On Functions of Integrable Mean Oscillation

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ABSTRACT

Given $f \in L^1(\mathbb{T})$ we denote by $w_{\rm mo}(f)$ the modulus of mean oscillation given by

$$w_{\rm mo}(f)(t) = \sup_{0<|I|\leq t} \frac{1}{|I|}\int_I |f(e^{i\theta}) - m_I(f)|\frac{d\theta}{2\pi}$$

where I is an arc of $\mathbb{T}$, $|I|$ stands for the normalized length of I, and $m_I(f) = \frac{1}{|I|}\int_I f(e^{i\theta})\frac{d\theta}{2\pi}$. Similarly we denote by $w_{\rm ho}(f)$ the modulus of harmonic oscillation given by

$$w_{\rm ho}(f)(t) = \sup_{1-t\leq |z|<1} \int_{\mathbb{T}} |f(e^{i\theta}) - P(f)(z)|P_z(e^{i\theta})\frac{d\theta}{2\pi}$$

where $P_z(e^{i\theta})$ and $P(f)$ stand for the Poisson kernel and the Poisson integral of f respectively.

It is shown that, for each $0 < p < \infty$, there exists $C_p > 0$ such that

$$\int_0^1 [w_{\rm mo}(f)(t)]^p \frac{dt}{t} \leq \int_0^1 [w_{\rm ho}(f)(t)]^p \frac{dt}{t} \leq C_p \int_0^1 [w_{\rm mo}(f)(t)]^p \frac{dt}{t}.$$

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1. Introduction.

As usual we denote by BMO the space of functions $f \in L^1(\mathbb{T})$ such that

$$\|f\|_* = \sup_{I \subseteq \mathbb{T}} \frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)| \frac{d\theta}{2\pi} < \infty,$$

where I is an arc of the circle $\mathbb{T}$, $|I|$ stands for the normalized length of I and $m_I(f) = \frac{1}{|I|} \int_I f(e^{i\theta}) \frac{d\theta}{2\pi}$. We write $\|f\|_{\mathrm{BMO}} = |\hat{f}(0)| + \|f\|_*$.

If $f \in L^1(\mathbb{T})$ and $0 < t \le 1$, we define the modulus of mean oscillation of f at the point t as

$$w_{\mathrm{mo}}(f)(t) = \sup_{0<|I|\le t} \frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)| \frac{d\theta}{2\pi}.$$

Clearly, for $0 < t \le s < 1$, one has

$$\sup_{t<|I|\le s} \frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)| \frac{d\theta}{2\pi} \le \frac{2}{t} \|f\|_1.$$

Hence, for $0 < t \le s < 1$, one has

$$w_{\mathrm{mo}}(f)(t) \le w_{\mathrm{mo}}(f)(s) \le \max\left\{ w_{\mathrm{mo}}(f)(t), \frac{2\|f\|_1}{t} \right\}. \tag{1}$$

In particular, $f \in \mathrm{BMO}$ if and only if $w_{\mathrm{mo}}(f)(t) < \infty$ for some (or for all) $0 < t \le 1$.

It is known that one can consider other equivalent moduli to define BMO. For instance, for $0 < q < \infty$,

$$w_{\mathrm{mo},q}(f)(t) = \sup_{0<|I|\le t} \left(\frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)|^q \frac{d\theta}{2\pi} \right)^{1/q}.$$

It is also well known, by the John-Nirenberg lemma (see [7, 8]), that there exist $C_1, C_2 > 0$ such that for all $\lambda > 0$ and any arc I with $|I| \le t$,

$$\frac{|\{\, \theta \in \mathbb{T} : |f(e^{i\theta}) - m_I| > \lambda \,\}|}{|I|} \le C_1 e^{-\frac{C_2 \lambda}{w_{\mathrm{mo}}(f)(t)}}.$$

From here one gets that, for all $t > 0$,

$$w_{\mathrm{mo}}(f)(t) \approx w_{\mathrm{mo},q}(f)(t). \tag{2}$$

One can also consider

$$w'_{\mathrm{mo}}(f)(t) = \sup_{|I|\le t} \left(\frac{1}{|I|^2} \int_I \int_I |f(e^{i\theta}) - f(e^{i\varphi})| \frac{d\theta}{2\pi} \frac{d\varphi}{2\pi} \right)$$

or

$$\tilde{w}_{\mathrm{mo}}(f)(t) = \sup_{|I|\leq t}\left(\inf_c\left(\frac{1}{|I|}\int_I |f(e^{i\theta}) - c|\frac{d\theta}{2\pi}\right)\right)$$

Clearly one gets

$$w_{\mathrm{mo}}(f)(t) \leq w'_{\mathrm{mo}}(f)(t) \leq 2w_{\mathrm{mo}}(f)(t) \tag{3}$$

and

$$\tilde{w}_{\mathrm{mo}}(f)(t) \leq w_{\mathrm{mo}}(f)(t) \leq 2\tilde{w}_{\mathrm{mo}}(f)(t).$$

A function f is said to have vanishing mean oscillation, in short $f \in \mathrm{VMO}$, if

$$\lim_{|I|\to 0}\frac{1}{|I|}\int_I |f(e^{i\theta}) - m_I(f)|\frac{d\theta}{2\pi} = 0.$$

This is a closed subspace of BMO, which can be characterized in many ways (see [7, 8, 15]).

