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Gauss words and the topology of finitely determined map germs from $\mathbb{R}^n$ to $\mathbb{R}^n$

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Tras un largo camino, en poco tiempo pondré fin, con la defensa oral de este documento que aquí presento, a una etapa de mi vida. Esta tesis, pese a haber sido creada por el trabajo de estos últimos 5 años, ha sido posible gracias a una etapa formativa mucho más extensa y, como consecuencia, a todas aquellas personas que me han hecho posible disfrutar de ella.

Evidentemente, mis padres han sido un pilar fundamental para mi, siendo severos cuando era necesario, pero al mismo tiempo dándome su apoyo incondicional y respetando todas mis decisiones. Mi familia en general. Mi novia, que pese a haber compartido conmigo sólo estos últimos 3 años, ha sido clave en la consecución de este trabajo, confiando en mí más que yo mismo en algunos momentos. Mi director de tesis, que, aparte de introducirme en el mundo de las singularidades y ser junto conmigo el principal artífice de este trabajo, siempre ha estado ahí cuando lo he necesitado y con su humildad y forma de ser me ha convertido no sólo en un mejor matemático sino, lo que es más importante, en una mejor persona.

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Everyone wants in life to find a job to be happy, to enjoy it. Work in mathematics gives me such happiness and personal satisfaction and, despite the difficulties that may arise in this new stage of my life that begins from this moment, I will not cease in the effort. Thank you very much to all of you.

Chapter 1

Introducción-Introduction

1.1 Introducción

El objetivo de este trabajo es clasificar gérmenes de aplicación finitamente determinados $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ en los casos $n = 2, 3$ desde un punto de vista topológico, es decir, que son diferentes salvo homeomorfismo en el origen y en la llegada. Esta clasificación en ambos casos se basará en la construcción de un invariante topológico completo de los links asociados a un respresentante adecuado de estos gérmenes. Prestaremos también especial atención a las familias de gérmenes a un parámetro en el caso del plano ($f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$) probando que, bajo ciertas condiciones, son topológicamente triviales.

Las singularidades estables de aplicaciones del plano en el plano fueron estudiadas por primera vez por H. Whitney en su famoso artículo [37]. Él mostró que para una aplicación C^∞ genérica $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, el germen de f en cualquier punto $p \in U$ es, o bien regular, o de tipo pliegue o de tipo cúspide. Esto significa que f es $\mathcal{A}$-equivalente en p a $(x, y) \mapsto (x, y)$, $(x, y) \mapsto (x, y^2)$ o $(x, y) \mapsto (x, y^3 + xy)$, respectivamente. Más aún, si consideramos también multigérmenes tenemos que añadir una singularidad estable más, denominada pliegue doble transversal. En la figura 1.1 tenemos una imagen típica que representa una aplicación estable del plano al plano.

Para las aplicaciones estables de $\mathbb{R}^3$ en $\mathbb{R}^3$, siguiendo las técnicas de clasificación de Mather (véase por ejemplo [12]) es fácil comprobar que localmente sólo podemos tener una de las seis situaciones que aparecen en la figura 1.2.

Cuando $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ no es estable pero es finitamente determinado, el origen es una inestabilidad aislada por el criterio de Mather-Gaffney (véase [36]). En particular, existe un representante suficientemente pequeño $f : U \to V$ donde U, V son subconjuntos abiertos de $\mathbb{R}^n$ tales que f es es-

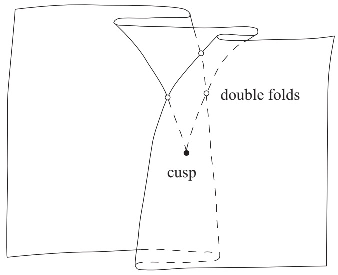


Figure 1.1: Singularidades estables de $\mathbb{R}^2$ en $\mathbb{R}^2$

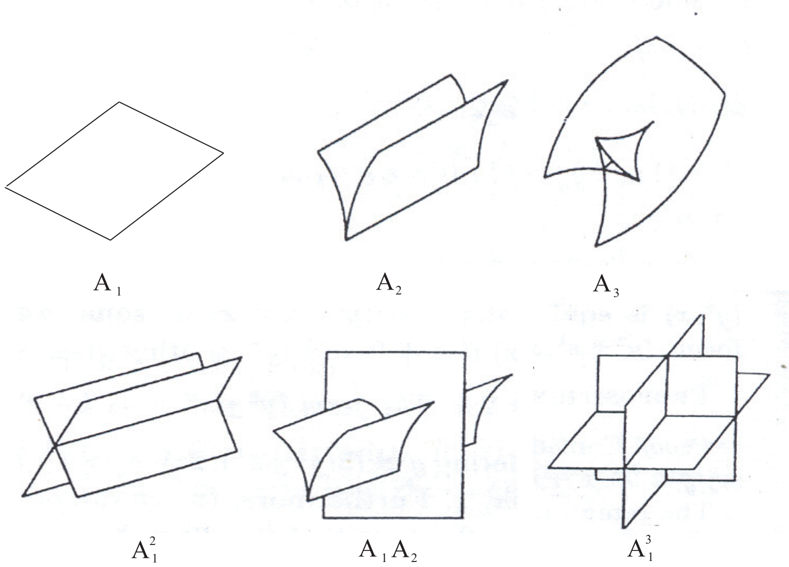


Figure 1.2: Singularidades estables de $\mathbb{R}^3$ en $\mathbb{R}^3$

table en $U \setminus \{0\}$. La estructura topológica de f viene determinada por el link de f, que se obtiene tomando la intersección de la imagen de f con una $(n-1)$-esfera suficientemente pequeña centrada en el origen S_ϵ^{n-1}. Usamos un teorema de Fukuda [5], que asegura que el link de f es una aplicación estable de S^{n-1} a S^{n-1} y que f es topológicamente equivalente al cono de su link. Nuestro trabajo será intentar clasificar topológicamente estas aplicaciones estables, y como consecuencia obtener la clasificación de los gérmenes de aplicación correspondientes.

La clasificación topológica de gérmenes de aplicación finitamente determinados del plano en el plano en el caso analítico complejo $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ ha sido hecha por Gaffney y Mond en [10, 9], restringiéndose a los polinomios casihomogéneos. En el caso analítico real, J.H.Rieger en [34] completa la clasificación $\mathcal{A}$-simple de los gérmenes de aplicación del plano en el plano de corrango 1. También Nishimura estudió en [30] la $\mathcal{K}$-equivalencia topológica de gérmenes finitos $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ y probó que el valor absoluto del grado topológico es un invariante topológico completo.

Para el caso equidimensional $n = 3$ debemos hacer referencia al trabajo de W.L.Marar y F.Tari([19]) donde estudian la $\mathcal{A}$-clasificación de estos gérmenes en el caso real.

Las técnicas principales usadas a lo largo de este trabajo ya han sido empleadas por otros autores([21], [18]) para obtener la clasificación completa de superficies regladas en $\mathbb{R}^3$ en el primer caso y una clasificación parcial de gérmenes de aplicación finitamente determinados de $\mathbb{R}^2$ en $\mathbb{R}^3$ en el último caso.

1.1.1 Organización del trabajo

Este trabajo está dividido en 7 capítulos, incluyendo éste que contiene una pequeña introducción al problema de estudio y un último donde el lector puede encontrar las conclusiones finales de los resultados obtenidos en este trabajo, así como algunas problemas abiertos motivadores.

En el capítulo 2 enunciamos algunos resultados preliminares, como la definición de una aplicación estable o de un germen de aplicación finitamente determinado, que se creen fundamentales para la correcta comprensión de los capítulos siguientes.

Los capítulos 3, 4 y 5 están enteramente dedicados al estudio de gérmenes de aplicación finitamente determinados $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. En el primero introducimos la definición de palabras de Gauss en el caso particular de estas aplicaciones, probando que son un invariante topológico completo y también damos una amplia clasificación topológica en el caso de corrango 1. En el capítulo 5 extendemos esta clasificación a gérmenes de aplicación de corrango 2 que son del tipo $\Sigma^{2,0}$ y en el capítulo 4 consideramos familias a 1 parámetro de gérmenes de aplicación finitamente determinados probando que bajo ciertas hipótesis son topológicamente triviales.

Finalmente en el capítulo 6 afrontamos la difícil tarea de tratar de extender los resultados del caso del plano a gérmenes de aplicación de $\mathbb{R}^3$ en $\mathbb{R}^3$. Aquí probamos que, con algunas restricciones en nuestros gérmenes, las palabras de Gauss son también un invariante topológico completo para este tipo de gérmenes y damos como aplicación la clasificación topológica de los gérmenes que pertenecen a la $\mathcal{A}^2$-clase (x, y, xz) en algunos casos particulares y la clasificación completa de gérmenes de aplicaciones regladas de $\mathbb{R}^3$ en $\mathbb{R}^3$.

1.2 Introduction

The aim of this work is to classify finitely determined map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ in the cases $n = 2, 3$ from a topological point of view, that is, they are different up to homeomorphism in the source and target. This classification in both cases will be based in the construction of a complete topological invariant of the links associated to a suitable representative of this map germs. We also pay special attention to the 1-parameter families in the planar case ($f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$) proving that, under certain conditions, they become topologically trivial.

The stable singularities of maps from the plane to the plane were studied for the first time by H. Whitney in his famous paper [37]. He showed that for a generic smooth map $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, the germ of f at any point $p \in U$ is either regular, of fold type or of cusp type. This means that f is $\mathcal{A}$-equivalent at p to either $(x, y) \mapsto (x, y)$, $(x, y) \mapsto (x, y^2)$ or $(x, y) \mapsto (x, y^3 + xy)$, respectively. Moreover, if we consider also multigerms, then we have to add one more stable singularity, namely the transverse double fold. In figure 1.3 we find a typical image which represents a stable map from the plane to the plane.

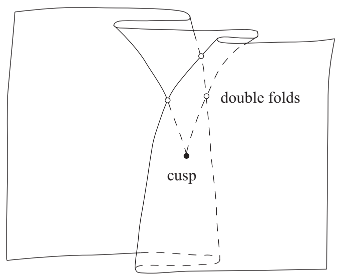


Figure 1.3: Stable singularities from $\mathbb{R}^2$ to $\mathbb{R}^2$

For stable maps from $\mathbb{R}^3$ to $\mathbb{R}^3$, following Mather techniques of classification (see for example [12]) is easy to see that locally we can have only one of the six situations that appear in figure 1.4.

When $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is not stable but it is finitely determined, then the origin is an isolated instability by the Mather-Gaffney criterion ([36]). In particular, there is a small enough representative $f : U \to V$ where U, V are open subsets of $\mathbb{R}^n$ such that f is stable in $U \setminus \{0\}$. The topological structure of f is determined by the so-called link of f, which is obtained by taking the intersection of the image of f with a small enough $(n-1)$-sphere centered at the origin S_ϵ^{n-1}. We use a theorem due to Fukuda [5], which

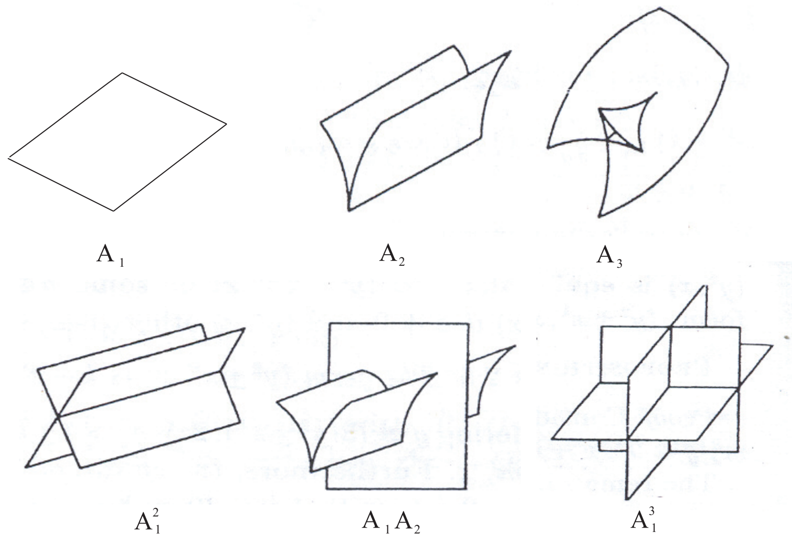


Figure 1.4: Stable singularities from $\mathbb{R}^3$ to $\mathbb{R}^3$

ensures that the link of f is a stable map from S^{n-1} to S^{n-1} and that f is topologically equivalent to the cone of its link. Our work will be trying to classify topologically this stable maps, and as a consequence obtain the classification of the correspondent map germs.

The topological classification of finitely determined map germs from the plane to the plane in the complex analytic case $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ has been done by Gaffney and Mond in [10, 9], restricting themselves to weighted homogeneous polynomials. In the analytic real case, J.H.Rieger in [34] fulfill the $\mathcal{A}$-simple classification of plane to plane map germs of corank 1. Also Nishimura studied in [30] the topological $\mathcal{K}$-equivalence of finite map germs $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ and he obtained that the absolute value of the degree is a complete topological invariant.

For equidimensional case $n = 3$ we should refer to the work of W.L.Marar and F.Tari([19]) where they study the $\mathcal{A}$-classification of these germs in the real case.

The main techniques used along this work has been already used by other authors ([21], [18]) to obtain the full classification of ruled surfaces in $\mathbb{R}^3$ in the first case and a partial classification of finitely determined map germs from $\mathbb{R}^2$ to $\mathbb{R}^3$ in the last case.

1.2.1 Organization of the work

This work is divided in 7 chapters, including this one containing a short introduction to the problem of study and a last one when the reader can find the final conclusions of the results achieved in this work as well as some motivating open problems.

In chapter 2 we state some preliminary results as the definition of a stable map or of a finitely determined map germ that are believed to be fundamental for the correct understanding of the subsequent chapters.

Chapters 3, 4 or 5 are entirely dedicated to the study of finitely determined map germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. In the first one we introduce the definition of Gauss words for the particular case of these maps, proving that they become a complete topological invariant and we also give a wide topological classification in the case of corank 1. In chapter 5 we extend this classification to map germs of corank 2 that are of type $\Sigma^{2,0}$ and in chapter 4 we consider 1-parameter families of finitely determined map germs, proving that under certain hypothesis they become topologically trivial.

Finally, in chapter 6 we face the difficult duty of trying to extend the results from the planar case to map germs from $\mathbb{R}^3$ to $\mathbb{R}^3$. Here we prove that, with some restrictions in our map germs, Gauss words are also a complete topological invariant for this kind of germs and we give as an application the topological classification of the germs that belong to the $\mathcal{A}^2$-class (x, y, xz) in some particular cases and the full classification of ruled map germs from $\mathbb{R}^3$ to $\mathbb{R}^3$.

Chapter 2

Preliminaries

In this chapter we remind the basic definitions and results that we are going to need along this work, including the characterization of stable maps, the Mather-Gaffney finite determinacy criterion and the link of a map germ. These results can be found in different books in the bibliography, such as [12], [13] and [20] for the part of stability and finite determinacy and in [5] in what concerns to the link of a map germ.

2.1 Stability

Definition 2.1.1. Given two smooth maps f, g between smooth manifolds X and Y defined on neighborhoods U and V of $x \in X$ respectively, we consider the equivalence relation given by $f \sim g$ if there exists a neighborhood W of x contained in $U \cap V$ such that $f|_W = g|_W$. The equivalence classes are called *map germs*.

Map germs are mainly used to study local properties of maps. We can locally take adequate charts so that f can be seen as a germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$.

Definition 2.1.2. We define the *k-jet* of a smooth map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $j^k f(0)$, as the Taylor expansion of f of order k at 0. We will denote by $J^k(\mathbb{R}^n, \mathbb{R}^p)$ the set of all *k-jets*.

Definition 2.1.3. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth map germ

- We call *rank of* f to the rank of the Jacobian matrix of a representative of the germ.

- If f has maximal rank we call it *regular*. Otherwise, we call it *singular*.

- We call *corank of f* to *min* $\{n,p\}-r$, where r is the rank of f. We denote the space of map germs with corank k by Σ^k.

- We denote by $S(f)=\{x\in\mathbb{R}^n : \text{rank}(Jf_x)<\min\{n,p\}\}$ the set of singular points of f, $\Delta(f)=f(S(f))$ and $X(f)=\overline{f^{-1}(f(S(f))\setminus S(f)}$.

Definition 2.1.4. A map germ $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ is said to be *weighted homogeneous* with weights $w_1,w_2,\dots,w_n$ if

$$\lambda^{k_i}f_i=f_i(\lambda^{w_1}x_1,\dots,\lambda^{w_n}x_n),$$

for some $k_1,\dots,k_p\in\mathbb{N}$ and for all λ, $i=1,\dots,p$.

Definition 2.1.5. Let $f,g:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be two smooth map germs.

- We say that f and g are $\mathcal{A}$-*equivalent*, and we denote it by $f\sim_{\mathcal{A}}g$, if there are germs of diffeomorphism $\alpha:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ and $\beta:(\mathbb{R}^p,0)\to(\mathbb{R}^p,0)$ such that the following diagram commutes:

$$\begin{array}{ccc}(\mathbb{R}^n,0) & \xrightarrow{f} & (\mathbb{R}^p,0)\\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta}\\ (\mathbb{R}^n,0) & \xrightarrow{g} & (\mathbb{R}^p,0)\end{array}$$

- In the case that α and β are germs of homeomorphism, we will say that f and g are *topologically equivalent*.

Definition 2.1.6. Let $f,g:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be two smooth map germs. We say that f and g are $\mathcal{K}$-*equivalent*, and we denote it by $f\sim_{\mathcal{K}}g$ if there are germs of diffeomorphism $\phi:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ and $H:(\mathbb{R}^n\times R^p,0)\to(\mathbb{R}^n\times\mathbb{R}^p,0)$ such that $H(\mathbb{R}^n\times\{0\})=\mathbb{R}^n\times\{0\}$ and the following diagram commutes:

$$\begin{array}{ccccc}(\mathbb{R}^n,0) & \xrightarrow{(\text{id},f)} & (\mathbb{R}^n\times\mathbb{R}^p,0) & \xrightarrow{\pi_n} & (\mathbb{R}^n,0)\\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle\phi}\\ (\mathbb{R}^n,0) & \xrightarrow{(\text{id},g)} & (\mathbb{R}^n\times\mathbb{R}^p,0) & \xrightarrow{\pi_n} & (\mathbb{R}^n,0)\end{array}$$

Definition 2.1.7. Given a map germ $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$, a r-*parameter unfolding* of f is a map germ $F:(\mathbb{R}^n\times\mathbb{R}^r,0)\to(\mathbb{R}^p\times\mathbb{R}^r,0)$ of the form $F(x,t)=(f_t(x),t)$, with $t=(t_1,t_2,\dots,t_r)$, and such that $f_0=f$.

Definition 2.1.8. Let F be a r-parameter unfolding of a smooth map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. We will say that F is *trivial* if there are germs of diffeomorphism $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^n \times \mathbb{R}^r, 0)$ and $\Phi : (\mathbb{R}^p \times \mathbb{R}^r, 0) \to (\mathbb{R}^p \times \mathbb{R}^r, 0)$ such that they are unfoldings of the identity and $F = \Phi^{-1} \circ (f \times \mathrm{id}) \circ \Psi$.

Definition 2.1.9. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth map germ. We will say that f is *stable* if any unfolding of f is trivial.

Definition 2.1.10. Let $S = \{x_1, \dots, x_r\}$ be a finite subset of $\mathbb{R}^n$. A *multigerm* $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$ is an equivalence class of smooth maps $\widetilde{f} : U \to \mathbb{R}^p$, where U is an open neighborhood of S and $\widetilde{f}(S) = \{y\}$, such that two maps are equivalent if they are equal in an open neighborhood of S. All the definitions of $\mathcal{A}$-equivalence, trivial unfolding and stability are generalized without problems to the case of multigerms.

Definition 2.1.11. We say that a smooth map $f : \mathbb{R}^n \to \mathbb{R}^p$ is (locally)*stable* if for any $y \in \mathbb{R}^p$, $S = f^{-1}(y) \cap S(f)$ is finite and $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$ is stable.

Now, by taking into account Whitney results [37] and Mather techniques of classification (see for example [12]) we can state the two following theorems, which will give us a characterization of stable maps for our particular cases of study.

