambda(x)1_{\mathcal{A}})$$

$$\begin{aligned} &= A(\beta x) - \lambda(\beta x)1_{\mathcal{A}} - (\beta A(x) - \beta\lambda(x)1_{\mathcal{A}}) \\ &= A(\beta x) - \lambda(\beta x)1_{\mathcal{A}} - A(\beta x) + \beta\lambda(x)1_{\mathcal{A}} \\ &= (\beta\lambda(x) - \lambda(\beta x))\ 1_{\mathcal{A}}. \end{aligned}$$

Since $\lambda \in \sigma_H^{left}(A)$, the first expression is not invertible. Then $(\beta\lambda(x) - \lambda(\beta\ x))\ 1_{\mathcal{A}} = 0$ and $\beta\lambda(x) = \lambda(\beta\ x)$. In the same way we get that $\lambda(x_1 + x_2) = \lambda(x_1) + \lambda(x_2)$.
We now prove that λ is continuous. Let $x \in E$ and suppose $x_\alpha \longrightarrow x$. We know that $a_{x_\alpha} \longrightarrow a_x$ and we want to show that $\lambda(x_\alpha) \longrightarrow \lambda(x)$.
Suppose that $(\lambda(x_\alpha))_\alpha$ does not converge to $\lambda(x)$. Going to a subnet, if necessary, we can assume that for all α,

$$|\lambda(x_\alpha) - \lambda(x)| \geq \delta > 0. \tag{3.5}$$

Obviously each $(\lambda(x_\alpha) - \lambda(x))1_{\mathcal{A}}$ is invertible. Taking into account (3.5) we can write

$$\begin{aligned} &(a_{x_\alpha} - \lambda(x_\alpha)1_{\mathcal{A}}) - (a_x - \lambda(x)1_{\mathcal{A}}) = (a_{x_\alpha} - a_x) + (\lambda(x) - \lambda(x_\alpha))1_{\mathcal{A}} \\ = \ &(\lambda(x) - \lambda(x_\alpha)) \left(\frac{a_{x_\alpha} - a_x}{\lambda(x) - \lambda(x_\alpha)} + 1_{\mathcal{A}} \right). \end{aligned}$$

Since $\mathcal{A}$ is a Q-algebra, $\mathcal{A}_{inv}$ is open and there exists an open neighbourhood of 0, U, such that $1_{\mathcal{A}} + U \subseteq \mathcal{A}_{inv}$. We have that

$$\frac{a_{x_\alpha} - a_x}{\lambda(x) - \lambda(x_\alpha)} \longrightarrow 0,$$

For α large enough we have that $\frac{a_{x_\alpha} - a_x}{\lambda(x) - \lambda(x_\alpha)} \in U$. Hence, $\frac{a_{x_\alpha} - a_x}{\lambda(x) - \lambda(x_\alpha)} + 1_{\mathcal{A}}$ is invertible and so is $(\lambda(x) - \lambda(x_\alpha)) \left(\frac{a_{x_\alpha} - a_x}{\lambda(x) - \lambda(x_\alpha)} + 1_{\mathcal{A}} \right)$. This means that we can find $b \in \mathcal{A}$ so that

$$\begin{aligned} 1_{\mathcal{A}} &= b\ \left[(\lambda(x) - \lambda(x_\alpha)) \left(\frac{a_{x_\alpha} - a_x}{\lambda(x) - \lambda(x_\alpha)} + 1_{\mathcal{A}} \right) \right] \\ &= b\ (a_{x_\alpha} - \lambda(x_\alpha)1_{\mathcal{A}}) - b\ (a_x \lambda(x)1_{\mathcal{A}}). \end{aligned}$$

But $\lambda \in \sigma_H^{left}(A)$, which gives a contradiction. Therefore $\lambda(x_\alpha) \longrightarrow \lambda(x)$ and λ is continuous.
Suppose now that E is a lmc algebra and A is an algebra homomorphism. Then

$$\begin{aligned} &(A(xy) - \lambda(xy)1_{\mathcal{A}}) - A(x)\,(A(y) - \lambda(y)1_{\mathcal{A}}) \\ - \ &\lambda(y)1_{\mathcal{A}}\,(A(x) - \lambda(x)1_{\mathcal{A}}) \\ = \ &(\lambda(x)\lambda(y) - \lambda(xy))\,1_{\mathcal{A}}. \end{aligned}$$

Since $\lambda \in \sigma_H^{left}$, the left hand side tells us that this element is not invertible. Therefore, it must be zero and hence $\lambda(x)\lambda(y) = \lambda(xy)$.
q.e.d.

Let us note that what is actually needed in this proposition is that $\mathcal{A}$ has a neighbourhood of $1_{\mathcal{A}}$ consisting of left invertible elements. The result is therefore true for any algebra, not necessarily Q-algebra, satisfying this condition. This situation was already considered by Kaplansky, who talked about Q_l-rings, in which the set of left invertible elements is open (see [38]). He wondered whether both concepts were equivalent. So far we do not know of any positive answer to this question, but neither of any counterexample.

Definition

Using Lemma 3.5.7, a Harte vector spectrum was defined in [18] for tensor products of Banach algebras and spaces. In the same way we define an equivalent one in or setting.

Definition 3.5.8 Let $\mathcal{A}$ be a unital lmc Q-algebra, E a locally convex space and $\mathcal{T}$ a uniform tensor topology on $\mathcal{A} \otimes E$. Given $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, we define its *left Harte spectrum* $\sigma_H^{left}(\mathsf{T})$ to be $\sigma_H^{left}\left([(I_{\mathcal{A}} \otimes x'](\mathsf{T}))_{x' \in E'}\right)$, that is

$$x'' \in \sigma_H^{left}(\mathsf{T}) \Leftrightarrow 1_{\mathcal{A}} \notin \{ \sum_{\substack{i\in F \\ F \text{ finite}}} a_i \ ([I_{\mathcal{A}} \otimes x_i'](\mathsf{T}) \ - \ x''(x_i')1_{\mathcal{A}}) \ : \ a_i \in \mathcal{A}, \ x_i' \in E'\}$$

The Harte spectrum lying in E

The Harte spectrum has been defined as a subset of the bidual E'' of E. Let us see now how can it actually be realized in E. Every locally convex E can be embedded into its bidual by means of the mapping $J_E : E \longrightarrow E''$ defined by $J_E(x)(x') = x'(x)$. In [18] the same result is proved for Banach algebras and uniform norms.

Proposition 3.5.9
Let $\mathcal{A}$ be a unital lmc Q-algebra, E a complete locally convex space and $\mathcal{T}$ a uniform tensor topology. Take $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$. Then:
For every $x'' \in \sigma_H^{left}(\mathsf{T})$ there exist some $x \in E$ such that $J_E(x) = x''$. Consequently, we can identify:

$$\sigma_H^{left}(\mathsf{T}) = \{x \in E : 1_{\mathcal{A}} \notin \{ \sum_{\substack{i\in F \\ F\text{finite}}} a_i([I_{\mathcal{A}} \otimes x_i'](\mathsf{T}) - x'(x)1_{\mathcal{A}}) : a_i \in \mathcal{A}, x_i' \in E'\}\}.$$

Proof.
We note first that if $\mathsf{S} = \sum_{i=1}^n b_i \otimes x_i \in \mathcal{A} \otimes E$, then the mapping $\varphi_{\mathsf{S}} : (E', \ \sigma(E', E)) \longrightarrow \mathcal{A}$, given by

$$x' \mapsto [I_{\mathcal{A}} \otimes x'](\mathsf{S}) = \sum_{i=1}^{n} x'(x_i) \ b_i$$

is continuous. Let $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$. We now claim see that the mapping $\varphi_{\mathsf{T}} : x' \mapsto [I_{\mathcal{A}} \otimes x'](\mathsf{T})$ is continuous on the equicontinuous subsets of E'.

Let $M \subseteq E'$ be equicontinuous Consider a net $(\mathsf{T}_\alpha)_\alpha \subseteq \mathcal{A}\otimes E$ converging to T. Now, $\{I_\mathcal{A}\}\otimes M$ is equicontinuous. Given any continuous seminorm p on $\mathcal{A}$ we can find a continuous seminorm ρ on $\mathcal{A}\hat{\otimes}_\mathcal{T} E$ such that $p([I_\mathcal{A}\otimes x'](\mathsf{Z})) \leq \rho(\mathsf{Z})$ for all $\mathsf{Z} \in \mathcal{A}\hat{\otimes}_\mathcal{T} E$ and all $x' \in M$. In particular,

$$\sup_{x'\in M} p([I_\mathcal{A}\otimes x'](\mathsf{T}_\alpha - \mathsf{T})) \leq \rho(\mathsf{T}_\alpha - \mathsf{T})$$

for all α. Hence

$$\sup_{x'\in M} p(\varphi_{\mathsf{T}_\alpha}(x') - \varphi_\mathsf{T}(x')) \longrightarrow 0.$$

This implies that $\varphi_{\mathsf{T}_\alpha} \longrightarrow \varphi_\mathsf{T}$ uniformly on M. Then φ_T is a uniform limit of continuous mappings and it is continuous on M with the $\sigma(E',E)$ topology. This proves our claim.
Take now $x'' \in E''$ such that $x'' \in \sigma_H^{left}(\mathsf{T})$. Let $(x'_\alpha)_\alpha$ be an equicontinuous net in E' converging to $x' \in E'$. If $x''(x'_\alpha)$ does not converge to $x''(x')$ then, going to a subnet if necessary, there exists $\delta > 0$ such that $|x''(x'_\alpha) - x''(x')| > \delta$ for all α.
Since $\mathcal{A}$ is a Q-algebra, the set of invertible elements, $\mathcal{A}_{inv}$, is open. Consider then $U \in \mathcal{U}$ so that $1_\mathcal{A}+U \subseteq \mathcal{A}_{inv}$. Since the mapping $x' \mapsto [I_\mathcal{A}\otimes x'](\mathsf{T})$ is continuous on the equicontinuous subsets of E', we can consider α_0 such that for all $\alpha \geq \alpha_0$

$$-[I_\mathcal{A}\otimes (x'_\alpha - x')](\mathsf{T}) \in \delta U.$$

If $\alpha \geq \alpha_0$ we have

$$[I_\mathcal{A}\otimes (x'_\alpha - x')](\mathsf{T}) - x''(x'_\alpha - x')1_\mathcal{A} = -x''(x'_\alpha - x')\left(1_\mathcal{A} - \frac{[I_\mathcal{A}\otimes (x'_\alpha - x')](\mathsf{T})}{x''(x'_\alpha - x')}\right).$$

But $1_\mathcal{A} - \dfrac{[I_\mathcal{A}\otimes (x'_\alpha - x')](\mathsf{T})}{x''(x'_\alpha - x')} \in 1_\mathcal{A}+U$ is invertible. This contradicts the fact that $x'' \in \sigma_H^{left}$.
Therefore, $x''(x'_\alpha) \longrightarrow x''(x')$.
Hence the linear functional x'' is $\sigma(E',E)$-continuous on the equicontinuous subsets of E'. Applying the Grothendieck's Completeness Theorem ([35], Chapter 4, section 11, Corollary 3), x'' is $\sigma(E',E)$-continuous and there exist some $x \in E$ such that $J_E(x) = x''$.
q.e.d.

Using the identity $x'(x)1_\mathcal{A} = [I_\mathcal{A}\otimes x'](1_\mathcal{A}\otimes x)$ we can rewrite the Harte spectrum in a more convenient way:

$$\sigma_H^{left}(\mathsf{T}) = \{x \in E:\ 1_\mathcal{A} \notin \{\sum_{\substack{i\in F\\ F\text{finite}}} a_i\ ([I_\mathcal{A}\otimes x'_i](\mathsf{T} - 1_\mathcal{A}\otimes x)) : a_i \in \mathcal{A}, x_i \in E'\}\} \qquad (3.6)$$

Now, if $\mathcal{A}$ is complete we can describe the spectrum in an even more convenient way. Given $a \in \mathcal{A}$, consider the multiplication mapping $M_a : \mathcal{A} \longrightarrow \mathcal{A}$ given by $M_a(b) = ab$. For each $x' \in E'$ we define $M_a \otimes x' : \mathcal{A}\hat{\otimes}_{\mathcal{T}} E \longrightarrow \mathcal{A}\hat{\otimes}_{\mathcal{T}} \mathbb{K} \cong \mathcal{A}$. Note that for elements of $\mathcal{A} \otimes E$ this mapping acts in the following way

$$(M_a \otimes x') \left(\sum_{i=1}^{n} b_i \otimes xi \right) = \sum_{i=1}^{n} x'(x_i) a b_i.$$

Writing $a \otimes x'$ in place of $M_a \otimes x'$ and extending $M_a \otimes x'$ by linearity and continuity we obtain an action of $\mathcal{A} \otimes E'$ on $\mathcal{A}\hat{\otimes}_{\mathcal{T}} E$. If $b \in \mathcal{A}$ and $x' \in E'$ we have

$$\begin{aligned} & b \left([I_{\mathcal{A}} \otimes x'] \left(\sum_{i=1}^{n} a_i \otimes xi - 1_{\mathcal{A}} \otimes x \right) \right) = b \left(\sum_{i=1}^{n} a_i x'(xi) - 1_{\mathcal{A}} x'(x) \right) \\ = & \sum_{i=1}^{n} b a_i x'(xi) - b 1_{\mathcal{A}} x'(x) \\ = & \sum_{i=1}^{n} < b \otimes x', a_i \otimes xi > - < b \otimes x', 1_{\mathcal{A}} \otimes x > \\ = & < b \otimes x', \sum_{i=1}^{n} a_i \otimes xi - 1_{\mathcal{A}} \otimes x > \end{aligned}$$

This shows that for every $b \in \mathcal{A}$, $x' \in E'$, $x \in E$ and $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$, then

$$b \left([I_{\mathcal{A}} \otimes x'] \left(\mathsf{T} - 1_{\mathcal{A}} \otimes x \right) \right) = < b \otimes x', \mathsf{T} - 1_{\mathcal{A}} \otimes x >$$

We thus have the following description, similar to the classical one.

Proposition 3.5.10
Let $\mathcal{A}$ be a complete unital lmc Q-algebra, E a complete locally convex space and $\mathcal{T}$ a uniform tensor topology. Then, for each $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ we have

$$\sigma_H^{left}(\mathsf{T}) = \{x \in E : \nexists \mathsf{Z} \in \mathcal{A} \otimes E' \text{ s.t. } < \mathsf{Z}, \mathsf{T} - 1_{\mathcal{A}} \otimes x >= 1_{\mathcal{A}}\}. \tag{3.7}$$

The Harte spectrum and the Waelbroeck spectrum

When $\mathcal{A}$ is a commutative lmc Q-algebra we have defined, for each element in $\mathcal{A}\hat{\otimes}_{\mathcal{T}} E$, two different spectra. We are going to show now that in many cases they are essentially the same set. If E is infrabarrelled the canonical embedding $J_E : E \longrightarrow E''$ of E into its bidual is an isomorphism onto its image and, if $\mathcal{A}$ is commutative, left and right inverses coincide and we can write $\sigma_H(\mathsf{T})$ instead of $\sigma_H^{left}(\mathsf{T})$.

Proposition 3.5.11
Let $\mathcal{A}$ be a commutative, unital lmc Q-algebra, E a complete locally convex space and $\mathcal{T}$ a uniform topology on $\mathcal{A} \otimes E$. Then, for every $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$,

$$J_E(\sigma_W(\mathsf{T})) = \sigma_H(\mathsf{T}).$$

If, moreover, E is infrabarrelled, then $J_E|_{\sigma_W(\mathsf{T})} : \sigma_W(\mathsf{T}) \longrightarrow \sigma_H(\mathsf{T})$ is a homeomorphism.
Proof.
The first and final parts of the proof coincide with those in [18], Proposition 6. For the sake of completeness we give a full proof. Take $\sum_{i=1}^n a_i \otimes x_i \in \mathcal{A} \otimes E$. Given $h \in \mathfrak{M}(\mathcal{A})$ and $x' \in E'$ we have

$$\begin{aligned} h\left([I_{\mathcal{A}} \otimes x']\left(\sum_{i=1}^n a_i \otimes x_i\right)\right) &= h\left(\sum_{i=1}^n a_i x'(x_i)\right) \\ &= \sum_{i=1}^n h(a_i)x'(x_i) \\ &= [h \otimes x']\left(\sum_{i=1}^n a_i \otimes x_i\right) \\ &= x'\left(\sum_{i=1}^n h(a_i)x_i\right) \\ &= x'\left([h \otimes I_E]\left(\sum_{i=1}^n a_i \otimes x_i\right)\right). \end{aligned}$$

Hence $h \circ [I_{\mathcal{A}} \otimes x'](\mathsf{T}) = [h \otimes x'](\mathsf{T}) = x' \circ [h \otimes I_E](\mathsf{T})$ for each $\mathsf{T} \in \mathcal{A} \otimes E$. By the condition (3) in Definition 3.2.9, $h \circ [I_{\mathcal{A}} \otimes x'],\ h \otimes x',\ x' \circ [h \otimes I_E] \in \left(\mathcal{A}\hat{\otimes}_{\mathcal{T}} E\right)'$.
Given any $\varepsilon > 0$, we can find a V neighbourhood of T such that, for every $\mathsf{S} \in V$:

$$\begin{aligned} |h \circ [I_{\mathcal{A}} \otimes x'](\mathsf{S}) - h \circ [I_{\mathcal{A}} \otimes x'](\mathsf{T})| &< \varepsilon/2, \\ |[h \otimes x'](\mathsf{S}) - [h \otimes x'](\mathsf{T})| &< \varepsilon/2. \end{aligned}$$

If $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ and $(\mathsf{T}_\alpha)_\alpha \subseteq \mathcal{A} \otimes E$ converges to T, there is some α_0 such that for every $\alpha \geq \alpha_0$, then $\mathsf{T}_\alpha \in V$. Therefore

$$\begin{aligned} & |h \circ [I_{\mathcal{A}} \otimes x'](\mathsf{T}) - [h \otimes x'](\mathsf{T})| \\ \leq\ & |h \circ [I_{\mathcal{A}} \otimes x'](\mathsf{T}) - h \circ [I_{\mathcal{A}} \otimes x'](\mathsf{T}_\alpha)| + |[h \otimes x'](\mathsf{T}_\alpha) - [h \otimes x'](\mathsf{T})| < \varepsilon. \end{aligned}$$

Thus, $h \circ [I_{\mathcal{A}} \otimes x'] = [h \otimes x']$ in $\mathcal{A}\hat{\otimes}_{\mathcal{T}} E$. Proceeding exactly in the same way, $[h \otimes x'] = x' \circ [h \otimes I_E]$ in $\mathcal{A}\hat{\otimes}_{\mathcal{T}} E$.
Let $x'' \in \sigma_H(\mathsf{T}) \subseteq E''$, then

$$1_{\mathcal{A}} \notin \{\sum_{i \in F} a_i\ ([I_{\mathcal{A}} \otimes x'](\mathsf{T})\ -\ x''(x_i')1_{\mathcal{A}})\ :\ a_i \in \mathcal{A},\ F \text{ finite},\ x_i' \in E'\} \tag{3.8}$$

Since $\mathcal{A}$ is a Q-algebra all its maximal ideals are closed (Proposition 3.2.4). The set in (3.8) forms a proper ideal in $\mathcal{A}$ and is thus contained in a closed maximal ideal and, hence,

in the kernel of some $h \in \mathfrak{M}(\mathcal{A})$. If $F = \{x'\}$ then $h([I_{\mathcal{A}} \otimes x'](\mathsf{T})) = x''(x')$. We have shown already that $h([I_{\mathcal{A}} \otimes x'](\mathsf{T})) = x'([h \otimes I_E](\mathsf{T}))$. Hence $J_E([h \otimes I_E](\mathsf{T})) = x''$. Thus $\sigma_H(\mathsf{T}) \subseteq J_E(\sigma_W(\mathsf{T})) \subseteq J_E(E)$.
Conversely, suppose that $x \in J_E(E) \setminus \sigma_H(\mathsf{T})$. This means that there exist $(a_i)_{i=1}^n \subseteq \mathcal{A}$ and $(x_i')_{i=1}^n \subseteq E'$ such that

$$1_{\mathcal{A}} = \sum_{i=1}^{n} a_i \; ([I_{\mathcal{A}} \otimes x_i'](\mathsf{T}) \; - \; x_i'(x)1_{\mathcal{A}}).$$

Since $h(1_{\mathcal{A}}) = 1$, we have

$$\begin{aligned} 1 \;&=\; \sum_{i=1}^{n} h(a_i) \; x_i'([h \otimes I_E](\mathsf{T}) \; - \; x) \\ &=\; \left[\sum_{i=1}^{n} h(a_i) \; x_i'\right] ([h \otimes I_E](\mathsf{T}) \; - \; x). \end{aligned}$$

Hence, $x \neq [h \otimes I_E](\mathsf{T})$ for $h \in \mathfrak{M}(\mathcal{A})$. This implies that $x \notin \sigma_W(\mathsf{T})$ and, hence, $J_E(\sigma_W(\mathsf{T})) \subseteq \sigma_H(\mathsf{T})$.
If E is infrabarrelled, then J_E is continuous. Hence, when restricted to $\sigma_W(\mathsf{T})$, we have a bijective, continuous mapping between a compact and a Hausdorff spaces. Then it is a homeomorphism.
q.e.d.

By this proposition the vector-valued Harte spectrum of an element of $\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, $\mathcal{A}$ commutative, is compact. We shall see later that it is compact for any $\mathcal{A}$ complete unital lmc Q-algebra.

Projective limits

We have seen that if $\mathcal{A}$ is a complete unital lmc algebra, it can be realized as a projective limit of Banach algebras. In the case when $\mathcal{A}$ is commutative we studied in Proposition 3.4.7 the relationship between the spectrum of an element of $\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$ and those of its projections. We now consider the non-commutative case. We use the notation in Proposition 3.4.7. Note that Lemma 3.4.4, Remark 3.4.5 and Lemma 3.4.6 hold for any algebra, not necessarily commutative. We need first the following lemma.

Lemma 3.5.12
Let $\mathcal{A}$ be a complete, unital lmc algebra, E a complete locally convex space and $\mathcal{T}$ a uniform tensor topology that satisfies the projective limit condition. Take $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, $j > i$ and $h \in \mathfrak{M}(\mathcal{A}_i)$. Then, for all $x' \in E'$,

$$\pi_{ij}([I_{\mathcal{A}_j} \otimes x'](\mathsf{T}_j)) = [I_{\mathcal{A}_i} \otimes x'](\mathsf{T}_i) = \pi_i([I_{\mathcal{A}} \otimes x'](\mathsf{T}))$$

Proof.
We have

$$\begin{aligned}
\pi_i\left([I_{\mathcal{A}}\otimes x']\left(\sum_{k=1}^{n} a_k\otimes x_k\right)\right) &= \pi_i\left(\sum_{k=1}^{n} a_k x'(x_k)\right)\\
&= \sum_{k=1}^{n}\pi_i(a_k)x'(x_k)\\
&= [I_{\mathcal{A}_i}\otimes x']\left(\sum_{k=1}^{n}\pi_i(a_k)\otimes x_k\right)\\
&= [I_{\mathcal{A}_i}\otimes x'](\mathsf{T}_i).
\end{aligned}$$

The proof is completed by a density argument. Proceeding in the same way we have the other equality.
q.e.d.

Proposition 3.5.13
Let $\mathcal{A}$ be a complete unital lmc Q-algebra, E a complete locally convex space and $\mathcal{T}$ a uniform tensor topology that satisfies the projective limit condition and let $\mathsf{T}\in\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$. Then,
(i) $\sigma_H^{left}(\mathsf{T}_i)\subseteq\sigma_H^{left}(\mathsf{T}_j)$ *for all* $j>i$.
(ii) $\sigma_H^{left}(\mathsf{T})=\bigcup_{i\in I}\sigma_H^{left}(\mathsf{T}_i)$.
Proof.
Let $x''\notin\sigma_H^{left}(\mathsf{T}_j)$. There exists a finite sum such that

$$1_{\mathcal{A}_j}=\sum_{\substack{k\in F\\ F \text{ finite}}} a_k^j\ ([I_{\mathcal{A}_j}\otimes x'_k](\mathsf{T}_j)\ -\ x''(x'_k)1_{\mathcal{A}_j})$$

where $a_k^j\in\mathcal{A}_j$ and $x'_k\in E'$ for all k. Using projections and Lemma 3.5.12,

$$\begin{aligned}
1_{\mathcal{A}_i} &= \pi_{ij}(1_{\mathcal{A}_j})=\sum_{k\in F}\pi_{ij}(a_k^j)\ (\pi_{ij}([I_{\mathcal{A}_j}\otimes x'_k](\mathsf{T}_j))\ -\ x''(x'_k)\pi_{ij}(1_{\mathcal{A}_j}))\\
&= \sum_{k\in F}\pi_{ij}(a_k^j)\ ([I_{\mathcal{A}_i}\otimes x'_k](\mathsf{T}_i)\ -\ x''(x'_k)1_{\mathcal{A}_i}).
\end{aligned}$$

Hence, $x''\notin\sigma_H^{left}(\mathsf{T}_i)$ and we proved (i).
For (ii), first take $x''\notin\sigma_H^{left}(\mathsf{T})$. Again we have a finite sum

$$1_{\mathcal{A}}=\sum_{\substack{k\in F\\ F \text{ finite}}} a_k\ ([I_{\mathcal{A}}\otimes x'_k](\mathsf{T})\ -\ x''(x'_k)1_{\mathcal{A}})$$

where $a_k \in \mathcal{A}$ and $x'_k \in E'$. Proceeding in the same way as before, i.e. taking π_i and applying Lemma 3.5.12 we have $x'' \notin \sigma_H^{left}(\mathsf{T}_i)$ for all $i \in I$. This implies that

$$\bigcup_{i \in I} \sigma_H^{left}(\mathsf{T}_i) \subseteq \sigma_H^{left}(\mathsf{T}).$$

For the converse let $x'' \in \sigma_H^{left}(\mathsf{T})$. We denote by $\mathcal{I}$ the left ideal generated in $\mathcal{A}$ by $\{[I_{\mathcal{A}} \otimes x'](\mathsf{T}) \;-\; x''(x')1_{\mathcal{A}}\}_{x' \in E'}$. We have $1_{\mathcal{A}} \notin \mathcal{I}$ and, since $\mathcal{A}$ is a Q-algebra, $1_{\mathcal{A}} \notin \overline{\mathcal{I}}$. Consider $\psi : \mathcal{A} \longrightarrow \mathbb{C}$ linear and continuous such that $\psi(1_{\mathcal{A}}) = 1$ and $\psi(\overline{\mathcal{I}}) = 0$. By Lemma 3.4.4, there exist some $i \in I$ and $\psi_i : \mathcal{A}_i \longrightarrow \mathbb{C}$ such that $\psi = \psi_i \circ \pi_i$. Then:

$$\psi_i(1_{\mathcal{A}_i}) = \psi_i(\pi_i(1_{\mathcal{A}})) = \psi(1_{\mathcal{A}}) = 1.$$

Denote by $\mathcal{I}_i$ the left ideal generated by $\{[I_{\mathcal{A}_i} \otimes x'](\mathsf{T}_i) \;-\; x''(x')1_{\mathcal{A}_i}\}_{x' \in E'}$ in $\mathcal{A}_i$. If $a \in \mathcal{I}$ then

$$\begin{aligned}
\pi_i(a) &= \pi_i\left(\sum_{k=1}^{n} a_k \; ([I_{\mathcal{A}} \otimes x'_k](\mathsf{T}) \;-\; x''(x'_k)1_{\mathcal{A}})\right) \\
&= \sum_{k=1}^{n} \pi_i(a_k) \; (\pi_i([I_{\mathcal{A}} \otimes x'_k](\mathsf{T})) \;-\; x''(x'_k)\pi_i(1_{\mathcal{A}})) \\
&= \sum_{k=1}^{n} \pi_i(a_k) \; ([I_{\mathcal{A}_i} \otimes x'_k](\mathsf{T}_i) \;-\; x''(x'_k)1_{\mathcal{A}_i}) \in \mathcal{I}_i.
\end{aligned}$$

Hence, $\pi_i(\mathcal{I}) \subseteq \mathcal{I}_i$ and $\overline{\pi_i(\mathcal{I})} \subseteq \overline{\mathcal{I}_i}$. Let

$$a^i = \sum_{k=1}^{n} a_k^i \; ([I_{\mathcal{A}_i} \otimes x'_k](\mathsf{T}_i) \;-\; x''(x'_k)1_{\mathcal{A}_i}) \in \mathcal{I}_i.$$

We know that $\pi_i(\mathcal{A})$ is dense in $\mathcal{A}_i$. Therefore we have $a_k^i = \lim_\alpha \pi_i(a_{k\alpha})$ where $a_{k\alpha} \in \mathcal{A}$. Since this is a finite sum,

$$\begin{aligned}
a^i &= \sum_{k=1}^{n} \lim_\alpha \pi_i(a_{k\alpha}) \; ([I_{\mathcal{A}_i} \otimes x'_k](\mathsf{T}_i) \;-\; x''(x'_k)1_{\mathcal{A}_i}) \\
&= \lim_\alpha \sum_{k=1}^{n} \pi_i(a_{k\alpha}) \; (\pi_i([I_{\mathcal{A}} \otimes x'_k](\mathsf{T})) \;-\; x''(x'_k)\pi_i(1_{\mathcal{A}})) \\
&= \lim_\alpha \pi_i\left(\sum_{k=1}^{n} a_{k\alpha} \; ([I_{\mathcal{A}} \otimes x'_k](\mathsf{T}) \;-\; x''(x'_k)1_{\mathcal{A}})\right) \in \overline{\pi_i(\mathcal{I})}.
\end{aligned}$$

This means that $\mathcal{I}_i \subseteq \overline{\pi_i(\mathcal{I})}$ hence $\overline{\mathcal{I}_i} \subseteq \overline{\pi_i(\mathcal{I})}$ and

$$\overline{\mathcal{I}_i} \subseteq \overline{\pi_i(\mathcal{I})} \subseteq \overline{\mathcal{I}_i}.$$

Hence

$$\overline{\mathcal{I}_i} = \overline{\pi_i(\mathcal{I})} = \pi_i(\overline{\mathcal{I}})$$

and $\psi_i(\overline{\mathcal{I}_i}) = \psi_i(\pi_i(\overline{\mathcal{I}})) = \psi(\overline{\mathcal{I}}) = 0$. On the other hand we already know that $\psi_i(1_{\mathcal{A}_i}) = 1$. This implies that $1_{\mathcal{A}_i} \notin \overline{\mathcal{I}_i}$ and $1_{\mathcal{A}_i} \notin \mathcal{I}_i$. Thus, $x'' \in \sigma_H^{left}(\mathsf{T}_i)$ and

$$\sigma_H^{left}(\mathsf{T}) \subseteq \bigcup_{i\in I} \sigma_H^{left}(\mathsf{T}_i).$$

This completes the proof.
q.e.d.

In view of Proposition 3.5.11, this result generalizes Proposition 3.4.7 and thus [8], Proposition 2.3.

The Harte spectrum and the classical one in $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$

If $\mathcal{A}, \mathcal{B}$ are lmc algebras and $\mathcal{T}$ is a uniform tensor topology such that $\mathcal{A} \otimes_{\mathcal{T}} \mathcal{B}$ is a lmc algebra, then $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is again an algebra. Therefore, given any $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ we can consider its classical algebraic left spectrum, $\sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}(\mathsf{T}) \subseteq \mathbb{C}$, given by those $\lambda \in \mathbb{C}$ such that $\mathsf{T} - \lambda 1_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}$ is not left invertible. There are, then, three a priori different spectra, two of them vector and one scalar. We have already seen that the two vector spectra (the Harte and the Waelbroeck ones) are under reasonably general conditions, essentially the same. We now study, following [18], the relationship between the classical algebraic scalar spectrum and the Waelbroeck spectrum. The proofs of Lemma 13 and Proposition 14 in [18] are purely algebraic and valid in our new setting. For the sake of completeness we state the results here without any proof.

Lemma 3.5.14
Let $\mathcal{A}$ be a unital, commutative lmc algebra, $\mathcal{B}$ a unital lmc algebra and $\mathcal{T}$ a uniform tensor topology such that $\mathcal{A} \otimes_{\mathcal{T}} \mathcal{B}$ is a lmc algebra. Take $(a_i)_{i\in I} \subseteq \mathcal{A}$, $(b_j)_{j\in J} \subseteq \mathcal{B}$; then

$$\sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}((a_i \otimes 1_{\mathcal{B}})_{i\in I}, (1_{\mathcal{A}} \otimes b_j)_{j\in J}) = \sigma_{\mathcal{A}}((a_i)_{i\in I}) \times \sigma^{left}_{\mathcal{B}}((b_j)_{j\in J}).$$

Proposition 3.5.15
Let $\mathcal{A}$ be a unital, commutative lmc algebra, $\mathcal{B}$ a unital lmc algebra. Let $\mathcal{T}$ be a uniform tensor topology such that $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is a lmc algebra. Take $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$, $\mu \in \mathbb{C}$, $h \in \mathfrak{M}(\mathcal{A})$. Then, the ideals generated by

$$\begin{aligned} &\{[(a - h(a)1_{\mathcal{A}}) \otimes 1_{\mathcal{B}}]_{a\in\mathcal{A}}, \mathsf{T} - \mu(1_{\mathcal{A}} \otimes 1_{\mathcal{B}})\} \\ \text{and} \quad &\{[(a - h(a)1_{\mathcal{A}}) \otimes 1_{\mathcal{B}}]_{a\in\mathcal{A}}, 1_{\mathcal{A}} \otimes ([h \otimes I_{\mathcal{B}}](\mathsf{T}) - \mu 1_{\mathcal{B}})\} \end{aligned}$$

coincide.

