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Jerarquies de models sigma: aplicacions a teories de Supergravetat i a teories conformes

PhD dissertation by

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Maria Antònia Lledó Barrena, Professora titular de la Universitat de València,

CERTIFICA:

Que la present memòria,

"Jerarquies de models sigma: aplicacions a teories de Supergravetat i a teories conformes" ha sigut realitzada sota la seua direcció al Departament de Física Teòrica de la Universitat de València-IFIC, per **Felip Alàez i Nadal** i constitueix la seua Tesi per optar al grau de Doctor en Física.

I per tal de què així conste, en compliment de la legislació vigent, presenta en el Departament de Física Teòrica la referida Tesi Doctoral, i signa el present certificat en València, a 2 de juliol de 2012.

Vist i plau tutor de tesi

Maria Antònia Lledó Barrena

Li ho direu a Pere Quart, li ho direu però amb moltíssima cura a poc a poc, com es deia aquell poeta que era fill d'un forner de Burjassot?

Vicent Andrés Estellés

Everything's gonna be alright.

Paul Butterfield

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A ma mare i al meu germà.

CONTENTS

1. INTRODUCTION

In this Thesis we investigate several aspects of the physical applications of Lie algebras. In particular, this work is divided in 3 parts: the construction of an associative, non commutative product on the Minkowski space [1, 2, 3], the development of sigma models [4] with left invariance under the action of the symmetry group and the study of the expansions of Lie algebras under discrete semigroups and their properties [5, 6].

The **first part** of this Thesis is developed in Chapters 3, 4 and 5.

The main objectives of this part of the work are:

- To define a non-commutative *star product* for the conformal complexification of Minkowski space.
- To give an explicit, analytic formula for the star product of two polynomials in Minkowski space.
- To show that the action of the star product on polynomials can be reproduced by a bidifferential operator.
- To define a coaction of the Poincaré group plus dilations on Minkowski space, compatible with the star product.
- To show that this coaction can be reproduced by a differential operator up to some order in the quantization paramether.
- To complete the construction of the non commutative Minkowski and Euclidian spaces, giving adequate real forms.

The **second part** is developed in Chapters 6 and 7. There our goals are:

- To define a class of sigma models invariant under a symmetry group (ISM).

- To study the differences between these models and the correspondent gauged WZW models.
- To show that, in general, these models do not present conformal invariance.
- To relate different ISM models by contraction of Lie groups.

The **third part** is developed in Chapters 8 and 9. Our objectives are:

- To study properties of Lie algebras which are preserved under the expansion of the Lie algebra with a finite semigroup (S-expansion).
- To perform a classification of the S-expansions of simple algebras.
- To use the S-expansion procedure to find relations between 2-dimensional and 3-dimensional Lie algebras.

The structure of spacetime at a fundamental level has been discussed since the discovery of General Relativity. This theory describes gravity as the metric of spacetime, being the matter the source of the metric. The success of General Relavity describing gravity is remarkable (think for example on GPS devices).

With the discovery of quantum mechanics at the beginning of XXth century, it became clear that the fundamental structure of spacetime should come out of the combination of these two theories. The quantization of General Relativity, seen as a quantum field theory, gives a non renormalizable theory. Several alternatives to solve this problem have been proposed, like string theory or loop quantum gravity, which try to quantize gravity in different ways. There is even a hope that Supergravity can actually be finite [7]. Unfortunately, the technical complexity of these theories makes it impossible today to have a definitive theory of quantum gravity. What seems to be clear is that, at a fundamental level, spacetime should have a fuzzy structure described by some, generically non commuting, unknown operators.

It is possible to ask about the structure of this spacetime without the introduction of a dynamical theory. Several attempts have been done [8, 9, 10, 11], defining non commutative products in field theories, to introduce the non commutativity effects of spacetime in diferent ways.

In Section 2.4 we introduce the conformal complexification of the usual Minkowski spacetime by means of the Grassmannian manifold $G(2,4)$, i.e., the space of complex 2-planes in $\mathbb{C}^4$ space (see, for example, refs.[12, 13]). In this manifold a natural action of the Poincaré plus dilations group exists, given by the lower parabolic subgroup, $P_l \subset \mathrm{SL}(4,\mathbb{C})$. It is convenient to work with this group in an algebraic way, i.e., with the algebra of polynomials in the group variables, $\mathcal{O}(P_l)$. In this formalism the group law is encoded as a coproduct (dual of the product) and the inverse is generalized to the antipode. The action of the Poincaré group on Minkowski space is given by a coaction defined on the generators of Minkowski space. We call $\mathcal{O}(\mathrm{M})$ the algebra of polynomials in the Minkowski generators. This formalism is convenient because it allows us to perform the quantization of the Minkowski space in a direct way.

Quantum groups [14] can be seen as deformations of Lie groups. The operations of product, coproduct and antipode are defined in terms of a non commutativity parameter q. The algebras involved here are defined as polynomials in terms of non commutative generators (non commutative variables). In the particular case where $q = 1$, we recover the commutative algebra. In Section 2.5 a deformation of the Grassmannian and Minkowski space in terms of quantum groups is given. Moreover, in ref.[1] a quantization of chiral super Minkowski space in terms of quantum groups is given. We denote it by $\mathcal{O}_q(M)$.

Working with fields defined on the quantum variables $\mathcal{O}_q(\mathrm{M})$ offers a great difficulty. We can define a map $\mathbb{C}_q$ between $\mathcal{O}(\mathrm{M})$ and $\mathcal{O}_q(\mathrm{M})$, which are isomorphic as modules, so we work with 'classical' objects (the fields defined on the usual Minkowski space) and introduce the non commutativity using a non commutative product for the fields. This map is the so called *quantization map* or *ordering rule*. In Chapter 3 we define a non commutative product (*star product*) in $G(2,4)$ comming from the gluing of star products on the *big cells* (the Minkowski space). In Minkowski space, an ordering rule is used. This product is associative by construction and defined for polynomials in $G(2,4)$, i.e., purely algebraic. To apply this product to a field theory it is mandatory to make a generalization to smooth functions defined in $G(2,4)$. For this we have to find a differential expression of the star product. Redefining $q = e^h$ it is possible to perform an expansion, which can be reproduced by the action of bidifferential operators on the classical polynomials (see Section 3.2). This result is non trivial, because the coefficients multiplying each monomial must match carefully. A careful analysis on the structure of the terms which appear in the star product allows to demostrate that all the polynomials that appear in the development of the star product have the correct structure, so the star product is differentiable and its expression by means of a bidifferential operator is unique. Thanks to this we define the star product on smooth functions in $\mathcal{O}(\mathrm{M})$ as the corresponding expansion in terms of bidifferential operators. To write down the differential operators to an arbitrary order in the non commutativity paramether the explicit calculation must be done. We compute them to order 2 and show that they exist to arbitrary order.

It is also possible to define a star product for the Poincaré group (Chapter 4). An ordering rule for the generators of the group is defined (see Appendix A.2.5) and we follow a procedure analogous to the one used for the star product in Minkowski space. Next, we define a *star coaction* compatible with the star product, using the quantization map. The star coaction, when acting on the generators of Minkowski space, is formally identical to the classical one, being all the non commutative effects due to the presence of the star product. This coaction is algebraic, so to be able to apply it to smooth functions we need to find how to express it in terms of differential operators. In this case we study the action of the group on Minkowski space as a differential operator which acts on a single argument: the classical result of the action. It is possible to find a first order expression for the action, which can be reproduced by a differential operator. In this way we define the action of the group on smooth functions (up to first order).

Up to here we have worked with complex Minkowski space and complex groups. In Chapter 5 we discuss the problem of finding the corresponding real forms. Classically the problem reduces to that of finding an *involution*, i.e., an automorphism with properties (5.1), whose set of fixed points is the real form. We give an involutive automorphism for Minkowski and Euclidean space, with the real forms of the groups which act on them.

The quantum case is different, because the involution must be consistent with the commutation rules, which forces it to be an antiisomorphism (i.e. an *antiinvolution*). This rules out the interpretation of the real form as the set of fixed points of the antiinvolution. Another consequence is that, when we equip the classical real Minkowski space with the star product that we have defined, the Poisson bracket is purely imaginary.

Non linear *sigma models* are built with a set of 2 dimensional fields taking values at the points of a differentiable manifold (the so called *target space*). Although they are described in terms of local coordinates, differential manifolds do not have privileged reference coordinates as the linear spaces. Sigma models then present a global invariance with respect to dipheomorphisms of the target space. The fields in a sigma model interact mainly due to a Riemannian metric in the target manifold, represented by a symmetric covariant 2-tensor. They can also interact by means of some other objects, like an antisymmetric tensor or a scalar field (the dilaton).

It is specially interesting when there is a group acting on the manifold. The global properties of the manifold are important to study the action of the group: the easiest examples are the 'coset' spaces of type G/H, with H a subgroup of G. H is the isotropy group or *little group*. The action of the group G is transitive in this case. The cases where the target manifold is itself a Lie group are also interesting; the action is the left and right multiplication of the group. The action is not only transitive: it does not have any fixed point.

When the group acts by isometries of the metric (or, for other kinds of interactions, the Lie derivative of the object is zero) this global symmetry can be made local introducing a non linear connection in the space. This is what is called a *gauged sigma model*, which appears in supersymmetric and Supergravity theories.

In supersymmetric and Supergravity theories, the sigma models appears because the supersymmetry representations (multiplets) generically contain scalars whose lagrangians are sigma models plus interaction terms with other fields.

Sigma models also appear in the context of 2-dimensional theories. In this case the worldsheet of a string plays the role of the spacetime and the target space is the spacetime where the string moves. If there is conformal invariance these theories show invariance under the action of the infinite-dimensional Virasoro algebra. A classical example are Wess-Zumino-Witten (WZW) models, which are also invariant under Kac-Moody algebras.

In ref.[15] certain hierarchies of sigma models appearing in Supergravity models were discovered. These hierarchies correspond to generalized contractions [16, 17] of the isometry group of the original model. The contractions decouple some fields and model exact truncations or integrations on massive modes. Modelling the integrations of massive modes by the geometric procedure of a contraction simplifies technically the problem.

The 2-dimensional Wess-Zumino-Witten models describe vacuum solutions for a string. The WZW action contains two parts: the integral of a

3-form in a 3-dimensional manifold whose compact border is a compactification of the string worldsheet, and the metric term which is an integral on the worldsheet. The forms are biinvariants (left and right invariants) under the action of the symmetry group. A WZW model can be, at least locally, written as a sigma model with a biinvariant metric and a 2-form which, under the action of the symmetry group, changes by the differential of a function. The relative constant between both terms is choosen to get a conformally invariant model.

The gauging of a WZW model [18] is performed by minimal coupling if the antisymmetric tensor is invariant under the gauged isometries. If it is not, gauging is still possible if we include additional terms to the model, provided that the symmetry subgroup which we wish to gauge is anomaly free [18].

In the series of coset spaces $\mathrm{SO}(2,n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$ (Chapter 6) it is possible to define a metric and a left invariant 2-form. With these objects we construct what we call invariant sigma models (ISM). One question that we address is if the result of gauging the $\mathrm{SO}(2)\times\mathrm{SO}(n)$ subgrup in a $\mathrm{SO}(2,n)$ WZW model is a ISM model. The result is negative.

For example we take the $\mathrm{SO}(2,1)/\mathrm{SO}(2)$ group (the simplest one). We use *solvable coordinates* in the coset. The solvable coordinates are convenient because they allow us to perform the calculation of the metric and the 2-form easily and also give simple outputs for these objects. They also make the comparison with ref.[15] possible. In Section 6.1 we check that this model is different than the $\mathrm{SO}(2,1)/\mathrm{SO}(2)^{\mathrm{R}}$ gauged WZW model, which is a free boson. The one loop beta equations tell us that the ISM model is not conformally invariant. Instead, it is invariant under the left action of $\mathrm{SO}(2,1)$ on the coset. In ref.[15] it is shown that gauging a subgroup (H) of the isometry group (G) of a sigma model consisting only in the metric term is a sigma model in the quotient manifold (G/H), invariant under the left action of G. We show that this is not true anymore for a WZW model due to the existence of the antisymmetric tensor.

The next example is the coset $\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)$ (see Section 6.2). As before, we compute the metric and the 2-form and show that the model is not conformally invariant at a quantum level. Analogous conclusions are valid for the group $\mathrm{SO}(2,3)/\mathrm{SO}(2)\times\mathrm{SO}(3)$ (see Section 6.3).

In the series of symmetric spaces $\mathrm{SO}(2,n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$ it is also possible to relate groups with different n using contractions of Lie algebras. The procedure to contract the metric was defined in ref.[15], but the generalization to contract any invariant tensor is new.

The contraction of the coset space $\mathrm{SO}(2,3)/\mathrm{SO}(2)\times\mathrm{SO}(3)$ with respect to $\mathrm{SO}(1,3)/\mathrm{SO}(3)$ has special interest. In this case it is possible the calculation of the 2-form to get a $(\mathrm{SO}(3,1)/\mathrm{SO}(3))\times\mathbb{R}^m$ model. Unfortunately the model is not left invariant under the full $\mathrm{SO}(1,3)$ group.

The deformation of Lie algebras is a procedure which has importance in Mathematics and Physics. In Chapter 7 we have studied how to relate ISM models with different symmetry group using Inönü-Wigner contractions. We can find contractions applied to Supergravity models in ref.[15]. A contraction of a Lie group is a procedure which changes the structure constants without changing the number of generators.

Expansions of Lie algebras by discrete semigroups (*S-expansions*, see Section 2.11) were introduced some years ago in refs.[19, 20, 21, 22, 23]. We take a discrete semigroup and a Lie algebra and define a new Lie bracket in the direct product space. It can be shown that this bracket is associative, antisymmetric and satisfies the Jacobi identity, so the result is a Lie algebra. An S-expansion changes the dimension of the algebra, since it goes from an n-dimensional algebra to a $n\times m$-dimensional one (being m the order of the semigroup).

It is possible to extract algebras of a smaller dimension from an S-expanded algebra. It is the case where there is a *resonant decomposition* of the semigroup. One can then extract the *resonant subalgebra* of the S-expanded algebra. In case the semigroup has a zero element, it is also possible to perform a reduction by zero. The reduced algebra is a quotient of the S-expanded one. Sometimes it is even possible to perform two reductions by zero. This is reviewed in Section 2.11.

There are certain properties of the algebras that are preserved under an S-expansion. We study them in Chapter 8. When we expand a solvable algebra the result is another solvable algebra. A consequence of this is the solvability of the resonant subalgebra. Finally, the 0-reduced algebra is also solvable. The same happens with the nilpotency.

When we expand a semisimple algebra we can not ensure the semisimplicity of the S-expanded algebra, its resonant subalgebras or a 0-reduced algebra. So happens with compactness. In Section 8.2 we use computer programs to S-expand $\mathfrak{sl}(2)$ and study the semisimplicity of the S-expanded algebras, its resonant subalgebras and the 0-reduced ones. This is an example of the kind of classification which can be performed for the S-expanded algebras. A complete study of all the S-expansions by semigroups up to order 6 of all the simple algebras (up to some dimension) must be performed in the future. We have developed a Java library for this [24]. This is a useful tool to study S-expansions. With it we can also look for resonant decompositions, get the resonant subalgebras, the 0-reduced ones and check them for semisimplicity. In the future we will implement the study of other properties of the algebra.

Using S-expansions it is possible to relate Lie algebras with different dimensions. The problem of knowing if two algebras can be related by means of an expansion is very interesting from both the physical and the mathematical points of view. In fact, many physical applications have been found in this context: for example, in ref.[20] the M-algebra (the maximal supersymmetric extension of the Poincaré algebra) is obtained as an expansion of the $\mathfrak{osp}(32/1)$ algebra. In ref.[25] this result was reobtained but via the S-expansion method which gives in addition the invariant tensors of the expanded algebra. This allowed the construction of an 11-dimensional gauge theory for the M-algebra. Invariant tensors are known for all the semisimple Lie algebras but this is not true for the non semisimple ones. Here the role of the expansion using semigroups is important because it gives the invariant tensors for the expanded algebra in terms of those of the original algebra (if they are known). So starting from a semisimple algebra it is possible to obtain invariant tensors for the expanded algebra even when it is not semisimple. This is the case of the simple algebra $\mathfrak{osp}(32/1)$ whose expansion yields the invariant tensors for the M algebra. Other interesting applications are in ref.[26] where (2+1)-dimensional Chern-Simons AdS gravity is obtained from the so called *exotic gravity*. In ref.[27], standard General Relativity is obtained from Chern-Simons Gravity. Finally, in ref.[28] a generalized action for $(2+1)$-dimensional Chern-Simons Gravity is found.

In Chapter 9 we explore the relations between the 2-dimensional and 3-dimensional Lie algebras, which were classified by Bianchi in ref.[29]. We find that only going to the resonant subalgebra we can find those relations. In fact, in the case when different resonant decompositions of the same semigroup exist, it is possible to relate different algebras performing the expansion with the same semigroup. Using an iterative procedure it is possible to deduce some conditions on the multiplication table of a semigroup and then look for all the possible ways to satisfy these conditions with different semigroups. This is done with the help of computer programs developed by us [24]. The programs used in Chapters 8 and 9 are reviewed in Appendix B.

Even when these algebras are well known in the literature [29], the non-trivial relations that we find between 2 and 3-dimensional algebras are new and interesting results. A possible generalization of this procedure to higher dimensions can be useful in physical applications.

2. PRELIMINARS

The mathematical and physical framework used in the Thesis is introduced in this Chapter. It is devoted to review some results in Mathematics and Physics which will be used through this work. It can be skipped to come back to it if needed when reading each Chapter.

The organization of this Chapter is as follows:

In Section 2.1 we define Hopf algebras.

In Section 2.2 we review the algebraic groups $\mathrm{SL}(4, \mathbb{C})$ and P_l.

In Section 2.3 we review the quantum groups $\mathrm{SL}_q(4, \mathbb{C})$ and P_{lq}.

In Section 2.4 we review the construction of the complex conformal compactification of Minkowski space [12, 13, 30]. Its quantization is performed in ref.[1]. We review it in Section 2.5.

In Section 2.6 we review the Cartan decomposition of a Lie group.

In Section 2.7 we introduce the Iwasawa decomposition of a Lie group.

In Section 2.8 we review the theory of invariant tensors on symmetric spaces.

In Section 2.9 we present the Inönü-Wigner contraction of Lie algebras and the procedure to calculate the contracted metric and 2-form.

In Section 2.10 we present discrete semigroups.

In Section 2.11 we review the S-expansion procedure.

In Section 2.12 we summarize Bianchi's classification of 2 and 3-dimensional Lie algebras.

In Section 2.13 we present the 2-dimensional WZW models.

2.1 Hopf algebras

Quantum groups are non commutative nor cocommutative Hopf algebras. We are going to start with a series of definitions that lead to the definition of Hopf algebras. Unless otherwise stated, the field under consideration can be understood as $\mathbb{R}$ or $\mathbb{C}$ [14].

Definition 2.1.1. An *algebra* (A, m, η) is a vector space A over the field k such that

1. The linear map $m : A \otimes A \to A$ (the *product*) is associative.
2. η is the linear map $\eta : k \to A$ by $\eta(1) = 1_A$ (the *unit* in the algebra).

In terms of commutative diagrams, these maps satisfy the properties listed in Figure 2.1.

■

$$\begin{tikzcd}
& A \otimes A \otimes A & \\
A \otimes A \arrow[ur, "m\otimes id"'] & & A \otimes A \arrow[ul, "id\otimes m"] \\
& A \arrow[ul, "m"] \arrow[ur, "m"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \arrow[d, "\eta\otimes id"'] & \\
k \otimes A = & A \arrow[ul, "m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[d, "id\otimes\eta"'] & \\
A \otimes k = & A \arrow[ul, "m"']
\end{tikzcd}$$

Fig. 2.1: Associativity and unit element expressed as commutative diagrams.

Definition 2.1.2. A *coalgebra* (C, Δ, ϵ) is a vector space C over a field k such that

1. The linear map $\Delta : C \to C \otimes C$ (the *coproduct*) is coassociative.
2. ϵ is a linear map $\epsilon : C \to k$ (the *counit*).

In terms of commutative diagrams, these maps satisfy the properties listed in Figure 2.2.

■

Definition 2.1.3. A *bialgebra* $(H, \Delta, \epsilon, m, \eta)$ is a vector space H over a field k such that

$$\begin{tikzcd}
 & C \otimes C \otimes C & \\
C \otimes C \arrow[ur, "\Delta\otimes id"'] & & C \otimes C \arrow[ul, "id\otimes\Delta"] \\
 & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
C \otimes C \arrow[d, "\epsilon\otimes id"'] & \\
k \otimes C = & C \arrow[ul, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes C \arrow[d, "id\otimes\epsilon"'] & \\
C \otimes k = & C \arrow[ul, "\Delta"']
\end{tikzcd}$$

Fig. 2.2: Coassociativity and counit element expressed as commutative diagrams.

1. (H, m, η) is an algebra.
2. (H, Δ, ϵ) is a coalgebra.
3. Δ and ϵ are algebra maps, where $H \otimes H$ has the tensor product algebra structure

$$(h \otimes g)(h^{'} \otimes g^{'}) = hh^{'} \otimes gg^{'}$$

for all $h, h^{'}, g, g^{'} \in H$.

$$\begin{tikzcd}
H \otimes H \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"] & H \arrow[r] & H \otimes^{\Delta} H \\
H \otimes H \otimes H \otimes H \arrow[rr, "id\otimes\tau\otimes id"'] & & H \otimes H \otimes H \otimes H \arrow[u, "m\otimes m"']
\end{tikzcd}$$

$$\begin{tikzcd}
H \arrow[r, "\epsilon"] & k \\
H \otimes H \arrow[u, "m"] \arrow[ur, "\epsilon\otimes\epsilon"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, "\eta"] \arrow[dr, "\eta\otimes\eta"'] & H \arrow[d, "\Delta"] \\
 & H \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, "\eta"] \arrow[dr, "id"'] & H \arrow[d, "\epsilon"] \\
 & k
\end{tikzcd}$$

Fig. 2.3: Compatibility between the maps of a bialgebra.

$$\begin{tikzcd}
H \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & k \arrow[r, "\eta"] & H \\
H \otimes H \arrow[rr, "id\otimes S, S\otimes id"'] & & H \otimes H \arrow[u, "m"']
\end{tikzcd}$$

Fig. 2.4: Additional axioms that make a bialgebra into a Hopf algebra.

In terms of commutative diagrams, property 3 can be seen as Figure 2.3, where τ is the map

$$H \otimes H \longrightarrow H \otimes H$$

$$a \otimes b \longrightarrow \tau(a \otimes b) = b \otimes a.$$

■

Finally we have all the needed ingredients to define a Hopf algebra.

Definition 2.1.4. A *Hopf algebra* $(H, \Delta, \epsilon, m, \eta, S)$ is a vector space H over a field k such that

1. $(H, \Delta, \epsilon, m, \eta)$ is a bialgebra.

2. The map $S : H \to H$ (the *antipode*) satisfies the compatibility conditions of Figure 2.4.

■

The antipode of a Hopf algebra is unique and it satisfies

$$S(gh) = S(h)S(g)$$

S^2 is an homomorphism. S does not need to have an inverse. If so, S^2 is an automorphism. If $S^2 = \mathbb{1}$ then the Hopf algebra is said to be involutive.

The algebraic groups that we use in this Thesis can be seen as commutative Hopf algebras. We study them in the next Section.

2.2 The algebraic groups $\mathrm{GL}(4,\mathbb{C})$, $\mathrm{SL}(4,\mathbb{C})$ and P_l

We first consider the algebraic group $\mathrm{GL}(4,\mathbb{C})$. The generators of $\mathrm{GL}(4,\mathbb{C})$ can be organized in matrix form

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}, \qquad \text{satisfying the condition} \qquad \det g \neq 0. \tag{2.1}$$

The algebra of polynomials of $\mathrm{GL}(4,\mathbb{C})$ is the algebra of polynomials in the entries of the matrix and an extra variable d, which is set to be the inverse of the determinant, thus forcing the determinant to be different from zero:

$$\mathcal{O}(\mathrm{GL}(4,\mathbb{C})) = \mathbb{C}[g_{AB}, d]/(d \cdot \det g - 1), \qquad A, B = 1, \dots, 4.$$

If we want to consider the algebra of $\mathrm{SL}(4,\mathbb{C})$ we will have simply

$$\mathcal{O}(\mathrm{SL}(4,\mathbb{C})) = \mathbb{C}[g_{AB}]/(\det g - 1), \qquad A, B = 1, \dots, 4. \tag{2.2}$$

We define the lower parabolic subgroup of $\mathrm{SL}(4,\mathbb{C})$ as all the matrices of the form

$$P_l = \left\{ \begin{pmatrix} x & 0 \\ Tx & y \end{pmatrix} \quad / \quad \det x \cdot \det y = 1 \right\}.$$

The bottom left entry is arbitrary but we have written it in this way for convenience.

In all cases the group law (matrix multiplication in notation (2.1)) is expressed algebraically as a coproduct, given on the generators as

$$\begin{array}{ccccc} \mathcal{O}(\mathrm{GL}(4,\mathbb{C})) & \xrightarrow{\Delta_c} & \mathcal{O}(\mathrm{GL}(4,\mathbb{C})) \otimes \mathcal{O}(\mathrm{GL}(4,\mathbb{C})) & & \\ g_{AB} & \longrightarrow & \sum_C g_{AC} \otimes g_{CB}, & & A, B, C = 1, \dots, 4, \\ d & \longrightarrow & d \otimes d & & \end{array} \tag{2.3}$$

and extended by multiplication to the whole $\mathcal{O}(\mathrm{GL}(4,\mathbb{C}))$. The coproduct is non cocommutative, since switching the two factors of $\Delta_c f$ does not leave the result unchanged.

Remark 2.2.1. Let us see intuitively why the coproduct corresponds to the matrix multiplication on the group itself. We try now to see an element of $\mathcal{O}(\mathrm{GL}(4,\mathbb{C}))$ as a function over the algebraic variety of the group itself. Let us denote the natural injection

$$\begin{array}{ccc} \mathcal{O}(\mathrm{GL}(4,\mathbb{C}))\otimes\mathcal{O}(\mathrm{GL}(4,\mathbb{C})) & \xrightarrow{\mu_G} & \mathcal{O}(\mathrm{GL}(4,\mathbb{C})\times\mathcal{O}(\mathrm{GL}(4,\mathbb{C})) \\ f_1\otimes f_2 & \longrightarrow & f_1\times f_2 \end{array}$$

such that $f_1\times f_2(g_1,g_2)=f_1(g_1)f_2(g_2)$. Then we have that

$$\mu_G\circ(\Delta_c f)(g_1,g_2)=f(g_1g_2), \qquad f\in\mathcal{O}(\mathrm{GL}(4,\mathbb{C})).$$

■

We also have the antipode S (which corresponds to the inverse on $\mathrm{GL}(4,\mathbb{C})$),

$$\begin{array}{ccc} \mathcal{O}(\mathrm{GL}(4,\mathbb{C})) & \xrightarrow{S} & \mathcal{O}(\mathrm{GL}(4,\mathbb{C})) \\ g_{AB} & \longrightarrow & {g^{-1}}_{AB}=d\,(-1)^{B-A}M_{BA} \\ d & \longrightarrow & \det g, \end{array} \tag{2.4}$$

where M_{BA} is the minor of the matrix g with the row B and the column A deleted. There is compatibility between the multiplication and comultiplication in $\mathcal{O}(\mathrm{GL}(4,\mathbb{C})$:

$$\Delta_c(f_1f_2)=\Delta_c f_1\Delta_c f_2, \tag{2.5}$$

and the properties of associativity and coassociativity of the product and the coproduct also hold. There is also a unit and a counit (see ref.[14], for example), and all this gives to $\mathcal{O}(\mathrm{GL}(4,\mathbb{C}))$ the structure of a commutative, non cocommutative Hopf algebra.

One can deal in the same way with the subgroups $\mathrm{SL}(4,\mathbb{C})$ and P_l, being the coproduct and the antipode well defined on their algebras, that is, on (2.2) and

$$\begin{aligned} &\mathcal{O}(P_l)=\mathbb{C}[x_{ij},y_{ab},T_{ai}]/(\det x\cdot\det y-1), \\ &i,j=1,2, \quad a,b=3,4. \end{aligned} \tag{2.6}$$

Since we have made a change of generators in P_l, we want to express the coproduct and the antipode in terms of x, y and T:

$$\begin{aligned}
\Delta x_{ij} &= x_{ik} \otimes x_{kj}, \\
\Delta y_{ab} &= y_{ac} \otimes y_{cb}, \\
\Delta T_{ai} &= T_{ai} \otimes 1 + y_{ac} S(x_{ji}) \otimes T_{cj}.
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
S(x_{ij}) &= {x^{-1}}_{ij} = \det y\, (-1)^{j-i} M_{ij}, \\
S(y_{ij}) &= {y^{-1}}_{ij} = \det x\, (-1)^{j-i} M_{ij}, \\
S(T_{ai}) &= -S(y_{ab}) T_{bj} x_{ji}.
\end{aligned} \tag{2.8}$$

2.3 *The quantum groups* $\mathrm{SL}_q(4,\mathbb{C})$ *and* $\mathcal{O}_q(P_l)$.

Remark 2.3.1. If k is a field, we denote by k_q the ring of formal power series in the indeterminates q and q^{-1}, with $qq^{-1} = 1$.

■

The quantum group $\mathrm{SL}_q(4,\mathbb{C})$ [14] is the free associative algebra over $\mathbb{C}_q$ with generators $\hat{g}_{AB}$, $A, B = 1, \dots, 4$ satisfying the commutation relations (Manin relations [31])

$$\begin{aligned}
&\hat{g}_{AB}\, \hat{g}_{CB} = q^{-1} \hat{g}_{CB}\, \hat{g}_{AB} \quad \text{if } A < C, \\
&\hat{g}_{AB}\, \hat{g}_{AD} = q^{-1} \hat{g}_{AD}\, \hat{g}_{AB}, \quad \text{if } B < D, \\
&\hat{g}_{AB}\, \hat{g}_{CD} = \hat{g}_{CD}\, \hat{g}_{AB} \quad \text{if } A < C \text{ and } D < B \text{ or } A > C \text{ and } D > B, \\
&\hat{g}_{AB}\, \hat{g}_{CD} - \hat{g}_{CD}\, \hat{g}_{AB} = (q^{-1} - q)\, \hat{g}_{AC}\, \hat{g}_{BD} \quad \text{if } A < C \text{ and } D > B,
\end{aligned} \tag{2.9}$$

and the condition on the *quantum determinant*

$$\det_q \hat{g} = \sum_{\sigma \in S_4} (-q)^{-l(\sigma)} \hat{g}_{4\sigma(4)} \cdots \hat{g}_{1\sigma(1)} = 1. \tag{2.10}$$

If we denote by $\mathcal{I}_{\mathrm{SL}_q(4,\mathbb{C})}$ the ideal generated by the relations (2.9) and (2.10), then

$$\mathrm{SL}_q(4,\mathbb{C}) = \mathbb{C}_q\langle \hat{g}_{AB} \rangle / \mathcal{I}_{\mathrm{SL}_q(4,\mathbb{C})}.$$

There is a coproduct in this algebra that, on the generators, is formally is the same coproduct than (2.3)

$$\Delta\, \hat{g}_{AB} = \sum_C \hat{g}_{AC} \otimes \hat{g}_{CB}$$

and it is extended to the whole $\mathrm{SL}_q(4,\mathbb{C})$ by multiplication. The antipode is a generalization of the formula (2.4)

$$S_q(\hat{g}_{AB}) = (-q)^{B-A} M^q_{BA},$$

where M^q_{BA} is the corresponding quantum minor. One can see that $S^2 \neq \mathbb{1}$, contrary to what happens in the commutative case. $\mathrm{SL}_q(4,\mathbb{C})$ is a non commutative, non cocommutative Hopf algebra.

We define $\mathcal{O}_q(P_l)$ to be the subalgebra of $\mathrm{SL}_q(4,\mathbb{C})$ generated by

$$g = \begin{pmatrix} \hat{x} & 0 \\ \hat{T}\hat{x} & \hat{y} \end{pmatrix} = \left(\begin{array}{cc|cc} \hat{g}_{11} & \hat{g}_{12} & 0 & 0 \\ \hat{g}_{21} & \hat{g}_{22} & 0 & 0 \\ \hline \hat{g}_{31} & \hat{g}_{32} & \hat{g}_{33} & \hat{g}_{34} \\ \hat{g}_{41} & \hat{g}_{42} & \hat{g}_{43} & \hat{g}_{44} \end{array}\right). \tag{2.11}$$

We introduce the notation

$$\hat{D}^{KL}_{IJ} = \hat{g}_{IK}\hat{g}_{JL} - q^{-1}\hat{g}_{IL}\hat{g}_{JK},$$

(that is, they are 2×2 quantum determinants). For simplicity, we will write

$$\hat{D}^{12}_{IJ} \equiv \hat{D}_{IJ}.$$

The condition (2.10) on the quantum determinant implies that $\det_q \hat{x} = \hat{D}_{12}$ and $\det_q \hat{y} = D^{34}_{34}$ are invertible and

$$\det{}_q\hat{x} \cdot \det{}_q\hat{y} = 1.$$

The generators $\hat{T}$ in (2.11) can be computed explicitly.

$$\hat{T} = \begin{pmatrix} -q^{-1}\hat{D}_{23}\det_q \hat{y} & \hat{D}_{13}\det_q \hat{y} \\ -q^{-1}\hat{D}_{24}\det_q \hat{y} & \hat{D}_{14}\det_q \hat{y} \end{pmatrix}.$$

We now give the commutation relations among the generators $\hat{x}_{ij}$, $\hat{y}_{ab}$, $\hat{T}_{ai}$. We have:

$$\begin{aligned} &\hat{x}_{11}\hat{x}_{12} = q^{-1}\hat{x}_{12}\hat{x}_{11}, && \hat{x}_{11}\hat{x}_{21} = q^{-1}\hat{x}_{21}\hat{x}_{11}, \\ &\hat{x}_{11}\hat{x}_{22} = \hat{x}_{22}\hat{x}_{11} + (q^{-1}-q)\hat{x}_{21}\hat{x}_{12}, && \hat{x}_{12}\hat{x}_{21} = \hat{x}_{21}\hat{x}_{12}, \\ &\hat{x}_{12}\hat{x}_{22} = q^{-1}\hat{x}_{22}\hat{x}_{12}, && \hat{x}_{21}\hat{x}_{22} = q^{-1}\hat{x}_{22}\hat{x}_{21}, \end{aligned} \tag{2.12}$$

$$
\begin{aligned}
&\hat{y}_{33}\hat{y}_{34} = q^{-1}\hat{y}_{34}\hat{y}_{33}, && \hat{y}_{33}\hat{y}_{43} = q^{-1}\hat{y}_{43}\hat{y}_{33},\\
&\hat{y}_{33}\hat{y}_{44} = \hat{y}_{44}\hat{y}_{33} + (q^{-1}-q)\hat{y}_{43}\hat{y}_{34}, && \hat{y}_{34}\hat{y}_{43} = \hat{y}_{43}\hat{y}_{34},\\
&\hat{y}_{34}\hat{y}_{44} = q^{-1}\hat{y}_{44}\hat{y}_{34}, && \hat{y}_{43}\hat{y}_{44} = q^{-1}\hat{y}_{44}\hat{y}_{43},
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
&\hat{T}_{42}\hat{T}_{41} = q^{-1}\hat{T}_{41}\hat{T}_{42}, && \hat{T}_{31}\hat{T}_{41} = q^{-1}\hat{T}_{41}\hat{T}_{31},\\
&\hat{T}_{32}\hat{T}_{41} = \hat{T}_{41}\hat{T}_{32} + (q^{-1}-q)\hat{T}_{42}\hat{T}_{31}, && \hat{T}_{31}\hat{T}_{42} = \hat{T}_{42}\hat{T}_{31},\\
&\hat{T}_{32}\hat{T}_{42} = q^{-1}\hat{T}_{42}\hat{T}_{32}, && \hat{T}_{32}\hat{T}_{31} = q^{-1}\hat{T}_{31}\hat{T}_{32}.
\end{aligned}
\tag{2.14}
$$

and for $i = 1, 2$, $a = 3, 4$

$$
\begin{aligned}
&\hat{x}_{1i}\hat{T}_{32} = \hat{T}_{32}\hat{x}_{1i}, && \hat{x}_{1i}\hat{T}_{42} = \hat{T}_{42}\hat{x}_{1i},\\
&\hat{x}_{1i}\hat{T}_{41} = q^{-1}\hat{T}_{41}\hat{x}_{1i}, && \hat{x}_{1i}\hat{T}_{31} = q^{-1}\hat{T}_{31}\hat{x}_{1i},\\
&\hat{x}_{21}\hat{T}_{a2} = q^{-1}\hat{T}_{a2}\hat{x}_{21} + q(q^{-1}-q)\hat{x}_{11}\hat{T}_{a1}, && \hat{x}_{2i}\hat{T}_{31} = \hat{T}_{31}\hat{x}_{2i},\\
&\hat{x}_{22}\hat{T}_{a2} = q^{-1}\hat{T}_{a2}\hat{x}_{22} + q(q^{-1}-q)\hat{x}_{12}\hat{T}_{a1}, && \hat{x}_{2i}\hat{T}_{41} = \hat{T}_{41}\hat{x}_{2i},
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
&\hat{y}_{33}\hat{T}_{3a} = q\hat{T}_{3a}\hat{y}_{33}, && \hat{y}_{34}\hat{T}_{3a} = q\hat{T}_{3a}\hat{y}_{34}, && \hat{y}_{43}\hat{T}_{4a} = q\hat{T}_{4a}\hat{y}_{43},\\
&\hat{y}_{33}\hat{T}_{4a} = \hat{T}_{4a}\hat{y}_{33}, && \hat{y}_{34}\hat{T}_{4a} = \hat{T}_{4a}\hat{y}_{34}, && \hat{y}_{43}\hat{T}_{3a} = \hat{T}_{3a}\hat{y}_{43},\\
&\hat{y}_{44}\hat{T}_{4a} = q\hat{T}_{4a}\hat{y}_{44},\\
&\hat{y}_{44}\hat{T}_{3a} = \hat{T}_{3a}\hat{y}_{44}.
\end{aligned}
\tag{2.16}
$$

This can be checked by direct computation [1, 2, 3]. If we denote by $\mathcal{I}_{P_l}$ the ideal generated by the relations (2.12, 2.13, 2.14, 2.15, 2.16), then

$$
\mathcal{O}_q(P_l) = \mathbb{C}_q\langle \hat{x}_{ij}, \hat{y}_{ab}, \hat{T}_{ai}\rangle / (\mathcal{I}_{P_l},\ \mathrm{det}_q\hat{x}\cdot \mathrm{det}_q\hat{y} - 1). \tag{2.17}
$$

The coproduct and the antipode are inherited form the ones in $\mathrm{SL}_q(4,\mathbb{C})$. It is instructive to compute the quantum antipode in terms of the variables $\hat{x}, \hat{y}, \hat{T}$. The coproduct is formally as in (2.7), while for the antipode one has to replace the minors by quantum minors. Explicitly,

$$
\begin{aligned}
S(\hat{x}) &= \mathrm{det}_q\hat{y}\begin{pmatrix} \hat{x}_{22} & -q\hat{x}_{12}\\ -q^{-1}\hat{x}_{21} & \hat{x}_{11}\end{pmatrix},\\
S(\hat{y}) &= \mathrm{det}_q\hat{x}\begin{pmatrix} y_{44} & -q\hat{y}_{34}\\ -q^{-1}\hat{y}_{43} & \hat{y}_{33}\end{pmatrix},\\
S(\hat{T}) &= -S(\hat{y})\hat{T}\hat{x}.
\end{aligned}
$$

2.4 The conformal complexification of Minkowski space

We give here the classical description of the conformal space as a Grassmannian variety and the Minkowski space as the big cell inside it. This description is well known (see for example refs.[12, 13]). We follow closely the notation of refs.[1, 13, 30].

The Grassmannian variety $G(2,4)$ is the set of 2-planes inside a four dimensional space $\mathbb{C}^4$ (the *twistor space*). A plane π can be given by two linearly independent vectors

$$\pi = (a,b) = \mathrm{span}\{a,b\}, \qquad a,b \in \mathbb{C}^4.$$

If $\mathrm{span}\{a,b\} = \mathrm{span}\{a',b'\}$ they define the same point of the Grassmannian. This means that we can take linear combinations of the vectors a and b

$$\begin{aligned} (a',b') &= (a,b)h, \qquad\qquad h \in \mathrm{GL}(2,\mathbb{C}),\\ \begin{pmatrix} a'_1 & b'_1 \\ a'_2 & b'_2 \\ a'_3 & b'_3 \\ a'_4 & b'_4 \end{pmatrix} &= \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \end{aligned} \tag{2.18}$$

to represent the same plane π.

What relates the Grassmannian to the conformal group is that there is a transitive action of $\mathrm{GL}(4,\mathbb{C})$ on $G(2,4)$,

$$g \in \mathrm{GL}(4,\mathbb{C}), \qquad g\pi = (ga, gb).$$

One can take $\mathrm{SL}(2,\mathbb{C})$ instead and the action is still transitive. Then, the Grassmannian is a homogeneous space of $\mathrm{SL}(4,\mathbb{C})$. Let us take the plane

$$\pi_0 = (e_1, e_2) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right),$$

expressed in the canonical basis of $\mathbb{C}^4$, $\{e_1, e_2, e_3, e_4\}$. The stability group of π_0 is the upper parabolic subgroup

$$P_0 = \left\{ \begin{pmatrix} L & M \\ 0 & R \end{pmatrix} \in \mathrm{SL}(4,\mathbb{C}) \right\},$$

with L, M, R being 2×2 matrices, and $\det L \cdot \det R = 1$. Then, one has that $G(2,4)$ is the homogeneous space

$$G(2,4) = \mathrm{SL}(4,\mathbb{C})/P_0.$$

The conformal group in dimension four and Minkowskian signature is the orthogonal group $\mathrm{SO}(2,4)$. Its spin group is $\mathrm{SU}(2,2)$. If we consider the complexification, $\mathrm{SO}(6,\mathbb{C})$ (later on we will study the real forms), the spin group is $\mathrm{SL}(4,\mathbb{C})$. We have then that the spin group of the complexified conformal group acts transitively on the Grassmannian $G(2,4)$.

One can see the complexified Minkowski space as the *big cell* inside the Grassmannian. This is a dense open set of $G(2,4)$. There is in fact an open covering of $G(2,4)$ by such sets. As we have seen, a plane $\pi = (a,b)$ can be represented by a matrix

$$\pi = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}.$$

This matrix has rank two, since the two vectors are independent. So at least one of the 2×2 blocks has to have determinant different from zero. We define the six open sets

$$U_{ij} = \Big\{(a,b) \in \mathbb{C}^4 \times \mathbb{C}^4 \quad / \quad a_i b_j - b_i a_j \neq 0\Big\}, \qquad i < j, \quad i,j = 1, \ldots 4. \tag{2.19}$$

This is an open covering of $G(2,4)$. The set U_{12} is called the *big cell* of $G(2,4)$. By using the freedom (2.18) we can always bring a plane in U_{12} to the form

$$\pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ t_{31} & t_{32} \\ t_{41} & t_{42} \end{pmatrix}, \tag{2.20}$$

with the entries of t totally arbitrary. So $U_{12} \approx \mathbb{C}^4$.

The subgroup of $\mathrm{SL}(4,\mathbb{C})$ that leaves invariant the big cell consists of all the matrices of the form

$$P_l = \left\{ \begin{pmatrix} x & 0 \\ Tx & y \end{pmatrix} \quad / \quad \det x \cdot \det y = 1 \right\}.$$

See Section 2.2. The action on U_{12} is then

$$t \mapsto ytx^{-1} + T, \tag{2.21}$$

so P_l has the structure of semidirect product $P_l = H \ltimes M_2$, where $M_2 = \{T\}$ is the set of 2×2 matrices, acting as translations, and

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad x, y \in \mathrm{GL}(2, \mathbb{C}), \quad \det x \cdot \det y = 1 \right\}.$$

The subgroup H is the direct product $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^\times$. But $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ is the spin group of $\mathrm{SO}(4,\mathbb{C})$, the complexified Lorentz group, and $\mathbb{C}^\times$ acts as a dilation. P_l is then the Poincaré group times dilations.

In the basis of the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.22}$$

an arbitrary matrix t can be written as

$$t = \begin{pmatrix} t_{31} & t_{32} \\ t_{41} & t_{42} \end{pmatrix} = x^0\sigma_0 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}.$$

Then

$$\det t = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

and (x^0, x^1, x^2, x^3) can be recognized as the ordinary coordinates of Minkowski space.

We have then that the Grassmannian $G(2,4)$ is the *complex conformal compactification* of the Minkowski space, or simply the *complex conformal space*. This compactification consists of adding a variety of points at infinity to the Minkowski space. In fact, the set of points that we add is the closure of a cone in $\mathbb{C}^4$ [30].

Algebraic approach. In the quantum theory the word quantization means changing (or deforming) the algebra of observables (usually functions over the phase space) to a non commutative one (usually operators over a Hilbert space). Also here, when talking about quantum spacetime we refer to a noncommutative deformation of a commutative algebra. The algebra of departure is the algebra of functions over spacetime. We will consider first polynomials (all the objects described above are algebraic varieties). In Section 3.2 we will see how the construction can be extended to smooth functions.

The Minkowski space is just the affine space $\mathbb{C}^4$, so its algebra of polynomials is

$$\mathcal{O}(\mathrm{M}) \approx \mathbb{C}[t_{ai}], \qquad a = 3,4, \quad i = 1,2.$$

The action of the Poincaré group on the Minkowski space is expressed as a coaction on its algebra

$$\begin{array}{ccc} \mathcal{O}(\mathrm{M}) & \xrightarrow{\tilde{\Delta}} & \mathcal{O}(P_l) \otimes \mathcal{O}(\mathrm{M}) \\ t_{ai} & \longrightarrow & y_{ab} S(x)_{ji} \otimes t_{bj} + T_{ai} \otimes 1. \end{array} \tag{2.23}$$

This corresponds to the standard action (2.21) and as in Section 2.1, if

$$\mathcal{O}(P_l) \otimes \mathcal{O}(\mathrm{M}) \xrightarrow{\mu_{G\times \mathrm{M}}} \mathcal{O}(P_l \times \mathrm{M}) \tag{2.24}$$

is the natural injection then

$$\mu_{G\times \mathrm{M}} \circ \tilde{\Delta}(f)(g,t) = f(ytS(x) + T), \qquad \text{where } g \approx (x,y,T).$$

We see then that in this formalism the coaction reproduces the standard action of the group on the space of functions on the variety,

$$\begin{array}{ccc} \mathcal{O}(\mathrm{M}) & \xrightarrow{g} & \mathcal{O}(\mathrm{M}) \\ f & \longrightarrow & gf \end{array}, \qquad \text{where}$$

$$gf(t) = f(g^{-1}t) = \mu \circ \tilde{\Delta}(f)(g^{-1}, t).$$

2.5 The quantum Minkowski space

The quantization of Minkowski and conformal spaces starts with the quantization of $\mathrm{SL}(4,\mathbb{C})$. We substitute the group by the corresponding quantum group $\mathrm{SL}_q(4,\mathbb{C})$, which is the quantization of the algebra $\mathcal{O}(\mathrm{SL}(4,\mathbb{C}))$ and then we quantize the rest of the structures in order to preserve the relations among them. This approach is followed in the series of papers [32, 33, 34] and we are not reproducing it here. We will only state the result for the quantization of the algebra of Minkowski space. For the proofs, we refer to those papers. It is nevertheless important to have in mind the structure of the quantum group $\mathrm{SL}_q(4,\mathbb{C})$ done in Section 2.3.

The complexified quantum Minkowski space is the free algebra in four generators

$$\hat{t}_{41}, \hat{t}_{42}, \hat{t}_{31} \text{ and } \hat{t}_{32},$$

satisfying the relations

$$\begin{aligned}
\hat{t}_{42}\hat{t}_{41} &= q^{-1}\hat{t}_{41}\hat{t}_{42},\\
\hat{t}_{31}\hat{t}_{41} &= q^{-1}\hat{t}_{41}\hat{t}_{31},\\
\hat{t}_{32}\hat{t}_{41} &= \hat{t}_{41}\hat{t}_{32} + (q^{-1} - q)\hat{t}_{42}\hat{t}_{31},\\
\hat{t}_{31}\hat{t}_{42} &= \hat{t}_{42}\hat{t}_{31},\\
\hat{t}_{32}\hat{t}_{42} &= q^{-1}\hat{t}_{42}\hat{t}_{32},\\
\hat{t}_{32}\hat{t}_{31} &= q^{-1}\hat{t}_{31}\hat{t}_{32}.
\end{aligned} \tag{2.25}$$

Formally, these relations are the same as (2.14)

The algebra of the complexified Minkowski space will be denoted as $\mathcal{O}_q(\mathrm{M})$. If we denote the ideal (2.25) by $\mathcal{I}_{\mathrm{M}_q}$, then we have that

$$\mathcal{O}_q(\mathrm{M}) \equiv \mathbb{C}_q\langle \hat{t}_{41}, \hat{t}_{42}, \hat{t}_{31}, \hat{t}_{32}\rangle / \mathcal{I}_{\mathrm{M}_q}.$$

It is not difficult to see that $\mathcal{O}_q(\mathrm{M})$ is isomorphic to the algebra of quantum matrices $M_q(2)$ defined by the relations (A.1) (see for example ref.[14]). The correspondence $M_q(2) \to \mathcal{O}_q(\mathrm{M})$ is given in terms of the respective generators:

$$\begin{pmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{pmatrix} \rightleftarrows \begin{pmatrix} \hat{t}_{32} & \hat{t}_{31} \\ \hat{t}_{42} & \hat{t}_{41} \end{pmatrix}.$$

Using this correspondence, one can check that the relations (2.25) become the relations satisfied by the generators of the quantum matrices $M_q(2)$.