Theorem 2.1.12. *(Whitney) Let $f : U \to V$ be a smooth proper map, where $U, V \subset \mathbb{R}^2$ are open subsets. We have that f is stable if and only if:*

1. *Its only singularities are folds and cusp points,*
2. *$f|_{S_{1,0}(f)}$ is an immersion with double transverse points, where we denote by $S_{1,0}(f)$ the set of fold points of f.*

Theorem 2.1.13. *(Mather) Let $f : U \to V$ be a smooth proper map, where $U, V \subset \mathbb{R}^3$ are open subsets. We have that f is stable if and only if:*

1. *Its only singularities are folds (A_1), cusps (A_2) and swallowtails (A_3).*
2. *$f|_{S_{1,0,0}(f)}$ is an immersion with double point curves (A_1^2) and isolated triple points (A_1^3), $f|_{S_{1,1,0}(f)}$ is an injective immersion and the images of both restrictions intersect transversally (A_1A_2), with $S_{1,0,0}(f)$ and $S_{1,1,0}(f)$ being the set of folds and the set of cusps respectively.*

2.2 Finite determinacy

Definition 2.2.1. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth map germ.

- We say that f is *k-determined* if for every smooth map germ $g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ such that $j^k f(0) = j^k g(0)$ we have that f and g are $\mathcal{A}$-equivalent.
- Our map germ f will be *finitely determined* if it is k-determined for some $k \geq 0$.

Theorem 2.2.2. *(Mather-Gaffney, [36]) Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a finitely determined map germ with $n \leq p$. Then, there is a representative $f : U \to V$, where $U \subset \mathbb{R}^n, V \subset \mathbb{R}^p$ are open sets, such that:*

1. *$f^{-1}(0) = \{0\}$,*
2. *$f : U \to V$ is proper,*
3. *the restriction $f|_{U \setminus \{0\}}$ is stable.*

Corollary 2.2.3. *($n = p = 2$) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. Then, there is a representative $f : U \to V$, where $U, V \subset \mathbb{R}^2$ are open sets, such that:*

1. *$f^{-1}(0) = \{0\}$,*
2. *$f : U \to V$ is proper,*
3. *the only singularities of $f|_{U \setminus \{0\}}$ are fold points and $f|_{(U \setminus \{0\}) \cap S_1(f)}$ is an injective immersion.*

Corollary 2.2.4. *($n = p = 3$) Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ. Then, there is a representative $f : U \to V$, where $U, V \subset \mathbb{R}^3$ are open sets, such that:*

1. *$f^{-1}(0) = \{0\}$,*
2. *$f : U \to V$ is proper,*
3. *the restriction $f|_{U \setminus \{0\}}$ is stable with only fold planes, cuspidal edges and double fold point curves.*

Definition 2.2.5. We say that $f : U \to V$ is a *good representative* for a finitely determined map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, with $n = 2$ or 3, if the conditions (1), (2) and (3) of corollary 2.2.3 or 2.2.4 hold.

2.3 Fukuda's theorem and the link of a map germ

We must remind an important result due to Fukuda, which tell us that any finitely determined map germ, $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, with $n \leq p$, has a conic structure over its link. The link is obtained by intersecting the image of a representative of f with a small enough sphere centered at the origin of $\mathbb{R}^p$.

Theorem 2.3.1. *([5]) Suppose $n \leq p$ and let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a finitely determined map germ. Then, up to $\mathcal{A}$-equivalence, there is a representative $f : U \to V$ and $\epsilon_0 > 0$, such that, for any ϵ with $0 < \epsilon \leq \epsilon_0$ we have:*

1. *$\widetilde{S}^{n-1}_\epsilon = f^{-1}(S^{p-1}_\epsilon)$ is a homotopy (n-1)-sphere which, if $n \neq 4, 5$ is diffeomorphic to the natural (n-1)-sphere S^{n-1},*

2. *the restricted map $f|_{\widetilde{S}^{n-1}_\epsilon} : \widetilde{S}^{n-1}_\epsilon \longrightarrow S^{p-1}_\epsilon$ is topologically stable (C^∞ stable if (n, p) is a "nice pair" (in Mather's sense)),*

3. *letting $\widetilde{D}^n_\epsilon = f^{-1}(D^p_\epsilon)$, the restricted map $f|_{\widetilde{D}^n_\epsilon} : \widetilde{D}^n_\epsilon \longrightarrow D^p_\epsilon$ is proper, topologically stable (C^∞ stable if (n, p) is a "nice pair") and topologically equivalent (C^∞ equivalent ($\mathcal{A}$- equivalent) if (n, p) is a "nice pair") to the product map*

$$\begin{array}{rccc} (f\mid_{\widetilde{S}^{n-1}_\epsilon}) \times id_{(0,\epsilon)} : & \widetilde{S}^{n-1}_\epsilon \times (0,\epsilon) & \longrightarrow & S^{p-1}_\epsilon \times (0,\epsilon) \\ & (x,t) & \longrightarrow & (f(x),t) \end{array}$$

   and

4. *consequently, $f|_{\widetilde{D}^n_\epsilon} : \widetilde{D}^n_\epsilon \longrightarrow D^p_\epsilon$ is topologically equivalent to the cone $C(f|_{\widetilde{S}^{n-1}_\epsilon}) : \frac{\widetilde{S}^{n-1}_\epsilon \times [0,\epsilon)}{\widetilde{S}^{n-1}_\epsilon \times \{0\}} \longrightarrow \frac{S^{p-1}_\epsilon \times [0,\epsilon)}{S^{p-1}_\epsilon \times \{0\}}$ of the stable map $f|_{\widetilde{S}^{n-1}_\epsilon} : \widetilde{S}^{n-1}_\epsilon \longrightarrow S^{p-1}_\epsilon$ defined by $C(f|_{\widetilde{S}^{n-1}_\epsilon})(x,t) = (f(x), t)$.*

Restricting ourselves to what are going to be our particular cases of study in this work, we have the following consequences.

Corollary 2.3.2. *($n = p = 2$) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. Then, up to $\mathcal{A}$-equivalence, there is a representative $f : U \to V$ and $\epsilon_0 > 0$, such that, for any ϵ with $0 < \epsilon \leq \epsilon_0$ we have:*

1. *$\widetilde{S}^1_\epsilon = f^{-1}(S^1_\epsilon)$ is diffeomorphic to S^1.*

2. *The map $f|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon$ is stable, in other words, it is a Morse function all of whose critical values are distinct.*

3. *f is topologically equivalent to the cone of $f|_{\widetilde{S}^1_\epsilon}$.*

Corollary 2.3.3. *($n = p = 3$) Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ. Then, up to $\mathcal{A}$-equivalence, there is a representative $f : U \to V$ and $\epsilon_0 > 0$, such that, for any ϵ with $0 < \epsilon \leq \epsilon_0$ we have:*

1. *$\widetilde{S}^2_\epsilon = f^{-1}(S^2_\epsilon)$ is diffeomorphic to S^2.*

2. *The map $f|_{\widetilde{S}^2_\epsilon} : \widetilde{S}^2_\epsilon \to S^2_\epsilon$ is stable.*

3. *f is topologically equivalent to the cone of $f|_{\widetilde{S}^2_\epsilon}$.*

As a consequence of this theorem we have the following definition.

Definition 2.3.4. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a finitely determined map germ, with $n \leq p$. We say that the stable map $f|_{\widetilde{S}^{n-1}_\epsilon} : \widetilde{S}^{n-1}_\epsilon \to S^{p-1}_\epsilon$ is the *link* of f, where f is a representative such that (1), (2), (3) and (4) of theorem 2.3.1 hold for any ϵ with $0 < \epsilon \leq \epsilon_0$. This link is well defined, up to $\mathcal{A}$-equivalence.

Remark 2.3.5. If we consider a multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$, with $n \leq p$ and $S = \{x_1, \ldots, x_r\}$, the construction of the link can be done in an analogous way. By reviewing carefully Fukuda's arguments, we see that the only difference is the condition (1) of theorem 2.3.1: now $\widetilde{S}^{n-1}_\epsilon$ is not diffeomorphic to S^{n-1} anymore, but it is diffeomorphic to a disjoint union of r copies $S^{n-1} \sqcup \ldots \sqcup S^{n-1}$. However, the other conditions (2), (3) and (4) are still valid in this case.

2.4 The topological degree of a map germ

Before finishing this chapter we must remember how we define the topological degree of a map germ, as well as its main properties.

Definition 2.4.1. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a smooth map germ such that $f^{-1}(0) = \{0\}$. We will call *topological degree of f* to the degree of the associated map $(f/\|f\|) : S^{n-1}_\epsilon \to S^{n-1}$ for $\epsilon > 0$ small enough. Let's notice that to compute the degree of f (see for example [22]) is enough to take a regular value $w \in \mathbb{R}^n$ small enough and we have

$$\deg(f) = \sum_{z_i \in f^{-1}(w)} \operatorname{ind}(f, z_i),$$

where

$$\operatorname{ind}(f, z_i) = \begin{cases} 1, \text{if } Jf(z_i) > 0, \\ -1, \text{if } Jf(z_i) < 0. \end{cases}$$

This definition doesn't depend on the chosen regular value w.

Proposition 2.4.2. *Let $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a smooth function germ such that $f^{-1}(0) = \{0\}$. Then,* $\deg(f) \in \{0, \pm 1\}$.

Proof. Let $w \in \mathbb{R}$ be a regular value of f, $f^{-1}(w) = \{z_1, \dots, z_t\}$, with $z_1 < \cdots < z_t$. We have two cases:

- If t is even, f intersects the straight line $f(x) = w$ an even number of times. Therefore, for any $i \in \{1, 2, \dots, t\}$ if $f'(z_i) > 0$, then $f'(z_{i+1}) < 0$ and viceversa. Then, $\deg(f) = 0$.

- If t is odd,

$$\deg(f) = \sum_{i=1}^{2k} \operatorname{ind}(f, z_i) + \operatorname{ind}(f, z_{2k+1}) = \operatorname{ind}(f, z_{2k+1}) =$$

$$= \begin{cases} 1, & \text{if } f'(z_{2k+1}) > 0; \\ -1, & \text{if } f'(z_{2k+1}) < 0. \end{cases}$$

Thus, $\deg(f) \in \{0, \pm 1\}$. □

Proposition 2.4.3. *Let $F : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^n \times \mathbb{R}^r, 0)$ be a r-parameter unfolding of a smooth map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f^{-1}(0) = \{0\}$, with $F(x, t) = (f_t(x), t)$, $f_0 = f$. Then,* $\deg(F) = \deg(f)$.

Proof. Firstly, let's take suitable representatives $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ such that $f^{-1}(0) = 0$ and $F : V \times U \to \mathbb{R}^r \times \mathbb{R}^n$.

We choose a regular value of f $y \in \mathbb{R}^n$, with $f^{-1}(y) = \{x_1, \dots, x_t\}$. From this, it follows that $F^{-1}(0, y) = \{(0, x_1), \dots, (0, x_t)\}$. Then:

$$DF(u, x) = \begin{pmatrix} Id & * \\ 0 & Df_u(x) \end{pmatrix}$$

If we compute DF in a point of the form $(0, x_i)$ we obtain

$$DF(0, x_i) = \begin{pmatrix} Id & * \\ 0 & Df_u(x_i) \end{pmatrix}$$

Therefore, $(0, y)$ is a regular value of F. Moreover, $JF(0, x_i) = Jf(x_i)$. We conclude $\deg(f) = \deg(F)$. □

Corollary 2.4.4. *Let $F : (\mathbb{R} \times \mathbb{R}^r, 0) \to (\mathbb{R} \times \mathbb{R}^r, 0)$ be a r-parameter unfolding of a smooth function germ $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $f^{-1}(0) = \{0\}$, with $F(x, t) = (f_t(x), t)$, $f_0 = f$. Then,* $\deg(F) \in \{0, \pm 1\}$.

Chapter 3

The link of a finitely determined map germ from $\mathbb{R}^2$ to $\mathbb{R}^2$

In this chapter we study the topological classification of finitely determined map germs, $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, by looking at the topological type of its link. The main tool will be an adapted version of Gauss word, which will be proved to be a complete topological invariant of our map germs. We will also take special attention to the case that f has corank 1. In this case, f can be written as $f(x, y) = (x, g_x(y))$ and gives a stabilization of $g_0 : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. The topology of f is now determined by the two stabilizations g_x^+, with $x > 0$ and g_x^- with $x < 0$. We obtain the topological classification up to multiplicity 5, provided that f is weighted homogeneous (theorem 3.3.13). In the last part we will give a couple of results (theorem 3.4.1 and remark 3.4.2) related to the cusps and double folds that appear in a finitely determined map germ when you consider a stable perturbation of it.

3.1 The link of a germ

We have defined in chapter 2 the link of a finitely determined map germ in the general case. Now, we adapt this concept to our particular case of study.

Definition 3.1.1. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. We say that the stable map $f|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon$ is the *link* of f, where f is a representative such that (1), (2) and (3) of corollary 2.3.2 hold for any ϵ with $0 < \epsilon \leq \epsilon_0$. Then, in this case f is a Morse function, all of whose critical values are distinct. This link is well defined, up to $\mathcal{A}$-equivalence. We also say that ϵ_0 is a *Milnor-Fukuda radius* for f.

Since any finitely determined map germ is topologically equivalent to the cone of its link, we have the following immediate consequence.

Corollary 3.1.2. *Two finitely determined map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ are topologically equivalent if their associated links are topologically equivalent.*

We will see the converse of this result at the end of the following section.

3.2 The Gauss word

We recall that a Gauss word is a word which contains each letter exactly twice, one with exponent +1 and another one with exponent -1. They were introduced originally by Gauss to describe the topology of closed curves in the plane $\mathbb{R}^2$ or in the sphere S^2. Here, we use the same terminology of Gauss word to represent a different type of word, adapted to our particular case of stable maps $S^1 \to S^1$.

Let S^1 be the unit 1-sphere with the anti-clockwise orientation and let us choose a base point $z_0 \in S^1$. Any point $x \in S^1$ can be written in a unique way as $x = z_0 e^{i\alpha}$, with $\alpha \in [0, 2\pi)$. Given $x = z_0 e^{i\alpha}$ and $y = z_0 e^{i\beta}$, with $\alpha, \beta \in [0, 2\pi)$, we denote $x \leq y$ if $\alpha \leq \beta$. If S^1 is considered with the clockwise orientation, then we write $x = z_0 e^{-i\alpha}$, with $\alpha \in [0, 2\pi)$ and the order relation is defined in an analogous way.

Definition 3.2.1. Let $\gamma : S^1 \to S^1$ be a stable map, that is, such that all its singularities are of Morse type and its critical values are distinct. We fix orientations in each S^1 and we also choose base points $z_0 \in S^1$ in the source and $a_0 \in S^1$ in the target.

Suppose that γ has r critical values labeled by r letters $a_1, \ldots, a_r \in S^1$ and let us denote their inverse images by $z_1, \ldots, z_k \in S^1$. We assume they are ordered such that $a_0 \leq a_1 < \cdots < a_r$ and $z_0 \leq z_1 < \cdots < z_k$ and following the orientation of each S^1.

We define a map $\sigma : \{1, \ldots, k\} \to \{a_1, \ldots, a_r, \overline{a}_1, \ldots, \overline{a}_r\}$ in the following way: given $i \in \{1, \ldots, k\}$, then $\gamma(z_i) = a_j$ for some $j \in \{1, \ldots, r\}$; we define $\sigma(i) = a_j$, if z_i is a regular point and $\sigma(i) = \overline{a}_j$, if z_i is a singular point. We call *Gauss word* to the sequence $\sigma(1) \ldots \sigma(k)$.

Example 3.2.2.

1. Let $\gamma : S^1 \to S^1$ be the link of the fold $f(x, y) = (x, y^2)$. There are only 2 critical values and 2 inverse images, one for each critical value. The Gauss word is $\overline{a}\overline{b}$ (figure 3.1).

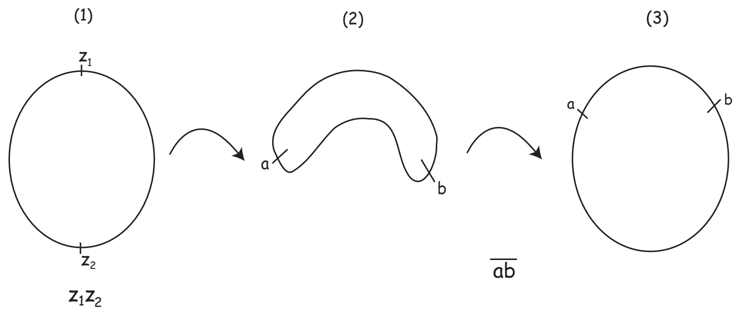


Figure 3.1

2. Let $\gamma : S^1 \to S^1$ be the link of the cusp $f(x,y) = (x, xy + y^3)$. There are 2 critical values and 4 inverse images, two for each critical value. The Gauss word in this case is $a\bar{b}\bar{a}b$ (figure 3.2).

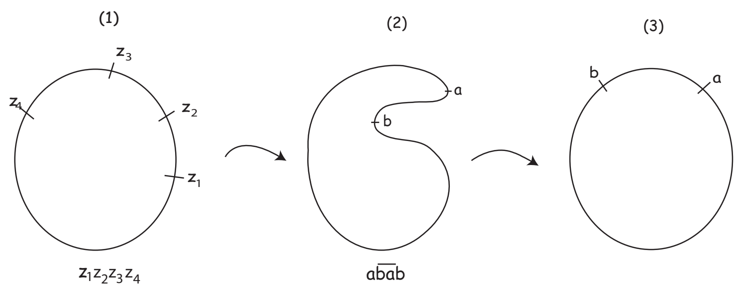


Figure 3.2

It is obvious that the Gauss word is not uniquely determined, since it depends on the chosen orientations and base points in each S^1. Different choices will produce the following changes in the Gauss word:

1. a cyclic permutation in the letters $a_1, \ldots, a_r$;
2. a cyclic permutation in the sequence $\sigma(1) \ldots \sigma(k)$;
3. a reversion in the set of the letters $a_1, \ldots, a_r$;
4. a reversion in the sequence $\sigma(1) \ldots \sigma(k)$.

We say that two Gauss words are equivalent if they are related through these four operations. Under this equivalence, the Gauss word is now well defined.

Moreover, we will have the following restrictions in their construction:

1. The number of folds (and as a consequence of distinct letters) have to be even. This is easily proved by considering Euler-Poincaré equality

$$0 = \chi(S^1) = \sum_{x \in S(\gamma)} \mathrm{ind}(\gamma, x)$$

with $\gamma : S^1 \longrightarrow S^1$ and

$$\mathrm{ind}(\gamma, x) = \begin{cases} 1, & \text{if } \gamma''(x) < 0; \\ -1, & \text{if } \gamma''(x) > 0. \end{cases}$$

2. We cannot have the same letter in two consecutive positions.

In order to simplify the notation, given a stable map $\gamma : S^1 \to S^1$, we denote by $w(\gamma)$ the associated Gauss word and by $\simeq$ the equivalence relation between Gauss words. We also denote by $\deg(\gamma)$ the topological degree. Then, we can state the main result of this section.

Theorem 3.2.3. *Let $\gamma, \delta : S^1 \to S^1$ be two stable maps. Then γ, δ are topologically equivalent if and only if*

$$\begin{cases} w(\gamma) \simeq w(\delta), & \text{if } \gamma, \delta \text{ are singular,} \\ & \\ |\deg(\gamma)| = |\deg(\delta)|, & \text{if } \gamma, \delta \text{ are regular.} \end{cases}$$

Proof. We choose orientations in the source and the target of $\gamma : S^1 \to S^1$ and we also choose base points $z_0 \in S^1$ and $a_0 \in S^1$. We denote by $a_1, \dots, a_r$ the critical values of γ and by $z_1, \dots, z_k$ their inverse images. Assume they are ordered such that $a_0 \leq a_1 < \cdots < a_r$ and $z_0 \leq z_1 < \cdots < z_k$ and following the orientation of each S^1. Let $\sigma(1) \dots \sigma(k)$ be the Gauss word of γ.

Suppose that $\delta : S^1 \to S^1$ is topologically equivalent to γ. Then, there are homeomorphisms $\phi, \psi : S^1 \to S^1$ such that $\delta = \psi \circ \gamma \circ \phi^{-1}$. We choose the orientations in the source and the target induced by the orientations of γ and the homeomorphisms ϕ, ψ. We denote $z'_i = \phi(z_i)$ with $i = 0, \dots, k$ and $a'_j = \psi(a_j)$ with $j = 0, \dots, r$. We take z'_0 and a'_0 as base points in the source and the target respectively. Then, $a'_1, \dots, a'_r$ are the critical values of δ and $z'_1, \dots, z'_k$ are their inverse images and all of them are well ordered with respect to the chosen base points and orientations. If we label the critical values also with the letters $a_1, \dots, a_r$, then δ has the same Gauss word $\sigma(1) \dots \sigma(k)$.