Theorem 1.1. *Let $f \in \mathrm{BMO}$. The following statements are equivalent:*

(i) $f \in \mathrm{VMO}$.

(ii) $\lim_{t\to 0^+}\|T_t f - f\|_{\mathrm{BMO}} = 0$, *where* $T_t f(e^{i\theta}) = f(e^{i(\theta - t)})$.

(iii) $\lim_{r\to 1}\|P_r * f - f\|_{\mathrm{BMO}} = 0$, *where* $P_r(e^{i\theta}) = \Re(\frac{1+re^{-i\theta}}{1-re^{-i\theta}})$.

(iv) *f belongs to the closure of $C(\mathbb{T})$ in* BMO.

(v) $\lim_{t\to 0^+} w_{\mathrm{mo}}(f)(t) = 0$.

A generalization of BMO is the space $\mathrm{BMO}(\rho)$, consisting of functions $f \in L^1(\mathbb{T})$ such that $w_{\mathrm{mo}}(f)(t) = O(\rho(t))$ for a fixed function ρ with certain properties. The space $\mathrm{BMO}(\rho)$ has been considered by various authors (see [10, 15, 17]).

Our aim will be to analyze spaces where the function ρ is not explicitly given, but we do know its behavior at the origin in terms of certain integrability conditions.

Given $0 < p < \infty$, we shall denote by $\mathrm{MO}^p(\mathbb{T})$ the space of integrable functions such that $\int_0^1 [w_{\mathrm{mo}}(f)(t)]^p \frac{dt}{t} < \infty$.

Due to (2), the spaces MO^p_q of functions such that $\int_0^1 [w_{\mathrm{mo},q}(f)(t)]^p \frac{dt}{t} < \infty$ are all the same for $0 < q < \infty$.

These spaces were considered in [12] (see page 74), under a different notation. Also some spaces $\mathrm{MO}^{\alpha}_{s,r}$, which are closely related to the ones considered in this paper, were introduced in [13].

We use the notations

$$\omega_\infty(f)(t) = \sup_{|\theta-\varphi|\leq t}|f(e^{i\theta}) - f(e^{i\varphi})|$$

and

$$\omega_q(f)(t) = \sup_{|u|\le t}\left(\int_{\mathbb{T}}|f(e^{i(\theta+u)}) - f(e^{i\theta})|^q \frac{d\theta}{2\pi}\right)^{1/q}$$

for $0 < q < \infty$.

Now, for $0 < s < 1$ and $0 < p, q \le \infty$, the Besov space $B^s_{q,p}(\mathbb{T})$ consists of functions in $L^q(\mathbb{T})$ such that $t^{-s}\omega_q(f) \in L^p((0, 2\pi), \frac{dt}{t})$. Of course the cases $B^s_{q,\infty}$ where $0 < q \le \infty$ correspond to Lipschitz or Hölder classes, to be denoted $\mathrm{Lip}_s(\mathbb{T})$ instead of $B^s_{\infty,\infty}(\mathbb{T})$.

We denote by X_p the space consisting of functions $L^\infty(\mathbb{T})$ such that $\omega_\infty(f) \in L^p((0, 2\pi), \frac{dt}{t})$.

From (3) we easily obtain, for any $t > 0$,

$$w_{\mathrm{mo}}(f)(t) \le C\omega_\infty(f)(t).$$

Hence $X_p \subset MO^p(\mathbb{T})$ for any $0 < p < \infty$.

On the other hand, if I, J are arcs on $\mathbb{T}$ such that $I \subset J$ then

$$|m_J(f) - m_I(f)| \le \frac{|J|}{|I|}\, w_{\mathrm{mo}}(f)(|J|). \tag{4}$$

Now, given I with $|I| \le t$, using the Lebesgue differentiation theorem, one gets

$$f(e^{i\theta}) = \lim_n m_{I_n} f, \quad \text{for a.a. } \theta \in I,$$

where I_n is a decreasing sequence of arcs containing θ such that $|I_n| = 2|I_{n+1}|$.