There is a coaction of $\mathcal{O}_q(P_l)$ on $\mathcal{O}_q(\mathrm{M})$, which on the generators has the same form as (2.23). At this stage, we have lost the interpretation in terms of functions over the Minkowski space. This will be recovered with the star product.

2.6 Cartan decomposition of a Lie algebra

As a brief explanation of what the Cartan decomposition is, let us cite Theorem 6.3 and 7.1 and Definitions from ref.[35]:

Theorem: *Every semisimple Lie algebra over $\mathbb{C}$ has a compact real form, which we will denote by $\mathfrak{g}_k$.*

■

In fact if $\{H_\alpha, E_\alpha, E_{-\alpha}\}$ is the Cartan-Weyl basis of a semisimple Lie algebra, then the compact real form $\mathfrak{g}_k$ is given by

$$\mathfrak{g}_k = \sum \mathbb{R}(iH_\alpha) + \sum \mathbb{R}(E_\alpha - E_{-\alpha}) + \sum \mathbb{R}(i(E_\alpha + E_{-\alpha})). \qquad (2.26a)$$

Theorem: *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, $\mathfrak{g}$ its complex form and $\mathfrak{u}$ any compact real form of $\mathfrak{g}$. Let σ and τ be conjugations of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and $\mathfrak{u}$ respectively. Then there exists an automorphism φ of $\mathfrak{g}$ such that the compact real form $\varphi(\mathfrak{u})$ is invariant under σ.*

■

So having these results, it is possible to give (see ref.[35]) the definition of a Cartan decomposition of a semisimple algebra:

Definition: *Let $\mathfrak{g}_0$ be a semisimple real Lie algebra, $\mathfrak{g}$ its complexification and σ a conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Then a decomposition*

$$\mathfrak{g}_0 = \mathfrak{h} + \mathfrak{p}, \qquad (2.27)$$

where $\mathfrak{h}$ is a subalgebra, is called a Cartan decomposition if there exists a compact real form, $\mathfrak{g}_k$, of $\mathfrak{g}$ such that

$$\sigma(\mathfrak{g}_k) \subset \mathfrak{g}_k \quad \text{and} \quad \begin{array}{l} \mathfrak{h} = \mathfrak{g}_0 \cap \mathfrak{g}_k, \\ \mathfrak{p} = \mathfrak{g}_0 \cap (i\mathfrak{g}_k). \end{array} \qquad (2.28)$$

■

The two first theorems cited imply that each real semisimple Lie algebra $\mathfrak{g}_0$ has a Cartan decomposition. It can be demonstrated also that $\mathfrak{h}$ is the maximal compactly imbedded subalgebra of $\mathfrak{g}_0$. From now on, we are going to use the symbol '0' to characterize structures related to real and semisimple Lie algebras[1]. One can prove also that $\mathfrak{g}_k = \mathfrak{h} + i\mathfrak{p}$.

[1] The only exeption to this convention will be when we consider an algebra $\mathfrak{g}$ and a semigroup S having a decomposition:

$$\mathfrak{g} = V_0 + V_1,$$
$$S = S_0 \cup S_1.$$

2.7 Iwasawa decomposition of a Lie group

Iwasawa decomposition. For non compact symmetric spaces, there is an alternative decomposition to the Cartan decomposition called the Iwasawa decomposition (see for example ref.[35]). Let

$$\mathfrak{g}_0 = \mathfrak{h} + \mathfrak{p}$$

be a Cartan decomposition. We consider another decomposition

$$\mathfrak{g}_0 = \mathfrak{h} + \mathfrak{s},$$

such that

$$[\mathfrak{h}, \mathfrak{s}] \subset \mathfrak{s}, \qquad [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s},$$

with $\mathfrak{s}$ a solvable algebra. It is constructed in the following way: we first choose a maximal abelian subalgebra of $\mathfrak{p}$ and a basis of it, $\{H_1, \dots H_n\}$. Then, we diagonalize the action of H_i over $\mathfrak{g}$. The non zero eigenvalues are the *restricted roots* of $\mathfrak{g}$ with respect to the Cartan decomposition. They come in pairs $\pm\lambda_1, \dots, \pm\lambda_p$. The restricted roots also have a positive system and they may have root spaces of dimension larger than 1. The solvable Lie algebra $\mathfrak{s}$ is then spanned by the maximal abelian subalgebra and the positive root spaces. The positive root spaces generate a nilpotent subalgebra.

There are several advantages for using the Iwasawa decomposition. The first one is that it exponentiates to the group so there exists a global decomposition $G \approx ANH$ where H is the maximal compact subgroup, A is an abelian Lie group with algebra the maximal abelian subalgebra of $\mathfrak{p}$ and N is a nilpotent Lie group whose Lie algebra is spanned by the positive roots. Then $G/H \approx AN$, and it has a solvable group structure.

In the following, we will denote by

$$\mathfrak{solv}(G/H)$$

the solvable Lie algebra associated to the coset space G/H by the Iwasawa decomposition.

2.8 Symmetric spaces with an invariant tensor field

The coset G/H and its tangent space. Let G be a Lie group and let

$$\begin{array}{ccc} G & \xrightarrow{L_g} & G \\ g' & \longrightarrow & gg', \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{R_g} & G \\ g' & \longrightarrow & g'g, \end{array}$$

be the left and right translations respectively. The Lie algebra of G, denoted by $\mathfrak{g}$ is the set of left invariant[2] vector fields

$$X^l_g = L^T_g X, \qquad X \in T_eG, \tag{2.29}$$

(e is the identity of G), with Lie bracket the usual Lie bracket of vector fields. The left invariant vector fields are non zero everywhere, and choosing a basis of T_eG one can construct a global frame on G. The manifold G is then parallelizable[3]. Equation (2.29) establishes an isomorphism of vector spaces $\mathfrak{g} \approx T_eG$, and the Lie bracket can then be defined in T_eG through this isomorphism.

We assume now that G is a semisimple group. We consider a Cartan decomposition of $\mathfrak{g}$ (see Section 2.6),

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p},$$

in terms of a Lie subalgebra $\mathfrak{h}$ and a subspace $\mathfrak{p}$ with the properties

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}. \tag{2.30}$$

The subspace $\mathfrak{p}$ then carries a representation of $\mathfrak{h}$. Let H be the subgroup of G whose Lie algebra is $\mathfrak{h}$. Then $\mathfrak{p}$ carries a representation of H that we will denote as

$$S(h) \equiv \mathrm{Ad}_h|_{\mathfrak{p}}. \tag{2.31}$$

We consider the distribution $\mathcal{F} \subset TG$ spanned by the left invariant vector fields in $\mathfrak{h}$

$$\mathcal{F} = \mathrm{span}\{X^l_g = L^T_g X \mid X \in T_eH\},$$

$\mathcal{F}$ is clearly an integrable distribution since $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Next, we can consider the bundle over G, $\mathcal{L} = TG/\mathcal{F}$, the equivalence classes meaning that two tangent vectors at the same point in G are identified if its difference belongs to the distribution $\mathcal{F}$.

Let us consider now the space of left cosets, $G/H = \{gH, \mid g \in G\}$, so $g' \in gH$ if and only if $g' = gh$ for some $h \in H$. G/H has a unique differentiable structure such that G acts as a group of transformations,

$$\begin{array}{ccc} G/H & \xrightarrow{\tilde{L}_g} & G/H \\ [g'] = g'H & \longrightarrow & [gg'] = gg'H. \end{array} \tag{2.32}$$

[2] We could use equally the right invariant vector fields.
[3] That is, its tangent bundle is trivial.

The isotropy group of this action at the identity coset is H since $hH = H$. At an arbitrary coset gH, the isotropy group is gHg^{-1}, so all the isotropy groups are conjugate.

To construct the tangent space to the coset manifold, we first identify the tangent spaces and the distribution subspaces at different points of $[g] = gH$ in the natural way,

$$
\begin{array}{ccccccc}
T_gG & \xrightarrow{R_h^T} & T_{gh}G & \qquad & \mathcal{F}_gG & \xrightarrow{R_h^T} & \mathcal{F}_{gh}G \\
X_g & \longrightarrow & R_h^T X_g, & \qquad & X_g & \longrightarrow & R_h^T X_g,
\end{array}
$$

so $X_g \approx R_h^T X_g$. The result of these identifications are two bundles over G/H, denoted as $\tilde{TG}$ and $\tilde{\mathcal{F}}$ respectively. The quotient

$$T(G/H) = \tilde{TG}/\tilde{\mathcal{F}},$$

can be identified with the tangent space of the coset manifold G/H.

Let $\tilde{X}_{[e]}$ be an equivalence class in $\tilde{TG}/\tilde{\mathcal{F}}$ at the identity coset, and let $p : \mathfrak{g} \to \mathfrak{p}$ be the natural projection. We can always choose a representative $X \in \mathfrak{p}$, since $X - p(X) \in \mathfrak{h}$. In fact we have that $T_{[e]}(G/H) \approx \mathfrak{p}$.

The Cartan-Killing form. The Cartan-Killing form is a symmetric, invariant, non degenerate bilinear form on $\mathfrak{g}$ defined by

$$\langle X, Y\rangle = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y), \qquad X, Y \in \mathfrak{g}.$$

The invariance means that if $s : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $\mathfrak{g}$ then

$$\langle s(X), s(Y)\rangle = \langle X, Y\rangle \qquad \forall\, X, Y \in \mathfrak{g}.$$

In infinitesimal form, the invariance becomes

$$\langle [Z, X], Y\rangle + \langle X, [Z, Y]\rangle = 0 \qquad \forall\, Z, X, Y \in \mathfrak{g}.$$

The subspaces $\mathfrak{h}$ and $\mathfrak{p}$ of a Cartan decomposition are orthogonal with respect to the Cartan-Killing form on $\mathfrak{g}$.

If G is compact (so G/H is a symmetric space of the compact type), then the Cartan-Killing form is negative definite. If G is noncompact, so H is its maximal compact subgroup (and G/H is a symmetric space of the non-compact type) then the Cartan-Killing form is definite negative on $\mathfrak{h}$ and positive definite on $\mathfrak{p}$.

2.8.1 Bi-invariant metric on G.

We can define a left invariant metric on G by transporting the Cartan-Killing form at any point of G with L_g^T. As before, we denote $X_g^l = {L_g}^T X$ and $Y_g^l = {L_g}^T Y$. Then we define

$$C_g(X_g^l, Y_g^l) \equiv \langle L_{g^{-1}}^T X_g^l, L_{g^{-1}}^T Y_g^l \rangle = \langle X, Y \rangle.$$

Since the left invariant vector fields span $T_g G$ at each g, C_g is well defined. By construction, C is left invariant:

$$C_{gg'}(X_{gg'}^l, Y_{gg'}^l) = \langle X, Y \rangle = C_{g'}(X_{g'}^l, Y_{g'}^l).$$

We consider now the right translation

$$\begin{aligned} G &\xrightarrow{R_g} G \\ g' &\longrightarrow g'g. \end{aligned}$$

Let us compute the right action on C,

$$\begin{aligned} C_{g'g}({R_g}^T X_{g'}^l, {R_g}^T Y_{g'}^l) = \; & \langle {L_{(g'g)^{-1}}}^T R_g^T X_{g'}^l, {L_{(g'g)^{-1}}}^T {R_g}^T Y_{g'}^l \rangle = \\ & \langle {L_{g^{-1}}}^T {L_{g'^{-1}}}^T {R_g}^T X_{g'}^l, {L_{g^{-1}}}^T {L_{g'^{-1}}}^T {R_g}^T Y_{g'}^l \rangle = \\ & \langle {L_{g^{-1}}}^T {R_g}^T X_e^l, {L_{g^{-1}}}^T {R_g}^T Y_e^l \rangle = \\ & \langle \mathrm{Ad}_{g^{-1}} X, \mathrm{Ad}_{g^{-1}} Y \rangle = \\ & \langle X, Y \rangle = C_{g'}(X_{g'}^l, Y_{g'}^l). \end{aligned}$$

We have used the facts that $R_g L_{g'} = L_{g'} R_g$, that $\mathrm{Ad}_{g^{-1}}$ is an automorphism of $\mathfrak{g}$ and that $\langle\ ,\ \rangle$ is invariant. So we have proven that C is invariant under left and right translations.

We note that the same construction can be done starting with right translations instead of left translations. The result is the same bi-invariant metric.

2.8.2 Invariant metric on G/H.

The Cartan-Killing form can be restricted to $\mathfrak{p}$, where it is definite (positive or negative). Using the action of the group $\tilde{L}_g$ on G/H (see (2.32)) inherited from the left translations, we can define a metric on G/H that has G as a group of isometries,

$$\tilde{C}_{[g]}(\tilde{X}_{[g]}, \tilde{Y}_{[g]}) \equiv \langle p(X), p(Y) \rangle,$$

where X_g is an arbitrary representative of $\tilde{X}_{[g]}$ and $X = L^T_{g^{-1}}X_g$.

Let us see if it is well defined. Let $X'_{g'} = L^T_{g'}X'$ be another representative with $g' = gh$, then $(X_g - R^T_{h^{-1}}X'_{g'}) \in \mathcal{F}_g$.

Let us compute it for $g = e$, so $g' = h$. We get

$$X'_h = L^T_h X', \qquad R^T_{h^{-1}}L^T_h X' - X = \mathrm{Ad}_h X' - X \in \mathfrak{h}.$$

The adjoint representation of G decomposes under H as

$$\mathrm{Ad}_G \xrightarrow[H]{} \mathrm{Ad}_H + S,$$

(see (2.31)). Then we have

$$p(\mathrm{Ad}_h X') = S(h)p(X') = p(X).$$

Since C is invariant under automorphisms,

$$\langle p(X), p(Y)\rangle = \langle S(h^{-1})p(X), S(h^{-1})p(Y)\rangle,$$

then $\tilde{C}_{[g]}$ is well defined. Notice that we have only used the invariance under H.

The invariance of $\tilde{C}_{[g]}$ under the left action of G on G/H

$$\begin{array}{ccc} G/H & \xrightarrow{\tilde{L}_{g'}} & G/H \\ [g] & \longrightarrow & [g'g] \end{array}$$

is straightforward: if $X_g = L^T_g X$ is a representative of $\tilde{X}_{[g]}$, $L^T_{g'}X_g$ is a representative of $\tilde{L}^T_{g'}\tilde{X}_{[g]}$. So we have

$$\tilde{C}_{[g'g]}(\tilde{L}^T_{g'}\tilde{X}_{[g]}, \tilde{L}^T_{g'}\tilde{Y}_{[g]}) = \langle p(X), p(Y)\rangle = \tilde{C}_{[g]}(\tilde{X}_{[g]}, \tilde{Y}_{[g]}),$$

as we wanted to show.

2.8.3 *Invariant tensors on G/H.*

Let B be a contravariant n-tensor on $\mathfrak{p}$, invariant under the action of H,

$$B(S(h)X^1, \dots, S(h)X^n) = B(X^1, \dots, X^n), \qquad X^i \in \mathfrak{p},\ h \in H.$$

Infinitesimally this means,

$$B([Y, X^1], \dots, X^n) + \cdots B(X^1, \dots, [Y, X^n]) = 0, \qquad X^i \in \mathfrak{p},\ Y \in \mathfrak{h}.$$

We want to define an invariant tensorial field over G/H. As before, if $X^i_g = L^T_g X^i$ is a representative of $\tilde{X}^i_{[g]}$ we can define

$$\tilde{B}_{[g]}(\tilde{X}^1_{[g]}, \dots, \tilde{X}^n_{[g]}) = B(p(X^1), \dots, p(X^n)). \tag{2.33}$$

Using the same arguments than in Subection 2.8.2, one can check that $\tilde{B}$ is well defined and that it is invariant under the action of G.

Notice that the requirements that we have made here are much softer than the ones in Subsection 2.8.2. We could have started with a symmetric invariant tensor on $\mathfrak{p}$, not necessarily on $\mathfrak{g}$ and invariant only under H. This defines an invariant tensorial field over G/H. With the Cartan-Killing form we have that the resulting tensor is a definite positive metric on G/H.

At the infinitesimal level, the invariance of $\tilde{B}$ means that the Lie derivative with respect to the fundamental vectors of the action of G on G/H is zero.

$$\mathcal{L}_X \tilde{B} = 0. \tag{2.34}$$

Invariant scalar functions are constant.

2.9 Inönü-Wigner contractions of Lie algebras

We will use the contraction (Inönü-Wigner and generalized contractions [16, 17]) of an algebra with respect to a subalgebra in order to relate different sigma models at the Lagrangian level. A similar problem was addressed in ref.[15] for higher dimensional (Supergravity related) sigma models without the WZ term. We will use the same technique here. We start by describing it.

Let $\mathfrak{g}$ be an arbitrary, finite dimensional Lie algebra with commutator $[\ ,\ ]$ and let $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, with $\mathfrak{g}_1$ a subalgebra. We define the following family of linear maps

$$\phi_\epsilon : \mathfrak{g} \to \mathfrak{g}$$

$$x = x_1 \oplus x_2 \to x = x_1 \oplus \epsilon x_2,$$

labeled by a real parameter ϵ whose interval of interest is $0 < \epsilon \leq 1$. In matrix form, the map and its inverse are block-diagonal

$$\phi_\epsilon = \begin{pmatrix} 1\!\!1_1 & 0 \\ 0 & \epsilon 1\!\!1_2 \end{pmatrix}, \qquad \phi_\epsilon^{-1} = \begin{pmatrix} 1\!\!1_1 & 0 \\ 0 & \epsilon^{-1} 1\!\!1_2 \end{pmatrix}.$$

We can define a new commutator

$$[X,Y]_\epsilon = \phi_\epsilon^{-1}\big([\phi_\epsilon(X),\phi_\epsilon(Y)]\big), \qquad X,Y\in\mathfrak{g}.$$

$[\ ,\ ]_\epsilon$ is a deformed bracket. For $\epsilon\neq 0$ it is, by construction, isomorphic to the bracket with $\epsilon=1$. But if the limit $\epsilon\to 0$ exists, then the bracket

$$[X,Y]_c = \lim_{\epsilon\to 0}[X,Y]_\epsilon\,, \qquad X,Y\in\mathfrak{g}. \tag{2.35}$$

is well defined but, since ϕ_0 is not invertible, $[\ ,\ ]_c$ will not be, in general, isomorphic to the original bracket. The Lie algebra with the same supporting space than $\mathfrak{g}$ and bracket $[\ ,\ ]_c$ is the Inönü-Wigner contraction [16] of $\mathfrak{g}$ with respect to the subalgebra $\mathfrak{g}_1$. We will denote it as $\mathfrak{g}_c$.

One can generalize this definition by allowing more general linear maps in the place of ϕ_ϵ. They should depend on a parameter ϵ and give a finite answer to the contracted bracket (2.35) [17].

Let $R:\mathfrak{g}\to\mathrm{End}(W)$ be a representation of $\mathfrak{g}$ on a finite dimensional vector space W. We assume that there is a decomposition $W=W_1\oplus W_2$ such that W_1 is an invariant subspace under the action of the subalgebra $\mathfrak{g}_1$. We define a one parameter family of linear maps

$$\begin{array}{ccc} W & \xrightarrow{\psi_\epsilon} & W \\ w=w_1\oplus w_2 & \longrightarrow & w=w_1\oplus\epsilon\, w_2, \end{array}$$

so

$$\psi_\epsilon=\begin{pmatrix}1\!\!1_1 & 0\\ 0 & \epsilon 1\!\!1_2\end{pmatrix}, \qquad \psi_\epsilon^{-1}=\begin{pmatrix}1\!\!1_1 & 0\\ 0 & \epsilon^{-1}1\!\!1_2\end{pmatrix}.$$

A representation of the deformed algebra is constructed as

$$R_\epsilon(X)=\psi_\epsilon^{-1}\circ R\big(\phi_\epsilon(X)\big)\circ\psi_\epsilon, \qquad X\in\mathfrak{g}. \tag{2.36}$$

It is easy to check that the map

$$R_c(X)=\lim_{\epsilon\to 0}R_\epsilon(X)$$

is well defined and that it is a representation of $\mathfrak{g}_c$ on W. Notice that $\psi_\epsilon=\phi_\epsilon$ for the adjoint representation.

2.9.1 Deformed metric and 2-form

To define the deformed metric on a symmetric space we are going to make use of the fact that G/H is a (solvable) group manifold. Let $\mathfrak{s} = \mathfrak{solv}(G/H) = \mathfrak{s}_1 + \mathfrak{s}_2$, where $\mathfrak{s}_1 = \mathfrak{solv}(G'/H')$. Notice that not necessarily $G' \subset G$ nor $H' \subset H$ (although it is so in our examples but see ref.[15] for some cases where it is not). We will denote by $\mathfrak{s}_\epsilon$ the solvable Lie algebra with the deformed bracket. The idea is to take the standard inner product in $\mathfrak{s}$,

$$\langle X|_{\mathfrak{p}}, Y|_{\mathfrak{p}}\rangle\,,\; X, Y \in \mathfrak{s}$$

and transport it with the left action of the deformed solvable group. We will have a coset representative L_ϵ in terms of some exponential coordinates,

$$L_\epsilon = \Pi_i \mathrm{e}^{s^i X_i},$$

where the dependence on ϵ is hidden in the group law. In a representation of the deformed algebra

$$R_\epsilon(L_\epsilon) = \psi_\epsilon^{-1} \circ \Pi_i \mathrm{e}^{s^i \phi_\epsilon(X_i)} \circ \psi_\epsilon,$$

so the deformed metric is

$$\mathrm{d}s^2_\epsilon(X_x, Y_x) = \langle L_\epsilon^{-1}\mathrm{d}L_\epsilon(X_x)|_{\mathfrak{p}}, L_\epsilon^{-1}\mathrm{d}L_\epsilon(Y_x)|_{\mathfrak{p}}\rangle.$$

The calculation will be done in the most convenient representation. Notice that the resulting metric is invariant by the left action of the solvable group. The rest of the symmetries are lost at this point, although some may be recovered after the contraction.

The calculation for the 2-form is the same: we transport the 2-form B from the tangent space at unity to the whole group by left translations,

$$B_\epsilon(X_x, Y_x) = B\big(L_\epsilon^{-1}\mathrm{d}L_\epsilon(X_x)|_{\mathfrak{p}}, L_\epsilon^{-1}\mathrm{d}L_\epsilon(Y_x)|_{\mathfrak{p}}\big).$$

2.10 Discrete semigroups

We consider a set of n elements (that we will call *generators*) $S = \{\lambda_\alpha, \alpha = 1, \cdots, n\}$. We say that S is a semigroup if it is equipped with an associative product

$$\cdot : S \times S \to S$$

■

Notice that:

- It does not exist necessarily the identity element $\lambda_\alpha \cdot \mathbb{1}_S = \lambda_\alpha \; \forall \alpha = 1, \ldots, n$
- The elements λ_α do not need to have an inverse.
- If there exists an element 0_S such that $\lambda_\alpha \cdot 0_S = 0_S \; \forall \alpha$ we will call it a *zero element*

n is the *order* of the semigroup. If $\lambda_\alpha \cdot \lambda_\beta = \lambda_\beta \cdot \lambda_\alpha$, the discrete semigroup is said to be *commutative* or *abelian.*

We can give the product by means of a multiplication table, a $n \times n$ matrix $\{a_{\alpha\beta}\}$ with entries in $\{\lambda_\alpha\}$.

	λ_1	$\cdots$	λ_β	$\cdots$	λ_n
λ_1	•	•	•	•	•
$\cdots$	•	•	•	•	•
λ_α	•	•	λ_κ	•	•
$\cdots$	•	•	•	•	•
λ_n	•	•	•	•	•

Fig. 2.5: Generic multiplication table for a semigroup of order n

Figure 2.5 is a very visual way to describe a semigroup. For instance, it allows us to check easily if a semigroup is commutative, because in that case its multiplication table is symmetric.

The quantities $\mathcal{K}_{\alpha\beta}^{\kappa}$, called *selectors* are defined in the following way

$$\mathcal{K}_{\alpha\beta}^{\kappa} = \begin{cases} 1 \text{ if } \lambda_\alpha \cdot \lambda_\beta = \lambda_\kappa, \\ 0 \text{ if } \lambda_\alpha \cdot \lambda_\beta \neq \lambda_\kappa. \end{cases}$$

An informal way of expressing the group law by means of the selectors is as follows:

$$\lambda_\alpha \cdot \lambda_\beta = \mathcal{K}_{\alpha\beta}^{\kappa} \lambda_\kappa .$$

If we have a subset of $m < n$ generators $S^{'} = \{\lambda_{\alpha_1} \cdots \lambda_{\alpha_m}\}$, $S^{'} \subset S$ such that the product closes on this subset, $\lambda_{\alpha_p} \cdot \lambda_{\alpha_q} = \mathcal{K}_{\alpha_p\,\alpha_q}^{\alpha_r} \lambda_{\alpha_r} \in S^{'}$, we say that $\{S^{'}, \cdot\}$ is a *discrete subsemigroup* of $\{S, \cdot\}$. A discrete subsemigroup is itself a semigroup, and it can be commutative and have a $0_{S'}$ element, even if S does not have this properties.

Isomorphisms of semigroups Consider the semigroups given by the multiplication tables Figure 2.6 and Figure 2.7.

	λ_1	λ_2
λ_1	λ_1	λ_1
λ_2	λ_1	λ_1

Fig. 2.6: Example of a semigroup.

	λ_1	λ_2
λ_1	λ_2	λ_2
λ_2	λ_2	λ_2

Fig. 2.7: Anoter example of a semigroup. Note that this has the same structure than Figure 2.6 if we change λ_1 by λ_2.

These two semigroups have exactly the same structure if we rename λ_1 by λ_2 in Figure 2.7 and viceversa. This is an example of an isomorphism of semigroups. The group of isomorphisms between semigroups of order n is isomorphic to the group of permutations of n elements, Σ_n [36].

Let $A = \{a_{\alpha\beta},\, \alpha, \beta = 1, \ldots, n\}$ and $B = \{b_{\alpha\beta},\, \alpha, \beta = 1, \ldots, n\}$ be the multiplication tables of two semigroups of order n. We say that A and B describe two isomorphic semigroups if a permutation $\sigma \in \Sigma_n$ exists such that

$$b_{\alpha\beta} = \sigma\big(a_{\sigma^{-1}(\alpha),\sigma^{-1}(\beta)}\big).$$

If, instead, we have

$$b_{\alpha\beta} = \alpha\big(a_{\sigma^{-1}(\beta),\sigma^{-1}(\alpha)}\big)$$

we say that A and B are related by an *antiisomorphism.*

Resonant decomposition of a semigroup. We consider a discrete semigroup $\{S, \cdot\}$ with two subsets $S_0, S_1 \subset S$. We say that (S_0, S_1) is a *resonant decomposition* of S if the following properties are satisfied:

1. $S_0 \cup S_1 = S$,
2. $S_0 \cdot S_0 \subset S_0$,
3. $S_0 \cdot S_1 \subset S_1$,

4. $S_1 \cdot S_1 \subset S_0$.

S_0 is a subsemigroup (S_1 is not). A semigroup can have more than one resonant decomposition. We explore this fact in Chapter 8.

2.11 The S-expansion procedure

In this Section we briefly describe the general abelian semigroup expansion procedure (S-expansion for short). We refer the interested reader to ref.[37] for further details.

We consider a Lie algebra $\mathfrak{g}$ with generators $\{X_i,\, i = 1, \dots, n\}$ and Lie bracket

$$[X_i, X_j] = C_{ij}^k X_k,$$

and a finite abelian semigroup $S = \{\lambda_\alpha,\, \alpha = 1, \dots, m\}$. According to Theorem 3.1 from ref.[37], the direct product

$$\mathfrak{g}_S = S \times \mathfrak{g} \tag{2.37}$$

is also a Lie algebra. The elements of this expanded algebra are denoted by

$$X_{(i,\alpha)} = \lambda_\alpha \times X_i, \tag{2.38}$$

where the product is understood as a direct product of the elements λ_α of the semigroup S and the generators X_i of $\mathfrak{g}$. The Lie bracket in $\mathfrak{g}_S$ is defined as

$$\left[X_{(i,\alpha)}, X_{(j,\beta)}\right] = \lambda_\alpha \cdot \lambda_\beta \times [X_i, X_j]\,. \tag{2.39}$$

Note that to get an antisymmetric bracket the semigroup must be abelian.[4] $\mathfrak{g}_S = S \times \mathfrak{g}$ is called the S-expansion of $\mathfrak{g}$. There are two cases when it is possible to systematically extract subalgebras from $S \otimes \mathfrak{g}$. For example, if we assume that we can decompose $\mathfrak{g}$ in a direct sum of subspaces, $\mathfrak{g} = V_0 \oplus V_1$, where

$$\begin{aligned} [V_0, V_0] &\subset V_0, \\ [V_0, V_1] &\subset V_1, \\ [V_1, V_1] &\subset V_0, \end{aligned}$$

and that the semigroup S has a resonant decomposition, $S = S_0 \cup S_1$, then we have that

$$\mathfrak{g}_{S,R} = (S_0 \otimes V_0) \oplus (S_1 \otimes V_1) \tag{2.40}$$

[4] The only exception is when the algebra $\mathfrak{g}$ is abelian.

is a Lie subalgebra of $\mathfrak{g}_S$ called a *resonant subalgebra* (see Theorem 4.2 of ref.[37]).

An even smaller algebra can be obtained when there is a zero element 0_S in the semigroup. When this is the case, the whole $0_S \otimes \mathfrak{g}$ sector can be removed from the resonant subalgebra. (see Definition 3.3 from ref.[37]). When the semigroup has a zero element the commutation relations of the expanded algebra $\mathfrak{g}_S$ are given by,

$$\begin{aligned}
\left[X_{(i,\alpha)}, X_{(j,\beta)}\right] &= C_{ij}^{k}\mathcal{K}_{\alpha\beta}^{\gamma}X_{(k,\gamma)} + C_{ij}^{k}\mathcal{K}_{\alpha\beta}^{0}X_{(k,0)},\ \lambda_\alpha, \lambda_\beta \neq 0_S,\\
\left[X_{(i,0)}, X_{(j,\beta)}\right] &= C_{ij}^{k}X_{(k,0)},\\
\left[X_{(i,0)}, X_{(j,0)}\right] &= C_{ij}^{k}X_{(k,0)}.
\end{aligned}$$

$0_S \times \mathfrak{g} \in \mathfrak{g}_S$ is an ideal so $\mathfrak{g}/0_S \times \mathfrak{g}$ is a Lie algebra. We denote this algebra by $\mathfrak{g}_{S,R}^{\mathrm{red}}$. The commutation relations of the 0_S-reduced algebra are

$$\left[X_{(i,\alpha)}, X_{(j,\beta)}\right] = C_{ij}^{k}\mathcal{K}_{\alpha\beta}^{\gamma}X_{(k,\gamma)}.$$

2.12 Bianchi's classification of 3-dimensional Lie algebras

In ref.[29] Bianchi proposed a procedure to classify the 3-dimensional spaces that admit a 3-dimensional isometry. He showed how to represent the generators as Killing vectors and how to get the corresponding metrics. The, he formulated Bianchi's theorem.

Here we are interested in studying the possibility to relate, by means of expansions, 2 and 3-dimensional isometry algebras. The 2-dimensional algebras are simply

$$[X_1, X_2] = 0 \ \text{ and} \tag{2.41}$$
$$[X_1, X_2] = X_1. \tag{2.42}$$

The $3-$dimensional algebras are given in Figure 2.8.

2.13 Wess-Zumino-Witten (WZW) models

In this Section we review Chapter 15 in ref.[38]. We follow closely ref.[39].

We are interested in models that exhibit conformal invariance. We consider 2-dimensional sigma models whose target space is a semisimple Lie group, from now on denoted by G. Let $g(z^0, z^1)$ be a map from a two

Group	**Algebra**
type I	$[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0$
type II	$[X_1, X_2] = [X_1, X_3] = 0, \quad [X_2, X_3] = X_1$
type III	$[X_1, X_2] = [X_2, X_3] = 0, \quad [X_1, X_3] = X_1$
type IV	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_1 + X_2$
type V	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_2$
type VI	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = hX_2,$ where $h \neq 0, 1$.
type VII$_1$	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1$
type VII$_2$	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1 + hX_2,$ where $h \neq 0$ $(0 < h < 2)$.
type VIII	$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3$
type IX	$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2$

Fig. 2.8: Bianchi's classification of 3-dimensional Lie algebras.

dimensional manifold M_2 with coordinates $(z_0,\, z_1)$ to G. We consider the following functional action:

$$S_0[g] = \frac{1}{4a^2}\int_{M_2} \mathrm{tr}(g^{-1}\mathrm{d}g g^{-1}\mathrm{d}g) = \frac{1}{4a^2}\int_{M_2} \mathrm{tr}(\mathrm{d}g g^{-1}\mathrm{d}g g^{-1}), \qquad (2.43)$$

where in (2.43) the notation means

$$g^{-1}\mathrm{d}g = g^{-1}(z^0, z^1)\partial_\mu g \mathrm{d}z^\mu, \qquad \mu = 0, 1,$$

that is, we are considering the pull-back by the map $g(z^0, z^1)$ of the 1-forms, and the integration is performed over the two dimensional manifold. In fact, we can write

$$S_0[g] = \frac{1}{4a^2}\int_{M_2} \mathrm{d}^2 z\, \mathrm{tr}(g^{-1}\partial_\mu g g^{-1}\partial^\mu g) = -\frac{1}{4a^2}\int_{M_2} \mathrm{d}^2 z\, \mathrm{tr}\left(\partial_\mu(g^{-1})\partial^\mu g\right).$$

The 1-form $g^{-1}\mathrm{d}g$ is valued in the Lie algebra $\mathfrak{g}$ of G, so it has $\dim(G)$ components that are ordinary, left invariant forms (Maurer-Cartan forms). The trace, taken over a representation of G, can be appropriately normalized. The 1-form $\mathrm{d}gg^{-1}$ is also Lie algebra valued, ant its components are right invariant forms. We conclude that the action (2.43) is invariant under independent left and right group translations, so the symmetry group is $G \times G$ or, as sometimes it is denoted, $G_L \times G_R$.

There is still another way of writing (2.43). Let us choose coordinates in G, ϕ^i , with $i = 1, \ldots, \dim(G)$, then

$$\partial_\mu g = \partial_i g \partial_\mu \phi^i,$$

and

$$S_0 = \frac{1}{2a^2} \int_{M_2} \mathrm{d}^2\sigma g_{ij} \partial^\mu \phi^i \partial_\mu \phi^j, \qquad g_{ij} = \frac{1}{2} \mathrm{tr}(g^{-1}\partial_i g g^{-1} \partial_j g). \tag{2.44}$$

The metric g_{ij} has then $G \times G$ as a group of isometries. In the form (2.44) the Wess-Zumino-Witten model is a standard sigma model in two dimensions.

To obtain the field equations, we compute the variation of S under an arbitrary change $g \to g + \partial g$,

$$\begin{aligned}
4a^2\delta\mathcal{L} \quad = \quad & \partial^\mu \delta g \partial_\mu g + \partial^\mu g^{-1} \partial_\mu \delta g = \\
& \partial^\mu(\delta g^{-1}\partial_\mu g) - \delta g^{-1} \Box g + \partial^\mu g^{-1} \partial_\mu \delta g = \\
& \partial^\mu(\delta g^{-1}\partial_\mu g) + g^{-1}\delta g g^{-1} \Box g + \partial^\mu(\delta g^{-1}\partial_\mu g) = \\
& \partial^\mu(\delta g^{-1}\partial_\mu g) + g^{-1}\delta g g^{-1} \Box g - g^{-1}\partial^\mu g g^{-1} \partial_\mu \delta g = \\
& \partial^\mu(\delta g^{-1}\partial_\mu g) + g^{-1}\delta g g^{-1} \Box g + \partial_\mu(g^{-1}\partial^\mu g g^{-1})\delta g - \\
& \partial_\mu(g^{-1}\partial^\mu g g^{-1}\delta g).
\end{aligned}$$

The first and fourth terms cancel each other using the cyclic property of the trace and $\partial_\mu g^{-1} = -g^{-1}\partial_\mu g g^{-1}$. We obtain

$$\begin{aligned}
4a^2\delta\mathcal{L} \quad = \quad & g^{-1}\delta g g^{-1}\Box g + \partial_\mu(g^{-1}\partial^\mu g g^{-1})\delta g = \\
& g^{-1}\delta g g^{-1}\Box g + g^{-1}\Box g g^{-1}\delta g + \partial_\mu g^{-1}\partial^\mu g g^{-1}\delta g + \\
& g^{-1}\partial^\mu g \partial_m u g^{-1}\delta g = g^{-1}\delta g g^{-1}\Box g + g^{-1}\Box g g^{-1}\delta g + \\
& \partial_\mu g^{-1}\partial^\mu g g^{-1}\delta g + g^{-1} g \partial_\mu g^{-1} g g^{-1}\partial_\mu g g^{-1}\delta g = \\
& g^{-1}\delta g g^{-1}\Box g + g^{-1}\Box g g^{-1}\delta g + \partial_\mu g^{-1}\partial^\mu g g^{-1}\delta g + \\
& g^{-1}\delta g \partial^\mu g^{-1}\partial_\mu g.
\end{aligned}$$

Reorganizing the terms we have

$$\begin{aligned}
\delta S_0[g] \quad = \quad & \frac{1}{2a^2}\int_{M_2} \mathrm{d}^2 z\, \mathrm{tr}(g^{-1}\delta g \partial^\mu g^{-1}\partial_\mu g + g^{-1}\delta g g^{-1}\Box g) = \\
& \frac{1}{2a^2}\int_{M_2} \mathrm{d}^2 z\, \mathrm{tr}(g^{-1}\delta g \partial^\mu(g^{-1}\partial_\mu g)).
\end{aligned}$$

Requiring that this variation is zero gives the field equations

$$\partial^\mu (g^{-1}\partial_\mu g) = 0. \tag{2.45}$$

If we want to use the formalism of complex variables we have to write the last equation in terms of holomorphic and antiholomorphic variables,

$$\partial_z J_{\bar{z}} + \bar{\partial}_{\bar{z}} J_{\bar{z}} = 0, \qquad J_z = g^{-1}\partial_z g, \qquad J_{\bar{z}} = g^{-1}\partial_{\bar{z}} g.$$

In general it is not true that the two currents, J_z and $J_{\bar{z}}$ are conserved separately. Due to the fact that the Virasoro algebra is spanned by two currents (the holomorphic and the antiholomorphic one), we should look for a theory where two independent currents are conserved.

To obtain the separate conservation of the currents, one can add a term due to Wess and Zumino [40],

$$\begin{aligned}\Gamma[\tilde{g}] &= \frac{-\mathrm{i}}{24\pi}\int_{M_3} \mathrm{tr}(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g) \\ &= \frac{-\mathrm{i}}{24\pi}\int_{M_3} \mathrm{d}^3y\, \epsilon_{\alpha\beta\gamma} \mathrm{tr}(\tilde{g}^{-1}\partial^\alpha \tilde{g}\tilde{g}^{-1}\partial^\beta \tilde{g}\tilde{g}^{-1}\partial^\gamma \tilde{g}).\end{aligned} \tag{2.46}$$

Some explanation is in order. The integration is done over a 3-dimensional space whose boundary is the compactification of our 2 dimensional space. For example, we can think on a solid ball in $\mathbb{R}^3$, delimited by the sphere S^2 which is the compactification of $\mathbb{R}^2$. Then we also compactify $\mathbb{R}^3$ to S^3. $\tilde{g}$ is an extension of the map g to that 3 dimensional space. It is easy to show, using the cyclic property of the trace and the antisymmetry of the wedge product that

$$\mathrm{d}\left(\mathrm{tr}(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g)\right) = 0,$$

so the expression under the integral sign in (2.46) is a closed, possibly non exact 3-form.

In S^3, the manifold delimited by a given cycle S^2 is not unique. Let us proceed by analogy in two dimensions, so one can visualize it better. Instead of S^3 we consider S^2, and we draw a circle S^1 on its surface. It delimits a disc, which plays the role of the solid ball. In Fig. 2.9 we can see the setting. If we use the right-hand rule in the frontier (the red circumference) to define the orientation of the manifolds A and B, then they have opposite orientations. So the difference between the integration in A and in B is the integration on all the surface of the sphere,

$$\Delta\Gamma = \Gamma\big|_A - \Gamma\big|_B = \Gamma\big|_{\text{sphere}}.$$

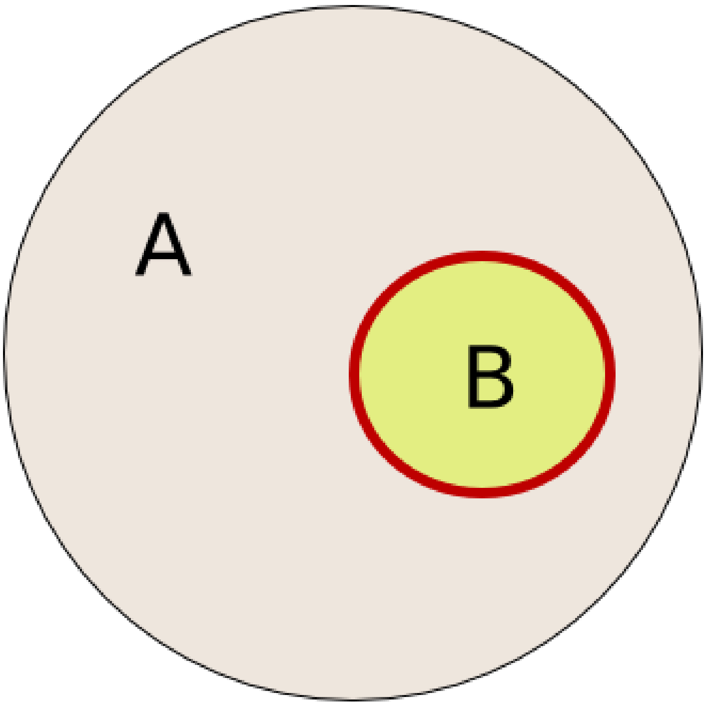

Fig. 2.9: Example of the setup needed to use the Wess-Zumino term

This quantity depends only on the cohomology class of the integrand, which is integer. This implies that it is an integer multiple of $2\pi\mathrm{i}$, (we choose a normalization of the trace so we get exactly this result), so the functional integral is well defined.

The same argument can be used to prove that the integral is independent of the extension $\tilde{g}$. We assume that we have an extension $\tilde{g}$ to the whole sphere. Let $\tilde{g}'$ a smooth deformation of $\tilde{g}$. Then

$$\Gamma\big|_A[\tilde{g}'] - \Gamma\big|_A[\tilde{g}] = \Gamma\big|_A[\tilde{g}'] + \Gamma\big|_B[\tilde{g}].$$

The right hand side of this equation is again $\Gamma\big|_{\text{sphere}}$ for some map defined by both, $\tilde{g}$ on A and $\tilde{g}'$ on B, on the whole sphere. It is again a multiple of $2\pi\mathrm{i}$, so, as before, the path integral with weight $\exp(-\Gamma)$ is well defined.

The action proposed by Wess and Zumino is then

$$S = S_0 + k\Gamma, \qquad k \in \mathbb{Z}.$$

Varying it with respect to g we obtain the field equations

$$(1 + \frac{a^2 k}{4\pi})\partial_z(g^{-1}\partial_{\bar{z}} g) + (1 - \frac{a^2 k}{4\pi})\partial_{\bar{z}}(g^{-1}\partial_z g) = 0. \tag{2.47}$$

One can choose $a^2 = \frac{4\pi}{k}$, and then the field equations reduce to the conservation law

$$\partial_z J_{\bar{z}} = 0, \qquad J_{\bar{z}} = g^{-1}\partial_{\bar{z}} g.$$

The general solution is

$$g(z,\bar{z}) = f(z)\bar{f}(\bar{z}), \qquad f(z), \bar{f}(\bar{z}) \in G.$$

This implies also the conservation of $J_z = \partial_z g g^{-1}$. The separate conservation of the two currents implies that the Virasoro algebra is an algebra of symmetries of the model. Let us see this.

As we mentioned at the beginning of this Section, the model has a global symmetry under $G_L \times G_R$. In fact, the symmetry is local in the following sense:

$$g(z,\bar{z}) \longrightarrow \Omega(z) g(z,\bar{z}) \bar{\Omega}(\bar{z}).$$

where $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ are two (different) group elements. The infinitesimal transformations are

$$\delta_w g = wg, \qquad \delta_{\bar{w}} g = -g\bar{w}, \qquad \delta_{w\bar{w}} g = \delta_w g + \delta_{\bar{w}} g. \tag{2.48}$$

We can actually compute the infinite dimensional algebra of the symmetry transformations (2.48). Let t^a be a basis of $\mathfrak{g}$, so

$$[t^a, t^b] = f^{ab}{}_c t^c,$$

where f^{ab}_c are the structure constants. The currents can be expressed as

$$J_z(z) = \sum_a J_a(z) t^a, \qquad \bar{J}_{\bar{z}}(\bar{z}) = \sum_a J_a(\bar{z}) t^a.$$

It is convenient to redefine the currents by multiplying them by the number k.

There are Ward identities with the currents $J(z)$ and $\bar{J}(\bar{z})$, which allow us to compute the *operator product expansion* (OPE) of the currents. This computation is done in detail in ref.[38], so we will just quote the result. Let us focus in the holomorphic current, with Laurent development

$$J^a(z) = \sum_n z^{-n-1} J^a_n.$$

Then, the OPE of two currents has the form

$$J^a(z) J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \sum_c \mathrm{i} f^{ab}{}_c \frac{J^c}{(z-w)},$$

one can prove that the modes J^a_n satisfy the Lie algebra

$$[J^a_n, J^b_m] = \sum_c i f_{abc} J^c_{n+m} + kn\delta^{ab}\delta_{n+m,0}.$$

This is a *Kac-Moody algebra.* It is an example of an *affine Lie algebra*, that can be constructed as a central extension of the *loop algebra* associated to $\mathfrak{g}$. The number k is called the *level* of the algebra.

The same procedure can be applied for the antiholomorphic current, giving another copy of the Kac-Moody algebra

$$[\bar{J}^a_n, \bar{J}^b_m] = \sum_c i f^{ab}{}_c \bar{J}^c_{n+m} + kn\delta^{ab}\delta_{n+m,0},$$

and vanishing mixed commutators,

$$[\bar{J}^a_n, J^b_m] = 0.$$

It can be proven [38] that the energy-momentum tensor of a WZW model has the form

$$T(z) = \frac{1}{2(k+\delta)} \sum_a : J^a(z) J^a)(z) :, \tag{2.49}$$

where we have introduced the normal ordered product of the currents. The number δ is a characteristic of the Kac-Moody algebra called the *dual Coxeter number* [5]. Then, the modes of the Laurent expansion of the energy-momentum tensor (2.49) can be expressed in terms of the modes J^a_n as

$$L_n = \frac{1}{2(k+\delta)} \sum_a \left\{ \sum_{m \le -1} J^a_m J^a_{n-m} + \sum_{m \ge 0} J^a_{n-m} J^a_m \right\} =$$
$$\frac{1}{2(k+\delta)} \sum_a \sum_m : J^a_m J^a_{n-m} : .$$

One can check that these modes satisfy the Virasoro algebra. Then, WZW models are a type of quantum field theory with conformal symmetry. The complete Kac-Moody and Virasoro algebras of this model are then

$$\begin{aligned} [L_n, L_m] &= (n-m) L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \\ [L_n, J^a_m] &= -m J^a_{n+m}, \\ \left[J^a_n, J^b_m\right] &= \sum_c i f_{abc} J^c_{n+m} + kn\delta_{ab}\delta_{n+m,0}. \end{aligned}$$

[5] We do not need to enter in more details on the dual Coxeter number, it will take us far away from our purpose.