If γ, δ are topologically equivalent, then we always have the equality $|\deg(\gamma)| = |\deg(\delta)|$. In fact, since any homeomorphism has degree ± 1, we obtain

$$\deg(\delta) = \deg(\psi \circ \gamma \circ \phi^{-1}) = \deg(\psi)\deg(\gamma)\deg(\phi^{-1}) = \pm\deg(\gamma).$$

We show now the converse. We divide the proof into several cases.

Case 1: γ, δ are singular and $w(\gamma) = w(\delta)$. We adopt the following notation:

1. $a_1, \ldots, a_r$ are the critical values of γ and $z_1, \ldots, z_k$ are their inverse images,

2. $a'_1, \ldots, a'_r$ are the critical values of δ and $z'_1, \ldots, z'_k$ are their inverse images.

We assume that all the points are well ordered with respect to the chosen base point and orientation in each corresponding S^1. The fact that $w(\gamma) = w(\delta)$ implies that $\gamma(z_i) = a_j$ if and only if $\delta(z'_i) = a'_j$.

We define the circle intervals

$$J_j = [a_j, a_{j+1}], \quad I_i = [z_i, z_{i+1}], \quad K_j = [a'_j, a'_{j+1}], \quad H_i = [z'_i, z'_{i+1}],$$

with $j = 1, \ldots, r$ and $i = 1, \ldots, k$ (we set $a_{r+1} = a_1$, $a'_{r+1} = a'_1$, $z_{k+1} = z_1$ and $z'_{k+1} = z'_1$).

For each $j = 1, \ldots, r$ we choose a homeomorphism $\psi_j : J_j \to K_j$ such that $\psi_j(a_j) = a'_j$ and we construct the homeomorphism $\psi : S^1 \to S^1$ by taking $\psi|_{J_j} = \psi_j$.

For each $i = 1, \ldots, k$, suppose that $\gamma(z_i) = a_j$ and $\delta(z'_i) = a'_j$. Then the restrictions $\gamma_i = \gamma|_{I_i} : I_i \to J_j$ and $\delta_i = \delta|_{H_i} : H_i \to K_j$ are also homeomorphisms. We define the homeomorphism $\phi_i : I_i \to H_i$ by $\phi_i = \delta_i^{-1} \circ \psi_j \circ \gamma_i$ (see figure 3.3). Finally, we construct the homeomorphism $\phi : S^1 \to S^1$ by taking $\phi|_{I_i} = \phi_i$. This homeomorphism verifies that $\delta = \psi \circ \gamma \circ \phi^{-1}$ and hence, γ, δ are topologically equivalent.

Case 2: γ, δ are singular and $w(\gamma) \simeq w(\delta)$.

In this case, we can define a new map $\tilde{\delta}$ which is topologically equivalent to δ and such that $w(\gamma) = w(\tilde{\delta})$. Then, the result follows from case 1.

In fact, given $\theta \in [0, 2\pi)$, we denote by $T_\theta : S^1 \to S^1$ the rotation with angle θ, that is, $T_\theta(z) = e^{i\theta}z$. We also denote the inversion $I : S^1 \to S^1$, where $I(z) = z^{-1}$.

1. If $w(\gamma), w(\delta)$ are related through a cyclic permutation in the letters $a_1, \ldots, a_r$, then there is θ such that $w(\gamma) = w(T_\theta \circ \delta)$.

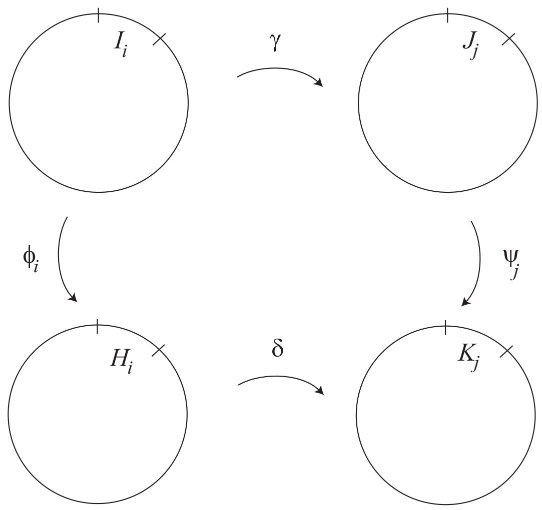


Figure 3.3: The commutative diagram between the stable maps γ and δ

2. If $w(\gamma), w(\delta)$ are related through a cyclic permutation in the sequence $\sigma(1) \dots \sigma(k)$, then there is θ such that $w(\gamma) = w(\delta \circ T_\theta)$.

3. If $w(\gamma), w(\delta)$ are related through a reversion in the set of the letters $a_1, \dots, a_r$, then $w(\gamma) = w(I \circ \delta)$.

4. If $w(\gamma), w(\delta)$ are related through a reversion in the sequence $\sigma(1) \dots \sigma(k)$, then $w(\gamma) = w(\delta \circ I)$.

Case 3: γ, δ are regular and $\deg(\gamma) = \deg(\delta)$.

We choose a point $a_0 \in S^1$ and let us denote $\gamma^{-1}(a_0) = \{z_1, \dots, z_k\}$ and $\delta^{-1}(a_0) = \{z'_1, \dots, z'_k\}$, where k is the absolute value of the topological degree of γ and δ. We assume that the points are well ordered in each corresponding S^1. We consider the intervals $I_i = [z_i, z_{i+1}]$ and $H_i = [z'_i, z'_{i+1}]$ and the restrictions $\gamma_i = \gamma|_{I_i} : I_i \to S^1$ and $\delta_i = \delta|_{H_i} : H_i \to S^1$, with $i = 1, \dots, k$.

For each $i = 1, \dots, k$, we define the homeomorphism $\phi_i : I_i \to H_i$ by taking $\phi_i = \delta_i^{-1} \circ \gamma_i$ on the interior of I_i, $\phi_i(z_i) = z'_i$ and $\phi_i(z_{i+1}) = z'_{i+1}$. We construct the homeomorphism $\phi : S^1 \to S^1$ by taking $\phi|_{I_i} = \phi_i$. This homeomorphism verifies that $\delta = \gamma \circ \phi^{-1}$ and γ, δ are topologically equivalent.

Case 4: γ, δ are regular and $\deg(\gamma) = -\deg(\delta)$.

We have that $\deg(\gamma) = \deg(\delta \circ I)$ and hence, this is a consequence of case 3. □

Given a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, we denote by $w(f)$ the Gauss word of its link and by $\deg(f)$ the local topological degree.

Remark 3.2.4. If $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a finitely determined map germ, then we can compute Gauss word of the link of f just by looking at the relative position of the branches of the three curves $S(f)$, $\Delta(f)$ and $X(f)$. This construction is useful sometimes because we do not need to compute explicitly the link of f. We take a small enough representative $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ such that

1. $f^{-1}(0) = \{0\}$,

2. the restriction $f|_{U \setminus \{0\}}$ is stable with only simple folds.

The three curves $S(f)$, $\Delta(f)$ and $X(f)$ are plane curves which are smooth outside the origin. By shrinking the neighbourhood U if necessary, we can assume that the three curves are simply connected.

The discriminant $\Delta(f)$ has a tree structure with one vertex at the origin and r adjacent edges labeled by r letters $a_1, \dots, a_r$. Analogously, $S(f) \cup X(f)$ has also a tree structure with one vertex at the origin and k adjacent edges labeled by $Z_1, \dots, Z_k$. We assume that the edges are well ordered $a_1 < \cdots < a_r$ and $Z_1 < \cdots < Z_k$ with respect to some chosen base points and orientations in the source and the target. We define the map $\sigma : \{1, \dots, k\} \to \{a_1, \dots, a_r, \overline{a}_1, \dots, \overline{a}_r\}$ in the following way: given $i \in \{1, \dots, k\}$, then $\gamma(Z_i) = a_j$ for some $j \in \{1, \dots, r\}$; we define $\sigma(i) = a_j$, if $Z_i \subset X(f)$ and $\sigma(i) = \overline{a}_j$, if $Z_i \subset S(f)$. Then, $\sigma(1) \dots \sigma(k)$ is equal to the Gauss word of the link of f.

As a direct consequence of the last remark and the theorem 3.2.3 we are in conditions of stating and proving the following result that will give us the converse of corollary 3.1.2

Corollary 3.2.5. *Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs. Then, if f and g are topologically equivalent, their links are topologically equivalent.*

Proof. If f and g are topologically equivalent, applying last remark we have that their respective Gauss words, $w(f)$ and $w(g)$, are equivalent. If we use now theorem 3.2.3 we arrive to the desired result. □

Then, we have the following immediate consequence of corollaries 3.1.2 and 3.2.5 and theorem 3.2.3.

Corollary 3.2.6. *Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs. Then f, g are topologically equivalent if and only if*

$$\begin{cases} w(f) \simeq w(g), & \text{if } f, g \text{ are singular outside the origin,} \\ \\ |\deg(f)| = |\deg(g)|, & \text{if } f, g \text{ are regular outside the origin.} \end{cases}$$

Remark 3.2.7. If f is regular outside the origin and $|deg(f)| = r$, then f is topologically equivalent to the germ $z \to z^r$, with $z = x + iy$.

Before finishing this section we should make the following remark, which will be very useful in the following chapter.

Remark 3.2.8. By following step by step the proof of theorem 3.2.3 we can observe the following fact: if $\gamma, \delta : S^1 \to S^1$ are stable maps with $w(\gamma) \simeq w(\delta)$ and if we fix any homeomorphism in the target $\psi : S^1 \to S^1$ such that $\psi(\Delta(\gamma)) = \Delta(\delta)$, then there is a unique homeomorphism in the source $\phi : S^1 \to S^1$ such that $\psi \circ \gamma \circ \phi^{-1} = \delta$.

By combining this observation with corollaries 3.1.2 and 3.2.5 we have an analogous result for map germs: let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs that are topologically equivalent. If we fix any homeomorphism in the target $\psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\psi(\Delta(f)) = \Delta(g)$, then there is a unique homeomorphism in the source $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\psi \circ f \circ \phi^{-1} = g$.

3.3 Topological classification of corank 1 map germs

In this section we study the topological classification of finitely determined map germs of corank 1. The main tool will be the Gauss word, which is a complete topological invariant, as we have seen in section 2.

First of all, we should remark that if $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ has corank ≤ 1, then after taking smooth changes of coordinates in the source and the target, we can write f in the form $f(x, y) = (x, g_x(y))$, in other words, f can be seen as a 1-parameter unfolding of the germ $g_0 : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$.

In addition to this, and taking in account the properties of the topological degree of a map germ f we have the following result.

Proposition 3.3.1. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finite germ of corank 1, with $f(x, y) = (x, g_x(y))$, $g_0(y) = a_n y^n + a_{n+1} y^{n+1} + \dots$ with $a_n \neq 0$. Then,*

$$\deg(f) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd and } a_n > 0, \\ -1, & \text{if } n \text{ is odd and } a_n < 0. \end{cases}$$

Proof. We know, by corollary 2.4.4 that $\deg(f) = \deg(g_0) \in \{0, 1, -1\}$.

On the other hand, since $g_0(y) = a_n y^n + a_{n+1} y^{n+1} + \dots$ with $a_n \neq 0$, we have that $g_0(y)$ is $\mathcal{A}$-equivalent to $\pm y^n$, depending on the sign of a_n.

Now, let's take a regular value $w \in \mathbb{R}$ of g_0 which will verify that $\pm y^n = w$. If n is even w will present two or none inverse images, depending on its sign, thus, in both cases $\deg(f) = 0$. If n is odd w will be present a single inverse image, obtaining that $\deg(f)$ will be equal to 1 or -1, depending on the sign of a_n. □

Another consequence is that the *multiplicity* of f is equal to n. In general, the multiplicity of $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is defined as

$$m(f) = \dim_{\mathbb{R}} \frac{\mathbb{R}\{x, y\}}{\langle f_1, f_2 \rangle},$$

where f_1, f_2 denote the components of f and $\mathbb{R}\{x, y\}$ is the local algebra of germs of analytic functions $(\mathbb{R}^2, 0) \to \mathbb{R}$. In this case, if f is written as in proposition 3.3.1,

$$m(f) = \dim_{\mathbb{R}}\{1, y, y^2, \ldots, y^{n-1}\} = n.$$

Written in the form $(x, g_x(y))$, f will be regular if $c_{0,1} = 0$ and singular otherwise, where we denote by $c_{i,j}$ the coefficient of the term $x^i y^j$ of Taylor polynomial of g_x. Clearly, $j^1 f(0)$ is $\mathcal{A}$-equivalent to (x, y) in the first case and to $(x, 0)$ in the last one.

Next, we state a result due to J.H.Rieger ([34]) which gives a classification of corank 1 map germs according to its 2-jet. We denote by $\Sigma^1 J^2(2, 2)$ the space of 2-jets of corank 1 map germs from $(\mathbb{R}^2, 0)$ to $(\mathbb{R}^2, 0)$ and $\mathcal{A}^2$ denotes the space of 2-jets of diffeomorphisms in the source and target.

Lemma 3.3.2. *There exist three orbits in $\Sigma^1 J^2(2, 2)$ under the action of $\mathcal{A}^2$, which are*

$$(x, y^2), \quad (x, xy), \quad (x, 0).$$

It is well known that the fold $f(x, y) = (x, y^2)$ is 2-determined. Thus, if a map germ has 2-jet equivalent to (x, y^2), then it is in fact $\mathcal{A}$-equivalent to the fold. Hence, we do not need to consider this case.

The rest of the section will be centered in the study of the two remaining cases.

3.3.1 Classification of germs with 2-jet of type (x, xy)

Now, we center our attention in germs with 2-jet $\mathcal{A}$-equivalent to (x, xy). We will prove that a map germ of this type is topologically equivalent to the fold, (x, y^2), or the cusp, $(x, xy + y^3)$. First of all, we state an important result, due to J. Damon [1].

Theorem 3.3.3. *Let $f_0 : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a weighted homogeneous finitely determined map germ. Then, any polynomial unfolding of f_0 with positive weighted degrees is topologically trivial.*

In our case, we will show that any finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with 2-jet of type (x, xy) is semi-weighted homogeneous, that is, we can write $f = f_0 + h$ where f_0 is weighted homogeneous and finitely determined and h has only terms of higher weighted degree. By Damon's result, this implies that f is topologically equivalent to the initial part f_0 and we can complete the topological classification.

Theorem 3.3.4. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ with 2-jet of type (x, xy) and with multiplicity n. Then, f is topologically equivalent either to the fold (x, y^2) if n is even, or to the cusp $(x, xy + y^3)$ if n is odd.*

Proof. We can assume, without loss of generality by the finite determinacy, that f is polynomial and that it is written in the form

$$f(x, y) = (x, xy + a_n y^n + \dots),$$

with $a_n \neq 0$ and where n is the multiplicity of f. We have that $f_0(x, y) = (x, xy + a_n y^n)$ is weighted homogeneous of weights $(n-1, 1)$ and weighted degrees $(n-1, n)$ and any other monomial appearing in $h = f - f_0$ has weighted degree $> n$.

If we define the unfolding $F(t, x, y) = (t, f_0(x, y) + th(x, y))$, then F is a polynomial unfolding of f_0 with positive weighted degrees in the sense of [1]. We will show that f_0 is finitely determined and by Damon's result, F is topologically trivial. In particular, we deduce that f is topologically equivalent to f_0.

Let $\hat{f}_0$ be the complexification of f_0. The jacobian determinant of $\hat{f}_0$ is $x + na_n y^{n-1}$ and thus, the singular curve $S(\hat{f}_0)$ is smooth. The restriction of $\hat{f}_0$ to $S(\hat{f}_0)$ is the map $y \mapsto (-na_n y^{n-1}, -(n-1)a_n y^n)$, which is an injective immersion outside the origin. By the stability criterion, $\hat{f}_0$ is stable outside the origin. Therefore, $\hat{f}_0$ (and hence f_0) is finitely determined by the Mather-Gaffney criterion (see corollary 2.2.3).

To finish the proof, it only remains to show that f_0 is topologically equivalent to the fold if the multiplicity n is even, or to the cusp if n is odd. Since the singular curve $S(f_0)$ is smooth, the discriminant $\Delta(f_0)$ has only one branch. Thus, the link of f_0 has only 2 critical values.

If n is even, then $\deg(f_0) = 0$. If we compute the number of roots of the polynomial $y^n + xy + w$ by computing the number of roots of its derivate

$ny^{n-1} + x$, we arrive to the conclusion that f can present at most 2 inverse images. Then, each critical value present a single inverse image and as a consequence the Gauss word of our link is $\overline{a}\overline{b}$. Hence, f_0 is topologically equivalent to the fold (x, y^2).

Analogously, if n is odd, then $\deg(f_0) = \pm 1$. Using an analogous procedure we arrive to the conclusion that f can present 1 or 3 inverse images in the real case. Thus each critical value is going to present two inverse images and the Gauss word of our link in this case will be $a\overline{b}\overline{a}b$. Then, f_0 is topologically equivalent to the cusp $(x, xy + y^3)$. □

3.3.2 Classification of germs with 2-jet of type $(x, 0)$

The germs with 2-jet of type $(x, 0)$ are the biggest class inside the corank 1 map germs and we cannot expect to obtain a complete classification. We will restrict ourselves to the weighted homogenous case and also to map germs with multiplicity ≤ 5, although the techniques can be also used to classify more degenerate singularities.

We assume that f is written in the form $f(x, y) = (x, g_x(y))$, and we look at f as a 1-parameter unfolding of the germ $g_0 : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. If f has multiplicity n, then g_0 has type A_{n-1} (i.e., it is $\mathcal{A}$-equivalent to y^n) and the $\mathcal{A}_e$-versal unfolding of the A_{n-1} singularity is

$$G(a_1, \ldots, a_{n-2}, y) = y^n + a_{n-2}y^{n-2} + \cdots + a_1 y.$$

As a consequence, after taking smooth changes of coordinates in the source and the target, we can assume that f is written in the following prenormal form:

$$f(x, y) = (x, y^n + a_{n-2}(x)y^{n-2} + \cdots + a_1(x)y),$$

for some germs $a_i : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, $i = 1, \ldots, n-2$. The germ $a = (a_1, \ldots, a_{n-2}) : (\mathbb{R}, 0) \to (\mathbb{R}^{n-2}, 0)$ defines a curve in the space of parameters of the versal unfolding. This will allow us to control the functions g_x by looking at the versal deformation.

Another important point in the classification is that if $f(x, y) = (x, g_x(y))$ is finitely determined, then f is a stabilization of g_0. This means that there is a representative $f : U = (-\epsilon, \epsilon) \times V \to \mathbb{R}^2$ such that for any x, with $0 < |x| < \epsilon$, $g_x : V \to \mathbb{R}$ is locally stable (that is, g_x is a Morse function with distinct critical values).

Proposition 3.3.5. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ given by $f(x, y) = (x, g_x(y))$. Then, f is a stabilization of g_0.*

Proof. By corollary 2.2.3, if f is finitely determined, we can choose a proper representative $f : U \to V$, where $U, V \subset \mathbb{R}^2$ are open sets, such that

1. $f^{-1}(0) = \{0\}$,

2. the restriction $f|_{U\setminus\{0\}}$ is stable with only simple folds.

We take V a neighbourhood of 0 in $\mathbb{R}$, and then take ϵ sufficiently small, so that we can assume that $U = (-\epsilon, \epsilon) \times V$. Let us take x, with $0 < |x| < \epsilon$. If g_x has a degenerate singularity at $y \in V$, then $g_x'(y) = g_x''(y) = 0$ and f should have a cusp at $(x, y) \in U \setminus \{0\}$. Analogously, if g_x is singular at two distinct point $y_1, y_2 \in V$ with $g_x(y_1) = g_x(y_2)$, then $g_x'(y_1) = g_x'(y_2) = 0$ and f should have a double fold at $(x, y_1), (x, y_2) \in U \setminus \{0\}$. So, we arrive in both cases to a contradiction with the hypothesis of the proposition. □

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ given by $f(x, y) = (x, g_x(y))$. We take a representative $f : U = (-\epsilon, \epsilon) \times V \to \mathbb{R}^2$ such that $g_x : V \to \mathbb{R}$ is stable for any x, with $0 < |x| < \epsilon$. Since g_0 has isolated singularity, by shrinking U if necessary, we can also assume that $g_0^{-1}(0) = \{0\}$ in V and that g_0 is regular in $V \setminus \{0\}$.

Because of the local stability, all the functions $g_x : V \to \mathbb{R}$ are $\mathcal{A}$-equivalent if $-\epsilon < x < 0$ and we will denote by g_x^- one of these functions. Analogously, all the functions $g_x : V \to \mathbb{R}$ are $\mathcal{A}$-equivalent if $0 < x < \epsilon$ and we will denote by g_x^+ one of these functions.