Proposition 3.5.16
Let $\mathcal{A}$ be a unital, commutative lmc algebra, $\mathcal{B}$ a unital lmc algebra and $\mathcal{T}$ a uniform tensor topology such that $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is a lmc algebra and $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is a Q-algebra.
If $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ then

$$\sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}(\mathsf{T}) = \bigcup_{h\in\mathfrak{M}(\mathcal{A})} \sigma^{left}_{\mathcal{B}}([h\otimes I_{\mathcal{B}}](\mathsf{T})).$$

Proof.
To begin with, $[h\otimes I_{\mathcal{B}}]$ is a non-zero algebra homomorphism and $[h\otimes I_{\mathcal{B}}](1_{\mathcal{A}}\otimes 1_{\mathcal{B}}) = 1_{\mathcal{B}}$. Since non-zero algebra homomorphisms send left invertible elements into left invertible elements this implies

$$\sigma^{left}_{\mathcal{B}}([h\otimes I_{\mathcal{B}}](\mathsf{T})) \subseteq \sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}(\mathsf{T})$$

for all $h \in \mathfrak{M}(\mathcal{A})$.
Conversely, take $\mu \in \sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}(\mathsf{T})$. Since $\mathcal{A}$ is commutative, the system $\{[a\otimes 1_{\mathcal{B}}]_{a\in\mathcal{A}}, \mathsf{T}\}$ is commutative (see Definition 3.5.5). Applying Proposition 3.5.4,

$$\sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}([a\otimes 1_{\mathcal{B}}]_{a\in\mathcal{A}}, \mathsf{T}) = \sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}([a\otimes 1_{\mathcal{B}}]_{a\in\mathcal{A}}) \times \sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}(\mathsf{T}).$$

The mapping $\mathcal{A} \longrightarrow \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ given by $a \mapsto a\otimes 1_{\mathcal{B}}$ is an algebra homomorphism. If $(\lambda_a)_{a\in\mathcal{A}} \in \sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}([a\otimes 1_{\mathcal{B}}]_{a\in\mathcal{A}})$, the mapping $\mathcal{A} \longrightarrow \mathbb{C}$, $a \mapsto \lambda_a$ is an algebra homomorphism (see Lemma 3.5.7). In other words, we can find $h \in \mathfrak{M}(\mathcal{A})$ such that $(h(a))_{a\in\mathcal{A}} \in \sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}([a\otimes 1_{\mathcal{B}}]_{a\in\mathcal{A}})$. Then

$$((h(a))_{a\in\mathcal{A}}, \mu) \in \sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}([a\otimes 1_{\mathcal{B}}]_{a\in\mathcal{A}}, \mathsf{T}).$$

By Lemma 3.5.15 and Proposition 3.5.14,

$$\begin{aligned} ((h(a))_{a\in\mathcal{A}}, \mu) &\in \sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}([a\otimes 1_{\mathcal{B}}]_{a\in\mathcal{A}}, 1_{\mathcal{A}}\otimes([h\otimes I_{\mathcal{B}}](\mathsf{T}))) \\ &= \sigma_{\mathcal{A}}((a)_{a\in\mathcal{A}}) \times \sigma^{left}_{\mathcal{B}}([h\otimes I_{\mathcal{B}}](\mathsf{T})). \end{aligned}$$

Therefore, $\mu \in \sigma^{left}_{\mathcal{B}}([h\otimes I_{\mathcal{B}}](\mathsf{T}))$.
q.e.d.

A natural question now is whether we can omit the condition that $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is Q-algebra and substitute it by conditions on $\mathcal{A}$ and $\mathcal{B}$. This leads directly to the problem of whether or not the tensor product of two Q-algebras is again a Q-algebra. This is a difficult open problem. It is a well known fact (see [49], Corollary 4.1 or [50], Section XII-1, Lemma 1.3) that if $\mathcal{A}$ and $\mathcal{B}$ are two complete commutative lmc Q-algebras, then so also is $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$. The non-commutative case is much harder and only few concrete results are known. Let X be a compact 2nd countable n-dimensional C^∞-manifold and $\mathcal{A}$ a unital finite dimensional Banach algebra (take e.g. $\mathcal{A} = M_n(\mathbb{C})$. Then $C^\infty(X,\mathcal{A}) = C^\infty(X)\hat{\otimes}_\pi\mathcal{A}$ is a Q-algebra (see [50], Section XI-2, (2.1) and (2.7)). Here we have that $C^\infty(X)$ is a commutative Q-algebra,

since X is compact and $\mathcal{A}$ is a non-commutative Banach algebra, therefore Q-algebra ([25]). Note that $C^\infty(X)$ can never be topologized to become a Banach algebra.
Another example where the tensor product of two Q-algebras, one of which is not commutative, is a Q-algebra is in [51], Lemma 1.3.

3.6 Invertibility theorems

It is a well known fact from the classical Gelfand theory that is $\mathcal{A}$ is a commutative Banach algebra, then $a \in \mathcal{A}$ is invertible if and only if $\hat{a}$ is invertible in $\mathcal{C}(\mathfrak{M}(\mathcal{A}))$,i.e. if and only if $\hat{a}(h) \neq 0$ for every $h \in \mathfrak{M}(\mathcal{A})$. In [18], a non-commutative tensor Gelfand theory and spectral theory is developed and the classical result is proved in a more general setting. We now generalize this result to a wider class of algebras. Essentially we have three different settings;
(1) $\mathcal{A}$ and $\mathcal{B}$ Fréchet algebras, in this case we can use a result by Arens ([1]).
(2) $\mathcal{A}$ a Fréchet Q-algebra, $\mathcal{B}$ any Fréchet algebra.
(3) $\mathcal{A}$ and $\mathcal{B}$ lmc Q-algebras.

Theorem 3.6.1
Let $\mathcal{A}$ be a commutative, unital, Fréchet algebra; $\mathcal{B}$ a unital, Fréchet algebra and $\mathcal{T}$ a uniform tensor topology such that $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is a lmc algebra and satisfies the projective limit condition. Take $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$; then, the following are equivalent:
(i) T is left invertible in $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$.
(ii) $\hat{\mathsf{T}}(h)$ is left invertible in $\mathcal{B}$ for every $h \in \mathfrak{M}(\mathcal{A})$.
Proof.
$(i) \Rightarrow (ii)$
If $h \in \mathfrak{M}(\mathcal{A})$, then $[h \otimes I_{\mathcal{B}}]$ is a non-zero homomorphism and sends left invertibles to left invertibles. Hence, if T is left invertible, then $\hat{\mathsf{T}}(h) = [h \otimes I_{\mathcal{B}}](\mathsf{T})$ is left invertible.
$(ii) \Rightarrow (i)$
Since $\mathcal{A}$ and $\mathcal{B}$ are Fréchet algebras, there are two dense projective systems of Banach algebras $(A_n)_n$ and $(B_m)_m$ so that $\mathcal{A} = \lim\limits_{\leftarrow} A_n$ and $\mathcal{B} = \lim\limits_{\leftarrow} B_m$ ([28], Theorem 3.3.7). Then, $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is again a Fréchet algebra and $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B} = \lim\limits_{\leftarrow} A_n\hat{\otimes}_{\mathcal{T}}B_m$.
Consider for each n and m, the projections $\Phi_n : \mathcal{A} \longrightarrow A_n$ and $\Psi_m : \mathcal{B} \longrightarrow B_m$.
We can represent $\mathsf{T} = (\mathsf{T}_{n,m})_{n,m}$, where each $\mathsf{T}_{n,m} = (\Phi_n \otimes \Psi_m)(\mathsf{T}) \in A_n\hat{\otimes}_{\mathcal{T}}B_m$. Fix n and m and let us see now that $\hat{\mathsf{T}}_{n,m}(g)$ is left invertible in B_m for every $g \in \mathfrak{M}(A_n)$. Take any $g \in \mathfrak{M}(A_n)$ and define $h = g \circ \Phi_n \in \mathfrak{M}(\mathcal{A})$. Then, by hypothesis, $\hat{\mathsf{T}}(h) = [h \otimes I_{\mathcal{B}}](\mathsf{T})$ is left invertible in $\mathcal{B}$. We now claim that

$$\Psi_m([h \otimes I_{\mathcal{B}}](\mathsf{T})) = [g \otimes I_{B_m}](\mathsf{T}_{n,m}).$$

Suppose $\mathsf{T} = \sum_{i=1}^{n} a_i \otimes b_i$. Then

$$\begin{aligned}\Psi_m([h \otimes I_{\mathcal{B}}](\mathsf{T})) &= \Psi_m\left(\sum_{i=1}^{n} h(a_i) b_i\right) = \sum_{i=1}^{n} h(a_i)\Psi_m(b_i)\\ &= \sum_{i=1}^{n} g(\Phi_n(a_i))\ \Psi_m(b_i) = [g \otimes I_{B_m}]((\Phi_n \otimes \Psi_m)(\mathsf{T}))\\ &= [g \otimes I_{B_m}](\mathsf{T}_{n,m}).\end{aligned}$$

A density argument proves our claim.
Since $\hat{\mathsf{T}}(h)$ is left invertible, there exists $b \in \mathcal{B}$ such that $b\ \hat{\mathsf{T}}(h) = 1_{\mathcal{B}}$. Hence, $\Psi_m(b\ \hat{\mathsf{T}}(h)) = 1_{B_m}$ and

$$1_{B_m} = \Psi_m(b) \cdot \Psi_m(\hat{\mathsf{T}}(h)) = \Psi_m(b) \cdot [g \otimes I_{B_m}](\mathsf{T}_{n,m}) = \Psi_m(b) \cdot \hat{\mathsf{T}}_{n,m}(g).$$

So, $\hat{\mathsf{T}}_{n,m}(g)$ is left invertible in A_n for every m. Applying [18], Proposition 17, we have that $\mathsf{T}_{n,m}$ is left invertible in $A_n \hat{\otimes}_{\mathcal{T}} B_m$ for every n and m. Now, by [1], Theorem 4.2, T is left invertible in $\mathcal{A} \hat{\otimes}_{\mathcal{T}} \mathcal{B}$.
q.e.d.

It is a well known fact that if f is a continuous mapping from some topological space X taking values in a Banach algebra, it is invertible if and only if $f(x)$ is invertible for every $x \in X$. Using the previous theorem we see that the same remains valid when we restrict ourselves to left invertibility. This is not so obvious now, since while an inverse of an element is unique, a left inverse is not necessarily.

Example 3.6.2
Let X be a completely regular space and $\mathcal{C}(X)$ the space of all complex-valued functions on X with the compact open topology. We know that $\mathcal{C}(X)$ is a Fréchet-algebra if and only if X is a hemicompact $k_{\mathbb{R}}$-space (see [28], chapter 3). Let, then X be such a space. Therefore we can identify $\mathfrak{M}(\mathcal{C}(X))$ with X ([28], 4.1.7) by means of the evaluation functionals $\delta_x(f) = f(x)$.
Consider now $\mathcal{B}$ a unital Fréchet algebra, not necessarily commutative. We have $\mathcal{C}(X, \mathcal{B}) \cong \mathcal{C}(X) \hat{\otimes}_{\varepsilon} \mathcal{B}$ ([37], 16.6.3). Given $\sum_{i=1}^{n} f_i \otimes b_i \in \mathcal{C}(X) \otimes_{\varepsilon} \mathcal{B}$ this defines a mapping from X into $\mathcal{B}$ by $\left(\sum_{i=1}^{n} f_i \otimes b_i\right)(x) = \sum_{i=1}^{n} f_i(x) b_i$.
If $\mathbf{f} = \sum_{i=1}^{n} f_i \otimes b_i \in \mathcal{C}(X) \hat{\otimes}_{\varepsilon} \mathcal{B}$ and $x \in X$ then $\hat{\mathbf{f}}(\delta_x) = [\delta_x \otimes I_{\mathcal{B}}](\mathbf{f}) = \sum_{i=1}^{n} f_i(x) b_i = \left(\sum_{i=1}^{n} f_i \otimes b_i\right)(x)$. By a density argument we have $\hat{\mathbf{f}}(\delta_x) = \mathbf{f}(x)$ for every $\mathbf{f} \in \mathcal{C}(X, \mathcal{B})$ and $x \in X$. Thus, applying Theorem 3.6.1 we have $\mathbf{f} \in \mathcal{C}(X, \mathcal{B})$ is left invertible if and only if $\mathbf{f}(x)$ is left invertible in $\mathcal{B}$ for all $x \in X$.

Example 3.6.3
Using different techniques we obtain a similar result for different classes of spaces and algebras. Take E a paracompact t.v.s. (each open covering has a countable subcover) and $\mathcal{B}$

a unital Q-algebra. Consider $f \in \mathcal{C}(E;\mathcal{B})$.
If f is left-invertible in $\mathcal{C}(E;\mathcal{B})$, then obviously $f(x)$ is left-invertible for all $x \in E$. Conversely suppose a_x is the left inverse of $f(x)$ for each $x \in X$, that is, $a_x f(x) = 1_{\mathcal{B}}$. Let $\mathcal{U}$ be a basis of neighbourhoods of 0 in $\mathcal{B}$ and take $U \in \mathcal{U}$ such that $1_{\mathcal{B}} + U \subseteq \mathcal{B}_{inv}$. Since f is continuous, there exists $N(x)$, a neighbourhood of x such that, if $y \in N(x)$, then $a_x(f(y) - f(x)) \in U$. Then, given $y \in N(x)$ we have

$$a_x f(y) = a_x f(x) + a_x(f(y) - f(x)) = 1_{\mathcal{B}} + a_x(f(y) - f(x)) \in 1_{\mathcal{B}} + U \subseteq \mathcal{B}_{inv}.$$

Therefore $a_x f(y)$ is invertible for all $y \in N(x)$. The inverse function is continuous, and hence we can find $g_x : N(x) \longrightarrow \mathcal{B}$ continuous such that, for each $y \in N(x)$

$$g_x(y) a_x f(y) = g_x(y)(1_{\mathcal{B}} + a_x(f(y) - f(x))) = 1_{\mathcal{B}}.$$

Since E is paracompact, the cover $(N(x))_{x \in X}$ contains a countable subcover $(N(x_n))_{n \in \mathbb{N}}$. Let $(\alpha_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}(E)$ denote a partition of unity subordinated to $(N(x_n))_{n \in \mathbb{N}}$. This means that $\operatorname{supp}(\alpha_n) \subseteq N(x_k)$ for some k, for all $x \in E$ $\sum_{n=1}^{\infty} \alpha_n(x) = 1$ and for each x only a finite number of $\alpha_n(x)$ are non-zero. Consider $\tilde{\alpha}_n : E \longrightarrow \mathcal{B}$ defined by $\tilde{\alpha}_n = \alpha_n 1_{\mathcal{B}}$. Clearly $\tilde{\alpha}_n \in \mathcal{C}(E;\mathcal{B})$ for all n and $\sum_{n=1}^{\infty} \tilde{\alpha}_n(x) = 1_{\mathcal{B}}$ for every $x \in E$ and only a finite number of $\tilde{\alpha}_n(x)$ are non-zero. Let $h = \sum_{n=1}^{\infty} \tilde{\alpha}_n g_{x_n} a_{x_n}$. Then

$$h(x) f(x) = \sum_{n=1}^{\infty} \tilde{\alpha}_n g_{x_n} a_{x_n} f(x) = 1_{\mathcal{B}} \sum_{n=1}^{\infty} \tilde{\alpha}_n(x) = 1_{\mathcal{B}}.$$

And h is a left inverse of f in $\mathcal{C}(E;\mathcal{B})$.

If $\mathcal{A}$ is a Q-algebra, the Gelfand mapping is a non-zero algebra homomorphism by Proposition 3.3.4 and it maps left invertible elements into left invertible elements and we have the following proposition.

Proposition 3.6.4
Let $\mathcal{A}$ be a commutative, unital, Fréchet Q-algebra; $\mathcal{B}$ a unital, Fréchet algebra and $\mathcal{T}$ a uniform tensor topology such that $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is a lmc algebra and that satisfies the projective limit condition. Take $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$; then, the following are equivalent:
(i) T is left invertible in $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$.
(ii) $\hat{\mathsf{T}}(h)$ is left invertible in $\mathcal{B}$ for every $h \in \mathfrak{M}(\mathcal{A})$.
(iii) $\hat{\mathsf{T}}$ is left invertible in $\mathcal{C}(\mathfrak{M}(\mathcal{A}),\mathcal{B})$.

Remark 3.6.5
In many cases if $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is a Q-algebra, then $\mathcal{A}$ is also a Q-algebra; for example when $\mathfrak{M}(\mathcal{B}) \neq \emptyset$ and $\mathcal{A}$ is complete. Indeed, in this case we consider the mapping $i : \mathbb{C} \longrightarrow \mathcal{B}$ given by $i(\lambda) = \lambda 1_{\mathcal{B}}$ and any $h \in \mathfrak{M}(\mathcal{B})$. This satisfies $h(1_{\mathcal{B}}) = 1$ and

$$\mathcal{A} \cong \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathbb{C} \overset{I_{\mathcal{A}}\otimes i}{\longrightarrow} \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B} \overset{I_{\mathcal{A}}\otimes h}{\longrightarrow} \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathbb{C} \cong \mathcal{A}$$

For $a \in \mathcal{A}$, $(I_{\mathcal{A}} \otimes h) \circ (I_{\mathcal{A}} \otimes i)(a \otimes \lambda) = (I_{\mathcal{A}} \otimes h)(a \otimes \lambda 1_{\mathcal{B}}) = a \otimes \lambda$. This means that $(I_{\mathcal{A}} \otimes h) \circ (I_{\mathcal{A}} \otimes i) = I_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathbb{C}}$. Hence, $(I_{\mathcal{A}} \otimes i) \circ (I_{\mathcal{A}} \otimes h)$ is an algebra homomorphism and a projection that makes $\mathcal{A}$ a complemented subspace of $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$. By Remark 3.2.5, $\mathcal{A}$ is a Q-algebra.

The main point in the proof of Theorem 3.6.1 is the use of [1], Theorem 4.2. This allows us to go to the Banach case, apply the result in [18] and go back to the general case. The problem is that Theorem in [1] is true only for countable projective limits and the proof cannot be adapted to the general case. This forced us to restrict ourselves to the case of Frechét algebras. Now, using Proposition 3.5.16, we can prove a similar result without using [1]. This allows us to widen the class of algebras that we can consider, but we must place conditions on the topologies. We assume that both $\mathfrak{M}(\mathcal{A})$ and $\mathfrak{M}(\mathcal{B})$ are not empty.

Theorem 3.6.6
Let $\mathcal{A}$ be a commutative, complete, unital, lmc algebra; $\mathcal{B}$ a complete, unital, lmc algebra and $\mathcal{T}$ a uniform tensor topology such that $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is a lmc algebra and such that $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ is a Q-algebra. Take $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$; then, the following are equivalent:
(i) T is left invertible in $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$.
(ii) $\hat{\mathsf{T}}(h)$ is left invertible in $\mathcal{B}$ for every $h \in \mathfrak{M}(\mathcal{A})$.
(iii) $\hat{\mathsf{T}}$ is left invertible in $\mathcal{C}(\mathfrak{M}(\mathcal{A}), \mathcal{B})$.
Proof.
$(i) \Leftrightarrow (ii)$
We have T left invertible in $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}$ if and only if (Proposition 3.5.16),

$$0 \notin \sigma^{left}_{\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathcal{B}}(\mathsf{T}) = \bigcup_{h \in \mathfrak{M}(\mathcal{A})} \sigma^{left}_{\mathcal{B}}(\hat{\mathsf{T}}(h)).$$

This is true if and only if $0 \notin \sigma^{left}_{\mathcal{B}}(\hat{\mathsf{T}}(h))$ for all $h \in \mathfrak{M}(\mathcal{A})$. But this is satisfied if and only if $\hat{\mathsf{T}}(h)$ is left invertible in $\mathcal{B}$ for all $h \in \mathfrak{M}(\mathcal{A})$.
Suppose now that $\mathcal{A}$ is a Q-algebra.
$(i) \Rightarrow (iii)$
The mapping $\mathsf{T} \mapsto \hat{\mathsf{T}}$ is a non-zero algebra homomorphism and, therefore, sends left invertible elements to left invertible elements. The implication $(iii) \Rightarrow (ii)$ is obvious.
q.e.d.

Example 3.6.7
We use Theorem 3.6.6 to characterize the left invertible holomorphic germs on a compact set with values in a (non-commutative) Banach algebra. Let E be Fréchet-Schwartz space whose topology is generated by an increasing sequence of seminorms $(p_n)_{n\in\mathbb{N}}$ such that each $\widehat{E}_n$, a Banach space, has the approximation property. Take $K \subseteq E$ compact, balanced and polynomially convex and $\mathcal{B}$ a unital Banach algebra. Under these conditions we have

the following representation (see [3])

$$(\mathcal{H}(K,\mathcal{B}),\tau_\omega) \cong (\mathcal{H}(K),\tau_\omega)\hat{\otimes}_\varepsilon \mathcal{B}.$$

Moreover, since K is polynomially convex, $\mathfrak{M}(\mathcal{H}(K)) \cong K$ by means of the identification $h(f) = f(k)$ (see [57], Theorem 28.2). Both $\mathcal{H}(K)$ and $\mathcal{H}(K,\mathcal{B})$ are Q-algebras, since they are inductive limits of Banach algebras ([57], Proposition 25.5).
Take $\mathsf{T} = \sum_{i=1}^n f_i \otimes x_i \in \mathcal{H}(K)\hat{\otimes}_\varepsilon \mathcal{B}$ and $h_k \in \mathfrak{M}(\mathcal{H}(K))$, then

$$[h_k \otimes I_{\mathcal{B}}](\mathsf{T}) = \sum_{i=1}^n h_k(f_i)x_i \sum_{i=1}^n f_i(k)x_i = \mathsf{T}(k).$$

By a density argument we have $[h_k \otimes I_{\mathcal{B}}](\mathsf{T}) = \mathsf{T}(k)$ for all $\mathsf{T} \in \mathcal{H}(K)\hat{\otimes}_\varepsilon \mathcal{B}$ and $k \in K$. By Theorem 3.6.6, $F \in \mathcal{H}(K,\mathcal{B})$ is left invertible if and only if $F(k)$ is left invertible in $\mathcal{B}$ for all $k \in K$.
But we have more. Given $F \in \mathcal{H}(K,\mathcal{B})$, its Gelfand transform $\hat{F} : K \longrightarrow \mathcal{B}$ acts in the following way, $\hat{F}(k) = [h_k \otimes I_{\mathcal{B}}](F) = F(k)$. Then, the Gelfand mapping $\widehat{\ } : \mathcal{H}(K,\mathcal{B}) \longrightarrow \mathcal{C}(K,\mathcal{B})$ is nothing other than the inclusion mapping. By Theorem 3.6.6 (iii), F is left invertible in $\mathcal{H}(K,\mathcal{B})$ if and only if it is left invertible in $\mathcal{C}(K,\mathcal{B})$.

Consider now $\mathcal{A} = \lim\limits_{\leftarrow} \mathcal{A}_i$, a complete Q-algebra, and $E = \mathbb{C}$. We know that $\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathbb{C} \cong \mathcal{A}$. Then, $\lambda \notin \sigma_H^{left}(a \otimes 1)$ if and only if there is a finite sum

$$\begin{aligned} 1_{\mathcal{A}} &= \sum_{j\in F} a_j \ ([I_{\mathcal{A}} \otimes \mu_j](a \otimes 1) \ - \ \lambda \mu_j 1_{\mathcal{A}}) \\ &= \sum_{j\in F} a_j \ (\mu_j a \ - \ \mu_j \lambda 1_{\mathcal{A}}) = \sum_{j\in F} a_j \ \mu_j (a \ - \ \lambda 1_{\mathcal{A}}) \\ &= b(a \ - \ \lambda 1_{\mathcal{A}}) \end{aligned}$$

and this is equivalent to $\lambda \notin \sigma_{\mathcal{A}}^{left}(a)$ (the classical spectrum). Then, $\sigma_H^{left}(a\otimes 1) = \sigma_{\mathcal{A}}^{left}(a)$. Writing $\pi_i(a) = a_i$ and applying Proposition 3.5.13 we obtain

$$\sigma_{\mathcal{A}}^{left}(a) = \bigcup_{i\in I} \sigma_{\mathcal{A}_i}^{left}(a_i).$$

By [49], Theorem 4.1 Section III-4 or [54], Theorem 5.2, $a \in \mathcal{A}$ is invertible if and only if each a_i is invertible in $\mathcal{A}_i$. The uniqueness of the inverse is essential and the proof cannot be adapted to the left invertible setting. In [1], Theorem 4.2, Arens proves a general result for Fréchet algebras that as a particular case has that $a \in \mathcal{A}$ is left invertible if and only if each a_n is left invertible in $\mathcal{A}_n$, we have used this result in the proof of Theorem 3.6.1. Arens' proof uses a sort of 'step method' based on the fact that the projective limit is countable. We present here a proof of the arbitrary (perhaps uncountable) using the spectral theory we have just stated for Q-algebras.

Theorem 3.6.8
Let $\mathcal{A} = \varprojlim \mathcal{A}_i$ be a complete Q-algebra; then:
$a \in \mathcal{A}$ is left invertible $\Leftrightarrow$ a_i is left invertible in $\mathcal{A}_i$ for all i.
Proof.
$a \in \mathcal{A}$ is left invertible if and only if $0 \notin \sigma_{\mathcal{A}}^{left}(a) = \bigcup_{i\in I} \sigma_{\mathcal{A}_i}^{left}(a_i)$, if and only if $0 \notin \sigma_{\mathcal{A}_i}^{left}(a_i)$ for all i. This is equivalent to each a_i being left invertible in $\mathcal{A}_i$ for all i.
q.e.d.

3.7 Polynomial extensions

Let E and F be two locally convex space, $\mathcal{A}$ a lmc algebra and $\mathcal{T}$ a uniform tensor topology. If $P \in \mathcal{P}(^nE; F)$, it seems natural to ask if we can define $P_{\mathcal{A}} \in \mathcal{P}(^n(\mathcal{A}\hat{\otimes}_{\mathcal{T}}E); \mathcal{A}\hat{\otimes}_{\mathcal{T}}F)$ so that

$$P_{\mathcal{A}}(a \otimes x) = a^n \otimes P(x).$$

This would allow us to define for any $P \in \mathcal{P}(E; F)$ a polynomial extension $P_{\mathcal{A}} \in \mathcal{P}(\mathcal{A}\hat{\otimes}_{\mathcal{T}}E; \mathcal{A}\hat{\otimes}_{\mathcal{T}}F)$. This problem was first considered by Dineen, Harte and Taylor in [19] in the context of Banach spaces and algebras.
The problem has to be approached in two steps. First, define a polynomial $P_{\mathcal{A}}$ from $\mathcal{A} \otimes E$ into $\mathcal{A} \otimes F$ satisfying the desired condition; second, extend, when possible, this polynomial to the completion. The first step is purely algebraical and was completely solved in [19]. The second part obviously depends on the topologies considered and was treated also in the mentioned paper for Banach spaces and algebras. We will study the situation in our new setting.

3.7.1 Algebraic extension

The extension of polynomials is achieved by extending homogeneous polynomials and these are extended by considering the associated multilinear mappings.
Let E and F be any two vector spaces over $\mathbb{C}$ and $\mathcal{A}$ any complex algebra. If $L \in \mathcal{L}_a(^nE; F)$, then we can define a $2n$-linear mapping $L_1 : \mathcal{A}^n \times E^n \longrightarrow \mathcal{A} \otimes F$ by letting

$$L_1(a_1, \ldots, a_n, x_1, \ldots, x_n) = a_1 \cdots a_n \otimes L(x_1, \ldots, x_n).$$

Universal properties of tensor product and associativity give a linear mapping

$$L_2 : \bigotimes_n (\mathcal{A} \otimes E) \longrightarrow \mathcal{A} \otimes F$$

satisfying

$$L_2((a_1 \otimes x_1) \otimes \cdots (a_n \otimes x_n)) = a_1 \cdots a_n \otimes L(x_1, \ldots, x_n).$$

If $P \in \mathcal{P}_a(^nE;F)$, we consider the associated symmetric n-linear mapping $L = \check{P} \in \mathcal{L}_a^s(^nE;F)$ and define L_2 as above. Let

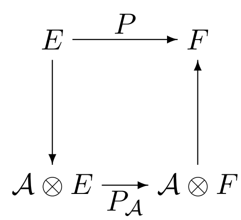
$$P_{\mathcal{A}}\left(\sum_{i=1}^{k} a_i \otimes x_i\right) = L_2\left((\sum_{i=1}^{k} a_i \otimes x_i) \otimes \cdots \otimes (\sum_{i=1}^{k} a_i \otimes x_i)\right)$$

and extending by linearity to define an element of $\mathcal{P}_a(^n\mathcal{A}\otimes E;\mathcal{A}\otimes F)$. The polynomial $P_{\mathcal{A}}$ defined in this way satisfies

$$P_{\mathcal{A}}(a \otimes x) = a^n \otimes P(x)$$

for every $a \in \mathcal{A}$ and $x \in E$. Since L_2 is unique (because of the definition of tensor product), $P_{\mathcal{A}}$ is unique.

If $\mathcal{A}$ has an identity $1_{\mathcal{A}}$, the polynomial $P_{\mathcal{A}}$ can be regarded as an extension of P in the following sense. The space E is embedded in $\mathcal{A}\otimes E$ by the mapping $x \mapsto 1_{\mathcal{A}} \otimes x$. We obtain the following commutative diagram.



3.7.2 Continuous extensions to the completion

Once $P \in \mathcal{P}_a(^nE;F)$ has been extended to $P_{\mathcal{A}} \in \mathcal{P}_a(^n\mathcal{A}\otimes E;\mathcal{A}\otimes F)$, we endow the tensor products with some topology $\mathcal{T}$ and ask whether $P_{\mathcal{A}}$ is continuous and, therefore, can be extended to the completion $\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$.

It is well known that any polynomial between locally convex spaces is continuous if and only if it is continuous at 0. In this case, since $\mathcal{A}\otimes_{\mathcal{T}} E$ is dense in $\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, the polynomial $P_{\mathcal{A}}$ admits a unique continuous extension (which we also denote $P_{\mathcal{A}}$), to the completions, i.e. $P_{\mathcal{A}} \in \mathcal{P}(^n\mathcal{A}\hat{\otimes}_{\mathcal{T}}E;\mathcal{A}\hat{\otimes}_{\mathcal{T}}F)$. Thus, the main point in defining an extension to the completion is that $P_{\mathcal{A}}$ be continuous at 0 (in $\mathcal{A}\otimes_{\mathcal{T}} E$). In particular this means that for all $\mathcal{T}$-neighbourhood of 0 in $\mathcal{A}\otimes F$, V, there exists U, a $\mathcal{T}$-neighbourhood of 0 in $\mathcal{A}\otimes E$, such that

$$P_{\mathcal{A}}(U) \subseteq V.$$

If the topologies of $\mathcal{A}$, E and F are generated by families of seminorms $\{p\}$, $\{q_1\}$ and $\{q_2\}$, respectively, and the topology $\mathcal{T}$ is generated by seminorms $\{p\otimes q_1\}$ for $\mathcal{A}\otimes_{\mathcal{T}} E$ and $\{p\otimes q_2\}$ for $\mathcal{A}\otimes_{\mathcal{T}} F$, then $P_{\mathcal{A}}$ can be continuously extended to $\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$ if and only if for p_2, q_2 there exist p_1, q_1 such that

$$(p_2 \otimes q_2)(P_{\mathcal{A}}(\mathsf{T})) \leq ((p_1 \otimes q_1)(\mathsf{T}))^n$$

for all $\mathsf{T} \in \mathcal{A}\otimes E$.

Examples

Clearly, whether or not $P_{\mathcal{A}}$ can be extended to $\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$ depends on the algebra $\mathcal{A}$, the space E, the topology $\mathcal{T}$ and the polynomial $P \in \mathcal{P}_a({}^nE;F)$.

Example 3.7.1
For any $\mathcal{A}$ lmc algebra, and any two locally convex spaces E and F, all $P \in \mathcal{P}({}^nE;F)$ can be extended to $P_{\mathcal{A}} \in \mathcal{P}({}^n\mathcal{A}\hat{\otimes}_\pi E;\mathcal{A}\hat{\otimes}_\pi F)$. Indeed, if $\mathsf{T} = \sum_{i=1}^k a_i \otimes x_i \in \mathcal{A}\otimes E$ and p is a continuous seminorm on $\mathcal{A}$ and q_2 is a continuous seminorm on F then since P is continuous we can find q_1, a continuous seminorm on E such that

$$\begin{aligned}
&(p\otimes_\pi q_2)(P_{\mathcal{A}}(\sum_{i=1}^k a_i\otimes x_i))\\
=\ &(p\otimes_\pi q_2)(\check{P}_{\mathcal{A}}(\sum_{i_1=1}^k a_{i_1}\otimes x_{i_1},\ldots,\sum_{i_n=1}^k a_{i_n}\otimes x_{i_n}))\\
=\ &(p\otimes_\pi q_2)(\sum_{i_1,\ldots,i_n=1}^k a_{i_1}\ldots a_{i_n}\otimes \check{P}(x_{i_1},\ldots,x_{i_n}))\\
\le\ &\sum_{i_1,\ldots,i_n=1}^k p(a_{i_1}\ldots a_{i_n})q_2(\check{P}(x_{i_1},\ldots,x_{i_n}))\\
\le\ &\sum_{i_1,\ldots,i_n=1}^k p(a_{i_1})\ldots p(a_{i_n})q_1(x_{i_1})\ldots q_1(x_{i_n})\\
=\ &\left(\sum_{i_1=1}^k p(a_i)\ q_1(x_i)\right)^n.
\end{aligned}$$

This is true for any representation of T. Hence

$$(p\otimes_\pi q_2)(P_{\mathcal{A}}\mathsf{T}) \le ((p\otimes_\pi q_1)(\mathsf{T}))^n$$

for all T. Thus, $P_{\mathcal{A}}$ is continuous at 0 and can be extended to $P_{\mathcal{A}} \in \mathcal{P}({}^n\mathcal{A}\hat{\otimes}_\pi E;\mathcal{A}\hat{\otimes}_\pi F)$.