3. STAR PRODUCT ON MINKOWSKI SPACE

In this Chapter we present a noncommutative product in Minkowski space (M) (see Section 2.4). In Section 3.1 we define this product (the *star product*) as an algebraic product in the algebra of polynomials on M, $\mathcal{O}(\mathrm{M})$. We give an explicit formula for the star product of two such polynomials. In Section 3.2 we show that it is possible to reproduce the star product with a bidifferential operator to any order in the noncommutativity parameter and we compute it up to order 2. Finally we define this bidifferential operator as the *star product on fields*. These results have been published in ref.[2].

3.1 Algebraic star product on Minkowski space

We consider now the algebra of the classical Minkowski space with the scalars extended to the ring $\mathbb{C}_q \approx \mathbb{C}[q, q^{-1}]$ (see Remark 2.3.1). With the generators used in (2.20)

$$\mathcal{O}(\mathrm{M})[q, q^{-1}] \equiv \mathbb{C}_q[t_{41}, t_{42}, t_{31}, t_{32}].$$

We consider also the non commutative algebra $O_q(M)$ on quantum Minkowski space defined in Section 2.5. There is an isomorphism $\mathcal{O}(\mathrm{M})[q, q^{-1}] \approx \mathcal{O}_q(\mathrm{M})$ as modules over $\mathbb{C}_q$. In fact, in ref.[41] it was proven that the map

$$\begin{array}{ccc} \mathbb{C}_q[t_{41}, t_{42}, t_{31}, t_{32}] & \xrightarrow{Q_{\mathrm{M}}} & \mathcal{O}_q(\mathrm{M}) \\ t_{41}^a t_{42}^b t_{31}^c t_{32}^d & \longrightarrow & \hat{t}_{41}^a \hat{t}_{42}^b \hat{t}_{31}^c \hat{t}_{32}^d. \end{array} \tag{3.1}$$

is a module isomorphism (so it has an inverse). A map like (3.1) is called an *ordering rule* or *quantization map*. So $\mathcal{O}_q(\mathrm{M})$ is a free module over $\mathbb{C}_q$, with basis the set of standard monomials.

We can pull back the product on $\mathcal{O}_q(\mathrm{M})$ to $\mathcal{O}(\mathrm{M})[q, q^{-1}]$. There, we can define the *star product* as

$$f \star g = Q_{\mathrm{M}}^{-1}\big(Q_{\mathrm{M}}(f) Q_{\mathrm{M}}(g)\big), \qquad f, g \in \mathcal{O}(\mathrm{M})[q, q^{-1}]. \tag{3.2}$$

By construction, the star product is associative.

The algebra $(\mathcal{O}(\mathrm{M})[q,q^{-1}],\star)$ is then isomorphic to $\mathcal{O}_q(\mathrm{M})$. Working on $\mathcal{O}(\mathrm{M})[q,q^{-1}]$ has the advantage of working with 'classical' objects (the polynomials), were one has substituted the standard pointwise product by the noncommutative star product. This is important for the physical applications. Moreover, we can study if this star product has an extension to all the C^∞ functions, and if the extension is differential. If so, Kontsevich's theory [42] would then be relevant.

We want to obtain a formula for the star product. We begin by computing the auxiliary relations

$$\begin{aligned}
\hat{t}_{42}^m \hat{t}_{41}^n &= q^{-mn}\hat{t}_{41}^n \hat{t}_{42}^m,\\
\hat{t}_{31}^m \hat{t}_{41}^n &= q^{-mn}\hat{t}_{41}^n \hat{t}_{31}^m,\\
\hat{t}_{31}^m \hat{t}_{42}^n &= \hat{t}_{42}^n \hat{t}_{31}^m,\\
\hat{t}_{32}^m \hat{t}_{42}^n &= q^{-mn}\hat{t}_{42}^n \hat{t}_{32}^m,\\
\hat{t}_{32}^m \hat{t}_{31}^n &= q^{-mn}\hat{t}_{31}^n \hat{t}_{32}^m.
\end{aligned}$$

After a (lengthy) computation we obtain

$$\hat{t}_{32}^m \hat{t}_{41}^n = \hat{t}_{41}^n \hat{t}_{32}^m + \sum_{k=1}^{\mu} F_k(q,m,n)\hat{t}_{41}^{n-k}\hat{t}_{42}^k \hat{t}_{31}^k \hat{t}_{32}^{m-k},$$

where $\mu = \min(m,n)$,

$$F_k(q,m,n) = \beta_k(q,m)\prod_{l=0}^{k-1} F(q,n-l) \qquad \text{with} \qquad F(q,n) = \left(\frac{1}{q^{2n-1}} - q\right) \tag{3.3}$$

and $\beta_k(q,m)$ is defined by the recursive relation

$$\beta_0(q,m) = \beta_m(q,m) = 1, \quad \text{and} \quad \beta_k(q,m+1) = \beta_{k-1}(q,m) + \beta_k(q,m)q^{-2k}.$$

Moreover, $\beta_k(q,m) = 0$ if $k < 0$ or if $k > m$. Using the above relations, we obtain the star product of two arbitrary polynomials:

$$\begin{aligned}
&(t_{41}^a t_{42}^b t_{31}^c t_{32}^d) \star (t_{41}^m t_{42}^n t_{31}^p t_{32}^r) = q^{-mc-mb-nd-dp} t_{41}^{a+m} t_{42}^{b+n} t_{31}^{c+p} t_{32}^{d+r}\\
&+ \sum_{k=1}^{\mu=min(d,m)} q^{-(m-k)c-(m-k)b-n(d-k)-p(d-k)} F_k(q,d,m)\\
&t_{41}^{a+m-k} t_{42}^{b+k+n} t_{31}^{c+k+p} t_{32}^{d-k+r}.
\end{aligned} \tag{3.4}$$

3.2 Differential star product on the big cell

In order to compare the algebraic star product obtained above with the differential star product approach, we consider a change in the parameter, $q = \exp h$. The classical limit is then obtained as $h \to 0$. We will expand (3.4) in powers of h and we will show that each term can be written as a bidifferential operator. Then, the extension of the star product to C^∞ functions is unique.

3.2.1 Explicit computation up to order 2

We first take up the explicit computation of the bidifferential operators up to order 2. Then we will argue that a differential operator can be found at each order.

We rewrite (3.4) as

$$f \star g = fg + \sum_{j=1}^{\infty} h^j C_j(f,g),$$

with

$$f = t_{41}^a t_{42}^b t_{31}^c t_{32}^d, \qquad g = t_{41}^m t_{42}^n t_{31}^p t_{32}^r.$$

At order 0 in h we recover the commutative product. At order n in h we have contributions from each of the terms with different k in (3.4).

$$C_n(f,g) = \sum_{k=0}^{\mu=min(d,m)} C_n^{(k)}(f,g),$$

(the terms with $k=0$ come from the first term in (3.4)).

Let us compute each of the contributions $C_1^{(k)}$:

- $k = 0$. We have

$$C_1^{(0)} = (-mc - mb - nd - dp)\; t_{41}^{a+m} t_{42}^{b+n} t_{31}^{c+p} t_{32}^{d+r}.$$

It is easy to see that this is reproduced by the bidifferential operator

$$C_1^{(0)}(f,g) = -(t_{41}t_{31}\partial_{31}f\partial_{41}g + t_{42}t_{41}\partial_{42}f\partial_{41}g + t_{32}t_{42}\partial_{32}f\partial_{42}g + t_{32}t_{31}\partial_{32}f\partial_{31}g).$$

We will denote the bidifferential operators by means of the tensor product (as it is customary). For example

$$C_1^{(0)} = -(t_{41}t_{31}\partial_{31}\otimes\partial_{41} + t_{42}t_{41}\partial_{42}\otimes\partial_{41} + t_{32}t_{42}\partial_{32}\otimes\partial_{42}+ t_{32}t_{31}\partial_{32}\otimes\partial_{31}),$$

so

$$C_1^{(0)}(f,g) = C_1^{(0)}(f\otimes g).$$

- $k=1$. Let us first compute the factor $F_1(q,d,m) = \beta_1(q,d)F(q,m)$. First, notice that

$$\beta_1(q,d) = 1 + q^{-2} + q^{-4} + \cdots + q^{-2(d-1)} = \frac{\mathrm{e}^{-2dh}-1}{\mathrm{e}^{-2h}-1} = d - d(d-1)h + \frac{1}{3}d((1-3d+2d^2)h^2 + \mathcal{O}(h^3),$$

and that

$$F(q,n) = -2nh + 2n(n-1)h^2 + \mathcal{O}(h^3),$$

so up to order h^2 we have

$$\beta_1(q,d)F(q,m) = -2mdh + 2md(d+m-2)h^2 + \mathcal{O}(h^3).$$

Finally, the contribution of the $k=1$ term to C_1 is

$$C_1^{(1)}(f,g) = -2mdt_{41}^{a+m-1}\, t_{42}^{b+n+1}t_{31}^{c+p+1}t_{32}^{d+r-1}.$$

This is reproduced by the bidifferential operator

$$C_1^{(1)} = -2t_{42}t_{31}\partial_{32}\otimes\partial_{41}.$$

- $k\geq 2$ We have the factor

$$\beta_k(q,d)F(q,m)F(q,m-1)\cdots F(q,m-k) = \mathcal{O}(h^k),$$

so the terms with $k\geq 2$ do not contribute C_1.

Summarizing,

$$C_1 = C_1^{(0)} + C_1^{(1)} = -(t_{41}t_{31}\partial_{31}\otimes\partial_{41} + t_{42}t_{41}\partial_{42}\otimes\partial_{41}+ t_{32}t_{42}\partial_{32}\otimes\partial_{42} + t_{32}t_{31}\partial_{32}\otimes\partial_{31} + 2t_{42}t_{31}\partial_{32}\otimes\partial_{41}), \tag{3.5}$$

so C_1 is extended to the C^∞ functions. If we antisymmetrize C_1 we obtain a Poisson bracket

$$\begin{aligned}\{f,g\} =& t_{41}t_{31}(\partial_{41}f\partial_{31}g-\partial_{41}g\partial_{31}f)+t_{42}t_{41}(\partial_{41}f\partial_{42}g-\partial_{41}g\partial_{42}f)+\\ & t_{32}t_{42}(\partial_{42}f\partial_{32}g-\partial_{42}g\partial_{32}f)+t_{32}t_{31}(\partial_{31}f\partial_{32}g-\partial_{31}g\partial_{32}f)+\\ & 2t_{42}t_{31}(\partial_{41}f\partial_{32}g-\partial_{41}g\partial_{32}f).\end{aligned} \tag{3.6}$$

We can express the Poisson bracket in terms of the usual variables in Minkowski space. Using (2.22), the change of coordinates is

$$\begin{pmatrix} t_{31} & t_{32}\\ t_{41} & t_{42}\end{pmatrix} = x^\mu\sigma_\mu = \begin{pmatrix} x^0+x^3 & x^1-\mathrm{i}x^2\\ x^1+\mathrm{i}x^2 & x^0-x^3\end{pmatrix},$$

and the inverse change is

$$x^0=\frac{1}{2}(t_{31}+t_{42}),\quad x^1=\frac{1}{2}(t_{32}+t_{41}),\quad x^2=\frac{i}{2}(t_{32}-t_{41}),\quad x^3=\frac{1}{2}(t_{31}-t_{42}).$$

In these variables the Poisson bracket is

$$\begin{aligned}\{f,g\} =& \mathrm{i}\Big(\big((x^0)^2-(x^3)^2\big)(\partial_1 f\partial_2 g-\partial_1 g\partial_2 f)+x^0x^1(\partial_0 f\partial_2 g-\partial_0 g\partial_2 f)-\\ & x^0x^2(\partial_0 f\partial_1 g-\partial_0 g\partial_1 f)-x^1x^3(\partial_2 f\partial_3 g-\partial_2 g\partial_3 f)+\\ & x^2x^3(\partial_1 f\partial_3 g-\partial_1 g\partial_3 f)\Big).\end{aligned} \tag{3.7}$$

We now compute the term C_2. We sum the contributions to the order h^2 of each term in (3.4)

- $k=0$. The contribution to the order h^2 is

$$C_2^{(0)}=\frac{1}{2}(mc+mb+nd+dp)^2\; t_{41}^{a+m}t_{42}^{b+n}t_{31}^{c+p}t_{32}^{d+r}.$$

This is reproduced by

$$
\begin{aligned}
C_2^{(0)} =& \frac{1}{2} t_{31}t_{41}\ \partial_{31}(t_{31}\partial_{31}) \otimes \partial_{41}(t_{41}\partial_{41}) + \\
& t_{42}t_{31}t_{41}\ \partial_{42}\partial_{31} \otimes \partial_{41}(t_{41}\partial_{41}) + t_{31}t_{32}t_{41}t_{42}\ \partial_{31}\partial_{32} \otimes \partial_{41}\partial_{42} + \\
& t_{31}^2 t_{32}t_{41}\ \partial_{31}\partial_{32} \otimes \partial_{41}\partial_{31} + \frac{1}{2} t_{42}t_{41}\ \partial_{42}(t_{42}\partial_{42}) \otimes \partial_{41}(t_{41}\partial_{41}) + \\
& t_{41}t_{42}^2 t_{32}\partial_{42}\partial_{32} \otimes \partial_{41}\partial_{42} + t_{41}t_{42}t_{31}t_{32}\partial_{42}\partial_{32} \otimes \partial_{41}\partial_{31} + \\
& \frac{1}{2} t_{32}t_{42}t_{31}\partial_{32}(t_{32}\partial_{32}) \otimes \partial_{42}\partial_{31} + \\
& \frac{1}{2} t_{32}t_{31}\ \partial_{32}(t_{32}\partial_{32}) \otimes \partial_{31}(t_{31}\partial_{31}) + \\
& t_{32}t_{42}t_{31}\ \partial_{32}(t_{32}\partial_{32}) \otimes \partial_{42}(t_{42}\partial_{42}).
\end{aligned}
$$

- $k = 1$. We have that

$$F_1(q, d, m) = \beta_1(q, d)F(q, m).$$

Expanding both factors we have

$$
\begin{aligned}
\beta_1(q, d) &= d - d(d-1)h + O(h^2), \\
F(q, m) &= -2mh - 2m(1-m)h^2 + O(h^3),
\end{aligned}
$$

so we get

$$\beta_1(q, d)F(q, m) \approx -2mdh + 2md((m-1) + (d-1))h^2,$$

and the contribution to order h^2 is

$$
h^2 \Big(2md\big((m-1) + (d-1) + (m-1)c + (m-1)b + n(d-1) + p(d-1)\big)\Big) \cdot t_{41}^{a+m-1} t_{42}^{b+n+1} t_{31}^{c+p+1} t_{32}^{d+r-1}.
$$

We reproduce this result with

$$
\begin{aligned}
C_2^{(1)} =& 2t_{32}t_{42}t_{31}\partial_{32}^2 \otimes \partial_{41} + 2t_{31}t_{42}t_{41}\partial_{32} \otimes \partial_{41}^2 + \\
& 2t_{31}t_{42}^2 t_{41}\partial_{42}\partial_{32} \otimes \partial_{41}^2 + 2t_{42}t_{31}^2 t_{41}\partial_{31}\partial_{32} \otimes \partial_{41}^2 + \\
& 2t_{31}t_{42}^2 t_{32}\partial_{32}^2 \otimes \partial_{41}\partial_{42} + 2t_{42}t_{31}^2 t_{32}\partial_{32}^2 \otimes \partial_{41}\partial_{31}.
\end{aligned}
$$

- $k = 2$. One can show that

$$\beta_2(q,d) = \frac{d(d-1)}{2} + O(h),$$

so

$$\beta_2(q,d)F(q,m)F(q,m-1) \approx 2d(d-1)m(m-1)h^2,$$

and the contribution of this term to the order h^2 is

$$h^2\, 2d(d-1)m(m-1)\; t_{41}^{a+m-2} t_{42}^{b+n+2} t_{31}^{c+p+2} t_{32}^{d+r-2}.$$

This is given by

$$C_2^{(2)} = 2t_{42}^2 t_{31}^2 \partial_{32}^2 \otimes \partial_{41}^2.$$

Summarizing we get

$$\begin{aligned}
C_2 =& \frac{1}{2} t_{31}t_{41}\ \partial_{31}(t_{31}\partial_{31}) \otimes \partial_{41}(t_{41}\partial_{41}) + t_{42}t_{31}t_{41}\ \partial_{42}\partial_{31} \otimes \partial_{41}(t_{41}\partial_{41}) + \\
& t_{31}t_{32}t_{41}t_{42}\ \partial_{31}\partial_{32} \otimes \partial_{41}\partial_{42} + t_{31}^2 t_{32}t_{41}\ \partial_{31}\partial_{32} \otimes \partial_{41}\partial_{31} + \\
& \frac{1}{2} t_{42}t_{41}\ \partial_{42}(t_{42}\partial_{42}) \otimes \partial_{41}(t_{41}\partial_{41}) + t_{41}t_{42}^2 t_{32}\partial_{42}\partial_{32} \otimes \partial_{41}\partial_{42} + \\
& t_{41}t_{42}t_{31}t_{32}\partial_{42}\partial_{32} \otimes \partial_{41}\partial_{31} + \frac{1}{2} t_{32}t_{42}t_{31}\partial_{32}(t_{32}\partial_{32}) \otimes \partial_{42}\partial_{31} + \\
& \frac{1}{2} t_{32}t_{31}\ \partial_{32}(t_{32}\partial_{32}) \otimes \partial_{31}(t_{31}\partial_{31}) + 2t_{42}^2 t_{31}^2 \partial_{32}^2 \otimes \partial_{41}^2 + \\
& t_{32}t_{42}t_{31}\ \partial_{32}(t_{32}\partial_{32}) \otimes \partial_{42}(t_{42}\partial_{42}) + 2t_{32}t_{42}t_{31}\partial_{32}^2 \otimes \partial_{41} + \\
& 2t_{31}t_{42}t_{41}\partial_{32} \otimes \partial_{41}^2 + 2t_{31}t_{42}^2 t_{41}\partial_{42}\partial_{32} \otimes \partial_{41}^2 + 2t_{42}t_{31}^2 t_{32}\partial_{32}^2 \otimes \partial_{41}\partial_{31} + \\
& 2t_{42}t_{31}^2 t_{41}\partial_{31}\partial_{32} \otimes \partial_{41}^2 + 2t_{31}t_{42}^2 t_{32}\partial_{32}^2 \otimes \partial_{41}\partial_{42}.
\end{aligned}$$

3.2.2 *Differentiability at arbitrary order*

We are going to prove now the differentiability of the star product. We keep in mind the expression (3.4), which has to be expanded in h. Our goal will be to show that, at each order, it can be reproduced by a bidifferential operator with no dependence on the exponents a, b, c, d, m, n, p, r.

Let us first argue on a polynomial function of one variable, say x. For example, we have

$$m\, x^{m-1} = \partial_x(x^m).$$

More generally, we have

$$\begin{aligned}
&m^b\, x^m = (x\partial_x)^b(x^m) \qquad \text{and}\\
&m^b(m-1)^c\cdots(m-k+1)^d\, x^{m-k} =\\
&\partial_x(x\partial_x)^{d-1}\dots\partial_x(x\partial_x)^{c-1}\partial_x(x\partial_x)^{b-1}(x^m).
\end{aligned} \tag{3.8}$$

Notice that in the last formula, we have $b, c, \dots, d \geq 1$, otherwise the formula makes no sense. In fact, an arbitrary polynomial

$$p(x) = \sum_{k\in\mathbb{Z}} f_k(m,x)x^{m-k},$$

is not generically obtainable from x^m by the application of a differential operator with coefficients that are independent of the exponents and polynomial in the variable x. One can try for example with $p(x) = x^{m-1}$. We then have that

$$x^{m-1} = \frac{1}{m}\partial_x(x^m), \qquad \text{or} \qquad x^{m-1} = \frac{1}{x}x^m.$$

So the right combinations should appear in the coefficients in order to be reproduced by a differential operator with polynomial coefficients.

Let us see the contribution of the terms with different k in (3.4). We start with the term $k=0$. From

$$q^{-mc-mb-nd-dp}\; t_{41}^{a+m}t_{42}^{b+n}t_{31}^{c+p}t_{32}^{d+r}$$

we only get terms of the form

$$b^{i_b}c^{i_c}d^{i_d}m^{i_m}n^{i_n}p^{i_p}\; t_{41}^{a+m}t_{42}^{b+n}t_{31}^{c+p}t_{32}^{d+r}.$$

Applying the rules (3.8), these terms can be easily reproduced by the bidifferential operators of the form

$$(t_{42}\partial_{42})^{i_b}(t_{31}\partial_{31})^{i_c}(t_{32}\partial_{32})^{i_d}\otimes(t_{41}\partial_{41})^{i_m}(t_{42}\partial_{42})^{i_n}(t_{31}\partial_{31})^{i_p},$$

applied to

$$t_{41}^a t_{42}^b t_{31}^c t_{32}^d \otimes t_{41}^m t_{42}^n t_{31}^p t_{32}^r.$$

We turn now to the more complicated case of $k\neq 0$. We have to consider the two factors in (3.4)

$$q^{-(m-k)c-(m-k)b-n(d-k)-p(d-k)}, \qquad \text{and} \qquad F_k(q,d,m).$$

Expanding both factors in powers of h it is easy to see that the coefficients at each order are polynomials in m, n, p, b, c, d, k. What we have to check is that these polynomials have a form that can be reproduced with a bidifferential operator using (3.8). Let us start with

$$F_k(q,d,m) = \beta_k(q,m) \prod_{l=0}^{k-1} F(q,m-l).$$

From the definition (3.3), we have that $F(q,j)|_{j=0} = 0$, so

$$F(q,j) = jG(q,j),$$

with $G(q,j)$ a series in h with coefficients that are polynomial in j. More generally, for the product we have

$$L_k(q,m) = \prod_{l=0}^{k-1} F(q,m-l) = m(m-1)(m-2)\cdots(m-k+1)L'(q,m).$$

The polynomials in $L'(q,m)$ are easily obtained with combinations of differential operators of the form

$$(t_{41}\partial_{41})^i(t_{41}^m).$$

The remaining factor $m(m-1)(m-2)\cdots(m-k+1)t_{41}^{m-k}$ is adjusted with the differential operator

$$\partial_{41}^k(t_{41}^m) = m(m-1)(m-2)\cdots(m-k+1)\, t_{41}^{m-k}.$$

Let us work now with $\beta_k(q,m)$. We have that

$$\beta_k(q,d) = 0 \qquad \text{for } d < k,$$

so

$$\beta_k(q,d) = d(d-1)(d-2)\cdots(d-k+1)\beta'_k(q,d),$$

with $\beta'_k(q,d)$ a series in h with coefficients that are polynomial in d. The differential operator that we need is of the form

$$\partial_{32}^k(t_{32}^d) = d(d-1)(d-2)\cdots(d-k+1)\, t_{32}^{d-k}.$$

Finally, the factor $q^{-(m-k)c-(m-k)b-n(d-k)-p(d-k)}$ introduces factors of the form

$$b^{i_b}c^{i_c}(d-k)^{i_d}(m-k)^{i_m}n^{i_n}p^{i_p}\; t_{41}^{a+m-k}t_{42}^{b+k+n}t_{31}^{c+k+p}t_{32}^{d-k+r},$$

which are reproduced by

$$t_{42}^k t_{31}^k (t_{42}\partial_{42})^{i_b} (t_{31}\partial_{31})^{i_c} (t_{32}\partial_{32})^{i_d} \otimes (t_{41}\partial_{41})^{i_m} (t_{42}\partial_{42})^{i_n} (t_{31}\partial_{31})^{i_p}$$

acting on

$$t_{41}^a t_{42}^b t_{31}^c t_{32}^{d-k} \otimes t_{41}^{m-k} t_{42}^n t_{31}^p t_{32}^r.$$

This completes the proof of differentiability of the star product at arbitrary order.

4. POINCARÉ COACTION

We can also define a star product on the groups. In this Chapter we show that it is responsible for the quantum corrections to the classical action of the Poincaré group on Minkowski space. We start by finding an *ordering rule* for the group which we use to define a *star coaction* compatible with the star product on Minkowski space. In Section 4.1 we calculate the order h contribution to the coaction and reproduce it by means of a differential operator which we define as the *order h star coaction on fields*. These results have been published in [2].

Algebraic star coaction In particular, we are interested in the quantum group $\mathcal{O}_q(P_l)$, the lower parabolic subgroup of $\mathrm{SL}(4, \mathbb{C})$ (see Section 2.3) which we call the *quantum Poincaré group plus dilations*[1]. We would like to see how the coaction on the Minkowski space looks in terms of the star product, and if it is also differential. First of all, we notice that the subalgebra generated by $\{\hat{x}_{ij}\}$ and $\{\hat{y}_{ab}\}$ are two copies of the algebra of 2×2 quantum matrices, which commute among them. The maps to the standard quantum matrices (A.1) are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightleftarrows \begin{pmatrix} \hat{x}_{11} & \hat{x}_{12} \\ \hat{x}_{21} & \hat{x}_{22} \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightleftarrows \begin{pmatrix} \hat{y}_{33} & \hat{y}_{34} \\ \hat{y}_{43} & \hat{y}_{44} \end{pmatrix},$$

as can be deduced from (2.12) and (2.13). One can chose the Manin order in each subset of variables,

$$\hat{y}_{44} < \hat{y}_{43} < \hat{y}_{34} < \hat{y}_{33}, \qquad \hat{x}_{22} < \hat{x}_{21} < \hat{x}_{12} < \hat{x}_{11}.$$

With this one can construct a quantization map (given by the standard monomials basis) for the quantum Lorentz plus dilations group. We have now to include the translations to have the complete quantization map for

[1] We will call it simply quantum Poincaré group, but we shall remember that there is this dilation factor.

the Poincaré group. It is clear that one can choose the Manin order also for the variables $\hat{T}$, but, since these variables do not commute with the $\hat{x}$'s and the $\hat{y}$'s, we have to be careful in choosing a full ordering rule. This is a non trivial problem, but it can be solved. In Appendix A we show that the ordering

$$\hat{y}_{44} < \hat{y}_{43} < \hat{y}_{34} < \hat{y}_{33} < \hat{x}_{22} < \hat{x}_{21} < \hat{x}_{12} < \hat{x}_{11} < \hat{T}_{41} < \hat{T}_{42} < \hat{T}_{31} < \hat{T}_{32}$$

gives standard monomials that form a basis for the quantum Poincaré group $\mathcal{O}_q(P_l)$. As for the Minkowski space star product (3.2), we extend the scalars of the commutative algebra to $\mathbb{C}_q$ and define a quantization map Q_G

$$\begin{aligned} \mathcal{O}(P_l)[q, q^{-1}] &\xrightarrow{Q_G} \mathcal{O}_q(P_l) \\ YXZ &\longrightarrow \hat{Y}\hat{X}\hat{Z}. \end{aligned}$$

where

$$\begin{aligned} Y &= y_{44}^a y_{43}^b y_{34}^c y_{33}^d, & \hat{Y} &= \hat{y}_{44}^a \hat{y}_{43}^b \hat{y}_{34}^c \hat{y}_{33}^d, \\ X &= x_{22}^e x_{21}^f x_{12}^g x_{11}^l, & \hat{X} &= \hat{x}_{22}^e \hat{x}_{21}^f \hat{x}_{12}^g \hat{x}_{11}^l, \\ Z &= T_{41}^m T_{42}^n T_{31}^p T_{32}^r, & \hat{Z} &= \hat{T}_{41}^m \hat{T}_{42}^n \hat{T}_{31}^p \hat{T}_{32}^r. \end{aligned}$$

If $f, g \in \mathcal{O}(P_l)[q, q^{-1}]$, then the star product is defined as for the Minkowski space,

$$f \star_G g = Q_G^{-1}(Q_G(f) \cdot Q_G(g)).$$

Let us now consider the coaction, formally as in (2.23). Using both quantization maps (Q_{M} and Q_G) we can define a star coaction,

$$\begin{aligned} \mathcal{O}(\mathrm{M})[q, q^{-1}] &\xrightarrow{\tilde{\Delta}_\star} \to \mathcal{O}(G)[q, q^{-1}] \otimes \mathcal{O}(\mathrm{M})[q, q^{-1}] \\ f &\longrightarrow \quad Q_G^{-1} \otimes Q_M^{-1}(\Delta(Q_M(f)), \end{aligned}$$

having the compatibility property (see (2.5))

$$\tilde{\Delta}_\star(f \star_M g) = \tilde{\Delta}_\star(f)(\star_G \otimes \star_M)\tilde{\Delta}_\star(g), \qquad f, g \in \mathcal{O}(\mathrm{M})[q, q^{-1}]. \tag{4.1}$$

4.1 The coaction as a differential operator

We will restrict to the Lorentz group times dilations, that is, we will consider only the generators x and y.

On the generators of Minkowski space the star coaction is simply

$$\tilde{\Delta}_\star(t_{ai}) = y_{ab}S(x_{ji}) \otimes t_{bj},$$

and using the notation

$$t_{mi}^{\star a} = \underbrace{t_{mi} \star_{\mathrm{M}} t_{mi} \star_{\mathrm{M}} \cdots \star_{\mathrm{M}} t_{mi}}_{a\,\text{times}}$$

for an arbitrary standard monomial the coaction is expressed as

$$\tilde{\Delta}_\star\left(t_{41}^a t_{42}^b t_{31}^c t_{32}^d\right) = \tilde{\Delta}_\star(t_{41}^{\star a} \star_{\mathrm{M}} t_{42}^{\star b} \star_{\mathrm{M}} t_{31}^{\star c} \star_{\mathrm{M}} t_{32}^{\star d}) =$$
$$(\tilde{\Delta}_\star t_{41})^{\star a}(\star_G \otimes \star_{\mathrm{M}})(\tilde{\Delta}_\star t_{42})^{\star b}(\star_G \otimes \star_{\mathrm{M}})(\tilde{\Delta}_\star t_{31})^{\star c}(\star_G \otimes \star_{\mathrm{M}})(\tilde{\Delta}_\star t_{32})^{\star d}.$$

We have used the symbol '$\star$' to indicate'$\star_{\mathrm{M}}$', '$\star_G$' or '$\star_{G\times \mathrm{M}}$' to simplify the notation. The meaning should be clear from the context. Contracting with $\mu_{G\times\mathrm{M}}$ (see (2.24)) we define

$$\tau_{ij} \equiv \mu_{G\times\mathrm{M}} \circ \tilde{\Delta}_\star(t_{ij}) = y_{ab}t_{bj}S(x_{ji}).$$

Applying $\mu_{G\times\mathrm{M}}$ to the coaction, we get

$$\mu_{G\times\mathrm{M}} \circ \Delta_\star(t_{41}^a t_{42}^b t_{31}^c t_{32}^d) = \tau_{41}^{\star a} \star_{G\times\mathrm{M}} \tau_{42}^{\star b} \star_{G\times\mathrm{M}} \tau_{31}^{\star c} \star_{G\times\mathrm{M}} \tau_{32}^{\star d}. \tag{4.2}$$

Notice that in each τ there is a sum of terms with factors $ytS(x)$ that generically do not commute. So we need to work out the star products in the right hand side of (4.2).

As we are going to see, the calculation is quite involved. We are going to make a change in the parameter $q = \exp h$ and expand the star product in power series of h. At the end, we will compute only the first order term in h of the star coaction.

The star product $\star_{G\times\mathrm{M}}$ is written, as usual,

$$f_1 \star_{G\times\mathrm{M}} f_2 = \sum_{m=0}^{\infty} h^m D_m(f_1, f_2), \qquad f_1, f_2 \in \mathcal{O}(g \times \mathrm{M})[[h]].$$

For our purposes it will be enough to consider functions f_1 and f_2 that are polynomials in τ. The generators x, y and t commute among themselves, so the star product in $G \times \mathrm{M}$ can be computed by reordering the generators in each set x, y, and t in the Manin ordering. The result will contain terms similar to the star product (see Section 3.4), and in particular, D_1 will contain three terms of the type C_1 (3.5), one for the variables x, another for the variables y and another for the variables t. But C_1 is a bidifferential operator of order 1 in each of the arguments, so it satisfies the Leibnitz rule

$$D_1(f_1, f_2u) = D_1(f_1, f_2)u + D_1(f_1, u)f_2, \qquad u \in \mathcal{O}(P_l)[q, q^{-1}],$$

and we have, for example,

$$D_1(\tau_{ij}, \tau_{kl}^a) = aD_1(\tau_{ij}, \tau_{kl})\tau_{kl}^{a-1}. \tag{4.3}$$

In general, we have

$$\tau_{41}^{\star a} \star \tau_{42}^{\star b} \star \tau_{31}^{\star c} \star \tau_{32}^{\star d} = \sum_{I \in \mathcal{I}} h^M D_{i_1}(\tau_{41}, D_{i_2}(\tau_{41}, \dots D_{i_{a-1}}(\tau_{41}, D_{j_1}(\tau_{42}, \\ D_{j_2}(\tau_{42}, \dots D_{j_{b-1}}(\tau_{42}, D_{l_1}(\tau_{31}, D_{l_2}(\tau_{31}, \dots D_{l_{c-1}}(\tau_{31}, D_{m_1}(\tau_{32}, \\ D_{m_2}(\tau_{32}, \dots D_{m_{d-1}}(\tau_{32}, \tau_{32}) \dots).$$

Here $M = i_1 + \ldots + i_a + j_1 + \ldots + j_b + l_1 + \ldots + l_c + m_1 + \ldots m_d$ and we sum over all the multiindices

$$I = (i_1, \dots, i_{a-1}, j_1, \dots, j_{b-1}, l_1, \dots, l_{c-1}, m_1, \dots, m_{d-1}).$$

We are interested in the first order in h, so $M = 1$. This means that for any term in the sum we have only one D_1 operator (the others are D_0, which is just the standard product of both arguments). So we have the sum

$$\sum_k \Big(\tau_{41}^k D_1(\tau_{41}, \tau_{41}^{a-k-1} \tau_{42}^b \tau_{31}^c \tau_{32}^d) + \tau_{41}^a \tau_{42}^k D_1(\tau_{42}, \tau_{42}^{b-k-1} \tau_{31}^c \tau_{32}^d) + \\ \tau_{41}^a \tau_{42}^b \tau_{31}^k D_1(\tau_{31}, \tau_{31}^{c-k-1} \tau_{32}^d) + \tau_{41}^a \tau_{42}^b \tau_{31}^c \tau_{32}^k D_1(\tau_{32}, \tau_{32}^{d-k-1}) \Big).$$

using (4.3) we get

$$\sum_{k=1}^{a-1} k\tau_{41}^{a-2}\tau_{42}^{b}\tau_{31}^{c}\tau_{32}^{d}D_1(\tau_{41},\tau_{41}) + \sum_{k=1}^{a} b\tau_{41}^{a-1}\tau_{42}^{b-1}\tau_{31}^{c}\tau_{32}^{d}D_1(\tau_{41},\tau_{42}) +$$
$$\sum_{k=1}^{a} c\tau_{41}^{a-1}\tau_{42}^{b}\tau_{31}^{c-1}\tau_{32}^{d}D_1(\tau_{41},\tau_{31}) + \sum_{k=1}^{a} d\tau_{41}^{a-1}\tau_{42}^{b}\tau_{31}^{c}\tau_{32}^{d-1}D_1(\tau_{41},\tau_{32}) +$$
$$\sum_{k=1}^{b-1} k\tau_{41}^{a}\tau_{42}^{b-2}\tau_{31}^{c}\tau_{32}^{d}D_1(\tau_{42},\tau_{42}) + \sum_{k=1}^{b} c\tau_{41}^{a}\tau_{42}^{b-1}\tau_{31}^{c-1}\tau_{32}^{d}D_1(\tau_{42},\tau_{31}) +$$
$$\sum_{k=1}^{b} d\tau_{41}^{a}\tau_{42}^{b-1}\tau_{31}^{c}\tau_{32}^{d-1}D_1(\tau_{42},\tau_{32}) + \sum_{k=1}^{c-1} k\tau_{41}^{a}\tau_{42}^{b}\tau_{31}^{c-2}\tau_{42}^{d}D_1(\tau_{31},\tau_{31}) +$$
$$\sum_{k=1}^{c} d\tau_{41}^{a}\tau_{42}^{b}\tau_{31}^{c-1}\tau_{32}^{d-1}D_1(\tau_{31},\tau_{32}) + \sum_{k=1}^{d-1} k\tau_{41}^{a}\tau_{42}^{b}\tau_{31}^{c}\tau_{32}^{d-2}D_1(\tau_{32},\tau_{32}).$$

These sums can be easily done. We then get the order h contribution to the action of the deformed Lorentz plus dilations group:

$$\frac{a(a-1)}{2}D_1(\tau_{41},\tau_{41})\tau_{41}^{a-2}\tau_{42}^{b}\tau_{31}^{c}\tau_{32}^{d} + abD_1(\tau_{41},\tau_{42})\tau_{41}^{a-1}\tau_{42}^{b-1}\tau_{31}^{c}\tau_{32}^{d} +$$
$$\frac{b(b-1)}{2}D_1(\tau_{42},\tau_{42})\tau_{41}^{a}\tau_{42}^{b-2}\tau_{31}^{c}\tau_{32}^{d} + bcD_1(\tau_{42},\tau_{31})\tau_{41}^{a}\tau_{42}^{b-1}\tau_{31}^{c-1}\tau_{32}^{d} +$$
$$\frac{c(c-1)}{2}D_1(\tau_{31},\tau_{31})\tau_{41}^{a}\tau_{42}^{b}\tau_{31}^{c-2}\tau_{32}^{d} + cdD_1(\tau_{31},\tau_{32})\tau_{41}^{a}\tau_{42}^{b}\tau_{31}^{c-1}\tau_{32}^{d-1} +$$
$$\frac{d(d-1)}{2}D_1(\tau_{32},\tau_{32})\tau_{41}^{a}\tau_{42}^{b}\tau_{31}^{c}\tau_{32}^{d-2} + acD_1(\tau_{41},\tau_{31})\tau_{41}^{a-1}\tau_{42}^{b}\tau_{31}^{c-1}\tau_{32}^{d} +$$
$$adD_1(\tau_{41},\tau_{32})\tau_{41}^{a-1}\tau_{42}^{b}\tau_{31}^{c}\tau_{32}^{d-1} + bdD_1(\tau_{42},\tau_{32})\tau_{41}^{a}\tau_{42}^{b-1}\tau_{31}^{c}\tau_{32}^{d-1}.$$

This is reproduced by the differential operator

$$\frac{1}{2}D_1(\tau_{41},\tau_{41})\partial_{\tau_{41}}^2 + D_1(\tau_{41},\tau_{42})\partial_{\tau_{41}}\partial_{\tau_{42}} + \frac{1}{2}D_1(\tau_{42},\tau_{42})\partial_{\tau_{42}}^2 +$$
$$D_1(\tau_{42},\tau_{31})\partial_{\tau_{42}}\partial_{\tau_{31}} + \frac{1}{2}D_1(\tau_{31},\tau_{31})\partial_{\tau_{31}}^2 + D_1(\tau_{31},\tau_{32})\partial_{\tau_{31}}\partial_{\tau_{32}} +$$
$$\frac{1}{2}D_1(\tau_{32},\tau_{32})\partial_{\tau_{32}}^2 + D_1(\tau_{41},\tau_{31})\partial_{\tau_{41}}\partial_{\tau_{31}} + D_1(\tau_{41},\tau_{32})\partial_{\tau_{41}}\partial_{\tau_{32}} +$$
$$D_1(\tau_{42},\tau_{32})\partial_{\tau_{42}}\partial_{\tau_{32}}.$$

Notice that the coefficients have to match in order to get a differential operator, so the result is again non trivial. For completeness, we write the values of $D_1(\tau_{ij},\tau_{kl})$ in terms of the original variables x, y, t:

$$
\begin{aligned}
D_1(\tau_{41},\tau_{41}) =& -2(y_{44}y_{43}s_{11}^2t_{41}t_{31}+y_{43}^2s_{11}s_{21}t_{31}t_{32}+y_{44}^2s_{11}s_{21}t_{41}t_{42}+\\
&y_{44}y_{43}s_{21}^2t_{42}t_{32}+2y_{44}y_{43}s_{11}s_{21}t_{42}t_{31}+y_{44}y_{43}s_{11}s_{21}t_{41}t_{32}),\\
D_1(\tau_{41},\tau_{42}) =& -(y_{43}^2s_{21}s_{12}t_{31}t_{32}+y_{44}^2s_{21}s_{12}t_{41}t_{42}+2y_{44}y_{43}s_{21}^2t_{42}t_{32}+\\
&2y_{44}y_{43}s_{11}s_{12}t_{41}t_{31}+2y_{44}y_{43}s_{21}s_{12}t_{42}t_{31}+y_{44}y_{43}s_{21}s_{12}t_{41}t_{32})+\\
&y_{44}y_{43}s_{11}s_{21}t_{42}t_{31}),\\
D_1(\tau_{42},\tau_{42}) =& -(y_{44}^2s_{21}s_{12}t_{41}t_{42}+2y_{44}y_{43}s_{12}^2t_{41}t_{31}+2y_{43}^2s_{12}s_{22}t_{31}t_{32}+\\
&y_{44}y_{43}s_{21}s_{12}t_{42}t_{31}+2y_{44}y_{43}s_{12}s_{22}t_{42}t_{31}+2y_{44}y_{43}s_{12}s_{22}t_{41}t_{32})+\\
&3y_{44}y_{43}s_{21}s_{22}t_{42}t_{32}),\\
D_1(\tau_{42},\tau_{31}) =& -(y_{43}y_{33}s_{11}s_{12}t_{31}^2+y_{44}y_{33}s_{11}s_{12}t_{41}t_{31}+2y_{43}y_{34}s_{11}s_{12}t_{41}t_{31}+\\
&y_{44}y_{34}s_{11}s_{12}t_{41}^2+y_{43}y_{34}s_{21}s_{12}t_{41}t_{32}+y_{43}y_{34}s_{21}s_{12}t_{41}t_{32}+\\
&y_{44}y_{33}s_{11}s_{21}t_{42}t_{31}+2y_{44}y_{34}s_{11}s_{21}t_{41}t_{42}+y_{43}y_{33}s_{21}s_{12}t_{31}t_{32}+\\
&y_{43}y_{33}s_{11}s_{22}t_{31}t_{32}-2y_{43}y_{34}s_{21}s_{12}t_{42}t_{31}+y_{43}y_{34}s_{11}s_{22}t_{42}t_{31}+\\
&2y_{43}y_{34}s_{21}s_{12}t_{42}t_{31}+y_{43}y_{34}s_{11}s_{22}t_{42}t_{31}+y_{43}y_{33}s_{21}s_{22}t_{32}^2+\\
&2y_{43}y_{34}s_{21}s_{22}t_{42}t_{32}),\\
D_1(\tau_{31},\tau_{31}) =& -2(y_{34}y_{33}s_{11}^2t_{41}t_{31}+y_{33}^2s_{11}s_{21}t_{31}t_{32}+y_{34}^2s_{11}s_{21}t_{41}t_{42}+\\
&y_{34}y_{33}s_{21}^2t_{42}t_{32}+2y_{34}y_{33}s_{11}s_{21}t_{42}t_{31}+y_{34}y_{33}s_{11}s_{21}t_{41}t_{32}),\\
D_1(\tau_{32},\tau_{32}) =& -2(y_{34}y_{33}s_{12}^2t_{41}t_{31}+y_{33}^2s_{12}s_{22}t_{31}t_{32}+y_{34}^2s_{12}s_{22}t_{41}t_{42}+\\
&+y_{34}y_{33}s_{22}^2t_{42}t_{32}+2y_{34}y_{33}s_{12}s_{22}t_{42}t_{31}+y_{34}y_{33}s_{12}s_{22}t_{41}t_{32}),\\
D_1(\tau_{41},\tau_{31}) =& -(y_{43}y_{34}s_{11}^2t_{41}t_{31}+y_{43}y_{34}s_{21}^2t_{42}t_{32}+2y_{43}y_{33}s_{11}s_{21}t_{31}t_{32}+\\
&y_{44}y_{33}s_{11}s_{21}t_{42}t_{31}+2y_{43}y_{34}s_{11}s_{21}t_{42}t_{31}+y_{43}y_{34}s_{11}s_{21}t_{41}t_{32}+\\
&2y_{44}y_{34}s_{11}s_{21}t_{41}t_{42}),\\
D_1(\tau_{41},\tau_{32}) =& -(y_{43}y_{34}s_{11}s_{12}t_{41}t_{31}+y_{43}y_{33}s_{21}s_{12}t_{31}t_{32}+y_{43}y_{34}s_{21}s_{12}t_{42}t_{31}+\\
&y_{43}y_{34}s_{21}s_{12}t_{42}t_{31}+y_{44}y_{34}s_{21}s_{12}t_{41}t_{42}+y_{43}y_{34}s_{21}s_{22}t_{42}t_{32}),\\
D_1(\tau_{42},\tau_{32}) =& -(y_{43}y_{34}s_{12}^2t_{41}t_{31}+y_{43}y_{34}s_{22}^2t_{42}t_{32}+y_{44}y_{34}s_{21}s_{12}t_{41}t_{42}+\\
&2y_{43}y_{33}s_{12}s_{22}t_{31}t_{32}+2y_{43}y_{34}s_{12}s_{22}t_{42}t_{31}+y_{43}y_{34}s_{12}s_{22}t_{41}t_{32}),\\
D_1(\tau_{31},\tau_{32}) =& -(y_{33}^2s_{21}s_{12}t_{31}t_{32}+y_{34}^2s_{21}s_{12}t_{41}t_{42}+2y_{34}y_{33}s_{11}s_{12}t_{41}t_{31}+\\
&2y_{34}y_{33}s_{21}s_{12}t_{42}t_{31}+y_{34}y_{33}s_{21}s_{12}t_{41}t_{32}+\\
&y_{34}y_{33}s_{11}s_{22}t_{42}t_{31}+2y_{34}y_{33}s_{21}s_{22}t_{42}t_{32}).
\end{aligned}
$$

5. THE REAL FORMS

All the work done up to this point has been realized in a complexification of the conformal Minkowski space, with complex groups acting on it. In this chapter we deal with the appropiate real forms. In Section 5.1 we give the real forms for the classical case and in 5.2 for the quantum one. Finally in Section 5.3 we say a few words about a quadratic invariant which generalizes the metric in Minkowski space. These results have been published in ref.[2].

5.1 The real forms in the classical case

Let $\mathcal{A}$ be a commutative algebra over $\mathbb{C}$. An involution ι of $\mathcal{A}$ is an antilinear map satisfying, for $f, g \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$

$$\iota(\alpha f + \beta g) = \alpha^* \iota f + \beta^* \iota g, \qquad \text{(antilinearity)} \tag{5.1}$$

$$\iota(fg) = \iota(f)\iota(g), \qquad \text{(automorphism)} \tag{5.2}$$

$$\iota \circ \iota = 1\!\!1. \tag{5.3}$$

Let us consider the set of fixed points of ι,

$$\mathcal{A}^\iota = \{f \in \mathcal{A} \;/\; \iota(f) = f\}.$$

It is easy to see that this is a real algebra whose complexification is $\mathcal{A}$. $\mathcal{A}^\iota$ is a *real form* of $\mathcal{A}$.

The real Minkowski space.