Next step will be asking ourselves what will happen if we have two finitely determined map germs f, f' such that g_x and g_x' are topologically equivalent for $x > 0$ and $x < 0$. In that case, will f and f' be topologically equivalent?

Motivated by this question let us associate a partial Gauss word to each of the functions g_x^-, g_x^+ in a similar way to definition 3.2.1.

Definition 3.3.6. Let $g : V \to \mathbb{R}$ be one of the functions g_x^- or g_x^+. Let $a_1, \ldots, a_r \in \mathbb{R}$ be the critical values of g and let $y_1, \ldots, y_k \in V$ their inverse images. Assume all of them are ordered such that $a_1 < \cdots < a_r$ and $y_1 < \cdots < y_k$. We define the *partial Gauss word* of g as $\sigma(1) \ldots \sigma(k)$, where

$$\sigma(i) = \begin{cases} a_j, & \text{if } g(y_i) = a_j \text{ and } y_i \text{ is regular;} \\ \overline{a}_j, & \text{if } g(y_i) = a_j \text{ and } y_i \text{ is singular.} \end{cases}$$

Definition 3.3.7. Assume that g_x^+ and g_x^- have r and s critical values respectively and let $\sigma^+(1) \ldots \sigma^+(k)$ and $\sigma^-(1) \ldots \sigma^-(\ell)$ be their respective partial Gauss words. We denote by ϕ the map $\phi(a_j) = a_{r+s-j+1}$ and $\phi(\overline{a}_j) = \overline{a}_{r+s-j+1}$, for $j = 1, \ldots, s$. Then we define the *union* of the partial Gauss words as the Gauss word with $r + s$ critical values defined by

$$\sigma^+(1) \ldots \sigma^+(k)\phi(\sigma^-(\ell)) \ldots \phi(\sigma^-(1)).$$

After giving both definitions we are now in conditions of stating and proving the following result.

Theorem 3.3.8. *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *be a finitely determined map germ given by* $f(x, y) = (x, g_x(y))$*. Then the Gauss word of* f *is equivalent to the union of the partial Gauss words of* g_x^+ *and* g_x^-*.*

Proof. We will compute the Gauss word of f by following the construction of remark 3.2.4. Take a small enough representative $f : U = (-\epsilon, \epsilon) \times V \to \mathbb{R}^2$ such that

1. $f^{-1}(0) = \{0\}$,
2. f is stable with only simple folds in $U \setminus \{0\}$,
3. the three curves $S(f)$, $\Delta(f)$ and $X(f)$ are simply connected.

The first two conditions imply that if $0 < |x| < \epsilon$, then $g_x : V \to \mathbb{R}$ is stable and that $g_0^{-1}(0) = \{0\}$.

We show that the three curves $S(f)$, $\Delta(f)$ and $X(f)$ are transverse to the vertical lines $\{x\} \times \mathbb{R}$ if $0 < |x| < \epsilon$. In fact, $S(f)$ is defined by equation $g_x'(y) = 0$ and the intersection with $\{x\} \times \mathbb{R}$ is not transverse if in addition $g_x''(y) = 0$, but this should imply that $(x, y) \in U \setminus \{0\}$ is a cusp of f.

Now, if $\alpha(t) = (x(t), y(t))$ is a local parametrization of $S(f)$ near a point $(x, y) \in S(f)$ with $x \neq 0$, then $f(\alpha(t)) = (x(t), g_{x(t)}(y(t)))$ gives a local parametrization of $\Delta(f)$ near the point $f(x, y) \in \Delta(f)$. Since $x'(t) \neq 0$, $\Delta(f)$ is also transverse to $\{x\} \times \mathbb{R}$ at $f(x, y)$. A similar argument shows that the same is true for $X(f)$.

The transversality of $S(f)$ with the vertical lines, together with the fact that $S(f)$ is simply connected, imply that $S(f) \cap (\{0\} \times \mathbb{R}) = \{0\}$. In particular, g_0 is regular in $V \setminus \{0\}$ and we have a good representative in order to define the partial Gauss words.

We take points x_1, x_2 with $-\epsilon < x_2 < 0 < x_1 < \epsilon$. We assume that:

1. $\Delta(f) \cap (\{x_1\} \times \mathbb{R}) = \{a_1, \dots, a_r\}$,
2. $f^{-1}(\Delta(f)) \cap (\{x_1\} \times \mathbb{R}) = \{y_1, \dots, y_k\}$,
3. $\Delta(f) \cap (\{x_2\} \times \mathbb{R}) = \{a_1', \dots, a_s'\}$,
4. $f^{-1}(\Delta(f)) \cap (\{x_2\} \times \mathbb{R}) = \{y_1', \dots, y_\ell'\}$.

We choose the indices such that all the points are well ordered and we denote the corresponding partial Gauss words by $\sigma^+(1) \dots \sigma^+(k)$ and $\sigma^-(1) \dots \sigma^-(\ell)$ respectively.

By transversality, the curve $\Delta(f)$ has $r+s$ edges $A_1, \ldots, A_{r+s}$ which we can label so that $a_1 \in A_1, \ldots, a_r \in A_r, a'_s \in A_{r+1}, \ldots, a'_1 \in A_{r+s}$. Moreover, the edges are well ordered following the standard orientation of $\mathbb{R}^2$. Analogously, $f^{-1}(\Delta(f))$ has $k+\ell$ edges $Z_1, \ldots, Z_{k+\ell}$ which can be labeled such that $y_1 \in Z_1, \ldots, y_k \in Z_k, y'_\ell \in Z_{k+1}, \ldots, y'_1 \in Z_{k+\ell}$ and they are well ordered. We deduce from definition 3.3.7 that the associated Gauss word is exactly the union of the two partial Gauss words (see figure 3.4).

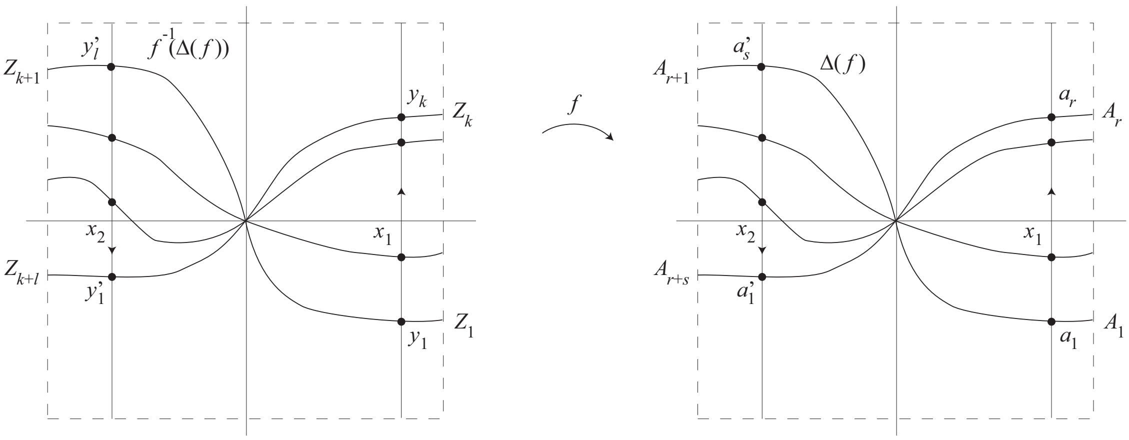


Figure 3.4

□

Corollary 3.3.9. *Let $f, \tilde{f} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs given by $f(x, y) = (x, g_x(y))$ and $\tilde{f}(x, y) = (x, \tilde{g}_x(y))$. If the two partial Gauss words of $f, \tilde{f}$ are equal, then $f, \tilde{f}$ are topologically equivalent.*

Remark 3.3.10. The condition that the partial Gauss words are equal is a necessary condition, because in general, the union of equivalent partial Gauss words does not give equivalent Gauss words. We will find examples of that in example 3.3.12.

Remark 3.3.11. The Gauss word of a stable map $\gamma : S^1 \to S^1$, with γ being the link of a corank 1 map germ, has always the following property: at two consecutive positions of the Gauss word, we must have two consecutive letters (either overlined or not). Hence, in the corank 1 case our Gauss words can be simplified in the two following ways: each time that we find a group of the form $a_i \overline{a_j} a_i$ in the Gauss word, then we substitute it by just $\overline{a_j}$. For instance, the Gauss word $\overline{a}bc\overline{d}c\overline{b}\overline{c}b$ can be simplified with this operation (see figure 3.5):

$$\overline{a}bc\overline{d}c\overline{b}\overline{c}b \to \overline{a}b\overline{d}\overline{b}\overline{c}b \to \overline{a}\overline{d}\overline{b}\overline{c}.$$

It is obvious that given a simplified Gauss word, we can recover the complete Gauss word just by adding the missing consecutive letters.

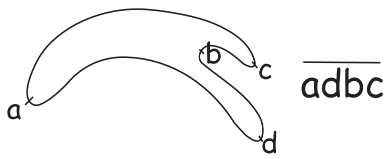


Figure 3.5: The link and its simplified Gauss word

The last remark is not true in the general case, where our map germs can't be seen as 1-parameter unfoldings of functions. We will give in chapter 5 several examples of map germs from $\mathbb{R}^2$ to $\mathbb{R}^2$ with the same simplified Gauss word that are not topologically equivalent.

Example 3.3.12. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map with 2-jet of type $(x, 0)$ and multiplicity 3. We assume that f is written in its prenormal form

$$f(x, y) = (x, y^3 + u(x)y),$$

where $u(x) = u_k x^k + \dots$ and $u_k \neq 0$. If $x \neq 0$, we have two possibilities for the stabilization $g_x(y) = y^3 + u(x)y$. If $u(x) > 0$, then g_x is regular and the partial Gauss word is $\emptyset$. Otherwise, if $u(x) < 0$, then g_x has 2 critical values and the partial Gauss word is $a\bar{b}\bar{a}b$ (see figure 3.6).

Figure 3.6

By taking the union of the partial Gauss words we get 3 possibilities for f:

1. If k is odd, then g_x^+ and g_x^- are the 2 stabilizations of y^3. Hence, the link of f has 2 critical values and the Gauss word is $a\bar{b}\bar{a}b$.

2. If k is even and $u_k > 0$, then both g_x^+ and g_x^- are regular. Hence, the link of f is regular and the Gauss word is $\emptyset$.

3. If k is even and $u_k < 0$, then both g_x^+ and g_x^- are singular. Hence, the link of f has 4 critical values and the Gauss word is $a\bar{b}\bar{a}bc\bar{d}\bar{c}d$.

The pictures and the normal forms for these three topological classes can be found in the first three entries of degree 1 in table 3.1.

Analogously, if f has multiplicity 4, a similar analysis can be done. We have three stabilizations of y^4 with partial Gauss words: (a) $\bar{a}$, (b) $c\overline{ac}\bar{b}c$ and (c) $c\bar{b}\overline{ca}c$ (see figure 3.7).

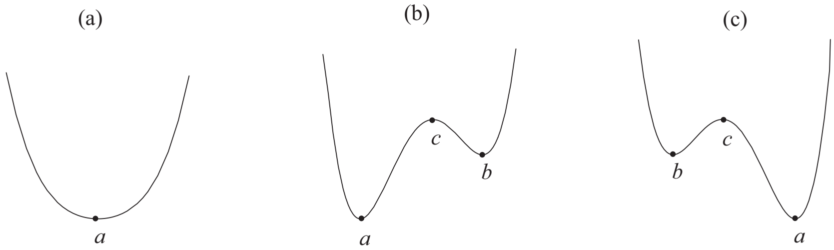


Figure 3.7

The possible Gauss words for f are obtained by taking all the possible combinations between these 3 stabilizations. We see that (a)+(b) is equivalent to (a)+(c) and that (b)+(b) is also equivalent to (c)+(c). Then, there are only 4 non-equivalent possibilities, namely (a)+(a), (a)+(b), (b)+(b) and (b)+(c). The corresponding Gauss words are respectively:

1. $\bar{a}\bar{b}$,

2. $\overline{ac}\overline{bd}$,

3. $c\bar{b}\overline{ca}cd\bar{e}\overline{df}d$,

4. $c\bar{b}\overline{ca}cd\overline{fd}\bar{e}d$.

The pictures and the normal forms for these four topological classes can be found in the first four entries in table 3.1.

A similar analysis can be done for higher multiplicity, just by looking at the stabilizations of the germ y^n and taking the possible unions which are not equivalent. For y^5, there are seven stabilizations of y^5 with partial Gauss words: (a) $\emptyset$, (b) $a\bar{b}\bar{a}b$, (c) $ab\overline{db}\overline{ca}cd$, (d) $a\bar{b}\bar{a}bc\bar{d}\bar{c}d$ (e) $ab\overline{ca}\overline{db}cd$, (f) $ab\bar{d}\overline{ac}\bar{b}cd$ and (g) $ab\bar{c}\overline{bd}\bar{a}cd$ (see figures 3.6 and 3.8).

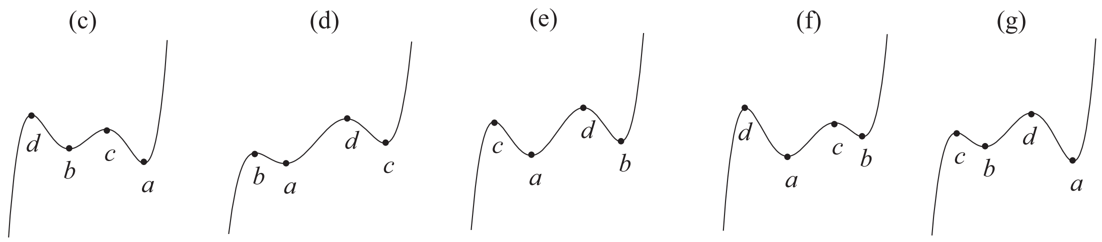


Figure 3.8

Now let's state the main theorem of this section, which will give us the complete classification of weighted homogeneous map germs of multiplicity ≤ 5.

Theorem 3.3.13. *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *be a weighted homogeneous finitely determined map germ given by* $f(x,y) = (x, g_x(y))$*, with* 2*-jet of type* $(x, 0)$ *and multiplicity* ≤ 5*. Then,* f *is topologically equivalent to one of the germs of tables 3.1 and 3.2, depending on the topological configuration of its associated link and its topological degree.*

Proof. The cases of multiplicity 3 and 4 have been analyzed in example 3.3.12. Hence, we assume that the multiplicity is 5. We distinguish 2 cases:

Case A: f *is homogeneous.* Since $g_x(y)$ is homogeneous of degree 5, we have the symmetry $g_{-x}(-y) = -g_x(y)$. Let us denote by $\sigma(1)\dots\sigma(k)$ the partial Gauss word of g_x^+ with r letters. Then the partial Gauss word of g_x^- is $\tau(\sigma(k))\dots\tau(\sigma(1))$, where $\tau(a_i) = a_{r-i+1}$. By theorem 3.3.8, the Gauss word of f is $\sigma(1)\dots\sigma(k)\phi(\sigma(1))\dots\phi(\sigma(k))$, where $\phi(a_i) = a_{r+i}$.

Consider the seven stabilizations of y^5 (a),...,(g) given in figures 3.6 and 3.8. The stabilization (f) is symmetric to (g), but each one of the remaining cases is its own symmetric. Thus, we obtain six possible combinations, namely, (a)+(a), (b)+(b), (c)+(c), (d)+(d), (e)+(e) and (f)+(g). The corresponding Gauss words are respectively:

1. $\emptyset$,
2. $a\bar{b}\bar{a}bc\bar{d}\bar{c}d$,
3. $ab\bar{d}\bar{b}\bar{c}\bar{a}cdef\bar{h}\bar{f}\bar{g}\bar{e}gh$,
4. $a\bar{b}\bar{a}bc\bar{d}\bar{c}de\bar{f}\bar{e}fg\bar{h}\bar{g}h$,
5. $ab\bar{c}\bar{a}\bar{d}\bar{b}cdef\bar{g}\bar{e}\bar{h}\bar{f}gh$,

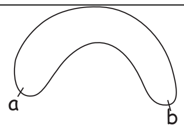
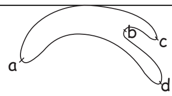
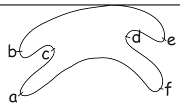
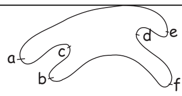
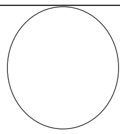
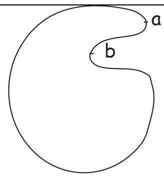
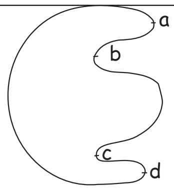
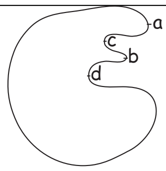
Degree	Germ	Associated link
0	$(x, y^4 + x^2y^2)$	 $\overline{ab}$
	$(x, y^4 - xy^2 - x^2y)$	 $\overline{adbc}$
	$(x, y^4 - x^4y^2 + \frac{1}{4}x^6y)$	 $c\overline{bca}cd\overline{fde}d$
	$(x, y^4 - x^2y^2 - \frac{1}{4}x^3y)$	 $c\overline{acb}cd\overline{fde}d$
1	$(x, y^3 + x^2y)$	 $\emptyset$
	$(x, y^3 + x^3y)$	 $a\overline{ba}b$
	$(x, y^3 - x^2y)$	 $a\overline{ba}bc\overline{dc}d$
	$(x, y^5 + 2xy^3 + \frac{1}{2}x^2y)$	 $ab\overline{dbca}cd$

Table 3.1

6. $ab\bar{d}\overline{ac}\bar{b}cdef\bar{h}\overline{eg}\bar{f}gh$.

Case B: f is weighted homogeneous. We suppose now that $g_x(y)$ is a weighted homogeneous polynomial of weights w_1 and w_2, with $w_1 \neq w_2$. We can write this polynomial in the form

$$g_x(y) = x^r y^s (a_0(x^{w_2})^d + a_1(x^{w_2})^{d-1}y^{w_1} + \cdots + a_d(y^{w_1})^d).$$

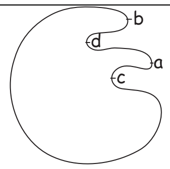
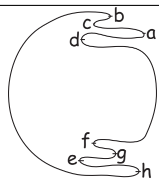
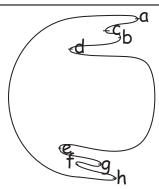
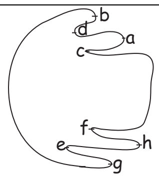
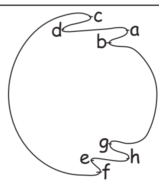
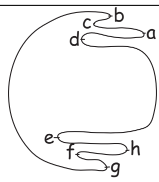
Degree	Germ	Associated link
1	$(x, y^5 + 3xy^3 + 2x^2y)$	 $ab\overline{cadb}cd$
	$(x, y^5 - 3x^2y^3 + \frac{5}{4}x^3y^2 + x^4y)$	 $ab\overline{dacb}cdef\overline{hegf}gh$
	$(x, y^5 - 5x^2y^3 + x^4y)$	 $ab\overline{dbca}cdef\overline{hfge}gh$
	$(x, y^5 - \frac{3}{2}x^2y^3 + \frac{1}{2}x^4y)$	 $ab\overline{cadb}cdef\overline{gehf}gh$
	$(x, y^5 - 3x^2y^3 + 3x^4y)$	 $a\overline{ba}bc\overline{dc}de\overline{fe}fg\overline{hg}h$
	$(x, y^5 - \frac{7}{2}x^4y^3 + 2x^6y^2 + x^8y)$	 $ab\overline{dacb}cdef\overline{gfhe}gh$

Table 3.2

Since f is finitely determined, we must have necessarily $r = 0$ and $s \leq 2$. We have two possible cases:

1. $f(x, y) = (x, y(a_0x^{dw_2} + a_1x^{(d-1)w_2}y^{w_1} + \cdots + a_dy^{dw_1}))$,
2. $f(x, y) = (x, y^2(a_0x^{dw_2} + a_1x^{(d-1)w_2}y^{w_1} + \cdots + a_{d-1}x^{w_2}y^{(d-1)w} + a_dy^{dw_1}))$,

where either $dw_1 + 1 = 5$ or $dw_1 + 2 = 5$. In addition, we can also take into account the following restrictions:

(a) As a consequence of the prenormal form of map germs of corank 1, if $w_1 = 1$, we can take $a_{d-1} = 0$.

(b) The functions $\phi_1(x) = x^{w_2}$ if w_2 is odd and $\phi_2(x^2) = x^{w_2}$ if w_2 is even are homeomorphisms. Thus, we only consider $w_2 = 1, 2$.