Example 3.7.2
We consider the injective topology. In this case need the algebra to be a uniform Q-algebra. Let $\mathcal{A}$ be a uniform lmc Q-algebra and take any two locally convex spaces E, F. Let p and q_2 be continuous seminorms on E and F and let $\mathsf{T} = \sum_{i=1}^k a_i \otimes x_i \in \mathcal{A}\otimes E$. We have

$$(p\otimes_\pi q_2)(P_{\mathcal{A}}(\mathsf{T})) \quad = \quad \sup_{\substack{\phi\in B_p^\circ\\ \psi\in B_{q_2}^\circ}} |(\phi\otimes\psi)(P_{\mathcal{A}}\mathsf{T})|$$

$$
\begin{aligned}
&= \sup_{\substack{\phi\in B_p^\circ\\ \psi\in B_{q_2}^\circ}} |(\phi\otimes\psi)(\sum_{i_1,\dots,i_n=1}^{k} a_{i_1}\dots a_{i_n}\otimes \check{P}(x_{i_1},\dots,x_{i_n}))| \\
&= \sup_{\substack{\phi\in B_p^\circ\\ \psi\in B_{q_2}^\circ}} |\phi(\sum_{i_1,\dots,i_n=1}^{k} a_{i_1}\dots a_{i_n}\ \psi(\check{P}(x_{i_1},\dots,x_{i_n})))| \\
&= \sup_{\substack{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})\\ \psi\in B_{q_2}^\circ}} |h(\sum_{i_1,\dots,i_n=1}^{k} a_{i_1}\dots a_{i_n}(\psi\circ\check{P})(x_{i_1},\dots,x_{i_n}))| \\
&= \sup_{\substack{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})\\ \psi\in B_{q_2}^\circ}} |\sum_{i_1,\dots,i_n=1}^{k} h(a_{i_1})\dots h(a_{i_n})(\psi\circ\check{P})(x_{i_1},\dots,x_{i_n}))| \\
&= \sup_{\substack{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})\\ \psi\in B_{q_2}^\circ}} |\psi(\sum_{i_1,\dots,i_n=1}^{k} h(a_{i_1})\dots h(a_{i_n})\ \check{P}(x_{i_1},\dots,x_{i_n}))| \\
&= \sup_{\substack{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})\\ \psi\in B_{q_2}^\circ}} |\psi(\check{P}(\sum_{i_1=1}^{k} h(a_{i_1})x_{i_1},\dots,\sum_{i_n=1}^{k} h(a_{i_n})x_{i_n}))| \\
&= \sup_{\substack{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})\\ \psi\in B_{q_2}^\circ}} |\psi(P(\sum_{i=1}^{k} h(a_i)x_i))| \\
&= \sup_{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})} q_2(P\circ[h\otimes I_E](\mathsf{T})).
\end{aligned}
$$

Since P is continuous, we can find a continuous seminorm $\tilde{q}_1$ on E such that $q_2(P(x)) \le (\tilde{q}_1(x))^n$ for all $x\in E$. On the other hand, since $\mathcal{A}$ is a Q-algebra, $\mathfrak{M}(\mathcal{A})$ is equicontinuous. The ε topology is uniform, then $\mathfrak{M}(\mathcal{A})\otimes\{I_E\}$ is also equicontinuous. Therefore, given $\tilde{q}_1$, we can find p_1 a continuous seminorm on $\mathcal{A}$ and q_1 a continuous seminorm on E such that, for all $h\in\mathfrak{M}(\mathcal{A})$ and all $\mathsf{T}\in\mathcal{A}\otimes E$,

$$\tilde{q}_1([h\otimes I_E](\mathsf{T})) \le (p_1\otimes_\varepsilon q_1)(\mathsf{T}).$$

Therefore

$$
\begin{aligned}
(p\otimes_\pi q_2)(P_{\mathcal{A}}(\mathsf{T})) &= \sup_{h\in B_p^\circ\cap\mathfrak{M}(\mathcal{A})} q_2(P\circ[h\otimes I_E](\mathsf{T})) \\
&\le \sup_{h\in\mathfrak{M}(\mathcal{A})} (\tilde{q}_1([h\otimes I_E](\mathsf{T})))^n \\
&\le ((p_1\otimes_\varepsilon q_1)(\mathsf{T}))^n.
\end{aligned}
$$

Hence $P_{\mathcal{A}}$ is continuous at 0 and can be extended continuously to $P_{\mathcal{A}}\in\mathcal{P}(^n\mathcal{A}\hat{\otimes}_\varepsilon E;\mathcal{A}\hat{\otimes}_\varepsilon F)$.

3.8 Spectral theorems

The question of the existence of spectral mapping theorems relating the tensor spectra with the extensions of polynomials was studied for Banach spaces and Banach algebras by Dineen, Harte and Taylor in [19]. We prove analogous results for the locally convex space case. An important fact in the proofs in [19] is that the left Harte spectrum, though defined to be contained in E'' can be identified in a canonical way with a subset of E. By Proposition 3.5.9 this is also true for Q-algebras, i.e. if $\mathcal{A}$ is a Q-algebra, E is any complete locally convex space and $\mathcal{T}$ is a uniform tensor topology then, for all $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, we can identify

$$\begin{aligned}\sigma_H^{left}(\mathsf{T}) &= \{x \in E \ : \ 1_{\mathcal{A}} \notin \{\sum_{i=1}^{n} a_i \ ([I_{\mathcal{A}} \otimes x_i'](\mathsf{T}) \ - \ x_i'(x)1_{\mathcal{A}}) \ : n \in \mathbb{N}\}\} \\ &= \{x \in E \ : \ 1_{\mathcal{A}} \notin \{\sum_{i=1}^{n} a_i \ ([I_{\mathcal{A}} \otimes x_i'](\mathsf{T} \ - \ 1_{\mathcal{A}} \otimes x)) \ : n \in \mathbb{N}\}\},\end{aligned}$$

where $a_i \in \mathcal{A}$ and $x_i' \in E'$. We use several times the fact that $\mathcal{A} \otimes E$ is an $\mathcal{A}$-module via the mapping $(a, b \otimes x) \mapsto (ab) \otimes x$. Our next result is purely algebraic.

Lemma 3.8.1 ([18] Lemma 17)
Let $\mathcal{A}$ be a lmc Q-algebra, E, F complete locally convex spaces and $\mathcal{T}$ a uniform tensor topology. Then for any $P \in \mathcal{P}(^nE; F)$ such that $P_{\mathcal{A}} \in \mathcal{P}(^n\mathcal{A}\hat{\otimes}_{\mathcal{T}}E; \mathcal{A}\hat{\otimes}_{\mathcal{T}}F)$ and all $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, $x \in E$,

$$P_{\mathcal{A}}(\mathsf{T}) - 1_{\mathcal{A}} \otimes P(x) = \sum_{j=0}^{n-1} \check{P}_{\mathcal{A}}(\overbrace{\mathsf{T}, \dots, \mathsf{T}}^{j \text{ times}}, \mathsf{T} - 1_{\mathcal{A}} \otimes x, \overbrace{1_{\mathcal{A}} \otimes x, \dots, 1_{\mathcal{A}} \otimes x}^{n-j-1 \text{ times}})$$

Proposition 3.8.2
Let $\mathcal{A}$ be a lmc Q-algebra, E, F complete locally convex spaces and $\mathcal{T}$ a uniform tensor topology. Then for any $P \in \mathcal{P}(E; F)$ such that the extension $P_{\mathcal{A}} \in \mathcal{P}(\mathcal{A}\hat{\otimes}_{\mathcal{T}}E; \mathcal{A}\hat{\otimes}_{\mathcal{T}}F)$ and all $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$,

$$P(\sigma_H^{left}(\mathsf{T})) \subseteq \sigma_H^{left}(P_{\mathcal{A}}(\mathsf{T})).$$

Proof.
Let $x \in E$ such that $P(x) \notin \sigma_H^{left}(P_{\mathcal{A}}(\mathsf{T}))$ and let us show that $x \notin \sigma_H^{left}(\mathsf{T})$. We can find $a_1, \dots, a_m \in \mathcal{A}$ and $y_1', \dots, y_m' \in F'$ such that

$$\begin{aligned}1_{\mathcal{A}} &= \sum_{i=1}^{m} a_i[I_{\mathcal{A}} \otimes y_i'](P_{\mathcal{A}}(\mathsf{T}) - 1_{\mathcal{A}} \otimes P(x)) \\ &= \sum_{i=1}^{m}\sum_{j=0}^{n-1} a_i[I_{\mathcal{A}} \otimes y_i'](\check{P}_{\mathcal{A}}(\overbrace{\mathsf{T}, \dots, \mathsf{T}}^{j}, \mathsf{T} - 1_{\mathcal{A}} \otimes x, \overbrace{1_{\mathcal{A}} \otimes x, \dots, 1_{\mathcal{A}} \otimes x}^{n-j-1})).\end{aligned}$$

Define the mapping $\Phi : \mathcal{A}\hat{\otimes}_{\mathcal{T}} E \longrightarrow \mathcal{A}$ by letting

$$\Phi(\mathsf{S}) = \sum_{i=1}^{m}\sum_{j=0}^{n-1} a_i [I_{\mathcal{A}} \otimes y_i'](\check{P}_{\mathcal{A}}(\overbrace{\mathsf{S},\ldots,\mathsf{S}}^{j}, \mathsf{T} - 1_{\mathcal{A}} \otimes x, \overbrace{1_{\mathcal{A}} \otimes x, \ldots, 1_{\mathcal{A}} \otimes x}^{n-j-1})).$$

This mapping is clearly continuous and $\Phi(\mathsf{T}) = 1_{\mathcal{A}}$. By continuity and density, given any neighbourhood U of $1_{\mathcal{A}}$ such that $U \subseteq \mathcal{A}_{inv}$ (this exists since $\mathcal{A}$ is a Q-algebra) we can find some $\mathsf{S} = \sum_{r=1}^{t} c_r \otimes x_r \in \mathcal{A} \otimes E$ such that $\Phi(\mathsf{S}) \in U$.
Let now $\mathsf{Z} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ be arbitrary; we have for each $i = 1, \ldots, m$,

$$\begin{aligned}
& [I_{\mathcal{A}} \otimes y_i'](\check{P}_{\mathcal{A}}(\overbrace{\mathsf{S},\ldots,\mathsf{S}}^{j}, \mathsf{Z}, \overbrace{1_{\mathcal{A}} \otimes x, \ldots, 1_{\mathcal{A}} \otimes x}^{n-j-1})) \\
= \;& [I_{\mathcal{A}} \otimes y_i'](\check{P}_{\mathcal{A}}(\sum_{r_1=1}^{t} c_{r_1} \otimes x_{r_1}, \ldots, \sum_{r_j=1}^{t} c_{r_j} \otimes x_{r_j}, \mathsf{Z}, \overbrace{1_{\mathcal{A}} \otimes x, \ldots, 1_{\mathcal{A}} \otimes x}^{n-j-1})) \\
= \;& [I_{\mathcal{A}} \otimes y_i'](\sum_{r_1,\ldots,r_j=1}^{t} c_{r_1} \ldots c_{r_j} \check{P}_{\mathcal{A}}(1_{\mathcal{A}} \otimes x_{r_1}, \ldots, 1_{\mathcal{A}} \otimes x_{r_j}, \mathsf{Z}, \overbrace{1_{\mathcal{A}} \otimes x, \ldots, 1_{\mathcal{A}} \otimes x}^{n-j-1})) \\
= \;& \sum_{r_1,\ldots,r_j=1}^{t} [I_{\mathcal{A}} \otimes y_i'](c_{r_1} \ldots c_{r_j} \check{P}_{\mathcal{A}}(1_{\mathcal{A}} \otimes x_{r_1}, \ldots, 1_{\mathcal{A}} \otimes x_{r_j}, \mathsf{Z}, \overbrace{1_{\mathcal{A}} \otimes x, \ldots, 1_{\mathcal{A}} \otimes x}^{n-j-1})).
\end{aligned}$$

Hence,

$$\begin{aligned}
\Phi(\mathsf{S}) \;&=\; \sum_{i=1}^{m}\sum_{j=0}^{n-1} a_i [I_{\mathcal{A}} \otimes y_i'](\check{P}_{\mathcal{A}}(\overbrace{\mathsf{S},\ldots,\mathsf{S}}^{j}, \mathsf{T} - 1_{\mathcal{A}} \otimes x, \overbrace{1_{\mathcal{A}} \otimes x, \ldots, 1_{\mathcal{A}} \otimes x}^{n-j-1})) \\
&=\; \sum_{i=1}^{m}\sum_{j=0}^{n-1} a_i \sum_{r_1,\ldots,r_j=1}^{t} [I_{\mathcal{A}} \otimes y_i'] \\
&\quad\; (c_{r_1} \ldots c_{r_j} \check{P}_{\mathcal{A}}(1_{\mathcal{A}} \otimes x_{r_1}, \ldots, 1_{\mathcal{A}} \otimes x_{r_j}, \mathsf{T} - 1_{\mathcal{A}} \otimes x, (1_{\mathcal{A}} \otimes x)^{n-j-1})) \\
&=\; \sum_{i=1}^{m}\sum_{j=0}^{n-1} \sum_{r_1,\ldots,r_j=1}^{t} a_i c_{r_1} \ldots c_{r_j} \\
&\quad\; [I_{\mathcal{A}} \otimes y_i'](\check{P}_{\mathcal{A}}(1_{\mathcal{A}} \otimes x_{r_1}, \ldots, 1_{\mathcal{A}} \otimes x_{r_j}, \mathsf{T} - 1_{\mathcal{A}} \otimes x, (1_{\mathcal{A}} \otimes x)^{n-j-1})).
\end{aligned}$$

Let us define now for $1 \leq i \leq m$, $0 \leq j \leq n-1$ and $1 \leq r_1, \ldots, r_j \leq t$, $x'_{(i,j,r_1,\ldots,r_j)} \in E'$ by

$$x'_{(i,j,r_1,\ldots,r_j)}(w) = y_i'(\check{P}(x_{r_1}, \ldots, x_{r_j}, w, \overbrace{x, \ldots, x}^{n-j-1})).$$

Therefore, if $\mathsf{R} = \sum_{s=1}^{s_0} d_s \otimes x_s$, then

$$\begin{aligned}
[I_{\mathcal{A}} \otimes x'_{(i,j,r_1,\ldots,r_j)}](\mathsf{R}) &= [I_{\mathcal{A}} \otimes x'_{(i,j,r_1,\ldots,r_j)}](\sum_{s=1}^{s_0} d_s \otimes x_s) \\
&= \sum_{s=1}^{s_0} [I_{\mathcal{A}} \otimes x'_{(i,j,r_1,\ldots,r_j)}](d_s \otimes x_s) \\
&= \sum_{s=1}^{s_0} d_s x'_{(i,j,r_1,\ldots,r_j)}(x_s) \\
&= \sum_{s=1}^{s_0} d_s y'_i(\check{P}(x_{r_1}, \ldots, x_{r_j}, w_s, \overbrace{x, \ldots, x}^{n-j-1})) \\
&= \sum_{s=1}^{s_0} [I_{\mathcal{A}} \otimes y'_i](d_s \otimes \check{P}(x_{r_1}, \ldots, x_{r_j}, w_s, \overbrace{x, \ldots, x}^{n-j-1})) \\
&= [I_{\mathcal{A}} \otimes y'_i](\sum_{s=1}^{s_0} d_s \otimes \check{P}(x_{r_1}, \ldots, x_{r_j}, w_s, \overbrace{x, \ldots, x}^{n-j-1})) \\
&= [I_{\mathcal{A}} \otimes y'_i](\check{P}_{\mathcal{A}}(1_{\mathcal{A}} \otimes x_{r_1}, \ldots, 1_{\mathcal{A}} \otimes x_{r_j}, \mathsf{R}, \overbrace{x, \ldots, x}^{n-j-1})).
\end{aligned}$$

By continuity and density this is true for all $\mathsf{R} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$. Thus

$$\Phi(\mathsf{S}) = \sum_{i=1}^{m} \sum_{j=0}^{n-1} \sum_{r_1,\ldots,r_j=1}^{t} a_i c_{r_1} \ldots c_{r_j} [I_{\mathcal{A}} \otimes x'_{(i,j,r_1,\ldots,r_j)}](\mathsf{T} - 1_{\mathcal{A}} \otimes x).$$

Since $\Phi(\mathsf{S})$ is invertible we have

$$1_{\mathcal{A}} = \sum_{i=1}^{m} \sum_{j=0}^{n-1} \sum_{r_1,\ldots,r_j=1}^{t} \Phi(\mathsf{S})^{-1} a_i c_{r_1} \ldots c_{r_j} [I_{\mathcal{A}} \otimes x'_{(i,j,r_1,\ldots,r_j)}](\mathsf{T} - 1_{\mathcal{A}} \otimes x).$$

This implies that $x \notin \sigma_H^{left}(\mathsf{T})$ and completes the proof.
q.e.d.

Our purpose now is to obtain the reverse inclusion. This would give a two way spectral theorem. Unfortunately we cannot prove it in the most general case.
We say that $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ is *commutative* if the set $\{[I_{\mathcal{A}} \otimes x'] : x' \in E'\} \subseteq \mathcal{A}$ is a commutative system. A locally convex space E has the *approximation property* (see [17], 2.7) if for every $K \subseteq E$ compact, every continuous seminorm q and every $\varepsilon > 0$ there is some $u \in \mathcal{L}_f(E; E)$ such that, for all $x \in K$,

$$q(x - ux) \leq \varepsilon.$$

If the set of u's is equicontinuous, E is said to have the *bounded* approximation property. Let $P \in \mathcal{P}({}^n E; F)$ be such that $P_{\mathcal{A}} \in \mathcal{P}({}^n \mathcal{A}\hat{\otimes}_{\mathcal{T}} E; \mathcal{A}\hat{\otimes}_{\mathcal{T}} F)$. Then since u is continuous, $P \circ u$ is also in $\mathcal{P}({}^n E; F)$ and $(P \circ u)_{\mathcal{A}} \in \mathcal{P}({}^n \mathcal{A}\hat{\otimes}_{\mathcal{T}} E; \mathcal{A}\hat{\otimes}_{\mathcal{T}} F)$. We also have

$$\begin{aligned} (P \circ u)_{\mathcal{A}} \left(\sum_{i=1}^{k} a_i \otimes x_i \right) &= \sum_{i_1,\ldots,i_n=1}^{k} a_{i_1} \cdots a_{i_n} \otimes (P \circ u)\check{}(x_{i_1}, \ldots, x_{i_n}) \\ &= \sum_{i_1,\ldots,i_n=1}^{k} a_{i_1} \cdots a_{i_n} \otimes \check{P}(u(x_{i_1}), \ldots, u(x_{i_n})) \\ &= P_{\mathcal{A}} \left([I_{\mathcal{A}} \otimes u](\sum_{i=1}^{k} a_i \otimes x_i) \right). \end{aligned}$$

By a density argument we obtain $(P \circ u)_{\mathcal{A}} = P_{\mathcal{A}} \circ [I_{\mathcal{A}} \otimes u]$.
Consider now a locally convex space E with the bounded approximation property. This gives us an equicontinuous set $\mathcal{U} \subseteq \mathcal{L}_f(E; E)$. Let $\mathcal{A}$ be a lmc Q-algebra and $\mathcal{T}$ a uniform tensor topology defined by a set of seminorms $\{p \otimes q : p \in \mathcal{P},\ q \in \mathcal{Q}\}$, where $\mathcal{P}$ and $\mathcal{Q}$ are two systems of seminorms defining the topologies of $\mathcal{A}$ and E respectively. Since $\mathcal{U}$ is equicontinuous, $\{I_{\mathcal{A}}\} \otimes \mathcal{U}$ is again equicontinuous and, given any two continuous seminorms p and q, there are continuous seminorms p_1 and q_1 such that,

$$(p \otimes q)([I_{\mathcal{A}} \otimes u](\mathsf{T})) \leq (p_1 \otimes q_1)(\mathsf{T}),$$

for every $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ and all $u \in \mathcal{U}$.
Fix $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}} E$ and $\varepsilon > 0$. Choose $\mathsf{S} = \sum_{i=1}^{k} a_i \otimes x_i$ such that

$$\max\{(p \otimes q)(\mathsf{T} - \mathsf{S}), (p_1 \otimes q_1)(\mathsf{T} - \mathsf{S}) \leq \frac{\varepsilon}{3}.$$

The set $\{x_1, \ldots, x_k\} \subseteq E$ is compact, so we can find u satisfying

$$q(x_i - ux_i) < \frac{\varepsilon}{3 \sum_{j=1}^{k} p(a_j)}$$

for all $i = 1, \ldots, k$. Hence

$$\begin{aligned} (p \otimes q)(\mathsf{T} - [I_{\mathcal{A}} \otimes u](\mathsf{T})) &\leq (p \otimes q)(\mathsf{T} - \mathsf{S}) \\ &\quad + (p \otimes q)(\sum_{i=1}^{k} a_i \otimes x_i - [I_{\mathcal{A}} \otimes u](\sum_{i=1}^{k} a_i \otimes x_i)) \\ &\quad + (p \otimes q)([I_{\mathcal{A}} \otimes u](\mathsf{T}) - [I_{\mathcal{A}} \otimes u](\mathsf{S})) \\ &= (p \otimes q)(\mathsf{T} - \mathsf{S}) + (p \otimes q)(\sum_{i=1}^{k} a_i \otimes (x_i - ux_i)) \end{aligned}$$

$$
\begin{aligned}
&+(p\otimes q)([I_{\mathcal{A}}\otimes u](\mathsf{T}-\mathsf{S}))\\
\leq\ &(p\otimes q)(\mathsf{T}-\mathsf{S})\\
&+\sum_{i=1}^{k}p(a_i)\otimes q(x_i-ux_i)+(p_1\otimes q_1)(\mathsf{S}-\mathsf{T})\\
<\ &\frac{\varepsilon}{3}+\sum_{i=1}^{k}p(a_i)\frac{\varepsilon}{3\sum_{j=1}^{k}p(a_j)}+\frac{\varepsilon}{3}=\varepsilon.
\end{aligned}
$$

That is, for any $\mathsf{T}\in\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, $\varepsilon>0$ and $p\otimes q$ there exists some u such that

$$(p\otimes q)(\mathsf{T}-[I_{\mathcal{A}}\otimes u](\mathsf{T}))<\varepsilon.$$

We are now ready to prove the following result, analogous to Proposition 19 in [19].

Theorem 3.8.3
Let $\mathcal{A}$ be a complete lmc Q-algebra and E a complete locally convex space with the bounded approximation property. Let $\mathcal{T}$ be a uniform tensor topology defined by a system of seminorms $\{p\otimes q\}$. Take $\mathsf{T}\in\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$ commutative. If $P\in\mathcal{P}(E;F)$ is such that $P_{\mathcal{A}}\in\mathcal{P}(^n\mathcal{A}\hat{\otimes}_{\mathcal{T}}E;\mathcal{A}\hat{\otimes}_{\mathcal{T}}F)$, then

$$P(\sigma_H^{left}(\mathsf{T}))=\sigma_H^{left}(P_{\mathcal{A}}(\mathsf{T})).$$

Proof.
From Proposition 3.8.2, it is enough to show that

$$\sigma_H^{left}(P_{\mathcal{A}}(\mathsf{T}))\subseteq P(\sigma_H^{left}(\mathsf{T})).$$

Fix $\mathsf{T}\in\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, assume that $P\in\mathcal{P}(^nE;F)$ and let $y\in\sigma_H^{left}(P_{\mathcal{A}}(\mathsf{T}))$. Let $y'\in F'$. We have

$$[I_{\mathcal{A}}\otimes y'](P_{\mathcal{A}}(\mathsf{T}))=(y'\circ P)_{\mathcal{A}}(\mathsf{T})\in\mathcal{A}\hat{\otimes}_{\mathcal{T}}\mathbb{C}\cong\mathcal{A}.$$

Let $Q=y'\circ P$. We want to show now that, for every $x'\in E'$,

$$[I_{\mathcal{A}}\otimes x'](\mathsf{T})\cdot Q_{\mathcal{A}}(\mathsf{T})=Q_{\mathcal{A}}(\mathsf{T})\cdot[I_{\mathcal{A}}\otimes x'](\mathsf{T}). \tag{3.9}$$

Note that this is a product in $\mathcal{A}$. To prove the equality let $\varepsilon>0$ and choose any continuous seminorm p on $\mathcal{A}$. By the previous discussion, there is some u such that

$$p((Q\circ u)_{\mathcal{A}}-Q_{\mathcal{A}}(\mathsf{T}))<\frac{\varepsilon}{2p([I_{\mathcal{A}}\otimes x'](\mathsf{T}))}.$$

If $R=Q\circ u=y'\circ P\circ u$, then, since u is of finite rank, we have $R\in\mathcal{P}_f(^nE;\mathbb{C})$. Thus, there are $x'_1,\ldots,x'_k\in E'$ so that $R=\sum_{i=1}^{k}(x'_i)^n$. Hence

$$R_{\mathcal{A}}(\mathsf{T})=\sum_{i=1}^{k}[(x'_i)^n]_{\mathcal{A}}(\mathsf{T})=\sum_{i=1}^{k}[(x'_i)_{\mathcal{A}}]^n(\mathsf{T})=\sum_{i=1}^{k}([I_{\mathcal{A}}\otimes x'_i](\mathsf{T}))^n.$$

But, since T is commutative,

$$\begin{aligned}[I_{\mathcal{A}} \otimes x'](\mathsf{T}) \cdot R_{\mathcal{A}}(\mathsf{T}) &= [I_{\mathcal{A}} \otimes x'](\mathsf{T}) \sum_{i=1}^{k} ([I_{\mathcal{A}} \otimes x_i'](\mathsf{T}))^n \\ &= \sum_{i=1}^{k} [I_{\mathcal{A}} \otimes x'](\mathsf{T}) \cdot \overbrace{[I_{\mathcal{A}} \otimes x_i'](\mathsf{T}) \cdots [I_{\mathcal{A}} \otimes x_i'](\mathsf{T})}^{n} \\ &= R_{\mathcal{A}}(\mathsf{T}) \cdot [I_{\mathcal{A}} \otimes x'](\mathsf{T}).\end{aligned}$$

Therefore,

$$\begin{aligned}& p([I_{\mathcal{A}} \otimes x'](\mathsf{T}) \cdot Q_{\mathcal{A}}(\mathsf{T}) - Q_{\mathcal{A}}(\mathsf{T}) \cdot [I_{\mathcal{A}} \otimes x'](\mathsf{T})) \\ \leq\ & p([I_{\mathcal{A}} \otimes x'](\mathsf{T}) \cdot Q_{\mathcal{A}}(\mathsf{T}) - [I_{\mathcal{A}} \otimes x'](\mathsf{T}) \cdot R_{\mathcal{A}}(\mathsf{T})) \\ & + p(R_{\mathcal{A}}(\mathsf{T}) \cdot [I_{\mathcal{A}} \otimes x'](\mathsf{T}) - Q_{\mathcal{A}}(\mathsf{T}) \cdot [I_{\mathcal{A}} \otimes x'](\mathsf{T}) \\ =\ & p([I_{\mathcal{A}} \otimes x'](\mathsf{T})(Q_{\mathcal{A}}(\mathsf{T}) - R_{\mathcal{A}}(\mathsf{T}))) + p((Q_{\mathcal{A}}(\mathsf{T}) - R_{\mathcal{A}}(\mathsf{T}))[I_{\mathcal{A}} \otimes x'](\mathsf{T})) \\ \leq\ & 2 \cdot p([I_{\mathcal{A}} \otimes x'](\mathsf{T})) \cdot p((Q_{\mathcal{A}}(\mathsf{T}) - R_{\mathcal{A}}(\mathsf{T}))) < \varepsilon.\end{aligned}$$

Since this is true for any continuous seminorm p and any $\varepsilon > 0$ we have proved (3.9). From now on the proof follows the same pattern as [19]. We continue for the sake of completeness. The collection $\{[I_{\mathcal{A}} \otimes x'](\mathsf{T}), [I_{\mathcal{A}} \otimes y'](P_{\mathcal{A}}(\mathsf{T}))\}_{x' \in E', y' \in F'}$ is a commutative system and we can apply Corollary 3.5.6 to find $x \in \sigma_H^{left}(\mathsf{T})$ such that

$$(x, y) \in \sigma_H^{left}(\mathsf{T}, P_{\mathcal{A}}(\mathsf{T})).$$

Let $F \in \mathcal{P}(E \times F; F)$ defined by $F(z, w) = w - P(z)$. Then $F_{\mathcal{A}}$ can be continuously extended to $\mathcal{A}\hat{\otimes}_{\mathcal{T}}(E \times F) \cong (\mathcal{A}\hat{\otimes}_{\mathcal{T}}E) \times (\mathcal{A}\hat{\otimes}_{\mathcal{T}}F)$. Density and a continuity argument show that for every $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$ and $\mathsf{S} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}F$, we have

$$F_{\mathcal{A}}(\mathsf{T}, \mathsf{S}) = \mathsf{S} - P_{\mathcal{A}}(\mathsf{T})$$

Proposition 3.8.2 implies

$$F(\sigma_H^{left}(\mathsf{T}, \mathsf{S})) \subseteq \sigma_H^{left}(F_{\mathcal{A}}(\mathsf{T}, \mathsf{S})).$$

for all T and S. Since $(x, y) \in \sigma_H^{left}(\mathsf{T}, P_{\mathcal{A}}(\mathsf{T}))$, this implies

$$F(x, y) \in \sigma_H^{left}(F_{\mathcal{A}}(\mathsf{T}, P_{\mathcal{A}}(\mathsf{T}))).$$

But $F_{\mathcal{A}}(\mathsf{T}, P_{\mathcal{A}}(\mathsf{T})) = P_{\mathcal{A}}(\mathsf{T}) - P_{\mathcal{A}}(\mathsf{T}) = 0$ and $\sigma_H^{left}(\{0\}) = 0$. Thus $F(x, y) = 0$ and, from this,

$$y = P(x) \in P(\sigma_H^{left}(\mathsf{T})).$$

This shows the desired inclusion

$$\sigma_H^{left}(P_{\mathcal{A}}(\mathsf{T})) \subseteq P(\sigma_H^{left}(\mathsf{T}))$$

and completes the proof.
q.e.d.

3.8.1 The vector Harte spectrum is compact

Using the last results we are now able to prove that the vector-valued Harte spectrum of an element of $\mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, where $\mathcal{A}$ is a complete unital lmc Q-algebra, is compact. This result for Banach spaces is due to Dineen, Harte and Taylor (see [20]). If $a \in \mathcal{A}$, its *spectral radius* is defined in the following way,

$$r_{\mathcal{A}}(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

This is a classical concept and it is well known (see [76], Theorem 4.2) that a lmc algebra $\mathcal{A}$ is a Q-algebra if and only if there is some seminorm p_0 such that, for all $a \in \mathcal{A}$, one has

$$r_{\mathcal{A}}(a) \leq p_0(a). \tag{3.10}$$

In the same way, considering left spectra we can define a *left spectral radius*, $r_{\mathcal{A}}^{left}(a)$. Clearly $r_{\mathcal{A}}^{left}(a) \leq r_{\mathcal{A}}(a)$.