Hence using (4), we have that for any $f \in \mathrm{MO}^1(\mathbb{T})$

$$\begin{aligned} |f(e^{i\theta}) - m_I| &\le \lim_n |m_{I_n} f - m_I f| \\ &\le \sum_{k=1}^{\infty} |m_{I_k} f - m_{I_{k-1}} f| \\ &\le C\sum_{k=1}^{\infty} w_{\mathrm{mo}}(f)(2^{-k}t) \\ &\le C\int_0^t w_{\mathrm{mo}}(f)(s)\frac{ds}{s}. \end{aligned}$$

Therefore we obtain, for any $t > 0$,

$$\omega_\infty(f)(t) \le 2C\int_0^t w_{\mathrm{mo}}(f)(s)\frac{ds}{s}. \tag{5}$$

This implies that $\mathrm{MO}^1(\mathbb{T}) \subset \mathrm{Lip}_\phi$, where Lip_ϕ stands for the space of continuous functions such that

$$|f(e^{i\theta}) - f(e^{i\varphi})| \le C\phi(|\theta - \varphi|),$$

for $\phi(t) = \sup\{\int_0^t w_{\text{mo}}(f)(s)\frac{ds}{s} : \int_0^1 w_{\text{mo}}(f)(t)\frac{dt}{t} \leq 1\}$.

BMO-type characterizations of these spaces have been extensively considered in the literature. The reader is referred to [2, 3, 9] for the case $p = \infty$ and to [4, 5] for the cases $0 < s$ and $1 < p < \infty$.

We shall consider a description of $\text{MO}^p(\mathbb{T})$ where the averages over arcs are replaced by averages with respect the Poisson kernel.

We denote by BMOH the space of functions $f \in L^1(\mathbb{T})$ such that

$$\|f\|_{**} = \sup_{z\in\Delta} \int_{\mathbb{T}} |f(e^{i\theta}) - P(f)(z)| P_z(e^{i\theta}) \frac{d\theta}{2\pi} < \infty,$$

where Δ denotes the open unit disc, $P(f)(z) = \int_{\mathbb{T}} f(e^{i\theta}) P_z(e^{i\theta}) \frac{d\theta}{2\pi}$ and $P_z(e^{i\theta}) = \Re(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}})$. We write $\|f\|_{\text{BMOH}} = |P(f)(0)| + \|f\|_{**}$.

It is not difficult to prove (see [7, 8]) that $f \in \text{BMO}$ if and only if $f \in \text{BMOH}$ with equivalent norms.

In this situation we define the modulus of harmonic oscillation of f at the point t as

$$w_{\text{ho}}(f)(t) = \sup_{1-t\leq|z|<1} \int_{\mathbb{T}} |f(e^{i\theta}) - P(f)(z)| P_z(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Hence, $f \in \text{BMO}$ (respect. BMOH) if and only if $w_{\text{mo}}(f)(1) < \infty$ (respect. $w_{\text{ho}}(f)(1) < \infty$).

For $0 < p < \infty$, we denote by $\text{HO}^p(\mathbb{T})$ the space of $f \in L^1(\mathbb{T})$ such that $\int_0^1 [w_{\text{ho}}(f)(t)]^p \frac{dt}{t} < \infty$.

Of course one can also use other moduli to define this space. For instance, for $0 < q < \infty$,

$$w_{\text{ho},q}(f)(t) = \sup_{1-t\leq|z|<1} \left(\int_{\mathbb{T}} |f(e^{i\theta}) - P(f)(z)|^q P_z(e^{i\theta}) \frac{d\theta}{2\pi} \right)^{1/q},$$

or

$$\tilde{w}_{\text{ho}}(f)(t) = \sup_{1-t\leq|z|<1} \inf_c \int_{\mathbb{T}} |f(e^{i\theta}) - c|\, P_z(e^{i\theta}) \frac{d\theta}{2\pi}.$$

The main objective of the paper is to show that, for $0 < p < \infty$, we have $\text{MO}^p(\mathbb{T}) = \text{HO}^p(\mathbb{T})$ and with equivalent "norms".

The paper is divided into two sections. The first one is devoted to introducing $\text{MO}^p(\mathbb{T})$ and proving some of its properties and the second one to introducing $\text{HO}^p(\mathbb{T})$ and to showing that $\text{HO}^p(\mathbb{T})$ coincides with $\text{MO}^p(\mathbb{T})$.

2. Integrable mean oscillation.

We will see first that the modulus of mean oscillation is continuous. We shall use the following lemma.

Lemma 2.1. *Let $f \in L^1(\mathbb{T})$. If $\{I_n\}$ is a sequence of arcs such that $\lim_{n\to\infty} I_n = I$ for some arc I with $|I| > 0$ then*

$$\lim_{n\to\infty} \frac{1}{|I_n|}\int_{I_n} |f(e^{i\theta}) - m_{I_n}(f)|\frac{d\theta}{2\pi} = \frac{1}{|I|}\int_I |f(e^{i\theta}) - m_I(f)|\frac{d\theta}{2\pi}.$$