Depending on the possible values of d and w_1 and with the above restrictions, we find only 4 possibilities for f:

$$f(x,y) = \begin{cases} (x, y^5 + ax^4y^3 + bx^6y^2 + cx^8y), \\ (x, y^5 + axy^3 + bx^2y), \\ (x, y^5 + ax^2y^2), \\ (x, y^5 + axy^2). \end{cases}$$

In the first and third cases, we have the symmetry $g_{-x}(y) = g_x(y)$. Thus, we will have one of the following combinations (a)+(a), (b)+(b), (c)+(c), (d)+(d), (e)+(e), (f)+(f) or (g)+(g). Since (f)+(f) is equivalent to (g)+(g), we only need to add one more topological type to our list, namely that with Gauss word $ab\bar{d}\bar{a}\bar{c}\bar{b}cdef\bar{g}\bar{f}\bar{h}\bar{e}gh$.

In the second case, f is finitely determined if and only if $(20b - 9a^2)(4b - a^2)(10b + 81a^2)b \neq 0$. This bifurcation set is obtained by looking for the values (a, b) such as f presents simple cusps or double fold points, that is

$$g'_x(y) = g''_x(y) = 0$$

or

$$g_x(y_1) = g_x(y_2), g'_x(y_1) = g'_x(y_2) = 0.$$

This gives a partition of the (a, b) plane into 8 connected components (see figure 3.9). By taking a point in each connected component we find all the possible topological types of f. We find two more new types corresponding to the combinations (a)+(c) and (a)+(e), with Gauss word $ab\bar{d}\bar{b}\bar{c}\bar{a}cd$ and $ab\bar{c}\bar{a}\bar{d}\bar{b}cd$ respectively. Finally, in the fourth case, f is finitely determined if and only if $a \neq 0$ and we get two connected components, but do not get any new topological type in this case.

We remark that all the types that appear in tables 3.1 and 3.2 can be realized by considering the normal forms listed there. We have used the software `singR2R2` developed by A. Montesinos [25] in order to check that each normal form gives the desired topological type. □

Before finishing this section, let's state a result that will give us a necessary condition that a stable map $\gamma : S^1 \longrightarrow S^1$ should verify to be the link of a corank 1 map germ.

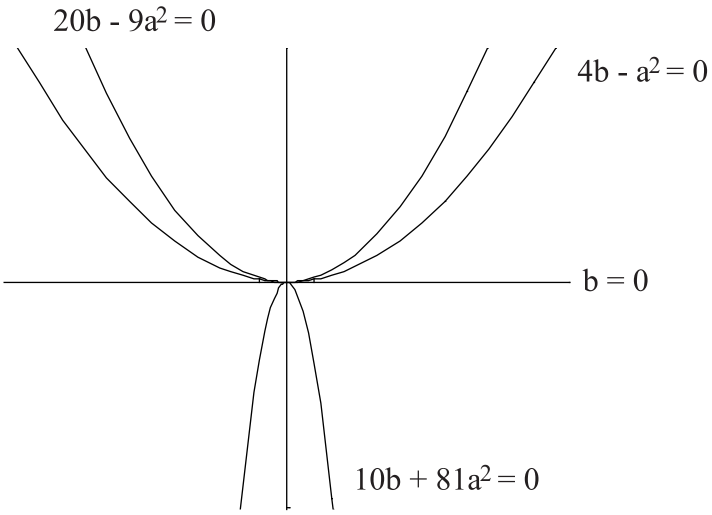


Figure 3.9: Bifurcation set of the map germ $(x, y^5 + axy^3 + bx^2y)$

Proposition 3.3.14. *Any finitely determined map germ* $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ *of corank 1, with link* γ*, verifies that*

$$\mathrm{mult}(\gamma) = \begin{cases} 0, & \textit{if } \deg(f) = 0, \\ 1, & \textit{if } \deg(f) = \pm 1. \end{cases}$$

Here, we define the multiplicity of a stable map $\gamma : S^1 \longrightarrow S^1$ *as* $\mathrm{mult}(\gamma) = min_{p \in S^1} \mathrm{mult}(p)$*, with* $\mathrm{mult}(p) = \#\gamma^{-1}(p)$*.*

Proof. The three possible values of the topological degree of f are a consequence of proposition 3.3.1. Let's suppose that $f(x, y) = (x, g_x(y))$, with $g_x(y) = y^n + a_{n-2}(x)y^{n-2} + \cdots + a_1(x)y$.

If $\deg(f) = 0$, n is even, $n - 1$ is odd and, as a consequence, the both curves g_x^+, g_x^-, that will form the link of f will have both an odd number of folds. Thus, γ will not be surjective and $\mathrm{mult}(\gamma) = 0$. If $\deg(f) = \pm 1$, n is odd, $n - 1$ is even and the union of both partial curves will completely fill S^1, so $\mathrm{mult}(\gamma) = 1$. □

3.4 The number of cusps and double folds of germs of corank 1

In this last section, we give some results related to the number of cusps and double folds that appear near the origin in a stable perturbation of a finitely determined map germ of corank 1.

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. We denote by $\hat{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ the complexification of f and we consider $\hat{F} = (t, \hat{f}_t)$ a stabilization of $\hat{f}$ (i.e., $\hat{f}_0 = \hat{f}$ and if $t \neq 0$, then $\hat{f}_t$ is stable in a small enough neighbourhood U). Then, we denote

$$\begin{aligned} c(f) &= \text{ the number of cusps of } \hat{f}_t \text{ in } U, \\ d(f) &= \text{ the number of double folds of } \hat{f}_t \text{ in } U. \end{aligned}$$

These two numbers $c(f), d(f)$ were introduced for the first time by Rieger [34] for the case of corank 1 and independently $c(f)$ was studied by Fukuda-Ishikawa in [6] for general case. Both numbers $c(f), d(f)$ are invariants of f which do not depend on the stabilization $\hat{F}$. Moreover, they can be computed algebraically in terms of the dimensions of some local algebras associated to f (see [9, 10]). In fact, if f has corank 1 and f has the form $f(x, y) = (x, g_x(y))$, then we have

$$c(f) = \dim_{\mathbb{R}} \frac{\mathbb{R}\{x, y\}}{\langle g'_x, g''_x \rangle},$$

where g'_x, g''_x denote the first and second partial derivatives of g_x with respect to y and $\mathbb{R}\{x, y\}$ is the local algebra of germs of analytic functions $(\mathbb{R}^2, 0) \to \mathbb{R}$.

If F is a real stabilization of f, then the numbers $c(f_t)$ and $d(f_t)$ of cusps and double folds of f_t respectively, depend on the stabilization F. However, $c(f_t)$ is congruent modulo 2 to the invariant $c(f)$ (see [6]).

Theorem 3.4.1. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of corank 1 given by $f(x, y) = (x, g_x(y))$. Then,*

$$\deg(g'_x, g''_x) = \frac{n(g_x^-) - n(g_x^+)}{2} \equiv c(f) \mod 2,$$

where $\deg(g'_x, g''_x)$ is the local topological degree of the map germ $(g'_x, g''_x) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $n(g_x^+)$ and $n(g_x^-)$ denote the number of critical values of g_x^+ and g_x^- respectively.

Proof. The equality is a direct consequence of Theorem C (2) in [31] applied to $g'_x : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. In fact, $n(g_x^+)$ is the number of branches of $(g'_x)^{-1}(0)$ which lie in the half region $x > 0$ and $n(g_x^-)$ is the number of branches of $(g'_x)^{-1}(0)$ which lie in the half region $x < 0$. Moreover, since f is finitely determined we have that

$$c(f) = \dim_{\mathbb{R}} \frac{\mathbb{R}\{x, y\}}{\langle g'_x, g''_x \rangle} < \infty.$$

Then, the result of Nishimura, Fukuda and Aoki implies that

$$n(g_x^-) - n(g_x^+) = 2\deg(g'_x, g''_x).$$

Finally, the congruence follows from the relation between the multiplicity and the local degree of $(g'_x, g''_x) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, taking into account that

$$c(f) = \text{mult}(g'_x, g''_x)$$

and

$$\text{mult}(g'_x, g''_x) \equiv \deg(g'_x, g''_x) \mod 2.$$

□

Remark 3.4.2. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of corank 1 and multiplicity $m(f)$. Then:

1. $c(f) = 0$ if and only if $m(f) \leq 2$,
2. $d(f) = 0$ if and only if $m(f) \leq 3$.

We can take a prenormal form

$$f(x, y) = (x, y^k + a_1(x)y^{k-2} + \cdots + a_{k-2}(x)y),$$

for some functions $a_i : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, where $k = m(f)$. We use a result of Gaffney and Mond [9]. We set

$B_C = \{u \in \mathbb{C}^{k-1} : g_u$ has a degenerate critical point $\}$,
$B_D = \{u \in \mathbb{C}^{k-1} : g_u$ has two critical points having the same critical value $\}$,

with $g_u(x) = y^k + u_1 y^{k-2} + \cdots + u_{k-2}y$. We denote by b_C and b_D the reduced equations of B_C and B_D respectively. Then,

$$c(f) = \nu(b_C \circ a), \quad d(f) = \nu(b_D \circ a),$$

where $a = (a_1, \ldots, a_{k-2}) : (\mathbb{C}, 0) \to (\mathbb{C}^{k-2}, 0)$ and $\nu(h)$ denotes the order of a function h.

As a consequence, $c(f) \geq 1$ if and only if the polynomial b_C is not constant, in other words, if the set B_C is not empty. But this means that $c(f) \geq 1$ if and only if there is a map germ f_0 such that $m(f_0) = k$ and $c(f_0) \geq 1$. Analogously, $d(f) \geq 1$ if and only if there is a map germ f_0 such that $m(f_0) = k$ and $d(f_0) \geq 1$.

By remark 2.2 of [10] we know that, if f is weighted homogeneous,

$$c(f) = (k-1)(k-2)\frac{w_2}{w_1},$$

$$d(f) = (k-1)(k-2)(k-3)\frac{w_2}{2w_1},$$

where k is the multiplicity of f and w_1 and w_2 are the weights of x and y respectively.

Therefore, if we consider $f_0(x,y) = (x, y^k + xy)$, it is easy to compute

$$c(f_0) = k-2, \quad d(f_0) = \frac{(k-2)(k-3)}{2},$$

which implies the desired result.

Chapter 4

Families of map germs from $\mathbb{R}^2$ to $\mathbb{R}^2$

In this chapter we consider a 1-parameter unfolding of f, that is, a map germ $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $F(x,t) = (f_t(x), t)$ and such that $f_0 = f$.

We are interested in the topological triviality of F, which means that it is topologically equivalent as an unfolding to the constant unfolding. Our main result is that F is topologically trivial if it is excellent in the sense of Gaffney [8] and moreover, the family of discriminant curves $\Delta(F)$ is a topologically trivial deformation of $\Delta(f)$. This can be seen as a real version of the same result obtained by Gaffney for complex analytic map germs [8, Theorem 9.9]. In fact, since $\Delta(f)$ is a plane curve, the topological triviality of F in the complex case is equivalent to the constancy of the Milnor number $\mu(\Delta(f_t))$. In the real case, we show that this is also a sufficient condition, although it is not necessary in general. In order to have a necessary and sufficient condition we should need an invariant which controls the topological triviality of a family of real plane curves. In the last section we consider unfoldings which are not topologically trivial and give a result about the number of cusps that appear in f_t.

The techniques used to prove this result have been already used by J.J.Nuño-Ballesteros in [32], where he gets a sufficient condition for the topological triviality in the case $\mathbb{R}^2$ to $\mathbb{R}^3$. The topological triviality of plane-to-plane has been also studied by Fukuda in [7]. We also refer to the work of Ikegami and Saeki [15] for related results.

4.1 Cobordism of links

We recall that a cobordism between two smooth manifolds M_0, M_1 is a smooth manifold with boundary W such that $\partial W = M_0 \sqcup M_1$. Analogously, a cobordism between smooth maps $f_0 : M_0 \to N_0$ and $f_1 : M_1 \to N_1$ is another smooth map $F : W \to Q$ such that W, Q are cobordisms between M_0, M_1 and N_0, N_1 respectively, and for each $i = 0, 1$, $F^{-1}(N_i) = M_i$ and the restriction $F|_{M_i} : M_i \to N_i$ is equal to f_i. In the case that f_0, f_1 belong to some special class of maps (for instance, immersions, embeddings, stable maps, etc.), then we also require that the cobordism F belongs to the same class.

Definition 4.1.1. Given two stable maps $\gamma_0, \gamma_1 : S^1 \to S^1$, a cobordism between γ_0 and γ_1 is a stable map $\Gamma : S^1 \times I \to S^1 \times I$, where $I = [0, 1]$ and such that for $i = 0, 1$,

$$\Gamma^{-1}(S^1 \times \{i\}) = S^1 \times \{i\}, \quad \Gamma|_{S^1 \times \{i\}} = \gamma_i \times \{i\}.$$

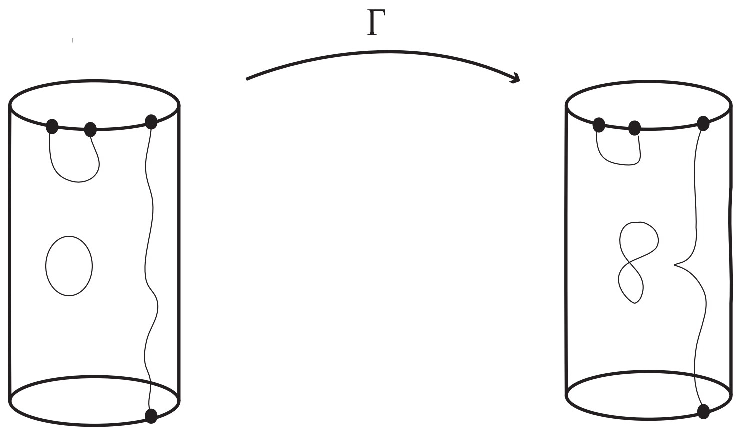


Figure 4.1: Example of a cobordism between stable maps $\gamma, \delta : S^1 \to S^1$

The first condition implies that $\Gamma(S^1 \times \{0\}) \subset S^1 \times \{0\}$, $\Gamma(S^1 \times \{1\}) \subset S^1 \times \{1\}$ and $\Gamma(S^1 \times (0, 1)) \subset S^1 \times (0, 1)$, but in general, Γ is not level preserving (see figure 4.1).

Lemma 4.1.2. *Let Γ be a cobordism between γ_0, γ_1. If $\Delta(\Gamma)$ is diffeomorphic to $\Delta(\gamma_0) \times I$, then γ_0, γ_1 are topologically equivalent.*

Proof. Since $\Delta(\Gamma)$ is diffeomorphic to $\Delta(\gamma_0) \times I$, Γ cannot have cusps or double folds. Thus, Γ restricted to $\Gamma^{-1}(\Delta(\Gamma))$ is a local diffeomorphism and it follows that $\Gamma^{-1}(\Delta(\Gamma))$ is also diffeomorphic to $\gamma_0^{-1}(\Delta(\gamma_0)) \times I$.

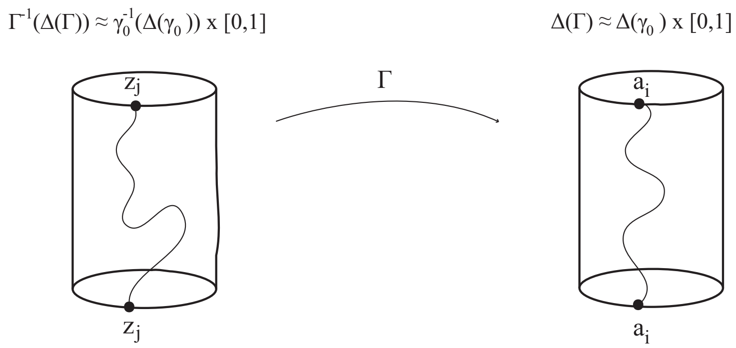


Figure 4.2: A cobordism with $\Delta(\Gamma)$ diffeomorphic to $\Delta(\gamma_0) \times I$

In particular, for each critical value or each inverse image of γ_0 there is a unique arc joining the point in $S^1 \times \{0\}$ with a point in $S^1 \times \{1\}$ corresponding to a critical value or an inverse image of γ_1 respectively. We choose the orientations and the base points of γ_0, γ_1 in such a way that if two critical values are joined by an arc, then they share the same label a_i and if two inverse images are joined by an arc, then they share the same label z_j (see figure 4.2).

With these choices, it follows that $w(\gamma_0) = w(\gamma_1)$ and hence γ_0 and γ_1 are topologically equivalent by theorem 3.2.3. □

Remark 4.1.3. If Γ is a cobordism between γ_0, γ_1 such that $\Delta(\Gamma)$ is diffeomorphic to $\Delta(\gamma_0) \times I$, then it can be shown that Γ is trivial, that is, Γ is $\mathcal{A}$-equivalent to the product cobordism $\gamma_0 \times \text{id} : S^1 \times I \to S^1 \times I$ by diffeomorphisms $\Phi, \Psi : S^1 \times I \to S^1 \times I$ such that $\Phi|_{S^1 \times \{0\}}, \Psi|_{S^1 \times \{0\}} = \text{id}$.

To show this, we first choose a diffeomorphism $\psi : \Delta(\gamma_0) \times I \to \Delta(\Gamma)$ such that $\psi(p, 0) = (p, 0)$, for all $p \in \Delta(\gamma_0)$. We denote by $\phi : \gamma_0^{-1}(\Delta(\gamma_0)) \times I \to \Gamma^{-1}(\Delta(\Gamma))$ the induced diffeomorphism by Γ in such a way that $\phi(s, 0) = (s, 0)$, for all $s \in \gamma_0^{-1}(\Delta(\gamma_0))$ and the following diagram is commutative:

$$
\begin{array}{ccc}
\Gamma^{-1}(\Delta(\Gamma)) & \xrightarrow{\Gamma} & \Delta(\Gamma) \\
\uparrow \phi & & \uparrow \psi \\
\gamma_0^{-1}(\Delta(\gamma_0)) \times I & \xrightarrow{\gamma_0 \times id} & \Delta(\gamma_0) \times I
\end{array}
$$

We extend the diffeomorphisms ϕ, ψ to $S^1 \times I$. This can be done by using standard arguments of extensions of vector fields. Details are left to the reader.

4.2 Extending the cone structure

Let $f : U \to V$ be a good representative of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. Since $\Delta(f)$ is a 1-dimensional analytic subset, we can also shrink the neighborhoods U, V so that this set is contractible. In this case $\Delta(f) \setminus \{0\}$ has a finite number of connected components, each one of them is an edge joining the origin with the boundary of V. We orient each one of this edges from 0 to ∂V. We denote by $X : \Delta(f) \setminus \{0\} \to \mathbb{R}^2$ the unit normal vector field of $\Delta(f) \setminus \{0\}$ with respect to this orientation (see figure 4.3).

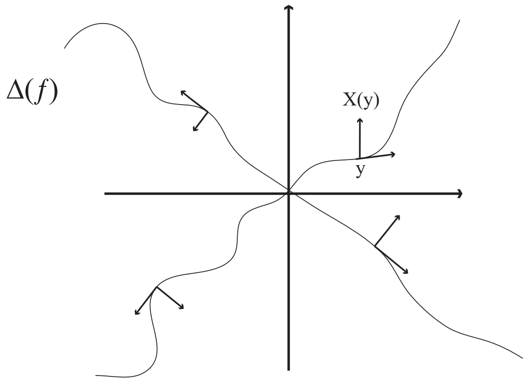


Figure 4.3

Definition 4.2.1. Let $f : U \to V$ be a good representative of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\Delta(f)$ is contractible. We say that $\epsilon > 0$ is a *convenient* radius for f if the following conditions hold:

1. S^1_ϵ is transverse to $\Delta(f)$,
2. $\widetilde{S}^1_\epsilon$ is diffeomorphic to S^1,
3. S^1_ϵ intersects $\Delta(f)$ *properly*, that is, S^1_ϵ intersects each connected component of $\Delta(f) \setminus \{0\}$ at exactly one point.

It is easy to see that S^1_ϵ intersects $\Delta(f)$ properly if and only if S^1_ϵ cuts each point of $\Delta(f)$ following the orientation of the outward-pointing normal of S^1_ϵ. In other words, S^1_ϵ cuts $\Delta(f)$ properly if and only if

$$\det(X(y), y) > 0, \quad \forall y \in S^1_\epsilon \cap \Delta(f).$$

If ϵ_0 is a Milnor-Fukuda radius (see definition 3.1.1), then S^1_ϵ intersects $\Delta(f)$ properly for any $0 < \epsilon \leq \epsilon_0$, but in general, this may not be true for a convenient radius, as we can see in figure 4.4.