Theorem 3.8.4
Let $\mathcal{A}$ be a complete, unital, lmc Q-algebra, E a complete locally convex space and $\mathcal{T}$ a uniform tensor topology. Then, for all $\mathsf{T} \in \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$, the vector Harte left spectrum $\sigma_H^{left}(\mathsf{T})$ is compact in the topology of E.
Proof.
Let us begin by showing that it is a closed set. Choose $x \in E$ such that $x \notin \sigma_H^{left}(\mathsf{T})$. By (3.7), we can find $\mathsf{Z} = \sum_{i=1}^n b_i \otimes x' \in \mathcal{A}\otimes E'$ such that $< \mathsf{Z}, \mathsf{T} - 1_{\mathcal{A}} \otimes x >= 1_{\mathcal{A}}$. We can regard Z as a continuous mapping $\mathcal{A}\hat{\otimes}_{\mathcal{T}}E \longrightarrow \mathcal{A}$. We can also consider the inclusion $E \hookrightarrow \mathcal{A}\hat{\otimes}_{\mathcal{T}}E$ given by $w \mapsto 1_{\mathcal{A}} \otimes w$ and the composition of these two mappings is obviously continuous. Since $\mathcal{A}_{inv}$ is open, we can find a neighbourhood V of $0_{\mathcal{A}}$ such that $1_{\mathcal{A}} + V \subseteq \mathcal{A}_{inv}$. From the continuity of the composition of the two previous mappings, there is a neighbourhood of 0_E U such that, for all $w \in U$,

$$b := 1_{\mathcal{A}} - < \mathsf{Z}, 1_{\mathcal{A}} \otimes w > \in 1_{\mathcal{A}} + V$$

and, consequently, b is invertible. Let

$$\tilde{\mathsf{Z}} = b^{-1}\mathsf{Z} = \sum_{i=1}^n b^{-1} b_i \otimes x' \in \mathcal{A} \otimes E'.$$

If $w \in U$ then

$$\begin{aligned} & < \tilde{\mathsf{Z}}, \mathsf{T} - 1_{\mathcal{A}} \otimes (x + w) >= b^{-1} < \mathsf{Z}, \mathsf{T} - 1_{\mathcal{A}} \otimes x - 1_{\mathcal{A}} \otimes w > \\ = \; & b^{-1}(< \mathsf{Z}, \mathsf{T} - 1_{\mathcal{A}} \otimes x > - < \mathsf{Z}, \mathsf{T} - 1_{\mathcal{A}} \otimes w >) \\ = \; & b^{-1}(1_{\mathcal{A}} - < \mathsf{Z}, \mathsf{T} - 1_{\mathcal{A}} \otimes w >) = b^{-1} b = 1_{\mathcal{A}}. \end{aligned}$$

This implies that $x + w \notin \sigma_H^{left}(\mathsf{T})$ and, since w was arbitrary, $(x + U) \cap \sigma_H^{left}(\mathsf{T}) = \emptyset$. Therefore, $\sigma_H^{left}(\mathsf{T})$ is closed.
Let $F : (\mathcal{A}', \sigma(\mathcal{A}', \mathcal{A})) \longrightarrow E$ be defined by $F(\phi) = [\phi \otimes I_E](\mathsf{T})$. If $\mathsf{T} = \sum_{i=1}^n a_i \otimes x_i \in \mathcal{A} \otimes E$, then

$$F(\phi) = \sum_{i=1}^n \phi(a_i) x_i$$

and F is continuous. Fix $\mathsf{T} \in \mathcal{A} \hat{\otimes}_{\mathcal{T}} E$ and an equicontinuous net $(\phi_i)_i \subseteq \mathcal{A}'$ converging to ϕ in the $\sigma(\mathcal{A}', \mathcal{A})$ topology. Choose any continuous seminorm q in E and any $\varepsilon > 0$. Since $(\phi_i)_i$ is equicontinuous, so also is $(\phi_i - \phi)_i$ and, as $\mathcal{T}$ is uniform, $\{(\phi_i - \phi) \otimes I_E\}_i$ is also equicontinuous. Then, given q, we can choose a seminorm ρ in $\mathcal{A} \hat{\otimes}_{\mathcal{T}} E$ such that, for all $\mathsf{R} \in \mathcal{A} \hat{\otimes}_{\mathcal{T}} E$ and all i,

$$q([(\phi_i - \phi) \otimes I_E](\mathsf{R})) \leq \rho(\mathsf{R}).$$

By density we can find $\mathsf{S} \in \mathcal{A} \otimes E$ such that $\rho(\mathsf{T} - \mathsf{S}) < \varepsilon/2$. As previously observed, since $\mathsf{S} \in \mathcal{A} \otimes E$, the mapping $\phi \mapsto [\phi \otimes I_E](\mathsf{S})$ is continuous. Hence there exists i_0 so that, for every $i \geq i_0$,

$$q([\phi_i \otimes I_E](\mathsf{S}) - [\phi \otimes I_E](\mathsf{S})) < \frac{\varepsilon}{2}.$$

This gives, for $i \geq i_0$,

$$\begin{aligned} q(F(\phi_i) - F(\phi)) &= q([\phi_i \otimes I_E](\mathsf{T}) - [\phi \otimes I_E](\mathsf{T})) \\ &= q([(\phi_i - \phi) \otimes I_E](\mathsf{T}) \\ &\leq q([(\phi_i - \phi) \otimes I_E](\mathsf{S})) + q([(\phi_i - \phi) \otimes I_E](\mathsf{T} - \mathsf{S})) \\ &< \frac{\varepsilon}{2} + \rho(\mathsf{T} - \mathsf{S}) < \varepsilon; \end{aligned}$$

and F is continuous on the equicontinuous subsets of $\mathcal{A}'$.
For any finite choice of scalars $\lambda_1, \ldots, \lambda_n$ and $x'_1, \ldots, x'_n \in E'$ we have

$$\sum_{i=1}^n \lambda_i([I_{\mathcal{A}} \otimes x'_i](\mathsf{T})) = [I_{\mathcal{A}} \otimes \textstyle\sum_{i=1}^n \lambda_i x'_i](\mathsf{T}).$$

This means that $H = \{[I_{\mathcal{A}} \otimes x'](\mathsf{T})\}_{x' \in E'}$ is a vector subspace of $\mathcal{A}$. Take $x_0 \in \sigma_H^{left}(\mathsf{T})$ and define $\alpha_{x_0} : H \longrightarrow \mathbb{K}$ by $\alpha_{x_0}([I_{\mathcal{A}} \otimes x'](\mathsf{T})) = x'(x_0)$. Since x' is a polynomial Proposition 3.8.2 implies

$$x'(\sigma_H^{left}(\mathsf{T})) \subseteq \sigma^{left}(x'_{\mathcal{A}}(\mathsf{T})).$$

Since $x'_{\mathcal{A}} = [I_{\mathcal{A}} \otimes x']$, this implies

$$x'(x_0) \in \sigma^{left}([I_{\mathcal{A}} \otimes x'](\mathsf{T}))$$

for all $x' \in E'$. Applying (3.10), there is now a seminorm on $\mathcal{A}$, p_0, such that

$$\alpha_{x_0}([I_{\mathcal{A}} \otimes x'](\mathsf{T})) = |x'(x_0)| \leq p_0([I_{\mathcal{A}} \otimes x'](\mathsf{T})).$$

By the Hahn-Banach Theorem, there is a linear extension $\tilde{\alpha}_{x_0} : \mathcal{A} \longrightarrow \mathbb{K}$ such that $\tilde{\alpha}_{x_0}(a) \leq p_0(a)$ for all $a \in \mathcal{A}$. In particular $\tilde{\alpha}_{x_0} \in \mathcal{A}'$ and we have

$$\begin{aligned} x'(F(\tilde{\alpha}_{x_0})) &= x'([\tilde{\alpha}_{x_0} \otimes I_E](\mathsf{T})) = \tilde{\alpha}_{x_0}([I_E \otimes x'](\mathsf{T})) \\ &= \alpha_{x_0}([I_E \otimes x'](\mathsf{T})) = x'(x_0). \end{aligned}$$

Since this is true for all $x' \in E'$, the Hahn-Banach Theorem implies $x_0 = F(\tilde{\alpha}_{x_0})$.
Let us now consider the set $B_{p_0} = \{a \in \mathcal{A} : p_0(a) \leq 1\}$ and its polar set $B_{p_0}^{\circ}$ of those $\alpha \in \mathcal{A}'$ such that $|\alpha(a)| \leq 1$ for all $a \in B_{p_0}$. Clearly $\tilde{\alpha}_{x_0} \in B_{p_0}^{\circ}$, and $x_0 \in F(B_{p_0}^{\circ})$. From this we have

$$\sigma_H^{left}(\mathsf{T}) \subseteq F(B_{p_0}^{\circ}).$$

By the Alaoglu-Bourbaki Theorem (see e.g. [44], Section 20) $B_{p_0}^{\circ}$ is compact with the $\sigma(\mathcal{A}', \mathcal{A})$ topology and equicontinuous. Since F is continuous on the equicontinuous subsets of $\mathcal{A}'$, $F(B_{p_0}^{\circ})$ is compact in E. Hence $\sigma_H^{left}(\mathsf{T})$ is a closed set contained in the compact set $F(B_{p_0}^{\circ})$ and $\sigma_H^{left}(\mathsf{T})$ is compact.
q.e.d.

Chapter 4

Cotype 2 estimates for spaces of polynomials on sequence spaces

4.1 Introduction

The study of type and cotype of Banach spaces started in the early 1970's, but its origins go back to the 1930's. W. Orlicz, while studying the unconditional convergence of a series of functions using Khinchin's inequality established in [60] the first type-cotype style inequality. In 1968 Kahane proved his generalization of Khinchin's inequality and these ideas were revisited and used for the study of the relations of strong p-summability and unconditional summability. But in 1972 Kwapień proved in [46] that Hilbert spaces are the only Banach spaces that simultaneously have type 2 and cotype 2, although he did not explicitly use those names. Shortly after that achievement the concepts of type and cotype were formulated and widely used in the study of limit theorems of probability, martingales in superreflexive spaces or the connections between the geometry of Banach spaces and the behaviour of random variables.
In 1995 Seán Dineen showed (see [16] and [17] Proposition 1.54) that if E is an infinite-dimensional Banach space, then ℓ_∞ is finitely representable in $\mathcal{P}(^m E)$ for all $m \geq 2$. This in particular means that $\mathcal{P}(^m E)$ does not have cotype 2. If X is a Banach sequence space (for example ℓ_p with $1 \leq p \leq \infty$) and we denote by X_n the subspace spanned by the first e_k vectors, $k = 1, \ldots, n$, this implies that the sequence $(\mathbf{C}_2(\mathcal{P}(^m X_n)))_n$ must tend to ∞. Our goal in this chapter is to give asymptotical descriptions of this divergence.

4.2 Banach spaces

4.2.1 Type and cotype

We begin by giving the definitions of type p and cotype q for arbitrary Banach spaces. These two concepts have been widely studied and there is a big literature on them. A careful study can be found in [11], [14], [75]. We shall see later how these sit in a more general frame. All through this chapter r_k will always denote the classical k-th Rademacher function defined as follows

Definition 4.2.1 The first Rademacher function, $r_1 : [0,1] \longrightarrow \mathbb{R}$, is defined as $r_1(t) = 1$ if $t \in [0, 1/2]$ and $r_1(t) = -1$ if $t \in]1/2, 1]$. From this, the *k-th Rademacher function* is defined by $r_k(t) = r_1(2^k(t-(j-1)2^{-k}))$ if $t \in](j-1)2^{-k}, j2^{-k}]$ for $j = 1, \ldots, 2^k$ and $r_k(0) = 1$.

What we do for the k-th function is divide the interval $[0,1]$ in subintervals of length $2^{-(k+1)}$ and the function takes alternatively in each subinterval the values 1 and -1.

Definition 4.2.2 Let E be any Banach space; it is said to have *type p* if there exist a finite constant $\kappa > 0$ such that for any finite choice of vectors $x_1, \ldots, x_n \in E$,

$$\left(\int_0^1 \| \sum_{k=1}^n r_k(t) x_k \|^2 dt \right)^{1/2} \leq \kappa \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p},$$

The best constant in this inequality is called the *type p constant* of E and is denoted by $\mathbf{T}_p(E)$.

Definition 4.2.3 Let E be any Banach space; it is said to have *cotype q* if there exist a finite constant $\kappa \geq 0$ such that for any finite choice $x_1, \ldots, x_n \in E$,

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leq \kappa \left(\int_0^1 \| \sum_{k=1}^n r_k(t) x_k \|^2 dt \right)^{1/2}.$$

To cover the case $q = \infty$ we have to consider in the right hand side, $\max_{k=1,\ldots,n} \|x_k\|$. Then we define the *cotype q constant* of E, as the best constant satisfying the previous inequality and denote it by $\mathbf{C}_q(E)$.

It is well known that every Banach space has type p for any $0 < p \leq 1$ and cotype ∞ and that no Banach space (apart from the trivial spaces) has type p for $p > 2$ or cotype q with $q < 2$ (see [14], Remarks 11.5 (c) and (d)). Therefore the only interesting cases are type p for $1 \leq p \leq 2$ and cotype q for $2 \leq q \leq \infty$.
A Banach space has both type 2 and cotype 2 if and only if it is a Hilbert space. In this case $\mathbf{T}_2(H) = \mathbf{C}_2(H) = 1$ (see [14], Corollary 11.8).

It is also well known that a Banach space has the same type and cotype as its bidual ([14], Corollary 11.9). If E has type p, then its dual E' has cotype p' ($\frac{1}{p}+\frac{1}{p'}=1$) and $\mathbf{C}_{p'}(E') \leq \mathbf{T}_p(E)$ ([14], Proposition 11.10).
On the other hand, ℓ_p has type p and cotype 2 for all $1 \leq p \leq 2$ and type 2 and cotype p when $2 \leq q < \infty$ ([14], Remark 11.5, (g)).

4.2.2 p-summing operators

The classes of p-summing operators were introduced by Pietsch in [61], although some particular cases had been studied before by Grothendieck. A good and detailed study can be found in [14], Chapter 2 and in [11], Section 11. Here we only present some basic definitions and facts. A natural generalization of p-summing operators are the (Y, X)-summing operators, used in [52], that will be presented and very useful later on.

Definition 4.2.4 Let E and F be two Banach spaces and $1 \leq p < \infty$. A linear operator $T : E \longrightarrow F$ is *p-summing* if there is some constant $\kappa > 0$ such that for any finite choice of vectors $x_1, \ldots, x_n \in E$,

$$\left(\sum_{k=1}^{n} \|Tx_k\|_F^p\right)^{1/p} \leq \kappa \sup_{\|x'\|_{E'}\leq 1} \left(\sum_{k=1}^{n} |<x', x_k>|^p\right)^{1/p}.$$

The best constant in this inequality is denoted by $\pi_p(T)$. The space of all p-summing operators between E and F is denoted by $\Pi_p(E, F)$.

It is well known that for each T, we have $\|T\| \leq \pi_p(T)$ for all p. Also, $(\Pi_p(E, F), \pi_p)$ is a Banach space for all p and (Π_p, π_p) is an injective Banach operator ideal (see [61], also [11] Section 11, and [14] 2.4, 2.5).
Useful characterizations can be given in terms of associated operators. First of all, a sequence $(x_n)_n \subseteq E$ is said to be *strongly p-summable* if the scalar sequence $(\|x_n\|_E)_n$ is in ℓ_p. The space of such sequences is denoted by $\ell_p^{\text{strong}}(E)$ and a norm with which it becomes a Banach space is defined by

$$\|(x_n)_n\|_p^{\text{strong}} = \left(\sum_{n=1}^{\infty} \|x_n\|_E^p\right)^{1/p}.$$

Now, a sequence $(x_n)_n \subseteq E$ is said to be *weakly p-summable* if for all $x' \in E'$, we have that $(<x', x_n>)_n \in \ell_p$. The space of all such sequences is denoted by $\ell_p^{\text{weak}}(E)$; with the norm

$$\|(x_n)_n\|_p^{\text{weak}} = \sup_{\|x'\|_{E'}\leq 1} \left(\sum_{n=1}^{\infty} |<x', x_n>|^p\right)^{1/p}$$

it is a Banach space (see [14], Chapter 2). Now, if $T \in \mathcal{L}(E; F)$, a correspondence between sequences, $\hat{T}$, can be defined by doing $\hat{T}((x_n)_n) = (Tx_n)_n$. Then it is well known (see [61],

also [14], Proposition 2.1) that T is p-summing if and only if $\hat{T} : \ell_p^{\text{weak}}(E) \longrightarrow \ell_p^{\text{strong}}(E)$ is well defined and, in this case, $\pi_p(T) = \|\hat{T} : \ell_p^{\text{weak}}(E) \to \ell_p^{\text{strong}}(E)\|$ (the operator norm).
We have another useful characterization in terms of tensor products. There is a natural embedding $\ell_p \otimes E \hookrightarrow \ell_p^{\text{strong}}(E)$ given by $(\xi_n)_n \otimes x \mapsto ((\xi_n x)_n$. This induces a norm on $\ell_p \otimes E$, denoted Δ_p satisfying

$$\|\sum_{k=1}^{n} e_k \otimes x_k\|_{\Delta_p} = \left(\sum_{n=1}^{\infty} \|x_n\|_E^p\right)^{1/p}.$$

Note that Δ_p is just a norm, not a tensor norm (see [11], 12.1), since it does not satisfy the metric mapping property. With this identification, $T \in \mathcal{L}(E;F)$ is p-summing if and only if $id \otimes T : \ell_p \otimes_\varepsilon E \longrightarrow \ell_p \otimes_{\Delta_p} E$ is continuous. In this case, $\pi_p(T) = \|id \otimes T : \ell_p \otimes_\varepsilon E \to \ell_p \otimes_{\Delta_p} E\|$ (see [11], Section 11).
We will see that these properties can be transferred to the setting of (Y,X)-summing operators.
We end this section with two well known facts. Firstly, if E has cotype 2, then $\Pi_1(E,F) = \Pi_2(E,F)$ holds isometrically for all Banach space F (see e.g. [14] Corollary 11.16). Secondly, if E is a normed space with $\dim(E) = n$, then (see e.g. [75] Proposition 9.11)

$$\pi_2(id_E) = \sqrt{n}. \tag{4.1}$$

4.2.3 The l norm of an operator

Definition 4.2.5 Let E be any Banach space and $T \in \mathcal{L}(\ell_2^n; E)$. Then for independent Gaussian random variables $g_1, \dots, g_n$ on a probability space (Ω, Σ, μ) the *l-norm* of T is defined to be

$$l(T) = \left(\int_\Omega \|\sum_j g_j T(e_j)\|_E^2 \, d\mu\right)^{1/2}.$$

This definition is independent from the choice of the orthonormal basis in ℓ_2^n and of the random variables $(g_j)_j$ (see e.g. [75], Section 12).

Relations between Gaussian and Rademacher averages

So far we have used averages involving both Rademacher functions and independent Gaussian random variables. These ideas are closely related and their relation is studied in e.g. [75] , Section 4.

Let E be any Banach space. Take $r_1, \dots, r_n$ the Rademacher functions and $g_1, \dots, g_n$ independent Gaussian random variables on a probability space (Ω, Σ, μ). There is a universal constant $c > 0$ such that for any finite choice $x_1, \dots, x_n \in E$,

$$\left(\int_0^1 \| \sum_j r_j(t) x_j \|_E^2 \, dt \right)^{1/2} \leq c \left(\int_\Omega \| \sum_j g_j x_j \|_E^2 \, d\mu \right)^{1/2} . \tag{4.2}$$

The converse situation is not so nice. Nevertheless, we get a fairly good result. If E is a Banach space of cotype q, then there is a universal constant $c > 0$ such that for every finite choice $x_1, \dots, x_n \in E$ we have

$$\left(\int_\Omega \| \sum_j g_j x_j \|_E^2 \, d\mu \right)^{1/2} \leq c\sqrt{q}\mathbf{C}_q(E) \left(\int_0^1 \| \sum_j r_j(t) x_j \|_E^2 \, dt \right)^{1/2} . \tag{4.3}$$

If E is an n-dimensional Banach space,

$$\left(\int_\Omega \| \sum_j g_j x_j \|_E^2 \, d\mu \right)^{1/2} \leq c\sqrt{\log(n+1)} \left(\int_0^1 \| \sum_j r_j(t) x_j \|_E^2 \, dt \right)^{1/2} \tag{4.4}$$

for all finite $x_1, \dots, x_n \in E$. We will use these inequalities several times along this chapter.

It is also of interest to study the relations between Rademacher and Gaussian averages of different order. The next inequality was proved by Kahane during the 1960's and is now classical; it can be found in, e.g., [14], Chapter 11, [48], Theorem 1.e.13 or [75], (4.7).
For $0 < p, q < \infty$ there is a universal constant $K_{p,q} > 0$ such that for all Banach space E and all finite choice $x_1, \dots, x_n \in E$ the following holds,

$$\left(\int_0^1 \| \sum_j r_j(t) x_j \|_E^p \, dt \right)^{1/p} \leq K_{p,q} \left(\int_0^1 \| \sum_j r_j(t) x_j \|_E^q \, dt \right)^{1/q} . \tag{4.5}$$

This, in particular, means that in the definition of type and cotype, any exponent can be taken in the integrals instead of 2. This would give equivalent definitions, at the only expense of rearranging the constants by some universal factor.
An analog inequality for Gaussian averages, due to Hoffmann-Jørgensen, can be found in e.g. [75], (4.8). Let $0 < q < p < \infty$. There exists a constant $\tilde{K}_{p,q} > 0$ such that for every Banach space E and every finite choice $x_1, \dots, x_n \in E$,

$$\left(\int_\Omega \| \sum_j g_j x_j \|_E^p \, d\mu \right)^{1/p} \leq \tilde{K}_{p,q} \left(\int_\Omega \| \sum_j g_j x_j \|_E^q \, d\mu \right)^{1/q} . \tag{4.6}$$

As before, this means that if we define the l_p-norm of an operator as the corresponding integral with Gaussian random variables and exponent p, then this norm is equivalent to the l-norm that has been already defined.

Chevét's inequality

This is an inequality that relates the l-norm of a tensor product of operators with their respective l-norms and operator norms. It was proved by Chevét in 1977 and can be found in [75], (43.2). For any two operators $T \in \mathcal{L}(\ell_2^n; E)$ and $S \in \mathcal{L}(\ell_2^m; F)$,

$$l(T \otimes S : \ell_2^n \otimes_2 \ell_2^m \to E \otimes_\varepsilon F) \le c(\|T\| l(S) + l(T)\|S\|), \tag{4.7}$$

being $c > 0$ a universal constant. In [10], Lemma 6, it is shown by induction of Chevét's inequality that for any $T \in \mathcal{L}(\ell_2^n; E)$ and $m \ge 2$,

$$l(\otimes^m T : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m E) \le c_m l(T)\|T\|^{m-1}, \tag{4.8}$$

being $c_m > 0$ a constant depending only on m.

4.3 Banach sequence spaces

4.3.1 Definitions

Symmetric Banach sequence spaces

From now on, X will denote a real or complex Banach space of functions $\phi : \mathcal{J} \longrightarrow \mathbb{K}$, where $\mathbb{K}$ is either the real or the complex field and $\mathcal{J}$ is a countable or finite set, such that
(i) If $|\psi(j)| \le |\phi(j)|$ for all $j \in \mathcal{J}$ and $\phi \in X$, then $\psi \in X$ and $\|\psi\| \le \|\phi\|$,
(ii) for all finite $\mathcal{I} \subseteq \mathcal{J}$, the characteristic function $\chi_{\mathcal{I}}$ belongs to X.
In other words, X will be a real or complex Köthe function space over $(\mathcal{J}, \Sigma, \mu)$, where μ is the counting measure on $\mathcal{J}$ (see [48], Definition 1.b.17). We say that X is a Banach sequence space whenever $\mathcal{J} = \mathbb{N}$ and $\ell_1 \hookrightarrow X \hookrightarrow \ell_\infty$ with embeddings of norm 1. In particular, all $e_n = (\delta_{nk})_k \in X$ and $\|e_n\| = 1$.
If X is a Banach sequence space, for each $\xi \in X$, the decreasing rearrangement of ξ, denoted $(\xi_n^\star)_{n\in\mathbb{N}}$, is defined by

$$\xi_n^\star := \inf\{ \sup_{i \in \mathbb{N}\setminus J} |\xi_i| \ : \ J \subseteq \mathbb{N} \ , \ \operatorname{card}(J) < n\}.$$

A Banach sequence space is *symmetric* if for every $\xi \in X$, $\|(\xi_n)_n\|_X = \|(\xi_n^\star)_n\|_X$. This definition is due to Schatten. It is equivalent to the fact that the norm is invariant under reordering or multiplication by absolute value 1 scalars (i.e., changing of the signs of the elements of the sequence in the real case or multiplication by $e^{i\theta_n}$ in the complex case); see [71], Chapter 1. From this we have that, for every $n \in \mathbb{N}$,

$$\| \sum_{i=1}^n \varepsilon_i \mu_i e_{\pi(i)} \| = \| \sum_{i=1}^n \mu_i e_i \|.$$

for all $(\varepsilon_i)_i$ with $|\varepsilon_i| = 1$, all scalars $(\mu_i)_i$ and all permutations π of $\{1, \ldots, n\}$ (the group of which we denote by Σ_n). Consider the operator $\phi : \ell_\infty^n \longrightarrow X$ given by $\phi(\lambda) = \sum_{i=1}^n \lambda_i \mu_i e_i$; then

$$\|\phi\| = \sup_{\|\lambda\|_\infty \leq 1} \| \sum_{i=1}^n \lambda_i \mu_i e_i \| = \sup_{|\varepsilon_i|=1} \| \sum_{i=1}^n \varepsilon_i \mu_i e_i \| = \| \sum_{i=1}^n \mu_i e_i \|.$$

This shows that if $W \subseteq \{1, \ldots, n\}$, given any $\mu_1, \ldots, \mu_n \in \mathbb{K}$, any $\pi \in \Sigma_n$ and any $|\varepsilon_i| = 1$ for $i \in W$, just by defining $\lambda_i = \varepsilon_i$ when $i \in W$ and 0 otherwise, we have

$$\| \sum_{i \in W} \varepsilon_i \mu_i e_{\pi(i)} \| \leq \| \sum_{i=1}^n \mu_i e_i \|.$$

Now, for each $n \in \mathbb{N}$ we define the space $X_n = \operatorname{span}\{e_1, \ldots, e_n\}$.

Following standard notation we define the *fundamental function* of X by $\lambda_X(n) = \| \sum_{k=1}^n e_k \|_X$ for $n \in \mathbb{N}$. Applying [47], 3.a.6, we have that $\| \sum_{i=1}^n e_i \|_{X_n} \| \sum_{i=1}^n e_i \|_{X_n'} = n$; in other words,

$$\lambda_{X_n}(n) \lambda_{X_n'}(n) = n. \tag{4.9}$$

Convexity and concavity

These are two concepts that are crucial in the theory of Banach lattices. They are closely related to those of type and cotype and have been widely studied; see e.g. [48], where all the results that are mentioned here can be found. Although all this is defined for general Banach lattices we only present it here for our Köthe function space setting.

Definition 4.3.1 Let X be a Köthe function space modeled on a countable or finite set; then X is *r-convex* (with $1 \leq r < \infty$) if there is a constant $\kappa > 0$ such that, for any finite choice $\xi_1, \ldots, \xi_n \in X$,

$$\| \left(\sum_{k=1}^n |\xi_k|^r \right)^{1/r} \|_X \leq \kappa \left(\sum_{k=1}^n \|\xi_k\|_X^r \right)^{1/r}.$$

For $r = \infty$,

$$\| \max_{k=1,\ldots,n} |\xi_k| \, \|_X \leq \kappa \max_{k=1,\ldots,n} \|\xi_k\|_X.$$

The *r-convexity constant* of X is defined to be the inifimum of all possible values of κ and is denoted by $\mathbf{M}^{(r)}(X)$.

Definition 4.3.2 We say that a Köthe function space modeled on a countable or finite set is *s-concave* (with $1 \leq s < \infty$) if there is a constant $\kappa > 0$ such that, for any finite choice $\xi_1, \ldots, \xi_n \in X$,

$$\left(\sum_{k=1}^{n} \|\xi_k\|_X^s \right)^{1/s} \leq \kappa \; \| \left(\sum_{k=1}^{n} |\xi_k|^s \right)^{1/s} \|_X .$$

For the case $s = \infty$,

$$\max_{k=1,\ldots,n} \|\xi_k\|_X \leq \kappa \; \| \max_{k=1,\ldots,n} |\xi_k| \; \|_X .$$

The *s-concavity constant* of X, $\mathbf{M}_{(s)}(X)$, is defined as the infimum over all possible values of κ.

When X is r-convex (s-concave) for some finite r (resp. s) we will say that it is *non-trivially convex* (concave) or that it has *non-trivial convexity* (concavity).

Some basic facts concerning convexity and concavity that will be used later are that X is r-convex if and only if $X^\times$ is s-concave (with $\frac{1}{r} + \frac{1}{s} = 1$) and $\mathbf{M}_{(s)}(X^\times) = \mathbf{M}^{(r)}(X)$; also X is s-concave if and only if $X^\times$ is r-convex and $\mathbf{M}^{(s)}(X^\times) = \mathbf{M}_{(r)}(X)$ ([48], Proposition 1.d.4).
Clearly, if X is r-convex, then it is r_1-convex for all $r_1 \leq r$; and if it is s-concave, then it is s_1-concave for all $s_1 \geq s$. Every space is 1-convex and ∞-concave.
It is also well known that a s-concave Banach sequence space with $s \geq 2$ has cotype s (and $\mathbf{C}_s(X) \leq \mathbf{M}_{(s)}(X) A_1^{-1}$, where A_1^{-1} comes from the Khinchin inequality). Also, an r-convex Banach sequence space with $1 < r \leq 2$ which is also non-trivially concave has type r ([48], Proposition 1.f.3). Conversely, if X has type (cotype) p for some $1 < p < \infty$, then it is r-convex (s-concave) for all $1 < r < p < s < \infty$. Furthermore, X has cotype 2 if and only if it is 2-concave. On the other hand, X has type 2 if and only if it is 2-convex and has non-trivial concavity.

4.3.2 Examples

Before giving examples, let us introduce some notation. Given any two real sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$, we write

$$(a_n) \prec (b_n)$$

whenever there exist a constant $K > 0$ such that, for all $n \in \mathbb{N}$,

$$a_n \leq K \; b_n .$$

If $(a_n) \prec (b_n)$ and $(b_n) \prec (a_n)$ we write $(a_n) \asymp (b_n)$.

ℓ_p spaces

The first immediate example of symmetric Banach sequence spaces are the ℓ_p spaces. It is well known that if $1 \leq p \leq 2$, ℓ_p has type p and cotype 2 and if $2 \leq p < \infty$, it has type 2 and cotype p. The space ℓ_∞ is 2-convex. When we consider the n-th dimensional spaces we have that their cotype 2 constants behave in the following way (see [62]),

$$\mathbf{C}_2(\ell_p^n) \asymp \begin{cases} 1 & \text{if} \quad 1 \leq q \leq 2 \\ n^{\frac{1}{2}-\frac{1}{p}} & \text{if} \quad 2 \leq q < \infty \\ \frac{n^{1/2}}{\sqrt{\log(n+1)}} & \text{if} \quad p = \infty \end{cases}$$

Orlicz spaces

This is another natural example of symmetric Banach sequence spaces (see [47], Chapter 4, for definitions and properties). Orlicz spaces are a natural generalization of the ℓ_p spaces and are defined as follows. We call *Orlicz function* to any continuous, non-decreasing, convex function $\varphi : [0,\infty[\longrightarrow [0,\infty[$ such that $\varphi(0) = 0$, $\varphi(t) \neq 0$ for all $t \neq 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. The associated *Orlicz space* is defined as

$$\ell_\varphi = \{\xi = (\xi_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} \ : \ \exists \rho > 0 \ , \ \sum_{n=1}^{\infty} \varphi\left(\frac{|\xi_n|}{\rho}\right) < \infty\}.$$

With the norm

$$\|\xi\| = \inf\{\rho > 0 \ : \ \sum_{n=1}^{\infty} \varphi\left(\frac{|\xi_n|}{\rho}\right) \leq 1\}$$

ℓ_φ is a symmetric Banach sequence space. Note that taking $\varphi(t) = t^p$, we have $\ell_\varphi = \ell_p$.
An Orlicz function satisfies the Δ_2 condition if

$$\limsup_{t\to 0} \frac{\varphi(2t)}{\varphi(t)} < \infty.$$

Equivalently, if there is a constant $C > 0$ such that $\varphi(2t) \leq C\varphi(t)$ for all $t \geq 0$.
It is well known that $\lambda_{\ell_\varphi}(n) = \frac{1}{\varphi^{-1}(1/n)}$ for all $n \in \mathbb{N}$.
It is known (see [40], Corollary 13 and Corollary 15) that ℓ_φ is s-concave ($2 \leq s < \infty$) if and only if there is a constant $K > 0$ such that $\varphi(\lambda t) \geq K\lambda^s\varphi(t)$ for all $0 \leq \lambda, t \leq 1$. On the other hand, ℓ_φ is r-convex ($1 < r \leq 2$) if and only if φ satisfies the Δ_2 condition and $\varphi(\lambda t) \leq K\lambda^r\varphi(t)$ for all $0 \leq \lambda, t \leq 1$ and some $K > 0$. The fact that φ satisfies the Δ_2 condition guarantees that ℓ_φ has non-trivial concavity (see [40], Proposition 7).

Lorentz spaces

Let $1 \leq p < \infty$ and $w = (w_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ non-increasing such that $w_1 = 1$, $\lim_n w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$. We define the corresponding *Lorentz space*, denoted by $d(w,p)$, to be the space of all sequences $(\xi_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ such that

$$\|\xi\| = \sup_{\pi\in\Sigma_{\mathbb{N}}} \left(\sum_{n=1}^{\infty} |\xi_{\pi(n)}|^p w_n \right)^{1/p} = \left(\sum_{n=1}^{\infty} |\xi_n^\star|^p w_n \right)^{1/p}.$$

Obviously, this is a symmetric Banach sequence space. It is also clear that $\lambda_{d(w,p)}(n) = (\sum_{k=1}^{n} w_k)^{1/p}$.
Concerning the case of when is it convex or concave, in [65] can be found that $d(w,p)$ is always p-convex, with $\mathbf{M}_{(p)}(d(w,p)) = 1$, and it is not r-convex for $r > p$. For the concavity, we say that w is p-regular if $w_n^p \asymp \frac{1}{n}\sum_{i=1}^{n} w_i^p$. Then ([65], Theorem 2), for $p < s < \infty$, $d(w,p)$ is s-concave if and only if w is $\frac{q}{p}$-regular, with $\frac{1}{q} = \frac{1}{p} - \frac{1}{s}$. The space $d(w,p)$ has non-trivial concavity if and only if w is 1-regular.

$\ell_{p,q}$ spaces

These spaces are also sometimes called Lorentz spaces. As we will see, in some circumstances the previous ones can be realized as a particular case of these. For the basics on definitions and properties, see [63]. Given $1 < p < \infty$ and $1 \leq q < \infty$, the space $\ell_{p,q}$ is that of those $\xi = (\xi_n)_n \subseteq \mathbb{K}$ such that

$$\|\xi\|_{p,q} = \left(\sum_{n=1}^{\infty} \left(\xi_n^\star \, n^{\frac{1}{p}-\frac{1}{q}} \right)^q \right)^{1/q} < \infty.$$

With this norm, $\ell_{p,q}$ is a symmetric Banach sequence space. It clearly satisfies, $\lambda_{\ell_{p,q}}(n) = n^{1/p}$. Also, if we take $w_n = n^{\frac{q}{p}-1}$ for each n, if $p \geq q$, then $\ell_{p,q} = d(w,q)$. Therefore, there are still some cases that are not covered by the previous result.
Regarding the concavity and convexity of these spaces, it is known (see [9]) that $\ell_{p,q}$ is r-convex if and only if $r < p$, $r \leq q$ and it is s-concave if and only if $p < s$, $q \leq s$.

4.3.3 Spaces of X-summable sequences

Definition and first properties

Definition 4.3.3 Let X be a Köthe sequence space modeled on $\mathcal{J}$, countable or finite, and E any Banach space. Then, the space

$$X(E) = \{(x_j)_{j\in\mathcal{J}} \subseteq E : (\|x_j\|_E)_{j\in\mathcal{J}} \in X\}$$

with the norm $\|x\|_{X(E)} = \|(\|x_j\|_E)_{j\in\mathcal{J}}\|_X$ is a Banach space.

Following [52], when $\mathcal{J} = \mathbb{N}$, we say that a sequence $x = (x_n)_{n\in\mathbb{N}} \subseteq E$ is *strongly X-summable* when $x \in X(E)$.