We consider the algebra of the complex Minkowski space $\mathcal{O}(\mathrm{M}) \approx [t_{31}, t_{32}, t_{41}, t_{42}]$ and the following involution,

$$\begin{pmatrix} \iota_{\mathrm{M}}(t_{31}) & \iota_{\mathrm{M}}(t_{32}) \\ \iota_{\mathrm{M}}(t_{41}) & \iota_{\mathrm{M}}(t_{42}) \end{pmatrix} = \begin{pmatrix} t_{31} & t_{41} \\ t_{32} & t_{42} \end{pmatrix},$$

which can be also written simply as

$$\iota_{\mathrm{M}}(t) = t^T.$$

Using the Pauli matrices (2.22)

$$t = \begin{pmatrix} t_{31} & t_{32} \\ t_{41} & t_{42} \end{pmatrix} = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix},$$

so

$$\begin{aligned} x^0 &= \frac{1}{2}(t_{31} + t_{42}), & x^1 &= \frac{1}{2}(t_{32} + t_{41}), \\ x^2 &= \frac{1}{2\mathrm{i}}(t_{41} - t_{32}), & x^3 &= \frac{1}{2}(t_{31} - t_{42}), \end{aligned}$$

are fixed points of the involution. In fact, it is easy to see that

$$\mathcal{O}(\mathrm{M})^{\iota_{\mathrm{M}}} = \mathbb{R}[x^0, x^1, x^2, x^3].$$

■

The Euclidean space. We consider now the following involution on $O(\mathrm{M})$

$$\begin{pmatrix} \iota_{\mathrm{E}}(t_{31}) & \iota_{\mathrm{E}}(t_{32}) \\ \iota_{\mathrm{E}}(t_{41}) & \iota_{\mathrm{E}}(t_{42}) \end{pmatrix} = \begin{pmatrix} t_{42} & -t_{41} \\ -t_{32} & t_{31} \end{pmatrix}.$$

Another way of expressing it is in terms of the matrix of cofactors,

$$\iota_{\mathrm{E}}(t) = \mathrm{cof}(t).$$

The combinations

$$\begin{aligned} z^0 &= \frac{1}{2}(t_{31} + t_{42}), & z^1 &= \frac{\mathrm{i}}{2}(t_{32} + t_{41}), \\ z^2 &= \frac{1}{2}(t_{41} - t_{32}), & z^3 &= \frac{\mathrm{i}}{2}(t_{31} - t_{42}), \end{aligned}$$

are fixed points of ι_{E}, and as before,

$$\mathcal{O}(\mathrm{M})^{\iota_E} = \mathbb{R}[z^0, z^1, z^2, z^3].$$

■

We are interested now in the real forms of the complex Poincaré plus dilations that have a coaction on the real algebras. So we start with (2.6)

$$\mathcal{O}(P_l) = \mathbb{C}[x_{ij}, y_{ab}, T_{ai}]/(\det x \cdot \det y - 1).$$

We then look for the appropriate involution in $\mathcal{O}(P_l)$, denoted as $\iota_{P_l,\mathrm{M}}$ or $\iota_{P_l,\mathrm{E}}$ 'preserving' the corresponding real form (Minkowskian, Euclidean) of the complex Minkowski space. This means that the involution has to satisfy

$$\begin{aligned}\tilde{\Delta} \circ \iota_{\mathrm{M}} &= \iota_{P_l,\mathrm{M}} \otimes \iota_{\mathrm{M}} \circ \tilde{\Delta},\\ \tilde{\Delta} \circ \iota_{\mathrm{E}} &= \iota_{P_l,\mathrm{E}} \otimes \iota_{\mathrm{E}} \circ \tilde{\Delta}.\end{aligned}$$

It is a matter of calculation to check that

$$\iota_{P_l,\mathrm{M}}(x) = S(y)^T, \qquad \iota_{P_l,\mathrm{M}}(y) = S(x)^T, \qquad \iota_{P_l,\mathrm{M}}(T) = T^T, \tag{5.4}$$

$$\iota_{P_l,\mathrm{E}}(x) = S(x)^T, \qquad \iota_{P_l,\mathrm{E}}(y) = S(y)^T, \qquad \iota_{P_l,\mathrm{E}}(T) = \mathrm{cof}(T), \tag{5.5}$$

are the correct expressions. It is not difficult to realize that in the Minkowskian case the real form of the Lorentz group (corresponding to the generators x and y) is $\mathrm{SL}(2,\mathbb{C})_{\mathbb{R}}$ and in the Euclidean case is $\mathrm{SU}(2) \times \mathrm{SU}(2)$. One can further check the compatibility of these involutions with the coproduct and the antipode

$$\begin{aligned}\Delta \circ \iota_{P_l,\mathrm{M}} &= \iota_{P_l,\mathrm{M}} \otimes \iota_{P_l,\mathrm{M}} \circ \Delta,\\ \Delta \circ \iota_{P_l,\mathrm{E}} &= \iota_{P_l,\mathrm{E}} \otimes \iota_{P_l,\mathrm{E}} \circ \Delta,\end{aligned} \tag{5.6}$$

$$\begin{aligned}S \circ \iota_{P_l,\mathrm{M}} &= \iota_{P_l,\mathrm{M}} \circ S,\\ S \circ \iota_{P_l,\mathrm{E}} &= \iota_{P_l,\mathrm{E}} \circ S.\end{aligned} \tag{5.7}$$

5.2 *The real forms in the quantum case*

We have to reconsider the meaning of 'real form' in the case of quantum algebras. We can try to extend the involutions (5.4, 5.5) to the quantum algebras. We will denote this extension with the same name since they cannot be confused in the present context.

The first thing that we notice is that property (5.2) has to be modified. In fact, the property that the involutions ι_{M}, ι_{E} satisfy with respect to the commutation relations (2.25) of the complex algebra $\mathcal{O}(\mathrm{M})$ is that they are antiautomorphisms, that is

$$\iota_{\mathrm{M}}(fg) = \iota_{\mathrm{M}}(g)\iota_{\mathrm{M}}(f),$$
$$\iota_{\mathrm{E}}(fg) = \iota_{\mathrm{E}}(g)\iota_{\mathrm{E}}(f).$$

This discards the interpretation of the real form of the non commutative algebra as the set of fixed points of the involution. The other two properties are still satisfied.

When considering the involutions $\iota_{P_l,\mathrm{M}}$ and $\iota_{P_l,\mathrm{E}}$ in the quantum group $\mathcal{O}_q(P_l)$, we also obtain an antiautomorphism of algebras, but now the involution has to be compatible with the Hopf algebra structure. The coproduct is formally the same and properties (5.6) and (5.7) are still satisfied (so the involutions are automorphisms of coalgebras). On the other hand, differently from the classical case, the involutions do not commute with the antipode. This is essentially due to the fact that $S^2 \neq 1$. One can explicitly check that

$$S^2 \circ \iota_{P_l\mathrm{M}} \circ S = S \circ \iota_{P_l\mathrm{M}},$$
$$S^2 \circ \iota_{P_l\mathrm{E}} \circ S = S \circ \iota_{P_l\mathrm{E}}. \tag{5.8}$$

Property (5.3) is still satisfied, ${\iota_{P_l,\mathrm{M}}}^2 = 1$ and ${\iota_{P_l,\mathrm{E}}}^2 = 1$. Using this fact, (5.8) can be written as

$$(\iota_{P_l\mathrm{M}} \circ S)^2 = 1\!\!1,$$
$$(\iota_{P_l,\mathrm{E}} \circ S)^2 = 1\!\!1.$$

All these properties define what is known as a Hopf $*$-algebra structure (see for example [14]).

Definition 5.2.1. *Hopf $*$-algebra structure.* Let $\mathcal{A}$ be a Hopf algebra. We say that it is a Hopf $*$-algebra if there exists an antilinear involution ι on $\mathcal{A}$ which is an antiautomorphism of algebras and an automorphism of coalgebras and such that

$$(\iota \circ S)^2 = 1\!\!1,$$

being S the antipode. ■

For example, each real form of a complex Lie algebra corresponds to a $*$-algebra structure in the enveloping algebra, seen as a Hopf algebra.

Remark 5.2.2. Real forms on the star product algebra. The involutions can be pulled back to the star product algebra using the quantization maps Q_M (see (3.1)), and Q_G (see (4.1)) and then extended to the algebra of smooth functions. The Poisson bracket in terms of the Minkowski space variables (x^μ) or the Euclidean ones (z^μ) is purely imaginary (see (3.7)), as a consequence of the antiautomorphism property of the involutions.

In the case of the quantum groups, the whole Hopf $*$-algebra structure is pulled back to the polynomial algebra and then extended to the smooth functions. ■

5.3 The deformed quadratic invariant.

Let us consider the quantum determinant in $\mathcal{O}_q(\mathrm{M})$

$$\hat{C}_q = \det_q \begin{pmatrix} \hat{t}_{32} & \hat{t}_{31} \\ \hat{t}_{42} & \hat{t}_{41} \end{pmatrix} = \hat{t}_{32}\hat{t}_{41} - q^{-1}\hat{t}_{31}\hat{t}_{42}.$$

Under the coaction of $\mathcal{O}_q(P_l)$ with the translations put to zero (that is for the quantum Lorentz times dilation group), the quantum determinant satisfies

$$\tilde{\Delta}(\hat{C}_q) = \det_q\hat{y}\, S(\det_q\hat{x}) \otimes \hat{C}_q,$$

so if we suppress the dilations, then $\det_q\hat{y} = 1$, $\det_q\hat{x} = 1$ and the determinant is a *quantum invariant*,

$$\tilde{\Delta}(\hat{C}_q) = 1 \otimes \hat{C}_q.$$

The invariant $\hat{C}_q$ can be pulled back to the star product algebra with the quantization map Q_{M}:

$$C_q = Q_{\mathrm{M}}^{-1}(\hat{C}_q) = t_{41}t_{32} - qt_{42}t_{31}. \tag{5.9}$$

We can now change to the Minkowski space variables, and the quadratic invariant in the star product algebra is

$$C_q = -q(x^0)^2 + q(x^3)^2 + (x^1)^2 + (x^2)^2. \tag{5.10}$$

C_q is the quantum star invariant. Notice that the expressions (5.9) and (5.10) depend upon the quantization map or ordering rule chosen.

6. INVARIANT SIGMA MODEL IN $\mathrm{SO}(2,N)/\mathrm{SO}(2)\times\mathrm{SO}(N)$.

In this Chapter we define sigma models on coset spaces that are invariant under the action of a Lie group G. In particular, we consider the series of cosets G/H $\mathrm{SO}(2,n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$, with H the maximal compact subgroup of G. We start by choosing a suitable parametrization of these cosets, then performing the Iwasawa decomposition (see Section 2.7) to work with *solvable coordinates* in the coset, which are particularly easy to use. We define a left-invariant 2-form in the coset and use it to write a left-invariant sigma model which we call an Invariant Sigma Model (ISM).

In Section 6.1 we write down the $\mathrm{SO}(2,1)$ WZW model (see Section 2.13) and gauge a right $\mathrm{SO}(2)$ isometry to get the $\mathrm{SO}(2,1)/\mathrm{SO}(2)^R$ gauged WZW model. This model results to be a free boson, which clearly is conformally invariant. We compute the ISM in $\mathrm{SO}(2,1)/\mathrm{SO}(2)$ and show that it is a different model than the gauged one. We use the 1-loop beta equations to test it for conformal invariance and show that it is not conformal invariant.

The ISM model in $\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)$ is built in Section 6.2. Finally we construct the ISM model in $\mathrm{SO}(2,3)/\mathrm{SO}(2)\times\mathrm{SO}(3)$ in Section 6.3.

ISM on $\mathrm{SO}(2,n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$. We consider the orthogonal group $G = \mathrm{SO}(2,n)$ with maximal compact subgroup $H=\mathrm{SO}(2)\times\mathrm{SO}(n)$. We choose the metric in $\mathbb{R}^{2+n}$ in the standard form,

$$\mathrm{diag}(+1,+1,\underbrace{-1,\ldots,-1}_{n}).$$

Then, a matrix in the Lie algebra $\mathfrak{so}(2,n)$ is of the form

$$\begin{pmatrix} A_{2\times 2} & b_{2\times n} \\ b^T_{n\times 2} & C_{n\times n} \end{pmatrix}, \tag{6.1}$$

where A and C are antisymmetric matrices and b is arbitrary. The diagonal blocs A and C span the maximal compact subalgebra $\mathfrak{h}=\mathfrak{so}(2)\oplus\mathfrak{so}(n)$, and the matrices $b \in M_{2\mathrm{xn}}(\mathbb{R})$ span the subspace $\mathfrak{p}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$. In terms of representations, we have that

$$\mathfrak{so}(2,n) = (\mathfrak{so}(2) \oplus \mathfrak{so}(n)) \ltimes \mathfrak{p}, \quad \mathfrak{p} = (2)_{\mathrm{SO}(2)} \otimes (n)_{\mathrm{SO}(n)}. \tag{6.2}$$

Let L be a local section $L : G/H \to G$. In the physics literature the section L is called a *coset representative.* L is an arbitrary section, and it will be chosen later on in order to simplify the calculations or to show explicitly some symmetry.

The invariant metric at a point $x \in G/H$ represented by $L(x)$ is then given by (see Section 2.8.2)

$$\mathrm{d}s^2(X_x, Y_x) = \langle \Big(L^{-1}\mathrm{d}L(X_x)\Big)\,|_{\mathfrak{p}}, \Big(L^{-1}\mathrm{d}L(Y_x)\Big)\,|_{\mathfrak{p}} \rangle, \qquad X_x, Y_x \in T_x(G/H). \tag{6.3}$$

with $\langle\ ,\ \rangle$ the Cartan-Killing metric on $\mathfrak{g}$.

Next, we are going to construct an invariant 2-form on G/H. We only need a 2-form on $\mathfrak{p}$ (the tangent space at the identity coset), invariant under H. Then, in the same way that it is done for the metric, we can construct an invariant tensor field over G/H, by translating the 2-form over $\mathfrak{p}$, with the action of the group G. As it is shown in the Section 2.8.3, the invariance of the 2-form over $\mathfrak{p}$ under H is enough to ensure the invariance under G of the tensor defined over G/H .

Notice that $\mathrm{SO}(2) \subset \mathrm{Sp}(2,\mathbb{R}) \approx \mathrm{SL}(2,\mathbb{R})$, so there is an invariant, antisymmetric form on the fundamental representation (2) of SO(2), let us call it Ω. By definition, $\mathrm{SO}(n)$ leaves invariant a symmetric form δ on the fundamental representation space (n). Then, on the tensor product $\mathfrak{p} = (2) \otimes (n)$ (see (6.2)) there is an antisymmetric 2-form $B = \Omega \otimes \delta$ that is invariant under the action of H.

We will give this form explicitly for a basis of $\mathfrak{p}$. We consider the basis of $M_{2\times\mathrm{n}}(\mathbb{R})$

$$\{\hat{e}_{a\alpha}, \quad a = 1,2, \quad \alpha = 1, \cdots, n\}, \qquad \text{where } (\hat{e}_{a\alpha})_{b\beta} = \delta_{ab}\delta_{\alpha\beta},$$

which in the notation (6.1) gives

$$e_{a\alpha} = \begin{pmatrix} 0 & \hat{e}_{a\alpha} \\ \hat{e}^T_{a\alpha} & 0 \end{pmatrix}. \tag{6.4}$$

It is then enough to give the bilinear form on the elements of the basis,

$$B(e_{a\alpha}, e_{b\beta}) = \delta_{\alpha\beta}\epsilon_{ab}, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.5}$$

Then, as for the metric,

$$B_x(X_x, Y_x) = B\left(\left(L^{-1}\mathrm{d}L(X_x)\right)|_{\mathfrak{p}}, \left(L^{-1}\mathrm{d}L(Y_x)\right)|_{\mathfrak{p}}\right). \tag{6.6}$$

Let $\phi^i, i = 1, \dots n$ be coordinates on the target space manifold $\mathcal{M} = G/H$. The Lagrangian of the ISM is defined as[1]

$$\mathcal{L} = \int_{\mathcal{M}} \mathrm{d}^2x \left(\frac{1}{4a^2} g_{ij}\partial_\mu\phi^i\partial_\nu\phi^i + \frac{b}{2}\epsilon_{\mu\nu}B_{ij}\partial_\mu\phi^i\partial_\nu\phi^j + \Phi R^{(2)}\right), \tag{6.7}$$

where a^2 and b are coupling constants which may be not independent (for example, to obtain conformal invariance in some cases), Φ is a scalar field (the dilaton), and $R^{(2)}$ is the curvature of the two dimensional world sheet. The Lagrangian is invariant under G if Φ is constant, in which case this term becomes a total derivative.

We are going to consider the three cases $n = 1, 2, 3$.

6.1 The SO(2,1)/SO(2) example

According to (6.1), a matrix $M \in \mathfrak{so}(2,1)$ can be decomposed as

$$M = \begin{pmatrix} A & p \\ p^T & 0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \qquad p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

The Cartan decomposition is then

$$\mathfrak{so}(2,1) = \mathfrak{so}(2) + \mathfrak{p}, \qquad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & 0 & p_1 \\ 0 & 0 & p_2 \\ p_1 & p_2 & 0 \end{pmatrix} \right\}.$$

[1] We use the conventions

$$\mathrm{d}\phi^i\mathrm{d}\phi^j = \frac{1}{2}(\mathrm{d}\phi^i \otimes \mathrm{d}\phi^j + \mathrm{d}\phi^j \otimes \mathrm{d}\phi^i), \qquad \mathrm{d}\phi^i \wedge \mathrm{d}\phi^j = \frac{1}{2}(\mathrm{d}\phi^i \otimes \mathrm{d}\phi^j - \mathrm{d}\phi^j \otimes \mathrm{d}\phi^i),$$

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j, \qquad B = B_{ij}\mathrm{d}\phi^i \wedge \mathrm{d}\phi^j,$$

$$\epsilon^{\mu\nu}\mathrm{d}^2x = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.$$

In $\mathfrak{p}$ we consider the basis

$$\mathrm{e}_{11} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathrm{e}_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We choose

$$\mathrm{H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

as a maximal abelian subalgebra of $\mathfrak{p}$ in order to perform the Iwasawa decomposition. There are two roots, and we choose the positive system

$$\mathrm{X} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The commutation rule is

$$[\mathrm{H}, \mathrm{X}] = \mathrm{X}. \tag{6.8}$$

This is the solvable algebra $\mathfrak{solv}(\mathrm{SO}(2,1)/\mathrm{SO}(2)) = \mathrm{span}\{H, X\}$.

Using the solvable group structure of G/H we use exponential coordinates, so the coset representative is

$$L = e^{uX} e^{\varphi H}.$$

These are called *solvable coordinates* and they are global [15]. Then,

$$L^{-1}\mathrm{d}L = e^{-\varphi}\mathrm{d}u\, X + \mathrm{d}\varphi\, \mathrm{e}_{21}.$$

Projecting over $\mathfrak{p}$ we get

$$L^{-1}\mathrm{d}L|_{\mathfrak{p}} = -e^{-\varphi}\mathrm{d}u\, \mathrm{e}_{11} + \mathrm{d}\varphi\, \mathrm{e}_{21}.$$

From (6.3) the metric is

$$\mathrm{d}s^2 = \mathrm{Tr}(L^{-1}\mathrm{d}L|_{\mathfrak{p}}, L^{-1}\mathrm{d}L|_{\mathfrak{p}}) = 2e^{-2\varphi}\mathrm{d}u^2 + 2\mathrm{d}\varphi^2,$$

and from (6.6) the 2-form is

$$B = -2e^{-\varphi}\mathrm{d}u \wedge \mathrm{d}\varphi.$$

The lagrangian is, then,

$$\mathcal{L}_{\mathrm{SO}(2,1)/\mathrm{SO}(2)} = \frac{1}{2a^2}(e^{-2\varphi}\partial_\mu u\partial^\mu u + \partial_\mu\varphi\partial^\mu\varphi) - be^{-\varphi}\epsilon^{\mu\nu}\partial_\mu u\partial_\nu\varphi, \quad (6.9)$$

and it is invariant under the infinitesimal transformations generated by the vector fields of the action of the group

$$\begin{aligned} X_u^L &= \partial_u, \\ X_\varphi^L &= u\partial_u + \partial_\varphi, \\ X_\phi^L &= e^\varphi\partial_\phi + \frac{1}{2}(1 - e^{2\varphi} + u^2)\partial_u + u\partial_\varphi. \end{aligned} \quad (6.10)$$

The denomination of the generators corresponds to the exponential coordinates in the group $\mathrm{SO}(2,1)$, $g = e^{uX}e^{\varphi H}e^{\phi A}$. One can check that they close the algebra of $\mathfrak{so}(2,1)$.

One can ask if this model is quantum-mechanically conformal invariant. We then must check the 1-loop beta equations, which are (see for example ref.[43], with appropriate redefinitions in the coupling constants)

$$\begin{aligned} 0 &= 4a^2 R_{\mu\nu} + \left(\frac{4a^2 b}{3}\right)^2 H_\mu^{\lambda\rho} H_{\nu\lambda\rho} - 8\pi^2 D_\mu D_\nu \Phi, \\ 0 &= D_\lambda H_{\mu\nu}^\lambda - 8\pi(D_\lambda\Phi)H_{\mu\nu}^\lambda, \\ 0 &= 4(4\pi)^2(D_\mu\Phi)^2 - 16\pi D_\mu D^\mu\Phi + R + \frac{1}{12}\left(\frac{4a^2 b}{3}\right)^2 H_{\mu\nu\rho}H^{\mu\nu\rho} + \\ &\quad \frac{(D-26)\pi}{a^2}, \end{aligned} \quad (6.11)$$

with

$$H = \frac{3}{2}\mathrm{d}B.$$

But these equations do not have solution (even with the dilaton $\neq 0$), so the model is not conformally invariant.

Gauging of the WZW model $\mathrm{SO}(1,2)$. We want to compare our model with the result of gauging a group valued WZW model. We consider the basis of $\mathfrak{so}(2,1)$ $\{H, X, A\}$ where A is the basis for $\mathfrak{so}(2)$,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and H and X are given above.We take exponential coordinates in the group

$$g = e^{uX} e^{\varphi H} e^{\phi A}, \qquad L \in G.$$

One can prove [15] that (u, φ) are global coordinates while ϕ is a cyclic (angular) coordinate parametrizing $S^1 \approx \mathrm{SO}(2)$. The WZW model on the group manifold $\mathrm{SO}(2,1)$ is defined as

$$\mathcal{L}_{\mathrm{SO}(2,1)} = \frac{1}{4a^2} \int_{M=\partial B} \mathrm{Tr}(g^{-1}\mathrm{d}g g^{-1}\mathrm{d}g) + \frac{b}{3} \int_B \mathrm{Tr}(\tilde{g}^{-1}\mathrm{d}\tilde{g}\tilde{g}^{-1}\mathrm{d}\tilde{g}\tilde{g}^{-1}\mathrm{d}\tilde{g}),$$

where B is a three dimensional manifold whose boundary is M, the compactification of the original 2-dimensional space. We have

$$\begin{aligned} \mathrm{Tr}(g^{-1}\mathrm{d}g g^{-1}\mathrm{d}g) &= -4e^{-\varphi}\mathrm{d}u\mathrm{d}\phi - 2\mathrm{d}\phi^2 + 2\mathrm{d}\varphi^2, \\ \mathrm{Tr}(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g) &= -6e^{-\varphi}\mathrm{d}u \wedge \mathrm{d}\phi \wedge \mathrm{d}\varphi = \mathrm{d}(6e^{-\varphi}\mathrm{d}u \wedge \mathrm{d}\phi)). \end{aligned}$$

Then the lagrangian can be written, up to boundary terms, in solvable coordinates as

$$\mathcal{L}_{\mathrm{SO}(2,1)} = \frac{1}{2a^2}\big(-2\mathrm{e}^{-\varphi}\partial_\mu u \partial^\mu \phi - \partial_\mu \phi \partial^\mu \phi + \partial_\mu \varphi \partial^\mu \varphi\big) + 2b\mathrm{e}^{-\varphi}\epsilon^{\mu\nu}\partial_\mu u \partial_\nu \phi.$$

Notice that the coupling constant of the WZ term does not become quantized, since there are no topologically different ways of mapping S^3 into $\mathrm{SO}(2,1)$ (namely, $\pi_3(\mathrm{SO}(2,1)) = 0$).

This model has invariance under left and right $\mathrm{SO}(2,1)$. The infinitesimal generators of the left and right actions of $\mathrm{SO}(1,2)$ on itself are

- Left invariant vector fields:

$$\begin{aligned} X^L_\phi &= e^{\varphi}\partial_\phi + \frac{1}{2}(1 - e^{2\varphi} + u^2)\partial_u + u\partial_\varphi, \\ X^L_u &= \partial_u, \\ X^L_\varphi &= u\partial_u + \partial_\varphi. \end{aligned}$$

- Right invariant vector fields:

$$\begin{aligned} X^R_\phi &= \partial_\phi, \\ X^R_u &= (1 - \cos\phi)\partial_\phi + e^{\varphi}\cos\phi\partial_u + \sin\phi\partial_\varphi, \\ X^R_\varphi &= \sin\phi\partial_\phi - e^{\varphi}\sin\phi\partial_u + \cos\phi\partial_\varphi. \end{aligned}$$

We are going to gauge the generator X^R_ϕ corresponding to the algebra element A. In principle, we are not touching the left symmetries, but we will have to see if they survive. We set

$$D_\mu\phi = \partial_\mu\phi + A_\mu, \qquad D_\mu\varphi = \partial_\mu\varphi, \qquad D_\mu u = \partial_\mu u.$$

We substitute the standard derivatives by covariant derivatives in the lagrangian. Then, the equation of motion for A_μ becomes

$$A^\mu = -\partial^\mu\phi - e^{-\varphi}\partial^\mu u + 2ba^2\epsilon^{\nu\mu}e^{-\varphi}\partial_\nu u,$$

and substituting back in the covariant derivatives and in the Lagrangian one gets

$$\mathcal{L}_{\text{gauged}} = \frac{1}{2a^2}\Big((1+(2ba^2)^2)\mathrm{e}^{-2\varphi}\partial_\mu u\partial^\mu u + \partial_\mu\varphi\partial^\mu\varphi\Big).$$

So if $2ba^2 = \mathrm{i}$ the model is a free boson, which is conformally invariant. Instead, if $b = 0$, one recovers the symmetric lagrangian without WZ term (6.9). Notice that the effect of a WZ term in the Lagrangian $\mathcal{L}_{\mathrm{SO}(2,1)}$ is not a WZ term in the gauged Lagrangian, but a deformation of the metric on $\mathrm{SO}(2,1)/\mathrm{SO}(2)$.

Under this deformation, the infinitesimal transformations associated to the action of the group (6.10) do not all preserve the character of symmetry. While X^L_u and X^L_φ are still symmetries, X^L_ϕ is not anymore.

■

What this example shows is that the invariant models that we are constructing do not arise from a gauging of a model with target space a group manifold.

6.2 *The invariant WZW model* SO(2, 2)/SO(2) × SO(2) *in solvable coordinates.*

We remind the general form of the matrices of $\mathfrak{so}(2,2)$ in (6.1). Let us perform the Iwasawa decomposition. We consider the following basis of $\mathfrak{h} = \mathfrak{so}(2)\oplus\mathfrak{so}(2)$:

$$A_1 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \qquad \text{and} \qquad A_2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{array}\right).$$

As a maximal abelian subalgebra of $\mathfrak{p}$ we choose

$$H_1 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad H_2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

There are four roots with one dimensional root spaces. We denote them as $\Omega_{\lambda_1,\lambda_2}$ with $\lambda_i = 1, -1$. We have the commutation rules

$$[H_i, \Omega_{\lambda_1,\lambda_2}] = \lambda_i \Omega_{\lambda_1,\lambda_2},$$

and the rest zero. We have the positive system $\{\Omega_{1,1}, \Omega_{1,-1}\}$, which explicitly is

$$\Omega_{1,1} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{array}\right), \qquad \Omega_{1,-1} = \left(\begin{array}{cc|cc} 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ \hline 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{array}\right).$$

So

$$\mathfrak{solv}\,(\mathrm{SO}(2,2)/\mathrm{SO}(2) \times \mathrm{SO}(2)) = \{H_1, H_2, \Omega_{1,1}, \Omega_{1,-1}\}, \tag{6.12}$$

with the commutation rules given above.

The coset representative in solvable coordinates is

$$L = e^{\varphi_1 H_1 + \varphi_2 H_2} e^{\alpha \Omega_{1,1}} e^{\gamma \Omega_{1,-1}},$$

from which, after projection over $\mathfrak{p}$, we get[2]

$$\begin{aligned} L^{-1}\mathrm{d}L|_{\mathfrak{p}} =& e_{11}\mathrm{d}\varphi_1 - e_{12}\Big(\mathrm{d}\alpha + \mathrm{d}\gamma + (\alpha+\gamma)\mathrm{d}\varphi_1 + (\alpha-\gamma)\mathrm{d}\varphi_2\Big) + \\ & e_{21}\Big(\mathrm{d}\alpha - \mathrm{d}\gamma + (\alpha-\gamma)\mathrm{d}\varphi_1 + (\alpha+\gamma)\mathrm{d}\varphi_2\Big) + e_{22}\mathrm{d}\varphi_2. \end{aligned}$$

Then the invariant metric in the symmetric space (6.3) in terms of the solvable coordinates is

$$\begin{aligned} \mathrm{d}s^2 =& \Big(2 + 4\alpha^2 + 4\gamma^2\Big)\,\mathrm{d}\varphi_1^2 + \Big(8\alpha^2 - 8\gamma^2\Big)\,\mathrm{d}\varphi_1\mathrm{d}\varphi_2 + 8\alpha\mathrm{d}\varphi_1\mathrm{d}\alpha + \\ & 8\gamma\mathrm{d}\varphi_1\mathrm{d}\gamma + \Big(2 + 4\alpha^2 + 4\gamma^2\Big)\,\mathrm{d}\varphi_2^2 + 8\alpha\mathrm{d}\varphi_2\mathrm{d}\alpha - 8\gamma\mathrm{d}\varphi_2\mathrm{d}\gamma + \\ & 4\mathrm{d}\alpha^2 + 4\mathrm{d}\gamma^2. \end{aligned} \tag{6.13}$$

[2] For this and the rest of the calculations of different metrics, coset representatives and Killing vectors we have used the program Wolfram Research, Inc., Mathematica, Version 5.1, Champaign, IL (2004).

As for the invariant 2-form, by equation (6.5), we have

$$B(e_{11}, e_{21}) = -B(e_{21}, e_{11}) = 1, \qquad B(e_{12}, e_{22}) = -B(e_{22}, e_{12}) = 1$$

and the rest zero, so the invariant 2-form $B(L^{-1}\mathrm{d}L|_{\mathfrak{p}}, L^{-1}\mathrm{d}L|_{\mathfrak{p}})$ in solvable coordinates becomes

$$\frac{1}{2}B = \mathrm{d}\varphi_1 \wedge \mathrm{d}\alpha - \mathrm{d}\varphi_1 \wedge \mathrm{d}\gamma + \mathrm{d}\varphi_2 \wedge \mathrm{d}\alpha + \mathrm{d}\varphi_2 \wedge \mathrm{d}\gamma. \tag{6.14}$$

For constant dilaton, the Lagrangian is invariant under the rigid symmetries

$$X^L_{\varphi_1} = \partial_{\varphi_1}, \qquad X^L_{\varphi_2} = \partial_{\varphi_2}, \qquad X^L_{\alpha} = e^{-\varphi_1-\varphi_2}\partial_\alpha, \quad X^L_{\gamma} = e^{\varphi_2-\varphi_1}\partial_\gamma,$$

$$\begin{aligned}
X^L_{\phi_1} =& \left(e^{\varphi_1+\varphi_2}\alpha - e^{\varphi_1-\varphi_2}\gamma\right)\partial_{\varphi_1} + e^{\varphi_1-\varphi_2}\left(e^{2\varphi_2}\alpha+\gamma\right)\partial_{\varphi_2} - \\
&\left(e^{\varphi_1+\varphi_2}\alpha^2 + \frac{1}{2}\sinh\left(\varphi_1+\varphi_2\right)\right)\partial_\alpha + \\
&\frac{1}{4}\left(-e^{-\varphi_1+\varphi_2} + e^{\varphi_1-\varphi_2}(1+4\gamma^2)\right)\partial_\gamma, \\
X^L_{\phi_2} =& -e^{\varphi_1-\varphi_2}\left(e^{2\varphi_2}\alpha+\gamma\right)\partial_{\varphi_1} + \left(-e^{\varphi_1+\varphi_2}\alpha + e^{\varphi_1-\varphi_2}\gamma\right)\partial_{\varphi_2} + \\
&\left(e^{\varphi_1+\varphi_2}\alpha^2 + \frac{1}{2}\sinh\left(\varphi_1+\varphi_2\right)\right)\partial_\alpha + \\
&\frac{1}{4}\left(-e^{-\varphi_1+\varphi_2} + e^{\varphi_1-\varphi_2}(1+4\gamma^2)\right)\partial_\gamma,
\end{aligned}$$

which close the algebra $\mathfrak{so}(2,2)$. The denomination of the generators corresponds to the exponential coordinates in the group

$$L = e^{\varphi_1 H_1 + \varphi_2 H_2} e^{\alpha\Omega_{1,1}} e^{\gamma\Omega_{1,-1}} e^{\phi_1 A_1} e^{\phi_2 A_2}.$$

As for the $\mathrm{SO}(2,1)/\mathrm{SO}(2)$ model, this model is not conformally invariant at the quantum mechanical level.

6.3 The SO(2,3)/SO(2)×SO(3) example

A maximal abelian subalgebra of $\mathfrak{p}$ is

$$\mathrm{H}_1 = \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \qquad \mathrm{H}_2 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0 \end{pmatrix}.$$

There are eight restricted roots, $\Omega_{\lambda_1,\lambda_2}$ with $\lambda_1, \lambda_2 = 1, 0, -1$ but not both simultaneously 0. A positive system is

$$\Omega_{1,1} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Omega_{1,0} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix},$$
$$\Omega_{1,-1} = \begin{pmatrix} 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Omega_{0,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix},$$

with commutation rules

$$\begin{aligned} &[\mathrm{H}_i, \Omega_{\lambda_1,\lambda_2}] = \lambda_i \Omega_{\lambda_1,\lambda_2}, \quad && i = 1, 2, \\ &[\Omega_{1,0}, \Omega_{0,1}] = -\Omega_{1,1}, && [\Omega_{1,-1}, \Omega_{0,1}] = 2\Omega_{1,0}, \end{aligned}$$

and the rest zero. We have

$$\mathfrak{solv}(\mathrm{SO}(2,3)/\mathrm{SO}(2) \times \mathrm{SO}(3)) = \mathrm{span}\{H_1, H_2, \Omega_{1,1}, \Omega_{1,0}, \Omega_{1,-1}, \Omega_{0,1}\}, \tag{6.15}$$

with the commutation rules given above.

In solvable coordinates, the coset representative becomes

$$L = e^{\varphi_1 H_1} e^{\varphi_2 H_2} e^{\alpha \Omega_{1,1}} e^{\beta \Omega_{1,0}} e^{\gamma \Omega_{1,-1}} e^{\rho \Omega_{0,1}}.$$

It is more convenient for the computations to have the Cartan-Killing and the antisymmetric forms in terms of the solvable basis. We can check

$$\begin{aligned} &H_1 = e_{11}, && H_2 = e_{22}, \\ &\Omega_{1,1}|_{\mathfrak{p}} = -e_{12} + e_{21}, && \Omega_{1,-1}|_{\mathfrak{p}} = -e_{12} - e_{21}, \\ &\Omega_{1,0}|_{\mathfrak{p}} = -e_{13}, && \Omega_{01}|_{\mathfrak{p}} = e_{23}. \end{aligned}$$

So then

$$\begin{aligned} &\langle \Omega_{11}|_p, \Omega_{11}|_p \rangle = 4, && \langle \Omega_{1,-1}|_p, \Omega_{1,-1}|_p \rangle = 4, \\ &\langle \Omega_{10}|_p, \Omega_{10}|_p \rangle = 2, && \langle \Omega_{01}|_p, \Omega_{10}|_p \rangle = 2, \\ &\langle H_1|_p, H_1|_p \rangle = 2, && \langle H_2|_p, H_2|_p \rangle = 2, \end{aligned}$$

$$
\begin{aligned}
&B(H_1,\Omega_{11}) = 1, && B(H_1,\Omega_{1,-1}) = -1,\\
&B(H_2,\Omega_{11}) = 1, && B(H_2 2,\Omega_{1,-1}) = 1,\\
&B(\Omega_{10},\Omega_{01}) = -1,
\end{aligned}
$$

and the rest 0. Then we obtain the metric and the 2-form:

$$
\begin{aligned}
\mathrm{d}s^2 = &\Big(2 + 4\alpha^2 + 2\beta^2 + 4\gamma^2 - 8\alpha\beta\rho + 8\beta\gamma\rho + 4\beta^2\rho^2 - 8\alpha\gamma\rho^2 + 8\gamma^2\rho^2 +\\
&8\beta\gamma\rho^3 + 4\gamma^2\rho^4\Big)\,\mathrm{d}\varphi_1^2 + \Big(8\alpha^2 - 8\gamma^2 - 8\alpha\beta\rho - 8\beta\gamma\rho - 16\gamma^2\rho^2 - 8\beta\gamma\rho^3 -\\
&8\gamma^2\rho^4\Big)\,\mathrm{d}\varphi_1\mathrm{d}\varphi_2 + \Big(8\alpha - 8\beta\rho - 8\gamma\rho^2\Big)\,\mathrm{d}\varphi_1\mathrm{d}\alpha + (4\beta - 8\alpha\rho + 8\rho\gamma +\\
&8\beta\rho^2 + 8\gamma\rho^3\Big)\,\mathrm{d}\varphi_1\mathrm{d}\beta + \Big(2 + 4\alpha^2 + 4\gamma^2 + 2\rho^2 + 8\alpha\gamma\rho^2 + 8\gamma^2\rho^2 +\\
&4\gamma^2\rho^4\Big)\,\mathrm{d}\varphi_2^2 + \Big(8\alpha + 8\gamma\rho^2\Big)\,\mathrm{d}\varphi_2\mathrm{d}\alpha + \Big(-8\alpha\rho - 8\gamma\rho - 8\gamma\rho^3\Big)\,\mathrm{d}\varphi_2\mathrm{d}\beta\\
&+ \Big(-8\gamma - 8\alpha\rho^2 - 16\gamma\rho^2 - 8\gamma\rho^4\Big)\,\mathrm{d}\varphi_2\mathrm{d}\gamma + 4\rho\mathrm{d}\varphi_2\mathrm{d}\rho + 4\mathrm{d}\alpha^2 -\\
&8\rho\mathrm{d}\alpha\mathrm{d}\gamma + \Big(2 + 4\rho^2\Big)\,\mathrm{d}\beta^2 + \Big(8\rho + 8\rho^3\Big)\,\mathrm{d}\beta\mathrm{d}\gamma + \Big(4 + 8\gamma^2 + 4\rho^4\Big)\,\mathrm{d}\gamma^2 +\\
&2\mathrm{d}\rho^2, \qquad (6.16)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}B =&\mathrm{d}\varphi_1 \wedge \mathrm{d}\alpha - \rho\mathrm{d}\varphi_1 \wedge \mathrm{d}\beta - (1+\rho^2)\mathrm{d}\varphi_1 \wedge \mathrm{d}\gamma - (\beta + 2\gamma\rho)\mathrm{d}\varphi_1 \wedge \mathrm{d}\rho +\\
&\mathrm{d}\varphi_2 \wedge \mathrm{d}\alpha + (1+\rho^2)\mathrm{d}\varphi_2 \wedge \mathrm{d}\gamma + 2\gamma\rho\mathrm{d}\varphi_2 \wedge \mathrm{d}\rho - \mathrm{d}\beta \wedge \mathrm{d}\rho - 2\rho\mathrm{d}\gamma \wedge \mathrm{d}\rho. \qquad (6.17)
\end{aligned}
$$

One can compute the rigid symmetry generators associated to the solvable Lie group:

$$
\begin{aligned}
&\mathrm{X}^L_{\varphi_1} = \partial_{\varphi_1}, && \mathrm{X}^L_{\varphi_2} = \partial_{\varphi_2},\\
&\mathrm{X}^L_{\beta} = e^{-\varphi_1}\partial_\beta, && \mathrm{X}^L_{\gamma} = e^{\varphi_2-\varphi_1}\partial_\gamma,\\
&\mathrm{X}^L_{\alpha} = e^{-\varphi_1-\varphi_2}\partial_\alpha, && \mathrm{X}^L_{\rho} = \beta e^{-\varphi_2}\partial_\alpha - 2\gamma e^{-\varphi_2}\partial_\beta + e^{-\varphi_2}\partial_\rho.
\end{aligned}
$$

There are 4 other symmetry generators, X_{ϕ_1}, X_{ϕ_2}, X_{ϕ_3} and X_{ϕ_4} but its computation exceeds the capabilities of our computers. As in the previous cases, the model is not conformally invariant.

7. CONTRACTION OF SIGMA MODELS

In Chapter 6 we have defined what we call *Invariant Sigma Models*. We have explicitely written down the $\mathrm{SO}(2,1)/\mathrm{SO}(2)$, $\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)$ and the $\mathrm{SO}(2,3)/\mathrm{SO}(2)\times\mathrm{SO}(3)$ models. The Lie algebras of these models can be related by contraction (see Section 2.9), so we can relate these models by contraction too.

In Section 7.1 we perform the contraction of the $\mathrm{SO}(2,3)/\mathrm{SO}(2)\times\mathrm{SO}(3)$ with respect to $\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)$ and in Section 7.2 we perform the contraction of $\mathrm{SO}(2,3)/\mathrm{SO}(2)\times\mathrm{SO}(3)$ with respect to $\mathrm{SO}(1,3)/\mathrm{SO}(3)$.

We will denote $\mathfrak{s}(n) = \mathfrak{solv}(\mathrm{SO}(2,n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$. In the basis that we have used, the following embeddings of the Lie algebras (6.8), (6.12) and (6.15) are quite obvious,

$$\mathfrak{s}(1) \subset \mathfrak{s}(2) \subset \mathfrak{s}(3).$$

7.1 $\mathrm{SO}(2,3)/\mathrm{SO}(2)\times\mathrm{SO}(3)$ *contracted with respect to* $\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)$

We consider the embedding (see (6.12) and (6.15))

$$\mathfrak{s}(2) \subset \mathfrak{s}(3),$$

with

$$\begin{aligned}
\mathfrak{s}(3) &= \mathrm{span}\{H_1, H_2, \Omega_{1,1}, \Omega_{1,0}, \Omega_{1,-1}, \Omega_{0,1}\},\\
\mathfrak{s}(2) &= \mathrm{span}\{H_1, H_2, \Omega_{1,1}, \Omega_{1,-1}\}.
\end{aligned}$$

One possible contraction is given by the map

$$\begin{array}{ccc}
\mathfrak{s}(3) & \xrightarrow{\phi_\epsilon} & \mathfrak{s}(3)\\[1ex]
H_1, H_2, \Omega_{1,1}, \Omega_{1,-1} & \longrightarrow & H_1, H_2, \Omega_{1,1}, \Omega_{1,-1},\\[1ex]
\Omega_{1,0}, \Omega_{0,1} & \longrightarrow & \epsilon\Omega_{1,0}, \epsilon\Omega_{0,1}.
\end{array}$$

The deformed algebra has only one bracket that changes

$$[\Omega_{1,0}, \Omega_{0,1}]_\epsilon = -\epsilon^2 \Omega_{1,1},$$

and it goes to zero after the contraction $\epsilon \to 0$. The other bracket remains unchanged

$$[\Omega_{1,-1}, \Omega_{0,1}]_\epsilon = 2\Omega_{1,0}.$$

In the fundamental representation of $\mathfrak{so}(2,3)$ one has two invariant subspaces under the action of $\mathfrak{s}(2)$:

$$V_1 = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad W_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e \end{pmatrix} \right\}.$$

We will do the case V_1. It is not difficult to check that the case W_1 gives the same metric and the same 2-form.

We have

$$\psi_\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \epsilon \end{pmatrix}.$$

In the deformed representation (2.36) we have

$$H_{1\epsilon} = H_1, \qquad H_{2\epsilon} = H_2,$$

$$\Omega_{1,1\epsilon} = \Omega_{1,1}, \qquad \Omega_{1,-1\epsilon} = \Omega_{1,-1},$$

$$\Omega_{1,0\epsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\epsilon^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon^2 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \Omega_{0,1\epsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^2 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

The metric is (see Section 2.9.1)

$$\begin{aligned}
\mathrm{d}s^2_\epsilon = &\Big(2 + 4\alpha^2 + 2\beta^2 + 4\gamma^2 + 8\beta\gamma\rho - 8\alpha\beta\rho\epsilon^2 + 8\gamma^2\rho^2 - 8\alpha\gamma\rho^2\epsilon^2 + \\
&4\beta^2\rho^2\epsilon^4 + 8\beta\gamma\rho^3\epsilon^4 + 4\gamma^2\rho^4\epsilon^4\Big)\,\mathrm{d}\varphi_1^2 + \Big(8\alpha^2 - 8\gamma^2 - 8\alpha\beta\rho\epsilon^2 - 8\beta\gamma\rho - \\
&16\gamma^2\rho^2 - 8\beta\gamma\rho^3\epsilon^4 - 8\gamma^2\rho^4\epsilon^4\Big)\,\mathrm{d}\varphi_1\mathrm{d}\varphi_2 + \Big(8\alpha - 8\beta\rho\epsilon^2 - \\
&8\gamma\rho^2\epsilon^2\Big)\,\mathrm{d}\varphi_1\mathrm{d}\alpha + \Big(4\beta - 8\alpha\rho\epsilon^2 - 8\gamma\rho + 8\beta\rho^2\epsilon^4 + 8\gamma\rho^3\epsilon^4\Big)\,\mathrm{d}\varphi_1\mathrm{d}\beta \\
&+ \Big(8\gamma + 8\beta\rho + 16\gamma\rho^2 - 8\alpha\rho^2\epsilon^2 + 8\beta\rho^3\epsilon^4 + 8\gamma\rho^4\epsilon^4\Big)\,\mathrm{d}\varphi_1\mathrm{d}\gamma \\
&+ \Big(2 + 4\alpha^2 + 4\gamma^2 + 2\rho^2 + 8\alpha\gamma\rho^2 + 8\alpha\gamma\rho^2\epsilon^2 + 4\gamma^2\rho^4\epsilon^4\Big)\,\mathrm{d}\varphi_2^2 + \\
&\Big(8\alpha + 8\gamma\rho^2\epsilon^2\Big)\,\mathrm{d}\varphi_2\mathrm{d}\alpha + \Big(-8\gamma\rho - 8\alpha\rho\epsilon^2 - 8\gamma\rho^3\epsilon^4\Big)\,\mathrm{d}\varphi_2\mathrm{d}\beta + \\
&\Big(-8\gamma - 16\gamma\rho^2 - 8\alpha\rho^2\epsilon^2 - 8\gamma\rho^4\epsilon^4\Big)\,\mathrm{d}\varphi_2\mathrm{d}\gamma + 4\rho\mathrm{d}\varphi_2\mathrm{d}\rho + 4\mathrm{d}\alpha^2 - \\
&8\epsilon^2\rho\mathrm{d}\alpha\mathrm{d}\beta - 8\epsilon^2\rho^2\mathrm{d}\alpha\mathrm{d}\gamma + \Big(2 + 4\epsilon^4\rho^2\Big)\,\mathrm{d}\beta^2 + \Big(8\rho + 8\rho^3\epsilon^4\Big)\,\mathrm{d}\beta\mathrm{d}\gamma \\
&+ \Big(4 + 8\rho^2 + 4\epsilon^4\rho^4\Big)\,\mathrm{d}\gamma^2 + 2\mathrm{d}\rho^2.
\end{aligned}$$

If we take $\epsilon = 1$ we recover the metric (6.16). The limit $\epsilon \to 0$ is well defined,

$$\begin{aligned}
\mathrm{d}s^2_{\epsilon=0} = &\Big(2 + 4\alpha^2 + 2\beta^2 + 4\gamma^2 + 8\beta\gamma\rho + 8\gamma^2\rho^2\Big)\,\mathrm{d}\varphi_1^2 + \Big(8\alpha^2 - 8\gamma^2 - \\
&8\beta\gamma\rho - 16\rho^2\gamma^2\Big)\,\mathrm{d}\varphi_1\mathrm{d}\varphi_2 + 8\alpha\mathrm{d}\varphi_1\mathrm{d}\alpha + (4\beta + 8\gamma\rho)\,\mathrm{d}\varphi_1\mathrm{d}\beta + \\
&\Big(8\gamma + 8\beta\rho + 16\gamma\rho^2\Big)\,\mathrm{d}\varphi_1\mathrm{d}\gamma + \Big(2 + 4\alpha^2 + 4\gamma^2 + 2\rho^2 + 8\gamma^2\rho^2\Big)\,\mathrm{d}\varphi_2^2 + \\
&8\alpha\mathrm{d}\varphi_2\mathrm{d}\alpha - 8\gamma\rho\mathrm{d}\varphi_2\mathrm{d}\beta + \Big(-8\gamma - 16\gamma\rho^2\Big)\,\mathrm{d}\varphi_2\mathrm{d}\gamma + 4\rho\mathrm{d}\varphi_2\mathrm{d}\rho + 4\mathrm{d}\alpha^2 + \\
&2\mathrm{d}\beta^2 + 8\rho\mathrm{d}\beta\mathrm{d}\gamma + \Big(4 + 8\rho^2\Big)\,\mathrm{d}\gamma^2 + \mathrm{d}\rho^2.
\end{aligned}$$

We can rearrange the terms so we can factor the metric (6.13) of the Lie group SO(2,2)/SO(2)×SO(2), which depends on the coordinates $\varphi_1, \varphi_2, \alpha, \gamma$ plus a series of terms involving the new coordinates β, ρ which give non trivial interactions.

$$
\begin{aligned}
\mathrm{d}s^2_{\epsilon=0} = {} & \mathrm{d}s^2_{\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)} + \mathrm{d}\varphi_1^2\left(2\beta^2+8\beta\gamma\rho+8\gamma^2\rho^2\right) - \\
& \mathrm{d}\varphi_1\mathrm{d}\varphi_2\left(8\beta\gamma\rho+16\rho^2\gamma^2\right)+\mathrm{d}\varphi_1\mathrm{d}\beta\left(4\beta+8\gamma\rho\right)+ \\
& \mathrm{d}\varphi_1\mathrm{d}\gamma\left(8\beta\rho+16\gamma\rho^2\right)+\mathrm{d}\varphi_2^2\left(2\rho^2+8\gamma^2\rho^2\right)-8\gamma\rho\mathrm{d}\varphi_2\mathrm{d}\beta- \\
& 16\gamma\rho^2\mathrm{d}\varphi_2\mathrm{d}\gamma+4\rho\mathrm{d}\varphi_2\mathrm{d}\rho+2\mathrm{d}\beta^2+8\rho\mathrm{d}\beta\mathrm{d}\gamma+8\rho^2\mathrm{d}\gamma^2+2\mathrm{d}\rho^2.
\end{aligned} \tag{7.1}
$$

We can do the same for the two form and obtain

$$
\begin{aligned}
\frac{1}{2}B_\epsilon = {} & \left(2\gamma\rho^2(\epsilon^2-1)+\rho\beta(\epsilon^2-1)\right)\mathrm{d}\varphi_1\wedge\mathrm{d}\varphi_2+\mathrm{d}\varphi_1\wedge\mathrm{d}\alpha- \\
& \epsilon^2\rho\mathrm{d}\varphi_1\wedge\mathrm{d}\beta-(\beta+2\gamma\rho)\mathrm{d}\varphi_1\wedge\mathrm{d}\rho-(\epsilon^2\rho^2+1)\mathrm{d}\varphi_1\wedge\mathrm{d}\gamma+ \\
& \mathrm{d}\varphi_2\wedge\mathrm{d}\alpha+\rho(1-\epsilon^2)\mathrm{d}\varphi_2\wedge\mathrm{d}\beta+\left(1+\rho^2(2-\epsilon^2)\right)\mathrm{d}\varphi_2\wedge\mathrm{d}\gamma+ \\
& 2\gamma\rho\mathrm{d}\varphi_2\wedge\mathrm{d}\rho-\mathrm{d}\beta\wedge\mathrm{d}\rho-2\rho\mathrm{d}\gamma\wedge\mathrm{d}\rho.
\end{aligned}
$$

For $\epsilon = 1$ we recover the 2-form (6.17), while for $\epsilon = 0$ we get (6.14) plus some interaction terms

$$
\begin{aligned}
\frac{1}{2}B_{\epsilon=0} = {} & \frac{1}{2}B_{\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)}-(2\gamma\rho^2+\rho\beta)\mathrm{d}\varphi_1\wedge\mathrm{d}\varphi_2+ \\
& (\beta+2\gamma\rho)\mathrm{d}\varphi_1\wedge\mathrm{d}\rho+\rho\mathrm{d}\varphi_2\wedge\mathrm{d}\beta\mathrm{d}\varphi_1\wedge\mathrm{d}\rho2\rho^2\mathrm{d}\varphi_2\wedge\mathrm{d}\gamma+ \\
& 2\gamma\rho\mathrm{d}\varphi_2\wedge\mathrm{d}\rho-\mathrm{d}\beta\wedge\mathrm{d}\rho-2\rho\mathrm{d}\gamma\wedge\mathrm{d}\rho.
\end{aligned} \tag{7.2}
$$

If we assume that the masses of the modes β and ρ are much higher than the rest, then in the equations of motion their kinetic terms can safely be ignored. The equations are then compatible with $\beta = 0$, $\rho = 0$, so they disappear from the lagrangian. So the contracted models admit a truncation to the light modes, which establishes a hierarchy among the models $\mathrm{SO}(2,3)/\mathrm{SO}(2)\times\mathrm{SO}(3)$ and $\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)$.