Proof. Let us first estimate

$$\begin{aligned}
\frac{1}{|I_n|}\int_{I_n} &|f(e^{i\theta}) - m_{I_n}(f)|\frac{d\theta}{2\pi} - \frac{1}{|I|}\int_I |f(e^{i\theta}) - m_I(f)|\frac{d\theta}{2\pi} \\
&\le \frac{1}{|I_n|}\int_{I_n} |f(e^{i\theta}) - m_I(f)|\frac{d\theta}{2\pi} + |m_{I_n}(f) - m_I(f)| \\
&\quad - \frac{1}{|I|}\int_I |f(e^{i\theta}) - m_I(f)|\frac{d\theta}{2\pi} \\
&\le \frac{1}{|I|}\Big(\int_{I_n} |f(e^{i\theta}) - m_I(f)|\frac{d\theta}{2\pi} - \int_I |f(e^{i\theta}) - m_I(f)|\frac{d\theta}{2\pi}\Big) \\
&\quad + |m_{I_n}(f) - m_I(f)| + 2\|f\|_1\Big(\frac{1}{|I_n|} - \frac{1}{|I|}\Big).
\end{aligned}$$

Notice that $\nu(A) = \int_A f(e^{i\theta})\frac{d\theta}{2\pi}$ and $\mu^I(A) = \int_A |f(e^{i\theta}) - m_I(f)|\frac{d\theta}{2\pi}$ are a complex and a positive measure respectively with integrable densities. Therefore the result follows passing to the limit as n goes to ∞. □

Proposition 2.2. *Let $f \in \mathrm{BMO}$. Then $w_{\mathrm{mo}}(f)$ is increasing and continuous in $(0, 1]$.*

Proof. Monotonicity has already been proved in our formula (1).

Let $0 < t_0 \le 1$ and let us prove that it is left continuous at t_0. Given $\varepsilon > 0$ we find $I_{t_0} \subset \mathbb{T}$ such that $0 < |I_{t_0}| \le t_0$ and

$$w_{\mathrm{mo}}(f)(t_0) \le \frac{1}{|I_{t_0}|}\int_{I_{t_0}} |f(e^{i\theta}) - m_{I_{t_0}}(f)|\frac{d\theta}{2\pi} + \frac{\varepsilon}{2}.$$

Let (t_n) be a sequence such that $t_n \le t_0$ for all $n \in \mathbb{N}$ and converges to t_0.

If $|I_{t_0}| = t_0$, we can find $I_n \subset I_{t_0}$ such that $\lim_{n\to\infty} I_n = I_{t_0}$. Hence

$$\begin{aligned}
w_{\mathrm{mo}}(f)(t_0) &- w_{\mathrm{mo}}(f)(t_n) \le \\
&\le \frac{1}{|I_{t_0}|}\int_{I_{t_0}} |f(e^{i\theta}) - m_{I_{t_0}}(f)|\frac{d\theta}{2\pi} - \frac{1}{|I_n|}\int_{I_n} |f(e^{i\theta}) - m_{I_n}(f)|\frac{d\theta}{2\pi} + \frac{\varepsilon}{2}.
\end{aligned}$$

Now use Lemma 2.1 to get $\lim_{n\to\infty} w_{\mathrm{mo}}(f)(t_0) - w_{\mathrm{mo}}(f)(t_n) = 0$.

If $|I_{t_0}| < t_0$ there exists n_0 such that $|I_{t_0}| \le t_n$ for $n \ge n_0$. Hence $w_{\mathrm{mo}}(f)(t_0) - w_{\mathrm{mo}}(f)(t_n) < \frac{\varepsilon}{2}$ for $n \ge n_0$.

To see that it is right continuous at t_0, we shall argue as follows: Let (t_n) be a sequence such that $t_n \geq t_0$ for all $n \in \mathbb{N}$ and converges to t_0. We shall find a subsequence (t_{n_k}) such that $\lim_{k\to\infty} w_{\mathrm{mo}}(f)(t_{n_k}) = w_{\mathrm{mo}}(f)(t_0)$.

Given $\varepsilon > 0$ we find $I_n \subset \mathbb{T}$ such that $0 < |I_n| \leq t_n$ and

$$w_{\mathrm{mo}}(f)(t_n) \leq \frac{1}{|I_n|} \int_{I_n} |f(e^{i\theta}) - m_{I_n}(f)| \frac{d\theta}{2\pi} + \varepsilon.$$

Let $\mathcal{F} = \{n \in \mathbb{N} : |I_n| > t_0\}$. If $\mathcal{F}$ is finite then $|I_n| \leq t_0$ for $n \geq n_0$ and

$$w_{\mathrm{mo}}(f)(t_n) - w_{\mathrm{mo}}(f)(t_0) < \varepsilon \quad \text{for } n \geq n_0.$$

Without loss of generality we assume $|I_n| > t_0$ for all $n \in \mathbb{N}$.