Now for X such a Köthe function space and E any Banach space we can embed $X \otimes E \hookrightarrow X(E)$ by means of $(\zeta \otimes x) \mapsto (\zeta_j x)_{j\in\mathcal{J}}$. With this, $X(E)$ induces a norm in $X \otimes E$ (we denote $X \otimes_X E$) satisfying that for each $\mathcal{I} \subseteq \mathcal{J}$ finite

$$\|\sum_{k\in\mathcal{I}} e_k \otimes x_k\|_X = \|(x_k)_{k\in\mathcal{I}}\|_{X(E)} = \|(\|x_k\|_E)_{k\in\mathcal{I}}\|_X.$$

Lemma 4.3.4
$\|\cdot\|_X \leq \pi$.
Proof.
Consider the bilinear mapping $X \times E \longrightarrow X(E)$ defined by $(\zeta, x) \mapsto (\zeta_j x)_j$; whose linearization is obviously the inclusion $X \otimes E \hookrightarrow X(E)$ and let us see that it is continuous,

$$\|(\zeta_j x)_j\|_{X(E)} = \|(\|\zeta_j x\|_E)_j\|_X = \|(\zeta_n)_j \|x\|_E\|_X = \|(\zeta_n)_j\|_X \cdot \|x\|_E.$$

Then $X \otimes_\pi E \hookrightarrow X(E)$ is continuous and $\|\cdot\|_X \leq \pi$.
q.e.d.

Remark 4.3.5
Given E and F any two Banach spaces, if $E_0 \subseteq E$ is dense, then $E_0 \otimes F$ is dense in $E \otimes_\alpha F$ for any norm $\alpha \leq \pi$. Indeed, for the π case let $z \in E \otimes F$ and take any representation $z = \sum_{i=1}^n x_i \otimes y_i$. For each $i = 1, \ldots, n$ there is some sequence $(x_i(k))_{k\in\mathbb{N}} \subseteq E_0$ converging to x_i. Let $\varepsilon > 0$ and take $k_0 \in \mathbb{N}$ such that $\|x_i^k - x_i\| < \frac{\varepsilon}{n \sup \|y_i\|}$ for all $k \geq k_0$ and all $i = 1, \ldots, n$. Hence, for $k \geq k_0$,

$$\begin{aligned} &\|\sum_{i=1}^n x_i \otimes y_i - \sum_{i=1}^n x_i^k \otimes y_i\|_\pi = \|\sum_{i=1}^n (x_i - x_i^k) \otimes y_i\|_\pi \\ \leq\ &\sum_{i=1}^n \|x_i - x_i^k\| \cdot \|y_i\| \leq \sup_i \|y_i\| \sum_{i=1}^n \|x_i - x_i^k\| < \varepsilon. \end{aligned}$$

Therefore $E_0 \otimes F$ is dense in $E \otimes_\pi F$. For $\alpha \leq \pi$, given any $z = E \otimes F$, there exist some sequence $(z_n)_{n\in\mathbb{N}} \subseteq E_0 \otimes F$ such that $z_n \stackrel{\pi}{\longrightarrow} z$. Since $\alpha \leq \pi$, this implies that $z_n \stackrel{\alpha}{\longrightarrow} z$.

If $\{e_k\}_k$ form a basis of X, then $\bigcup_{n\in\mathbb{N}} X_n$ is dense in X. Remark 4.3.5 implies that in order to check that some mapping is continuous in $X \otimes_X E$, it is enough to check it in $\bigcup_{n\in\mathbb{N}} X_n \otimes E$ or, what is the same, for tensors of the form $\sum_{k=1}^n e_k \otimes x_k$.

Lemma 4.3.6
Let X be such that $\{e_k\}_k$ form a basis of X; then $\varepsilon \leq \|\cdot\|_X$.

Proof.
Let us see that the identity mapping $X \otimes_X E \longrightarrow X \otimes_\varepsilon E$ is continuous. We have

$$\begin{aligned} \|\sum_{k=1}^n e_k \otimes x_k\|_\varepsilon &= \sup_{\|x'\|_{E'}\leq 1} \sup_{\|\psi\|_{X'}\leq 1} |(\psi \otimes x')(\sum_{k=1}^n e_k \otimes x_k)| \\ = \sup_{\|x'\|_{E'}\leq 1} \sup_{\|\psi\|_{X'}\leq 1} |\psi(\sum_{k=1}^n e_k x'(x_k))| &= \sup_{\|x'\|_{E'}\leq 1} \|\sum_{k=1}^n e_k x'(x_k)\|_X \\ = \sup_{\|x'\|_{E'}\leq 1} \|(x'(x_k))_{k=1}^n\|_X. \end{aligned}$$

For each $\|x'\|_{E'} \leq 1$ we have $|x'(x_k)| \leq \|x'\|_{E'} \cdot \|x_k\|_E \leq \|x_k\|_E$. Then $\|(x'(x_k))_{k=1}^n\|_X \leq \|(\|x_k\|_E)_{k=1}^n\|_X$. Hence

$$\|\sum_{k=1}^n e_k \otimes x_k\|_\varepsilon = \sup_{\|x'\|_{E'}\leq 1} \|(x'(x_k))_{k=1}^n\|_X \leq \|(\|x_k\|_E)_{k=1}^n\|_X = \|\sum_{k=1}^n e_k \otimes x_k\|_X.$$

This proves our claim.
q.e.d.

If X, Y are Köthe function spaces modeled on $\mathcal{J}$ and $\mathcal{K}$ (each of them finite or countable) respectively we can then consider the space,

$$Y(X) = \{x = (x_j)_{j\in\mathcal{J}} : (\|x_j\|_X)_{j\in\mathcal{J}} \in Y\}.$$

This can be regarded as a Köthe function space over $\mathcal{J} \times \mathcal{K}$ in the following way. Let us write $x_j = (x_j(k))_{k\in\mathcal{K}} \in X$. We can identify $x = (x_j)_j$ with a mapping

$$\begin{array}{rccc} \phi: & \mathcal{J} \times \mathcal{K} & \longrightarrow & \mathbb{K} \\ & (j,k) & \mapsto & \phi(j,k) = x_j(k) \end{array}$$

And

$$Y(X) = \{\phi : \mathcal{J} \times \mathcal{K} \to \mathbb{K} : (\phi(j,k))_k \in X \ \forall j \in \mathcal{J}, \ (\|(\phi(j,k))_k\|_X)_j \in Y\}.$$

With the norm $\|\phi\| = \|(\|(\phi(j,k))_k\|_X)_j\|_Y$ this becomes a Köthe function space space. Indeed, if $|\psi(j,k)| \leq |\phi(j,k)|$ for all (j,k), being X is a Banach sequence space implies $\|(\psi(j,k))_k\|_X \leq \|(\phi(j,k))_k\|_X$ for all $j \in \mathcal{J}$. Now Y is also a Köthe function space, then $\|\psi\| \leq \|\phi\|$. Let $\mathcal{I} \subseteq \mathcal{J} \times \mathcal{K}$ be finite. Clearly, $\chi_\mathcal{I} \in Y(X)$. Thus, we can look at $Y(X)$ as a Köthe function space modeled on $\mathcal{J} \times \mathcal{K}$. When $X = Y$ is a Banach sequence space this process can be iterated in two different ways.

Iteration of the process

Let X be a Köthe function space modeled on $\mathcal{J}$, finite or countable and let us define $[X]^2$ to be the space of functions $\phi : \mathcal{J} \times \mathcal{J} \to \mathbb{K}$ such that $(\phi(i,j))_j \in X$ for all i and $(\|(\phi(i,j))_j\|_X)_i \in X$. With the norm $\|\phi\|_{[X]^2} = \|(\|(\phi(i,j))_j\|_X)_i\|_X$, we have that $[X]^2$ is a Köthe function space modeled on $\mathcal{J}^2$. Note that $[X]^2 \cong X(X)$ holds isometrically.
Suppose that $[X]^{m-1}$ has been defined and let us define $[X]^m$ as those functions $\phi : \mathcal{J}^m \to \mathbb{K}$ such that $(\phi(i_1, \dots, i_{m-1}, i_m))_{i_m \in \mathcal{J}} \in X$ for all $(i_1, \dots, i_{m-1}) \in \mathcal{J}^{m-1}$ and $(\|(\phi(i_1, \dots, i_{m-1}, i_m))_{i_m}\|_X)_{(i_1,\dots,i_{m-1})} \in [X]^{m-1}$. We endow it with the norm

$$\|\phi\|_{[X]^m} = \|(\|(\phi(i_1, \dots, i_{m-1}, i_m))_{i_m}\|_X)_{(i_1,\dots,i_{m-1})}\|_{[X]^{m-1}}.$$

With this norm $[X]^m$ is a Köthe function space modeled on $\mathcal{J}^m$ and $[X]^m \cong [X]^{m-1}(X)$. As we noted after Definition 4.3.3, we can embed $\otimes^m X \hookrightarrow [X]^m$. The situation gets particularly nice in the finite dimensional case, in which we have $\mathcal{J} = \{1, \dots, n\}$ and both spaces can be even identified. Let us consider the following mapping

$$\begin{array}{ccc} \otimes^m X_n & \longrightarrow & [X_n]^m \\ e_{i_1} \otimes \cdots \otimes e_{i_m} & \mapsto & \delta_{i_1,\dots,i_m} \end{array}$$

where $\delta_{i_1 \dots i_m} : (\{1, \dots, n\})^m \longrightarrow \mathbb{K}$ is given by $\delta_{i_1,\dots,i_m}(j_1, \dots, j_m) = 1$ if $(j_1, \dots, j_m) = (i_1, \dots, i_m)$ and 0 otherwise. This defines an isomorphism that allows us to make the algebraic identification $\otimes^m X_n \cong [X_n]^m$. This induces a norm from $[X_n]^m$ on $\otimes^m X_n$. We sometimes consider $\otimes^m X_n$ with this induced norm. Keeping in mind this identification we will simply use the notation $[X_n]^m$.
When $m = 2$ this norm is exactly the one that we had already defined. The space X_n has a basis and so also have all $[X_n]^m$. By Lemma 4.3.6 we have $\|[X_n]^2 \to X_n \otimes_\varepsilon X_n\| \leq 1$. Suppose that $\|[X_n]^{m-1} \to \otimes_\varepsilon^{m-1} X_n\| \leq 1$. Then

$$\begin{aligned} [X_n]^m &= [X_n]^{m-1}(X_n) = [X_n]^{m-1} \otimes_{[X_n]^{m-1}} X_n \to [X_n]^{m-1} \otimes_\varepsilon X_n \to \\ &\to (\otimes_\varepsilon^{m-1} X_n) \otimes_\varepsilon X_n = \otimes_\varepsilon^m X_n \end{aligned}$$

and $\|[X_n]^m \to \otimes_\varepsilon^m X_n\| \leq 1$.

We can also define $[X]_2$ as those functions $\phi : \mathcal{J} \times \mathcal{J} \to \mathbb{K}$ such that $(\phi(i,j))_i \in X$ for all j and $(\|(\phi(i,j))_i\|_X)_j \in X$. With the norm defined in the obvious way we have $[X]_2 \cong X(X)$ isometrically. Let us suppose that $[X]_{m-1}$ has been defined and define $[X]_m$ to be the space of all $\phi : \mathcal{J}^m \to \mathbb{K}$ such that $(\phi(i_1, \dots, i_{m-1}, i_m))_{(i_1,\dots,i_{m-1}) \in \mathcal{J}^{m-1}} \in [X]_{m-1}$ for all i_m and $(\|(\phi(i_1, \dots, i_{m-1}, i_m))_{(i_1,\dots,i_{m-1})}\|_{[X]_{m-1}})_{i_m} \in X$. We define the norm

$$\|\phi\|_{[X]_m} = \|(\|(\phi(i_1, \dots, i_{m-1}, i_m))_{(i_1,\dots,i_{m-1})}\|_{[X]_{m-1}})_{i_m}\|_X.$$

In this case, $[X]_m \cong X([X]_{m-1})$ holds isometrically. As before, we can identify $\otimes^m X_n \cong [X_n]_m$algebraically and, in this way, induce a topology on the tensor product. When we consider $\otimes^m X_n$ endowed with this topology we simply write $[X_n]_m$. We also have that $\|[X_n]_m \to \otimes_\varepsilon^m X_n\| \leq 1$.

4.3.4 The Köthe dual

The Köthe dual of a general Köthe function space is a well known object (see [48], [52]). We give here a definition adapted to our particular framework.

Definition 4.3.7 Let X be a Köthe function space modeled on a countable or finite set $\mathcal{J}$. The *Köthe dual* of X is the space

$$X^{\times} = \{\xi \in \mathbb{K}^{\mathcal{J}} : \xi\zeta \in \ell_1(\mathcal{J}) \text{ for all } \zeta \in X\}.$$

We define the norm $\|\xi\|_{X^\times} = \sup_{\|\zeta\|_X \leq 1} \|\zeta\xi\|_{\ell_1(\mathcal{J})}$.

This is again a Köthe function space modeled on $\mathcal{J}$. If X is a Banach sequence space, $X^\times$ is symmetric whenever X is so. The Köthe dual also satisfies that $(X_n)' = (X^\times)_n$. Indeed, given $\xi = (\xi_1, \ldots, \xi_n) \in (X_n)'$ we have

$$\|\xi\|_{X_n'} = \sup_{\|(\zeta_k)_{k=1}^n\|_{X_n} \leq 1} |\sum_{k=1}^n \xi_k \zeta_k| = \sup_{\|(\zeta_k)_{k=1}^\infty\|_X \leq 1} \|\xi\zeta\|_{\ell_1} = \|\xi\|_{(X^\times)_n}.$$

Example 4.3.8
It is well known in the literature that $(\ell_\infty)^\times = \ell_1$. The proof is very simple and we reproduce it here. If $(\xi_n)_n \in \ell_1$ and $(\zeta_n)_n \in \ell_\infty$ we have

$$\sum_{n=1}^N |\xi_n \zeta_n| \leq \|\zeta\|_\infty \sum_{n=1}^N |\xi_n| \leq \|\zeta\|_\infty \cdot \|\xi\|_1.$$

This holds for all N, hence $\xi\zeta \in \ell_1$ and $\xi \in (\ell_\infty)^\times$.
On the other hand, let $(\xi_n)_n$ be such that $\sum_{n=1}^\infty |\xi_n \zeta_n| < \infty$ for all $\zeta \in \ell_\infty$. Taking the sequence $\zeta_n = 1$ we have $\xi \in \ell_1$. This shows that the Köthe dual can be strictly smaller than the algebraic dual.

4.3.5 Some general facts

Let us now establish some basic facts on Banach sequence spaces that will be repeatedly used over the whole chapter. We begin with some basic remark on the behaviour of the l norm of the identity between ℓ_2^n and X_n, being X a Banach sequence space.

Remark 4.3.9
We will frequently use the fact that if X has non-trivial concavity, then

$$l(id : \ell_2^n \to X_n) \asymp \lambda_X(n). \tag{4.10}$$

Indeed, from (4.3), (4.5) and (4.6), we can take $r_1, \ldots, r_n$ the classical Rademacher functions to get

$$l(id : \ell_2^n \to X_n) \asymp \int_\Omega \|\sum_{k=1}^n g_k e_k\|_X \, d\mu \asymp \int_0^1 \|\sum_{k=1}^n r_k(t) e_k\|_X \, dt = \|\sum_{k=1}^n e_k\|_X.$$

The next property, concerning the cotype 2 constant of vector valued Köthe function spaces modeled on a countable or finite set, is well known in the literature. A proof of it can be found, e.g., in [55], Lemma 1.12. For the sake of completeness we give here an adapted proof.

Lemma 4.3.10
Let X be a 2-concave Köthe function space modeled on a countable or finite set $\mathcal{J}$ and E a Banach space with cotype 2. Then, $X(E)$ has cotype 2 and there is a universal constant K such that

$$\mathbf{C}_2(X(E)) \leq K\mathbf{M}_{(2)}(X)\mathbf{C}_2(E). \tag{4.11}$$

Proof.
Take finitely many $x_1, \ldots, x_n \in X(E)$. Each one of them can be represented as $x_i = (x_i(j))_{j\in\mathcal{J}}$. Then,

$$\begin{aligned}
\left(\sum_{i=1}^n \|x_i\|_{X(E)}^2\right)^{1/2} &= \left(\sum_{i=1}^n \|(\|x_i(j)\|_E)_j\|_X^2\right)^{1/2} \\
\leq \; & \mathbf{M}_{(2)}(X) \cdot \| \left(\left(\sum_{i=1}^n \|x_i(j)\|_E^2\right)^{1/2}\right)_j \|_X \\
\leq \; & K\mathbf{M}_{(2)}(X)\mathbf{C}_2(E) \cdot \| \left(\int_0^1 \|\sum_{i=1}^n r_i(t)x_i(j)\|_E \, dt\right)_j \|_X \\
\leq \; & K\mathbf{M}_{(2)}(X)\mathbf{C}_2(E) \cdot \int_0^1 \| \left(\|\sum_{i=1}^n r_i(t)x_i(j)\|_E\right)_j \|_X \, dt \\
\leq \; & K\mathbf{M}_{(2)}(X)\mathbf{C}_2(E) \cdot \int_0^1 \|\sum_{i=1}^n r_i(t)x_i\|_{X(E)} \, dt.
\end{aligned}$$

This completes the proof.
q.e.d.

Remark 4.3.11
By straightforward induction of (4.11) we get that for every m there is a constant $K > 0$ such that, for each 2-concave Köthe function space X,

$$\mathbf{C}_2([X]^m) \leq K\mathbf{M}_{(2)}(X)^m \tag{4.12}$$

and

$$\mathbf{C}_2([X]_m) \leq K\mathbf{M}_{(2)}(X)^m. \tag{4.13}$$

4.4 $\mathfrak{A}$-properties

Let $(\mathfrak{A}, A)$ be an operator ideal. For each normed vector space E we say that E *has the* $\mathfrak{A}$*-property* if $id_E \in \mathfrak{A}$. Then we define the $\mathfrak{A}$*-constant* of E by $A(E) = A(id_E)$. Allowing $A(E) = \infty$ we include the case when E does not have the $\mathfrak{A}$ property. In this way we have a normed space invariant, in the sense that if E and F are isometrically isomorphic then $A(E) = A(F)$.

Example 4.4.1
(1) We have a first example considering the type p operators. An operator $T : E \longrightarrow F$ has type p if there exists a finite constant $c > 0$ such that for any finite choice $x_1, \dots, x_n \in E$,

$$\left(\int_0^1 \| \sum_{k=1}^n r_k(t) T x_k \|^2 dt \right)^{1/2} \leq c \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p} .$$

Then we define the type p constant of T, $\mathbf{T}_p(T)$, as the best constant in this inequality. We denote by $\mathfrak{T}_p(E, F)$ the space of all type p operators between E and F. Then $[\mathfrak{T}_p, \mathbf{T}_p]$ is a Banach operator ideal. A space E has the $\mathfrak{T}_p$-property (has type p) if $id_E \in \mathfrak{T}_p(E, E)$; in other words, if we can find a constant $c > 0$ such that for any finite choice $x_1, \dots, x_n \in E$,

$$\left(\int_0^1 \| \sum_{k=1}^n r_k(t) x_k \|^2 dt \right)^{1/2} \leq c \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p} .$$

This is the classical definition of 'space with type p' that we gave in Definition 4.2.2. The constant we get via the operator ideal is obviously the usual type p constant of E. See also [11], Section 7.7.

(2) Another example is the cotype q operators and spaces (see [11], [14]); this will be studied with more detail later.

(3) We can also recover the Gordon-Lewis property in this way.

(4) Let us consider now the operator ideal Γ_2, defined in the following way. An operator $T : E \longrightarrow F$ is in $\Gamma_2(E, F)$ if when we consider $\tilde{T} : E \longrightarrow F \hookrightarrow F''$, this factorizes through some Hilbert space. Then, a space E has the Γ_2-property if and only if it is homeomorphic to a Hilbert space. The constant that we get in this case, $\gamma_2(E)$, is the Banach-Mazur distance from E to the Hilbert space of the same dimension (see [75], Section 13).

(5) Let Γ_∞ be the operator ideal of those $T : E \longrightarrow F$ such that, when extended to $\tilde{T} : E \longrightarrow F \hookrightarrow F''$, they factorize through some $L_\infty(\mu)$. Then, E has the Γ_∞-property if and only if it is injective and $\gamma_\infty(E)$ is the projection constant of E ([75] Section, 34).

We see that many important invariants can be defined in this way. It makes sense, thus, to try to treat them with a global point of view.
Let F be a complemented subspace of E with inclusion i and projection P. Then, a simple application of the ideal property shows that if E has the $\mathfrak{A}$ property, then so also has F and $A(F) \leq \|i\|A(E)\|P\|$. If $(\mathfrak{A}, A)$ is a maximal injective operator ideal, then the $\mathfrak{A}$-property is a superproperty. In this case we therefore have that ℓ_∞ does not have any non-trivial $\mathfrak{A}$-property.

A particularly interesting space is the space of continuous homogeneous polynomials of a Banach space. In [17], Proposition 1.54, (see also [16]) the author shows that if E is an infinite-dimensional Banach space, then ℓ_∞ is finitely represented in $\mathcal{P}(^m E)$ for all $m \geq 2$. This in particular implies that $\mathcal{P}(^m E)$ has no non-trivial $\mathfrak{A}$ property when E is infinite-dimensional and $(\mathfrak{A}, A)$ is maximal and injective. When X is a symmetric Banach sequence space, this affects $A(\mathcal{P}(^m X_n))$ as n tends to ∞. It is of interest, then, to study the behaviour of this sequence.

We write $\otimes^m E$ for the m-th full tensor product of a Banach space E, and $\otimes_\varepsilon^m E$ whenever we endow this space with the injective norm ε. Similarly we denote by $\otimes_{\varepsilon_s}^{m,s} E$ the m-th symmetric tensor product of E endowed with the symmetric injective norm ε_s. By the symmetrization map

$$S_E^m : \otimes^m E \longrightarrow \otimes^m E \ \ , \ \ S_E^m(x_1 \otimes \ldots \otimes x_m) = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(m)},$$

where Σ_m stands for the group of permutations of $\{1, \ldots, m\}$, the space $\otimes_{\varepsilon_s}^{m,s} E$ can be considered as a complemented subspace of $\otimes_\varepsilon^m E$. Recall that the natural embedding has norm $\leq m^m/m!$ and the projection $P = S_E^m$ has norm 1 (see e.g. [21], 3.1).
It is well known that, when M is a finite dimensional Banach space, we can represent the space of m-homogeneous polynomials on M as a symmetric tensor product of M' (see [21], 5.3),

$$\otimes_{\varepsilon_s}^{m,s} M' = \mathcal{P}(^m M) \ \ , \ \ \otimes^m x' \mapsto [x \mapsto x'(x)^m]. \tag{4.14}$$

In our case we can go even a little bit further and work with the full tensor product. We have the following result concerning the asymptotical behaviour of the spaces $\mathcal{P}(^m X_n)$.

Theorem 4.4.2
Let $(\mathfrak{A}, A)$ be a Banach operator ideal, X a symmetric Banach sequence space and $m \in \mathbb{N}$. Let $(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ such that $a_{mn} \prec a_n$ (resp. $a_n \prec a_{mn}$); then the following are equivalent:
(i) $A(\mathcal{P}(^m X_n)) \prec a_n$ *(resp.* $a_n \prec A(\mathcal{P}(^m X_n))$*).*
(ii) $A(\otimes_{\varepsilon_s}^{m,s} X_n') \prec a_n$ *(resp.* $a_n \prec A(\otimes_{\varepsilon_s}^{m,s} X_n')$*).*
(iii) $A(\otimes_\varepsilon^m X_n') \prec a_n$ *(resp.* $a_n \prec A(\otimes_\varepsilon^m X_n')$*).*

Before proving this Theorem we need some considerations.

Let X be a Banach sequence space. Consider $n, m, k \in \mathbb{N}$. For each $i = 1, \ldots, m$ let us define mappings $I_i : X_n \longrightarrow X_{mn+k}$ and $P_i : X_{mn+k} \longrightarrow X_n$ by

$$I_i(\sum_{j=1}^{n} \lambda_j e_j) = \sum_{j=1}^{n} \lambda_j e_{n(i-1)+j} \ , \ \ P_i(\sum_{j=1}^{mn+k} \lambda_j e_j) = \sum_{j=1}^{n} \lambda_{n(i-1)+j} e_j.$$

For each fixed i we clearly have $id_{X_n} = P_i \circ I_i$. Also,

$$\begin{aligned} \|I_i(\sum_{j=1}^{n} \lambda_j \ e_j)\| &= \|\sum_{j=1}^{n} \lambda_j \ e_{n(i-1)+j}\| = \|\sum_{j=1}^{n} \lambda_j \ e_j\| \\ \|P_i(\sum_{j=1}^{mn+k} \lambda_j \ e_j)\| &= \|\sum_{j=1}^{n} \lambda_{n(i-1)+j} \ e_j\| = \ \|\sum_{j=1}^{n} \lambda_j \ e_j\| \leq \|\sum_{j=1}^{mn+k} \lambda_j \ e_j\|. \end{aligned}$$

Hence $\|I_i\| = 1$ and $\|P_i\| \leq 1$. Using these mappings we can represent the full tensor product as a complemented subspace of some symmetric tensor product.

Lemma 4.4.3
Let X be a Banach sequence space and $m \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, the space $\otimes_{\varepsilon}^{m} X_n$ is a complemented subspace of $\otimes_{\varepsilon_s}^{m,s} X_{mn+k}$,

$$\otimes_{\varepsilon}^{m} X_n \underset{P}{\overset{i}{\leftrightarrows}} \otimes_{\varepsilon_s}^{m,s} X_{mn+k},$$

with $\|i\| \leq 1$ and $\|P\| \leq m^m$.
Proof.
Fix n, k and for each i consider the mappings I_i, P_i that we have just defined. Take the canonical symmetrization mapping $S_{X_{mn+k}}^{m}$ and the embedding $I_{X_{mn+k}}^{m} : \otimes_{\varepsilon_s}^{m,s} X_{mn+k} \longrightarrow \otimes_{\varepsilon}^{m} X_{mn+k}$. By [21], 1.10 we have

$$\otimes_{\varepsilon}^{m} X_n \xrightarrow{\otimes_j I_j} \otimes_{\varepsilon}^{m} X_{mn+k} \xrightarrow{S_{X_{mn+k}}^{m}} \otimes_{\varepsilon_s}^{m,s} X_{mn+k} \xrightarrow{I_{X_{mn+k}}^{m}} \otimes_{\varepsilon}^{m} X_{mn+k} \xrightarrow{m! \otimes_j P_j} \otimes_{\varepsilon}^{m} X_n$$

gives the identity mapping $id_{\otimes_{\varepsilon}^{m} X_n}$. This clearly proves our claim. Regarding the norms of the inclusion and the projection we have,

$$\begin{aligned} \|P\| &= \|m! \ (P_1 \otimes \ldots \otimes P_m) \circ i_{X_{mn+k}}^{m}\| \leq \|m! \ (P_1 \otimes \ldots \otimes P_m)\| \cdot \|i_{X_{mn+k}}^{m}\| \\ &\leq m! \ c(m, X_{mn+k}) \leq m^m \end{aligned}$$

and

$$\|i\| = \|S_{X_{mn+k}}^{m} \circ (I_1 \otimes \ldots \otimes I_m)\| \leq \|S_{X_{mn+k}}^{m}\| \cdot \|(I_1 \otimes \ldots \otimes I_m)\| \leq 1.$$

q.e.d.

Corollary 4.4.4
Let X be a Banach sequence space. For each $n > m$ define the number $\left[\frac{n}{m}\right] = \max\{k \in \mathbb{N} : k \leq \frac{n}{m}\}$. Then, $\otimes_{\varepsilon}^{m} X_{\left[\frac{n}{m}\right]}$ is a complemented subspace of $\otimes_{\varepsilon_s}^{m,s} X_n$,

$$\otimes_{\varepsilon}^{m} X_{\left[\frac{n}{m}\right]} \underset{P}{\overset{i}{\rightleftarrows}} \otimes_{\varepsilon_s}^{m,s} X_n,$$

with $\|i\| \leq 1$ and $\|P\| \leq m^m$.

Remark 4.4.5
Following the same steps of the proof of Lemma 4.4.3 the last two results can be obtained in a more general setting. Precisely, when β is a s-tensor norm (a tensor norm for symmetric tensor products) and α is a full tensor norm that is symmetric (i.e. $\alpha(\cdot; E_1, \ldots E_m) = \alpha(\cdot; E_{\sigma(1)}, \ldots E_{\sigma(m)})$ for every choice $E_1, \ldots E_m$ of normed spaces and any permutation σ) such that its restriction to symmetric tensor products, $\alpha|_s$, and β are equivalent Given any s-tensor norm, a full tensor norm satisfying this can always be generated (see [22]).

With this we are now ready to give the **Proof of Theorem 4.4.2**.
The equivalence $(i) \Leftrightarrow (ii)$ follows in both cases clearly from the representation $\mathcal{P}(^m X_n) = \otimes_{\varepsilon_s}^{m,s} X_n'$.
In order to prove the equivalence $(ii) \Leftrightarrow (iii)$, let us begin by assuming that $a_{mn} \prec a_n$. In this case the implication $(ii) \Rightarrow (iii)$ follows from Lemma 4.4.3. Indeed, we have

$$A(\otimes_{\varepsilon}^{m} X_n') \leq m^m A(\otimes_{\varepsilon_s}^{m,s} X_{mn}') \prec a_{mn} \prec a_n.$$

The implication $(iii) \Rightarrow (ii)$ follows from $\otimes_{\varepsilon_s}^{m,s} X_n' \rightleftarrows \otimes_{\varepsilon}^{m} X_n'$, with the norms obtained in Lemma 4.4.3. Then

$$A(\otimes_{\varepsilon_s}^{m,s} X_n') \leq \frac{m^m}{m!} A(\otimes_{\varepsilon}^{m} X_n') \prec a_n.$$

If $a_n \prec a_{mn}$, then we have $(ii) \Rightarrow (iii)$ using $\otimes_{\varepsilon_s}^{m,s} X_n' \rightleftarrows \otimes_{\varepsilon}^{m} X_n'$. For the converse implication, apply Corollary 4.4.4.
q.e.d.

4.5 A particular case, cotype 2

4.5.1 Conjecture

A particular case of the general framework presented in the previous section is the cotype 2 constant.

Definition 4.5.1 A linear operator $T : E \longrightarrow F$ has *cotype 2* if there exist a finite constant $\kappa > 0$ such that for any finite choice of vectors $x_1, \ldots, x_n \in E$,

$$\left(\sum_{k=1}^{n} \|Tx_k\|^2 \right)^{1/2} \leq \kappa \left(\int_0^1 \| \sum_{k=1}^{n} r_k(t) x_k \|^2 dt \right)^{1/2}.$$

The smallest constant in this inequality is called the *cotype 2 constant* of T and denoted $\mathbf{C}_2(T)$. We write $\mathfrak{C}_2(E,F)$ for the space of all cotype 2 operators between E and F.

It is well known that $(\mathfrak{C}_2, \mathbf{C}_2)$ is a Banach operator ideal (see [11]). A space E has then cotype 2 if $id_E \in \mathfrak{C}_2$, namely if there exists a finite constant $\kappa > 0$ such that for any finite choice $x_1, \dots, x_n \in E$,

$$\left(\sum_{k=1}^{n} \|x_k\|^2 \right)^{1/2} \leq \kappa \; \left(\int_0^1 \| \sum_{k=1}^{n} r_k(t) x_k \|^2 dt \right)^{1/2} .$$

The cotype 2 constant of E, denoted $\mathbf{C}_2(E)$, is then the smallest possible constant in this inequality.

Remark 4.5.2
Let E_n be n-dimensional vector spaces such that $E_n \subseteq E_{n+1}$. Dineen's result ([16], [17] Proposition 1.54) implies that for every $m \geq 2$ the sequence $(\mathbf{C}_2(\mathcal{P}(^m E_n)))_n$ tends to ∞ as $n \to \infty$. Indeed, let E be the completion of the vector space generated by $\bigcup_n E_n$ (in fact, $E = \bigcup_n E_n$). What Dineen shows is in fact that ℓ_∞ is finitely representable in $\mathcal{P}_f(^m F)$ (the finite type polynomials) for every infinite dimensional space F. This in particular means that $\mathcal{P}_f(^m E)$ does not have cotype 2. We claim that

$$(\mathbf{C}_2(\mathcal{P}_f(^m E_n)))_n \longrightarrow \infty.$$

This obviously implies our assumption. Suppose that

$$\sup_{n \in \mathbb{N}} \mathbf{C}_2(\mathcal{P}_f(^m E_n)) = K < \infty.$$

Let $P_1, \dots, P_k \in \mathcal{P}_f(^m E)$. For each $j = 1, \dots, k$ and $n \in \mathbb{N}$ let $Q_{j,n} = P_j|_{E_n} \in \mathcal{P}_f(^m E_n)$. Clearly

$$\|Q_{j,n}\| = \sup_{\substack{\|x\| \leq 1 \\ x \in E_n}} |P_j(x)| \leq \sup_{\substack{\|x\| \leq 1 \\ x \in E}} |P_j(x)| = \|P_j\|.$$

Then, for all n,

$$\begin{aligned} \left(\sum_{j=1}^{k} \|Q_{j,n}\|_{E_n}^2 \right)^{1/2} &\leq K \left(\int_0^1 \| \sum_{j=1}^{k} r_j(t) Q_{j,n} \|_{E_n}^2 dt \right)^{1/2} \\ &\leq K \left(\int_0^1 \| \sum_{j=1}^{k} r_j(t) P_j \|_{E_n}^2 dt \right)^{1/2} . \end{aligned}$$

Now, given any P_j and $\varepsilon > 0$, there is $x_\varepsilon \in E$ such that $|P_j(x_\varepsilon)| \geq \|P_j\| - \varepsilon/2$. By density and continuity we can find n_0 and $x_{n_0} \in E_{n_0}$ with

$$|P_j(x_\varepsilon)| \leq |P_j(x_{n_0})| + \frac{\varepsilon}{2} = |Q_{j,n_0}(x_{n_0})| + \frac{\varepsilon}{2}.$$

Hence

$$|Q_{j,n_0}(x_{n_0})| \geq |P_j(x_\varepsilon)| - \frac{\varepsilon}{2} \geq \|P_j\| - \varepsilon.$$

This implies $\|Q_{j,n_0}\| \geq \|P_j\| - \varepsilon$ for all $n \geq n_0$. Therefore $\|Q_{j,n}\| \to \|P_j\|$. We finally obtain

$$\left(\sum_{j=1}^{k} \|P_j\|_{E_n}^2 \right)^{1/2} \leq K \left(\int_0^1 \| \sum_{j=1}^{k} r_j(t) P_j \|_{E_n}^2 dt \right)^{1/2}.$$

This leads to a contradiction and shows that $\mathbf{C}_2(\mathcal{P}_f({}^m E_n))$ must tend to ∞.