We can still perform another contraction (a generalized contraction) in which this phenomenon is even more natural. Let us consider the map

$$
\begin{array}{ccc}
\mathfrak{s}(3) & \xrightarrow{\phi_\epsilon} & \mathfrak{s}(3) \\
H_1, H_2, \Omega_{1,1}, \Omega_{1,-1} & \longrightarrow & H_1, H_2, \Omega_{1,1}, \Omega_{1,-1}, \\
\Omega_{1,0} & \longrightarrow & \epsilon^c\,\Omega_{1,0}, \\
\Omega_{0,1} & \longrightarrow & \epsilon^d\,\Omega_{0,1},
\end{array}
$$

where c and d are real numbers that we will chose conveniently. This changes two brackets

$$
\begin{aligned}
[\Omega_{1,0}, \Omega_{0,1}]_\epsilon &= -\epsilon^{c+d}, \\
[\Omega_{1,-1}, \Omega_{0,1}]_\epsilon &= 2\epsilon^{d-c}\Omega_{10}.
\end{aligned} \tag{7.3}
$$

We need $d \geq c$. The case $d = c = 1$ is the one that we studied before. For case $d > c$, (7.3) goes to zero when $\epsilon \to 0$, so the contracted algebra is even simpler.

We use the adjoint representation of the solvable algebra, with the ordered basis (6.15). Then $\psi_\epsilon = \phi_\epsilon$,

$$
\psi_\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^c & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon^d \end{pmatrix}.
$$

In the deformed adjoint representation the generators become

$$
H_{1\epsilon} = H_1, \qquad H_{2\epsilon} = H_2, \qquad \Omega_{1,1\epsilon} = \Omega_{1,1},
$$

$$
\Omega_{1,-1\epsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\epsilon^{d-c} \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\Omega_{1,0\epsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\epsilon^{d+c} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$\Omega_{0,1\epsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^{c+d} & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\epsilon^{d-c} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The metric is

$$\begin{aligned}
\mathrm{d}s^2 = & \Big(2 + 4\alpha^2 + 2\beta^2 + 4\gamma^2 + 8\beta\gamma\rho\epsilon^{-c+d} - 8\alpha\beta\rho\epsilon^{c+d} - 8\alpha\gamma\rho^2\epsilon^{2d}+ \\
& 8\gamma^2\rho^2\epsilon^{-2c+2d} + 4\beta^2\rho^2\epsilon^{2c+2d} + 8\beta\gamma\rho^3\epsilon^{c+3d} + 4\gamma^2\rho^4\epsilon^{4d}\Big)\,\mathrm{d}\varphi_1^2+ \\
& \Big(8\alpha^2 - 8\gamma^2 - 8\beta\gamma\rho\epsilon^{-c+d} - 8\alpha\beta\rho\epsilon^{c+d} - 16\gamma^2\rho^2\epsilon^{-2c+2d} - 8\beta\gamma\rho^3\epsilon^{c+3d}- \\
& 8\gamma^2\rho^4\epsilon^{4d}\Big)\,\mathrm{d}\varphi_1\mathrm{d}\varphi_2 + \Big(8\alpha - 8\beta\epsilon^{c+d}\rho - 8\gamma\rho^2\epsilon^{2d}\Big)\,\mathrm{d}\varphi_1\mathrm{d}\alpha \\
& + \Big(4\beta + 8\gamma\rho\epsilon^{-c+d} - 8\alpha\rho\epsilon^{c+d} + 8\beta\rho^2\epsilon^{2c+2d} + 8\gamma\rho^3\epsilon^{c+3d}\Big)\,\mathrm{d}\varphi_1\mathrm{d}\beta+ \\
& \Big(8\gamma + 8\beta\rho\epsilon^{-c+d} - 8\alpha\rho\epsilon^{2d} + 16\gamma\rho^2\epsilon^{-2c+2d} + 8\beta\rho^3\epsilon^{c+3d} + 8\gamma\rho^4\epsilon^{4d}\Big)\,\mathrm{d}\varphi_1\mathrm{d}\gamma+ \\
& \Big(2 + 4\alpha^2 + 4\gamma^2 + 2\rho^2 + 8\alpha\gamma\rho^2\epsilon^{2d} + 8\gamma^2\rho^2\epsilon^{-2c+2d} + 4\gamma^2\rho^4\epsilon^{4d}\Big)\,\mathrm{d}\varphi_2^2+ \\
& \Big(8\alpha + 8\gamma\rho^2\epsilon^{2d}\Big)\,\mathrm{d}\varphi_2\mathrm{d}\alpha + \Big(-8\gamma\rho\epsilon^{-c+d} - 8\alpha\rho\epsilon^{c+d} - 8\gamma\rho^3\epsilon^{c+3d}\Big)\,\mathrm{d}\varphi_2\mathrm{d}\beta+ \\
& \Big(-8\gamma - 8\alpha\rho^2\epsilon^{2d} - 16\gamma\rho^2\epsilon^{-2c+2d} - 8\gamma\rho^4\epsilon^{4d}\Big)\,\mathrm{d}\varphi_2\mathrm{d}\gamma + 4\rho\mathrm{d}\varphi_2\mathrm{d}\rho + 4\mathrm{d}\alpha^2 \\
& - 8\rho\epsilon^{c+d}\mathrm{d}\alpha\mathrm{d}\beta - 8\rho^2\epsilon^{2d}\mathrm{d}\alpha\mathrm{d}\gamma + \Big(2 + 4\rho^2\epsilon^{2c+2d}\Big)\,\mathrm{d}\beta^2 + \Big(8\rho\epsilon^{-c+d}+ \\
& 8\rho^3\epsilon^{c+3d}\Big)\,\mathrm{d}\beta\mathrm{d}\gamma + \Big(4 + 8\rho^2\epsilon^{-2c+2d} + 4\rho^4\epsilon^{4d}\Big)\,\mathrm{d}\gamma^2 + 2\mathrm{d}\rho^2.
\end{aligned}$$

Taking $\epsilon = 1$ we recover the metric (6.16). For $\epsilon \to 0$ we get the metric (6.13) plus interaction terms

$$\begin{aligned}
\mathrm{d}s^2 = & \mathrm{d}s^2_{\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)} + \Big(2\beta^2 + \Big(8\beta\gamma\rho + 8\gamma^2\rho^2\Big)\,\delta_{c,d}\Big)\,\mathrm{d}\varphi_1^2 - (8\beta\gamma\rho+ \\
& 16\gamma^2\rho^2\Big)\,\delta_{c,d}\mathrm{d}\varphi_1\mathrm{d}\varphi_2 + (4\beta + 8\gamma\rho\delta_{c,d})\,\mathrm{d}\varphi_1\mathrm{d}\beta + (8\beta\rho\delta_{c,d}+ \\
& 16\gamma\rho^2\delta_{c,d}\Big)\,\mathrm{d}\varphi_1\mathrm{d}\gamma + \Big(2 + 8\gamma^2\rho^2\delta_{c,d}\Big)\,\mathrm{d}\varphi_2^2 - 8\gamma\delta_{c,d}\mathrm{d}\varphi_2\mathrm{d}\beta- \\
& 16\gamma\rho^2\delta_{c,d}\mathrm{d}\varphi_2\mathrm{d}\gamma + 4\rho\mathrm{d}\varphi_2\mathrm{d}\rho + 2\mathrm{d}\beta^2 + 8\rho\delta_{c,d}\mathrm{d}\beta\mathrm{d}\gamma + 8\rho^2\delta_{c,d}\mathrm{d}\gamma^2 + 2\mathrm{d}\rho^2.
\end{aligned}$$

If $d = c$ we recover (7.1). If $d > c$ several interaction terms cancel and we get a simpler expression

$$\begin{aligned}
\mathrm{d}s^2 = & \mathrm{d}s^2_{\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)} + 2\beta^2\mathrm{d}\varphi_1^2 + 4\beta + \mathrm{d}\varphi_1\mathrm{d}\beta + 2\mathrm{d}\varphi_2^2 + 4\rho\mathrm{d}\varphi_2\mathrm{d}\rho+ \\
& 2\mathrm{d}\beta^2 + 2\mathrm{d}\rho^2.
\end{aligned}$$

For the two form we obtain

$$\frac{1}{2}B_\epsilon = \Big(2\gamma\rho^2(\epsilon^{2d} - \epsilon^{d-c}) + \rho\beta(\epsilon^{c+d} - 1)\Big)\,\mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 + \mathrm{d}\varphi_1 \wedge \mathrm{d}\alpha -$$
$$\epsilon^{c+d}\rho\mathrm{d}\varphi_1 \wedge \mathrm{d}\beta - (\beta + 2\gamma\rho\epsilon^{d-c})\mathrm{d}\varphi_1 \wedge \mathrm{d}\rho - (\epsilon^{2d}\rho^2 + 1)\mathrm{d}\varphi_1 \wedge \mathrm{d}\gamma +$$
$$\mathrm{d}\varphi_2 \wedge \mathrm{d}\alpha + \rho(1 - \epsilon^{c+d})\mathrm{d}\varphi_2 \wedge \mathrm{d}\beta + \Big(1 + \rho^2(2\epsilon^{d-c} - \epsilon^{2d})\Big)\,\mathrm{d}\varphi_2 \wedge \mathrm{d}\gamma +$$
$$2\gamma\rho\epsilon^{d-c}\mathrm{d}\varphi_2 \wedge \mathrm{d}\rho - \mathrm{d}\beta \wedge \mathrm{d}\rho - 2\rho\epsilon^{d-c}\mathrm{d}\gamma \wedge \mathrm{d}\rho,$$

recovering (6.17) for $\epsilon = 1$. For $\epsilon \to 0$ we get (6.14) plus interaction terms

$$\frac{1}{2}B = \frac{1}{2}B_{\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)} - \Big(2\gamma\rho^2\delta_{c,d} + \rho\beta\Big)\,\mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 -$$
$$(\beta + 2\gamma\delta_{c,d})\,\mathrm{d}\varphi_1 \wedge \mathrm{d}\rho + \rho\mathrm{d}\varphi_2 \wedge \mathrm{d}\beta + 2\rho^2\delta_{c,d}\mathrm{d}\varphi_2 \wedge \mathrm{d}\gamma + 2\gamma\rho\delta_{c,d}\mathrm{d}\varphi_2 \wedge \mathrm{d}\rho -$$
$$\mathrm{d}\beta \wedge \mathrm{d}\rho - 2\rho\delta_{c,d}\mathrm{d}\gamma \wedge \mathrm{d}\rho.$$

As for the metric, if $d = c$ we recover (7.2), and if $d > c$ some terms cancel and we get the simpler form

$$\frac{1}{2}B = \frac{1}{2}B_{\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)} - \rho\beta\mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 - \beta\mathrm{d}\varphi_1 \wedge \mathrm{d}\rho + \rho\mathrm{d}\varphi_2 \wedge \mathrm{d}\beta -$$
$$\mathrm{d}\beta \wedge \mathrm{d}\rho.$$

7.2 SO(2,3)/SO(2) × SO(3) *contracted with respect to* SO(1,3)/SO(3)

We want now to contract the ISM SO(2,3)/SO(2) × SO(3) with respect to SO(3,1)/SO(3). We first identify the algebra

$$\mathfrak{t} = \mathfrak{solv}(\mathrm{SO}(3,1)/\mathrm{SO}(3))$$

as a subalgebra of SO(2,3). We can take the lower diagonal 4×4 block of (6.1)

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & b_{1\times 3} \\ b^t_{1\times 3} & C_{3\times 3} \end{pmatrix} \right\}, \qquad C^T = -C,$$

being the Cartan decomposition

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & C_{3\times 3} \end{pmatrix} \right\}, \qquad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & b_{1\times 3} \\ b^t_{1\times 3} & 0 \end{pmatrix} \right\}.$$

It is convenient to take as maximal abelian subalgebra of $\mathfrak{p}$

$$\mathrm{span}\left\{ H = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

in which case the generalized root spaces are

$$\mathfrak{n}^+ = \mathrm{span}\left\{ n_1^+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ n_2^+ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \right\},$$

$$\mathfrak{n}^- = \mathrm{span}\left\{ n_1^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ n_2^- = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \right\},$$

with

$$[H, n_i^+] = n_i^+, \qquad [H, n_i^-] = -n_i^-.$$

Finally we get the algebra

$$\mathfrak{t} = \mathrm{span}\{H\} + \mathfrak{n}^+ = \mathrm{span}\{H, n_1^+, n_2^+\} \approx \mathrm{span}\{\mathrm{H}_2, \Omega_{1,1}, \Omega_{0,1}\},$$

where we have already identified the generators with a subalgebra of $\mathfrak{s}_3$. Notice that the identification corresponds to the lower diagonal 4×4 blocks. In ref.[15] the choice of the abelian subalgebra is different but it gives the same result.

We consider the coset representative for the Lie group $\mathrm{SO}(1,3)/\mathrm{SO}(3)$, $L = e^{\varphi_2 H_2} e^{\alpha\Omega_{1,1}} e^{\rho\Omega_{0,1}}$ and compute the metric as usual[1]

$$\mathrm{d}s^2_{\mathrm{SO}(1,3)/\mathrm{SO}(3)} = (2 + 4\alpha^2 + 2\rho^2)\mathrm{d}\varphi_2^2 + 8\alpha \mathrm{d}\varphi_2 \mathrm{d}\alpha + 4\rho \mathrm{d}\varphi_2 \mathrm{d}\rho + 4\mathrm{d}\alpha^2 + 2\mathrm{d}\rho^2. \tag{7.4}$$

[1] If we choose instead the coset representative $L = e^{\alpha'\Omega_{1,1}} e^{\rho'\Omega_{0,1}} e^{\varphi_2' H_2}$ the metric is

$$\mathrm{d}s^2_{\mathrm{SO}(1,3)/\mathrm{SO}(3)} = 2\mathrm{d}{\varphi'}_2^2 + 4e^{-2\varphi_2'}\mathrm{d}\alpha'^2 + 2e^{-2\varphi'_2}\mathrm{d}\rho'^2,$$

which can be compared with the results in [15].

We are going to perform a generalized contraction of $\mathfrak{s}(3)$ with respect to $\mathfrak{t}$. We define ϕ_ϵ as

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\phi_\epsilon} & \mathfrak{g} \\
H_2,\, \Omega_{1,1},\, \Omega_{0,1} & \longrightarrow & H_2,\, \Omega_{1,1},\, \Omega_{0,1}, \\
H_1,\, \Omega_{1,0},\, \Omega_{1,-1} & \longrightarrow & \epsilon^a H_1,\, \epsilon^b\Omega_{1,0},\, \epsilon^c\Omega_{1,-1},
\end{array}$$

where $a, b, c > 0$. In the ordered basis (6.15) we have

$$\phi_\epsilon = \begin{pmatrix} \epsilon^a & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^b & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The deformed algebra is

$$\begin{aligned}
[H_1, \Omega_{1,1}] &= \epsilon^a\Omega_{1,1}, \\
[H_1, \Omega_{1,-1}] &= \epsilon^{a+c}\Omega_{1,-1}, \\
[H_1, \Omega_{1,0}] &= \epsilon^a\Omega_{10}, \\
[H_2, \Omega_{1,1}] &= \Omega_{1,1}, \\
[H_2, \Omega_{1,-1}] &= \Omega_{1,-1}, \\
[H_2, \Omega_{0,1}] &= \Omega_{0,1}, \\
[\Omega_{1,0}, \Omega_{0,1}] &= -\epsilon^b\Omega_{1,1}, \\
[\Omega_{1,-1}, \Omega_{0,1}] &= \epsilon^{c-b}2\Omega_{10},
\end{aligned}$$

and the rest 0. We see that $c \geq b$ in order to have a well defined contracted bracket ($\epsilon \to 0$). The case $c = b$ gives a different result when $\epsilon \to 0$.

The deformed representation that we get is given by the matrices

$$H_{1\epsilon} = \epsilon^a H_1, \qquad H_{2\epsilon} = H_2,$$

$$\Omega_{1,1\epsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\epsilon^a & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega_{1,-1\epsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\epsilon^{c-b} \\ -\epsilon^a & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega_{1,0\epsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\epsilon^b \\ -\epsilon^a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega_{0,1\epsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^b & 0 & 0 \\ 0 & 0 & 0 & -2\epsilon^{c-b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The deformed metric then becomes

$$\begin{aligned}
\mathrm{d}s^2 = {} & \mathrm{d}\varphi_1^2 \left(2 + 4\alpha^2\epsilon^{2a} + 2\beta^2\epsilon^{2a} + 4\gamma^2\epsilon^{2a} - 8\alpha\beta\rho\epsilon^{2a+b} + 8\beta\gamma\rho\epsilon^{a+c-b} + \right. \\
& 4\beta^2\rho^2\epsilon^{2(a+b)} - 8\alpha\gamma\rho^2\epsilon^{2a+c} + 8\gamma^2\rho^2\epsilon^{a+2(c-b)} + 8\beta\gamma\rho^3\epsilon^{2a+b+c} + \\
& \left. 4\gamma^2\rho^4\epsilon^{2(a+c)}\right) + \mathrm{d}\varphi_1\mathrm{d}\varphi_2 \left(8\alpha^2\epsilon^a - 8\gamma^2\epsilon^a - 8\alpha\beta\rho\epsilon^{a+b} - 8\beta\gamma\rho\epsilon^{a+c-b} - \right. \\
& \left. 16\gamma^2\rho^2\epsilon^{a+2(c-b)} - 8\beta\gamma\rho^3\epsilon^{a+b+c} - 8\gamma^2\rho^4\epsilon^{a+2c}\right) + \mathrm{d}\varphi_1\mathrm{d}\alpha \left(8\alpha\epsilon^a - \right. \\
& \left. 8\beta\rho\epsilon^{a+b} - 8\gamma\rho^2\epsilon^{a+c}\right) + \mathrm{d}\varphi_1\mathrm{d}\beta \left(4\beta\epsilon^a - 8\alpha\rho\epsilon^{a+b} + 8\gamma\rho\epsilon^{a+c-b} + 8\beta\rho^2\epsilon^{a+2b} + \right. \\
& \left. 8\gamma\rho^3\epsilon^{a+b+c}\right) + \mathrm{d}\varphi_1\mathrm{d}\gamma \left(8\gamma\epsilon^a + 8\beta\rho\epsilon^{a+c-b} - 8\alpha\rho^2\epsilon^{a+c} + 16\gamma\rho^2\epsilon^{a+2(c-b)} + \right. \\
& \left. 8\beta\rho^3\epsilon^{a+b+c} + 8\gamma\rho^4\epsilon^{a+2c}\right) + \mathrm{d}\varphi_2^2 \left(2 + 4\alpha^2 + 4\gamma^2 + 2\rho^2 + 8\alpha\gamma\rho^2\epsilon^c + \right. \\
& \left. 8\gamma^2\rho^2\epsilon^{2(c-b)} + 4\gamma^2\rho^4\epsilon^{2c}\right) + \mathrm{d}\varphi_2\mathrm{d}\alpha \left(8\alpha + 8\gamma\rho^2\epsilon^c\right) + \mathrm{d}\varphi_2\mathrm{d}\beta \left(-8\alpha\rho\epsilon^b - \right. \\
& \left. 8\gamma\rho\epsilon^{c-b} - 8\gamma\rho^3\epsilon^{b+c}\right) + \mathrm{d}\varphi_2\mathrm{d}\gamma \left(-8\gamma - 8\alpha\rho^2\epsilon^c - 16\gamma\rho^2\epsilon^{2(c-b)} - 8\gamma\rho^4\epsilon^{2c}\right) + \\
& 4\rho\mathrm{d}\varphi_2\mathrm{d}\rho + 4\mathrm{d}\alpha^2 - 8\rho\epsilon^b\mathrm{d}\alpha\mathrm{d}\beta - 8\rho^2\epsilon^c\mathrm{d}\alpha\mathrm{d}\gamma + \mathrm{d}\beta^2 \left(2 + 4\rho^2\epsilon^{2b}\right) + \\
& \mathrm{d}\beta\mathrm{d}\gamma \left(8\rho\epsilon^{c-b} + 8\rho^3\epsilon^{b+c}\right) + \mathrm{d}\gamma^2 \left(4 + 8\rho^2\epsilon^{2(c-b)} + 4\rho^4\epsilon^{2c}\right) + 2\mathrm{d}\rho^2.
\end{aligned}$$

For $\epsilon = 1$ we obtain (6.16). For $\epsilon \to 0$ we get

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\mathrm{SO}(1,3)/\mathrm{SO}(3)} + 2\mathrm{d}\varphi_1^2 + \left(2\rho^2 + 8\gamma^2\rho^2\delta_{c,b}\right)\mathrm{d}\varphi_2^2 - 8\gamma\rho\delta_{c,b}\mathrm{d}\varphi_2\mathrm{d}\beta -$$
$$\mathrm{d}\varphi_2\mathrm{d}\gamma\left(8\gamma + 16\gamma\rho^2\delta_{c,b}\right) + 2\mathrm{d}\beta^2 + 8\rho\delta_{c,b}\mathrm{d}\beta\mathrm{d}\gamma + \left(4 + 8\rho^2\delta_{c,b}\right)\mathrm{d}\gamma^2,$$

and for $c > b$ it is simpler,

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\mathrm{SO}(1,3)/\mathrm{SO}(3)} + 2\mathrm{d}\varphi_1^2 + 2\rho^2\mathrm{d}\varphi_2^2 - 8\gamma\mathrm{d}\varphi_2\mathrm{d}\gamma + 2\mathrm{d}\beta^2 + 4\mathrm{d}\gamma^2.$$

The deformed 2-form is

$$\frac{1}{2}B = \Big(\alpha(1-\epsilon^a) + \gamma\rho^2(\epsilon^c + \epsilon^{a+c} - 2\epsilon^{a+c-b}) + \gamma(1-\epsilon^a) - \beta\rho(\epsilon^a - \epsilon^{a+b})\Big)$$
$$\mathrm{d}\varphi_1\wedge\mathrm{d}\varphi_2 + \mathrm{d}\varphi_1\wedge\mathrm{d}\alpha - \rho\epsilon^b\mathrm{d}\varphi_1\wedge\mathrm{d}\beta - (1+\rho^2\epsilon^c)\mathrm{d}\varphi_1\wedge\mathrm{d}\gamma -$$
$$(\beta\epsilon^a + 2\gamma\rho\epsilon^{a+c-b})\mathrm{d}\varphi_1\wedge\mathrm{d}\rho + \mathrm{d}\varphi_2\wedge\mathrm{d}\alpha + \rho(1-\epsilon^b)\mathrm{d}\varphi_2\wedge\mathrm{d}\beta +$$
$$\Big(1 - \rho^2(\epsilon^c - 2\epsilon^{c-b})\Big)\mathrm{d}\varphi_2\wedge\mathrm{d}\gamma + 2\gamma\rho\epsilon^{c-b}\mathrm{d}\varphi_2\wedge\mathrm{d}\rho - \mathrm{d}\beta\wedge\mathrm{d}\rho -$$
$$2\rho\epsilon^{c-b}\mathrm{d}\gamma\wedge\mathrm{d}\rho.$$

For $\epsilon = 1$ we get (6.17) and for $\epsilon \to 0$ we get

$$\frac{1}{2}B = (\alpha+\gamma)\mathrm{d}\varphi_1\wedge\mathrm{d}\varphi_2 + \mathrm{d}\varphi_1\wedge\mathrm{d}\alpha - \mathrm{d}\varphi_1\wedge\mathrm{d}\gamma + \mathrm{d}\varphi_2\wedge\mathrm{d}\alpha + \rho\mathrm{d}\varphi_2\wedge\mathrm{d}\beta$$
$$+\left(1 + 2\rho^2\delta_{c,b}\right)\mathrm{d}\varphi_2\wedge\mathrm{d}\gamma + 2\gamma\rho\delta_{c,b}\mathrm{d}\varphi_2\wedge\mathrm{d}\rho - \mathrm{d}\beta\wedge\mathrm{d}\rho - 2\rho\delta_{c,b}\mathrm{d}\gamma\wedge\mathrm{d}\rho.$$

Truncating to the coset SO(1, 3)/SO(3) corresponds to take the coordinates $\varphi_1 = \gamma = \beta = 0$. Substituting back we get a 2 form on SO(1, 3)/SO(3)

$$\frac{1}{2}B_{\mathrm{SO}(1,3)/\mathrm{SO}(3)} = \mathrm{d}\varphi_2\wedge\mathrm{d}\alpha.$$

By construction, this 2-form is invariant under the action of the solvable group, but one can explicitly check that it is not invariant under the whole SO(1, 3). So, although it is defined on the coset manifold SO(1, 3)/SO(3), the lagrangian (or the model) is not invariant under the whole isometry group SO(1, 3) (see Sec. 2.8.3).

8. GENERAL PROPERTIES OF THE S-EXPANSION METHOD

The expansion of a Lie algebra by a discrete semigroup that we have described in Sections 2.10 and 2.11 has some applications in Physics. We have said some words about them in the introduction and in Chapter 9 we will apply it to relate the 2-dimensional and 3-dimensional Lie algebras in Bianchi's classification. In this Chapter we discuss the conservation of solvability, nilpotency and semisimplicity of Lie algebras under the S-expansion procedure. We have collected these results in ref.[5], to appear soon.

History of finite semigroups programs The number of finite non-isomorphic semigroups of order n are given in the following table:

order	$Q = \#$ semigroups	
1	1	
2	4	
3	18	
4	126	[Forsythe '54]
5	1,160	[Motzkin, Selfridge '55]
6	15,973	[Plemmons '66]
7	836,021	[Jurgensen, Wick '76]
8	1,843,120,128	[Satoh, Yama, Tokizawa '94]
9	**52,989,400,714,478**	[Distler, Kelsey, Mitchell '09]

Fig. 8.1: Number of semigroups of each order, up to order 9.

As shown in the table the problem of enumerating all the discrete semigrups of a certain order up to isomorphism is a nontrivial problem. In fact, the number Q of semigroups increases very quickly with the order of the semigroup.

In ref.[44] a set of algorithms is given that allow us to make certain calculations with finite semigroups. The first program, *gen.f*, gives, up to isomorphism all the semigroups of order n for $n = 1, 2, \dots, 8$. The input is the order of the semigroups that we want to obtain and the *output* is a list of all the isomorphism classes of semigroups that exist at this order. We denote each semigroup as $S_{(n)}^{\alpha}$ where the superindex $\alpha = 1, \dots, Q$ runs over all the semigroups of order n. In Appendix C all the semigroups of order 3 and some of order 4 used in this work are shown.

We have developed a Java library to be able to automatize the S-expansion of Lie algebras and then easily check properties of the S-expanded algebras. Our starting point is the output of the program *gen.f.* We review how the library works in Appendix B [24].

In Section 8.2 we are going to use this library to perform all the possible expansions of the algebra $\mathfrak{sl}(2)$ that can be made (with semigroups up to the order 6) and check explicitly the theoretical properties that we will find in the next sections.

■

This Chapter is organized as follows: in Section 8.1 we study the question of the preservation (or not) of the properties of solvability, nilpotency, semisimplicity and compactness under expansions. In Section 8.2 we make an exhaustive study of the possible expansions that can be made for the algebra $\mathfrak{sl}(2)$ and check our theoretical results.

8.1 Properties preserved under the S-expansion procedure

When looking for a semigroup to get a given Lie algebra from the expansion of another one, it is convenient to know which properties are going to be preserved after the expansion. In this way some expansions can be rejected a priori.

8.1.1 Solvable Lie algebras

A solvable algebra $\mathfrak{g}$ is one for which the sequence

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(1)} = \left[\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}\right], \quad \dots, \quad \mathfrak{g}^{(n)} = \left[\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}\right] \tag{8.1}$$

terminates, i.e., $\mathfrak{g}^{(n)} = 0$ for some n (see ref.[45] for details).

In order to study the expansions of solvable algebras, it will be useful to have an expression of the solvability condition in terms of the structure constants. This is done in the following way:

Let $\{X_i\}, i = 1, \ldots, n$ be a basis of an algebra $\mathfrak{g}$. The fact that $\mathfrak{g} = \mathfrak{g}^{(0)}$ is solvable implies that in its commutation relations

$$[X_i, X_j] = C_{ij}^k X_k \tag{8.2}$$

there is at least one value of k for which $C_{ij}^k = 0$. So, let $k^{(1)}$ represent the set of values for which $C_{ij}^{k^{(1)}} \neq 0$. Then $k^{(1)}$ runs for all values of the basis elements $\{X_k\}$ except for those values for which

$$C_{ij}^{k \neq k^{(1)}} = 0. \tag{8.3}$$

It is clear then that the set $\{X_{k^{(1)}}\}$ is smaller than $\{X_k\}$ and $\{X_{k^{(1)}}\} \subset \{X_k\}$.

Let us consider $\mathfrak{g}^{(2)} = \left[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}\right]$ where $\mathfrak{g}^{(1)} = \left[\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}\right] =: \text{span}\,\{X_{k^{(1)}}\}$ according to the notation established above. We have

$$\left[X_{i^{(1)}}, X_{j^{(1)}}\right] = C_{i^{(1)}j^{(1)}}^{k^{(2)}} X_{k^{(2)}}. \tag{8.4}$$

As $\mathfrak{g}$ is solvable the index $k^{(2)}$ must run in a smaller subset with respect to that where $k^{(1)}$ runs. So as the algebra is solvable, there exists some n for which $\mathfrak{g}^{(n)} = 0$ and $\mathfrak{g}^{(n-1)}$ is abelian, i.e.,

$$\left[X_{i^{(n-1)}}, X_{j^{(n-1)}}\right] = C_{i^{(n-1)}j^{(n-1)}}^{k^{(n)}} X_{k^{(n)}} = 0. \tag{8.5}$$

The solvability of an algebra can then be expressed in terms of its structure constants as the condition that there is some n for which $C_{i^{(n-1)}j^{(n-1)}}^{k^{(n)}} = 0$.

Proposition 8.1.1. ***(For the expanded algebra)***

Let $\{X_i\}, i = 1, \ldots, n$ be a basis of a solvable Lie algebra $\mathfrak{g}$, $S = \{\lambda_\alpha\}, \alpha = 1, \ldots, m$ a finite abelian semigroup and

$$\mathfrak{g}_S = S \otimes \mathfrak{g} = \{\lambda_\alpha \otimes X_i\} = \left\{X_{(i,\alpha)}\right\} \tag{8.6}$$

the S-expanded algebra, which satisfies

$$\left[X_{(i,\alpha)}, X_{(j,\beta)}\right] = C_{(i,\alpha)(j,\beta)}^{(k,\gamma)} X_{(k,\gamma)} = C_{ij}^k \mathcal{K}_{\alpha\beta}^\gamma X_{(k,\gamma)}. \tag{8.7}$$

Then the expanded algebra $\mathfrak{g}_S = S \otimes \mathfrak{g}$ of a solvable algebra $\mathfrak{g}$ is solvable.

Proof. Let us consider the following sequence for the expanded algebra

$$\mathfrak{g}_S^{(0)} = \mathfrak{g}_S, \quad \mathfrak{g}_S^{(1)} = \left[\mathfrak{g}_S^{(0)}, \mathfrak{g}_S^{(0)}\right], \quad \ldots, \quad \mathfrak{g}_S^{(n)} = \left[\mathfrak{g}_S^{(n-1)}, \mathfrak{g}_S^{(n-1)}\right] . \tag{8.8}$$

For $\mathfrak{g}_S^{(n)}$ we have

$$\left[X_{\left(i^{(n-1)},\alpha^{(n-1)}\right)}, X_{\left(j^{(n-1)},\beta^{(n-1)}\right)}\right] = C_{i^{(n-1)}j^{(n-1)}}^{k^{(n)}} \mathcal{K}_{\alpha^{(n-1)}\beta^{(n-1)}}^{\gamma^{(n)}} X_{\left(k^{(n)},\gamma^{(n)}\right)}. \tag{8.9}$$

So as $\mathfrak{g}$ is solvable by hypothesis, then there exists some n for which the sequence (8.8) terminates, i.e., for which $\mathfrak{g}_S^{(n)} = 0$ or, in terms of the structure constants, for which

$$C_{\left(i^{(n-1)},\alpha^{(n-1)}\right)\left(j^{(n-1)},\beta^{(n-1)}\right)}^{\left(k^{(n)},\gamma^{(n)}\right)} = C_{i^{(n-1)}j^{(n-1)}}^{k^{(n)}} \mathcal{K}_{\alpha^{(n-1)}\beta^{(n-1)}}^{\gamma^{(n)}} = 0.$$

■

Proposition 8.1.2. ***(For the resonant subalgebra)***

The resonant subalgebra $\mathfrak{g}_{S,R}$ (defined in Section 2.11) of the expanded algebra $\mathfrak{g}_S$ is always solvable if the original algebra $\mathfrak{g}$ is solvable.

Proof. By the theory of classification of Lie algebras, any subalgebra of a solvable algebra must be solvable ■

Proposition 8.1.3. ***(For the 0_S-reduced algebra)***

Consider the expansion of a solvable Lie algebra $\mathfrak{g}$ with a zero element. Then the reduced algebra $\mathfrak{g}_S^{\mathrm{red}}$ is always solvable.

Proof. According with the S-expansion procedure (of ref. [37]) when the semigroup has a zero element 0_S, $S = \{\lambda_\alpha, 0_S\}$ the commutation relations of the expanded algebra $\mathfrak{g}_S$ are given by,

$$\begin{aligned}
\left[X_{(i,\alpha)}, X_{(j,\beta)}\right] &= C_{ij}^{k} \mathcal{K}_{\alpha\beta}^{\gamma} X_{(k,\gamma)} + C_{ij}^{k} \mathcal{K}_{\alpha\beta}^{0} X_{(k,0)}, \\
\left[X_{(i,0)}, X_{(j,\beta)}\right] &= C_{ij}^{k} X_{(k,0)}, \\
\left[X_{(i,0)}, X_{(j,0)}\right] &= C_{ij}^{k} X_{(k,0)}
\end{aligned}$$

and the 0_S-reduced algebra $\mathfrak{g}_S^{\mathrm{red}}$ is given by

$$\left[X_{(i,\alpha)}, X_{(j,\beta)}\right] = C_{ij}^{k} \mathcal{K}_{\alpha\beta}^{\gamma} X_{(k,\gamma)}.$$

So if $\mathfrak{g}$ is solvable then there exists some n for which $C_{i^{(n-1)}j^{(n-1)}}^{k^{(n)}} = 0$. Therefore, for the same n

$$C_{\left(i^{(n-1)},\alpha^{(n-1)}\right)\left(j^{(n-1)},\beta^{(n-1)}\right)}^{\left(k^{(n)},\gamma^{(n)}\right)} = C_{i^{(n-1)}j^{(n-1)}}^{k^{(n)}} \mathcal{K}_{\alpha^{(n-1)}\beta^{(n-1)}}^{\gamma^{(n)}} = 0.$$

and we can conclude that if $\mathfrak{g}$ is solvable then the reduced algebra $\mathfrak{g}_S^{\text{red}}$ is solvable, too. ■

8.1.2 Nilpotent Lie algebras

A nilpotent algebra $\mathfrak{g}$ is an algebra for which the sequence

$$\mathfrak{g}_{(0)} = \mathfrak{g}, \quad \mathfrak{g}_{(1)} = \left[\mathfrak{g}_{(0)}, \mathfrak{g}\right], \quad \ldots, \quad \mathfrak{g}_{(n)} = \left[\mathfrak{g}_{(n-1)}, \mathfrak{g}\right]$$

terminates, i.e., $\mathfrak{g}_{(n)} = 0$ for some n (see ref.[45] for details). In terms of the structure constants this means that for some n, $C_{i^{(n-1)}j}^{k^{(n)}}$ must vanish.

Proposition 8.1.4. ***(For the expanded algebra)***
The S-expansion of a nilpotent algebra $\mathfrak{g}$, $\mathfrak{g}_S = S \otimes \mathfrak{g}$ is always nilpotent.

Proof. For the expanded algebra $\mathfrak{g}_S = S \otimes \mathfrak{g}$, let us consider the sequence

$$\mathfrak{g}_{S,(0)} = \mathfrak{g}_S, \quad \mathfrak{g}_{S,(1)} = \left[\mathfrak{g}_{S,(0)}, \mathfrak{g}_S\right], \quad \ldots, \quad \mathfrak{g}_{S,(n)} = \left[\mathfrak{g}_{S(n-1)}, \mathfrak{g}_S\right].$$

For

$$\mathfrak{g}_{S,(n)} = \left\{X_{\left(i_{(n)},\alpha_{(n)}\right)}\right\}$$

we have

$$\left[X_{\left(i_{(n-1)},\alpha_{(n-1)}\right)}, X_{(j,\beta)}\right] = C_{\left(i_{(n-1)},\alpha_{(n-1)}\right)(j,\beta)}^{\left(k_{(n)},\gamma_{(n)}\right)} X_{\left(k_{(n)},\gamma_{(n)}\right)}.$$

But as $\mathfrak{g}$ is supposed to be nilpotent, so $C_{i_{(n-1)}j}^{k_{(n)}} = 0$ for some n, and consequently we have

$$C_{\left(i_{(n-1)},\alpha_{(n-1)}\right)(j,\beta)}^{\left(k_{(n)},\gamma_{(n)}\right)} = C_{i_{(n-1)}j}^{k_{(n)}} \mathcal{K}_{\alpha_{(n-1)}\beta}^{\gamma_{(n)}} = 0.$$

which means that $\mathfrak{g}_{S,(n)} = 0$ for this n. Therefore, the expansion $\mathfrak{g}_S = S \otimes \mathfrak{g}$ of a nilpotent algebra $\mathfrak{g}$ must be nilpotent, too. ■

Proposition 8.1.5. ***(For the resonant subalgebra)***

The resonant subalgebra $\mathfrak{g}_{S,R}$ (defined in Section 2.11) of the expanded algebra $\mathfrak{g}_S$ is always nilpotent if the original algebra $\mathfrak{g}$ is nilpotent.

Proof. As in the case of a solvable algebra, from the theory of Lie algebras, we know that a subalgebra of a nilpotent algebra must be nilpotent. As a consequence the resonant subalgebra $\mathfrak{g}_{S,R}$ of the expanded algebra $\mathfrak{g}_S$ of a nilpotent algebra $\mathfrak{g}$ must be nilpotent, too. ■

Proposition 8.1.6. ***(For the 0_S-reduced algebra)***

Consider the expansion of a nilpotent Lie algebra $\mathfrak{g}$ with a zero element. Then the reduced algebra $\mathfrak{g}_S^{red}$ is always nilpotent.

Proof. When the semigroup has a zero element 0_S, $S = \{\lambda_\alpha, 0_S\}$ the commutation relations of the 0_S-reduced algebra $\mathfrak{g}_S^{\mathrm{red}}$ are given by

$$\left[X_{(i,\alpha)}, X_{(j,\beta)}\right] = C_{ij}^{k} \mathcal{K}_{\alpha\beta}^{\gamma} X_{(k,\gamma)}.$$

So if $\mathfrak{g}$ is nilpotent then there exists some n for which $C_{i_{(n-1)}j}^{k_{(n)}} = 0$. Therefore, for the same n

$$C_{\left(i_{(n-1)},\alpha_{(n-1)}\right)(j,\beta)}^{\left(k_{(n)},\gamma_{(n)}\right)} = C_{i_{(n-1)}j}^{k_{(n)}} \mathcal{K}_{\alpha_{(n-1)}\beta}^{\gamma_{(n)}} = 0.$$

and we can conclude that if $\mathfrak{g}$ is nilpotent, then the reduced algebra $\mathfrak{g}_S^{\mathrm{red}}$ is nilpotent, too. ■

8.1.3 Semisimple Lie algebras

The Cartan-Killing metric of a Lie algebra is defined as

$$g(X,Y) = \mathrm{Tr}(\mathrm{ad}_X, \mathrm{ad}_Y)$$

which in a basis-dependent notation gives the symmetric matrix

$$g_{ij} = C_{ik}^{l} C_{jl}^{k}.$$

An algebra $\mathfrak{g}$ is semisimple if its Cartan-Killing metric is not degenerated ($\det(g_{ij}) \neq 0$, see ref.[45] for details).

If we perform the expansion of a semisimple algebra $\mathfrak{g}$, the Cartan-Killing metric of the expanded algebra $\mathfrak{g}_S$ is

$$\begin{aligned} g_{(i,\alpha)(j,\beta)} &= C_{(i,\alpha)(k,\gamma)}^{(l,\lambda)} C_{(j,\beta)(l,\lambda)}^{(k,\gamma)} \\ &= \mathcal{K}_{\alpha\gamma}^{\lambda} \mathcal{K}_{\beta\lambda}^{\gamma} C_{ik}^{l} C_{jl}^{k} \end{aligned}$$

and here we can see that there is no argument to know if det $\left(g_{(i,\alpha)(j,\beta)}\right)$ vanishes or not, because the information is mixed with the semigroup selectors $\mathcal{K}_{\alpha\beta}^{\gamma}$ which take values in the set $\{0,1\}$. So depending on the semigroup, the selectors $\mathcal{K}_{\alpha\beta}^{\gamma}$ can make the Cartan-Killing metric $g_{(i,\alpha)(j,\beta)}$ degenerated or not.

This fact agrees with the results obtained in different applications of the S-expansion method. For example, in ref.[28] the semisimple extension of the Poincaré algebra was obtained as an expansion of the semisimple AdS algebra. On the other hand, in ref.[25] the non semisimple *superalgebra M* (the maximal supersymmetric extension of the Poincaré algebra in $d = 11$ dimensions) was obtained as an expansion of the semisimple superalgebra $\mathfrak{osp}(32/1)$[1]. So in general, and depending on the semigroup used in the expansion procedure, starting from a semisimple algebra we can obtain eitter

- semisimple Lie algebras,
- non-semisimple Lie algebras.

We can find examples of both in Figure 8.2.

8.1.4 Expansion of compact Lie algebras

If we perform the expansion of a compact algebra $\mathfrak{g}_k = \{X_i\}$ with a semigroup $S = \{\lambda_\alpha\}$ what we obtain is an algebra $\mathfrak{g}_{k,S}$ with a Cartan-Killing metric given by

$$\begin{aligned} g_{(i,\alpha)(j,\beta)} &= C_{(i,\alpha)(k,\gamma)}^{(l,\lambda)} C_{(j,\beta)(l,\lambda)}^{(k,\gamma)} \\ &= \mathcal{K}_{\alpha\gamma}^{\lambda} \mathcal{K}_{\beta\lambda}^{\gamma} g_{ij}. \end{aligned} \tag{8.10}$$

[1] The S-expansion procedure can be extended to the case of superalgebras, as shown in ref.[37].

We know that g_{ij} is definite negative, because $\mathfrak{g}_k$ is compact (see ref.[45] for details). We cannot say the same for $g_{(i,\alpha)(j,\beta)}$, because the selectors $\mathcal{K}_{\alpha\gamma}^{\lambda}$ can change this property.

Therefore, compactness is not preserved in general by the S-expansion procedure. Even if the metric (8.10) is non degenerate and the semisimplicity is preserved, the result may be a metric that is not negative definite.

8.2 Expansions of $\mathfrak{sl}(2,\mathbb{R})$, an instructive example

In this section we are going to study expansions with the abelian semigroups that are in the lists given by the computer programs *gen.f*. We identify each semigroup by $S_{(n)}^{\alpha}$, being n the order of the semigroup and α the identifier assigned to the semigroup by the program *gen.f*. We have developed a Java library to perform the expansions (see Appendix B or [24]).

8.2.1 Classification of the different kinds of expansions

The real semisimple Lie algebra $\mathfrak{sl}(2,)$ is given in the Cartan Weyl (CW) basis by

$$\begin{aligned}\mathfrak{sl}(2,\mathbb{R}) &= \{\sigma_3\} + \{\sigma_+,\sigma_-\}\\ &= V_0 + V_1,\end{aligned}$$

with $V_0 = \{\sigma_3\}$ and $V_1 = \{\sigma_+,\sigma_-\}$ having the subspace structure

$$\begin{aligned}[V_0,V_0] &\subset V_0,\\ [V_0,V_1] &\subset V_1,\\ [V_1,V_1] &\subset V_0.\end{aligned}$$

We are going to study the properties of all the possible expansions that can be made using semigroups of order $n = 1, 2, \dots, 6$. From the list of all non-isomorphic semigroups of order n we can perform the following kinds of expansions:

- expansions with a generic abelian semigroup. As it has been seen in Section 2.11 we need abelian semigroups to get a well defined Lie bracket in the S-expanded algebra.
- Expansions with a generic abelian semigroup with a zero element.

- Expansions with a generic abelian semigroup with a resonant decomposition of the form

$$S = S_0 \cup S_1, \text{ such that, } \begin{array}{l} S_0 \cdot S_0 \in S_0, \\ S_0 \cdot S_1 \in S_1, \\ S_1 \cdot S_1 \in S_0. \end{array} \tag{8.11}$$

- Expansions with a generic abelian semigroup with a zero element and simultaneously with a resonant decomposition of the form (8.11)[2].

To perform all the possible expansions we use an algorithm that will be described in what follows. First we identify all the semigroups of a certain order n (we have limited this to up to order 6) satisfying the conditions enumerated above. For example, for $n = 3$ the results are given in figure 8.2. The interested reader can check a complete list of the semigroups of order 3 in Appendix C.