Call $I_0 = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} I_k$. It is easy to see that I_0 is an arc and that $|I_0| = t_0$. Take a subsequence n_k such that (I_{n_k}) converges to I_0. We have

$$\begin{aligned} w_{\mathrm{mo}}(f)(t_{n_k}) - w_{\mathrm{mo}}(f)(t_0) &\leq w_{\mathrm{mo}}(f)(t_{n_k}) - \frac{1}{|I_0|} \int_{I_0} |f(e^{i\theta}) - m_{I_0}(f)| \frac{d\theta}{2\pi} \\ &\leq \frac{1}{|I_{n_k}|} \int_{I_{n_k}} |f(e^{i\theta}) - m_{I_{n_k}}(f)| \frac{d\theta}{2\pi} - \frac{1}{|I_0|} \int_{I_0} |f(e^{i\theta}) - m_{I_0}(f)| \frac{d\theta}{2\pi} + \varepsilon. \end{aligned}$$

The proof is complete invoking Lemma 2.1. □

Remark 2.3. Let $f \in \mathrm{BMO}$ and take $a(f) = \lim_{t\to 0^+} w_{\mathrm{mo}}(f)(t)$. Hence $f \in \mathrm{VMO}$ if and only if $a(f) = 0$.

For each $0 < p < \infty$, we define the quasi-norm (norm for $p \geq 1$) on $\mathrm{MO}^p(\mathbb{T})$ by

$$\|f\|_{\mathrm{MO}^p} = \|f\|_{L^1(\mathbb{T})} + \left(\int_0^1 [w_{\mathrm{mo}}(f)(t)]^p \frac{dt}{t} \right)^{1/p}.$$

Although the next result is probably known, we include a proof for the sake of completeness.

Theorem 2.4. *Let $0 < p < \infty$. Then $(\mathrm{MO}^p(\mathbb{T}), \ \|.\|_{\mathrm{MO}^p})$ is a complete space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $\mathrm{MO}^p(\mathbb{T})$. In particular, there exists $f \in \mathrm{BMO}$ such that $\{f_n\}$ converges to f.

Let $|I| \leq t$, $0 < t \leq 1$. Using that $f_n \to f$ in $L^1(\mathbb{T})$ we get that $m_I(f_n) \to m_I(f)$ and that there exists a subsequence (n_k), such that $f_{n_k} \to f$ a.e.

Now

$$\begin{aligned}
\frac{1}{|I|}\int_I &|f_n(e^{i\theta}) - f(e^{i\theta}) - m_I(f_n - f)|\frac{d\theta}{2\pi}\\
&= \frac{1}{|I|}\int_I |f_n(e^{i\theta}) - \lim_k f_{n_k}(e^{i\theta}) - \lim_k \ m_I(f_n - f_{n_k})|\frac{d\theta}{2\pi}\\
&= \frac{1}{|I|}\int_I \lim_k |f_n(e^{i\theta}) - f_{n_k}(e^{i\theta}) - m_I(f_n - f_{n_k})|\frac{d\theta}{2\pi}\\
&\leq \liminf_k \frac{1}{|I|}\int_I |f_n(e^{i\theta}) - f_{n_k}(e^{i\theta}) - m_I(f_n - f_{n_k})|\frac{d\theta}{2\pi}\\
&\leq \liminf_k w_{\text{mo}}(f_n - f_{n_k})(t).
\end{aligned}$$

Therefore

$$w_{\text{mo}}(f_n - f)(t) \leq \liminf_k w_{\text{mo}}(f_n - f_{n_k})(t).$$

Hence

$$\begin{aligned}
\int_0^1 \Big[w_{\text{mo}}(f_n - f)(t)\frac{dt}{t}\Big]^p &\leq \int_0^1 \liminf_k [w_{\text{mo}}(f_n - f_{n_k})(t)]^p \frac{dt}{t}\\
&\leq \liminf_k \int_0^1 [w_{\text{mo}}(f_n - f_{n_k})(t)]^p \frac{dt}{t}
\end{aligned}$$

Finally, using that f_n is a Cauchy sequence we get $\lim_{n\to\infty}\|f_n - f\|_{\text{MO}^p} = 0$ and that $f \in \text{MO}^p$. □

Proposition 2.5. *Let* $0 < p \leq q < \infty$ *and* $s > 0$*.*

(i) $\text{MO}^p(\mathbb{T}) \subseteq \text{MO}^q(\mathbb{T})$.

(ii) $\text{Lip}_s(\mathbb{T}) \subset \bigcap_{p>0} \text{MO}^p(\mathbb{T}) \subset \text{MO}^1(\mathbb{T}) \subset C(\mathbb{T})$.

(iii) $\bigcup_{p>0} \text{MO}^p(\mathbb{T}) \subset \text{VMO}$.

Proof. (i) It is a consequence of the following fact:

$$\Big(\int_0^1 [w_{\text{mo}}(f)(t)]^p \frac{dt}{t}\Big)^{1/p} \approx \Big(\sum_{k=0}^{\infty} [w_{\text{mo}}(f)(2^{-k})]^p\Big)^{1/p}.$$

(ii) Note that $f \in \text{Lip}_s$ if and only if $w_{\text{mo}}(f)(t) \leq Ct^s$. This gives the first inclusion.