If X is a symmetric Banach space and we consider the X_n defined before, Remark 4.5.2 shows that $\mathbf{C}_2(\mathcal{P}_f({}^m X_n))$ tend to ∞. Our interest is to estimate the asymptotical behaviour of these cotype 2 constants. We conjecture that for any Banach symmetric sequence space X and any $m \geq 2$,

$$\mathbf{C}_2(\mathcal{P}({}^m X_n)) \asymp (n^{1/2})^{m-1} \mathbf{C}_2(X_n'). \tag{4.15}$$

Although we cannot prove this conjecture in the most general case we give positive answers for some important cases.
Knowing the relationship between convexity, cotype and the dual spaces, we can rewrite the conjecture in the following terms

$$\mathbf{C}_2(\mathcal{P}({}^m X_n)) \asymp (n^{1/2})^{m-1} \mathbf{M}^{(2)}(X_n). \tag{4.16}$$

Let us check that the sequence $a_n = (n^{1/2})^{m-1} \mathbf{C}_2(X_n')$ in our conjecture satisfies that, for any fixed m, $(a_{mn}) \asymp (a_n)$. This is obvious in view of the following result.

Lemma 4.5.3
Let $(\mathfrak{A}, A)$ a Banach operator ideal, X a symmetric Banach sequence space; then, for any $m, n \in \mathbb{N}$, $A(X_{mn}) \leq m A(X_n)$.
Proof.
Using the injections and projections that we defined right before Lemma 4.4.3 we have,

$$\begin{CD} X_{mn} @>>> X_{mn} \\ @V{P_i}VV @AA{I_i}A \\ X_n @>>> X_n \end{CD}$$

We can write $id_{X_{mn}} = \sum_{i=1}^{m} I_i id_{X_n} P_i$ and then,

$$\begin{aligned} A(id_{X_{mn}}) &\leq \sum_{i=1}^{m} A(I_i id_{X_n} P_i) \leq \sum_{i=1}^{m} \|I_i\| A(id_{X_n}) \|P_i\| \\ &\leq \sum_{i=1}^{m} A(id_{X_n}) = mA(id_{X_n}). \end{aligned}$$

q.e.d.

We know that $\mathbf{C}_2(Y_n) \asymp \mathbf{M}_{(2)}(Y_n)$ for any symmetric Banach sequence space Y. By Theorem 4.4.2 our conjecture (4.15) is true for every symmetric Banach sequence space if and only if

$$\mathbf{C}_2(\otimes_{\varepsilon}^{m} X_n) \asymp (n^{1/2})^{m-1} \mathbf{M}_{(2)}(X_n) \tag{4.17}$$

holds for every symmetric Banach sequence space X. We prove (4.17) for some classes of spaces and then apply Theorem 4.4.2 to obtain (4.15) for their Köthe dual spaces. Therefore, from now on we will work with full tensor products and translate our results to spaces of m-homogeneous polynomials.

4.5.2 A first case, ℓ_1

We have a first immediate positive answer to our conjecture (4.15) when $X = \ell_1$. Then $X^{\times} = \ell_\infty$ is 2-concave and $X_n' = \ell_\infty^n$. In this case $\mathbf{C}_2(\ell_\infty^n) \asymp n^{1/2}/\sqrt{\log(n+1)}$ ([75], Section 4) and $\otimes_{\varepsilon}^{m} \ell_\infty^n = \ell_\infty^{n^m}$. Also,

$$\sqrt{\log(n+1)} \leq \sqrt{\log(n^m+1)} \leq \sqrt{\log(n+1)^m} = \sqrt{m}\sqrt{\log(n+1)}.$$

Hence

$$\begin{aligned} \mathbf{C}_2(\otimes_{\varepsilon}^{m} \ell_\infty^n) &= \mathbf{C}_2(\ell_\infty^{n^m}) \asymp \frac{(n^m)^{1/2}}{\sqrt{\log(n+1)}} = \frac{(n^{1/2})^m}{\sqrt{\log(n+1)}} \\ &\asymp (n^{1/2})^{m-1} \frac{n^{1/2}}{\sqrt{\log(n+1)}} = (n^{1/2})^{m-1} \mathbf{C}_2(\ell_\infty^n). \end{aligned}$$

Thus

$$\mathbf{C}_2(\mathcal{P}(^m \ell_1^n)) \asymp \frac{(n^{1/2})^m}{\sqrt{\log(n+1)}} \asymp (n^{1/2})^{m-1} \mathbf{M}^{(2)}(\ell_1^n). \tag{4.18}$$

4.6 General upper and lower bounds

In $\ell_2^n \otimes \ell_2^n$ we define a Hilbert norm in the following natural way. if $(e_i)_{i=1}^n$ is the canonical basis of ℓ_2^n, then $(e_i \otimes e_j)_{i,j=1}^n$ is a basis of $\ell_2^n \otimes \ell_2^n$. Each element in $\ell_2^n \otimes \ell_2^n$ has a representation

$x = \sum_{i,j} a_{ij}\ e_i \otimes e_j$ (note that this is nothing else but a matrix). We define the norm $\|x\| = (\sum_{i,j} |a_{ij}|^2)^{1/2}$. With this, $\ell_2^n \otimes_2 \ell_2^n$ is a Hilbert space (thus isometric to $\ell_2^{n^2}$). Moreover, if $x = \sum_i x_i\ e_i$ and $y = \sum_j y_j\ e_j \in \ell_2^n$, we have

$$\begin{aligned} \|x \otimes y\|_2 &= \sum_{i,j} x_i y_j\ e_i \otimes e_j = (\sum_{i,j} |x_i y_j|^2)^{1/2} \\ &= (\sum_i |x_i|^2)^{1/2} (\sum_j |y_j|^2)^{1/2} = \|x\| \cdot \|y\|. \end{aligned}$$

Iterating this we can define $\otimes_2^m \ell_2^n$. This is a Hilbert space of dimension n^m and therefore equal to $\ell_2^{n^m}$.

4.6.1 Groups of symmetries

Let X be a symmetric Banach sequence space. Fix $n \in \mathbb{N}$. For each choice of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, +1\}^n$ and any permutation $\pi \in \Sigma_n$ we define the mappings

$$\begin{array}{rccl} M_\varepsilon : & \mathbb{K}^n & \longrightarrow & \mathbb{K}^n \\ & (x_k)_{k=1}^n & \mapsto & M_\varepsilon((x_k)_{k=1}^n) = (\varepsilon_k x_k)_{k=1}^n \end{array}$$

$$\begin{array}{rccl} T_\pi : & \mathbb{K}^n & \longrightarrow & \mathbb{K}^n \\ & (x_k)_{k=1}^n & \mapsto & T_\pi((x_k)_{k=1}^n) = (x_{\pi(k)})_{k=1}^n \end{array}$$

Each one of these mappings is obviously an isometry on X_n. Let $S(\mathbb{K}^n)$ and $S(\otimes^m \mathbb{K}^n)$ be respectively the groups generated by the sets

$$\begin{gathered} \{M_\varepsilon \ :\ \varepsilon \in \{-1, +1\}^n\} \cup \{T_\pi \ :\ \pi \in \Sigma_n\} \\ \{T_1 \otimes \cdots \otimes T_m \ :\ T_j \in S(X_n),\ j = 1, \ldots, m\} \end{gathered}$$

Lemma 4.6.1
Let X be a symmetric Banach sequence space and α a norm on $\otimes^m X_n$ such that for all $T \in S(\otimes^m \mathbb{K}^n)$, $T : \otimes_\alpha^m X_n \longrightarrow \otimes_\alpha^m X_n$ is an isometry; then
(i) All $T \in S(\otimes^m \mathbb{K}^n)$ satisfy that $T : \otimes_2^m \ell_2^n \longrightarrow \otimes_2^m \ell_2^n$ is an isometry.
(ii) Let $u : \otimes^m X_n \longrightarrow \otimes^m X_n$ be a linear mapping. If $uT = Tu$ for all $T \in S(\otimes^m \mathbb{K}^n)$, then there exist a constant $\lambda \in \mathbb{K}$ such that $u = \lambda id_{\otimes^m X_n}$.
(iii) For all $u \in \mathcal{L}(\otimes_\alpha^m X_n; \otimes_\alpha^m X_n)$ and every $T_1, T_2 \in S(\otimes^m \mathbb{K}^n)$, the equality $\|T_1 u T_2\| = \|u\|$ holds.
Proof.
(i) Clearly each $T \in S(\mathbb{K}^n)$ is an isometry in ℓ_2^n. Thus $(T(e_j))_{j=1}^n$ is an orthonormal basis of ℓ_2^n. Then $(T_1(e_{j_1}) \otimes \cdots \otimes T_m(e_{j_m}))_{j_1, \ldots, j_m = 1, \ldots, n}$ is an orthonormal basis of $\otimes_2^m \ell_2^n$. Hence

$$\|(T_1 \otimes \cdots \otimes T_m)(\sum_{j_1, \ldots, j_m = 1}^n a_{j_1, \ldots, j_m}\ e_{j_1} \otimes \cdots \otimes e_{j_m})\|_2$$

$$
\begin{aligned}
&= \| \sum_{j_1,\dots,j_m=1}^{n} a_{j_1,\dots,j_m} \ (T_1 \otimes \cdots \otimes T_m)(e_{j_1} \otimes \cdots \otimes e_{j_m}) \|_2 \\
&= \| \sum_{j_1,\dots,j_m=1}^{n} a_{j_1,\dots,j_m} T_1(e_{j_1}) \otimes \cdots \otimes T_m(e_{j_m}) \|_2 \\
&= \| \sum_{j_1,\dots,j_m=1}^{n} a_{j_1,\dots,j_m} \ e_{j_1} \otimes \cdots \otimes e_{j_m} \|_2 .
\end{aligned}
$$

(ii) We use an argument that was first used by Gordon and Lewis in [29]. We proceed by induction on m. It is known that X_n satisfies this condition. Suppose that this is also true for $\otimes^{m-1} X_n$. We can write a set of generators of $S(\otimes^m \mathbb{K}^n)$ in the following more convenient way

$$
\{\tilde{T} \otimes T_m \ : \ \tilde{T} \in S(\otimes^{m-1}\mathbb{K}^n), \ T_m \in S(\mathbb{K}^n)\}.
$$

Take a linear mapping $u : \otimes^m X_n \longrightarrow \otimes^m X_n$ such that $Tu = uT$ for all $T \in S(\otimes^m \mathbb{K}^n)$. Fix $\xi \in X_n$ and $\xi^* \in X_n^*$. Let

$$
\begin{array}{rcll}
v : \otimes^{m-1} X_n & \longrightarrow & \left(\otimes^{m-1} X_n\right)^{**} = & \otimes^{m-1} X_n \\
\eta & \mapsto & v(\eta) &
\end{array}
$$

be given by $< v(\eta), \eta^* >=< u(\xi \otimes \eta), \xi^* \otimes \eta^* >$. This satisfies that $\tilde{T}v = v\tilde{T}$ for all $\tilde{T} \in S(\otimes^{m-1}\mathbb{K}^n)$. Indeed, for all $\eta \in \otimes^{m-1} X_n$ and $\eta^* \in \left(\otimes^{m-1} X_n\right)^*$ we have

$$
\begin{aligned}
< v\tilde{T}\eta, \eta^* > &= < u(\tilde{T}\eta \otimes \xi), \eta^* \otimes \xi^* > \\
&= < u(\tilde{T} \otimes id_{X_n})(\eta \otimes \xi), \eta^* \otimes \xi^* > \\
&= < (\tilde{T} \otimes id_{X_n}) u(\eta \otimes \xi), \eta^* \otimes \xi^* > \\
&= < u(\eta \otimes \xi), (\tilde{T}^* \otimes id_{X_n^*})(\eta^* \otimes \xi^*) > \\
&= < u(\eta \otimes \xi), \tilde{T}^*\eta^* \otimes \xi^* > \\
&= < v\eta, \tilde{T}^*\eta^* >=< \tilde{T}v\eta, \eta^* > .
\end{aligned}
$$

Then for every ξ, ξ^* there is a unique scalar $\lambda(\xi, \xi^*)$ such that

$$
< v\eta, \eta^* >= \lambda(\xi, \xi^*) < \eta, \eta^* >
$$

for all $\eta \in \otimes^{m-1} X_n$ and all $\eta^* \in \left(\otimes^{m-1} X_n\right)^*$. Define now another mapping by

$$
\begin{array}{rcllcl}
w : X_n & \longrightarrow & X_n^{**} = & X_n & & \\
\xi & \mapsto & w\xi & : \ X_n^* & \longrightarrow & \mathbb{K} \\
 & & & \quad \xi^* & \mapsto & < w\xi, \xi^* >= \lambda(\xi, \xi^*)
\end{array}
$$

Let $\eta_0 \in \otimes^{m-1} X_n$ and $\eta_0^* \in \otimes^{m-1} X_n^*$ be such that $< \eta_0, \eta_0^* >= 1$. Hence $\lambda(\xi, \xi^*) =< v\eta_0, \eta_0^* >=< u(\eta_0 \otimes \xi), \eta_0^* \otimes \xi^* >$. Proceeding in the same way as before we have $wT_m = T_m w$

for all $T_m \in S(\mathbb{K}^n)$. Therefore, we can find t such that $t\ < \xi, \xi^* > = < w\xi, \xi^* > = \lambda(\xi, \xi^*)$ for all $\xi \in X_n$ and all $\xi^* \in X_n^*$. Hence, for every η, ξ, η^*, ξ^*,

$$< u(\eta \otimes \xi), \eta^* \otimes \xi^* > = t\ < \xi, \xi^* >< \eta, \eta^* > = t\ < \eta \otimes \xi, \eta^* \otimes \xi^* > .$$

This implies $u = t\ id_{\otimes^m X_n}$.
(iii) This fact follows easily from the fact that both T_1 and T_2 are isometries for the norm α. Indeed,

$$\begin{aligned} \|T_1 u T_2\| &= \sup_{\|\xi\|_\alpha \le 1} \|T_1 u T_2(\xi)\|_\alpha = \sup_{\|\xi\|_\alpha \le 1} \|u(T_2(\xi))\|_\alpha \\ &= \sup_{\|T_2(\xi)\|_\alpha \le 1} \|u(T_2(\xi))\|_\alpha = \|u\|. \end{aligned}$$

q.e.d.

With this we can prove the following interesting result. It has been proved in the case $m = 2$ in [7]. We were not able to extend the proof by induction, and we have had to develop this alternative proof.

Proposition 4.6.2
Let X, Y be symmetric Banach sequence spaces; α, β two norms in $\otimes^m X_n$ and $\otimes^m Y_n$ respectively so that all $T \in S(\otimes^m \mathbb{K}^n)$ and all $R \in S(\otimes^m \mathbb{K}^n)$ are isometries when the spaces are endowed with α and β. Then,

$$\pi_2(\otimes_\alpha^m X_n \to \otimes_\beta^m Y_n) = (n^m)^{1/2} \frac{\| \otimes_2^m \ell_2^n \to \otimes_\beta^m Y_n\|}{\| \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n\|}.$$

Proof.
Note that all X_n, Y_n and ℓ_2^n are nothing else but $\mathbb{K}^n$ with some norms. Therefore $X_n = Y_n = \ell_2^n$ algebraically. Hence $\otimes^m X_n = \otimes^m Y_n = \otimes^m \ell_2^n = \mathbb{K}^{n^m}$ as vector spaces. Consider a linear isomorphism $\Psi : \otimes^m \mathbb{K}^n \longrightarrow \mathbb{K}^{n^m}$ such that, when the Hilbert norms are considered, $\Psi : \otimes_2^m \ell_2^n \longrightarrow \ell_2^{n^m}$ is an isometry. With this mapping we can define two norms in $\mathbb{K}^{n^m}$ by $\|x\|_\alpha = \alpha(\Psi^{-1}(x))$ and $\|x\|_\beta = \beta(\Psi^{-1}(x))$ for each $x \in \mathbb{K}^{n^m}$. In this way, considering the norms,

$$\Psi : \otimes_\alpha^m X_n \longrightarrow (\mathbb{K}^{n^m}, \| \ \|_\alpha) \quad , \quad \Psi : \otimes_\beta^m Y_n \longrightarrow (\mathbb{K}^{n^m}, \| \ \|_\beta)$$

Ψ is again an isometry. We factorize

$$\begin{array}{ccc} \otimes_\alpha^m X_n & \longrightarrow & \otimes_\beta^m Y_n \\ {\scriptstyle \Psi}\big\downarrow & & \big\uparrow{\scriptstyle \Psi^{-1}} \\ (\mathbb{K}^{n^m}, \| \ \|_\alpha) & \longrightarrow & (\mathbb{K}^{n^m}, \| \ \|_\beta) \end{array}$$

Define now the group

$$\Sigma(\mathbb{K}^{n^m}) = \{\Psi T \Psi^{-1} \ : \ T \in S(\otimes^m \mathbb{K}^n)\}.$$

If $\tilde{T} \in \Sigma(\mathbb{K}^{n^m})$, since it is a composition of isometries, it is an isometry in $\ell_2^{n^m}$.
Choose $u : \mathbb{K}^{n^m} \longrightarrow \mathbb{K}^{n^m}$ linear such that $u\tilde{T} = \tilde{T}u$ for all $\tilde{T} \in \Sigma(\mathbb{K}^{n^m})$. Then $u\Psi T\Psi^{-1} = \Psi T\Psi^{-1}u$ for all $T \in S(\otimes^m \mathbb{K}^n)$. Multiplying by Ψ^{-1} from the left and by Ψ from the right we have $\Psi^{-1}u\Psi T = T\Psi^{-1}u\Psi$ for all $T \in S(\otimes^m \mathbb{K}^n)$. From Lemma 4.6.1 there is a $\lambda \in \mathbb{K}$ such that $\Psi^{-1}u\Psi = \lambda id_{\otimes^m X_m}$. Hence $u = \lambda id_{\mathbb{K}^{n^m}}$.
Let $\tilde{T}_1, \tilde{T}_2 \in \Sigma(\mathbb{K}^{n^m})$. Since they are isometries for $\| \ \|_\alpha$ we have $\|\tilde{T}_1 u \tilde{T}_2\|_\alpha = \|u\|_\alpha$ for all $u \in \mathcal{L}(\mathbb{K}^{n^m}, \| \ \|_\alpha)$. The same is true for the $\| \ \|_\beta$ norm.
Applying [55], Lemma 2.5 and using that Ψ is an isometry we obtain

$$\begin{aligned}
\pi_2(\otimes_\alpha^m X_n \to \otimes_\beta^m Y_n) &= \pi_2((\mathbb{K}^{n^m}, \| \ \|_\alpha) \to (\mathbb{K}^{n^m}, \| \ \|_\beta)) \\
&= \sqrt{n^m} \frac{\|\ell_2^{n^m} \to (\mathbb{K}^{n^m}, \| \ \|_\beta)\|}{\|\ell_2^{n^m} \to (\mathbb{K}^{n^m}, \| \ \|_\alpha)\|} \\
&= (n^m)^{1/2} \frac{\| \otimes_2^m \ell_2^n \to \otimes_\beta^m Y_n\|}{\| \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n\|}.
\end{aligned}$$

q.e.d.

s-numbers

Definition 4.6.3 Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$. For each $k \in \mathbb{N}$ we define the k-th *approximation number* of T by

$$a_k(T) = \inf\{\|T - S\| : S \in \mathcal{L}(E, F), \ \ \text{rank } S < k\},$$

and the k-th *Weyl number* of T by

$$x_k(T) = \sup\{a_k(TA) : A \in \mathcal{L}(\ell_2, E), \ \ \|A\| = 1\}.$$

These numbers have been widely studied; see e.g. [43], [62], [63]. It is well known that these numbers form decreasing sequences. Clearly from the definition, $\|T\| = a_1(T) = x_1(T)$ for all operator T and $a_k(T) \geq x_k(T)$ for all k and all T. If both E and F are Hilbert spaces, then $a_k(T) = x_k(T)$ for all k and all T.
Take now the identity mapping $id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n$ where α is like in Proposition 4.6.2. Obviously $a_k(id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n) = x_k(id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n) = 0$ for all $k \geq n^m + 1$. Then we only have n^m non-zero of each of the numbers. But of those only the 'second half' is significant, in the sense that the first $\left[\frac{n^m}{2}\right]$ are essentially like $\|id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n\|$ and give us no information More precisely, following the ideas in the proof of Lemma 2.5 in [55], we get

Lemma 4.6.4
Let α be a norm in $\otimes^m X_n$ like in the statement of Proposition 4.6.2. Then for all $1 \leq k \leq$

$\left[\frac{n^m}{2}\right]$ *we have*

$$\begin{aligned} &\|id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n\| \geq a_k(id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n) \\ \geq \quad & x_k(id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n) \geq \frac{1}{\sqrt{2}} \|id : \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n\|. \end{aligned}$$

Proof.
The first two inequalities are clear. Let us show the last one. It is well known that $n^{1/2} x_n(T) \leq \pi_2(T)$ for every 2-summing operator T (see [43], 2.a.3 or [63], 2.7.3). With this fact and Proposition 4.6.2 we have, for all $1 \leq k \leq n^m$,

$$\begin{aligned} 1 &= x_{n^m}(id_{\otimes_2^m \ell_2^n}) = x_k(\otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n)\ x_{n^m-k+1}(\otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n) \\ &\leq x_k(\otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n)(n^m - k + 1)^{1/2} \pi_2(\otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n) \\ &= x_k(\otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n) \frac{n^{1/2}}{(n^m - k + 1)^{1/2}} \frac{\| \otimes_2^m \ell_2^n \to \otimes_2^m \ell_2^n \|}{\| \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n \|} \end{aligned}$$

Hence

$$x_k(\otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n) \geq \left(\frac{n}{n^m - k + 1} \right)^{1/2} \| \otimes_2^m \ell_2^n \to \otimes_\alpha^m X_n \|$$

for all $1 \leq k \leq n^m$. Doing $k = \left[\frac{n^m}{2}\right]$ we get

$$\left(\frac{n^m - \left[\frac{n^m}{2}\right] + 1}{n} \right)^{1/2} \geq \frac{1}{\sqrt{2}}.$$

This shows our claim.
q.e.d.

With this we are ready to give a lower bound for $\mathbf{C}_2(\otimes_\varepsilon^m X_n)$.

4.6.2 A general lower estimate

The following result, due to Milman and Pisier ([56], also [64], Chapter 10), is well known.

Proposition 4.6.5
Let E be any Banach space of cotype q and $T : \ell_2^n \longrightarrow E$ a linear operator. Then,

$$\sup_{k \in \mathbb{N}} k^{1/q} a_k(T) \leq \mathbf{C}_q(E)\ l(T).$$

Let $q = 2$, $E = \otimes_\varepsilon^m X_n$ and $T = id : \otimes_2^m \ell_2^n \longrightarrow \otimes_\varepsilon^m X_n$. From Proposition 4.6.5 we get

$$k^{1/2}\ a_k(id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n) \leq \mathbf{C}_2(\otimes_\varepsilon^m X_n)\ l(id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n)$$

for all $k \in \mathbb{N}$. If $1 \leq k \leq \left[\frac{n^m}{2}\right]$, Lemma 4.6.4 implies

$$k^{1/2}\ a_k(id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n) \geq \left(\frac{k}{2}\right)^{1/2} \|id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n\|.$$

Let $k = \left[\frac{n^m}{2}\right]$ and we get

$$\left(\left[\frac{n^m}{2}\right]\frac{1}{2}\right)^{1/2} \geq \left(\frac{1}{2}\frac{n^m}{2}\frac{1}{2}\right)^{1/2} \geq \left(\frac{n^m}{8}\right)^{1/2}.$$

Hence

$$\left(\frac{n^m}{8}\right)^{1/2} \|id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n\| \leq \mathbf{C}_2(\otimes_\varepsilon^m X_n)\ l(id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n). \tag{4.19}$$

Consider the index set, $\mathcal{M}(m,n) = \{(i_1,\dots,i_m) : i_1,\dots,i_m \in \{1,\dots,n\}\}$. Let (Ω,μ) be a probability space and $(g_{\mathrm{i}})_{\mathrm{i}\in\mathcal{M}(m,n)}$ a family of independent gaussian random variables. In [10] the authors define, for any finite set of vectors $x_1,\dots,x_n \in X_n$,

$$l(m;(x_k)_{k=1}^n) = \int_\Omega \| \sum_{\mathrm{i}\in\mathcal{M}(m,n)} g_{\mathrm{i}}\ x_{i_1} \otimes \cdots \otimes x_{i_m} \|_{\otimes_\varepsilon^m X_n} d\mu.$$

For each fixed $m \in \mathbb{N}$ there exists a constant $d > 0$ such that (see [10], Lemma 6)

$$l(m;(x_k)_{k=1}^n) \leq d\ l(1;(x_k)_{k=1}^n) \sup_{\|x'\|\leq 1} (\sum_{k=1}^n |x'(x_k)|^2)^{(m-1)/2}.$$

Therefore

$$\begin{aligned} l(id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n) &= (\int_\Omega \| \sum_{\mathrm{i}\in\mathcal{M}(m,n)} g_{\mathrm{i}}\ e_{i_1} \otimes \cdots \otimes e_{i_m} \|_{\otimes_\varepsilon^m X_n}^2 d\mu)^{1/2} \\ &\prec \int_\Omega \| \sum_{\mathrm{i}\in\mathcal{M}(m,n)} g_{\mathrm{i}}\ e_{i_1} \otimes \cdots \otimes e_{i_m} \|_{\otimes_\varepsilon^m X_n} d\mu \\ &= l(m;(e_i)_{i=1}^n) \\ &\prec l(1;(e_i)_{i=1}^n) \sup_{\|x'\|\leq 1} (\sum_{i=1}^n |x'(e_i)|^2)^{(m-1)/2}. \end{aligned}$$

Consider the mapping $id : \ell_2^n \longrightarrow X_n$. Then

$$l(1;(e_i)_{i=1}^n) = \int_\Omega \| \sum_{\mathrm{i}\in\mathcal{M}(1,n)} g_{\mathrm{i}}\ e_{i_1} \otimes \cdots \otimes e_{i_m} \|_{\otimes_\varepsilon^m X_n} d\mu$$

$$\begin{aligned} &= \int_\Omega \|\sum_{i=1}^n g_i e_i\|_{X_n} d\mu \\ &\prec (\int_\Omega \|\sum_{i=1}^n g_i e_i\|^2_{X_n} d\mu)^{1/2} \\ &= l(id : \ell_2^n \to X_n). \end{aligned}$$

On the other hand we have

$$\begin{aligned} \sup_{\|x'\|\le 1} (\sum_{i=1}^n |x'(e_i)|^2)^{1/2} &= \sup_{\|x'\|\le 1} \|\sum_{i=1}^n x'(e_i)\|_2 \\ &= \sup_{\|x'\|\le 1} \sup_{\sum |\lambda_i|^2 \le 1} |\sum_{i=1}^n \lambda_i x'(e_i)| \\ &= \sup_{\sum |\lambda_i|^2 \le 1} \sup_{\|x'\|\le 1} |x'\left(\sum_{i=1}^n \lambda_i e_i\right)| \\ &= \sup_{\sum |\lambda_i|^2 \le 1} \|\sum_{i=1}^n \lambda_i id(e_i)\|_{X_n} \\ &= \sup_{\sum |\lambda_i|^2 \le 1} \|id\left(\sum_{i=1}^n \lambda_i e_i\right)\|_{X_n} \\ &= \|id : \ell_2^n \longrightarrow X_n\|. \end{aligned}$$

Hence, for each $m \in \mathbb{N}$

$$l(id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n) \prec \|id : \ell_2^n \longrightarrow X_n\|^{m-1} l(id : \ell_2^n \to X_n).$$

From this and (4.19)

$$\begin{aligned} \left(\frac{n^m}{8}\right)^{1/2} \|id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n\| \prec \; & \mathbf{C}_2(\otimes_\varepsilon^m X_n) \cdot \\ & \cdot \|id : \ell_2^n \longrightarrow X_n\|^{m-1} l(id : \ell_2^n \to X_n). \end{aligned} \tag{4.20}$$

Let us now estimate $\|id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n\|$. We need the following.

Remark 4.6.6
Let X_1, X_2, Y_1, Y_2 be Banach spaces and α, β be two norms on $X_1 \otimes Y_1$ and $X_2 \otimes Y_2$ respectively. Suppose that $\|x_1 \otimes y_1\|_\alpha = \|x_1\| \cdot \|y_1\|$ for all $x_1 \in X_1$ and all $y_1 \in Y_1$ and $\|x_2 \otimes y_2\|_\beta = \|x_2\| \cdot \|y_2\|$ for all $x_2 \in X_2$ and all $y_2 \in Y_2$. Then for any two $T \in \mathcal{L}(X_1, X_2)$, $S \in \mathcal{L}(Y_1, Y_2)$,

$$\|T\| \cdot \|S\| \le \|T \otimes S : X_1 \otimes_\alpha Y_1 \longrightarrow X_2 \otimes_\beta Y_2\|.$$

Indeed, let $\varepsilon > 0$. Choose $x \in X_1$ and $y \in Y_1$ with $\|x\|, \|y\| \leq 1$ such that $\|T\| \leq (1+\varepsilon)\|Tx\|$ and $\|S\| \leq (1+\varepsilon)\|Sy\|$. Then

$$\begin{aligned} \|T\| \cdot \|S\| &\leq (1+\varepsilon)^2 \|Tx\| \cdot \|Sy\| = (1+\varepsilon)^2 \|Tx \otimes Sy\|_\beta \\ &= (1+\varepsilon)^2 \|(T \otimes S)(x \otimes y)\|_\beta \leq (1+\varepsilon)^2 \|T \otimes S\| \cdot \|x \otimes y\|_\alpha \\ &= (1+\varepsilon)^2 \|T \otimes S\| \cdot \|x\| \cdot \|y\| \leq (1+\varepsilon)^2 \|T \otimes S\|. \end{aligned}$$

Since this is true for all $\varepsilon > 0$, we have what we want.

By Remark 4.6.6, $\|id : \ell_2^n \to X_n\|^2 \leq \|id : \ell_2^n \otimes_2 \ell_2^n \to X_n \otimes_\varepsilon X_n\|$. Now, with an easy induction,

$$\|id : \ell_2^n \to X_n\|^m \leq \|id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n\|.$$

In the proof of Lemma 6 in [10] a proof of the converse inequality can be found. Therefore,

$$\|id : \ell_2^n \to X_n\|^m = \|id : \otimes_2^m \ell_2^n \to \otimes_\varepsilon^m X_n\|.$$

With this fact and (4.20) we finally get that for any symmetric Banach sequence space X and any fixed m,

$$(n^m)^{1/2} \frac{\|id : \ell_2^n \to X_n\|}{l(id : \ell_2^n \to X_n)} \prec \mathbf{C}_2(\otimes_\varepsilon^m X_n). \tag{4.21}$$

4.6.3 A general estimate

The lower bound that we have obtained in (4.21) is true for any symmetric Banach sequence space. With it we can give another estimate, valid for any symmetric Banach sequence space.

Lemma 4.6.7
Let X be any symmetric Banach sequence space and $m \in \mathbb{N}$. Then,

$$\frac{(n^{1/2})^{m-1}}{\sqrt{\log(n+1)}} \prec \mathbf{C}_2(\otimes_\varepsilon^m X_n) \prec (n^{1/2})^m.$$

Proof.
We get the lower estimate from (4.21). From Definition 4.2.5, applying (4.4) and (4.6), we get

$$\begin{aligned} l(\ell_2^n \to X_n) &= \int_\Omega \|\sum_{k=1}^n g_k e_k\|_X^2 d\mu)^{1/2} \\ &\prec \sqrt{\log(n+1)} (\int_0^1 \|\sum_{k=1}^n r_k(t) e_k\|_X^2 dt)^{1/2} \\ &\prec \sqrt{\log(n+1)} \|\sum_{k=1}^n e_k\|_X. \end{aligned}$$

On the other hand

$$\|\sum_{k=1}^{n} e_k\|_{X_n} \leq \|\ell_2^n \to X_n\| \cdot \|\sum_{k=1}^{n} e_k\|_{\ell_2^n} \leq \|\ell_2^n \to X_n\|\sqrt{n}.$$

Hence

$$\|\ell_2^n \to X_n\| \geq \frac{\|\sum_{k=1}^n e_k\|_{X_n}}{\sqrt{n}}.$$

From all these estimates and (4.21) we obtain

$$\frac{(n^{1/2})^{m-1}}{\sqrt{\log(n+1)}} \prec \mathbf{C}_2(\otimes_\varepsilon^m X_n).$$

If E_n is any n-dimensional Banach space, we can factorize the identity to get

$$\begin{array}{ccc} E_n & \xrightarrow{\;id\;} & E_n \\ \downarrow & & \uparrow \\ \ell_2^n & \xrightarrow[\;id\;]{} & \ell_2^n \end{array}$$

Therefore

$$\mathbf{C}_2(E_n) \leq d(E_n, \ell_2^n)\mathbf{C}_2(\ell_2^n) \leq c\sqrt{n}$$

where $c > 0$ is a universal constant. Now, since $\otimes^m X_n$ has dimension n^m we have the upper estimate

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \prec (n^m)^{1/2}.$$

q.e.d.