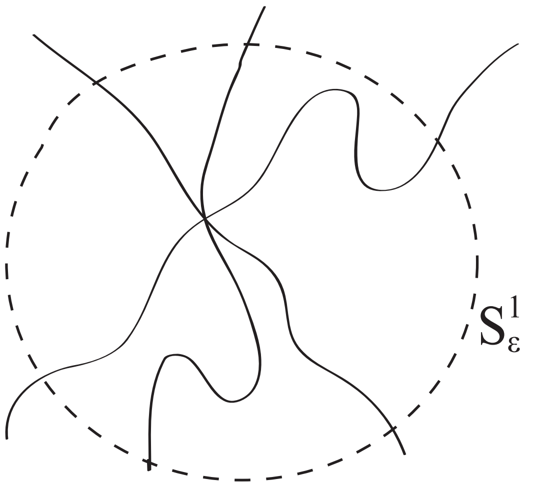


Figure 4.4

Theorem 4.2.2. *Let $f : U \to V$ be a good representative of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\Delta(f) \subset V$ is contractible and let $\epsilon > 0$ be a convenient radius for f. Then,*

1. *$f|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon$ is topologically equivalent to the link of f.*
2. *$f|_{\widetilde{D}^2_\epsilon} : \widetilde{D}^2_\epsilon \to D^2_\epsilon$ is topologically equivalent to the cone of $f|_{\widetilde{S}^1_\epsilon}$.*

Proof. Let $\epsilon_0 > 0$ be a Milnor-Fukuda radius for f. If $\epsilon \leq \epsilon_0$, then the result follows from corollary 2.3.2 . We assume $\epsilon > \epsilon_0$ and take $0 < \delta < \epsilon_0$. We consider the two associated links $\gamma_0 = f|_{\widetilde{S}^1_\delta}$ and $\gamma_1 = f|_{\widetilde{S}^1_\epsilon}$ and we denote by

$$C^2_{\delta,\epsilon} = \{y \in \mathbb{R}^2 : \delta \leq \|y\|^2 \leq \epsilon\}, \quad \widetilde{C}^2_{\delta,\epsilon} = f^{-1}(C^2_{\delta,\epsilon}),$$

and $\Gamma = f|_{\widetilde{C}^2_{\delta,\epsilon}} : \widetilde{C}^2_{\delta,\epsilon} \to C^2_{\delta,\epsilon}$, which defines a cobordism between γ_0 and γ_1. We only need to show that γ_0 and γ_1 are topologically equivalent, since in this case we have that the cone structure of $f|_{\widetilde{D}^2_\delta}$ can be extended to $f|_{\widetilde{D}^2_\epsilon}$.

Let $\Delta_1, \ldots, \Delta_r$ be the connected components of $\Delta(f) \setminus \{0\}$. Since $\Delta(f) \subset V$ is closed, contractible and regular outside the origin, we have that each Δ_i is diffeomorphic to an open interval, whose end points are the origin and another point of ∂V. Now, both S^1_δ and S^1_ϵ intersect $\Delta(f)$ properly, so that $S^1_\delta \cap \Delta_i = \{x_i\}$ and $S^1_\epsilon \cap \Delta_i = \{x'_i\}$ for each $i = 1, \ldots, r$. It follows that

$$\Delta(\Gamma) = \overline{x_1 x'_1} \cup \cdots \cup \overline{x_r x'_r},$$

where $\overline{x_i x'_i}$ is the closed interval in Δ_i joining the points x_i and x'_i. Therefore, $\Delta(\Gamma)$ is diffeomorphic to $\{x_1, \ldots, x_r\} \times [\delta, \epsilon]$ and γ_0 and γ_1 are topologically equivalent by lemma 4.1.2.

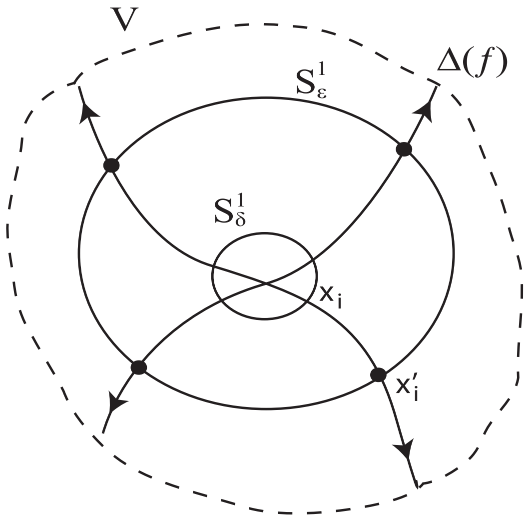


Figure 4.5: Scheme of $C^2_{\delta,\epsilon}$

□

4.3 Topological triviality of families

Given a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, a 1-parameter unfolding is a map germ $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $F(x, t) = (f_t(x), t)$ and such that $f_0 = f$. Here, we consider that the unfolding is origing preserving, that is, $f_t(0) = 0$ for any t. Hence, we have a 1-parameter family of map germs $f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

Definition 4.3.1. Let F be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

1. We say that F is *excellent* if there is a representative $F : U \to V \times I$, where U, V, I are open neighborhoods of the origin in $\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}$ respectively, such that for any $t \in I$, $f_t : U_t \to V$ is a good representative in the sense of definition 2.2.5.

2. We say that F has *constant topological type* if for any $t \neq t'$, the map germs f_t and $f_{t'}$ are topologically equivalent.

3. We say that F is *topologically trivial* if there are homeomorphism germs $\Psi, \Phi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ such that they are unfoldings of the identity and $F = \Psi \circ (f \times \mathrm{id}) \circ \Phi$.

Example 4.3.2. Any topologically trivial unfolding F has constant topological type, but the converse is not true in general. Let us consider $h_t : \mathbb{R}^2 \to \mathbb{R}$ the equation of $S(f_t)$ for each t, given by

$$h_t(x, y) = (x + 3y)(5x - 2y)s_t(x, y),$$

with $s_t(x, y) = ((x - 2)^2 + (y - 3)^2)t - \epsilon^2 t$ (see figure 4.6). Then, we set:

$$f_t(x, y) = (x, \int h_t(x, y)dy).$$

It is not difficult to check that the Gauss word is constant $w(f_t) = a\overline{b}\overline{a}bc\overline{d}\overline{c}d$. As a consequence, the map germs f_t and $f_{t'}$ are topologically equivalent for any $t \neq t'$. However, it is clear that our family is not topologically trivial.

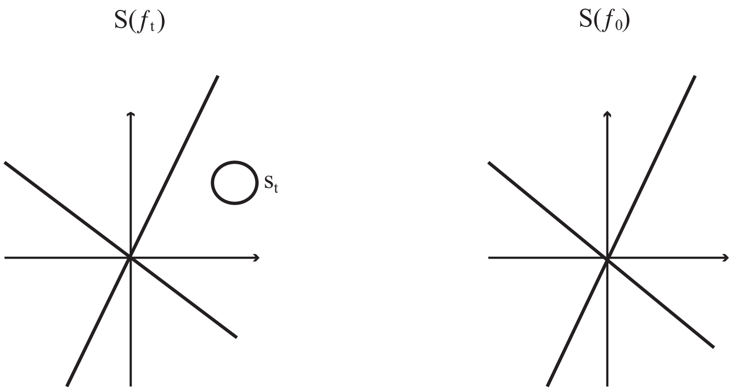


Figure 4.6: Singular sets of f_t (left) and f_0 (right)

Theorem 4.3.3. *Let F be an excellent unfolding of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. If $\Delta(F)$ is topologically trivial, then F is topologically trivial.*

Proof. Let $F : U \to V \times I$ be a representative of the unfolding F, where U, V, I are open neighborhoods of the origin in $\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}$ respectively, and such that $f_t : U_t \to V$ is a good representative of the map germ f_t, for any $t \in I$. We can shrink the neighborhoods if necessary and assume that $\Delta(f_0) \subset V$ is contractible.

On the other hand, since $\Delta(F)$ is topologically trivial, by shrinking again the neighbourhoods if necessary, there is a homeomorphism $\Psi : V \times I \to V \times I$ of the form $\Psi = (\psi_t, t)$ such that $\psi_0 = \text{id}$ and $\psi_t(\Delta(f_t)) = \Delta(f)$, for any $t \in I$. In particular, $\Delta(f_t)$ is homeomorphic to $\Delta(f_0)$ and it is also contractible.

We take $X : (V \setminus \{0\}) \times I \to \mathbb{R}^2$ such that $X_t(y) = X(y,t)$ is the unit normal vector at each point $y \in \Delta(f_t) \setminus \{0\}$ as in definition 4.2.1. We also denote by $g_t : U_t \to \mathbb{R}$ the function $g_t(x) = \|f(x)\|^2$ and $G : U \to \mathbb{R}$, given by $G(x,t) = g_t(x)$.

Let $\epsilon_0 > 0$ be a Milnor-Fukuda radius for f and let $0 < \epsilon \leq \epsilon_0$. We have that ϵ is a regular value of g_0, $\widetilde{S}^1_\epsilon = g_0^{-1}(\epsilon)$ is diffeomorphic to S^1 and that S^1_ϵ intersects properly to $\Delta(f)$, that is,

$$\det(X_0(y), y) > 0, \ \forall y \in S^1_\epsilon \cap \Delta(f).$$

Once ϵ is fixed, we can choose $\delta > 0$ such that for any $t \in (-\delta, \delta)$, by a continuity argument, ϵ is also a regular value of g_t and

$$\det(X_t(y), y) > 0, \ \forall y \in S^1_\epsilon \cap \Delta(f_t).$$

By the fibration theorem, we have that $\widetilde{S}^1_{\epsilon,t} = g_t^{-1}(\epsilon)$ is diffeomorphic to $\widetilde{S}^1_\epsilon$, and hence to S^1. Moreover, the above condition gives that S^1_ϵ is transverse to $\Delta(f_t)$ and that S^1_ϵ intersects $\Delta(f_t)$ properly. In conclusion, we have shown that ϵ is a convenient radius for f_t, for any $t \in (-\delta, \delta)$. By theorem 4.2.2, $\gamma_{\epsilon,t} = f_t|_{\widetilde{S}^1_{\epsilon,t}}$ is the link of f_t and $f_t|_{\widetilde{D}^2_{\epsilon,t}}$ is topologically equivalent to the cone of $\gamma_{\epsilon,t}$.

Since $\gamma_{\epsilon,t} : \widetilde{S}^1_{\epsilon,t} \to S^1_\epsilon$, with $t \in (-\delta, \delta)$, is stable, we have that this family of links is trivial. Hence, each $f_t|_{\widetilde{D}^2_{\epsilon,t}}$ is topologically equivalent to $f|_{\widetilde{D}^2_\epsilon}$. By remark 3.2.8, there is a unique homeomorphism in the source ϕ_t such that $\psi_t \circ f_t \circ \phi_t^{-1} = f$. Note that the unicity of ϕ_t implies that it depends continuously on t. We consider now $\Phi = (\phi_t, t) : F^{-1}(D^2_\epsilon \times (-\delta, \delta)) \to \widetilde{D}^2_\epsilon \times (-\delta, \delta)$. Then Φ is a homeomorphism, it is an unfolding of the identity and $\Psi \circ F \circ \Phi^{-1} = f \times \ id$. □

Before stating an immediate consequence of this result we should remember what the Milnor number of a curve is.

Definition 4.3.4. In the complex case, given a plane curve $(X, 0)$ with reduced equation $h(u,v) = 0$ in $(\mathbb{C}^2, 0)$, its Milnor number is the colength of the ideal generated by the partial derivatives h_u, h_v, that is,

$$\mu(X,0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle h_u, h_v\rangle}.$$

If $f : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ is a finitely determined map germ, we denote by $\mu(\Delta(f))$ the Milnor number of the discriminant $\Delta(\hat{f})$ of the complexification $\hat{f} : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$.

Example 4.3.5. Let us consider $f : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ given by $f(x,y) = (x, x^2y + y^3/3)$. We have that $S(f)$ has defining equation $x^2 + y^2 = 0$ and hence, $\Delta(f)$ is given by $4u^6 + 9v^2 = 0$. Although $\Delta(f) = \{0\}$ as set germs, we have that $\mu(\Delta(f)) = 5$, which is the Milnor number of the complex curve given by this equation.

Definition 4.3.6. Let F be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$. We say that F is *μ-constant* if the Milnor number $\mu(\Delta(f_t))$ is independent of t.

Corollary 4.3.7. *Any μ-constant unfolding F of a finitely determined map germ $f : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ is topologically trivial.*

Proof. Any μ-constant unfolding F is excellent. This is known to be true in the complex case by the results of Gaffney [8]. Since F is analytic we are able to consider its complexification $\widehat{F}$ and we have that $\mu(\Delta(\widehat{f_t})) = \mu(\Delta(f_t))$ is constant. Then, $\widehat{F}$ is excellent, and as a consequence, F is also excellent. On the other hand, the μ-constant condition in the family of plane curves $\Delta(F)$ implies its topological triviality by the results of [16]. By theorem 4.3.3, F is topologically trivial. □

It is well known that in the complex case, any family of plane curves is topologically trivial if and only if the Milnor number is constant in the family. Hence, the converse of corollary 4.3.7 is also true in the complex case. In the real case, this is not true in general, as shown in the following example.

Example 4.3.8. Consider the family $f_t(x,y) = (x, x^4y + y^5 + t^2y^3)$. We have $f_t^{-1}(0) = \{0\}$, $Jf = x^4 + 5y^4 + 3t^2y^2 = 0$ and $S(f_t) = \{0\}$, for any $t \in \mathbb{R}$. Thus, the unfolding $F = (f_t, t)$ is excellent. Moreover, $\Delta(f_t) = \{0\}$ for any $t \in \mathbb{R}$, and hence F is topologically trivial by theorem 4.3.3.

On the other hand, the discriminant $\Delta(\hat{f}_t)$ of the complexification $\hat{f}_t$ is given by equation:

$$108t^{10}v^2 + 16t^8u^{12} - 900t^6u^4v^2 - 128t^4u^{16} + 2000t^2u^8v^2 + 256u^{20} + 3125v^4 = 0.$$

We have that $\mu(\Delta(f_t)) = 11$ for $t \neq 0$, but $\mu(\Delta(f_0)) = 57$. The computations have been done with the aid of Mathematica and Singular.

4.4 The number of cusps of an unfolding

In this last section, we follow the arguments of the proof of theorem 4.3.3 to give a formula for the parity of the number of cusps of an unfolding $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of a finitely determined map germ f. Here, we do not assume that F is excellent, but we only assume the condition given by the following definition.

Definition 4.4.1. Let $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ be an unfolding of a finitely determined map germ f. We say that F verifies the condition $(*)$ if there is a representative $F : U \to V \times I$, where U, V, I are open neighbourhoods of the origin in $\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2, \mathbb{R}$ respectively, such that $f_t : U_t \to V$ is proper and its restriction to $f_t^{-1}(V \setminus \{0\})$ is stable.

Given an unfolding satisfying this condition $(*)$, we introduce the following notation:

1. $c(f_t^+)$ (respectively $c(f_t^-)$) is the number of cusps of f_t on $f_t^{-1}(V \setminus \{0\})$ for $t > 0$ (respectively $t < 0$).
2. $r(f_t^+)$ (respectively $r(f_t^-)$) is the number of points of $f_t^{-1}(0)$ for $t > 0$ (respectively $t < 0$).
3. $\#S(f_t^+)$ (respectively $S(f_t^-)$) is the number of branches of $S(f_t)$ at $f_t^{-1}(0)$ for $t > 0$ (respectively $t < 0$).
4. $\#S(f_0)$ is the number of branches of $S(f_0)$ at 0.

If the neighbourhoods U, V, I are small enough, then these numbers are well defined. We also denote the multiplicity of a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ by

$$m(f) = \dim_{\mathbb{R}} \frac{\mathcal{E}_2}{\langle f_1, f_2 \rangle}.$$

We have the following congruences, which can be also deduced from the arguments of [9, Proof of Theorem 1.12]

Proposition 4.4.2. *Let $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ satisfying condition $(*)$. Then,*

$$c(f_t^{\pm}) \equiv 1 - r(f_t^{\pm}) + \#S(f_0) + \#S(f_t^{\pm}) \mod 2.$$

Moreover, if $m(f_t)$ is constant for each $t \in \mathbb{R}$ we have that

$$c(f_t^{\pm}) \equiv \#S(f_0) + \#S(f_t^{\pm}) \mod 2.$$

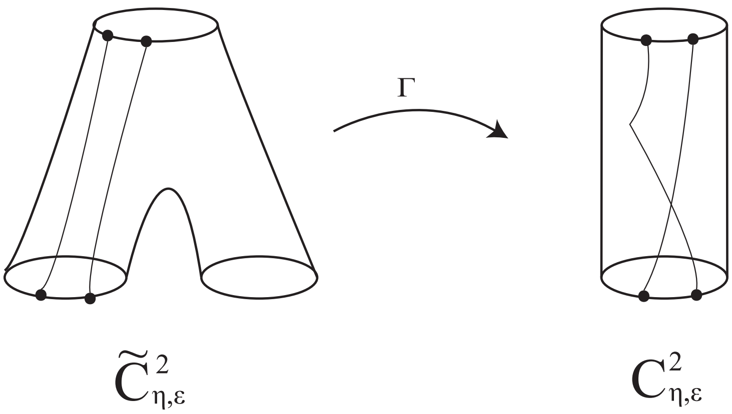


Figure 4.7: An example of the cobordism $\Gamma = f_t|_{\widetilde{C}^2_{\eta,\epsilon}}$

Proof. Let $\epsilon_0 > 0$ be a Milnor-Fukuda radius for f and take $0 < \epsilon \leq \epsilon_0$. There is $\delta > 0$ such that if $t \in (-\delta, \delta)$, then ϵ is a convenient radius for the multigerm $f_t : (\mathbb{R}^2, Z_t) \to (\mathbb{R}^2, 0)$, where $f_t^{-1}(0) = Z_t$.

We fix $0 < t < \delta$, the case $-\delta < t < 0$ being analogous. Take $0 < \eta < \epsilon$, where $\eta \leq \eta_0$, a Milnor Fukuda radius for f_t. We denote:

$$\begin{aligned} \gamma_0 =& f_t|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon, \\ \gamma_1 =& f_t|_{\widetilde{S}^1_\eta} : \widetilde{S}^1_\eta \to S^1_\eta, \\ \Gamma =& f_t|_{\widetilde{C}^2_{\eta,\epsilon}} : \widetilde{C}^2_{\eta,\epsilon} \to C^2_{\eta,\epsilon}. \end{aligned}$$

We have that γ_0 is $\mathcal{A}$-equivalent to the link of the map germ f, γ_1 is the link of the multigerm f_t and Γ is a cobordism between γ_0, γ_1. Since Γ is a stable map between compact oriented connected surfaces with boundary, we can apply a result due to Fukuda - Ishikawa [6]:

$$c(\Gamma) \equiv \chi(\widetilde{C}^2_{\eta,\epsilon}) + \deg(\Gamma|_{\partial\widetilde{C}^2_{\eta,\epsilon}})\chi(C^2_{\eta,\epsilon}) + \frac{1}{2}\#(S(\Gamma|_{\partial\widetilde{C}^2_{\eta,\epsilon}})) \mod 2,$$

where $c(\Gamma)$ is the number of cusps of Γ. We have $c(\Gamma) = c(f_t^+)$, $\chi(\widetilde{C}^2_{\eta,\epsilon}) = 1 - r(f_t^+)$, $\chi(C^2_{\eta,\epsilon}) = 0$ and

$$\frac{1}{2}\#(S(\Gamma|_{\partial\widetilde{C}^2_{\eta,\epsilon}})) = \#S(f_0) + \#S(f_t^+).$$

Thus, we arrive to

$$c(f_t^+) \equiv 1 - r(f_t^+) + \#S(f_0) + \#S(f_t^{\pm}) \mod 2.$$

If $m(f_t)$ is constant, we have that $\{f_t^{-1}(0)\} = \{0\}$, $r(f_t^+) = 1$ and hence,

$$c(f_t^+) \equiv \#S(f_0) + \#S(f_t^+) \mod 2.$$

$\square$

Chapter 5

Corank 2 map germs from $\mathbb{R}^2$ to $\mathbb{R}^2$

5.1 The corank 2 case

After having defined a complete topological invariant for a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and having used it to classify map germs of this kind of corank 1, the following logical step would be trying to extend this classification to germs of corank 2. This classification is also motivated by the fact that, taking into account proposition 3.3.14, some examples of links are not realizable by corank 1 map germs, even if $|\deg(f)| \leq 1$ (see figure 5.1).