The fact that $\text{MO}^1(\mathbb{T}) \subset C(\mathbb{T})$ follows by (5).

(iii) Observe that, for any $p > 0$ and $t > 0$, one has

$$w_{\text{mo}}(f)^p(t)\log\frac{1}{t} \ \leq \ \int_t^1 w_{\text{mo}}(f)^p(u)\frac{du}{u} \leq \|f\|_{\text{MO}^p}.$$

Hence $\lim_{t\to 0^+} w_{\text{mo}}(f)(t) = 0$ for $f \in \cup_{p>0} \text{MO}^p(\mathbb{T})$. □

Let us point out some properties of BMO that are shared by these spaces.

Proposition 2.6. *If* $f \in \mathrm{MO}^p(\mathbb{T})$ *then* $|f| \in \mathrm{MO}^p(\mathbb{T})$.

Proof. Let $t \in (0,1)$ and $I \subset \mathbb{T}$ with $|I| \leq t$. Then

$$\begin{aligned}\frac{1}{|I|}\int_I \big||f(e^{i\theta})| - m_I(|f|)\big|\frac{d\theta}{2\pi} &\leq \frac{1}{|I|}\int_I \big||f(e^{i\theta})| - |m_I(f)|\big|\frac{d\theta}{2\pi} \\ &\quad + \big|m_I(|f|) - |m_I(f)|\big| \\ &\leq \frac{2}{|I|}\int_I \big||f(e^{i\theta})| - |m_I(f)|\big|\frac{d\theta}{2\pi} \\ &\leq \frac{2}{|I|}\int_I |f(e^{i\theta}) - m_I(f)|\frac{d\theta}{2\pi}\end{aligned}$$

This shows that $w_{\mathrm{mo}}(|f|)(t) \leq 2\, w_{\mathrm{mo}}(f)(t)$ and the proof is complete. □

Recall that T_t denotes the translation operator, that is $T_t f(e^{i\theta}) = f(e^{i(\theta - t)})$. We have the following result.

Theorem 2.7. *Let* $0 < p < \infty$ *and* $f \in \mathrm{MO}^p(\mathbb{T})$. *Then*

$$\lim_{s\to 0^+} \|T_s f - f\|_{\mathrm{MO}^p} = 0.$$

Proof. Due to (iii) in Proposition 2.5 $f \in \mathrm{VMO}$. Now Theorem 1.1 gives that $\lim_{s\to 0^+}\|T_s f - f\|_{\mathrm{BMO}} = 0$.

Note that $w_{\mathrm{mo}}(T_s f - f)(t) \leq \|T_s f - f\|_{\mathrm{BMO}}$ for all $0 < t \leq 1$.

On the other hand

$$\begin{aligned} w_{\mathrm{mo}}(T_s f - f)(t) &= \sup_{|I|\leq t} \frac{1}{|I|}\int_I \big|(T_s f - f)(e^{i\theta}) - m_I(T_s f - f)\big|\,\frac{d\theta}{2\pi} \\ &\leq \sup_{|I|\leq t} \frac{1}{|I|}\int_I \big|T_s f(e^{i\theta}) - m_I(T_s f)\big|\,\frac{d\theta}{2\pi} \\ &\quad + \sup_{|I|\leq t} \frac{1}{|I|}\int_I \big|f(e^{i\theta}) - m_I(f)\big|\,\frac{d\theta}{2\pi} \\ &= 2\, w_{\mathrm{mo}}(f)(t)\end{aligned}$$

The Lebesgue dominated convergence theorem gives $\lim_{s\to 0^+}\|T_s f - f\|_{\mathrm{MO}^p} = 0$. □

3. Integrable harmonic oscillation.

Throughout this section, given $z \in \Delta \setminus \{0\}$, we denote by I_z the open arc in $\mathbb{T}$ with midpoint $\frac{z}{|z|}$ and length $|I_z| = 1 - |z|$. Given an arc $I \subset \mathbb{T}$ and $\lambda \leq |I|^{-1}$ we shall write λI for the arc with the same midpoint and length $\lambda |I|$.

Let us collect several known facts to be used later on.

Lemma 3.1. *There exist constants* $0 < C, C_1, C_2, C_3 < \infty$ *such that*

(i) $1 - |z| \leq |e^{i\theta} - z| \leq C\ (1 - |z|)$, $e^{i\theta} \in I_z$, *and* $z \in \Delta$.

(ii) $C_1 \frac{1}{|I_z|} \leq P_z(e^{i\theta}) \leq C_2 \frac{1}{|I_z|}$, $e^{i\theta} \in I_z$, *and* $z \in \Delta$.