Remark 4.6.8
If X has non-trivial concavity we know from (4.10) that $l(id : \ell_2^n \to X_n) \asymp \lambda_X(n)$. Then

$$(n^{1/2})^{m-1} \prec \mathbf{C}_2(\otimes_\varepsilon^m X_n) \prec (n^{1/2})^m.$$

This condition is not very restrictive. In fact it is equivalent to the fact that the ℓ_∞^n are not uniformly embedded into X. In other words, we get this last estimate except when we are 'very close' to ℓ_∞ (we already know this for ℓ_∞).

A straightforward application of Lemma 4.6.7 and Remark 4.6.8 jointly with Theorem 4.4.2 gives

Theorem 4.6.9
Let X be any symmetric Banach sequence space and $m \in \mathbb{N}$; then

$$\frac{(n^{1/2})^{m-1}}{\sqrt{\log(n+1)}} \prec \mathbf{C}_2(\mathcal{P}(^m X_n)) \prec (n^{1/2})^m.$$

If moreover X has non-trivial convexity, we have that

$$(n^{1/2})^{m-1} \prec \mathbf{C}_2(\mathcal{P}(^m X_n)) \prec (n^{1/2})^m.$$

This result shows directly that the cotype 2 constants of the spaces $\mathcal{P}(^m X_n)$ tend to infinity.

4.7 The tensor conjecture for 2-concave spaces

Our aim in this section is to prove our conjecture for tensor products (4.17) when X is a 2-concave symmetric Banach sequence. In this case the sequence $\mathbf{M}_{(2)}(X_n)$ is bounded and, thus, it behaves asymptotically like the constant sequence 1. Then what we want to prove is the following

Proposition 4.7.1
Let X be a symmetric 2-concave Banach sequence space and $m \in \mathbb{N}$; then

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \asymp (n^{1/2})^{m-1}.$$

This will allow us later to prove the conjecture for polynomials for 2-convex spaces.

4.7.1 Weakly summable sequences

Let X be any Köthe function space modeled on a finite or countable set $\mathcal{J}$ and E any Banach space. Following [52] we define

Definition 4.7.2 We say that $x = (x_j)_{j\in\mathcal{J}} \subseteq E$ is *weakly X-summable* if $(x'(x_j))_{n\in\mathcal{J}} \in X$ for all $x' \in X'$.

We write $X^\omega(E)$ for the space of weakly X-summable sequences in E. Let us define now a quasi norm for $X^\omega(E)$. First, we observe the following.

Remark 4.7.3
Let E and F be Banach spaces. Suppose that there is a topological vector space G and a continuous inclusion $i : F \hookrightarrow G$. If $T : E \longrightarrow F$ is such that $i \circ T : E \longrightarrow G$ is continuous, then T has closed graph and therefore it is continuous. Indeed, consider $x_n \xrightarrow{E} x$ and $Tx_n \xrightarrow{F} y$. Since i is continuous, we have $i(Tx_n) \xrightarrow{F} i(y)$. Once again, $i \circ T$ is continuous, which implies $i(Tx_n) = i(y)$. But i is injective and this implies $Tx = y$.

The natural candidate to be our quasi norm is $\sup_{\|x'\|_{E'}\leq 1}\|(x'(x_j))_{j\in\mathcal{J}}\|_X$. Let $(x_j)_{j\in\mathcal{J}} \in X^\omega(E)$ and define the operator $T : E' \longrightarrow X$ given by $T(x') = (x'(x_j))_{j\in\mathcal{J}}$.
Consider now $\mathbb{K}^{\mathcal{J}}$, the space of all real or complex functions ϕ defined on $\mathcal{J}$. With the coordinatewise convergence, $\mathbb{K}^{\mathcal{J}}$ is a topological vector space. Then we have

$$\begin{array}{cccc} E' & \overset{T}{\longrightarrow} & X & \overset{i}{\hookrightarrow} \mathbb{K}^{\mathcal{J}} \\ x' & \mapsto & (x'(x_j))_j & \end{array}$$

Since the norm convergence in X implies the convergence coordinatewise, the inclusion $X \overset{i}{\hookrightarrow} \mathbb{K}^{\mathcal{J}}$ is continuous. Let $x'_m \longrightarrow x'$ in E'. In particular, for all $j \in \mathcal{J}$, $x'_m(x_j) \longrightarrow x'(x_j)$ as m tends to ∞. Therefore $(x'_m(x_j))_{j\in\mathcal{J}} \overset{\mathbb{K}^{\mathcal{J}}}{\longrightarrow} (x'(x_j))_{j\in\mathcal{J}}$ and $i\circ T$ is continuous. By Remark 4.7.3, T has closed graph and is continuous. Thus we can define, for each $(x_j)_{j\in\mathcal{J}} \in X^\omega(E)$,

$$w_{X,E}((x_j)_{j\in\mathcal{J}}) = \sup_{\|x'\|_{E'}\leq 1}\|(x'(x_j))_{j\in\mathcal{J}}\|_X < \infty.$$

This fact is also proved in [52].

4.7.2 (Y, X)-summing operators

Definition

The following definition was introduced in [52] and is a generalization of the classical concept of (p, q)-summing operators.

Definition 4.7.4 Let X, Y be any two Köthe function spaces modeled on some finite or countable set $\mathcal{J}$ and E, F Banach spaces. An operator $T \in \mathcal{L}(E; F)$ is (Y, X)-*summing* if there exists a constant $\kappa > 0$ such that for any finite $\mathcal{I} \subseteq \mathcal{J}$ and $(x_i)_{i\in\mathcal{I}} \subseteq E$, we have

$$\|(\|Tx_i\|_F)_{i\in\mathcal{I}}\|_Y \leq \kappa \sup_{\|x'\|_{E'}\leq 1}\|(x'(x_i))_{i\in\mathcal{I}}\|_X;$$

the smallest constant in this inequality is denoted by $\pi_{Y,X}(T)$ and called the (Y, X)-*summing constant of* T.

We write $\Pi_{Y,X}(E, F)$ for the space of (Y, X)-summing operators between E and F. It is easily seen that $\Pi_{Y,X}(E, F)$ with $\pi_{Y,X}$ is a normed space.

Remark 4.7.5
In [52] X and Y are always Banach sequence spaces. Then (Y, X)-summing operators are defined as those $T \in \mathcal{L}(E; F)$ such that

$$\begin{array}{rccl} \hat{T}: & X^\omega(E) & \longrightarrow & Y(F) \\ & (x_n)_{n\in\mathbb{N}} & \mapsto & \hat{T}((x_n)_{n\in\mathbb{N}}) = (Tx_n)_{n\in\mathbb{N}} \end{array}$$

is bounded. The (Y,X)-summing constant of T is defined as the operator norm $\|\hat{T}\|_{X^\omega(E)\to Y(F)}$. This definition implies ours, in the sense that every (Y,X)-summing operator in the sense of [52] is (Y,X)-summing in our sense and $\pi_{Y,X}(T) \leq \|\hat{T}\|$. This is easily checked just by taking a finite sequence in $X^\omega(E)$ and applying that $\hat{T}$ is continuous.
Both definitions are in fact equivalent in a wide range of cases. Namely when Y satisfies the Fatou property, that is if $\xi_n \uparrow \xi$ (coordinatewise) a.e. with $(\xi_n)_{n\in\mathbb{N}} \subseteq Y$, $\xi_n \geq 0$ a.e. and $\sup_n \|\xi_n\|_Y < \infty$, then $\xi \in Y$ and $\|\xi\| = \lim_n \|\xi_n\|_Y$ (see [48], Sect. 1.c). Indeed, let Y have the Fatou property. If T is (Y,X)-summing (in our sense), given any $(x_n)_{n\in\mathbb{N}} \in X^\omega(E)$, we can consider for each $m \in \mathbb{N}$, $(\|Tx_n\|_F)_{n=1}^m \in Y$. This generates a sequence in Y converging coordinatewise to $(\|Tx_n\|_F)_{n\in\mathbb{N}}$ satisfying the conditions of the Fatou property. Hence $(\|Tx_n\|_F)_{n\in\mathbb{N}} \in Y$ and $\hat{T}$ is well defined. Moreover,

$$\begin{aligned}
\|(\|Tx_n\|_F)_{n\in\mathbb{N}}\|_Y &= \lim_m \|(\|Tx_n\|_F)_{n=1}^m\|_Y \\
&= \sup_m \|(\|Tx_n\|_F)_{n=1}^m\|_Y \\
&\leq \kappa \sup_m \sup_{\|x'\|_{E'}\leq 1} \|(x'(x_n))_{n=1}^m\|_X \\
&= \kappa \sup_{\|x'\|_{E'}\leq 1} \sup_m \|(x'(x_n))_{n=1}^m\|_X \\
&\leq \kappa \sup_{\|x'\|_{E'}\leq 1} \|(x'(x_n))_{n\in\mathbb{N}}\|_X .
\end{aligned}$$

Thus, $\hat{T}$ is continuous and $\|\hat{T}\| \leq \pi_{Y,X}(T)$.

Some of the following results are slight modifications of some others in [52].

Remark 4.7.6
Let Y be a Banach sequence space with the Fatou property. Suppose that there exists a non-zero operator $T : E \longrightarrow F$ that is (Y,X)-summing. Take any $x \in E$ with $\|x\|_E = 1$ and $Tx \neq 0$ and some sequence $(\zeta_n)_{n\in\mathbb{N}} \in X$. Then for any $n \in \mathbb{N}$ we have,

$$\begin{aligned}
\|(\|T(\zeta_k x)\|_F)_{k=1}^n\|_Y &\leq \pi_{Y,X}(T) \sup_{\|x'\|_{E'}\leq 1} \|x'(\zeta_k x)_{k=1}^n\|_X \\
&\leq \pi_{Y,X}(T) \sup_{\|x'\|_{E'}\leq 1} \|(\zeta_n x'(x))_{n\in\mathbb{N}}\|_X \\
&= \pi_{Y,X}(T)\ \|(\zeta_n)_{n\in\mathbb{N}}\|_X \sup_{\|x'\|_{E'}\leq 1} x'(x) \\
&= \pi_{Y,X}(T)\ \|(\zeta_n)_{n\in\mathbb{N}}\|_X .
\end{aligned}$$

Since Y has the Fatou property we have

$$\|(\|\zeta_n T(x)\|_F)_{n\in\mathbb{N}}\|_Y \leq \pi_{Y,X}(T)\ \|(\zeta_n)_{n\in\mathbb{N}}\|_X$$

Hence

$$\|(\zeta_n)_{n\in\mathbb{N}}\|_Y \le \frac{\pi_{Y,X}(T)}{\|Tx\|_F}\|(\zeta_n)_{n\in\mathbb{N}}\|_X.$$

In other words, if there is some non-trivial (Y, X)-summing operator we automatically have that $X \hookrightarrow Y$.

Characterization

As we have recalled in Section 4.2.2, p-summing operators are characterized in terms of tensor products and norms of operators (see [11] Section 11). We give now an analogous characterization for (Y, X)-summing operators.

Proposition 4.7.7
Let $X \hookrightarrow Y$ be two Köthe function spaces modeled on some countable set $\mathcal{J}$. Let E, F be Banach spaces. Then, for all $T \in \mathcal{L}(E;F)$,

$$T \in \Pi_{Y,X}(E,F) \Leftrightarrow i\otimes T : X\otimes_\varepsilon E \longrightarrow Y\otimes_Y F \text{ is continuous.}$$

In this case $\pi_{Y,X}(T) = \|i\otimes T : X\otimes_\varepsilon E \longrightarrow Y\otimes_Y F\|$.
Proof.
Assume first that $i\otimes T$ is continuous. Let $\mathcal{I}$ be a finite subset of $\mathcal{J}$. Choose vectors $(x_k)_{k\in\mathcal{I}} \subseteq E$ and consider $\sum_{k\in\mathcal{I}} e_k\otimes x_k \in X\otimes E$. Then,

$$\|(i\otimes T)(\sum_{k\in\mathcal{I}} e_k\otimes x_k)\|_Y \le \|i\otimes T\|\cdot\|\sum_{k\in\mathcal{I}} e_k\otimes x_k\|_\varepsilon.$$

On the left hand side we have

$$\|(i\otimes T)(\sum_{k\in\mathcal{I}} e_k\otimes x_k)\|_Y = \|\sum_{k\in\mathcal{I}} e_k\otimes Tx_k\|_Y = \|(Tx_k)_{k\in\mathcal{I}}\|_Y.$$

On the right hand side, $\|\sum_{k\in\mathcal{I}}\otimes x_k\|_\varepsilon = \sup_{\|x'\|_{E'}\le 1}\|(x'(x_k))_{k\in\mathcal{I}}\|_X$ (see the proof of Lemma 4.3.6). Hence

$$\|(Tx_k)_{k\in\mathcal{I}}\|_Y \le \|i\otimes T\| \sup_{\|x'\|_{E'}\le 1}\|(x'(x_k))_{k\in\mathcal{I}}\|_X.$$

Therefore $T\in\Pi_{Y,X}(E,F)$ and $\pi_{Y,X}(T)\le\|i\otimes T : X\otimes_\varepsilon E\longrightarrow Y\otimes_Y F\|$.
To prove the converse implication we have

$$\begin{aligned}\|(i\otimes T)(\sum_{k\in\mathcal{I}} e_k\otimes x_k)\|_Y &= \|(Tx_k)_{k\in\mathcal{I}}\|_Y\\ \le\ \pi_{Y,X}(T)\sup_{\|x'\|_{E'}\le 1}\|(x'(x_k))_{k\in\mathcal{I}}\|_X &= \pi_{Y,X}(T)\|\sum_{k\in\mathcal{I}} e_k\otimes x_k\|_\varepsilon.\end{aligned}$$

Hence $i\otimes T$ is continuous and $\|i\otimes T : X\otimes_\varepsilon E\longrightarrow Y\otimes_Y F\|\le\pi_{Y,X}(T)$.
q.e.d.

Operator ideal

Proposition 4.7.8
Let X, Y be two Köthe function spaces modeled on a countable space $\mathcal{J}$ such that $\|X \hookrightarrow Y\| = 1$. Then $[\Pi_{Y,X}, \pi_{Y,X}]$ is an operator ideal.
Proof.
To see this we are going to use the following criterion ([11], Section 9.4): $[\Pi_{Y,X}, \pi_{Y,X}]$ is an operator ideal if and only if
(i) $id_{\mathbb{K}} \in \Pi_{Y,X}$ and $\pi_{Y,X}(id_{\mathbb{K}}) = 1$.
(ii) If STR is defined and $T \in \Pi_{Y,X}$, then $STR \in \Pi_{Y,X}$ and $\pi_{Y,X}(STR) \leq \|S\|\pi_{Y,X}(T)\|R\|$.
(iii) If $T_n \in \Pi_{Y,X}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \pi_{Y,X}(T_n) < \infty$, then $T = \sum_{n=1}^{\infty} T_n \in \Pi_{Y,X}$ and $\pi_{Y,X}(T) \leq \sum_{n=1}^{\infty} \pi_{Y,X}(T_n)$.
First we have $X \otimes_\varepsilon \mathbb{K} \cong X$ and $Y \otimes_Y \mathbb{K} \cong Y$. Then $i \otimes id_{\mathbb{K}} = i : X \hookrightarrow Y$ is clearly continuous. Hence $id_{\mathbb{K}} \in \Pi_{Y,X}$ and we have $\pi_{Y,X}(id_{\mathbb{K}}) = \|i \otimes id_{\mathbb{K}}\| = \|X \hookrightarrow Y\| = 1$.
Let now $R \in \mathcal{L}(E_0; E)$, $T \in \Pi_{Y,X}(E, F)$ and $S \in \mathcal{L}(F; F_0)$. Consider $STR : E_0 \longrightarrow F_0$. Then

$$i \otimes (STR) : X \otimes_\varepsilon E_0 \stackrel{id_X \otimes R}{\longrightarrow} X \otimes_\varepsilon E \stackrel{i \otimes T}{\longrightarrow} Y \otimes_Y F \stackrel{id_Y \otimes S}{\longrightarrow} Y \otimes_Y F_0$$

Both $id_X \otimes R$ and $i \otimes T$ are clearly continuous. Let $\mathcal{I} \subseteq \mathcal{J}$ be finite and take $(y_i)_{i \in \mathcal{I}} \subseteq Y$. Then

$$\begin{aligned} \|\sum_{i \in \mathcal{I}} e_i \otimes Sy_i\|_Y &= \|(\|Sy_i\|_{F_0})_{i \in \mathcal{I}}\|_Y \\ &\leq \|S\| \cdot \|(\|y_i\|_F)_{i \in \mathcal{I}}\|_Y = \|S\| \cdot \|\sum_{i \in \mathcal{I}} e_i \otimes y_i\|_Y. \end{aligned}$$

Then $id_Y \otimes S$ is continuous and $\|id_Y \otimes S\| \leq \|S\|$. Thus $i \otimes (STR)$ is continuous and

$$\begin{aligned} \pi_{Y,X}(STR) &= \|i \otimes (STR)\| \\ &\leq \|id_X \otimes R\| \cdot \|X \otimes_\varepsilon E \stackrel{i \otimes T}{\longrightarrow} Y \otimes_Y F\| \cdot \|id_Y \otimes S\| \\ &\leq \|S\| \cdot \pi_{Y,X}(T) \cdot \|R\|. \end{aligned}$$

Before checking the last condition, let us observe that if $T \in \Pi_{Y,X}(E, F)$, then $\|T\| \leq \pi_{Y,X}(T)$. Indeed, just take $x \in E$ and we have from the definition,

$$\|Tx\|_F \leq \pi_{Y,X}(T) \sup_{\|x'\| \leq 1} |x'(x)| = \pi_{Y,X}(T)\ \|x\|_E.$$

Let $(T_n)_{n \in \mathbb{N}} \subseteq \Pi_{Y,X}(E, F)$ such that $\sum_{n=1}^{\infty} \pi_{Y,X}(T_n) < \infty$. For each $m \in \mathbb{N}$ we have $\sum_{n=1}^{m} \|T_n\| \leq \sum_{n=1}^{m} \pi_{Y,X}(T_n) \leq \sum_{n=1}^{\infty} \pi_{Y,X}(T_n)$. Then the series $\sum_{n=1}^{\infty} T_n$ is absolutely convergent. Consider $T = \sum_{n=1}^{\infty} T_n \in \mathcal{L}(E; F)$. Let us see that $T \in \Pi_{Y,X}(E, F)$. Given $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that for $n, m \geq n_0$, $\sum_{k=n}^{m} \pi_{Y,X}(T_k) < \varepsilon$. Then,

$$\|i \otimes \sum_{k=n}^{m} T_k : X \otimes_\varepsilon E \to Y \otimes_Y F\| \leq \sum_{k=n}^{m} \|i \otimes T_k\| < \varepsilon.$$

Thus, the sequence $(i \otimes \sum_{k=1}^{n} T_k)_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{L}(X \otimes_\varepsilon E; Y \otimes_Y F)$ and converges to $i \otimes T \in \mathcal{L}(X \otimes_\varepsilon E; Y \otimes_Y F)$. Hence $T \in \Pi_{Y,X}(E,F)$. Clearly $\pi_{Y,X}(T) \leq \sum_{n=1}^{\infty} \pi_{Y,X}(T_n)$.
q.e.d.

Further properties

Definition 4.7.9 Given X, Y two Köthe function spaces modeled on a countable or finite set $\mathcal{J}$, the space of *multipliers* from X to Y, $M(X,Y)$, is the space of all $(\xi_j)_{j\in\mathcal{J}} \in \mathbb{K}^{\mathcal{J}}$ such that the operator $X \longrightarrow Y$ given by $(\zeta_j)_j \mapsto (\zeta_j \xi_j)_j$ is well defined and continuous. We define a norm in $M(X,Y)$ by $\|\xi\|_{M(X,Y)} = \sup_{\|\zeta\|_X \leq 1} \|\zeta\xi\|_Y$.

Note that $X^\times = M(X, \ell_1)$. The following proposition and its proof are an adapted version of Lemma 1.6 in [52].

Proposition 4.7.10
Let X, Y be two Köthe function spaces modeled on a countable or finite set $\mathcal{J}$. Then, for all Banach spaces E, F,

$$\Pi_{Y,\ell_1(\mathcal{J})}(E,F) \subseteq \Pi_{M(X,Y),X^\times}(E,F)$$

and $\pi_{M(X,Y),X^\times} \leq \pi_{Y,\ell_1(\mathcal{J})}$.
Proof.
Let $T \in \Pi_{Y,\ell_1(\mathcal{J})}(E,F)$ and $\mathcal{I} \subseteq \mathcal{J}$ finite. Take $(x_i)_{i\in\mathcal{I}} \subseteq E$. Then

$$\|(\|Tx_i\|_F)_{i\in\mathcal{I}}\|_Y \leq \pi_{Y,\ell_1(\mathcal{J})}(T) \sup_{\|x'\|_{E'}\leq 1} \|(x'(x_i))_{i\in\mathcal{I}}\|_{\ell_1(\mathcal{J})}.$$

Hence

$$\begin{aligned} \|(\|Tx_i\|_F)_{i\in\mathcal{I}}\|_{M(X,Y)} &= \sup_{\|\zeta\|_X\leq 1} \|(\zeta_i \|Tx_i\|_F)_{i\in\mathcal{I}}\|_Y \\ &\leq \pi_{Y,\ell_1(\mathcal{J})}(T) \sup_{\|\zeta\|_X\leq 1} \sup_{\|x'\|_{E'}\leq 1} \|(x'(\zeta_i x_i))_{i\in\mathcal{I}}\|_{\ell_1} \\ &= \pi_{Y,\ell_1(\mathcal{J})}(T) \sup_{\|x'\|_{E'}\leq 1} \sup_{\|\zeta\|_X\leq 1} \|(\zeta_i x'(x_i))_{i\in\mathcal{I}}\|_{\ell_1} \\ &= \pi_{Y,\ell_1(\mathcal{J})}(T) \sup_{\|x'\|_{E'}\leq 1} \|(x'(x_i))_{i\in\mathcal{I}}\|_{X^\times}. \end{aligned}$$

q.e.d.

With this we can give the following useful result.

Corollary 4.7.11
Let Y be a Köthe function space modeled on $\mathcal{J}$, finite or countable, such that $Y = Y^{\times\times}$. Then for all Banach spaces E, F,

$$\Pi_{\ell_1(\mathcal{J}),\ell_1(\mathcal{J})}(E,F) \hookrightarrow \Pi_{Y,Y}(E,F).$$

Proof.
The proof is immediate from the last proposition, since

$$\Pi_{\ell_1(\mathcal{J}),\ell_1(\mathcal{J})} \hookrightarrow \Pi_{M(Y^\times,\ell_1(\mathcal{J})),Y^{\times\times}} = \Pi_{M(\ell_\infty(\mathcal{J}),Y),Y} = \Pi_{Y,Y}.$$

q.e.d.

4.7.3 Proof of the conjecture

It has been known for long that in some cases $\pi_p(id_{\ell_q^n}) \asymp \sqrt{n}$ (see [62], Section 22.4); in other words, $\|id_{\ell_p} \otimes id_{\ell_q^n} \ell_p \otimes_\varepsilon \ell_q^n \to \ell_p \otimes_{\Delta_p} \ell_q^n\| \asymp \sqrt{n}$. It makes sense, then, to expect that something similar happens in our new setting. We get a partly satisfactory answer, shown in the following result of independent interest.

Lemma 4.7.12
Let Y be as in Lemma 4.7.11. Then for every Banach sequence space X and $n \in \mathbb{N}$

$$\pi_{Y,Y}(id_{X_n}) = \|id : Y \otimes_\varepsilon X_n \longrightarrow Y \otimes_Y X_n\| \leq K_G \mathbf{M}_{(2)}(X_n)\; n^{1/2},$$

where $K_G \geq 1$ is Grothendieck's constant.
Proof.
Let $T \in \mathcal{L}(\mathcal{C}(K); X_n)$ and finitely many $x_1, \ldots, x_m \in \mathcal{C}(K)$. By the Grothendieck-Krivine inequality (see [48], Theorem 1.f.14, the proof is for real lattices, but it can be adapted to the complex case),

$$\left\| \left(\sum_{k=1}^m |Tx_k|^2 \right)^{1/2} \right\|_X \leq K_G \|T\| \cdot \left\| \left(\sum_{k=1}^m |x_k|^2 \right)^{1/2} \right\|_{\mathcal{C}(K)}.$$

Hence,

$$\begin{aligned}
\left(\sum_{k=1}^m \|Tx_k\|_X^2 \right)^{1/2} &\leq \mathbf{M}_{(2)}(X_n)\; \| \left(\sum_{k=1}^m |Tx_k|^2 \right)^{1/2} \|_X \\
&\leq K_G \mathbf{M}_{(2)}(X_n)\; \|T\| \cdot \| \left(\sum_{k=1}^m |x_k|^2 \right)^{1/2} \|_{\mathcal{C}(K)} \\
&= K_G \mathbf{M}_{(2)}(X_n)\; \|T\| \sup_{\|x'\| \leq 1} \left(\sum_{k=1}^m |x'(x_k)|^2 \right)^{1/2}.
\end{aligned}$$

Thus $\pi_2(T) \leq K_G \mathbf{M}_{(2)}(X_n)\|T\|$. By [75], Proposition 10.17, and the well known fact that $\pi_2(id_{X_n}) = \sqrt{n}$ (see e.g. [14] Theorem 4.17 or [75] Proposition 9.11) we obtain

$$\pi_1(id_{X_n}) \leq K_G \mathbf{M}_{(2)}(X_n) \pi_2(id_{X_n}) = K_G \mathbf{M}_{(2)}(X_n)\sqrt{n}.$$

Applying Lemma 4.7.11, we finally get

$$\begin{aligned}\|Y \otimes_\varepsilon X_n \to Y \otimes_Y X_n\| &= \pi_{Y,Y}(id_{X_n}) \le \pi_{\ell_1,\ell_1}(id_{X_n}) \\ &= \pi_1(id_{X_n}) \le K_G \mathbf{M}_{(2)}(X_n)\sqrt{n}.\end{aligned}$$

q.e.d.

Proposition 4.7.13
Let X be a Banach sequence space and $m \in \mathbb{N}$. Then for all n

$$\|id : \otimes_\varepsilon^m X_n \longrightarrow [X_n]^m\| \le K_G^{m-1} \mathbf{M}_{(2)}(X_n)^{m-1}\ (n^{1/2})^{m-1}.$$

Proof.
We prove it by induction. The case $m = 2$ follows from Lemma 4.7.12 (put $Y = X_n$). Suppose that the result holds for $m-1$. Consider the following commutative diagram with the natural mappings

$$\begin{array}{ccc} \otimes_\varepsilon^m X_n = (\otimes_\varepsilon^{m-1} X_n) \otimes_\varepsilon X_n & \longrightarrow & [X_n]^m = [X_n]^{m-1}(X_n) \\ \downarrow & \nearrow & \\ [X_n]^{m-1} \otimes_\varepsilon X_n & & \end{array}$$

Then

$$\begin{aligned}\| \otimes_\varepsilon^m X_n \longrightarrow [X_n]^m\| \ \le\ & \|(\otimes_\varepsilon^{m-1} X_n) \otimes_\varepsilon X_n \to [X_n]^{m-1} \otimes_\varepsilon X_n\| \cdot \\ & \|[X_n]^{m-1} \otimes_\varepsilon X_n \to [X_n]^{m-1} \otimes_{[X_n]^{m-1}} X_n\|.\end{aligned}$$

From Lemma 4.7.12 we have $\|[X_n]^{m-1} \otimes_\varepsilon X_n \to [X_n]^{m-1} \otimes_{[X_n]^{m-1}} X_n\| \le K_G \mathbf{M}_{(2)}(X_n)\, n^{1/2}$. Since $\|(\otimes_\varepsilon^{m-1} X_n) \otimes_\varepsilon X_n \to [X_n]^{m-1} \otimes_\varepsilon X_n\| \le \| \otimes_\varepsilon^{m-1} X_n \to [X_n]^{m-1}\|$, by the Induction Hypothesis we obtain

$$\| \otimes_\varepsilon^m X_n \longrightarrow [X_n]^m\| \le K_G^{m-1} \mathbf{M}_{(2)}(X_n)^{m-1}\ (n^{1/2})^{m-1}.$$

q.e.d.

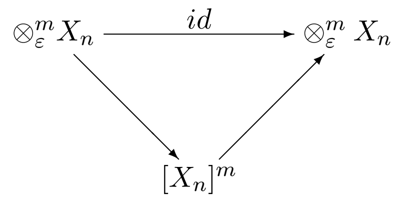
Note that when X is 2-concave, $\mathbf{M}_{(2)}(X_n) \le \mathbf{M}_{(2)}(X)$ for all n.

We are now ready to give the desired positive answer to our conjecture (4.17).

Proposition 4.7.1
Let X be a symmetric 2-concave Banach sequence space and $m \in \mathbb{N}$. Then

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \asymp (n^{1/2})^{m-1}.$$

Proof.
To get the upper estimate we factorize



From Lemma 4.7.13 and the fact that $\|[X_n]^m \to \otimes_\varepsilon^m X_n\| \leq 1$ we have

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \prec (n^{1/2})^{m-1}\mathbf{C}_2([X_n]^m).$$

By (4.12), $\mathbf{C}_2([X_n]^m) \prec \mathbf{M}_{(2)}(X_n)^m \leq \mathbf{M}_{(2)}(X)^m$. Hence

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \prec (n^{1/2})^{m-1}.$$

Since X is non-trivially concave the lower bound follows from Remark 4.6.8.
q.e.d.

It is well know that ℓ_p is 2-concave if and only if $1 \leq p \leq 2$ ([14], 11.5). Then from the previous Theorem we get the following result.

Corollary 4.7.14
Let $1 \leq p \leq 2$. Then for every $m \in \mathbb{N}$

$$\mathbf{C}_2(\otimes_\varepsilon^m \ell_p^n) \asymp (n^{1/2})^{m-1}.$$

4.8 The tensor conjecture for 2-convex spaces

We are now interested in proving our conjecture for tensor products (4.17) for another class of symmetric Banach sequence spaces, those that are 2-convex and non-trivially concave. In this section X will always be such a space. This implies that X has type 2 (see [48], Proposition 1.f.3). Our aim is to prove the following.

Proposition 4.8.1
Let X be a 2-convex symmetric Banach sequence space with finite concavity and fix $m \in \mathbb{N}$. Then,

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \asymp (n^{1/2})^{m-1}\mathbf{M}_{(2)}(X_n) \asymp \frac{(n^{1/2})^m}{\lambda_X(n)}.$$

We begin by giving some estimates for the 2-concavity constant of the X_n.

Remark 4.8.2
Let X be a 2-convex symmetric Banach sequence space; then we have $1 \leq \|id : \ell_2^n \to$

$X_n\| \leq \mathbf{M}^{(2)}(X)$. Indeed, since $\|e_k\|_{\ell_2^n} = 1 = \|e_k\|_{X_n}$ for all $k = 1, \ldots, n$ the first inequality is trivial. For the second inequality we have

$$\begin{aligned}\|\sum_{k=1}^{n} \xi_k e_k\|_{X_n} &= \|\left(\sum_{k=1}^{n} |\xi_k e_k|^2\right)^{1/2}\|_X \leq \mathbf{M}^{(2)}(X)\left(\sum_{k=1}^{n} \|\xi_k e_k\|_X^2\right)^{1/2} \\ &= \mathbf{M}^{(2)}(X)\left(\sum_{k=1}^{n} |\xi_k|^2\right)^{1/2} = \mathbf{M}^{(2)}(X)\|\sum_{k=1}^{n} \xi_k e_k\|_{\ell_2^n}.\end{aligned}$$

Proposition 4.8.3
Let X be a 2-convex symmetric Banach sequence space with finite concavity; then,

$$\mathbf{M}_{(2)}(X_n) \asymp \frac{n^{1/2}}{\lambda_X(n)}.$$

Proof.
For the upper bound we factorize as usual,

$$\begin{array}{ccc} X_n & \longrightarrow & X_n \\ \downarrow & & \uparrow \\ \ell_2^n & \longrightarrow & \ell_2^n \end{array}$$

Since ℓ_2^n is a Hilbert space, $\mathbf{M}_{(2)}(\ell_2^n) = 1$. Let $\zeta_1, \ldots, \zeta_k \in X_n$. By Remark 4.8.2

$$\begin{aligned}\left(\sum_{i=1}^{k} \|\zeta_i\|_{X_n}^2\right)^{1/2} &\leq \|\ell_2^n \to X_n\| \left(\sum_{i=1}^{k} \|\zeta_i\|_{\ell_2^n}^2\right)^{1/2} \\ &\leq \mathbf{M}^{(2)}(X)\ \|\left(\sum_{i=1}^{k} |\zeta_i|^2\right)^{1/2}\|_{\ell_2^n} \\ &\leq \mathbf{M}^{(2)}(X)\ \|X_n \to \ell_2^n\| \cdot \|\left(\sum_{i=1}^{k} |\zeta_i|^2\right)^{1/2}\|_{X_n}.\end{aligned}$$

Hence

$$\mathbf{M}_{(2)}(X_n) \leq \mathbf{M}^{(2)}(X)\|X_n \to \ell_2^n\| = \mathbf{M}^{(2)}(X)\|\ell_2^n \to X_n'\|.$$

Since X is 2-convex, $X^\times$ is 2-concave (see [48] Proposition 1.d.4) and $X_n' = (X^\times)_n$. By [72], Proposition 2.2, $\|\ell_2^n \to E_n\| \asymp \lambda_{E_n}(n)/n^{1/2}$ whenever E is a 2-concave Banach sequence space. From this and (4.9) we get

$$\begin{aligned}\mathbf{M}_{(2)}(X_n) &\leq \mathbf{M}^{(2)}(X)\|\ell_2^n \to X_n'\| \prec \frac{\lambda_{X_n'}(n)}{n^{1/2}} \\ &= \frac{n}{\lambda_{X_n}(n)n^{1/2}} = \frac{n^{1/2}}{\lambda_{X_n}(n)}.\end{aligned}$$

For the lower estimate take $k = \left[\frac{n}{2}\right]$ in Proposition 4.6.5. Then

$$\left[\frac{n}{2}\right]^{1/2} a_{\left[\frac{n}{2}\right]}(\ell_2^n \to X_n) \prec \mathbf{M}_{(2)}(X_n)\ l(\ell_2^n \to X_n).$$

Since X has finite concavity we can apply (4.10) to get

$$\left[\frac{n}{2}\right]^{1/2} a_{\left[\frac{n}{2}\right]}(\ell_2^n \to X_n) \prec \mathbf{M}_{(2)}(X_n)\ \lambda_X(n). \tag{4.22}$$

Clearly $\left[\frac{n}{2}\right]^{1/2} \leq \left(\frac{n}{4}\right)^{1/2}$. Then (4.22) and Lemma 4.6.4 give

$$\mathbf{M}_{(2)}(X_n) \succ \frac{\|\ell_2^n \to X_n\|}{\lambda_X(n)} n^{1/2} \geq \frac{n^{1/2}}{\lambda_X(n)}.$$

This completes the proof.
q.e.d.