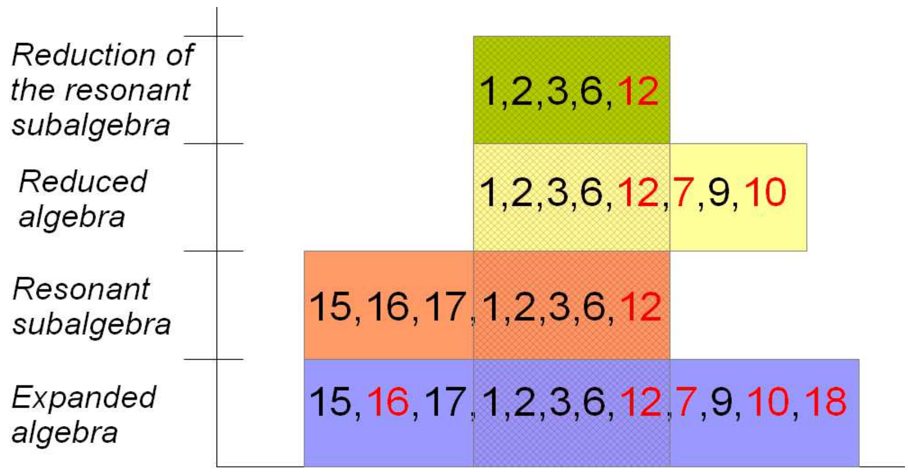

Fig. 8.2: Expansions of $\mathfrak{sl}(2,\mathbb{R})$ with abelian semigroups of order 3

The are 18 different semigroups of order 3, listed explicitly in Appendix C as $S^{\alpha}_{(3)}$, $\alpha = 1, \ldots, 18$. In Figure 8.2 we write only the label a for brevity. The horizontal axis represents the set of semigroups used in some specific expansion while the vertical axis represents the different kinds of expansions that can be performed. So,

[2] A semigroup can have more than one resonant decomposition leading then to different expanded algebras.

- in the first level we list all the abelian semigroups that allow us to perform a general expansion $S \otimes \mathfrak{g}$. Those are the abelian semigroups: $S^1_{(3)}$, $S^2_{(3)}$, $S^3_{(3)}$, $S^6_{(3)}$, $S^7_{(3)}$, $S^9_{(3)}$, $S^{10}_{(3)}$, $S^{12}_{(3)}$, $S^{15}_{(3)}$, $S^{16}_{(3)}$, $S^{17}_{(3)}$, $S^{18}_{(3)}$.
- In the second level we find all the abelian semigroups that contain at least one resonant decomposition so that a resonant subalgebra can be extracted from the expanded one. Those are the abelian semigroups: $S^1_{(3)}$, $S^2_{(3)}$, $S^3_{(3)}$, $S^6_{(3)}$, $S^{12}_{(3)}$, $S^{15}_{(3)}$, $S^{16}_{(3)}$, $S^{17}_{(3)}$.
- In the third level we see all the abelian semigroups that contain a zero element so that a reduced algebra can be extracted from the expanded one. Those are the semigroups: $S^1_{(3)}$, $S^2_{(3)}$, $S^3_{(3)}$, $S^6_{(3)}$, $S^7_{(3)}$, $S^9_{(3)}$, $S^{10}_{(3)}$, $S^{12}_{(3)}$.
- In the fourth level we find all the abelian semigroups that contain at least one resonant decomposition and also a zero element. So a reduced algebra can be obtained from the resonant subalgebra. The semigroups that allow us to do that are: $S^1_{(3)}$, $S^2_{(3)}$, $S^3_{(3)}$, $S^6_{(3)}$, $S^{12}_{(3)}$.

Then, for all these expansions, we identify the semigroups that preserve the semisimplicity of the original algebra. In the graphic of Figure 8.2 those semigroups are labeled with a red number.

8.2.2 General properties of the expansions with $n = 3, \dots, 6$

For higher orders $n \geq 4$ it is not possible to show a graphic like the one given in Figure 8.2. Instead we have Figure 8.3 that gives the number of abelian semigroups that lead to the different kinds of expansions that we have mentioned in the previous section. We also give in each case the number of semigroups preserving semisimplicity.

In the different rows we see the various kinds of expansions that can be done (the expanded algebra, the resonant subalgebra, the reduced algebra and the reduction of the resonant subalgebra) for each order of the semigroups. In each case the number of semigroups with which the expansions can be performed is specified. The number of semigroups preserving semisimplicity is also given. However, it may happen that a certain semigroup has more than one resonance, which then leads to different expanded algebras. We summarize all this information:

Order $n = 3$: There are 8 semigroups with resonant decomposition, 9 different resonant decompositions (so this is the number of different kinds

order	3	4	5	6
expanded $S\otimes\mathfrak{g}$ / preserving semisimplicity	#12 / #4	#58 / #16	#325 / #51	#2,143 / #201
expanded and reduced / preserving semisimplicity	#8 / #3	#39 / #9	#226 / #34	#1,538 / #135
resonant subalgebra / preserving semisimplicity	#8 / #1	#48 / #4	#299 / #7	#2,059 / #23
reduction of resonant subalg. / preserving semisimplicity	#5 / #1	#32 / #1	#204 / #6	#1,465 / #12

Fig. 8.3: Expansions of $\mathfrak{sl}(2,\mathbb{R})$ with abelian semigroups of order 3, 4, 5 and 6.

of expansions that can be made for $n = 3$) and 1 expansion that gives a semisimple Lie algebra with one of its resonances.

Order $n = 4$**:** There are 48 semigroups with resonant decomposition, 124 different resonant decompositions (so this is the number of different kinds of expansions that can be made for $n = 4$) and 4 expansions that give a semisimple Lie algebra.

Order $n = 5$**:** There are 299 semigroups with resonant decomposition, $1,653$ different resonant decompositions (so this is the number of different kinds of expansions that can be made for $n = 5$) and 7 expansions that give a semisimple Lie algebra.

Order $n = 6$**:** There are $2,059$ semigroups with resonant decomposition, $25,512$ different resonant decompositions (so this is the number of different kinds of expansions that can be made for $n = 6$) and 23 expansions that give a semisimple Lie algebra.

In general all these expansions with $n = 3, 4, 5, 6$ share the following property:

Consider a semigroup having more than one resonance. Then if it preserves semisimplicity, this happens just for one of its resonances. There is no semigroup preserving semisimplicity with more than one of its resonances.

It has been explicitly verified that starting from a semisimple algebra the expanded algebras are not necesarily semisimple, as was suggested in Section 8.1. In fact most of the expansions do not preserve semisimplicity.

9. S-RELATED LIE ALGEBRAS IN DIM $= 2, 3$

This Chapter is devoted to the study of the possibility to relate the 3-dimensional Lie algebras with the 2-dimensional ones in Bianchi's classification (see Section 2.12) through the S-expansion method. The results in this Chapter are new and have been collected in ref.[6], to appear soon. The programs used in the second part of the Chapter have been reviewed in ref.[24], which will also appear soon.

This Chapter is organized as follows:

In Section 9.1 it is shown in an instructive way how some types of 3-dimensional Lie algebras are related with 2-dimensional Lie algebras using known semigroups. We also use other semigroups that have not been used before in the applications of the S-expansion procedure. We then define an iterative procedure to relate 2-dimensional and 3-dimensional Lie algebras with a S-expansion[1]. In Subsection 9.1.4 we briefly summarize the results obtained by this procedure.

In Section 9.2 it is shown why is it not possible to obtain, by expansions, the other 3-dimensional isometries from the 2-dimensional algebras.

Finally in Section 9.3 we check the results using computer programs and solve the problem entirely.

9.1 The 3-dimensional algebras related with 2-dimensional Lie algebras

We start with a 2-dimensional Lie algebra and we are looking for a 3-dimensional one. The S-expanded algebra with a semigroup of order 4 is 8-dimensional and the reduced is 6-dimensional. So to obtain a 3-dimensional algebra we need to extract one smaller algebra. This is possible if there is a resonant subalgebra.

[1] In the first part of this Chapter we use a notation for the semigroups different to the one proposed in Chapter 8, due to historical reasons. In the end of this Chapter we identify these semigroups using our standard notation. The correspondence can be checked in Figure 9.4.

It is useful to remind Bianchi's classification of 3-dimensional Lie algebras in this point:

Group	**Algebra**
type I	$[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0$
type II	$[X_1, X_2] = [X_1, X_3] = 0, \quad [X_2, X_3] = X_1$
type III	$[X_1, X_2] = [X_2, X_3] = 0, \quad [X_1, X_3] = X_1$
type IV	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_1 + X_2$
type V	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_2$
type VI	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = hX_2,$ where $h \neq 0, 1$.
type VII$_1$	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1$
type VII$_2$	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1 + hX_2,$ where $h \neq 0$ $(0 < h < 2)$.
type VIII	$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3$
type IX	$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2$

Fig. 9.1: Bianchi's classification of 3-dimensional Lie algebras.

9.1.1 *The type III Lie algebra*

We want to perform the S-expansion of the Lie algebra

$$[X_1, X_2] = X_1$$

and study the 3-dimensional algebras that we can get.

Let us begin with the semigroup $S = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ defined by the following conditions:

I) λ_4 is a zero of the semigroup, so the table of multiplication law is of the generic form

	λ_1	λ_2	λ_3	λ_4
λ_1				λ_4
λ_2				λ_4
λ_3				λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

,

where the empty spaces must be filled in a way such that it is an abelian semigroup, i.e., closed, associative and commutative.

In order to get a smaller algebra we also demand

II) that it contains a resonant decomposition

$$\begin{aligned} S_0 &= \{\lambda_2, \lambda_3, \lambda_4\}\,, \\ S_1 &= \{\lambda_1, \lambda_4\}\,. \end{aligned} \tag{9.1}$$

Then, the resonant subalgebra of $\mathfrak{g}_{S,R} = S \times \mathfrak{g}$ is generated by

$$\begin{aligned} \mathfrak{g}_{S,R} &= (S_0 \times V_0) \oplus (S_1 \times V_1) \\ &= \{\lambda_2 \times X_2,\ \lambda_3 \times X_2,\ \lambda_4 \times X_2\} \oplus \{\lambda_1 \times X_1,\ \lambda_4 \times X_1\} \\ &= \{\lambda_2 \times X_2,\ \lambda_3 \times X_2,\ \lambda_4 \times X_2,\ \lambda_1 \times X_1,\ \lambda_4 \times X_1\}\,. \end{aligned} \tag{9.2}$$

Now we extract an even smaller algebra by means of a 0_S-reduction. This is done by just cancelling from (9.2) the generators that contain the zero element, λ_4. Therefore, the reduction of the resonant subalgebra is given by

$$\mathfrak{g}^{\text{red}}_{S,R} = \{\lambda_2 \times X_2,\ \lambda_3 \times X_2,\ \lambda_1 \times X_1\}$$

with the following commutation relations

$$\begin{aligned} [\lambda_2 \times X_2, \lambda_3 \times X_2] &= 0, \\ [\lambda_2 \times X_2, \lambda_1 \times X_1] &= -\lambda_1 \cdot \lambda_2 \times X_1, \\ [\lambda_3 \times X_2, \lambda_1 \times X_1] &= -\lambda_1 \cdot \lambda_3 \times X_1. \end{aligned}$$

In order to close the algebra we should choose:

(a) $\lambda_1 \cdot \lambda_2 = \lambda_1$ or $\lambda_1 \cdot \lambda_2 = \lambda_4$ and

(b) $\lambda_1 \cdot \lambda_3 = \lambda_1$ or $\lambda_1 \cdot \lambda_3 = \lambda_4$.

So we are left with four possibilities to construct a closed algebra:

(i) $\lambda_1 \cdot \lambda_2 = \lambda_1 \cdot \lambda_3 = \lambda_4$,

(ii) $\lambda_1 \cdot \lambda_2 = \lambda_1$ and $\lambda_1 \cdot \lambda_3 = \lambda_4$,

(iii) $\lambda_1 \cdot \lambda_2 = \lambda_4$ and $\lambda_1 \cdot \lambda_3 = \lambda_1$,

(iv) $\lambda_1 \cdot \lambda_2 = \lambda_1 \cdot \lambda_3 = \lambda_1$.

The case (i) will lead to translations in 3 dimensions, i.e., to the type I algebra

$$[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0.$$

It can be checked that the case (*iv*) is not useful, because it leads to a non associative product. On the other hand, it can be seen that both (*ii*) and (*iii*) will lead to the type III algebra. In fact, in case (*ii*) we have

$$
\begin{aligned}
&[\lambda_2 \times X_2, \lambda_3 \times X_2] = 0, \\
&[\lambda_2 \times X_2, \lambda_1 \times X_1] = -\lambda_1 \times X_1, \\
&[\lambda_3 \times X_2, \lambda_1 \times X_1] = -\lambda_4 \times X_1 = 0. \ (\lambda_4 \text{ is a zero element})
\end{aligned}
$$

Renaming the generators as

$$
\begin{aligned}
Y_1 &= \lambda_1 \times X_1, \\
Y_2 &= \lambda_3 \times X_2, \\
Y_3 &= \lambda_2 \times X_2,
\end{aligned}
$$

we immediately recognize the type III algebra (see table (2.8)).

In case (*iii*) we would have

$$
\begin{aligned}
&[\lambda_2 \times X_2, \lambda_3 \times X_2] = 0, \\
&[\lambda_2 \times X_2, \lambda_1 \times X_1] = -\lambda_4 \times X_1 = 0 \ (\lambda_4 \text{ is a zero element}), \\
&[\lambda_3 \times X_2, \lambda_1 \times X_1] = -\lambda_1 \times X_1,
\end{aligned}
$$

and we recover again the type III algebra by renaming the generators as

$$
\begin{aligned}
Y_1 &= \lambda_1 \times X_1, \\
Y_2 &= \lambda_2 \times X_2, \\
Y_3 &= \lambda_3 \times X_2.
\end{aligned}
$$

We will study now case (*iii*) to construct a semigroup that leads to the type III algebra. The case (*ii*) includes other semigroups leading to the same result. The table describing the multiplication law case (*iii*) is of the form

	λ_1	λ_2	λ_3	λ_4
λ_1		λ_4	λ_1	λ_4
λ_2	λ_4			λ_4
λ_3	λ_1			λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

, (9.3)

where the empty spaces must be filled in such a way that associativity is satisfied and the decomposition (9.1) satisfies the resonant condition.

There are semigroups that fit the multiplication table (9.3) and the resonant condition (9.1). We give several examples of semigroups of this type.

Semigroup $S_K^{(3)}$

Consider the series of semigrous $S_K^{(n)}$, with the multiplication law defined by

$$\begin{aligned}\lambda_\alpha \cdot \lambda_\beta &= \lambda_{\min\{\alpha,\beta\}}, \quad \alpha+\beta > n,\\ \lambda_\alpha \cdot \lambda_\beta &= \lambda_{n+1}, \qquad\quad \alpha+\beta \le n,\\ \alpha,\beta &= 1,2,\ldots,n.\end{aligned} \tag{9.4}$$

It is directly seen that for $n = 3$ the multiplication table

	λ_1	λ_2	λ_3	λ_4
λ_1	λ_4	λ_4	λ_1	λ_4
λ_2	λ_4	λ_2	λ_2	λ_4
λ_3	λ_1	λ_2	λ_3	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

fits with the form of the table (9.3). Therefore, an expansion with the semigroup $S_K^{(3)}$ reproduces the type III algebra after a 0_S-reduction of the resonant subalgebra.

Semigroup S_{N1}

Let us consider the following multiplication table:

	λ_1	λ_2	λ_3	λ_4
λ_1	λ_4	λ_4	λ_1	λ_4
λ_2	λ_4	λ_2	λ_4	λ_4
λ_3	λ_1	λ_4	λ_3	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

.

Here we have filled the empty spaces of (9.3) in another way, obtaining another semigroup that verifies the required properties. It can be directly shown that this multiplication table is associative, satisfies the resonant condition and fits the form of the table (9.3). The proof is direct but a little tedious. Therefore the semigroup S_{N1} also reproduces the type III algebra, although it is not isomorphic to the previous semigroup, $S_K^{(3)}$.

The semigroup $S_E^{(2)}$, another way to obtain the type III algebra

We consider now the series of semigroups $S_E^{(n)}$ introduced in ref.[37] with $n = 2$. Its multiplication law is given by the following table

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	λ_3
λ_2	λ_2	λ_3	λ_3	λ_3
λ_3	λ_3	λ_3	λ_3	λ_3

and its resonant decomposition is

$$\begin{aligned} S_0 &= \{\lambda_0, \lambda_2, \lambda_3\}\,, \\ S_1 &= \{\lambda_1, \lambda_3\}\,. \end{aligned}$$

The 0_S-reduction of the resonant subalgebra is given by

$$\mathfrak{g}_{S,R}^{\mathrm{red}} = \{\lambda_0 \times X_2,\ \lambda_2 \times X_2,\ \lambda_1 \times X_1\}$$

with commutation relations

$$\begin{aligned} &[\lambda_0 \times X_2, \lambda_2 \times X_2] = 0, \\ &[\lambda_0 \times X_2, \lambda_1 \times X_1] = -\lambda_1 \times X_1, \\ &[\lambda_2 \times X_2, \lambda_1 \times X_1] = 0. \end{aligned}$$

Renaming the generators as

$$\begin{aligned} Y_1 &= \lambda_1 \times X_1, \\ Y_2 &= \lambda_2 \times X_2, \\ Y_3 &= \lambda_0 \times X_2, \end{aligned}$$

we obtain again the type III algebra

$$[Y_1, Y_2] = [Y_2, Y_3] = 0, \qquad [Y_1, Y_3] = Y_1.$$

9.1.2 The type II and V algebras

The natural question here is if it is possible to generate other algebras starting from the 2-dimensional Lie algebras. To answer this question we will continue the procedure of Subsection 9.1.1, considering a semigroup $S = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ where λ_4 is a zero element, but we will use a different resonant decomposition:

$$\begin{aligned} S_0 &= \{\lambda_2, \lambda_4\}\,, \\ S_1 &= \{\lambda_1, \lambda_3, \lambda_4\}\,. \end{aligned} \tag{9.5}$$

The reduction of the resonant subalgebra is given by

$$\mathfrak{g}^{\text{red}}_{S,R} = \{\lambda_2 \times X_2,\ \lambda_1 \times X_1,\ \lambda_3 \times X_1\}$$

and the commutation relations

$$\begin{aligned} [\lambda_2 \times X_2, \lambda_1 \times X_1] &= -\lambda_1 \cdot \lambda_2 \times X_1, \\ [\lambda_2 \times X_2, \lambda_3 \times X_1] &= -\lambda_2 \cdot \lambda_3 \times X_1, \\ [\lambda_1 \times X_1, \lambda_3 \times X_1] &= 0. \end{aligned} \tag{9.6}$$

The resonant condition guarantees that (9.6) is a closed algebra. Here we have different possibilities.

Type II algebra and the S_{N2} semigroup

To reproduce the type II algebra we have to choose, for example,

$$\lambda_1 \cdot \lambda_2 = \lambda_3 \qquad \text{and} \qquad \lambda_2 \cdot \lambda_3 = \lambda_4. \tag{9.7}$$

In that case the commutation relations (9.6) take the form

$$\begin{aligned} [\lambda_2 \times X_2, \lambda_1 \times X_1] &= -\lambda_3 \times X_1, \\ [\lambda_2 \times X_2, \lambda_3 \times X_1] &= 0, \\ [\lambda_1 \times X_1, \lambda_3 \times X_1] &= 0, \end{aligned}$$

and renaming the generators as

$$\begin{aligned} Y_1 &= \lambda_3 \times X_1, \\ Y_2 &= \lambda_1 \times X_1, \\ Y_3 &= \lambda_2 \times X_2, \end{aligned}$$

we obtain the type II algebra

$$[Y_1, Y_2] = [Y_1, Y_3] = 0, \qquad [Y_2, Y_3] = Y_1.$$

But in order for this result to be true, we must provide an explicit semigroup that satisfies the conditions (9.7). Until now our table has the form

	λ_1	λ_2	λ_3	λ_4
λ_1		λ_3		λ_4
λ_2	λ_3		λ_4	λ_4
λ_3		λ_4		λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

(9.8)

and the empty spaces must be filled in such a way that it defines an associative, commutative product and such that the decomposition (9.5) satisfies the resonant condition.

After looking for different possibilities we have found one way to fill the multiplication table. The proposed semigroup is S_{N2}:

	λ_1	λ_2	λ_3	λ_4
λ_1	λ_2	λ_3	λ_4	λ_4
λ_2	λ_3	λ_4	λ_4	λ_4
λ_3	λ_4	λ_4	λ_4	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

. (9.9)

This multiplication table represents in fact an abelian semigroup. The associativity is proved by a tedious but direct calculation.

Note that there may be other semigroups that can also lead to the type II algebra. Those correspond to other ways to fill the empty spaces in table (9.8).

Type V and the S_{N3} semigroup

If we choose

$$\lambda_1 \cdot \lambda_2 = \lambda_1 \qquad \text{and} \qquad \lambda_2 \cdot \lambda_3 = \lambda_3 \tag{9.10}$$

the commutation relations (9.6) take the form

$$\begin{aligned}
[\lambda_2 \times X_2, \lambda_1 \times X_1] &= -\lambda_1 \times X_1, \\
[\lambda_2 \times X_2, \lambda_3 \times X_1] &= -\lambda_3 \times X_1, \\
[\lambda_1 \times X_1, \lambda_3 \times X_1] &= 0,
\end{aligned}$$

and renaming the generators as

$$\begin{aligned} Y_1 &= \lambda_1 \times X_1, \\ Y_2 &= \lambda_3 \times X_1, \\ Y_3 &= \lambda_2 \times X_2, \end{aligned}$$

we obtain the type V algebra

$$[Y_1, Y_2] = 0, \qquad [Y_1, Y_3] = Y_1, \qquad [Y_2, Y_3] = Y_2.$$

We must now provide of an explicit semigroup that satisfies the conditions (9.10). Until now our table has the form

	λ_1	λ_2	λ_3	λ_4
λ_1		λ_1		λ_4
λ_2	λ_1		λ_3	λ_4
λ_3		λ_3		λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

(9.11)

and the empty spaces must be filled in a way that respects the required conditions. Note that there are $4^4 = 256$ possibilities to fill this table in a closed way. This number is reduced by imposing associativity, commutativity and the resonant condition for the decomposition (9.5). There are semigroups that fit multiplication table (9.11) and resonant condition (9.5).

One way to fill the multiplication table (9.11) is the S_{N3} semigroup:

	λ_1	λ_2	λ_3	λ_4
λ_1	λ_4	λ_1	λ_4	λ_4
λ_2	λ_1	λ_2	λ_3	λ_4
λ_3	λ_4	λ_3	λ_4	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

.

We point out again that there may be other semigroups that can also lead to the type V algebra. Those correspond to other ways to fill the empty spaces in the multiplication table (9.11).

9.1.3 The type I algebra

Starting from the abelian 2-dimensional algebra

$$[X_1, X_2] = 0 \tag{9.12}$$

we note that it also possesses a graded subspace structure where $V_0 = \{X_2\}$ and $V_1 = \{X_1\}$. So, for example, by choosing the semigroup $S_K^{(3)}$ or S_{N1}, both having a resonant decomposition of the form

$$\begin{aligned} S_0 &= \{\lambda_2, \lambda_3, \lambda_4\}, \\ S_1 &= \{\lambda_1, \lambda_4\}, \end{aligned} \tag{9.13}$$

we obtain the reduction of the resonant subalgebra

$$\mathfrak{g}_{S,R}^{\mathrm{red}} = \{\lambda_2 \times X_2,\ \lambda_3 \times X_2,\ \lambda_1 \times X_1\}, \tag{9.14}$$

with the following commutation relations:

$$\begin{aligned} [\lambda_2 \times X_2, \lambda_3 \times X_2] &= \lambda_2 \cdot \lambda_3 \times [X_2, X_2] = 0, \\ [\lambda_2 \times X_2, \lambda_1 \times X_1] &= \lambda_1 \cdot \lambda_2 \times [X_2, X_1] = 0, \\ [\lambda_3 \times X_2, \lambda_1 \times X_1] &= \lambda_1 \cdot \lambda_3 \times [X_2, X_1] = 0. \end{aligned} \tag{9.15}$$

We see that it does not matter if we use the semigroup $S_K^{(3)}$ or S_{N1}, the result is always an abelian algebra in 3 dimensions because the original algebra is abelian. The same result can be reached with the semigroup $S_E^{(2)}$ whose semigroup decomposition is similar to (9.13).

Also, by using the semigroups S_{N2}, S_{N3} and probably others that have a resonant decomposition of the form

$$\begin{aligned} S_0 &= \{\lambda_2, \lambda_4\}, \\ S_1 &= \{\lambda_1, \lambda_3, \lambda_4\}, \end{aligned} \tag{9.16}$$

we obtain a reduction of the resonant subalgebra

$$\mathfrak{g}_{S,R}^{\mathrm{red}} = \{\lambda_2 \times X_2,\ \lambda_1 \times X_1,\ \lambda_3 \times X_1\},$$

whose commutation relations

$$\begin{aligned} [\lambda_2 \times X_2, \lambda_1 \times X_1] &= \lambda_1 \cdot \lambda_2 \times [X_2, X_1] = 0, \\ [\lambda_2 \times X_2, \lambda_3 \times X_1] &= \lambda_2 \cdot \lambda_3 \times [X_2, X_1] = 0, \\ [\lambda_1 \times X_1, \lambda_3 \times X_1] &= \lambda_1 \cdot \lambda_3 \times [X_1, X_1] = 0, \end{aligned}$$

are again the ones of a 3-dimensional abelian algebra.

So we conclude that starting from the abelian Lie algebra (9.12) whatever semigroup with a zero element and that has a resonant decomposition of the form (9.13) or (9.16) will lead to the type I algebra. Moreover, this result can be generalized:

An abelian Lie algebra in d dimensions can be obtained as an expansion of the abelian algebra in 2-dimensions by using an abelian semigroup with probably a zero element and a suitable resonant decomposition. By suitable we mean that using the resonant subalgebra plus the reduction by the zero elements gives the correct dimension.

Note that a crucial property to relate a 3-dimensional algebra (whichever type I, II, III and V) with a 2-dimensional algebra is the existence of the resonant subalgebra and the 0_S-reduction. This is the only way to obtain three generators starting from two.

9.1.4 Brief summary

Starting from

$$[X_1, X_2] = 0 \tag{9.17}$$

it is possible to obtain the type I abelian algebra in three dimensions using many semigroups as for example $S_E^{(2)}$, $S_K^{(3)}$, S_{N1}, S_{N2}, S_{N3} and probably others. Now starting from

$$[X_1, X_2] = X_1, \tag{9.18}$$

it is also possible to obtain the type I abelian algebra in three dimensions using for example a semigroup whose multiplication satisfies the condition (i) of Section 9.1.1, i.e., whose table has the form

	λ_1	λ_2	λ_3	λ_4
λ_1		λ_4	λ_4	λ_4
λ_2	λ_4			λ_4
λ_3	λ_4			λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

,

where the empty spaces must be filled with the corresponding conditions of associativity, resonant condition and reduction condition.

The semigroups which can be used to get the type I, II, III and V algebra starting from the 2-dimensional algebra (9.18) appear in the following table:

Algebra	**Semigroup used**
Type I	any semigroup with a 0-element and a resonant decomposition
Type II	S_{N2} and probably others
Type III	$S_E^{(2)}$, $S_K^{(3)}$, S_{N1} and probably others
Type V	S_{N3} and probably others

. (9.19)

The above semigroups are described in Figure 9.2.

9.2 *The Bianchi spaces not-related with 2-dimensional isometries*

9.2.1 *Type IV, VI, VII$_2$, VIII and IX algebras*

Let us consider for example the type IV algebra

$$[Y_1, Y_2] = 0, \tag{9.20}$$
$$[Y_1, Y_3] = Y_1, \tag{9.21}$$
$$[Y_2, Y_3] = Y_1 + Y_2. \tag{9.22}$$

As the S-expansion method uses an induced bracket

$$[\lambda_\alpha \times X_i, \lambda_\beta \times X_j] = \lambda_\alpha \cdot \lambda_\beta \times [X_i, X_j] = \lambda_{\gamma(\alpha,\beta)} \times [X_i, X_j]$$

for the expanded algebra, and considering that our original algebra has $i, j = 1, 2$ and commutation relation

$$[X_1, X_2] = X_1,$$

we have that the first two relations (9.20, 9.21) can be easily reproduced, but to reproduce (9.22) we must have a relation like

$$[\lambda_\alpha \times X_1, \lambda_\beta \times X_2] = \lambda_\alpha \cdot \lambda_\beta \times [X_1, X_2] = \lambda_{\gamma(\alpha,\beta)} \times X_1.$$

And here we can see that no matter which semigroup we choose, $\lambda_{\gamma(\alpha,\beta)}$ will always be an element of the semigroup and therefore we will be able to reproduce a sum of two generators.

Semigroup	Multiplication table					Res. decomposition	0_S
$S_E^{(2)}$		λ_0	λ_1	λ_2	λ_3	$S_0 = \{\lambda_0, \lambda_2, \lambda_3\}$, $S_1 = \{\lambda_1, \lambda_3\}$	λ_3
	λ_0	λ_0	λ_1	λ_2	λ_3		
	λ_1	λ_1	λ_2	λ_3	λ_3		
	λ_2	λ_2	λ_3	λ_3	λ_3		
	λ_3	λ_3	λ_3	λ_3	λ_3		
$S_K^{(3)}$		λ_1	λ_2	λ_3	λ_4	$S_0 = \{\lambda_2, \lambda_3, \lambda_4\}$, $S_1 = \{\lambda_1, \lambda_4\}$	λ_4
	λ_1	λ_4	λ_4	λ_1	λ_4		
	λ_2	λ_4	λ_2	λ_2	λ_4		
	λ_3	λ_1	λ_2	λ_3	λ_4		
	λ_4	λ_4	λ_4	λ_4	λ_4		
S_{N1}		λ_1	λ_2	λ_3	λ_4	$S_0 = \{\lambda_2, \lambda_3, \lambda_4\}$, $S_1 = \{\lambda_1, \lambda_4\}$	λ_4
	λ_1	λ_4	λ_4	λ_1	λ_4		
	λ_2	λ_4	λ_2	λ_4	λ_4		
	λ_3	λ_1	λ_4	λ_3	λ_4		
	λ_4	λ_4	λ_4	λ_4	λ_4		
S_{N2}		λ_1	λ_2	λ_3	λ_4	$S_0 = \{\lambda_2, \lambda_4\}$, $S_1 = \{\lambda_1, \lambda_3, \lambda_4\}$	λ_4
	λ_1	λ_2	λ_3	λ_4	λ_4		
	λ_2	λ_3	λ_4	λ_4	λ_4		
	λ_3	λ_4	λ_4	λ_4	λ_4		
	λ_4	λ_4	λ_4	λ_4	λ_4		
S_{N3}		λ_1	λ_2	λ_3	λ_4	$S_0 = \{\lambda_2, \lambda_4\}$, $S_1 = \{\lambda_1, \lambda_3, \lambda_4\}$	λ_4
	λ_1	λ_4	λ_1	λ_4	λ_4		
	λ_2	λ_1	λ_2	λ_3	λ_4		
	λ_3	λ_4	λ_3	λ_4	λ_4		
	λ_4	λ_4	λ_4	λ_4	λ_4		

Fig. 9.2: Description of some of the semigroups used in this Chapter.

Now consider the type VI algebra:

$$[Y_1, Y_2] = 0,$$
$$[Y_1, Y_3] = Y_1,$$
$$[Y_2, Y_3] = hY_2, \quad h \neq 0, 1.$$

Again the first two brackets could be reproduced by a certain semigroup, but for the third one we would have something like

$$[\lambda_\alpha \times X_1, \lambda_\beta \times X_2] = \lambda_{\gamma(\alpha,\beta)} \times X_1,$$

and again, no matter which semigroup we choose, $\lambda_{\gamma(\alpha,\beta)}$ will always be an element of the semigroup and we will never be able to reproduce a semigroup element multiplied by a numeric factor. A similar argument can be used to show that type VII_1 algebra cannot be obtained by the S-expansion procedure.

A mixture of the above arguments also explain why it is impossible to obtain the type VII_2 and VIII algebras as an expansion of a 2-dimensional algebra.

Finally, to show why it is also impossible to reproduce the type IX algebra

$$[Y_1, Y_2] = Y_3, \quad [Y_2, Y_3] = Y_1, \quad [Y_3, Y_1] = Y_2,$$

we have to realize that the candidate for the expanded algebra will have three commutation relations of the form

$$[\lambda_\alpha \times X_i, \lambda_\beta \times X_j] = \lambda_{\gamma(\alpha,\beta)} \times [X_i, X_j]$$

but where i, j takes the values 1 and 2. Therefore, in one of the three commutation relations one index will always be repeated leading to a vanishing bracket. So it is impossible to generate, by means of an S-expansion, a 3-dimensional algebra with the three brackets having a nonzero value.

Thus we conclude that these types of algebra, that cannot be obtained by an expansion of the 2-dimensional algebras, are in some sense intrinsic to 3 dimensions.

9.3 Checking with computer programs

A common question when working with semigroups in Section 9.1 is that of the existence of diverse semigroups given some elements of the multiplication table. We have, for example, a table like this one:

	λ_1	λ_2	λ_3	λ_4
λ_1		λ_4	λ_1	λ_4
λ_2	λ_4			λ_4
λ_3	λ_1			λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

.

In principle there are 256 different symmetric matrices which fill this template, but not all of them will be semigroups because the multiplication table will not always be associative. Moreover, we have to select only those that satisfy a certain *resonant condition*. Finally, many of these associative tables will be isomorphic, so we have to select one representant in each isomorphism class.

In what follows we find all the non isomorphic ways to fill the tables (9.3), (9.8) and (9.11) with the mentioned conditions and show that all the semigroups given in table (9.19) (those semigroups that we have constructed by hand) are isomorphic to one of the semigroups given by the computer program *com.f* of ref.[44].

We have developed a Java library [24] to answer these questions. The programs used in this Section are reviewed in Appendix B.

9.3.1 Type II

The template is:

	λ_1	λ_2	λ_3	λ_4
λ_1		λ_3		λ_4
λ_2	λ_3		λ_4	λ_4
λ_3		λ_4		λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

.

By using computer programs, we have found that there are two non isomorphic ways to fill this template such that: *a*) the resulting table is an abelian semigroup and *b*) the resonant decomposition is given by

$$S_0 = \{\lambda_2, \lambda_4\}, \qquad S_1 = \{\lambda_1, \lambda_3, \lambda_4\}. \tag{9.23}$$

Those are:

$$\begin{array}{|c||c|c|c|c|}\hline S_{II}^1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \hline\hline \lambda_1 & \lambda_4 & \lambda_3 & \lambda_4 & \lambda_4 \\ \hline \lambda_2 & \lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 \\ \hline \lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ \hline \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ \hline \end{array}, \qquad \begin{array}{|c||c|c|c|c|}\hline S_{II}^2 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \hline\hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_4 \\ \hline \lambda_2 & \lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 \\ \hline \lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ \hline \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ \hline \end{array}. \tag{9.24}$$

Each of them is isomorphic to one of the semigroups of the list given by the program *com.f* of ref.[44] for $n = 4$. We give this information in the following table,

$$\begin{array}{ccc|c} & \text{Isomorphic to} & & \text{Isomorphism} \\ \hline S_{II}^1 & \approx & S_{(4)}^{10} & (\lambda_4\ \lambda_3\ \lambda_1\ \lambda_2) \\ S_{II}^2 & \approx & S_{(4)}^{12} & (\lambda_4\ \lambda_3\ \lambda_2\ \lambda_1) \end{array}. \tag{9.25}$$

where the isomorphism denoted by $(\lambda_a\ \lambda_b\ \lambda_c\ \lambda_d)$ means: change λ_1 by λ_a, λ_2 by λ_b, λ_3 by λ_c and λ_4 by λ_d. The semigroups $S_{(4)}^{10}$ and $S_{(4)}^{12}$ of the list given by the program *com.f* for $n = 4$ are:

$$\begin{array}{|c||c|c|c|c|}\hline S_{(4)}^{10} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \hline\hline \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\ \hline \lambda_2 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\ \hline \lambda_3 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_2 \\ \hline \lambda_4 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_1 \\ \hline \end{array}, \qquad \begin{array}{|c||c|c|c|c|}\hline S_{(4)}^{12} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \hline\hline \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\ \hline \lambda_2 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\ \hline \lambda_3 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_2 \\ \hline \lambda_4 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_3 \\ \hline \end{array}. \tag{9.26}$$

It can be checked directly that applying the isomorphism $(\lambda_4\ \lambda_3\ \lambda_1\ \lambda_2)$ to S^{10} one obtains S_{II}^1 and applying the isomorphism $(\lambda_4\ \lambda_3\ \lambda_2\ \lambda_1)$ to S^{12} one obtains S_{II}^2.

9.3.2 Type III

The template is:

$$\begin{array}{|c||c|c|c|c|}\hline & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \hline\hline \lambda_1 & & \lambda_4 & \lambda_1 & \lambda_4 \\ \hline \lambda_2 & \lambda_4 & & & \lambda_4 \\ \hline \lambda_3 & \lambda_1 & & & \lambda_4 \\ \hline \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ \hline \end{array}.$$

We have found that there are 7 non isomorphic ways to fill this template such that: *a*) the resulting table is an abelian semigroup and *b*) the resonant decomposition is given by

$$S_0 = \{\lambda_2, \lambda_3, \lambda_4\}, \qquad S_1 = \{\lambda_1, \lambda_4\}. \tag{9.27}$$

Those ways are:

S_{III}^1	λ_1	λ_2	λ_3	λ_4
λ_1	λ_4	λ_4	λ_1	λ_4
λ_2	λ_4	λ_4	λ_4	λ_4
λ_3	λ_1	λ_4	λ_3	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

,

S_{III}^2	λ_1	λ_2	λ_3	λ_4
λ_1	λ_3	λ_4	λ_1	λ_4
λ_2	λ_4	λ_4	λ_4	λ_4
λ_3	λ_1	λ_4	λ_3	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

,

S_{III}^3	λ_1	λ_2	λ_3	λ_4
λ_1	λ_4	λ_4	λ_1	λ_4
λ_2	λ_4	λ_4	λ_2	λ_4
λ_3	λ_1	λ_2	λ_3	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

,

S_{III}^4	λ_1	λ_2	λ_3	λ_4
λ_1	λ_2	λ_4	λ_1	λ_4
λ_2	λ_4	λ_4	λ_2	λ_4
λ_3	λ_1	λ_2	λ_3	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

,

S_{III}^5	λ_1	λ_2	λ_3	λ_4
λ_1	λ_4	λ_4	λ_1	λ_4
λ_2	λ_4	λ_2	λ_4	λ_4
λ_3	λ_1	λ_4	λ_3	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

,

S_{III}^6	λ_1	λ_2	λ_3	λ_4
λ_1	λ_4	λ_4	λ_1	λ_4
λ_2	λ_4	λ_2	λ_2	λ_4
λ_3	λ_1	λ_2	λ_3	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

,

S_{III}^7	λ_1	λ_2	λ_3	λ_4
λ_1	λ_4	λ_4	λ_1	λ_4
λ_2	λ_4	λ_2	λ_4	λ_4
λ_3	λ_1	λ_4	λ_3	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

.

As before, each of these semigroups is isomorphic to one of the semigroups in the list given by the program *com.f* of ref.[44] for $n = 4$. The semigroups and the corresponding isomorphisms are given in the following table:

$$\begin{array}{ccc|l} & \text{Isomorphic to} & & \text{Isomorphism} \\ \hline S^1_{III} & \approx & S^{13}_{(4)} & (\lambda_4\ \lambda_2\ \lambda_1\ \lambda_3) \\ S^2_{III} & \approx & S^{28}_{(4)} & (\lambda_4\ \lambda_2\ \lambda_3\ \lambda_1) \\ S^3_{III} & \approx & S^{42}_{(4)} & (\lambda_4\ \lambda_1\ \lambda_2\ \lambda_3) \\ S^4_{III} & \approx & S^{43}_{(4)} & (\lambda_4\ \lambda_2\ \lambda_1\ \lambda_3) \\ S^5_{III} & \approx & S^{44}_{(4)} & (\lambda_4\ \lambda_1\ \lambda_2\ \lambda_3) \\ S^6_{III} & \approx & S^{45}_{(4)} & (\lambda_4\ \lambda_1\ \lambda_2\ \lambda_3) \\ S^7_{III} & \approx & S^{64}_{(4)} & (\lambda_4\ \lambda_2\ \lambda_3\ \lambda_1) \end{array}, \tag{9.28}$$

where the semigroups $S^{13}_{(4)}$, $S^{28}_{(4)}$, $S^{42}_{(4)}$, $S^{43}_{(4)}$, $S^{44}_{(4)}$, $S^{45}_{(4)}$ y $S^{64}_{(4)}$, of the list generated by the program *com.f* for $n = 4$, are explicitly given in Appendix C.

9.3.3 Type V

The template is:

	λ_1	λ_2	λ_3	λ_4
λ_1		λ_1		λ_4
λ_2	λ_1		λ_3	λ_4
λ_3		λ_3		λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

.

In this case we have found that there is just one way to fill this template such that: *a*) the resulting table is an abelian semigroup and *b*) the resonant decomposition is given by

$$S_0 = \{\lambda_2, \lambda_4\}, \qquad S_1 = \{\lambda_1, \lambda_3, \lambda_4\}. \tag{9.29}$$

This is:

S_V	λ_1	λ_2	λ_3	λ_4
λ_1	λ_4	λ_1	λ_4	λ_4
λ_2	λ_1	λ_2	λ_3	λ_4
λ_3	λ_4	λ_3	λ_4	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

.

This table is isomorphic to the semigroup $S^{42}_{(4)}$ given in Appendix C. The isomorphism is given by

$$(\lambda_4\ \lambda_1\ \lambda_3\ \lambda_2). \tag{9.30}$$

Algebra	Semigroup used
Type I	many semigroups (see Subsection 9.1.3)
Type II	$S_{(4)}^{10}$, $S_{(4)}^{12}$
Type III	$S_{(4)}^{13}$, $S_{(4)}^{28}$, $S_{(4)}^{42}$, $S_{(4)}^{43}$, $S_{(4)}^{44}$, $S_{(4)}^{45}$ and $S_{(4)}^{64}$
Type V	$S_{(4)}^{42}$

Fig. 9.3: Summary of the results in this Chapter.

Note that the semigroup $S_{(4)}^{42}$ also allows us to obtain the type III algebra. So we can ask, *how can the same semigroup lead at the same time to type III and V algebras?* The reason is that this semigroup has two different resonant decompositions, (9.23) and (9.27). Each of them leads, after the reduction, to completely different algebras.

9.3.4 Isomorphisms and consistency of the procedure

In the following table we summarize our results by specifying *all* the non isomorphic semigroups that allow us to generate the type I, II, III and the V algebra starting from the 2-dimensional algebra (9.18):

For consistency we should prove that each semigroup of the table (9.19) (which we have constructed by hand in Section 9.1) is isomorphic to one of the semigroups of table (9.3) that we have found by using computer programs. This information is given in the following table:

Isomorphic to			Isomorphism
S_{N1}	$\approx$	$S_{(4)}^{44}$	$(\lambda_4\ \lambda_1\ \lambda_2\ \lambda_3)$
S_{N2}	$\approx$	$S_{(4)}^{12}$	$(\lambda_4\ \lambda_3\ \lambda_2\ \lambda_1)$
S_{N3}	$\approx$	$S_{(4)}^{42}$	$(\lambda_4\ \lambda_1\ \lambda_3\ \lambda_2)$
$S_E^{(2)}$	$\approx$	$S_{(4)}^{43}$	$(\lambda_4\ \lambda_3\ \lambda_2\ \lambda_1)$
$S_K^{(3)}$	$\approx$	$S_{(4)}^{45}$	$(\lambda_4\ \lambda_2\ \lambda_1\ \lambda_3)$

Fig. 9.4: Identification of the semigroups used in this Chapter with the semigroups generated by *gen.f*.

All the semigroups used in this Chapter are shown in Appendix C.

10. CONCLUSIONS

Here we summarize the results presented in this Thesis.

We have computed an explicit formula for that star product on Minkowski space which has several properties:

- It can be extended to a star product on the conformal space $G(2,4)$. This is done by gluing the star products computed in each open set (2.19).
- It can be extended to act on smooth functions as a differential star product.
- The Poisson bracket is quadratic in the coordinates.
- There is a coaction of the quantum Poincaré group (or the conformal group in the case of the conformal spacetime) on the star product algebra.
- It has at least two real forms corresponding to the Euclidean and Minkowskiam signatures.
- It can be extended to the superspace (to chiral and real superfields).

Since fields are smooth functions, the differentiability of the star product gives a hope that one can develop a quantum deformed field theory, that is, a field theory on the quantum deformed Minkowski space. The departure point will be to find a generalization of the Laplacian and the Dirac operator associated to the quantum invariant C_q.

One advantage of using the quantum group $\mathrm{SL}_q(4,\mathbb{C})$ is that the coalgebra structure is isomorphic to the coalgebra of the classical group $\mathrm{SL}(4,\mathbb{C})$ (see for example Theorem 6.1.8 in ref.[46]). This means that the group law is unchanged, so the Poincaré symmetry principle of the field theory would be preserved in the quantum deformed case. Those results have been published in ref.[2].

We have defined what we call *Invariant Sigma Models* (ISM). These models can be defined in the series of coset spaces $\mathrm{SO}(2,n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$ due to the existence of a left-invariant 2-form. We have built explicitely the $\mathrm{SO}(2,1)/\mathrm{SO}(2)$, $\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)$ and $\mathrm{SO}(2,3)/\mathrm{SO}(2)\times\mathrm{SO}(3)$ models.

We have discussed exhaustively the $\mathrm{SO}(2,1)/\mathrm{SO}(2)$, comparing it to the $\mathrm{SO}(2,1)/\mathrm{SO}(2)^R$ gauged Wess-Zumino-Witten model. We have showed that, although these models coincide if there is no antisymmetric form, in general they are different. The $\mathrm{SO}(2,1)/\mathrm{SO}(2)$ ISM does not show conformal invariance at a quantum level, even adding a non trivial dilaton.

We discuss the contraction of ISM. We have defined the method to deform invariant tensors and we have applied it to the contraction of $\mathrm{SO}(2,3)/\mathrm{SO}(2)\times\mathrm{SO}(3)$ with respect to $\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)$ and with respect to $\mathrm{SO}(3,1)/\mathrm{SO}(3)$. We have performed both, usual and generalized contractions which can be interpreted as truncations of massive modes. Those results will be published in ref.[4].

We show that the properties of commutativity, solvability and nilpotency of the Lie algebras are preserved under a S-expansion. Other properties like semisimplicity and compactness are not necessarily preserved. This depends on the semigroup used to perform the expansion procedure [5].

These results are summarized in Figure 10.1.

Original $\mathfrak{g}$	Expanded $\mathfrak{g}_S$	Resonant $\mathfrak{g}_{S,R}$	Reduced & res. $\mathfrak{g}_{S,R}^{red}$
Abelian	Abelian	Abelian	Abelian
Solvable	Solvable	Solvable	Solvable
Nilpotent	Nilpotent	Nilpotent	Nilpotent
Compact	Arbitrary	Arbitrary	Arbitrary
Semisimple	Arbitrary	Arbitrary	Arbitrary
Arbitrary	Arbitrary	Arbitrary	Arbitrary

Fig. 10.1: Properties preserved under an S-expansion.

Finally, we gave an interesting example studying all the possible expansions of the semisimple algebra $\mathfrak{sl}(2,\mathbb{R})$ by abelian semigroups. All our theoretical results were verified by using this example. We must point out that we could not get a simple algebra expanding $\mathfrak{sl}(2)$, but we do not have any theoretical result forbiding the obtention of a simple algebra as the result of an S-expansion. Finding such result is a work to do in the future. We have collected these results in ref.[5], to appear soon.

Finally, we have presented a complete study about the possibility of relating, by means of an expansion, two and three dimensional Lie algebras, specifically those of type I, II, III and V (according to Bianchi's classification), as expansions of the Lie algebras in 2 dimensions. It can happen that different semigroups lead to the same expanded algebra. Also, it is shown that the other Bianchi algebras, types IV, VI-IX, cannot be obtained as an expansion from the algebras in 2 dimensions. This means that there are some algebras that have properties in some sense intrinsic to 3 dimensions. This work has been published in ref.[6].

There are some undergoing works based on the calculations in this Thesis. This includes:

- We can try to S-expand $\mathfrak{so}(2,1)$ and get $\mathfrak{so}(2,2)$ and $\mathfrak{so}(2,3)$ to find the inverse construction of the contractions described in Chapter 7. In ref.[6] no simple Lie algebra was found by expanding $\mathfrak{sl}(2)$. Succeding to find $\mathfrak{so}(2,3)$ would also give us an example of a simple algebra obtained as the result of an S-expansion.

- To complete a catalog with the properties of all the S-expansions of all the simple algebras up to some dimension. For instance, one could say which expansions give semisimple or simple algebras, which ones give compact algebras and identify the resulting algebras. The programs in Appendix B can be extended to fit these purposes.

- To compute the real quantum Minkowski space with a flag manifold. Using a flag manifold is mandatory if one wants to find the real form of complex quantum Minkowski space. In ref.[47] we calculate the classical case. The quantization of the flag manifold is an extremely complicated work which must be performed with the aid of computer programs. We are enhancing the programs developed in ref.[47] to try to perform the quantization.