(iii) $\frac{1}{4^k|I_z|} \leq P_z(e^{i\theta}) \leq C_3 \frac{1}{4^k|I_z|}$, $e^{i\theta} \in 2^k I_z \setminus 2^{k-1} I_z$, $k \in \{1, 2, \dots, N+1\}$, *where* $N = [\log_2 \frac{1}{|I_z|}]$ *and* $z \in \Delta$.

Proof. All the statements follow from the estimates

$$1 - |z| \leq |e^{i\theta} - z| \leq \left| e^{i\theta} - \frac{z}{|z|} \right| + (1 - |z|)$$

and

$$\left| e^{i\theta} - \frac{z}{|z|} \right| \leq |e^{i\theta} - z| + (1 - |z|). \qquad \square$$

For $0 < p < \infty$ we define

$$\|f\|_{\mathrm{HO}^p} = \|f\|_{L^1(\mathbb{T})} + \left(\int_0^1 [w_{\mathrm{ho}}(f)(t)]^p \frac{dt}{t} \right)^{1/p}$$

to get a quasi-norm in the space $\mathrm{HO}^p(\mathbb{T})$.

Proposition 3.2. *If* $f \in L^1(\mathbb{T})$ *and* $0 < t \leq 1$ *then* $w_{\mathrm{mo}}(f)(t) \leq c\, w_{\mathrm{ho}}(f)(t)$.

Proof. Let $I \subseteq \mathbb{T}$ be an arc such that $|I| \leq t$. Consider $z \in \Delta$ for which $I = I_z$. From $|I_z| = 1 - |z| \leq t$ we have $1 - t \leq |z| < 1$.

Using (ii) in Lemma 3.1 we have

$$\begin{aligned}
\frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)| \frac{d\theta}{2\pi} &\leq \frac{1}{|I_z|} \int_{I_z} |f(e^{i\theta}) - P(f)(z)| \frac{d\theta}{2\pi} \\
&\quad + |m_I(f) - P(f)(z)| \\
&\leq \frac{2}{|I_z|} \int_{I_z} |f(e^{i\theta}) - P(f)(z)| \frac{d\theta}{2\pi} \\
&\leq C \left(\int_{-\pi}^{\pi} |f(e^{i\theta}) - P(f)(z)| P_z(\theta) \frac{d\theta}{2\pi} \right) \\
&\leq C w_{\mathrm{ho}}(f)(t)
\end{aligned}$$

Now taking the supremum over all arcs we get $w_{\mathrm{mo}}(f)(t) \leq C\ w_{\mathrm{ho}}(f)(t)$. $\square$

Theorem 3.3. *Let* $0 < p < \infty$. *Then* $\mathrm{HO}^p(\mathbb{T}) = \mathrm{MO}^p(\mathbb{T})$ *with equivalent quasi-norms.*

Proof. $\mathrm{HO}^p(\mathbb{T}) \subseteq \mathrm{MO}^p(\mathbb{T})$ follows from Proposition 3.2.

Assume now that $f \in \mathrm{MO}^p(\mathbb{T})$. Let us show that $f \in \mathrm{HO}^p(\mathbb{T})$ and $\|f\|_{\mathrm{HO}^p} \leq C\|f\|_{\mathrm{MO}^p}$.

Let $t \in (0,1]$. For any $z \in \Delta$ with $|z| = 1-t$ we consider the arc $I = I_z$, which gives $|I_z| = 1 - |z| = t$. Take N and I_k ($k = 0, 1, \dots, N$) so that $I_k = 2^k I_z$ where $I_0 = I$, $I_{N-1} \subsetneq \mathbb{T}$, and $I_N = \mathbb{T}$.

Using (iii) in Lemma 3.1 we have

$$\begin{aligned}
\int_{\mathbb{T}} &|f(e^{i\theta}) - P(f)(z)| P_z(e^{i\theta}) \frac{d\theta}{2\pi} \\
&\leq 2 \int_{\mathbb{T}} |f(e^{i\theta}) - m_I(f)| P_z(e^{i\theta}) \frac{d\theta}{2\pi} \\
&\leq C\Big(\int_I |f(e^{i\theta}) - m_I(f)| P_z(e^{i\theta}) \frac{d\theta}{2\pi} \\
&\quad + \sum_{k=1}^N \int_{I_k \setminus I_{k-1}} |f(e^{i\theta}) - m_I(f)| P_z(e^{i\theta}) \frac{d\theta}{2\pi}\Big) \\
&\leq C\bigg(\frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)| \frac{d\theta}{2\pi} \\
&\quad + \sum_{k=1}^N \frac{1}{4^k|I|} \int_{I_k \setminus I_{k-1}} |f(e^{i\theta}) - m_I(f)| \frac{d\theta}{2\pi} \bigg) \\
&\leq C\bigg(w_{\mathrm{mo}}(f)(t) + \sum_{k=1}^N \frac{1}{2^k|I_k|} \int_{I_k} |f(e^{i\theta}) - m_I(f)| \frac{d\theta}{2\pi} \bigg).
\end{aligned}$$