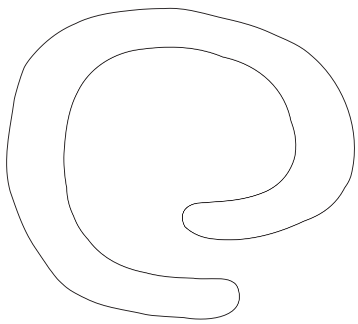


Figure 5.1

This classification was completed for the $\Sigma^{2,0}$ class in the case of $\mathcal{K}$ - equivalence following Mather's techniques of classification (see for example [12]) and Nishimura proved in [30] that, dealing with $\mathcal{K}$ - $\mathcal{C}^0$ - classes, the absolute value of $\deg(f)$ becomes a complete topological invariant. In the complex case, we can find related results in [24] and a full clasification for weighted homogeneous map germs from $\mathbb{C}^2$ to $\mathbb{C}^2$ in an article of T.Gaffney and D.Mond in [10].

The fact that we are not able to consider our germs as 1-parameter unfoldings of functions, as we did in the corank 1 case, makes things to become much more complex. The absolute value of the topological degree doesn't have to be necessarily less or equal than 1 and although our Gauss words continue being a complete topological invariant, since their links are not constituted as the union of 2 curves (as we did in chapter 2) the simplifications of letters are not allowed anymore.

Therefore, in this chapter we will classify corank 2 map germs but putting ourselves the convenient restrictions in their form that we believe are necessary to reach this goal. Firstly we will suppose that f is of type $\Sigma^{2,0}$, that is, all its partial derivatives vanish at $(0,0)$ and one of the minors of the derivatives of second order is distinct from 0. Departing from this point, we will establish a prenormal form of this kind of germs by using their $\mathcal{A}^2$-classes and Eisenburd-Levine formula([4]) will let us to compute their topological degree. As final step we will consider particular cases and we will try to obtain the different topological classes that we have in each case.

5.2 Topological classification of map germs of type $\Sigma^{2,0}$

In this section of the chapter we will classify corank 2 map germs, $f : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0)$ which are of type $\Sigma^{2,0}$.

First of all we will state a result that will give us two prenormal forms of map germs of this type.

Theorem 5.2.1. *Let $f : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0)$ a corank 2 map germ of type $\Sigma^{2,0}$. Then, f can be written in one of the following prenormal forms:*

1. $(xy, g(x,y))$
2. $(x^2+y^2, h(x,y))$,

where $g,h \in \mathcal{M}_2^2$

Proof. Firstly, we know that if we consider a map germ f of type $\Sigma^{2,0}$, its 2-jet $j^2f(0)$ is situated in one of the following $\mathcal{A}^2$-classes (see for example [12]):

$$(xy, x^2+y^2), \quad (xy, x^2), \quad (xy, 0), \quad (x^2+y^2, 0).$$

Therefore, f will present one of the following forms:

1. $(xy + a(x,y), b(x,y))$, with $a(x,y) \in \mathcal{M}_2^3$, $b(x,y) \in \mathcal{M}_2^2$
2. $(x^2 + y^2 + c(x,y), d(x,y))$, with $c(x,y) \in \mathcal{M}_2^3$, $d(x,y) \in \mathcal{M}_2^2$

By applying Morse's lemma we know that if we consider a function germ $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ of the form $f(x,y) = u(x,y) + v(x,y)$, with $u(x,y)$ being a non degenerate quadratic form and with $v(x,y) \in \mathcal{M}_2^3$, we can choose a suitable change of coordinates

$$\begin{array}{rccc} \alpha : & (\mathbb{R}^2, 0) & \longrightarrow & (\mathbb{R}^2, 0) \\ & (x,y) & \rightarrow & (X,Y) \end{array}$$

such that $u = f \circ \alpha^{-1}$.

As we have a non degenerate quadratic form in the first component, if we apply this change of coordinates in (1) and (2), we arrive to the desired result. □

The first step to classify topologically this kind of germs will be to compute their topological degree. Taking it into account, we state and prove the following result.

Proposition 5.2.2. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of type $\Sigma^{2,0}$.*

1. *If $f(x,y) = (xy, g(x,y))$, f can have degree $0, \pm 1$ or ± 2.*
2. *If $f(x,y) = (x^2 + y^2, h(x,y))$, f has degree 0.*

Proof. Let's prove first (2). If our germ f has as first component $x^2 + y^2$ it is not surjective. Then, $\deg(f) = 0$.

For (1), we can suppose, without loss of generality, that

$$g(x,y) = ax^p + by^q + k(x,y)$$

where

$$p, q \geq 2,$$
$$a, b > 0$$

and

$$k(x,y) \in \langle xy \rangle.$$

As we know that $(xy, g(x,y))$ is $\mathcal{K}$-equivalent to $(xy, ax^p + by^q)$ and that the topological degree is a $\mathcal{K}$-invariant we only need to compute the topological degree of $(xy, ax^p + by^q)$. We will do it by applying Eisenburd-Levine's formula ([4]), given by

$$\deg(f) = \operatorname{sign}\langle , \rangle_\varphi,$$

the signature of the quadratic form associated to a linear function

$$\varphi : Q(f) \to \mathbb{R}$$

defined conveniently, with

$$Q(f) = \frac{\mathcal{E}_2}{\langle f_1, f_2 \rangle}.$$

Thus, we have that

$$Q(f) = \frac{\mathcal{E}_2}{\langle xy, ax^p + by^q \rangle}$$

and a basis of this space will be given by

$$\{1, x, x^2, \ldots, x^{p-1}, y, y^2, \ldots, y^{q-1}, J(f)\}$$

with

$$J(f) = qby^q - pax^p.$$

We define the map

$$\begin{array}{rccc} \varphi : & Q(f) & \longrightarrow & \mathbb{R} \\ & J(f) & \to & 1 \\ & [1] & \to & 0 \\ & [x] & \to & 0 \\ & [y] & \to & 0 \\ & \vdots & \vdots & \vdots \\ & [x^{p-1}] & \to & 0 \\ & [y^{q-1}] & \to & 0 \end{array}$$

We will suppose that $a = b = \pm 1$, generalizing the result later.

The matrix of

$$\begin{array}{rccc} \langle , \rangle_\varphi : & Q(f) \times Q(f) & \longrightarrow & \mathbb{R} \\ & (p, q) & \to & \varphi(pq) \end{array}$$

with respect to the basis of $Q(f)$ is

$$A = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ \vdots \\ x^{p-1} \\ y \\ y^2 \\ \vdots \\ y^{q-1} \\ J \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & x & x^2 & \cdots & x^{p-1} & y & y^2 & \cdots & y^{q-1} & J \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & \mp\frac{1}{p+q} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \mp\frac{1}{p+q} & 0 & \cdots & 0 & 0 & 0\cdots & 0 & 0 & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \pm\frac{1}{p+q} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \pm\frac{1}{p+q} & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array}$$

taking into account the following facts:

- Each element of the form $x^i y^j \in \langle xy \rangle$ and as a consequence is 0 in $Q(f)$
- Each element of the form $Jx^i, Jy^j, x^{p+i}, y^{q+j}$ can be written as linear combination of the components of f, that is, they are 0 in $Q(f)$
- The elements x^p and y^q can be written in the following form:
  - $x^p = \frac{\mp q y^q \pm p x^p \pm q(\pm x^p \pm y^q)}{\pm(p+q)} = \frac{\mp 1}{p+q} J \pm \frac{q}{p+q}(\pm x^p \pm y^q)$, with $\varphi(x^p) = \mp\frac{1}{p+q}$
  - $y^q = \frac{\pm q y^q \mp p x^p \pm p(\pm x^p \pm y^q)}{\pm(p+q)} = \frac{\pm 1}{p+q} J \pm \frac{p}{p+q}(\pm x^p \pm y^q)$, with $\varphi(y^q) = \pm\frac{1}{p+q}$

Therefore, by computing the determinant of the matrix $(xI - A)$ we obtain the following characteristic polynomials, depending on the parity of p and q:

- If p and q are odd, $\det(xI - A) = (x^2 - 1)(x^2 - \frac{1}{(p+q)^2})^{\frac{p+q-2}{2}}$ and, as a consequence, $\deg(f) = \operatorname{sign}\langle , \rangle_\varphi = 0$.
- If p and q are even, $\det(xI - A) = (x^2 - 1)(x^2 - \frac{1}{(p+q)^2})^{\frac{p+q-4}{2}}(x \mp \frac{1}{p+q})(x \pm \frac{1}{p+q})$ and, as a consequence, $\deg(f) = \operatorname{sign}\langle , \rangle_\varphi = \begin{cases} 0, & \text{if } ab > 0, \\ \pm 2, & \text{if } ab < 0. \end{cases}$
- If p and q have different parity, $\det(xI - A) = (x^2 - 1)(x^2 - \frac{1}{(p+q)^2})^{\frac{p+q-3}{2}}(x \pm \frac{1}{p+q})$ and, as a consequence, $\deg(f) = \operatorname{sign}\langle , \rangle_\varphi = \pm 1$.

Let's see now that we are able to generalize this result for any $a, b \in \mathbb{R}$, with $a, b \neq 0$. We will prove this by constructing a homotopy.

Let $f_0(x,y) = (xy, ax^p + by^q)$, with $a > 0$ (analogous for $a < 0$), $f_1(x,y) = (xy, x^p + by^q)$ and we consider the family

$$f_t(x,y) = (xy, ((1-t)a + t)x^p + by^q),$$

with $t \in [0,1]$.

If we prove that for any t, $f_t^{-1}(0) = \{0\}$ and that if $t = 0$, $f_t = f_0$ and if $t = 1$, $f_t = f_1$, we will have that f_0 and f_1 are homotopic and, as a consequence, $\deg(f_0) = \deg(f_1)$.

As $(1-t)a + t \neq 0$ for any t, we will have that if we want that both terms of f_t vanish, x and y must be 0. Then, for any t, $f_t^{-1}(0) = \{0\}$. On the other hand, by substituting, if $t = 0$, $f_t(x,y) = (xy, ax^p + by^q) = f_0(x,y)$ and if $t = 1$, $f_t(x,y) = (xy, x^p + by^q) = f_1(x,y)$. Then, f_0 and f_1 are homotopic and $\deg(f_0) = \deg(f_1)$.

Analogously, we will have that $\deg(xy, x^p + by^q) = \deg(xy, x^p + y^q)$ if $b > 0$. Then, $\deg(xy, ax^p + by^q) = \deg(xy, x^p + y^q)$. □

Now, putting together theorem 5.2.1 and proposition 5.2.2, we have the following corollary.

Corollary 5.2.3. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of type $\Sigma^{2,0}$. Then, $|\deg(f)| \leq 2$.*

Proof. If f is of type $\Sigma^{2,0}$, by theorem 5.2.1 it can be written in the form $(xy, g(x,y))$ or in the form $(x^2 + y^2, h(x,y))$, that we have just seen that the absolute value of their topological degree is less or equal than 2. □

Before starting to compute the different topological classes of this kind of germs, we should remember the concepts of admissible weights and weighted degrees of a weighted homogeneous map germ which were introduced by Gaffney and Mond in [10] and will be very helpful for us in our classification.

Definition 5.2.4. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a weighted homogeneous map germ. We will say that its weights w_1, w_2 and its weighted degrees d_1, d_2 are *admissible* if they verify the two following conditions:

1. $(w_1, w_2) = (d_1, d_2) = 1$
2. $w_1 = w_2 = 1$ (homogeneous case) or $d_1 = k_1 w_1 w_2$, $d_2 = k_2 w_1 w_2 + w_1 + w_2$ (type 1) or $d_1 = k_1 w_1 w_2 + w_1$, $d_2 = k_2 w_1 w_2 + w_2$ (*type* 2).

Once we have introduced this concept, let's see its relation with finitely determined map germs.

Theorem 5.2.5. *([10]) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a weighted homogeneous finitely determined map germ. Then, w_1, w_2, d_1, d_2 must be admissible.*

Remark 5.2.6. Let's see how we apply this result to a finitely determined map germ in our particular case of study.

- $f(x, y) = (xy, g(x, y))$

  If f is weighted homogeneous, that is,

  $$g(x, y) = \sum_{i=0}^{p} a_i (x^{w_2})^i (y^{w_1})^{p-i}$$

  we must have that $(w_1, w_2) = (w_1 + w_2, pw_1w_2) = 1$. Departing from the basis that w_1, w_2 must be relatively primes, we have the following consequences, according to the value of p.

  - If $p = 1$, f is generically finitely determined.
  - If $p = 2$, f won't be finitely determined if w_1 and w_2 are odd.
  - If $p = 3$, f won't be finitely determined if $w_1 + w_2 = 3k$, with $k \in \mathbb{N}$.
  - In general, if $p = t_1^{\alpha_1} \dots t_m^{\alpha_m}$, f won't be finitely determined if there exists i such that $w_1 + w_2 = kt_i$, with $1 \leq i \leq m$ and $k \in \mathbb{N}$.

- $f(x, y) = (x^2 + y^2, h(x, y))$

  Because of the first component, we are only able to study this kind of germ in the homogeneous case $w_1 = w_2 = 1$, with

  $$h(x, y) = \sum_{i=0}^{p} a_i x^i y^{p-i}$$

  and $(2, p) = 1$. We arrive quickly to the conclusion that if $p = 2k$, f won't be finitely determined.

5.2.1 Germs with prenormal form $(xy, g(x, y))$

We consider the special case of weighted homogeneous map germs, that is,

$$g(x, y) = \sum_{i=0}^{p} a_i (x^{w_2})^i (y^{w_1})^{p-i},$$

with $(w_1 + w_2, pw_1w_2)$ being the weighted degrees of our germ and $(w_1, w_2) = 1$. We also suppose that $p \leq 3$. Then, the following results will give us a complete topological classification of these particular cases.

Theorem 5.2.7. *($p = 1$) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of corank 2 of the form $f(x, y) = (xy, ax^{w_2} + by^{w_1})$. Then,*

1. *if w_1, w_2 are odd, f is topologically equivalent to the fold (x, y^2),*
2. *if w_1, w_2 have different parity, f is topologically equivalent to the cusp $(x, xy + y^3)$.*

Proof. Let's prove first (1). If w_1, w_2 are odd, we know by the proof of theorem 5.2.2 that $\deg(f) = 0$. In addition to this, if we compute its singular set, we get the equation $w_1 b y^{w_1} - w_2 a x^{w_2} = 0$. Since this equation is irreducible, we can conclude that $S(f)$, and, as a consequence $\Delta(f)$, only present a single branch.

Let's see that we are going to have a single topological class which is the class of the fold. To prove this is enough to see that for any $a, b \in \mathbb{R} \setminus \{0\}$ there are points where f doesn't have any inverse image.

Let's consider the point $(1, 0)$. We get the equations

$$xy = 1$$

$$ax^{w_2} + by^{w_1} = 0,$$

obtaining that

$$y = \sqrt[w_1+w_2]{\frac{-a}{b}}.$$

Thus, if $ab > 0$ f doesn't have any inverse image and the result is proved (see figure 5.2). Analogously, if we take now the point $(-1, 0)$ we have that

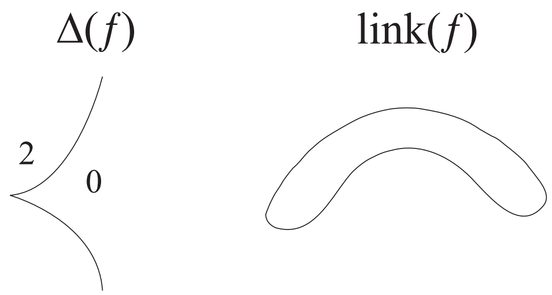


Figure 5.2

every map germ f with $ab < 0$ doesn't have any inverse image either and we arrive to the conclusion again that we have a single configuration of inverse images in the discriminant curve, which is the one of the fold.

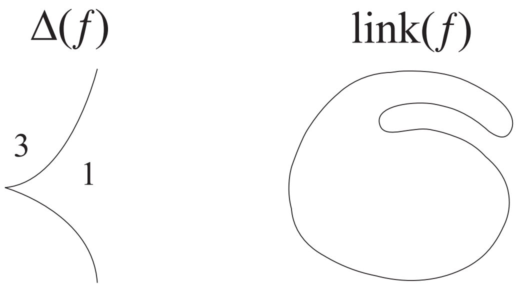


Figure 5.3

If w_1 and w_2 are of distinct parity, applying an analogous procedure as in (1) to prove the existence of points with a single inverse image, we obtain the desired result (see figure 5.3).

□

Theorem 5.2.8. *($p = 2$) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of corank 2 of the form $f(x, y) = (xy, ax^{2w_2} + bx^{w_2}y^{w_1} + cy^{2w_1})$. Then,*

1 if w_1, w_2 have the same parity, f is not finitely determined,

2 if w_1, w_2 have distinct parity, we have three cases,

- *if $(w_1 - w_2)^2b^2 + 16w_1w_2ac > 0$,*
  - *f is topologically equivalent to the map germ $(xy, x^2+xy^2+y^4)$ if $ac > 0$*
  - *f is topologically equivalent to the map germ $(xy, x^2+20xy^2-y^4)$ if $ac < 0$*
- *if $(w_1-w_2)^2b^2+16w_1w_2ac < 0$, f is topologically equivalent to the map germ $(xy, x^2 + xy^2 - y^4)$*
- *if $(w_1 - w_2)^2b^2 + 16w_1w_2ac = 0$, f is not finitely determined.*

Proof. If w_1, w_2 are both even or odd the result follows from remark 5.2.6. Let's suppose that w_1 and w_2 have different parity. The Jacobian determinant is given by

$$J(f) = -2w_2ax^{2w_2} + b(w_1 - w_2)x^{w_2}y^{w_1} + 2w_1cy^{2w_1},$$

that can be factorized in the form

$$-2w_2(x^{w_2} - \lambda_1y^{w_1})(x^{w_2} - \lambda_2y^{w_1}),$$

with $\lambda_i = \lambda_i(a, b, c, w_1, w_2) \in \mathbb{C}$, $i = 1, 2$. These λ_i are obtained by solving the quadratic equation given by the Jacobian determinant, whose discriminant is

$$(w_1 - w_2)^2 b^2 + 16 w_1 w_2 ac = 0.$$

Then, if this discriminant is positive we have two different real solutions for λ_i and as a consequence two branches in our singular set $S(f)$, if it is negative our singular set is empty outside of the origin and in the case that the discriminant vanishes, $\lambda_1 = \lambda_2$ and f won't be finitely determined. If the discriminant is negative, by remark 3.2.7 and proposition 5.2.2, taking into account that ac must be necessarily negative, we have that f will be topologically equivalent to the germ $(xy, x^2 - y^2)$. Since this germ is not finitely determined we can choose another member of this topological class that is finitely determined. Let's take, for example, $(xy, x^2 + xy^2 - y^4)$.

Thus, we center our attention in the case $(w_1 - w_2)^2 b^2 + 16 w_1 w_2 ac > 0$. If we call

$$C_i \equiv x^{w_2} - \lambda_i y^{w_1} = 0$$

for $i = 1, 2$, and apply the coordinate changes

$$\begin{cases} x = \alpha t^{w_1} \\ y = \beta t^{w_2} \end{cases}$$

we have that

$$f|_{C_i}(t) = (\alpha\beta t^{w_1 + w_2}, (a\lambda_i^2 + b\lambda_i + c) t^{2 w_1 w_2}),$$

whose derivative never vanishes out of 0 and it will present double folds if and only if $\alpha\beta = 0$, which is impossible. Let's observe that these curves are going to be symmetrical with respect to the y-axis (figure 5.4). From this

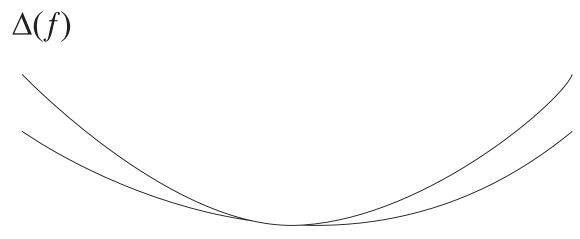


Figure 5.4

point, we must consider two different cases:

- If $ac > 0$, by proposition 5.2.2 we know that $\deg(f) = 0$. Taking into account that our discriminant set has 2 branches and the link of f can't have more than one connected component, if we are able to prove that for any b we have points with no inverse images, we finish.

  If we consider the point $(0, -1)$ we obtain the equations

$$xy = 0$$

$$ax^{2w_2} + bx^{w_2}y^{w_1} + cy^{2w_1} = -1,$$

  getting the equality $y = \sqrt[2w_1]{\frac{-1}{c}}$ if $x = 0$ and $x = \sqrt[2w_2]{\frac{-1}{a}}$ if $y = 0$. In both cases if a and c are positive the equalities don't have any real solution. Thus, f doesn't present any inverse image (see figure 5.5).