Some of these works will be published in the next months.

11. METODOLOGIA

En aquesta Tesi hem seguit la metodologia de recerca habitual en física teòrica.

En els Capítols 3, 4 i 5 s'han definit estructures algebraiques com el producte 'star' i la coacció 'star'. S'han realitzat programes amb Mathematica per a poder calcular algorítmicament aquests objectes. S'han trobat expressions analítiques que han permés realitzar a mà desenvolupaments complexes en potències de la constant de no commutativitat, ajudant-se amb programes fets amb Mathematica i Maxima, per tal de trobar operadors diferencials que reproduïren aquests resultats.

En el Capítol 6 s'han definit tensors invariants en espais coset i realitzat càlculs analítics tant a mà com mitjançant el programa Mathematica per als casos en què la dificultat dels càlculs era massa elevada. Per a cada model s'han calculat tots els objectes importants: generadors de l'acció del grup, corrents associats i tensors invariants. Els càlculs de les contraccions que apareixen en el Capítol 7 s'han realitzat amb l'ajut de Mathematica.

En el Capítol 8 s'han estudiat de forma analítica algunes propietats preservades sota l'expansió S d'una àlgebra de Lie. Aquest estudi s'ha realitzat a mà, mentres que per a la seua aplicació a les expansions S de l'àlgebra de Lie $\mathfrak{sl}(2)$ s'ha desenvolupat una llibreria en Java [24]. En el Capítol 9 s'estudia a mà com obtindre les àlgebres tridimensionals en la classificació de Bianchi a partir de les bidimensionals. Una vegada obteses les condicions sobre els semigrups a usar, la busca es realitza mitjançant programes escrits en Java [24].

12. RESUM

En aquesta Tesi investiguem distints aspectes de l'aplicació a la física de les àlgebres de Lie. En particular, aquest treball es divideix en tres parts: la construcció d'un product no commutatiu per a l'espai de Minkowski [1, 2, 3], el desenvolupament de models sigma [4] amb invariància sota l'acció del grup de simetria i l'estudi de la possiblitat de relacionar distintes àlgebres de Lie mitjançant l'expansió amb semigrups discrets [6].

La **primera part** d'aquesta Tesi es desenvolupa en els Capítols 3, 4 i 5.

Els principals objectius d'aquesta part del treball són:

- Definir un producte 'star' no commutatiu per a la compactificació conforme de l'espai de Minkowski.
- Donar una fórmula analítica explícita per al producte 'star' de dos polinomis en l'espai de Minkowski.
- Mostrar que l'acció del producte 'star' en polinomis es pot reproduir mitjançant un operador bidiferencial i per tant el producte 'star' es pot estendre a l'espai de les funcions C^∞.
- Definir una coacció del grup de Poincaré més dilatacions en l'espai de Minkowski de forma que siga compatible amb el producte 'star'.
- Mostrar que aquesta coacció es pot reproduir mitjançant un operador diferencial fins a un cert ordre en el paràmetre de quantització.
- Completar la construcció dels espais de Minkowski i Euclidià quàntics donant formes reals adequades.

La **segona part** es desenvolupa en els Capítols 6 i 7. Allí:

- Definim una classe de models sigma invariants sota un grup de simetria (ISM).

- Estudiem les diferències entre aquests models i els corresponents models ‘gauged’ WZW.
- Mostrem que, en general, aquests models no presenten invariància conforme.
- Relacionem distints ISM per contracció de grups de Lie.

La **tercera part** es desenvolupa en els Capítols 8 i 9. Els nostres objectius són:

- Estudiar propietats que es preserven sota el procediment d’expansió per semigrups discrets.
- Realitzar una classificació de les expansions S d’àlgebres simples.
- Usar el procediment d’expansió S per a trobar relacions entre àlgebres bidimensionals i tridimensionals.

L’estructura de l’espai-temps a un nivell fonamental ha sigut discutida des del descobriment de la Relativitat General. Aquesta teoria descriu la gravetat com la mètrica de l’espai-temps, mentre que la matèria és la font de dita mètrica. L’èxit de la Relativitat General descrivent la gravetat és remarcable (per exemple, penseu en els dispositius GPS).

Amb el descobriment de la mecànica quàntica al principi del segle XX es feu palés que l’estructura fonamental de l’espai-temps deuria sortir de la combinació d’aquestes dos teories. La quantització de la Relativitat General, vista com una teoria quàntica de camps, dóna una teoria no renormalitzable. S’han proposat diverses alternatives per a resoldre aquest problema, com la teoria de cordes o la gravetat quàntica de bucles, les quals intenten quantitzar la gravetat de formes distintes. Inclús existeix l’esperança què la Supergravetat podria ser finita [7]. Desafortunadament, la complexitat tècnica d’aquestes teories fa impossible tindre avui una teoria definitiva de la gravetat quàntica. El què sembla clar és que, a un nivell fonamental, l’espai-temps deuria tindre una estructura no localitzada o ‘fuzzy’ descrita per un àlgebra d’operadors que en general no commutarien.

És possible estudiar l’estructura de l’espai-temps sense introduir-hi una teoria dinàmica. S’han realitzat distints intents en aquest sentit [8, 9, 10, 11], definint productes no commutatius en teories de camps, per a introduir els efectes de la no commutativitat de l’espai-temps de diferents maneres.

En la Secció 2.4 presentem la complexificació conforme de l'espai-temps de Minkowski usual mitjançant la varietat Grassmaniana $G(2,4)$, és a dir, l'espai de plans bidimensionals en l'espai $\mathbb{C}^4$. En aquesta varietat existeix una acció natural del grup de Poincaré més dilatacions, donada per l'acció del subgrup parabòlic inferior, $P_l \subset \mathrm{SL}(4,\mathbb{C})$. Convé treballar amb aquest grup de forma algebraica, és a dir, amb l'àlgebra de polinomis en les variables del grup, $\mathcal{O}(P_l)$. En aquest formalisme la llei de grup es codifica com un coproducte (dual del producte) i la inversa es generalitza a l'antípoda. L'acció del grup de Poincaré en l'espai de Minkowski està donada per una coacció definida en els generadors de l'espai de Minkowski. Anomenem $\mathcal{O}(\mathrm{M})$ a l'àlgebra de polinomis en l'espai de Minkowski. Aquest formalisme és convenient perquè ens permet realitzar la quantització de Minkowski de forma directa.

Es pot veure els grups quàntics [14] com deformacions dels grups de Lie. Les operacions de producte, coproducte i antípoda es defineixen en termes d'un paràmetre de no commutativitat (q). Les àlgebres involucrades ací es defineixen com polinomis en termes de generadors no commutatius (variables no commutatives). En el cas particular $q = 1$, recuperem l'àlgebra commutativa. En la Secció 2.5 es dóna una deformació de la Grassmanniana i l'espai de Minkowski en termes de grups quàntics. A més a més, en ref.[1] es dóna una quantització de l'espai de superMinkowski quiral en termes de grups quàntics.

Treballar amb camps definits en les variables quàntiques $\mathcal{O}_q(\mathrm{M})$ presenta una gran dificultat. Podem definir un mapa Q_M entre $\mathcal{O}(\mathrm{M})$ i $\mathcal{O}_q(\mathrm{M})$, els quals són isormorfs com a mòduls, de forma que treballem amb funcions C^∞ (els camps definits en l'espai de Minkowski usual) i introduïm la no commutativitat usant un producte no commutatiu per als camps. Aquest mapa és l'anomenat *mapa de quantització* o *regla d'ordre*. En el Capítol 3 definim un producte no commutatiu (producte 'star') en $G(2,4)$ que ve del 'gluing' de productes 'star' en la 'big cell' (l'espai de Minkowski). En l'espai de Minkowski s'usa una regla d'ordre. Aquest producte és associatiu per construcció i definit per a polinomis en $G(2,4)$, és a dir, purament algebraic. Per a aplicar aquest producte a una teoria de camps és precís fer una generalització a funcions suaus definides en $G(2,4)$. Per a açò hem de trobar una expressió diferencial per al producte 'star'. Redefinint $q = e^h$ és possible realitzar una expansió que pot ser reproduïda per l'acció d'operadors diferencials en els polinomis clàssics (veure la Secció 3.2). Aquest resultat no és trivial, perquè els coeficients que multipliquen cada terme s'han de reproduir exactament. Una anàlisi acurada de l'estructura dels termes que apareixen en el producte 'star' permet demostrar que tots els polinomis que apareixen en el seu desenvolupament en termes de h tenen l'estructura correcta, de forma que el producte 'star' és diferencial i la seua expressió per un operador bidiferencial és única. Gràcies a açò definim el *producte 'star' per a funcions suaus* en $\mathcal{O}(\mathrm{M})$ com l'expansió corresponent en termes d'operadors bidiferencials. Per a escriure els operadors bidiferencials a un ordre arbitrari en el paràmetre de no commutativitat s'ha de fer el càlcul explícit. Els calculem fins a ordre 2 i mostrem que existeixen a ordre arbitrari.

També és possible definir un producte 'star' per al grup de Poincaré (Capítol 4). Es defineix una regla d'ordre per al grup (veure Apèndix A.2.5) i seguim un procediment anàleg a l'usat per al producte 'star' en l'espai de Minkowski. Seguidament definim una *coacció 'star'* del grup quàntic en $O_q(M)$ compatible amb el producte 'star', usant el mapa de quantització. La coacció 'star', quan actua sobre els generadors de l'espai de Minkowski, és formalment idéntica a la clàssica, éssent deguts els efectes no commutatius a la presència del producte 'star'. Aquesta coacció és algebraica, així que per a poder aplicar-la a funcions suaus necessitem expressar-la en termes d'operadors diferencials. En aquest cas estudiem l'acció del grup en l'espai de Minkowski com un operador diferencial que actua sobre un sol argument: el resultat clàssic de l'acció. L'acció pot ser reproduida per un operador diferencial i el trobem a primer ordre.

Fins a aquest punt hem treballat amb una complexificació de Minkowski i grups complexes. En el Capítol 5 discutim el problema de trobar les formes reals corresponents. Clàssicament el problema es redueix a trobar una *involució*, és a dir, un automorfisme amb les propietats (5.1), el conjunt de punts fixes del qual és la forma real que estem buscant. Donem les involucions per a l'espai de Minkowski i Euclidià, amb les formes reals dels grups que actuen en ells.

El cas quàntic és distint, perquè la involució ha de ser consistent amb les regles de commutació, cosa que la força a ser un antiisomorfisme, (és a dir, una *antiinvolució*). Açò descarta la interpretació d'una forma real com el conjunt de punts fixes d'un mapa. Una altra conseqüència és que, quan equipem l'espai de Minkowski real amb el producte 'star' que hem definit, el parèntesi de Poisson és purament imaginari.

Els models sigma no linials consisteixen en un conjunt de camps que prenen valors en els punts d'una varietat diferenciable (l' anomenat espai 'target'). Encara que es descriuen en termes de coordenades locals, les varietats diferenciables no tenen sistemes de coordinades privilegiats com els espais linials. Els models sigma presenten una invariància global respecte a difeomorfismes de l'espai target. Els camps en un model sigma interaccionen principalment degut a una mètrica Riemanniana en la varietat 'target', representada per un tensor simètric 2-covariant. També poden interaccionar mitjançant altres objectes, com un tensor antisimètric o un camp escalar (dilató).

És especialment interessant quan hi ha un grup actuant en la varietat. Les propietats globals de la varietat són importants per a estudiar l'acció del grup: els exemples més senzills són els espais quocient de tipus G/H, amb H un subgrup de G. H és el grup d'isotropia o *grup menut*. L'acció del grup G és transitiva en aquest cas. Els casos en què la varietat 'target' és ella mateixa un grup de Lie són també interessants, l'acció és la multiplicació per l'esquerra i per la dreta en el grup. L'acció no només és transitiva, sinò que amés no té cap punt fix.

Quan el grup actua mitjançant isometries de la mètrica (o, per a altres classes d'interaccions, la derivada de Lie de l'objecte és zero) aquesta simetria global es pot fer local introduint una conexió no linial en l'espai. Açò és el que s'anomena un model sigma 'gauged', els quals apareixen en teories supersimètriques i de Supergravetat.

En les teories supersimétriques i de Supergravetat els models sigma apareixen perquè les representacions de supersimetria (multiplets) genèricament contenen escalars, el lagrangià dels quals són models sigma més termes d'interacció amb altres camps.

Els models sigma també apareixen en el context de teories de cordes. En aquest cas la *fulla món* d'una corda juga el paper de l'espai-temps i l'espai 'target' és l'espai-temps on es mou la corda. Si hi ha invariància conforme aquestes teories mostren invariància sota l'acció de l'àlgebra infinit-dimensional de Virasoro. Un exemple clàssic són els models de Wess-Zumino-Witten (WZW), què són a més invariants sota àlgebres de Kac-Moddy.

En ref.[15] es descobriren jerarquies de models sigma que apareixen en models de Supergravetat. Aquestes jerarquies corresponen a contraccions generalitzades [16, 17] del grup d'isometria del model original. Aquestes contraccions desacoblen alguns camps i modelen truncacions exactes o integracions de modes massius. Modelar les integracions de modes massius pel procediment geomètric d'una contracció simplifica tècnicament el problema.

Els models WZW bidimensionals descriuen solucions de buit per a una corda. L'acció WZW conté dos parts: la part de la mètrica i la integral d'una 3-forma en una varietat tridimensional, la frontera de la qual és una compactificació de la fulla món de la corda. Aquestes formes són biinvariants (invariants esquerres i dretes) sota l'acció del grup mateix. Un model de WZW pot ser, almenys localment, escrit com un model sigma amb una mètrica biinvariant i una 2-forma la qual, sota l'acció del grup de simetria, canvia per la diferencial d'una funció. Les constants relatives entre ambdós termes s'elegeixen de forma que el model siga invariant conforme.

El 'gauging' d'un model WZW [18] es fa per acoblament mínim si el 2-tensor antisimètric és invariant sota les isometries 'gaugeades'. Si no ho és, encara és possible 'gaugear' si afegim termes adicionals al model, sempre que el subgrup de simetria que volem gaugear estiga lliure d'anomalies [18].

En la sèrie d'espais 'coset' $\mathrm{SO}(2,n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$ (Capítol 6) és possible definir una mètrica i una 2-forma invariants sota $\mathrm{SO}(2,n)$. Amb aquests objectes podem construir el que anomenem invariant sigma models (ISM). Cal esbrinar si el resultat de gaugear el subgrup $\mathrm{SO}(2)\times\mathrm{SO}(n)$ en un model WZW $\mathrm{SO}(2,n)$ és un ISM. El resultat és negatiu en els casos que hem estudiat.

Prenem per exemple el grup $\mathrm{SO}(2,1)/\mathrm{SO}(2)$ (el més simple). Usem *coordenades solubles* en el quocient. Les coordenades solubles són convenients perquè ens permeten realitzar el càlcul de la mètrica i la 2-forma fàcilment, donen formes simples per a aquests objectes i, a més, fan la comparació amb ref.[15] possible. En la Secció 6.1 comprovem que aquest model és diferent del model WZW 'gauged' $\mathrm{SO}(2,1)/\mathrm{SO}(2)^{\mathrm{R}}$, què és un bosó lliure. Les equacions de la funció beta a un loop ens diuen que l'ISM no és invariant conforme. En canvi, és invariant sota l'acció esquerra de $\mathrm{SO}(2,1)$ en el coset. En ref.[15] es mostra que el 'gauging' d'un subgrup (H) del grup d'isometria (G) d'un model sigma consistent només en el terme de la mètrica és un model sigma en la varietat quocient (G/H) invariant sota l'acció de G en G/H. Mostrem que açò no és cert en un model WZW degut a l'existència del tensor antisimètric.

El següent exemple és el grup $\mathrm{SO}(2,2)/\mathrm{SO}(2)\times\mathrm{SO}(2)$ (veure Secció 6.2). Com adés, calculem la mètrica i la 2-forma i mostrem que el model no és invariant conforme a nivell quàntic. Conclussions anàlogues són vàlides per al grup $\mathrm{SO}(2,3)/\mathrm{SO}(2)\times\mathrm{SO}(3)$ (veure Secció 6.3).

En la sèrie d'espais simètrics $\mathrm{SO}(2,n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$ és possible relacionar grups amb diferent n usant contraccions d'àlgebres de Lie. El procediment per a contraure la mètrica sigué definit en la ref.[15] i ací el generalitzem per a qualsevol tensor invariant.

La contracció de $\mathrm{SO}(2,3)/\mathrm{SO}(2) \times \mathrm{SO}(3)$ respecte a $\mathrm{SO}(1,3)/\mathrm{SO}(3)$ és d'especial interés. En aquest cas és possible obtindre un model de tipus $(\mathrm{SO}(3,1)/\mathrm{SO}(3)) \times \mathbb{R}^m$. Desafortunadament el model no és invariant esquerre respecte al grup complet $\mathrm{SO}(3,1)$, només respecte a la part soluble.

La deformació d'àlgebres de Lie és un procediment que té importància en Matemàtiques i Física. En el Capítol 7 hem estudiat com relacionar ISMs amb distints grups de simetria usant un procediment de contracció. Podem trobar contraccions aplicades a models de Supergravetat en ref.[15]. Una contracció d'àlgebres de Lie és un procediment que canvia les constants d'estructura sense canviar el nombre de generadors.

L'expansió d'àlgebres de Lie per semigrups discretes (des d'ara, *expansió S*, veure Secció 2.11) sigué introduïda fa alguns anys en refs.[19, 20, 21, 22, 23]. Prenem un semigrup discret i una àlgebra de Lie i definim un nou parèntesi de Lie en l'espai producte directe. Es pot demostrar que aquest parèntesi és associatiu, antisimètric i que satisfà la identitat de Jacobi, per la qual cosa el resultat és una àlgebra de Lie. Una expansió S canvia la dimensió de l'àlgebra, ja que va d'una àlgebra n-dimensional algebra a una $n \times m$-dimensional (éssent m l'ordre del semigrup).

És possible extraure àlgebres de dimensió menor a partir d'una àlgebra expandida S. És el cas quan hi ha una *descomposició ressonant* del semigrup. Hom pot extraure la subàlgebra ressonant de l'àlgebra expandida S. En el cas què el semigrup tinga un element zero, és possible realitzar una reducció per l'element zero. L'àlgebra reduïda és un quocient de l'expandida S. De vegades és inclús possible realitzar dos reduccions per zero. Açò es tracta en la Secció 2.11.

Existeixen certes propietats que es preserven sota una expansió S. Les estudiem en el Capítol 8. Quan expandim una àlgebra soluble el resultat és una altra àlgebra soluble. Una conseqüència d'açò és la solubilitat de la subàlgebra ressonant i de l'àlgebra 0-reduïda. El mateix passa amb la nilpotència.

Quan expandim una àlgebra semisimple no podem assegurar la semisimplicitat de l'àlgebra expandida S, les seues subàlgebres ressonants o la 0-reduïda. El mateix succeeix amb la compacitat. En la Secció 8.2 usem programes d'ordinador per a estudiar la semisimplicitat de l'àlgebra expandida S, les seues subàlgebres ressonants i la 0-reduïda. Açò és un exemple del tipus de classificació que es pot realitzar. Un estudi complet de totes les expansions S per semigrups fins a ordre 6 de totes les àlgebres simples (fins a una certa dimensió) es deu realitzar en el futur. Les ferramentes computacionals desenvolupades per a aquest treball ho fan possible.

En la Secció 8.2 discutim algunes expansions S interessants trobades usant els nostres programes. Amb aquest objectiu hem desenvolupat una llibreria Java [24]. Açò mostra la utilitat dels programes com a ferramenta per a l'estudi de les expansions S. Amb ells podem buscar descomposicions ressonants, obtindre les subàlgebres ressonants, les 0-reduïdes i comprovar si són semisimples. En el futur implementarem la busca d'altres propietats de les àlgebres.

En el Capítol 9 explorem les relacions entre les àlgebres de Lie bidimensionals i tridimensionals en la classificació de Bianchi [29]. Trobem que només podem establir eixes relacions gràcies a les subàlgebres ressonants. De fet, quan distintes descomposicions ressonants del mateix semigrup existeixen és possible relacionar distintes àlgebres mitjançant el mateix semigrup. Mitjançant un procediment iteratiu és possible deduïr algunes condicions en la taula de multiplicació d'un semigrup donant una certa relació i aleshores buscar totes les possibles formes de satisfer aquestes condicions amb distints semigrups usant programes desenvolupats per nosaltres.

A continuació presentem les conclusions d'aquesta Tesi.

Hem calculat una fórmula explícita per al producte 'star', que té les propietats següents:

- Es pot extendre a productes 'star' en l'espai conforme $G(2,4)$. Açò es fa enganxant els productes 'star' calculats en cada conjunt obert (2.19).
- Es pot extendre a actuar en funcions suaus com un producte diferencial.
- El parèntesi de Poisson és quadràtic en les coordinades.
- Existeix una coacció del grup de Poincaré quàntic (o el grup conforme en cas de l'espai conforme) en l'àlgebra del producte 'star'.
- Té almenys dos formes reals corresponent a les signatures Euclidiana i Minkowskiana.
- Es pot extendre al superespai (per a supercamps quirals i reals).

Donat què els camps són funcions suaus, la diferenciabilitat del producte 'star' ens dona esperança què hom puga desenvolupar una teoria de camps sobre la deformació quàntica de l'espai de Minkowski. El punt de partida és trobar una generalització del Laplacià i l'operador de Dirac associats a l'invariant quàntic C_q.

Un avantatge d'usar el grup quàntic $\mathrm{SL}_q(4, \mathbb{C})$ és que l'estructura de coàlgebra és isomorfa a la coàlgebra del grup clàssic $\mathrm{SL}(4, \mathbb{C})$ (veure per exemple el Teorema 6.1.8 en ref.[46]). Açò significa que la llei de grup roman inalterada, de forma que el principi de simetria de Poincaré de la teoria de camps seria preservat en el cas deformat quàntic.

Hem definit el que anomenem *models sigma invariants* (ISM). Aquests models es poden definir en la sèrie d'espais coset $\mathrm{SO}(2, n)/\mathrm{SO}(2) \times \mathrm{SO}(n)$ degut a l'existència d'una 2-forma invariant. Hem construït explícitament els models basats en els espais coset $\mathrm{SO}(2, 1)/\mathrm{SO}(2)$, $\mathrm{SO}(2, 2)/\mathrm{SO}(2) \times \mathrm{SO}(2)$ i $\mathrm{SO}(2, 3)/\mathrm{SO}(2) \times \mathrm{SO}(3)$.

Hem discutit exhaustivament el model $\mathrm{SO}(2, 1)/\mathrm{SO}(2)$, comparant-lo amb el model WZW 'gauged' $\mathrm{SO}(2, 1)/\mathrm{SO}(2)^R$. Hem fet veure què, encara que aquests models coincideixen si no afegim el terme de la 2-forma antisimètrica, quan prenem els models complets són dos models fonamentalment distints. L'ISM $\mathrm{SO}(2, 1)/\mathrm{SO}(2)$ no posseeix invariància conforme a nivell quàntic.

Discutim la contracció de models ISM. Hem definit el mètode per a deformar un tensor invariant arbitrari i l'hem aplicat a la contracció de $\mathrm{SO}(2, 3)/\mathrm{SO}(2) \times \mathrm{SO}(3)$ respecte a $\mathrm{SO}(2, 2)/\mathrm{SO}(2) \times \mathrm{SO}(2)$ i respecte a $\mathrm{SO}(3, 1)/\mathrm{SO}(3)$. Hem realitzant tant contraccions al mode usual com generalitzades, les quals podem interpretar com contraccions de modes massius.

Mostrem que les propietats de commutativitat, solubilitat i nilpotència de les àlgebres de Lie es preserven sota l'acció del procés d'expansió S a tots els nivells. Per altra part, altres propietats com ara la semisimplicitat i la compacticitat no es preserven necessàriament, fet que depén del semigrup usat per a realitzar l'expansió S. Aquests resultats es resumeixen en la Figura 12.1.

Presentem un exemple interessant estudiant totes les expansions possibles de l'àlgebra semisimple $\mathfrak{sl}(2, \mathbb{R})$. Tots els nostres resultats teòrics han sigut verificats gràcies a aquest exemple. Hem d'assenyalar què no hem pogut obtindre cap àlgebra simple expandint $\mathfrak{sl}(2)$, però no tenim cap resultat teòric que prohibisca l'obtenció d'una àlgebra simple com a resultat

Original $\mathfrak{g}$	Expandida $\mathfrak{g}_S$	Ressonant $\mathfrak{g}_{S,R}$	Reduïda i res. $\mathfrak{g}_{S,R}^{red}$
Abeliana	Abeliana	Abeliana	Abeliana
Soluble	Soluble	Soluble	Soluble
Nilpotent	Nilpotent	Nilpotent	Nilpotent
Compacta	Arbitrària	Arbitrària	Arbitrària
Semisimple	Arbitrària	Arbitrària	Arbitrària
Arbitrària	Arbitrària	Arbitrària	Arbitrària

Fig. 12.1: Propietats preservades sota l'expansió S.

d'una expansió S. Trobar aquest resultat és un treball a realitzar en el futur.

Finalment, hem presentat un estudi complet sobre la possibilitat de relacionar, mitjançant una expansió S, les àlgebres de Lie en 2 i 3 dimensions. Hem trobat que algunes àlgebres tridimensionals, específicament les de tipus I, II, III i V (segons la classificació feta per Bianchi), es poden obtindre com a expansions d'àlgebres bidimensionals. Pot ocòrrer que distints semigrups conduïsquen a la mateixa algebra expandida. A més, es mostra que els tipus IV, VI-IX en la classificació de Bianchi no es poden obtindre com a expansió d'àlgebres bidimensionals. Açò vol dir que aquestes àlgebres en un cert sentit són intrínseques a 3 dimensions.

Actualment estem realitzant alguns treballs basats en els resultats presentats en aquesta Tesi. Açò inclou:

- Intentar trobar les àlgebres $\mathfrak{so}(2,2)$ i $\mathfrak{so}(2,3)$ mitjançant l'expansió S de l'àlgebra de Lie $\mathfrak{so}(2,1)$ per tal de trobar la construcció inversa de les contraccions descrites en el Capítol 7. En la ref.[6] no es trobà cap àlgebra de Lie simple expandint $\mathfrak{sl}(2)$. Conseguir trobar $\mathfrak{so}(2,3)$ ens donaria un example d'una àlgebra simple obtesa com a resultat d'una expansió S.

- Completar un catàleg amb les propietats de totes les expansions S de totes les àlgebres simples fins a una certa dimensió. Per exemple, hom podria dir quines expansions donen àlgebres semisimples o simples, quines donen àlgebres compactes i identificar les àlgebres resultants. Hom pot extendre els programes en l'Apèndix B per a aquesta finalitat.

- Calcular l'espai de Minkowski quàntic real amb una varietat 'flag'. Cal usar una varietat 'flag' si hom vol trobar la forma real de l'espai de

Minkowski quàntic complex. En la ref.[47] calculem el cas clàssic. La quantització de la varietat flag és un treball extremadament complicat que s'ha de realitzar amb l'ajut de programes d'ordinador. Estem millorant els programes desenvolupats en la ref.[47] per tal d'intentar realitzar la quantització.

Alguns d'aquests treballs seran publicats en els pròxims mesos.

APPENDIX

A. A BASIS FOR THE POINCARÉ QUANTUM GROUP

In this Appendix we prove that, given a certain specific ordering on the generators of the Poincaré quantum group, the ordered monomials form a basis for its quantum algebra. This is a non trivial result based on the classical work by Bergman [48].

A.1 Generators and relations for the Poincaré quantum group

Let us consider $\mathrm{SL}_q(n,\mathbb{C})$ the quantum complex general linear group with indeterminates g_{IJ} subject to the Manin relations (2.9) and (2.10)[1] (see ref.[31])[2]. Inside $\mathrm{SL}_q(n,\mathbb{C})$ we consider the following elements which we write, as usual, in a matrix form:

$$\begin{pmatrix} g_{ij} \end{pmatrix} = \begin{pmatrix} x & 0 \\ Tx & y \end{pmatrix}$$

with

$$x = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad T = \begin{pmatrix} -q^{-1}D_{23}D_{12}^{-1} & D_{13}D_{12}^{-1} \\ -q^{-1}D_{24}D_{12}^{-1} & D_{14}D_{12}^{-1} \end{pmatrix}$$

$$y = \begin{pmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{pmatrix}.$$

As in (2.11), let us define the *quantum Poincaré group*, $\mathcal{O}_q(P_l)$ as the subring of $\mathrm{SL}_q(n,\mathbb{C})$ generated by the elements in the matrices x, y, T

[1] In this appendix we write the noncommutative generators without the hat to simplify the notation.

[2] All of the arguments in this appendix hold replacing $\mathrm{SL}_q(n,\mathbb{C})$ with the general linear quantum group and the complex field with any field of characteristic zero.

defined above. In order to give a presentation for $\mathcal{O}_q(P_l)$ we need to consider all of the commutation relations between the generators x, y, T given in (2.12, 2.13, 2.14, 2.15, 2.16). We call I_{P_l} the ideal generated by these relations.

The entries in x (resp. y) satisfy the Manin commutation relations in dimension 2, that is,

$$x = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad y = \begin{pmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{pmatrix} \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{aligned} &ba = qab, \qquad ca = qac, \qquad db = qbd, \qquad dc = qcd, \\ &cb = bc \qquad da = ad - (q^{-1} - q)bc. \end{aligned} \tag{A.1}$$

Moreover, they commute with each other:

$$x_{IJ} y_{KL} = y_{KL} x_{IJ}.$$

Similarly one can show that the entries in T_{IJ} satisfy the Manin relations, with the order

$$T = \begin{pmatrix} T_{32} & T_{31} \\ T_{42} & T_{41} \end{pmatrix} \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

but they do not commute with x and y (2.15, 2.16).

This provides a presentation of $\mathcal{O}_q(P_l)$ in terms of generators and relations (2.17) (see ref.[1] for more details),

$$\mathcal{O}_q(P_l) = \mathbb{C}_q\langle x_{IJ}, y_{KL}, T_{RS}\rangle / (\mathcal{I}_{P_l},\ \mathrm{det}_q x \cdot \mathrm{det}_q y - 1),$$

where $\mathcal{I}_{P_l}$ is the ideal generated by the commutation relations (2.12, 2.13, 2.14, 2.15, 2.16).

A.2 The Diamond Lemma

Let us recall some definitions and theorems from the fundamental work by Bergman [48] (see also ref.[49], pg. 103) [3].

[3] All of our arguments hold more in general replacing $\mathbb{C}_q$ with a commutative ring with unity.

Definition A.2.1. Let $\mathbb{C}_q\langle x_i\rangle$ be the free associative algebra over $\mathbb{C}_q$ with generators $x_1, \dots, x_n$ and let

$$X := \{X_I = x_{i_1} \cdots x_{i_s} \;/\; I = (i_1, \dots, i_s),\; i_j \in \{1, \dots, n\}\}$$

be the set of all (unordered) monomials. X is clearly a basis for $\mathbb{C}_q\langle x_i\rangle$. We define on X an order, $<$, such that given two monomials x and y, then $x < y$ if the length of x is less than the length of y and for equal lengths we apply the lexicographical ordering. ■

Let $\Pi = \{(X_{I_k}, f_k) \mid k = 1, \dots, s\}$ be a certain set of pairs $X_{I_k} \in X$ and $f_k \in \mathbb{C}_q\langle x_i\rangle$. We denote by $\mathcal{J}_\Pi$ the ideal

$$\mathcal{J}_\Pi = (X_{I_k} - f_k,\; k = 1, \dots, s) \subset \mathcal{O}_q(P_l).$$

In our application Π will yield the ideal of the commutation relations for the quantum Poincaré group.

Definition A.2.2. We say that Π is *compatible* with the ordering $<$ if f_k consists of a linear combination of ordered monomials. ■

For example if $M_q(2) = \mathbb{C}_q\langle a, b, c, d\rangle/\mathcal{I}_M$, where $\mathcal{I}_M$ is the ideal of the Manin relations, we have that

$$\begin{aligned}\Pi_M = \{&(ba, qab),\; (ca, qac),\; (cb, bc),\; (dc, qcd),\\ &(db, qbd),\; (da, ad - (q^{-1} - q)bc)\,\}\end{aligned}$$

is compatible with the ordering $a < b < c < d$.

We want to find a basis consisting of ordered monomials for a $\mathbb{C}_q$-module $\mathbb{C}_q\langle x_i\rangle/\mathcal{J}_\Pi$. Clearly this is not possible for any chosen total order. However, when Π is compatible with the order, that is, when the relations $X_{I_k} - f_k$ behave nicely with respect to the given order, then we can implement an algorithm to reduce any monomial to a *standard form* (namely to write it as a combination of ordered monomials). This is essentially the content of the Diamond Lemma for ring theory that we shall describe below.

We have two problems to solve: first, one has to make sure that any procedure to reduce a monomial to the standard form terminates, and then one has to make sure that the chosen procedure gives a unique result.

Definition A.2.3. Assume that we fix a generic set Π as above. Let $x, y \in X$ and let r_{xky} be the linear map of $\mathbb{C}_q\langle x_i\rangle$ sending the elements of the form $xx_{i_k}y$ to xf_ky and leaving the rest unchanged. r_{xky} is called a *reduction* and an element $x \in X$ (or more generally in $\mathbb{C}_q\langle x_i\rangle$) is *reduced* if $r(x) = x$ for all reductions r. ■

In general more than one reduction can be applied to an element. For example if we take the quantum matrices $M_q(2)$ and Π_M as above, we see that *dcba* is not reduced, and we have several ways to proceed to reduce it. We want to make sure that that there are no ambiguities, or, in other words, we want to make sure there is a unique reduced element associated with it.

Definition A.2.4. Let $x, y, z \in X$ and x_{i_k}, x_{i_l} be the first elements of two pairs in Π. We say that $(x, y, z, x_{i_k}, x_{i_l})$ form an *overlapping ambiguity* if $x_{i_k} = xy$, $x_{i_l} = yz$. The ambiguity is *resolvable* if there are two reductions r and r' such that $r(x_{i_k}z) = r'(xx_{i_l})$. In other words, if we can reduce xyz in two different ways, we must obtain the same result. Similarly (x, y, x_{i_k}, x_{i_l}) form an *inclusion ambiguity* if $x_{i_k} = xx_{i_l}y$. The inclusion ambiguity is *solvable* if there are two reductions r and r' such that $r(x_{i_k}) = r'(xx_{i_l}y)$.

■

Theorem A.2.5. *(Diamond Lemma). Let R be the ring defined by generators and relations as:*

$$R := \mathbb{C}_q\langle x_i\rangle/(X_{I_k} - f_k, k = 1 \dots s)$$

If $\Pi = \{X_{I_k}, f_k\}_{k=1,\dots,s}$ is compatible with the ordering $<$ and all ambiguities are resolvable, then the set of ordered monomials is a basis for R. Hence R is a free module over $\mathbb{C}_q$.

Proof. See ref.[48].

■

A.3 A basis for the Poincaré quantum group

In this section, we want to apply the Diamond Lemma, to obtain an explicit basis for the quantum algebra of the Poincaré quantum group. Let us fix a total order on the variables x, y, t as follows:

$$t_{32} > t_{31} > t_{42} > t_{41} > x_{11} > x_{12} > x_{21} > x_{22} > y_{33} > y_{34} > y_{43} > y_{44}.$$

One sees right away that the relations in $\mathcal{I}_M$ as described in (2.12, 2.13, 2.14, 2.15, 2.16) give raise to a Π compatible with the given order. Furthermore, notice that this order is the Manin ordering (see ref.[8]) in two dimensions when restricted to each of the sets $\{x_{IJ}\}$, $\{y_{KL}\}$, $\{t_{RS}\}$.

As one can readily see, the fact that Π is compatible with the given order ensures that any reordering procedure terminates.

Theorem A.3.1. *Let $\mathcal{O}_q(P_l) = \mathbb{C}_q\langle x_{ij}, y_{kl}, t_{il}\rangle/\mathcal{I}_{P_l}$ be the algebra corresponding to the quantum Poincaré group. Then, the monomials in the order:*

$$t_{32} > t_{31} > t_{42} > t_{41} > x_{11} > x_{12} > x_{21} > x_{22} > y_{33} > y_{34} > y_{43} > y_{44}.$$

are a basis for $\mathcal{O}_q(P_l)$.

Proof. By the Diamond Lemma A.2.5 we only need to show that all ambiguities are resolvable. We notice that when two generators a, b, q-commute, that is $ab = q^s ba$, they behave, as far the reordering is concerned, exactly as commutative indeterminates. Hence we only take into consideration ambiguities where no q-commuting relations appear. The proof consists in checking directly that all such ambiguities are resolvable.

Let us see, as an example of the procedure to follow, how to show that the ambiguity $x_{22}x_{11}t_{32}$ is resolvable. All the other cases follow the same pattern since the relations have essentially the same form as far as the reordering procedure is concerned.

We shall indicate the application of a reduction with an arrow, as it is customary to do.

$$\begin{aligned}(x_{22}x_{11})t_{32} \longrightarrow\ & (x_{11}x_{22} - (q^{-1} - q)x_{12}x_{21})t_{32} \longrightarrow x_{11}(q^{-1}t_{32}x_{22}+ \\ & + (q^{-1} - q)t_{31}x_{12}) - (q^{-1} - q)[x_{12}(q^{-1}t_{32}x_{21}+ \\ & (q^{-1} - q)t_{31}x_{11})] \longrightarrow q^{-1}t_{32}x_{11}x_{22} + q^{-1}(q^{-1} - q)t_{31}x_{11}x_{12}+ \\ & - q^{-1}(q^{-1} - q)t_{32}x_{12}x_{21} - q(q^{-1} - q)t_{31}x_{11}x_{12} = \\ & = q^{-1}t_{32}x_{11}x_{22} - q^{-1}(q^{-1} - q)t_{32}x_{12}x_{21} + (1 - q^2)t_{31}x_{11}x_{12}.\end{aligned}$$

Similarly

$$\begin{aligned}x_{22}(x_{11}t_{32}) \quad & \longrightarrow x_{22}t_{32}x_{11} \longrightarrow (q^{-1}t_{32}x_{22} + (q^{-1} - q)t_{32}x_{12})x_{11} \\ & \longrightarrow q^{-1}t_{32}(x_{11}x_{22} - (q^{-1} - q)x_{12}x_{21}) + (1 - q^2)t_{31}x_{11}x_{12}.\end{aligned}$$

As one can see the two expressions are the same and reduced, hence we obtain that this ambiguity is resolvable. ■

Remark A.3.2. We end the discussion by noticing that the Theorem A.3.1 holds also for the order:

$$x_{11} > x_{12} > x_{21} > x_{22} > y_{33} > y_{34} > y_{43} > y_{44} > t_{32} > t_{31} > t_{42} > t_{41}$$

the proof being the same. ∎

B. A JAVA LIBRARY TO PERFORM S-EXPANSIONS OF LIE ALGEBRAS

This Appendix is about the programming tools developed to perform the calculations in Chapters 8 and 9. We recommend its reading only to those readers interested into Java programming.

We present a Java library developed to perform expansions of Lie algebras by discrete semigroups. It is able to look for resonant decompositions of any semigroup up to order 6, identify automatically the zero element of a semigroup, check if two semigroups are isomorphic or antiisomorphic and look for semigroups satisfying certain conditions (like a given resonance condition). With this we can perform S-expansions of any Lie algebra, find its resonant subalgebras and perform reductions by the zero element. We show how the library works and we offer some examples of use.

As a preliminar step, we used the semigroup generating program *gen.f* listed in ref.[44] to generate files sem.2, sem.3, sem.4, sem.5 and sem.6, which contain all the semigroups up to isomorphism of orders 2 to 6 (the content of sem.3 and a partial content of sem.4 can be seen in Appendix C). We reproduce this program here:

```
C Semigroup generator program
      INTEGER*2 A(8,8),P(40320,8),Q(40320,8)
      OPEN(6,FILE='sem.7')
 3    FORMAT(1X,I6,I3)
 4    FORMAT(1X, I4)
      N=7
      ID=0
      DO 10 K=1,N
      DO 10 L=1,N
      A(K,L)=1
 10   CONTINUE
      A(N,N)=0
      CALL PERM(P,Q,N,NF)
```

```
      I=N
      J=N
20    CONTINUE
      A(I,J)=A(I,J)+1
      IF(A(I,J).LE.N) GO TO 50
      A(I,J)=0
      IF(J.EQ.1) GO TO 30
      J=J-1
      GO TO 40
30    CONTINUE
      J=N
      I=I-1
40    CONTINUE
      IF(I.EQ.1.AND.J.EQ.1) STOP
      GO TO 20
50    CONTINUE
      CALL ASTEST(A,N,KA)
      IF(KA.EQ.0) GO TO 20
      CALL ISOTEST(A,P,Q,N,NF,KI)
      IF(KI.EQ.0) GO TO 20
      IF(I.EQ.N.AND.J.EQ.N) GO TO 70
      IF(J.EQ.N) GO TO 60
      J=J+1
      GO TO 20
60    J=1
      I=I+1
      GO TO 20
70    CONTINUE
      ID=ID+1
      WRITE(6,3) ID,N
      DO 80 K=1,N
      WRITE(6,4) (A(K,L),L=1,N)
80    CONTINUE
      GO TO 20
      END

      SUBROUTINE ISOTEST(A,P,Q,N,NF,KI)
      INTEGER*2 A(8,8),B(8,8),C(8,8)
```

```
      INTEGER*2 P(40320,8),Q(40320,8)
      KI=0
      L=0
10    L=L+1
      IF(L.GT.NF) GO TO 50
      DO 30 I=1,N
      DO 30 J=1,N
      NA=A(Q(L,I),Q(L,J))
      IF(NA.EQ.0) GO TO 20
      B(I,J)=P(L,NA)
      GO TO 30
20    B(I,J)=0
30    CONTINUE
      CALL COMPARE(A,B,N,KC)
      IF(KC.EQ.0) RETURN
      DO 40 I=1,N
      DO 40 J=1,N
      C(I,J)=B(J,I)
40    CONTINUE
      CALL COMPARE(A,C,N,KC)
      IF(KC.EQ.0) RETURN
      GO TO 10
50    CONTINUE
      KI=1
      RETURN
      END

      SUBROUTINE ASTEST(A,N,KA)
      INTEGER*2 A(8,8)
      KA=1
      I=0
10    I=I+1
      IF(I.GT.N) RETURN
      J=0
20    J=J+1
      IF(J.GT.N) GO TO 10
      MF=A(I,J)
      IF(MF.EQ.0) RETURN
```

```
      K=0
30    K=K+1
      IF(K.GT.N) GO TO 20
      MS=A(J,K)
      IF(MS.EQ.0) GO TO 10
      IF(A(I,MS).EQ.0) GO TO 30
      IF(A(MF,K).EQ.0) GO TO 30
      IF(A(I,MS).NE.A(MF,K)) GO TO 40
      GO TO 30
40    KA=0
      RETURN
      END

      SUBROUTINE COMPARE(A,B,N,KC)
      INTEGER*2 A(8,8),B(8,8)
      KC=0
      I=0
10    I=I+1
      IF(I.GT.N) GO TO 30
      J=0
20    J=J+1
      IF(J.GT.N) GO TO 10
      IF(B(I,J).EQ.0) GO TO 30
      IF(A(I,J).GT.B(I,J)) RETURN
      IF(A(I,J).EQ.B(I,J)) GO TO 20
30    CONTINUE
      KC=1
      RETURN
      END

      SUBROUTINE PERM(P,Q,N,NF)
      INTEGER*2 P(40320,8),Q(40320,8),T(8)
      J=0
      M=N
      NF=N
10    M=M-1
      IF(M.EQ.0) GO TO 20
      NF=M*NF
```

```
      GO TO 10
 20   CONTINUE
      DO 30 L=1,N
      T(L)=0
 30   CONTINUE
      K=1
 40   CONTINUE
      T(K)=T(K)+1
      IF(T(K).LE.N) GO TO 50
      T(K)=0
      K=K-1
      IF(K.EQ.0) RETURN
      GO TO 40
 50   CONTINUE
      I=0
 60   I=I+1
      IF(I.EQ.K) GO TO 70
      IF(T(I).EQ.T(K)) GO TO 40
      GO TO 60
 70   CONTINUE
      IF(K.EQ.N) GO TO 80
      K=K+1
      GO TO 40
 80   CONTINUE
      J=J+1
      DO 90 L=1,N
         P(J,L)=T(L)
 90   CONTINUE
      DO 100 L=1,N
      M=P(J,L)
      Q(J,M)=L
 100  CONTINUE
      IF(J.EQ.NF) RETURN
      GO TO 40
      END
```

The files *sem.X* contain all the semigroups up to isomorphism of order ‘X’, labelled by a number which univocally identifies each semigroup. This number runs from 1 up to the number of semigroups of that order. This is the input data of our library. In our case, we were only able to compute the semigroups up to order 6 because we only had acces to our personal computers. With the program listed above it is possible to generate semigroups up to order 8. The order 9 is non trivial and was solved in 2009 in ref.[50].

We define the Semigroup class to represent a discrete semigroup and all the operations which can be performed with them.

```
public class Semigroup {

        int[][] data;
        int order;  //The order of the semigroup
        int ID; // This is the number of semigroups
                // for a given order.
```

There is obviously more code but we can not reproduce it completely here. We see that we use three objects to save the information of a semigroup: an integer '*ID*' which contains the number which identifies the semigroup, a second integer '*order*' which tells us the order of the semigroup and a matrix of integers '*data*' where we save the multiplication table of the semigroup. By convention we label the generators of the semigroup, λ_α, by the integer α and define its multiplication table

$$A = \left(a_{\alpha\beta}\right) \equiv \left(\lambda_\alpha \cdot \lambda_\beta\right)$$

B.1 Preliminars

The first thing that we need to do is loading the semigroups generated by *gen.f*. The method loadFile loads all the semigroups of a given order, returning us an array of Semigroup objects:

```
private static Semigroup[] loadFile (int order ) {
 int N = 0 ;
 int NSemigroups ;
 int number,id;
 BufferedReader theReader ;
 String str ;
 String strNumero;
 StringTokenizer st ;
 int [][] matrix ;
```

```
Semigroup [] result;
String fileName = null;
int elements = 0 ;
switch( order ) {
 case 2 :
  fileName = "src/semigroups/datos/sem.2" ;
  elements = 4;
  break;
 case 3 :
  fileName = "src/semigroups/datos/sem.3" ;
  elements= 18;
 break;
 case 4 :
  fileName = "src/semigroups/datos/sem.4" ;
  elements= 126;
  break;
 case 5 :
  fileName = "src/semigroups/datos/sem.5" ;
  elements= 1160;
  break;
 case 6 :
  fileName = "src/semigroups/datos/sem.6" ;
  elements= 15973;
  break;
}
result = new Semigroup[elements];
File arxiu = new File( fileName );
try {
 NSemigroups = elements ;
 FileReader reader = new FileReader(arxiu);
 theReader = new BufferedReader(reader);
 for ( N = 0 ; N < NSemigroups ; ++N ){
  matrix = new int[order][order];
  //The first we have to read is the
  // order and ID of the semigroup
  str = theReader.readLine();
  st = new StringTokenizer(str);
  strNumber = st.nextToken();
```

```
    number = Integer.parseInt(strNumber);
    id = number ;
    strNumber = st.nextToken();
    number = Integer.parseInt(strNumber);
    int i  , j  ;
    //Now we can start to read the multiplication table
    for ( i = 0 ; i < order ; ++ i ){
     for ( j = 0 ; j < order ; ++j ) {
      str = theReader.readLine();
      st = new StringTokenizer(str);
      strNumber = st.nextToken();
      number = Integer.parseInt(strNumber);
      matrix[i][j] = number;
      }
     }
    result[N] = new Semigroup(matrix) ;
    result[N].ID = id ;
   }
  reader.close();
  } catch( IOException excepcio  ) {
   System.out.println("Error");
  }
  return result;
}
```

This method just loads the semigroups of a given order. To load all the avaliable semigroups we use the method loadFromFile. It uses the method loadFile to load the semigroups from order 2 to 6:

```
public static Semigroup[] loadFromFile( ) {
 Semigroup [] result;
 Semigroup [] aux ;
 int i = 0 , j = 0 ;
 int alreadySaved = 0 ;
 result = new Semigroup[4+18+126+1160+15973];
 int[] elements = {4,18,126,1160,15973};
 for ( i = 2 ; i < 7 ; ++i) {
  aux = loadFile(i) ;
  for ( j = 0 ; j < elements[i - 2] ; ++j) {
   result[alreadySaved + j] = aux[j];
```

```
  }
  alreadySaved += elements[i-2];
 }
 return result;
}
```

B.1.1 Associativity

A semigroup must be associative, i.e. its generators must satisfy

$$(\lambda_\alpha \cdot \lambda_\beta) \cdot \lambda_\gamma = \lambda_\alpha \cdot (\lambda_\beta \cdot \lambda_\gamma)\,.$$

In terms of the multiplication table the condition above is written as

$$a_{a_{\alpha,\beta},\gamma} = a_{\alpha,a_{\beta,\gamma}}\,.$$

The method isAssociative returns true if a given multiplication table is associative, i.e. if it is really a semigroup.

```
public boolean isAssociative() {
 int i , j , k ;
 for ( i = 0 ; i < order ; ++ i) {
  for ( j = 0 ; j < order ; ++j) {
   for ( k = 0 ; k < order ; ++ k) {
    if( ! (data[i][data[j][k]-1]
          == data[data[i][j]-1][k])) {
        return false ;
    }
   }
  }
 }
 return true;
}
```

B.1.2 Commutativity

To perform an S-expansion, the semigroup used must be commutative. The method isCommutative returns true if a given semigroup is commutative, i.e.

$$\lambda_\alpha \cdot \lambda_\beta = \lambda_\beta \cdot \lambda_\alpha.$$

```
public boolean isCommutative(){
 int i, j;
 for ( i = 0 ; i < order ; ++i) {
  for( j = 0 ; j < order ; ++j ) {
   if( ! (data[i][j] == data[j][i])) {
    return false;
   }
  }
 }
 return true;
}
```

B.1.3 The zero element

A semigroup has a zero element, 0_S, when there is an element satisfying

$$\lambda_\alpha \cdot 0_S = 0_S \qquad \forall \alpha.$$

When a commutative semigroup has a zero element it is possible to perform a *reduction by the zero element* of the expanded algebra. To be able to automatize that procedure we need a method that can look for the zero element, in case it exists. The method findZero returns -1 in case that a given semigroup does not have a zero element or the zero element othercase.

```
public int findZero() {
 int i, j ;
 boolean isZero = false ;
 for ( i = 0 ; i < order ; ++ i ) {
  if ( isZero == true ) {
   return i ;
  }
  j = 0 ;
  isZero = true;
  while ( isZero && (j < order )) {
   if ( data[j][i] != i+1) {
    isZero = false ;
   }
  ++j;
  }
```

```
 }
 if ( isZero == true ) {
  return order ;
 }
 return -1 ;
}
```

B.1.4 Equality

Sometimes we need an auxiliary method to check for the equality of two given semigroups. The method isEqualTo returns true if they are equal.

```
public boolean isEqualTo ( Semigroup B ) {
 int i,j;
 if ( this.order != B.order ) {
  return false;
 }
 for ( i = 0 ; i < this.order ; ++i ) {
  for ( j = 0 ; j < B.order ; ++j) {
   if ( this.data[i][j] != B.data[i][j]){
   return false;
   }
  }
 }
 return true;
}
```

B.2 Isomorphisms and permutations

Two isomorphic semigroups lead, after S-expansion, to two isomorphic Lie algebras. It is important, then, to be able to check if two given semigroups are isormorphic. In ref.[44] it was shown that the group of isomorphisms of the semigroups of order n is the group of permutations of n elements, Σ_n.