On the other hand, by (4),

$$\begin{aligned}
\frac{1}{|I_k|} &\int_{I_k} |f(e^{i\theta}) - m_I(f)| \frac{d\theta}{2\pi} \\
&\leq \frac{1}{|I_k|} \int_{I_k} \left| f(e^{i\theta}) - m_{I_k}(f) + \left(\sum_{j=1}^k m_{I_j}(f) - m_{I_{j-1}}(f) \right) \right| \frac{d\theta}{2\pi} \\
&\leq \frac{1}{|I_k|} \int_{I_k} |f(e^{i\theta}) - m_{I_k}(f)| \frac{d\theta}{2\pi} + \sum_{j=1}^k |m_{I_j}(f) - m_{I_{j-1}}(f)| \\
&\leq \frac{1}{|I_k|} \int_{I_k} |f(e^{i\theta}) - m_{I_k}(f)| \frac{d\theta}{2\pi} + \sum_{j=1}^k \frac{|I_j|}{|I_{j-1}|} w_{\mathrm{mo}}(f)(|I_j|) \\
&\leq w_{\mathrm{mo}}(f)(|I_k|) + \sum_{j=1}^k 2 w_{\mathrm{mo}}(f)(|I_j|) \\
&\leq (1+2k) w_{\mathrm{mo}}(f)(|I_k|).
\end{aligned}$$

Combining both estimates one gets

$$\int_{\mathbb{T}} |f(e^{i\theta}) - P(f)(z)| P_z(e^{i\theta}) \frac{d\theta}{2\pi} \leq C\bigg(w_{\mathrm{mo}}(f)(t) + \sum_{k=1}^{N} \frac{1+2k}{2^k} w_{\mathrm{mo}}(f)(|I_k|)\bigg).$$

Taking the supremum over $\{z : |z| = 1-t\}$ and using $|I_k| = 2^k t$ we obtain

$$\sup_{|z|=1-t} \int_{\mathbb{T}} |f(e^{i\theta}) - P(f)(z)| P_z(e^{i\theta}) \frac{d\theta}{2\pi} \leq C\bigg(w_{\mathrm{mo}}(f)(t) + \sum_{k=1}^{N_t} \frac{1+2k}{2^k} w_{\mathrm{mo}}(f)(2^k t)\bigg)$$

where $N = N_t = [\log_2 \frac{1}{t}] + 1$. This implies that

$$w_{\mathrm{ho}}(f)(t) \leq C\bigg(w_{\mathrm{mo}}(f)(t) + \sum_{k=1}^{N_t} \frac{1+2k}{2^k} w_{\mathrm{mo}}(f)(2^k t)\bigg).$$

For $0 < p < 1$ we have

$$[w_{\mathrm{ho}}(f)(t)]^p \leq C_p\bigg([w_{\mathrm{mo}}(t)(t)]^p + \sum_{k=1}^{N_t} \frac{(1+2k)^p}{2^{pk}} [w_{\mathrm{mo}}(f)(2^k t)]^p\bigg).$$

For $p \geq 1$ we apply Hölder's inequality to obtain

$$[w_{\mathrm{ho}}(f)(t)]^p \leq C_p\bigg([w_{\mathrm{mo}}(t)(t)]^p + \sum_{k=1}^{N_t} \frac{(1+2k)^p}{2^{k}} [w_{\mathrm{mo}}(f)(2^k t)]^p\bigg).$$

Now integrating, and taking into account that $1 \leq k \leq N_t = [\log_2 \frac{1}{t}] + 1$ is equivalent to $0 < t \leq 2^{-k}$, we get

$$\begin{aligned}
\int_0^1 [w_{\mathrm{ho}}(f)(t)]^p \frac{dt}{t} &\leq C_p \int_0^1 [w_{\mathrm{mo}}(t)(t)]^p \frac{dt}{t} + C_p \int_0^1 \sum_{k=1}^{N_t+1} \frac{(1+2k)^p}{2^{k\min\{p,1\}}} [w_{\mathrm{mo}}(f)(2^k t)]^p \frac{dt}{t} \\
&\leq C_p \|f\|^p_{MO^p} + C_p \sum_{k=1}^{\infty} \frac{(1+2k)^p}{2^{k\min\{p,1\}}} \int_0^{2^{-k}} [w_{\mathrm{mo}}(f)(2^k t)]^p \frac{dt}{t} \\
&\leq C_p \|f\|^p_{MO^p} + C_p \sum_{k=1}^{\infty} \frac{(1+2k)^p}{2^{k\min\{p,1\}}} \int_0^{1} [w_{\mathrm{mo}}(f)(t)]^p \frac{dt}{t} \\
&\leq C \|f\|^p_{MO^p}.
\end{aligned}$$

Putting together all the estimates we have the result. □

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