Note that the second estimate in Proposition 4.8.1 follows immediately from Proposition 4.8.3. The next lemma corresponds to Lemma 4.7.12.

Lemma 4.8.4
Let X, Y be any two Banach sequence spaces; then for every $n \in \mathbb{N}$,

$$\|id : X_n \otimes_\varepsilon Y \longrightarrow X_n(Y)\| \leq \lambda_X(n).$$

Proof.
We factorize in the following way,

$$\begin{array}{ccc} X_n \otimes_\varepsilon Y & \longrightarrow & X_n(Y) \\ & \searrow \quad \nearrow & \\ & \ell_\infty^n \otimes_\varepsilon Y = \ell_\infty^n(Y) & \end{array}$$

Then $\|X_n \otimes_\varepsilon Y \to X_n(Y)\| \leq \|\ell_\infty^n \otimes_\varepsilon Y \to X_n(Y)\|$. We estimate the right-hand-side term of this inequality.

$$\begin{aligned} \|\sum_{k=1}^n e_k \otimes \zeta_k\|_{X_n(Y)} &= \|(\|\zeta_k\|_Y)_{k=1}^n\|_{X_n} \leq \|\ell_\infty^n \to X_n\| \ \sup_k \|\zeta_k\|_Y \\ &= \|\ell_\infty^n \to X_n\| \cdot \|\sum_{k=1}^n e_k \otimes \zeta_k\|_{\ell_\infty^n(Y)}. \end{aligned}$$

Hence

$$\|\ell_\infty^n \otimes_\varepsilon Y \to X_n(Y)\| \le \|\ell_\infty^n \to X_n\| = \sup_{|\lambda_k|\le 1} \|\sum_{k=1}^n \lambda_k e_k\|_{X_n} \le \|\sum_{k=1}^n e_k\|_X.$$

This proves our claim.
q.e.d.

Proposition 4.8.5
Let X be any Banach sequence space. Then for all $n, m \in \mathbb{N}$,

$$\|id : \otimes_\varepsilon^m X_n \longrightarrow [X_n]_m\| \le \lambda_X(n)^{m-1}.$$

Proof.
We prove it by induction. The case $m = 2$ is clear from Lemma 4.8.4. Assume the result true for $m-1$, that is $\|\otimes_\varepsilon^{m-1} X_n \longrightarrow [X_n]_{m-1}\| \le \lambda_X(n)^{m-2}$. For the m-th case we factorize

$$\begin{array}{ccc} \otimes_\varepsilon^m X_n = X_n \otimes_\varepsilon (\otimes_\varepsilon^{m-1} X_n) & \longrightarrow & [X_n]_m = X_n([X_n]_{m-1}) \\ \downarrow & \nearrow & \\ X_n \otimes_\varepsilon [X_n]_{m-1} & & \end{array}$$

By the metric mapping property and the Induction Hypothesis we have

$$\|X_n \otimes_\varepsilon (\otimes_\varepsilon^{m-1} X_n) \to X_n \otimes_\varepsilon [X_n]_{m-1}\| \le \| \otimes_\varepsilon^{m-1} X_n \to [X_n]_{m-1}\| \le \lambda_X(n)^{m-2}.$$

On the other hand, by Lemma 4.8.4, $\|X_n \otimes_\varepsilon [X]_{m-1} \to X_n([X]_{m-1})\| \le \lambda_X(n)$. Hence

$$\begin{aligned} \| \otimes_\varepsilon^m X_n \longrightarrow [X_n]_m\| &\le \|X_n \otimes_\varepsilon (\otimes_\varepsilon^{m-1} X_n) \to X_n \otimes_\varepsilon [X_n]_{m-1}\| \cdot \\ &\quad \cdot \|X_n \otimes_\varepsilon [X]_{m-1} \to X_n([X]_{m-1})\| \\ &\le \lambda_X(n)^{m-1}. \end{aligned}$$

q.e.d.

We give another positive answer to our conjecture.

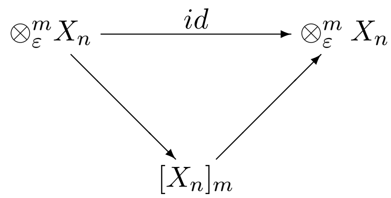
Proposition 4.8.1
Let X be a 2-convex symmetric Banach sequence space with finite concavity and $m \in \mathbb{N}$. Then,

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \asymp (n^{1/2})^{m-1} \mathbf{M}_{(2)}(X_n) \asymp \frac{(n^{1/2})^m}{\lambda_X(n)}.$$

Proof.
In view of Proposition 4.8.3, it is enough to show

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \asymp \frac{(n^{1/2})^m}{\lambda_X(n)}.$$

For the upper bound we factorize



We know $\|[X_n]_m \to \otimes_\varepsilon^m X_n\| \leq 1$. With this fact, together with (4.13), Proposition 4.8.5 and Proposition 4.8.3 we obtain

$$\begin{array}{rcl} \mathbf{C}_2(\otimes_\varepsilon^m X_n) & \leq & \| \otimes_\varepsilon^m X_n \to [X_n]_m\| \; \mathbf{C}_2([X_n]_m) \\ & \prec & (\lambda_X(n))^{m-1} \mathbf{M}_{(2)}(X_n)^m \\ & \prec & \dfrac{(n^{1/2})^m}{\lambda_X(n)}. \end{array}$$

For the lower estimate we already have in (4.21)

$$(n^{1/2})^m \frac{\|\ell_2^n \to X_n\|}{l(\ell_2^n \to X_n)} \prec \mathbf{C}_2(\otimes_\varepsilon^m X_n).$$

Since X has non-trivial concavity, $l(\ell_2^n \to X_n) \asymp \lambda_X(n)$ (by Remark 4.3.9). On the other hand, $\|\ell_2^n \to X_n\| \geq 1$. Hence

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n) \succ \frac{(n^{1/2})^m}{l(\ell_2^n \to X_n)} \succ \frac{(n^{1/2})^m}{\lambda_X(n)}.$$

This completes the proof.
q.e.d.

It is well known that if $2 \leq p < \infty$, then ℓ_p is 2-convex and p-concave. It is also well known that in this case $\mathbf{C}_2(\ell_p^n) = n^{\frac{1}{2}-\frac{1}{p}}$ (see [75], Section 4). We immediately have the following important corollary.

Corollary 4.8.6
Let $2 \leq p < \infty$. Then for every $m \in \mathbb{N}$

$$\mathbf{C}_2(\otimes_\varepsilon^m \ell_p^n) \asymp \frac{(n^{1/2})^m}{n^{1/p}}.$$

With this result we complete the study of the situation for ℓ_p for all p.

4.9 Results for spaces of polynomials

4.9.1 General results

Our main interest are the spaces of polynomials and they are the main goal of this chapter. We proved in the introduction to this chapter (Theorem 4.4.2) the equivalence of the study of cotype constants of $\mathcal{P}(^m X_n)$ and those of $\otimes_\varepsilon^m X_n$. We have obtained some results for full tensor products. Let us see now how are those results translated to the polynomial case. Let us recall that the estimate for $\mathbf{C}_2(\mathcal{P}(^m X_n))$ involves that of $\mathbf{C}_2(\otimes_\varepsilon^m X_n')$. Therefore, we have to apply the results we have obtained to the Köthe dual space.
If X is 2-convex, its Köthe dual $X^\times$ is 2-concave. Applying Proposition 4.7.1 we get

Theorem 4.9.1
Let X be a symmetric 2-convex Banach sequence space and $m \in \mathbb{N}$. Then

$$\mathbf{C}_2(\mathcal{P}(^m X_n)) \asymp (n^{1/2})^{m-1}.$$

Note that this result covers the case of ℓ_p when $2 \leq p \leq \infty$ whereas the following one covers the case when $1 < p \leq 2$.

Theorem 4.9.2
Let X be a 2-concave symmetric Banach sequence space with finite convexity and $m \in \mathbb{N}$. then

$$\mathbf{C}_2(\mathcal{P}(^m X_n)) \asymp n^{\frac{m}{2}-1}\lambda_X(n).$$

Proof.
Since X is 2-concave and has finite convexity, $X^\times$ is 2-convex and has finite concavity. From Proposition 4.8.1 and (4.9),

$$\mathbf{C}_2(\otimes_\varepsilon^m X_n') \asymp \frac{(n^{1/2})^m}{\lambda_{X^\times}(n)} = \frac{(n^{1/2})^m \lambda_X(n)}{n} = (n^{1/2})^{m-2}\lambda_X(n).$$

This proves our claim.
q.e.d.

4.9.2 Particular cases

ℓ_p spaces

For the case of ℓ_1 we apply (4.18) and we obtain

Proposition 4.9.3
For each $m \in \mathbb{N}$,

$$\mathbf{C}_2(\mathcal{P}(^m \ell_p^n)) \asymp \begin{cases} \frac{(n^{1/2})^m}{\sqrt{\log(n+1)}} & \text{if} \quad p = 1 \\ n^{\frac{m}{2}-1} n^{1/p} & \text{if} \quad 1 < p \leq 2 \\ (n^{1/2})^{m-1} & \text{if} \quad 2 < p \leq \infty \end{cases}$$

Orlicz spaces

In Section 4.3.2 we already mentioned some results from [40] concerning the concavity and convexity of Orlicz spaces. Then we have the corresponding estimate of the cotype 2 constants of spaces of polynomials. We write ℓ_φ^n for X_n.

Proposition 4.9.4
Let φ be a non-degenerated Orlicz function, ℓ_φ its associated Orlicz sequence space and m fixed. Then
(i) If φ satisfies the Δ_2 condition and is such that $\varphi(\lambda t) \leq K\lambda^2\varphi(t)$ for all $0 \leq \lambda, t \leq 1$ and some $K > 0$, then

$$\mathbf{C}_2(\mathcal{P}(^m\ell_\varphi^n)) \asymp (n^{1/2})^{m-1}.$$

(ii) If φ is such that $\varphi(\lambda t) \geq K\lambda^2\varphi(t)$ for all $0 \leq \lambda, t \leq 1$ and some $K > 0$, then

$$\mathbf{C}_2(\mathcal{P}(^m\ell_\varphi^n)) \asymp \frac{n^{\frac{m}{2}-1}}{\varphi^{-1}(1/n)}.$$

Lorentz spaces

As we know (see Section 4.3.2), $d(w,p)$ is 2-convex for all $2 \leq p < \infty$. If $1 \leq p < 2$ and $nw_n^q \asymp \sum_{i=1}^n w_i^q$ with $q = \frac{2}{2-p}$, $d(w,p)$ is 2-concave (and clearly p-convex). Note that if $d(w,p)$ is 2-concave, then w is 1-regular. Denoting X_n by $d^n(w,p)$, we get the following.

Proposition 4.9.5
Let $X = d(w,p)$ be a Lorentz space and $m \in \mathbb{N}$ fixed. Then
(i) If $2 \leq p < \infty$, then

$$\mathbf{C}_2(\mathcal{P}(^m d^n(w,p))) \asymp (n^{1/2})^{m-1}.$$

(ii) If $1 \leq p < 2$ and $nw_n^q \asymp \sum_{i=1}^n w_i^q$ with $q = \frac{2}{2-p}$,

$$\mathbf{C}_2(\mathcal{P}(^m d^n(w,p))) \asymp n^{\frac{m}{2}-1}(\sum_{i=1}^n w_i)^{1/p} \asymp n^{\frac{m}{2}-1} n^{\frac{1}{p}} w_n^{\frac{1}{p}}.$$

$\ell_{p,q}$ spaces

Once again, applying well known results we have that,

Proposition 4.9.6
Let $1 < p < \infty$, $1 \leq q < \infty$ and fix $m \in \mathbb{N}$; then

$$\mathbf{C}_2(\mathcal{P}(^m\ell_{p,q}^n)) \asymp \begin{cases} n^{\frac{m}{2}-1}n^{\frac{1}{p}} & \text{if} \quad 2 > p\ ,\ 2 \geq q \\ (n^{1/2})^{m-1} & \text{if} \quad 2 < p\ ,\ 2 \leq q. \end{cases}$$

4.10 The complex and the real case

All the results we have obtained so far are valid both for real and complex Banach Köthe spaces modeled on countable or finite sets. We have been able to prove our conjecture for some spaces, but not in general. For further developments the real theory of Banach lattices may be helpful. This would only give results for real Banach sequence spaces. But we are also interested in the complex case. For this reason the following results linking the real and the complex situation are of interest.

There exists a theory of complex Banach lattices, developed by Schaeffer, where complex lattices are derived from the real ones. Unfortunately this theory is not useful for our purposes, since for instance the complex lattice defined from the real ℓ_2^n is not the complex ℓ_2^n.

4.10.1 Complexifications

Let $(E, \| \ \|)$ be any real Banach space. A *complexification* of E is a complex Banach space $E + iE = E \times E$ together with a norm $\| \ \|_{\mathbb{C}}$ satisfying that for all $x, y \in E$

$$\max\{\|x\|, \|y\|\} \leq \|x + iy\|_{\mathbb{C}} = \|x - iy\|_{\mathbb{C}} \leq \|x\| + \|y\|$$

and

$$\|x + i0\|_{\mathbb{C}} = \|x\|.$$

Given any polynomial $P \in \mathcal{P}(^m E)$ let $\check{P}$ denote the symmetric linear mapping associated to P (see [17] Section 1.1). Define the complexification of P, a new polynomial $P^{\mathbb{C}} \in \mathcal{P}(^m(E + iE))$, by

$$P^{\mathbb{C}}(x + iy) = \sum_{k=0}^{m} \binom{m}{k} i^{m-k} \check{P}(x^k, y^{m-k}).$$

The polynomial $P^{\mathbb{C}}$ extends P. The polarization formula gives $\|P^{\mathbb{C}}\| \leq (2m)^m/m! \ \|P\|$.

Conversely, if F is a complex Banach space we denote the underlying real Banach space by $F_{\mathbb{R}}$. In this situation $\mathcal{P}(^m F_{\mathbb{R}})$ denotes the real Banach space of continuous real m-homogeneous polynomials from $F_{\mathbb{R}}$ into $\mathbb{R}$. Let $Q \in \mathcal{P}(^m(F + iF))$. For each $z \in F + iF$ we can write

$$Q(z) = R(z) + iS(z),$$

where $R, S : F + iF \longrightarrow \mathbb{R}$ are real m-homogeneous polynomials. Hence, for every $\lambda \in \mathbb{C}$ we have

$$R(\lambda z) + iS(\lambda z) = Q(\lambda z) = \lambda^m Q(z) = \lambda^m R(z) + i\lambda^m S(z).$$

Doing $\lambda^m = i$ we obtain $\lambda = e^{\frac{\pi i}{2m}}$ and $S(z) = -e^{\frac{\pi i}{2m}} R(z)$. Therefore there is a unique $R \in \mathcal{P}(^m(F + iF)_{\mathbb{R}})$ such that

$$Q(x + iy) = R(x + iy) - iR(e^{\frac{\pi i}{2m}}(x + iy)). \tag{4.23}$$

This trivially satisfies $\|R\| \leq \|Q\|$. Both norms are in fact equal. Assume that there exists z_0 such that $|Q(z_0)| = \|Q\|$ (in case there is not such z_0, we can consider a sequence $(z_n)_n$ with $(|Q(z_n)|)_n$ converging to $\|Q\|$). Choosing an appropriate λ with $|\lambda| = 1$ we can find $\tilde{z}_0$ with $\|Q\| = Q(\tilde{z}_0) = R(\tilde{z}_0)$. Then $\|R\| \geq R(\tilde{z}_0) = \|Q\|$. This gives the equality.
From this $\mathcal{P}(^m(F + iF)) \hookrightarrow \mathcal{P}(^m(F + iF)_{\mathbb{R}})$ isometrically. This mapping is not onto.

Let X be a complex Banach sequence space. Define

$$X(\mathbb{R}) = \{y \in X \ : \ y_n \in \mathbb{R} \ \text{ for all } \ n\}$$

and endow it with the induced norm from X. Then $X(\mathbb{R})$ is a real Banach sequence space and is symmetric, or 2-convex, or 2-concave whenever X is so.

4.10.2 Relation between the cotype constants

What we intend to do now is try to connect the cotype constants of $\mathcal{P}(^m X_n)$ with those of $\mathcal{P}(^m X(\mathbb{R})_n)$. Before giving the concrete result we have the following lemma, interesting by itself.

Lemma 4.10.1
Let E and F be real or complex Banach spaces such that the Banach-Mazur distance between them, $d(E, F) < \infty$. Then, for each $m = 1, 2, \ldots$,

$$\mathbf{C}_2(\mathcal{P}(^m F)) \leq d(E, F)^m \mathbf{C}_2(\mathcal{P}(^m E)).$$

Proof.
Let $T : E \longrightarrow F$ be a topological isomorphism. Define $T^* : \mathcal{P}(^m F) \longrightarrow \mathcal{P}(^m E)$ by $Q \mapsto Q \circ T$. We claim $\|T^*\| \leq \|T\|^m$. Indeed

$$\|Q \circ T\| = \sup_{\|x\| \leq 1} |Q(T(x))| = \|T\|^m \sup_{\|x\| \leq 1} \left| Q\left(\frac{T(x)}{\|T\|}\right) \right| \leq \|T\|^m \cdot \|Q\|.$$

Hence $\|T^*\| = \sup_{\|Q\| \leq 1} \|Q \circ T\| \leq \|T\|^m$. On the other hand, clearly

$$\left((T^{-1})^* \circ T^*\right)(Q) = (T^* Q) \circ T^{-1} = Q \circ T \circ T^{-1} = Q;$$

Therefore $(T^{-1})^* = (T^*)^{-1}$. If $Q_1, \ldots, Q_k \in \mathcal{P}(^m F)$ let $P_j = T^* Q_j$. Then

$$\begin{aligned} \left(\sum_{j=1}^{k} \|Q_j\|^2\right)^{1/2} &= \left(\sum_{j=1}^{k} \|(T^{-1})^* P_j\|^2\right)^{1/2} \\ &\leq \left(\sum_{j=1}^{k} \|(T^{-1})^*\|^2 \|P_j\|^2\right)^{1/2} \end{aligned}$$

$$
\begin{aligned}
&= \|(T^{-1})^*\| \left(\sum_{j=1}^k \|P_j\|^2 \right)^{1/2} \\
&\leq \|(T^{-1})^*\| \mathbf{C}_2(\mathcal{P}(^m E)) \left(\int_0^1 \| \sum_{j=1}^k r_j(t) P_j \|^2 dt \right)^{1/2} \\
&= \|(T^{-1})^*\| \mathbf{C}_2(\mathcal{P}(^m E)) \left(\int_0^1 \|T^*(\sum_{j=1}^k r_j(t) P_j) \|^2 dt \right)^{1/2} \\
&\leq \|(T^{-1})^*\| \mathbf{C}_2(\mathcal{P}(^m E)) \left(\int_0^1 \|T^*\|^2 \cdot \| \sum_{j=1}^k r_j(t) P_j) \|^2 dt \right)^{1/2} \\
&\leq \|(T^{-1})^*\| \cdot \|T^*\| \mathbf{C}_2(\mathcal{P}(^m E)) \left(\int_0^1 \| \sum_{j=1}^k r_j(t) P_j) \|^2 dt \right)^{1/2} .
\end{aligned}
$$

Hence

$$\mathbf{C}_2(\mathcal{P}(^m F)) \leq \|(T^{-1})^*\| \cdot \|T^*\| \mathbf{C}_2(\mathcal{P}(^m F)) \leq \|T^{-1}\|^m \cdot \|T\|^m \mathbf{C}_2(\mathcal{P}(^m F)).$$

Since T was arbitrary we have what we wanted.
q.e.d.

We can now relate the real and the complex cases.

Proposition 4.10.2
Let X be a complex symmetric Banach sequence space. Then for each m,

$$\mathbf{C}_2(\mathcal{P}(^m X(\mathbb{R})_n)) \prec \mathbf{C}_2(\mathcal{P}(^m X_n)) \prec \mathbf{C}_2(\mathcal{P}(^m X(\mathbb{R})_{2n})).$$

In particular, if $(a_n) \asymp (a_{2n})$ and $(b_n) \asymp (b_{2n})$, then

$$(a_n) \prec \mathbf{C}_2(\mathcal{P}(^m X(\mathbb{R})_n)) \prec (b_n)$$

if and only if

$$(a_n) \prec \mathbf{C}_2(\mathcal{P}(^m X_n)) \prec (b_n).$$

Proof.
For each choice $P_1, \ldots, P_M$ of polynomials in $\mathcal{P}(^m X(\mathbb{R})_n)$ we have

$$\left(\sum_{j=1}^M \|P_j\|^2 \right)^{1/2} \quad \leq \quad \left(\sum_{j=1}^M \|P_j^{\mathbb{C}}\|^2 \right)^{1/2}$$

$$\begin{aligned} &\leq \mathbf{C}_2(\mathcal{P}(^m X_n)) \left(\int_0^1 \| \sum_{j=1}^M r_j(t) P_j^{\mathbb{C}} \|^2 dt \right)^{1/2} \\ &= \mathbf{C}_2(\mathcal{P}(^m X_n)) \left(\int_0^1 \| \left(\sum_{j=1}^M r_j(t) P_j \right)^{\mathbb{C}} \|^2 dt \right)^{1/2} \\ &\leq \frac{(2m)^m}{m!} \mathbf{C}_2(\mathcal{P}(^m X_n)) \left(\int_0^1 \| \sum_{j=1}^M r_j(t) P_j \|^2 dt \right)^{1/2} ; \end{aligned}$$

This gives the first inequality. For the second inequality we first check

$$\mathbf{C}_2(\mathcal{P}(^m X_n)) \leq \mathbf{C}_2(\mathcal{P}(^m (X_n)_{\mathbb{R}})).$$

Indeed, for $Q_1, \ldots, Q_M \in \mathcal{P}(^m X_n)$ let $R_1, \ldots, R_M \in \mathcal{P}(^m (X_n)_{\mathbb{R}})$ be as in (4.23). Hence

$$\begin{aligned} \left(\sum_{j=1}^M \|Q_j\|^2 \right)^{1/2} &= \left(\sum_{j=1}^M \|R_j\|^2 \right)^{1/2} \\ &\leq \mathbf{C}_2(\mathcal{P}(^m (X_n)_{\mathbb{R}})) \left(\int_0^1 \| \sum_{j=1}^M r_j(t) R_j \|^2 dt \right)^{1/2} \\ &= \mathbf{C}_2(\mathcal{P}(^m (X_n)_{\mathbb{R}})) \left(\int_0^1 \| \sum_{j=1}^M r_j(t) Q_j \|^2 dt \right)^{1/2} . \end{aligned}$$

Define now a mapping $i : (X_n)_{\mathbb{R}} \longrightarrow X(\mathbb{R})_{2n}$ by doing $i(x_1, \ldots, x_n) = (\mathrm{Re}x_1, \mathrm{Im}x_1, \ldots, \mathrm{Re}x_n, \mathrm{Im}x_n)$. The symmetry of X gives $\|i\| \cdot \|i^{-1}\| \leq 4$. Applying Lemma 4.10.1 we obtain

$$\mathbf{C}_2(\mathcal{P}(^m X_n)) \leq \mathbf{C}_2(\mathcal{P}(^m (X_n)_{\mathbb{R}})) \leq 4^m \mathbf{C}_2(\mathcal{P}(^m X(\mathbb{R})_{2n})).$$

This completes the proof.
q.e.d.

4.11 More results for full tensor products

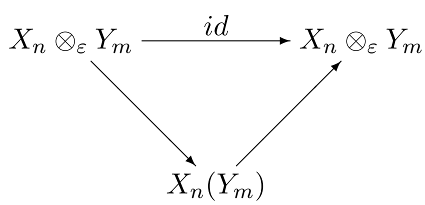
We give here a last result that, although it is not in the main trend of this chapter, is based on some of the techniques that we have been using all through the chapter. Note first of all that simply by joining Lemma 4.7.12 and Lemma 4.8.4 we have that if X and Y are two Banach sequence spaces, then for all n, m

$$\|id : X_n \otimes_\varepsilon Y_m \longrightarrow X_n(Y_m)\| \leq \min(\lambda_X(n),\ K_G \mathbf{M}_{(2)}(Y_m)\ m^{1/2}). \tag{4.24}$$

Proposition 4.11.1
Let X be either 2-concave or 2-convex with non-trivial concavity and let Y be either 2-concave or 2-convex with non-trivial concavity. Then

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \asymp \min(\sqrt{n}\mathbf{M}_{(2)}(Y_m),\ \sqrt{m}\mathbf{M}_{(2)}(X_n)),$$

where this means that we can find upper and lower bounds with constants depending neither on n nor on m.
This is a proper improvement of a result on cotype 2 estimates for injective tensor products of ℓ_p^n's given in [7], Proposition in Section 5.
Proof.
Let us assume first that both spaces are 2-concave. We factorize,



From this factorization and (4.24),

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \leq K_G \mathbf{M}_{(2)}(Y_m)\ m^{1/2} \mathbf{C}_2(X_n(Y_m)).$$

Using (4.11) we have a universal constant $K > 0$ such that $\mathbf{C}_2(X_n(Y_m)) \leq K\mathbf{M}_{(2)}(X_n)\mathbf{M}_{(2)}(Y_m)$. Since Y is 2-concave, $\mathbf{M}_{(2)}(Y_m) \leq \mathbf{M}_{(2)}(Y) < \infty$ for all m. Hence

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \prec\ m^{1/2}\mathbf{M}_{(2)}(X_n).$$

By the symmetry of ε, $\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \prec\ n^{1/2}\mathbf{M}_{(2)}(Y_m)$. This gives the upper estimate. To get the lower bound we have, from Proposition 4.6.5,

$$\sqrt{nm}\|\ell_2^n \otimes_2 \ell_2^m \to X_n \otimes_\varepsilon Y_m\| \prec \mathbf{C}_2(X_n \otimes_\varepsilon Y_m) l(\ell_2^n \otimes_2 \ell_2^m \to X_n \otimes_\varepsilon Y_m).$$

We know that $\|\ell_2^n \otimes_2 \ell_2^m \to X_n \otimes_\varepsilon Y_m\| = \|\ell_2^n \to X_n\| \cdot \|\ell_2^m \to Y_m\|$. By Chevét's inequality (4.7),

$$\begin{aligned} l(\ell_2^n \otimes_2 \ell_2^m \to X_n \otimes_\varepsilon Y_m) &\leq c\Big(l(\ell_2^n \to X_n)\|\ell_2^m \to Y_m\| \\ &\quad +\|\ell_2^n \to X_n\| l(\ell_2^m \to Y_m)\Big). \end{aligned}$$

Hence

$$1 \prec \mathbf{C}_2(X_n \otimes_\varepsilon Y_m)\left(\frac{l(\ell_2^n \to X_n)}{\sqrt{nm}\ \|\ell_2^n \to X_n\|} + \frac{l(\ell_2^m \to Y_m)}{\sqrt{nm}\ \|\ell_2^m \to Y_m\|}\right). \tag{4.25}$$

Since X has non-trivial concavity (4.10) implies $l(\ell_2^n \to X_n) \prec \lambda_X(n) \leq \sqrt{n}\|\ell_2^n \to X_n\|$. The same holds for Y. Therefore

$$1 \prec \mathbf{C}_2(X_n \otimes_\varepsilon Y_m)\left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}}\right) \prec \mathbf{C}_2(X_n \otimes_\varepsilon Y_m)\frac{1}{\min(\sqrt{m}, \sqrt{n})}.$$

This shows our claim.
Assume now that both X and Y are 2-convex and have non-trivial concavity. For the upper estimate we factorize as we did before. From Proposition 4.8.3 we have $\mathbf{M}_{(2)}(X_n) \asymp \sqrt{n}/\lambda_X(n)$. Using this, jointly with (4.11) and (4.24) we get

$$\begin{aligned} \mathbf{C}_2(X_n \otimes_\varepsilon Y_m) &\leq \lambda_X(n)\mathbf{C}_2(X_n(Y_m)) \prec \lambda_X(n)\mathbf{M}_{(2)}(X_n)\mathbf{M}_{(2)}(Y_m) \\ &\prec \sqrt{n}\mathbf{M}_{(2)}(Y_m). \end{aligned}$$

By the symmetry of ε, $\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \prec \sqrt{m}\mathbf{M}_{(2)}(X_n)$. For the lower bound we start, as before, from Proposition 4.6.5 and apply Chevét's inequality to arrive to (4.25). Since X is 2-convex and has non-trivial concavity, by (4.10) and Proposition 4.8.3, $l(\ell_2^n \to X_n) \asymp \lambda_X(n) \asymp \sqrt{n}/\mathbf{M}_{(2)}(X_n)$ (the same is true for Y). Now, $\|\ell_2^n \to X_n\| \geq 1$. Hence

$$\begin{aligned} 1 &\prec \mathbf{C}_2(X_n \otimes_\varepsilon Y_m)\left(\frac{\sqrt{n}}{\sqrt{nm}\mathbf{M}_{(2)}(X_n)} + \frac{\sqrt{m}}{\sqrt{nm}\mathbf{M}_{(2)}(Y_m)}\right) \\ &\prec \mathbf{C}_2(X_n \otimes_\varepsilon Y_m)\frac{1}{\min(\sqrt{m}\mathbf{M}_{(2)}(X_n), \sqrt{n}\mathbf{M}_{(2)}(Y_m))}. \end{aligned}$$

This proves the second case.
For the last case, let us assume that Y is 2-concave and X is 2-convex with finite concavity. For the upper bound we factorize in the same way as we did before and use (4.24) to get

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \leq K_G\mathbf{M}_{(2)}(Y_m)\sqrt{m}\mathbf{M}_{(2)}(X_n)\mathbf{M}_{(2)}(Y_m).$$

Since Y is 2-concave,

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \prec \sqrt{m}\mathbf{M}_{(2)}(X_n).$$

On the other hand we have

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \leq \lambda_X(n)\mathbf{M}_{(2)}(X_n)\mathbf{M}_{(2)}(Y_m).$$

Since X is 2-convex with finite concavity, Proposition 4.8.3 implies

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \prec \sqrt{n}\mathbf{M}_{(2)}(Y_m).$$

We have the upper estimate. For the lower one, starting from Proposition 4.6.5 and applying (4.7) we get to (4.25). Since X is 2-convex with finite concavity by (4.10) and Proposition 4.8.3, $l(\ell_2^n \to X_n) \asymp \lambda_X(n) \asymp \sqrt{n}/\mathbf{M}_{(2)}(X_n)$ holds. Also that $\|\ell_2^n \to X_n\| \geq 1$. Hence

$$\frac{l(\ell_2^n \to X_n)}{\sqrt{nm}\ \|\ell_2^n \to X_n\|} \prec \frac{\sqrt{n}}{\sqrt{nm}\ \mathbf{M}_{(2)}(X_n)} = \frac{1}{\sqrt{m}\ \mathbf{M}_{(2)}(X_n)}.$$

On the other hand, since Y is 2-concave $l(\ell_2^m \to Y_m) \asymp \lambda_Y(m) \leq \sqrt{m}\|\ell_2^m \to Y_m\|$ and $\mathbf{M}_{(2)}(Y_m) \leq \mathbf{M}_{(2)}(Y) < \infty$ for all m. Hence

$$\frac{l(\ell_2^m \to Y_m)}{\sqrt{nm}\ \|\ell_2^m \to Y_m\|} \prec \frac{\mathbf{M}_{(2)}(Y_m)}{\sqrt{n}\mathbf{M}_{(2)}(Y_m)} \prec \frac{1}{\sqrt{n}\mathbf{M}_{(2)}(Y_m)}.$$

This gives the lower estimate and completes the proof.
q.e.d

As a straightforward consequence we have the following characterization.

Corollary 4.11.2
Let X, Y be any two Banach sequence space; then X, Y are both 2-concave if and only if

$$\mathbf{C}_2(X_n \otimes_\varepsilon Y_m) \asymp \min(\sqrt{n}, \sqrt{m}).$$

Proof.
The 'only if' follows from Proposition 4.11.1. The 'if' implication follows from the fact that fixing $m = 1$ we have $X_n \otimes_\varepsilon Y_m = X_n \otimes_\varepsilon \mathbb{K} = X_n$ and

$$\mathbf{C}_2(X_n) \asymp \min(\sqrt{n}, 1) = 1.$$

In other words, the $\mathbf{C}_2(X_n)$ are bounded and X is not isometric to ℓ_∞. Therefore X has cotype 2. Hence X is 2-concave. The same proof is valid for Y.
q.e.d.

It is well known that, if either E or F are finite dimensional, $\mathcal{L}(E;F) = E' \otimes_\varepsilon F$ holds isometrically. We obtain the following corollary.

Corollary 4.11.3
Let X be either 2-concave with non-trivial convexity or 2-convex with non-trivial concavity. Then

$$\mathbf{C}_2(\mathcal{L}(X_n;X_n)) \asymp \sqrt{n}.$$

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