We represent a permutation by

$$(\lambda_{i_1}\,\lambda_{i_2}\cdots\lambda_{i_n})$$

which means change λ_1 by λ_{i_1}, change λ_2 by λ_{i_2}, etc, and finally change λ_n by λ_{i_n}. We define the Set class to represent a permutation.

```
public class Set {

 int [] list;
 int nElements;
```

We do not reproduce its complete definition here. We see that a Set object containts an integer '*nElements*' which is the number of generators which we want to apply the permutation to and a '*list*' where we save the permutation in the format explained above. These objects will also serve us to save any set of non repeated integers, like the one we will use for the resonant decomposition of a discrete semigroup.

B.2.1 Creation of sets

We can create a set just from an array of integers

```
public Set( int [] elements ) {
 list = elements;
 nElements = elements.length;
}
```

or we can create a Set of n elements containing the identity permutation in Σ_n with

```
public Set( int n ){
 nElements = n;
 list = new int[n];
 int i ;
 for ( i = 0 ; i < n ; ++i) {
  list[i] = i +1 ;
 }
}
```

B.2.2 Getting all the elements in Σ_n

To check if two given semigroups are isomorphic we have to try all the existent isormorphisms for a given order. This means using all the elements in Σ_n. This is performed by the method AllPermutations, which returns an array of Set objects containing all the elements in Σ_n.

```
Set [] AllPermutations( ) {
 Set result = new Set(0) ;
```

```
 return this.PermutationsAux( this , result);
}
```

This method uses the auxiliary method PermutationsAux, which actually performs most of the work. This is a recursive method which takes a Set object and reorders its elements in all the possible ways. That is, given the identity permutation

$$(1\,2\,\cdots\,n)$$

it returns all the elements of the permutation group Σ_n.

```
Set [] PermutationsAux( Set original, Set result ) {
 int i , j;
 Set original2 , result2 ;
 Set [] list = null;
 Set [] totalList = null;
 Set [] previousList = null;
 int N = 0 ;
 if ( original.nElements == 0 ) {
  list = new Set[1];
  list[0] = result ;
  return list ;
 }
 for ( i = 0 ; i < original.nElements ; i++ ){
  result2
    = result.addElement( original.elementAt(i) );
  original2 = original.eraseElement(i) ;
  list = PermutationsAux( original2 , result2) ;
  N = N + list.length ;
  previousList = totalList ;
  totalList = new Set[N];
  for ( j = 0 ; j < N - list.length ; ++j) {
   totalList[j] = previousList[j];
  }
  for ( j = 0 ; j < list.length ; ++j) {
   totalList[ j + N - list.length ] = list[j];
  }
 }
 return totalList;
}
```

PermutationsAux uses the methods addElement and eraseElement whose definition is obvious and we do not reproduce here.

B.2.3 Isomorphisms of semigroups

Given a permutation $\alpha \in \Sigma_n$, we say that the semigroups $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$ are isomorphic if

$$A\alpha = B \longleftrightarrow b_{ij} = \alpha(a_{\alpha^{-1}(i),\alpha^{-1}(j)}) \qquad \forall i, j.$$

The method which applies a given isormorphism to a semigroup is PermuteWith.

```
public Semigroup PermuteWith( Set s ) {
 int i,j;
 int [][] matrix = new int[this.order][ this.order];
 Set inverse  = s.inversePermutation() ;
 for ( i = 0 ; i < this.order ; ++i) {
  for ( j = 0 ; j < this.order ; ++j) {
   matrix[i][j] = this.data[ inverse.elementAt(i) - 1]
     [ inverse.elementAt(j) -1] ;
  }
 }
 for ( i = 0 ; i < this.order ; ++i) {
  for ( j = 0 ; j < this.order ; ++j) {
   if ( matrix[i][j] != -1 ) {
    matrix[i][j] = s.elementAt( matrix[i][j] - 1 );
   }
  }
 }
 return new Semigroup(matrix);
}
```

Sometimes what we need is finding all the isomorphic forms of a given semigroup, i.e., apply all the possible permutations to a given semigroup. In that case, we use the method Permute, which returns an array of Semigroup objects containing all the permutations of a given one.

```
public Semigroup [] Permute() {
 Set identity = new Set( this.order );
 Set [] permutations = identity.AllPermutations() ;
```

```
 int k;
 Semigroup [] result =
     new Semigroup [permutations.length];
 for ( k = 0 ; k < permutations.length ; ++k) {
  result[k ] = this.PermuteWith(permutations[k]);
 }
 return result;
}
```

B.3 Resonant decomposition of discrete semigroups

When the semigroup S can be decomposed in two subsets $S = S_0 \cup S_1$, such that they satisfy the *resonant condition*

$$S_0 \cdot S_0 \subset S_0, \qquad S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_1 \subset S_0,$$

it is said that the semigroup has a *resonant decomposition*.

A previous step to check if two given sets satisfy the resonance condition is being able to check if, as sets, $S_0 \cup S_1 = S$. This is done by the method fillTheSpace of the Set class.

```
public static boolean fillTheSpace
     ( Set s1 , Set s2 , int order) {
 int i ;
 for ( i = 0 ; i < order ; ++i) {
  if ( ! s1.find( i+1) && ! s2.find(i +1) ){
   return false;
  }
 }
 return true;
}
```

The parameter *order* tells the method the order of the semigroup which has a resonant decomposition ‘s1’ and ‘s2’.

The method isResonant returns true if the two Set objects ‘s0’ and ‘s1’ represent a resonant decomposition for the current Semigroup object.

```
public boolean isResonant( Set s0 , Set s1) {
 int i,j, n0 = s0.nElements, n1 = s1.nElements ;
 if ( Set.fillTheSpace(s0, s1, order) ){
```

```
  for ( i = 0 ; i < n0 ; ++i ) {
   for ( j = 0 ; j < n0 ; ++j ) {
    if ( ! s0.find(this.data[s0.elementAt(i) -1]
         [s0.elementAt(j)-1] ) ) {
     return false ;
    }
   }
  }
  for ( i = 0 ; i < n0 ; ++i ){
   for ( j = 0 ; j < n1 ; ++j) {
    if ( ! s1.find( this.data[s0.elementAt(i)-1]
         [s1.elementAt(j)-1]))  {
     return false;
    }
   }
  }
  for ( i = 0 ; i < n1 ; ++i) {
   for ( j = 0 ; j < n1 ; ++j) {
    if (! s0.find( this.data[s1.elementAt(i)-1]
          [s1.elementAt(j)-1])) {
     return false;
    }
   }
  }
 }else {
  return false;
 }
 return true;
}
```

Once we are able to check if a given decomposition of a semigroup is resonant, we want to be able to look for resonant decompositions. The method findResonances looks for all the possible resonances of a semigroup, having S_0 ‘n1’ elements and S_1 ‘n2’ elements.

```
public Set [][] findResonances( int n1, int n2 ) {
 Set total = new Set(this.order) ;
 Set [] list1 = total.SubSets(n1) ;
 Set [] list2 = total.SubSets(n2) ;
 Set [][] result = null;
```

```
 Set [][] auxiliar = null ;
 int foundResonances = 0 ;
 int i , j , k = 0;
 for ( i = 0 ; i < list1.length ; ++ i ) {
  for ( j = 0 ; j < list2.length ; ++j ) {
   if ( this.isResonant( list1[i], list2[j])
         && Set.fillTheSpace
             ( list1[i], list2[j], this.order))
         {
    foundResonances = foundResonances + 1 ;
    auxiliar = result ;
    result = new Set[foundResonances ] [2] ;
    for ( k = 0 ; k < foundResonances -1 ; ++k) {
      result[k][0] = auxiliar[k][0];
      result[k][1] = auxiliar[k][1];
    }
    result[ foundResonances - 1 ] [0] = list1[i];
    result[ foundResonances - 1] [1] = list2[j];
   }
  }
 }
 return result;
}
```

In case it finds a resonant decomposition, this method returns a 2-dimensional array whose element result[i][0] is S_0 and result[i][1] is S_1 for the ith decomposition found. This method uses the auxiliary method SubSets which returns all the subsets with n elements of a given set.

```
public Set [] SubSets( int n ) {
 Set result = new Set();
 return Set.cleanDuplicates
      (AuxSubset( this , result , n ));
}
```

This method is just a more convenient way to use the recursive method AuxSubset

```
Set[] AuxSubset( Set original , Set resultat , int n) {
 int i;
 Set [] list = null;
```

```
  Set [] totalList = null;
  if ( n == 0 ) {
   totalList = new Set[1];
   totalList[0] = resultat;
   return totalList ;
  }
  for ( i = 0 ; i < original.nElements ; ++i ) {
   list = AuxSubset( original.eraseElement(i),
   resultat.addElement( original.elementAt(i)) , n-1);
   totalList = Set.add(list , totalList );
  }
  return totalList ;
}
```

The auxiliary method cleanDuplicates just cleans possible duplicates in an array of Set objects:

```
public static Set[] cleanDuplicates( Set [] lst) {
  int i, j ;
  int n = lst.length ;
  int elements = n;
  Set [] newList ;
  for ( i = 0 ; i < n ; ++i) {
   lst[i] = Set.sort(lst[i]);
  }
  for ( i = 0 ; i < n ; ++i) {
   for ( j = i +1 ; j < n ; ++j){
    if (lst[i] != null && lst[j] != null
        && lst[i].equalTo(lst[j]) ) {
     lst[j] = null ;
     elements = elements - 1 ;
    }
   }
  }
  newList = new Set[elements] ;
  j = 0 ;
  for ( i = 0 ;  i < n ; ++i) {
   if ( lst[i] != null ) {
    newList[j] = lst[i];
    ++j;
```

```
  }
 }
 return newList ;
}
```

To find all the possible resonant decompositions of a given semigroup we define the method findAllResonances.

```
public Set [][] findAllResonances() {
 int i , j , k;
 Set [][] result = null;
 Set [][] auxiliar ;
 Set [][] intermediateResult;
 int N = 0;
 for ( i = 1 ; i < this.order ; ++i) {
  for ( j = 1 ; j < this.order ; ++j) {
   intermediateResult = this.findResonances( i, j) ;
   if ( intermediateResult != null ) {
    auxiliar = result ;
    N = N + intermediateResult.length ;
    result = new Set[ N][2];
    for ( k = 0 ;
      k < N - intermediateResult.length ; ++k ) {
     result[k][0] = auxiliar[k][0];
     result[k][1] = auxiliar[k][1];
     }
     for ( k = 0 ;
       k < intermediateResult.length ; ++k) {
         result[ N - intermediateResult.length + k]
           [0 ] = intermediateResult[k][0];
         result[ N - intermediateResult.length + k]
           [ 1 ] =       intermediateResult[k][1];
     }
    }
   }
  }
 return result;
}
```

B.4 S-expansions

In this section we explain the Java classes and methods developed to perform the S-expansion method reviewed in Section 2.11. We also explain the methods to get the resonant subalgebras and perform reductions by the zero element.

B.4.1 The selectors

We need a class to represent the selectors. This is the Selector class:

```
public class Selector {
 int order ;
 int [][][] data ;
```

It has 2 variables: an integer to save the order of the semigroup and an array to save all the selectors in the semigroup. We choose

$$\text{data}[a][b][c] \equiv \mathcal{K}_{ab}^{c}$$

B.4.2 Representing the Lie algebra

We define the StructureConstantSet class to represent a Lie algebra.

```
public class StructureConstantSet {
 double [][][] constants ;
 int N ;
```

With the convention

$$[X_i, X_j] = C_{ij}^{k} X_k$$

we choose

$$\text{constants}[i][j][k] \equiv C_{ij}^{k}.$$

We can perform several simple operations with this set, like computing the Cartan-Killing metric of the algebra:

```
public Matrix cartanKillingMetric(){
 int a , b , c , d ;
 double sum = 0 ;
 double [][] metric = new double[N][N] ;
 for ( a = 0 ; a < N ; ++a) {
  for ( b = 0 ; b < N ; ++b ) {
   sum = 0 ;
```

```
   for ( c = 0 ; c < N ; ++c ) {
    for ( d = 0 ; d < N ; ++d ) {
     sum = sum + this.structureConstant(a , c , d )
        * this.structureConstant(b , d , c ) ;
    }
   }
   metric[a][b] = sum ;
  }
 }
 return new Matrix( metric );
}
```

Obviously we can set the values of any structure constant. The method setStructureConstant sets the values of C_{ab}^{c} and C_{ba}^{c}.

```
public void setStructureConstant
    ( int a , int b , int c, double fabc ) {
 constants[a ][b ][c ] = fabc ;
 constants[b ][a ][c ] = - fabc;
}
```

B.4.3 The S-expanded algebra

In double index notation, the Lie bracket of the S-expanded algebra is written as

$$[X_{i,\alpha}, X_{j,\beta}] = \mathcal{K}_{\alpha\beta}^{\gamma} C_{ij}^{k} X_{k,\gamma} \equiv C_{(i\alpha),(j\beta)}^{(k\gamma)} X_{k,\gamma}.$$

We are going to use this double index notation internally in our library. We create the ExpandedStructureConstantSet class:

```
public class ExpandedStructureConstantSet {
 // n is the number of generators of the algebra,
 // m is the order of the semigroup
 int n,m ;
 double [][][][][][] data ;
```

An object of this class has information about the dimension of the original Lie algebra and the order of the semigroup used to perform the S-expansion. We use a 6-dimensional array to save the structure constants, in a way such that

$$\mathrm{data}[a][\alpha][b][\beta][c][\gamma] \equiv C_{(a\alpha)(b\beta)}^{(c\gamma)}.$$

To get the S-expanded algebra we must follow the next steps:

1. To create a Semigroup object to store the semigroup which we want to use for the S-expansion.

2. To create a StructureConstantSet to store the original Lie algebra which we want to S-expand.

3. To ask to the Semigroup object to perform the S-expansion of the Lie algebra.

An example of a program performing the three steps above would be

```
public static void main(String[] args) {
 Matrix metric ;
 StructureConstantSet sl2
     = new StructureConstantSet(3) ;
 sl2.setStructureConstant(0, 1, 1, 2) ;
 sl2.setStructureConstant(0, 2, 2, -2) ;
 sl2.setStructureConstant(1, 2, 0, 1) ;
 metric = sl2.cartanKillingMetric() ;
 int [][] matrix = {{1,2,3,4},{2,3,4,4},
        {3,4,4,4}, {4,4,4,4}};
 Semigroup group = new Semigroup(matrix) ;
 ExpandedStructureConstantSet expandedAlgebra ;
 expandedAlgebra
      = group.getExpandedStructureConstant( sl2 );
 metric = expandedAlgebra.cartanKillingMetric() ;
}
```

The program above performs the expansion of $\mathfrak{sl}(2)$, written in the basis

$$
\begin{aligned}
[X_0, X_1] &= 2X_1, \\
[X_0, X_2] &= -2X_2, \\
[X_1, X_2] &= X_0.
\end{aligned}
$$

by the semigroup in Figure B.4.3;

B.4.4 The resonant subalgebra

To get a resonant subalgebra of an S-expanded algebra, we define the ResonantExpandedStructureConstantSet class, which extends the ExpandedStructureConstantSet class.

	λ_1	λ_2	λ_3	λ_4
λ_1	λ_1	λ_2	λ_3	λ_4
λ_2	λ_2	λ_3	λ_4	λ_4
λ_3	λ_3	λ_4	λ_4	λ_4
λ_4	λ_4	λ_4	λ_4	λ_4

Fig. B.1: Semigroup used in the example in Subsection B.4.3.

```
public class ResonantExpandedStructureConstantSet
    extends ExpandedStructureConstantSet {
  Set S0, S1 , V0 , V1 ;
```

'S0' and 'S1' represent the resonant decomposition of the semigroup, whilst 'V0' and 'V1' give the graded decomposition of the Lie algebra. To get the resonant subalgebra of an S-expanded algebra we must:

1. Get the correspondent S-expanded algebra, following the steps in the previous section.
2. Introduce the resonant decomposition S_0 and S_1.
3. Introduce the graded decomposition V_0 and V_1.
4. Use all these objects to create a ResonantExpandedStructureConstantSet object.

The following piece of code gets the resonant subalgebra of $\mathfrak{sl}(2)$ S-expanded by the semigroup in the previous section, given by the resonant decomposition $S_0 = \{1, 3, 4\}$ and $S_1 = \{2, 4\}$. Note that, with the generators that we have choosen for $\mathfrak{sl}(2)$, its graded decomposition is given by $V_0 = \{X_1\}$ and $V_1 = \{X_2, X_3\}$.

```
public static void main(String[] args) {
  //We introduce the structure constants of sl2
  StructureConstantSet sl2
    = new StructureConstantSet(3) ;
  sl2.setStructureConstant(0, 1, 1, 2) ;
  sl2.setStructureConstant(0, 2, 2, -2) ;
  sl2.setStructureConstant(1, 2, 0, 1) ;
  //Now we introduce the semigroup
  int [][] matrix = {{1,2,3,4},{2,3,4,4},
```

```
    {3,4,4,4}, {4,4,4,4}};
 Semigroup group = new Semigroup(matrix) ;
 //Next the resonant decomposition
 int[] mS0 = {1,3,4} ;
 Set S0 = new Set(mS0) ;
 int[] mS1 = {2,4} ;
 Set S1 = new Set(mS1) ;
 //And now the graded decomposition
 //of the original Lie algebra
 int[] mV0 = {1};
 Set V0 = new Set(mV0);
 int[] mV1 = {2,3};
 Set V1 = new Set(mV1);
 //We first calculate the S-expanded algebra
 ExpandedStructureConstantSet c =
   group.getExpandedStructureConstant( sl2 );
 //Next we use it to calculate its resonant subalgebra
 ResonantExpandedStructureConstantSet rc =
  new ResonantExpandedStructureConstantSet
     ( c.data , S0 , S1 , V0 , V1 ) ;
}
```

B.4.5 S-expanded algebra followed by a reduction by the zero element

To perform the reduction by the zero element of an S-expanded Lie algebra we define the ReducedExpandedStructureConstantSet class, which extends the ExpandedStructureConstantSet class.

```
public class ReducedExpandedStructureConstantSet
    extends ExpandedStructureConstantSet {
 int zero ;
```

With respect to the ExpandedStructureConstantSet class, it only adds an integer variable to save the zero element.

To perform the reduction by zero of an S-expanded algebra we have to:

1. Get the S-expanded algebra in a ExpandedStructureConstantSet.
2. Use it to create a ReducedExpandedStructureConstantSet.

The semigroup which we have been using has the zero element λ_4. The following code performs the reduction by zero.

```
public static void main(String[] args) {
 Matrix metrica ;
 //We introduce the structure constants
 // of sl2
 StructureConstantSet sl2
    = new StructureConstantSet(3) ;
 sl2.setStructureConstant(0, 1, 1, 2) ;
 sl2.setStructureConstant(0, 2, 2, -2) ;
 sl2.setStructureConstant(1, 2, 0, 1) ;
 //Now we introduce the semigroup
 int [][] matriz = {{1,2,3,4},{2,3,4,4},
      {3,4,4,4}, {4,4,4,4}};
 Semigroup grupo = new Semigroup(matriz) ;
 //We perform the reduction by the
 // zero element, which is \lambda_4
ReducedExpandedStructureConstantSet
     algebraExpandidayReducida =
       new ReducedExpandedStructureConstantSet(
   (grupo.getExpandedStructureConstant(sl2).data ),4);
}
```

B.4.6 Resonant subalgebra followed by a reduction by the zero element

We define the class ReducedResonantExpandedStructureConstantSet, which inherits from the class ResonantExpandedStructureConstantSet.

```
public
  class ReducedResonantExpandedStructureConstantSet
   extends ResonantExpandedStructureConstantSet {
 int zero;
```

We just add an integer to save the zero element. Its use is analogous to that of the ResonantExpandedStructureConstantSet.

B.5 Isomorphisms and templates

Imagine that we have to find a semigroup with some multiplications fixed, as for example

$$\begin{pmatrix} \_ & 3 & \_ & 4 \\ 3 & \_ & 4 & 4 \\ \_ & 4 & \_ & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}.$$

This is a semigroup of order 4. If we think of this as a template, there are $4^5 = 1024$ different ways to fill it, where only some of them are associative so they are really semigroups. If we ask for a commutative multiplication table, there are 256 different ways to fill it. In general, for a semigroup of order n with x given elements, there are $(n^2 - x)^n$ different ways to fill it. Then, filling in a random way the template and checking for associativity is not a convenient way to look for semigroups filling the template.

We propose an alternative. We have a list with all the non isormorphic semigroups of a given order. What we do is checking if, for any of the semigroups, any of its $n!$ isomorphic semigroups fills the template. Depending on the number of fixed elements in the multiplication table, this can be much faster than the previous approach. It has the advantadge that we are sure that all the results that we get are non isomorphic semigroups, and we obtain its unique identifier in the list generated by the program **gen.f**.

As in Java an empty space can not be left in an array, we choose to represent them by a -1 in its place. For example, the template above will be represented by

$$\begin{pmatrix} -1 & 3 & -1 & 4 \\ 3 & -1 & 4 & 4 \\ -1 & 4 & -1 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}.$$

The method isTemplateFor checks if a semigroup can fill a given template.

```
public boolean isTemplateFor( Semigroup B) {
 int i , j ;
 if ( this.order != B.order ) {
  return false ;
 }
 else {
```

```
  for ( i = 0 ; i < this.order ; ++ i) {
   for ( j = 0 ; j < this.order ; ++j) {
    if ( this.data[i][j]!= -1) {
     if ( this.data[i][j] != B.data[i][j]) {
      return false;
     }
    }
   }
  }
 }
 return true;
}
```

In some applications we may need to know if a semigroup fills a given template. What we could do in this case is taking the list of non isomorphic semigroups of a given order and check if any of his isomorphic groups fills the template. Essentially we do this, but getting a list of isomorphic and antiisomorphic templates and checking if any of the semigroups in the lists fits any of the templates. We do this with the method isIsoTemplateFor.

```
public boolean isIsoTemplateFor( Semigroup B) {
 Semigroup [] isos = this.Permute() ;
 Semigroup [] antis = this.AntiPermute() ;
 int i ;
 for ( i = 0 ; i < isos.length ; ++i) {
  if ( isos[i].isTemplateFor(B)) {
   //Isomorphism
   return true;
  }
 }
 for ( i = 0 ; i < antis.length ; ++i){
  if ( antis[i].isTemplateFor(B)) {
   //Antiisomorphism
   return true;
  }
 }
 return false;
}
```

Next we show a Java program to look for ways to fill (modulo isomorphism) the template used in this examples.

```
public static void main(String[] args) {
 Semigroup[] list = Semigroup.loadFromFile() ;
 int i , k = 0 ;
 Semigroup [] perms = null;
 int [][] mTemplate = {{-1 , 3 , -1 , 4},
    { 3, -1, 4, 4},{ - 1, 4, -1 ,4},{ 4, 4, 4, 4}};
 Semigroup template = new Semigroup(mTemplate);
 for ( i = 0 ; i < list.length ; ++i) {
  if ( list[i].order == 4 ) {
   if ( template.isIsoTemplateFor(list[i])
               && list[i].isCommutative()) {
     System.out.println
("A semigroup filling the template has been found.");
         System.out.print("#");
         System.out.println(list[i].ID);
         list[i].show();
         System.out.println
           ("Let us see its isomorphic semigroups:");
         perms = list[i].Permute() ;
         for ( k = 0 ; k < perms.length ; ++k) {
          System.out.println(k);
          perms[k].show();
     }
    }
   }
  }
}
```

B.6 Applications

B.6.1 All the semigroups of a given order with a zero element

In some applications it is interesting to have a list of all the semigroups of a given order which have a zero element, i.e., if we want to perform all the possible S-expansions followed by a reduction by zero of a given Lie algebra.

With the library that we have developed, it is trivial to write a Java program to do this.

```
public static void main(String[] args) {
 Semigroup[] list = Semigroup.loadFromFile();
 int zeroElement ;
 int i;
 for ( i = 0 ; i < list.length ; ++i){
  zeroElement = list[i].findZero();
  if ( zeroElement != -1 && list[i].isCommutative()){
   System.out.print("#");
   System.out.println(list[i].ID);
   list[i].show();
   System.out.print("The zero element is ");
   System.out.println(zeroElement);
  }
 }
}
```

A sample output is shown next:

```
#14295
1 1 1 1 1 1
1 2 3 3 5 5
1 3 5 5 2 2
1 3 5 5 2 2
1 5 2 2 3 3
1 5 2 2 3 4
The zero element is 1
#14297
1 1 1 1 1 1
1 2 3 3 5 6
1 3 2 2 6 5
1 3 2 2 6 5
1 5 6 6 2 3
1 6 5 5 3 2
The zero element is 1
#14298
1 1 1 1 1 1
1 2 3 3 5 6
```

```
1 3 2 2 6 5
1 3 2 2 6 5
1 5 6 6 3 2
1 6 5 5 2 3
The zero element is 1
#14301
1 1 1 1 1 1
1 2 3 3 5 6
1 3 5 5 6 2
1 3 5 5 6 2
1 5 6 6 2 3
1 6 2 2 3 5
The zero element is 1
```

B.6.2 All the resonant decomposition of order N semigroups

A list containing all the possible resonant decomposition of all the semigroups of a given order is useful, for example, if we want to study all the possible resonant subalgebras of all the possible S-expanded algebras of a given Lie algebra. Creating a program to get this list is trivial with the Java classes that we have developed.

```java
public static void main(String[] args) {
 Semigroup[] list = Semigroup.loadFromFile();
 Set [][] resonances;
 int i, k;
 int [] N = {0,0,0,0,0} ;
 for ( i = 0 ; i < list.length ; ++i){
  if ( list[i].isCommutative()){
   resonances = list[i].findAllResonances();
   if ( resonances != null && list[i].order == 4) {
    System.out.print("Group #");
    System.out.print(list[i].ID);
    System.out.print(" , order ");
    System.out.println(list[i].order);
    list[i].show();
    System.out.print( resonances.length) ;
    System.out.println(" resonances found");
    N[list[i].order - 1 ] = N[list[i].order - 1 ] + 1;
    for ( k = 0 ; k < resonances.length ; ++k) {
```

```
      System.out.print( "Resonance #");
      System.out.println(k +1 );
      System.out.println("S0 : ");
      resonances[k][0].show();
      System.out.println("S1 : ");
      resonances[k][1].show();
     }
    }
   }
  }
  System.out.print("Order=2: ");
  System.out.println(N[1]);
  System.out.print("Order=3: ");
  System.out.println(N[2]);
  System.out.print("Order=4: ");
  System.out.println(N[3]) ;
  System.out.print("Order=5: ");
  System.out.println(N[4]);
}
```

Next we show a sample output:

```
Group #42 , order 4
1 1 1 1
1 1 1 2
1 1 1 3
1 2 3 4
5 resonances found
Resonance #1
S0 :
1 4
S1 :
1 2 3
Resonance #2
S0 :
1 3 4
S1 :
1 2
Resonance #3
S0 :
```

```
1 2 4
S1 :
1 3
Resonance #4
S0 :
1 3 4
S1 :
1 2 3
Resonance #5
S0 :
1 2 4
S1 :
1 2 3
Group #43 , order 4
1 1 1 1
1 1 1 2
1 1 2 3
1 2 3 4
2 resonances found
Resonance #1
S0 :
1 2 4
S1 :
1 3
Resonance #2
S0 :
1 2 4
S1 :
1 2 3
```

B.6.3 S-related algebras in Bianchi's classification

In Chapter 9 it has been studied the possibility to obtain the 3-dimensional Lie algebras classified by Bianchi by an S-expansion of a 2-dimensional one.

To find Bianchi's type II algebra

$$\begin{aligned}
&[Y_1, Y_2] = [Y_1, Y_3] = 0, \\
&[Y_2, Y_3] = Y_1,
\end{aligned}$$

after the S-expansion of the 2 dimensional Lie algebra

$$[X_1, X_2] = X_1,$$

the semigroup used must have a resonant decomposition with $S_0 = \{2, 4\}$ and $S_1 = \{1, 3, 4\}$ and 4 must be its zero element. In the end, the problem is reduced to find a semigroup filling the template

$$\begin{pmatrix} \_ & 3 & \_ & 4 \\ 3 & \_ & 4 & 4 \\ \_ & 4 & \_ & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}$$

in a such way that it satisfies the resonant decomposition given by $S_0 = \{2, 4\}$ and $S_1 = \{1, 3, 4\}$.

With the Java library that we have developed it is not difficult to implement a program to solve this problem.

```
public static void main(String[] args) {
 Semigroup[] list = Semigroup.loadFromFile() ;
 int i , k = 0 ;
 Semigroup [] perms = null;
 int [][] mTemplate = {{-1, 3, -1,4},{3 ,-1,4,4},
  { - 1 , 4 , -1 , 4},{ 4 , 4 , 4 , 4 }};
 Semigroup template = new Semigroup(mTemplate);
 int[] mS0 = { 2,4};
 int[] mS1 = {1,3,4};
 Set S0 = new Set( mS0);
 Set S1 = new Set( mS1 ) ;
 for ( i = 0 ; i < list.length ; ++i) {
  if (list[i].order == 4 && list[i].isCommutative()){
   if ( template.isIsoTemplateFor(list[i]) ) {
    System.out.println
 ("A semigroup filling the template has been found:");
    System.out.print("#");
    System.out.println(list[i].ID);
    list[i].show();
    //Let's see if this semigroup
    //also has the resonant decomposition
    perms = list[i].Permute() ;
    for ( k = 0 ; k < perms.length ; ++k) {
     if ( perms[k].isResonant(S0, S1)
```

```
        && template.isTemplateFor(perms[k]) ) {
        System.out.println("This one is resonant!!!");
        System.out.println(k);
        perms[k].show();
        }
      }
    System.out.println("To compare with:");
    template.show();
    }
  }
 }
}
```

Next, we can see a sample output of this program:

```
A semigroup filling the template has been found:
#12
1 1 1 1
1 1 1 1
1 1 1 2
1 1 2 3
This one is resonant!!!
23
2 3 4 4
3 4 4 4
4 4 4 4
4 4 4 4
```

B.6.4 Semisimplicity of S-expanded algebras

In Chapter 8 a study of the properties of Lie algebras preserved under the S-expansion procedures was performed. One of the properties studied was the semisimplicity of the S-expanded algebras.

The program that we need must calculate all the possible S-expansions of a given Lie algebra ($\mathfrak{sl}(2)$ in this case) and check if the determinant of its Cartan-Killing metric is different than zero.

Next we show a program which calculates all the S-expansions of $\mathfrak{sl}(2)$ by semigroups of order 3.

```
public static void main(String[] args) {
```

```
Matrix metric ;
Semigroup [] listOfSemigroups
  = Semigroup.loadFromFile() ;
// We introduce the structure constants
// of sl2
StructureConstantSet sl2
  = new StructureConstantSet(3) ;
sl2.setStructureConstant(0, 1, 1, 2) ;
sl2.setStructureConstant(0, 2, 2, -2) ;
sl2.setStructureConstant(1, 2, 0, 1) ;
metric = sl2.cartanKillingMetric() ;
System.out.println("We show the metric of sl2");
metric.print(2, 2) ;
System.out.println("Now its determinant");
System.out.println(metric.det());
Semigroup group = null ;
ExpandedStructureConstantSet expandedAlgebra ;
int i ;
for ( i = 0 ; i < listOfSemigroups.length ; ++i) {
 group = listOfSemigroups[i] ;
 if ( group.order == 3 && group.isCommutative()) {
  System.out.print("Expanding by the semigroup #");
  System.out.println(group.ID);
  expandedAlgebra
    = group.getExpandedStructureConstant( sl2 );
  metric = expandedAlgebra.cartanKillingMetric() ;
  if (  metric.det() !=  0 ) {
  System.out.println
    ("We have found a semisimple algebra");
  System.out.println
    ("We have expanded by the semigroup:");
  group.show();
  System.out.println
    ("We show the metric of the S-expanded algebra");
  metric.print(2, 2);
  System.out.print("Determinant of the metric:");
  System.out.println(metric.det());
  }
```

```
    }
  }
}
```

We show a sample of output of the program.

```
Expanding by the semigroup #16
We have found a semisimple algebra
We have expanded by the semigroup:
1 1 3
1 2 3
3 3 1
We show the metric of the S-expanded algebra

 16.00 16.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00
 16.00 24.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00
 0.00 0.00 16.00 0.00 0.00 0.00 0.00 0.00 0.00
 0.00 0.00 0.00 0.00 0.00 0.00 8.00 8.00 0.00
 0.00 0.00 0.00 0.00 0.00 0.00 8.00 12.00 0.00
 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 8.00
 0.00 0.00 0.00 8.00 8.00 0.00 0.00 0.00 0.00
 0.00 0.00 0.00 8.00 12.00 0.00 0.00 0.00 0.00
 0.00 0.00 0.00 0.00 0.00 8.00 0.00 0.00 0.00

Determinant of the metric: -1.34217728E8
```

Similar programs have been written for the rest of the calculations proposed in Chapter 8.

C. SEMIGROUPS

In this Appendix we give explicitly the multiplication tables of the semigroups of order 3. This information is shown in Figure C.1.

Next we give the multiplication table of the semigroups that we have used in this paper and that belong to the list generated by the program *com.f* of [44] for $n = 4$. Those semigroups are given in Figure C.2.

$S^1_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1
λ_3	λ_1	λ_1	λ_1

$S^2_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1
λ_3	λ_1	λ_1	λ_2

$S^3_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1
λ_3	λ_1	λ_1	λ_3

$S^4_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1
λ_3	λ_1	λ_2	λ_3

$S^5_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1
λ_3	λ_3	λ_3	λ_3

$S^6_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_2
λ_3	λ_1	λ_2	λ_3

$S^7_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_2	λ_1
λ_3	λ_1	λ_1	λ_3

$S^8_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_2	λ_1
λ_3	λ_3	λ_3	λ_3

$S^9_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_2	λ_2
λ_3	λ_1	λ_2	λ_2

$S^{10}_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_2	λ_2
λ_3	λ_1	λ_2	λ_3

$S^{11}_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_2	λ_2
λ_3	λ_1	λ_3	λ_3

$S^{12}_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_2	λ_3
λ_3	λ_1	λ_3	λ_2

$S^{13}_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_2	λ_3
λ_3	λ_3	λ_3	λ_3

$S^{14}_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_1
λ_2	λ_2	λ_2	λ_2
λ_3	λ_3	λ_3	λ_3

$S^{15}_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_3
λ_2	λ_1	λ_1	λ_3
λ_3	λ_3	λ_3	λ_1

$S^{16}_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_1	λ_3
λ_2	λ_1	λ_2	λ_3
λ_3	λ_3	λ_3	λ_1

$S^{17}_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_2
λ_2	λ_2	λ_1	λ_1
λ_3	λ_2	λ_1	λ_1

$S^{18}_{(3)}$	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3
λ_2	λ_2	λ_3	λ_1
λ_3	λ_3	λ_1	λ_2

Fig. C.1: All the semigroups of order 3

$S_{(4)}^{10}$	λ_1	λ_2	λ_3	λ_4
λ_1	λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1	λ_1
λ_3	λ_1	λ_1	λ_1	λ_2
λ_4	λ_1	λ_1	λ_2	λ_1

$S_{(4)}^{12}$	λ_1	λ_2	λ_3	λ_4
λ_1	λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1	λ_1
λ_3	λ_1	λ_1	λ_1	λ_2
λ_4	λ_1	λ_1	λ_2	λ_3

$S_{(4)}^{13}$	λ_1	λ_2	λ_3	λ_4
λ_1	λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1	λ_1
λ_3	λ_1	λ_1	λ_1	λ_3
λ_4	λ_1	λ_1	λ_3	λ_4

$S_{(4)}^{28}$	λ_1	λ_2	λ_3	λ_4
λ_1	λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1	λ_1
λ_3	λ_1	λ_1	λ_3	λ_4
λ_4	λ_1	λ_1	λ_4	λ_3

$S_{(4)}^{42}$	λ_1	λ_2	λ_3	λ_4
λ_1	λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1	λ_2
λ_3	λ_1	λ_1	λ_1	λ_3
λ_4	λ_1	λ_2	λ_3	λ_4

$S_{(4)}^{43}$	λ_1	λ_2	λ_3	λ_4
λ_1	λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1	λ_2
λ_3	λ_1	λ_1	λ_2	λ_3
λ_4	λ_1	λ_2	λ_3	λ_4

$S_{(4)}^{44}$	λ_1	λ_2	λ_3	λ_4
λ_1	λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1	λ_2
λ_3	λ_1	λ_1	λ_3	λ_1
λ_4	λ_1	λ_2	λ_1	λ_4

$S_{(4)}^{45}$	λ_1	λ_2	λ_3	λ_4
λ_1	λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_1	λ_1	λ_2
λ_3	λ_1	λ_1	λ_3	λ_3
λ_4	λ_1	λ_2	λ_3	λ_4

$S_{(4)}^{64}$	λ_1	λ_2	λ_3	λ_4
λ_1	λ_1	λ_1	λ_1	λ_1
λ_2	λ_1	λ_2	λ_1	λ_1
λ_3	λ_1	λ_1	λ_3	λ_4
λ_4	λ_1	λ_1	λ_4	λ_3

Fig. C.2: Review of the semigroups of order 4 used in Chapter 9.

BIBLIOGRAPHY

[1] D. Cervantes, R. Fioresi, M. A. Lledó. *The quantum chiral Minkowski and conformal superspaces.* Adv. Math. Phys.
D. Cervantes, R. Fioresi, M. A. Lledó. *On chiral quantum superspaces.* Contribution to 'Supersymmetry in Mathematics and Physics'. Lecture Notes in Mathematics Vol 2027. S. Ferrara, R. Fioresi, V. S. Varadarajan, eds. (2011).

[2] D. Cervantes, R. Fioresi, M. A. Lledó, F. Nadal. *Quantum twistors.* hep-th/1109.4438 (2011).

[3] D. Cervantes, R. Fioresi, M. A. Lledó, F. Nadal. *Quadratic deformation of Minkowski space.* Fortschr. Phys. 1-7 (2012).

[4] M. A. Lledó, F. Nadal, M. P. Petropoulos. *Invariant sigma models and group contractions.* In preparation.

[5] L. Andrianopoli, R. D'Auria, N. Merino, F. Nadal, M. Trigiante. *Formal aspects of expansion methods and Lie theory.* Work in progress.

[6] R. Caroca, I. Kondrashuk, N. Merino, F. Nadal. *Bianchi spaces and their 3-dimensional isometries as S-expansions of 2-dimensional isometries*, (2012). arXiv:1104.3541v2 [math-ph].

[7] Z. Bern, L. J. Dixon, R. Roiban. *Is $N = 8$ supergravity ultraviolet finite?*. Phys. Lett. B **644** 265-271 (2007).

[8] J. Madore. *The fuzzy sphere*. Class. Quantum Grav. **9** 69-87 (1992).

[9] S. Doplicher, K. Fredenhagen and J. E. Roberts. *The quantum structure of space-time at the Planck scale and quantum fields.* Commun. Math. Phys. **172** 187-220 (1995).

[10] A. Connes. *Non Commutative Geometry.* Academic Press. (1994).

[11] A. Connes, M. R. Douglas and A. Schwarz. *Non commutative geometry and Matrix theory:Compactification on tori*. JHEP **9802** 003 (1998).

[12] R. Penrose. *Twistor algebra*. J. Math. Phys. **8**, 345-366 (1967).
R. Penrose and M. A. H. MacCallum. *Twistor theory: an approach to the quantisation of fields and space-time*. Phys. Rep. **6** n. 4 241-316 (1972).
R. S. Ward and R. O. Wells. *Twistor geometry and field theory*. Cambridge University Press (1990).

[13] V. S. Varadarajan. *Supersymmetry for mathematicians: an introduction*. Courant Lecture Notes, **1**. AMS (2004).

[14] C. Kassel. *Quantum groups*. Springer Verlag, (1995).
S. Majid. *A quantum groups primer.* Cambridge University Press, Cambridge (2002).
V. Chari and A. Pressley. *A guide to quantum groups*. Cambridge University Press, (1994).

[15] L. Andrianopoli, S. Ferrara, M. A. Lledó, O. Maciá. *Integration of massive states as contractions of non linear sigma models.* J. Math. Phys. **46**, 072307. 1-20. (2005)
M. A. Lledó, O. Maciá. *Contractions of sigma models and integration of massive modes*. Proceedings of the IV International Symposium of Quantum Theory and Symmetries. V.K. Dobrev ed. Heron Press, Bulgaria. Vol. 1 311-319 (2006).

[16] E. Inönü and E. P. Wigner. *On the contractions of groups and their representations*. Proc. Nat. Acad. Sci. USA **39** 510 (1953).

[17] E. Weimar-Woods. *Contractions of Lie algebras: Generalized Inönü-Wigner contractions versus graded contractions*. J. Math. Phys. **36** 4519 (1995).

[18] C. M. Hull, B. Spence. *The gauged nonlinear sigma model with Wess-Zumino term*. Phys. Lett. B **232** 204-210, (1989).

[19] M. Hatsuda and M. Sakaguchi. *Wess-Zumino term for the AdS Superstring and generalized Inönü-Wigner contraction*. Prog. Theor. Phys. **109**, 853 (2003).

[20] J. A. de Azcárraga, J. M. Izquierdo, M. Picón, and O. Varela. *Generating Lie and gauge free differential (Super)algebras by expanding Maurer-Cartan forms and Chern-Simons Supergravity.* Nucl. Phys. B **662**, 185 (2003).

[21] J. A. de Azcárraga, J. M. Izquierdo, M. Picón and O. Varela. *Extensions, expansions, Lie algebra cohomology and enlarged superspaces.* Class. Quant. Grav. **21**, 1375 (2004).

[22] J. A. de Azcárraga, J. M. Izquierdo, M. Picón and O. Varela. *Expansions of algebras and superalgebras and some applications.* Int. Jor. Theor. Phys. **46** 2738-2752 (2007).

[23] F. Izaurieta, E. Rodríguez and P. Salgado. *Expanding Lie (super)algebras through abelian semigroups.* J. of Math. Phys. **47** 123512 (2006).

[24] N. Merino, F. Nadal *A Java library to perform S-expansions of Lie algebras.* In preparation.

[25] F. Izaurieta, E. Rodríguez, P. Salgado. *Eleven-dimensional gauge theory for the M algebra as an abelian semigroup expansion of* $\mathfrak{osp}(32/1)$. Eur. Phys. Jour. C **10**. 1140 (2008).

[26] F. Izaurieta, A. Pérez, E. Rodríguez, P. Salgado. *Dual formulation of the Lie algebra S-expansion procedure.* Jour. Math. Phys. **50** 7 073511 (2009).

[27] F. Izaurieta, P. Minning, A. Pérez, E. Rodríguez, Patricio Salgado. *Standard General Relativity from Chern-Simons Gravity.* Phys. Lett. B **678** 213-217 (2009).

[28] J. Díaz, O. Fierro, F. Izaurieta, N. Merino, E. Rodríguez, P. Salgado, O. Valdivia. *A generalized action for* $(2{+}1)$*-dimensional Chern-Simons Gravity.* Submitted to J. Phys. A. January (2012).

[29] L. Bianchi. *Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti.* Memorie di Matematica e di Fisica della Societa Italiana delle Scienze, Serie Terza, Tomo XI, pp. 267–352 (1898).

[30] R. Fioresi, M. A. Lledó and V. S. Varadarajan. *The Minkowski and conformal superspaces.* J. Math. Phys. **48**, 113505 1-27 (2007).

[31] Y. Manin. *Multiparametric quantum deformation of the general linear supergroup*. Comm. Math. Phys. **123**, 163-175, (1989).

[32] R. Fioresi. *Quantizations of flag manifolds and conformal space time.* Rev. Math. Phys. **9**, n. 4, 453-465, (1997).

[33] R. Fioresi. *A deformation of the big cell inside the Grassmannian manifold $G(r,n)$*. Rev. Math. Phys. **11**, 25-40 (1999).

[34] R. Fioresi. *Quantum deformation of the flag variety*. Comm. Alg. **27**, n. 11 (1999).

[35] S. Helgason. *Differential geometry, Lie groups and symmetric spaces.* Academic Press. USA (1978).

[36] R. Plemmons. *A survey of computer applications to semigroups and related structures*. ACM SIGSAM **12**, 28-39 (1969).

[37] F. Izaurieta, E. Rodríguez and P. Salgado. *Expanding Lie (super)algebras through abelian semigroups*. J. Math. Phys. **47** 123512 (2006).

[38] P. Di Francesco, P. Mathieu, D. Sénéchal. *Conformal field theory.* Springer Verlag, New York (1996).

[39] F. Nadal. *Conformal field theory: from symmetry to gauged Wess-Zumino-Witten models*. Master thesis. Universitat de València (2009).

[40] J. Wess and B. Zumino. *Consequences of anomalous Ward identities.* Phys. Lett. B **37**, 95-97 (1971).

[41] C. Kassel. *Quantum groups*. Springer Verlag, (1995).

[42] M. Kontsevich. *Deformation quantization of Poisson manifolds 1.* Lett. Math. Phys. **66** 157-216, (2003).

[43] M. B. Green, J. M. Schwarz, E. Witten. *Superstring theory.* Cambridge University Press (1987).

[44] J. Hildebrant. *Handbook of finite semigroup programs*. LSU Mathematics Electronic Preprint Series 2003-9 (2003).

[45] A. O. Barut and R. Ratzka. *Theory of group representations and applications.* World Scientific, 1986; R. Gilmore. *Lie groups, Lie algebras, and some of their applications.* Wiley-Interscience Publication, Johan Wiley Sons (1974).

[46] V. Chari and A. Pressley. *A guide to quantum groups.* Cambridge University Press, (1994).

[47] R. Fioresi, M. A. Lledó, F. Nadal. *The superconformal and Minkowski spaces as homogeneous spaces.* In preparation.

[48] G. Bergman. *The diamond lemma for ring theory.* Adv. in Math. **29** 178-218, (1978).

[49] A. Klimyk, K. Schmüdgen. *Quantum groups and their representations.* Berlin, Springer, (1997).

[50] A. Distler. *Classification and enumeration of finite semigroups.* PhD Thesis. Pure Mathematics Thesis, University of St Andrews (2010).