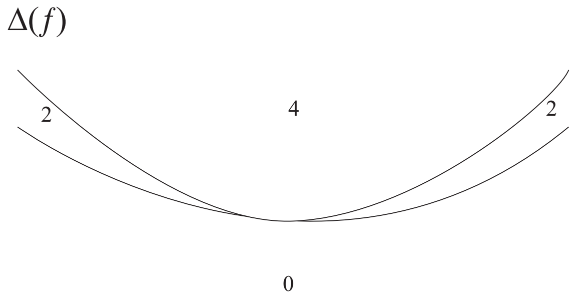


Figure 5.5

  Considering the point $(0, 1)$ and applying a totally analogous procedure we arrive to the conclusion that if $a, c < 0$ f doesn't present any inverse image either (see figure 5.6). Then, we have in both cases a single con-

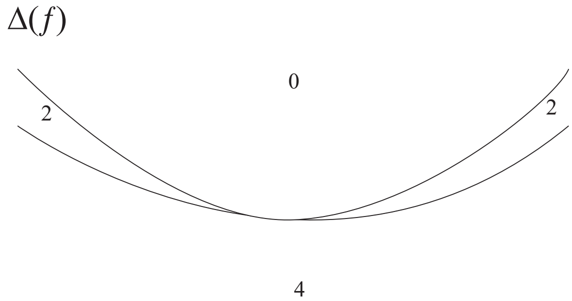


Figure 5.6

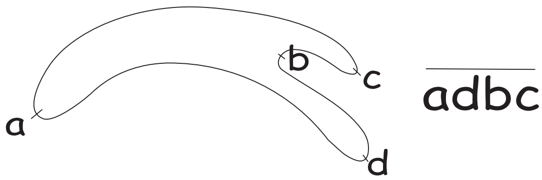


Figure 5.7

figuration of inverse images in the discriminant, obtaining the associated link and Gauss word that appear in figure 5.7. Thus, f is topologically equivalent to the known corank 1 normal form $(x, y^4 - xy^2 - x^2y)$. If we want to take a normal form of corank 2 we can choose, for example, $(xy, x^2 + xy^2 + y^4)$.

- If $ac < 0$, using again proposition 5.2.2, we know that $\deg(f) = \pm 2$. Taking into account that we are dealing with a map germ whose discriminant only has two branches if we are able to prove that the maximum number of inverse images of f is 4 we finish.

  Let's consider the equations

$$xy = d,$$

$$ax^{2w_2} + bx^{w_2}y^{w_1} + cy^{2w_1} = e,$$

with $(d, e) \in \mathbb{R}^2$. From here, we get the equality

$$cy^{2(w_1+w_2)} + bd^{w_2}y^{w_1+w_2} - ey^{2w_2} + ad^{2w_2} = 0.$$

Applying Descartes method and using the hypothesis $ac < 0$ we arrive to the conclusion that we can have three sign changes for $y > 0$ in the best of the cases and since all the exponents are even except $w_1 + w_2$, this is the only term whose sign is going to change when we consider $y < 0$. Then, we will have in this last case a single inverse image and a total of 4 inverse images, as we wanted to prove.

Thus, the only possible configuration of the inverse images in the discriminant of a map germ of this type will be the one that appears in figure 5.8, having its correspondent associated link and Gauss word (figure 5.9).

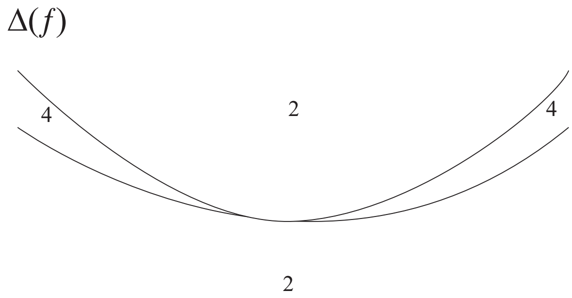


Figure 5.8

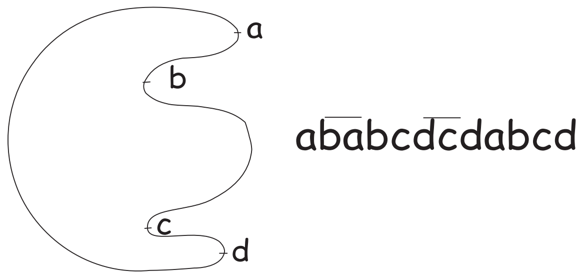


Figure 5.9

To finish, let's choose a representative of this topological class, for example, $(xy, x^2 + 20xy^2 - y^4)$. □

Theorem 5.2.9. *($p = 3$) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of corank 2 of the form $f(x, y) = (xy, ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1})$. Let's denote by*

$$A = -3w_2a\frac{(2w_1 - w_2)c}{3} - (\frac{(w_1 - 2w_2)b}{3})^2,$$

$$B = -3w_2a3w_1d - \frac{(w_1 - 2w_2)b}{3}\frac{(2w_1 - w_2)c}{3},$$

$$C = \frac{(w_1 - 2w_2)b}{3}3w_1d - (\frac{(2w_1 - w_2)c}{3})^2.$$

Then:

1. *Let's suppose that w_1, w_2 have different parity,*

   - *if $B^2 - 4AC > 0$, f is topologically equivalent to the simple cusp $(x, xy + y^3)$,*

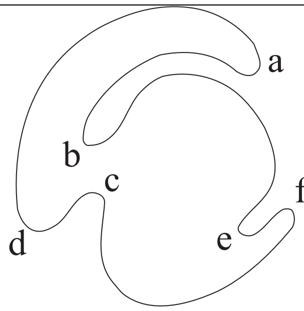
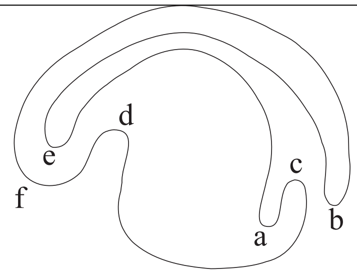
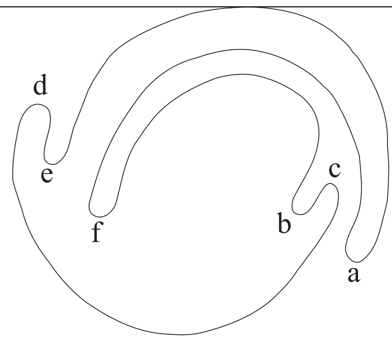
Degree	Germ	Associated link
1	$(xy, x^6 + 7x^4y^3 + 8x^2y^6 + y^9)$	 $a\overline{ba}bc\overline{dc}de\overline{fe}f$
	$(xy, x^6 + 2x^4y^3 + 9x^2y^6 + y^9)$	 $ab\bar{c}b\bar{a}bcd\bar{e}dc\bar{b}cde\bar{f}e\bar{d}ef$
	$(xy, x^6 - x^4y^3 + 7x^2y^6 + y^9)$	 $ab\overline{cb}cde\bar{f}edcb\bar{a}bcd\overline{ed}ef$

Table 5.1

- *if $B^2 - 4AC < 0$, f is topologically equivalent to one of the map germs that appear in table 5.1:*
- *if $B^2 - 4AC = 0$, f is not finitely determined.*

2. In the case that w_1, w_2 are both odd,

- *if $B^2 - 4AC > 0$, f is topologically equivalent to one of the map germs that appear in the table 5.2:*

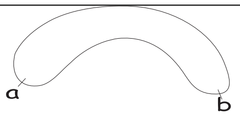
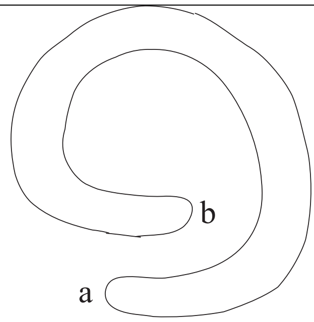
Degree	Germ	Associated link
0	(x, y^2)	 $\overline{ab}$
	$(xy, x^3 - x^2y^3 - xy^6 + y^9)$	 $\bar{a}ba\bar{b}ab$

Table 5.2

- *if $B^2 - 4AC < 0$, f is topologically equivalent to one of the map germs that appear in table 5.3:*
- *if $B^2 - 4AC = 0$, f is not finitely determined.*

Proof. If we compute the Jacobian determinant of f we get

$$Jf(x,y) = -3w_2ax^{3w_2} + (w_1 - 2w_2)bx^{2w_2}y^{w_1} + (2w_1 - w_2)cx^{w_2}y^{2w_1} + 3w_1dy^{3w_1}.$$

Let's realize that if we make the coordinate changes

$$\begin{cases} \overline{x} = x^{w_2} \\ \overline{y} = y^{w_1} \end{cases}$$

we get the cubic form

$$Jf(\overline{x},\overline{y}) = -3w_2a\overline{x}^3 + (w_1 - 2w_2)b\overline{x}^2\overline{y} + (2w_1 - w_2)c\overline{x}\overline{y}^2 + 3w_1d\overline{y}^3.$$

From this point we apply a known result (see for example [12]) which tell us that a cubic form will be of symbolic, hyperbolic, parabolic or elliptic type if

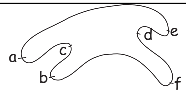
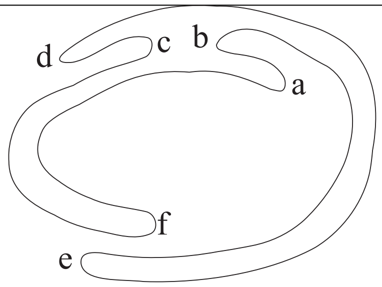
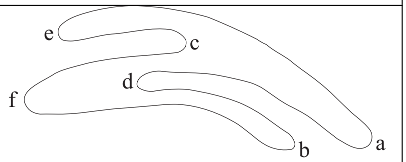
Degree	Germ	Associated link
0	$(xy, x^3 - x^2y^3 + 3xy^6 + y^9)$	 c$\overline{acb}$cd$\overline{fde}$d
	$(xy, x^3 - 6x^2y^3 + 4xy^6 + y^9)$	 abc$\overline{dc}$de$\overline{f}$edcb$\overline{ab}$af$\overline{e}$f
	$(xy, x^3 + 6x^2y^3 + 6xy^6 + y^9)$	 $\overline{a}$bcd$\overline{e}$d$\overline{c}$de$\overline{f}$edc$\overline{b}$c$\overline{d}$cb

Table 5.3

and only if its associated quadratic form obtained by computing the Hessian determinant is of symbolic, hyperbolic, parabolic or elliptic type respectively. Thus, if we compute the Hessian determinant of $Jf(\overline{x}, \overline{y})$ we get the quadratic form $A\overline{x}^2 + B\overline{xy} + C\overline{y}^2$ with A, B, C depending on the values of the initial coefficients a, b, c, d and of the weights w_1, w_2 and undoing the coordinate changes we made earlier we get the function $Ax^{2w_2} + Bx^{w_2}y^{w_1} + Cy^{2w_1}$ which we will use to determine the different cases of study. Therefore, we have the following possibilities:

1. Let's suppose that w_1 and w_2 have different parity. Firstly, if we consider as we did in the case $p = 2$ the coordinate changes

$$\begin{cases} x = \alpha t^{w_1} \\ y = \beta t^{w_2} \end{cases}$$

together with the image of the restriction of f to each one of the curves of the singular set, C_i, we get

$$f_{|C_i(t)}(t) = (\alpha\beta t^{w_1+w_2}, (a\lambda_i^3 + b\lambda^2 + c\lambda_i + d)t^{3w_1w_2}),$$

realizing that each one of these branches is symmetric with respect to the y-axis. Now, let's see the different configurations of inverse images that we can have in the discriminant, in order to obtain the distinct topological classes. As first step we will prove that $\# f^{-1}(z) \leq 5$, $\forall z \in \mathbb{R}^2$.

Let's take a point $(e, f) \in \mathbb{R}^2$ and let's consider the equations

$$\begin{cases} xy = e \\ ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1} = f. \end{cases}$$

Taking in the first equation $x = \frac{e}{y}$, with $y \neq 0$ and substituting we get

$$a(\frac{e}{y})^{3w_2} + b(\frac{e}{y})^{2w_2}y^{w_1} + c(\frac{e}{y})^{w_2}y^{2w_1} + dy^{3w_1} = f.$$

As last step we multiply both sides of the equation by y^{3w_2}, obtaining the final equation

$$dy^{3(w_1+w_2)} + ce^{w_2}y^{2(w_1+w_2)} - fy^{3w_2} + be^{2w_2}y^{w_1+w_2} + ae^{3w_2} = 0.$$

Now, putting in order the monomials according to their weighted degree and taking into account that the order of appearance of $(c, -f, b)$ can suffer variations due to the different values of (w_1, w_2), we apply Descartes rule of signs. Since we are working with a a polynomial consisting of 5 monomials, the worst configuration (with a biggest number of inverse images) will be given by $+-+-+$. Then, we will have at most 4 inverse images for $y > 0$ or $y < 0$ indistinctly(let's take $y > 0$). If $y < 0$, taking into account the parity of the weighted degrees of the monomials, we have the configuration $---++$ (or $--+++$, depending on the parity of w_2), obtaining a single inverse image and a total of 5 inverse images as we wanted to prove. If $(c, -f, b)$ would appear in a distinct order, by applying an analogous procedure we would arrive to the same result.

Secondly, we are going to prove that our germ f is always going to have points with a single inverse image and points with 3 inverse images. To do this we take a point $(0, f) \in \mathbb{R}^2$ and consider the equations

$$\begin{cases} xy = 0 \\ ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1} = f. \end{cases}$$

Since xy vanishes, x or y must be 0 and using the second equation we get in the first case $y = \sqrt[3w_1]{\frac{f}{d}}$ and in the second case $x = \sqrt[3w_2]{\frac{f}{a}}$.

Therefore, if w_1 is even and w_2 is odd we will have 3 inverse images if $fd > 0$ and a single one if $fd < 0$; analogously, if w_1 is odd and w_2 is even we will have 3 inverse images if $fa > 0$ and a single one if $fa < 0$. Then, from this point, what we know for sure is that the sectors of our bifurcation set in the image of f created by the discriminant curves that contain the y-axis are going to have one of them 3 inverse images and the other, a single one.

With all these previous calculations we are now in conditions to obtain the different topological classes.

- If $B^2 - 4AC > 0$, we have a single branch in our singular set and as a consequence, the only possible configuration of inverse images in its single discriminant curve is the one that appear in figure 5.10, which is clearly identified with the Gauss word and the link of the simple cusp (figure 5.11). Then, f is topologically equivalent to the simple cusp.

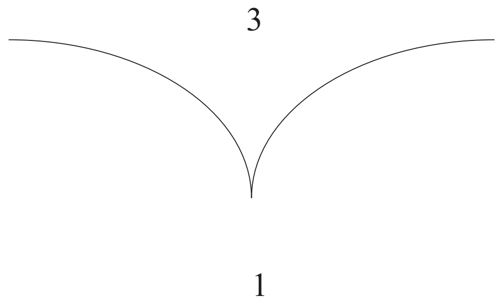


Figure 5.10

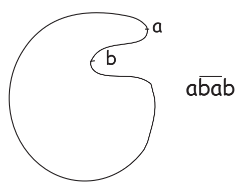


Figure 5.11

- If $B^2 - 4AC < 0$ we have three branches and two possible configurations of inverse images in the discriminant curves (figure 5.12), obtaining in the first case the associated link and Gauss word that appears in figure 5.13 , with normal form $(xy, x^6 +$

$7x^4y^3+8x^2y^6+y^9)$ and in the second case the two different topological classes that appear in figure 5.14, having as normal forms $(xy, x^6+2x^4y^3+9x^2y^6+y^9)$ and $(xy, x^6-x^4y^3+7x^2y^6+y^9)$ respectively.

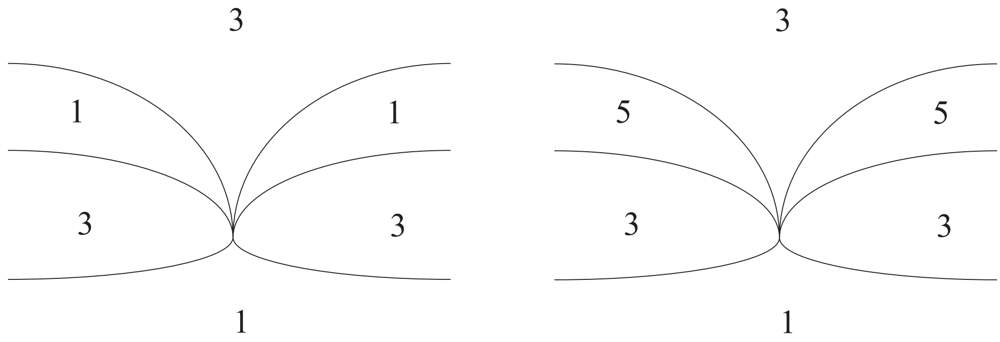


Figure 5.12

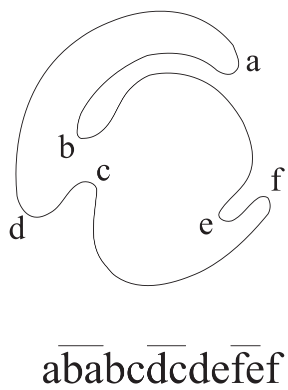


Figure 5.13

- If $B^2-4AC=0$ we will have, at least, a curve of double points and as a consequence f won't be finitely determined.

2. If w_1 and w_2 are odd, we consider again the coordinate changes

$$\begin{cases} x=\alpha t^{w_1} \\ y=\beta t^{w_2} \end{cases}$$

together with the image of the restriction of f to each one of the curves of the singular set, C_i. In this case, these images are symmetric with

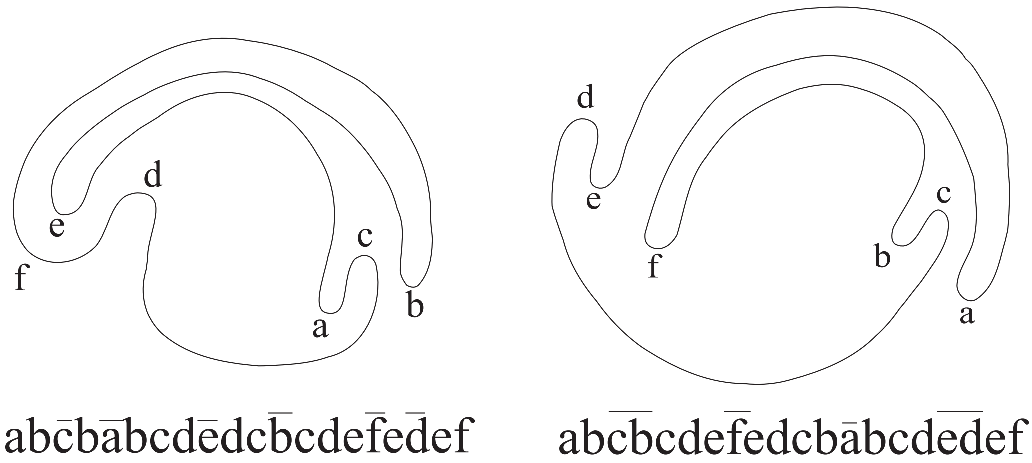


Figure 5.14

respect to the x-axis. Now, let's see the different configurations of inverse images that we can have in the discriminant, in order to obtain the distinct topological classes. Firstly, we will prove that $\# f^{-1}(z) \leq 6, \forall z \in \mathbb{R}^2$.

Following a totally analogous procedure to the case of weights with different parity, taking a point $(e, f) \in \mathbb{R}^2$ we arrive to the equation

$$dy^{3(w_1+w_2)} + ce^{w_2}y^{2(w_1+w_2)} - fy^{3w_2} + be^{2w_1}y^{w_1+w_2} + ae^{3w_2} = 0$$

and applying Descartes method we conclude that points situated in the image of f are going to present 6 inverse images at most.

Let's see now that f is always going to have points with 2 inverse images in the y-axis. To prove this, we consider a point $(0, f) \in \mathbb{R}^2$. Since the first component of f must vanish we get the equalities $y = \sqrt[3w_1]{\frac{f}{d}}$, with a single inverse image $(0, \sqrt[3w_1]{\frac{f}{d}})$ and $x = \sqrt[3w_2]{\frac{f}{a}}$, with a single inverse image $(\sqrt[3w_2]{\frac{f}{a}}, 0)$, getting a total of 2 inverse images, as we wanted to prove.

With all these previous remarks we are in conditions of giving a restricted list of the possible distribution of inverse images that we can have in the discriminant curves.

- If $B^2 - 4AC > 0$ our singular set and as consequence the discriminant has a single real branch. Therefore, we only have two possible distributions of inverse images (figure 5.15), getting in the first case the link and Gauss word that appear in the left hand side of figure 5.16, with the associated normal form of the fold (x, y^2), and in the last case the one that appear in the right hand side of figure 5.16, with the associated normal form $(xy, x^3 - x^2y^3 - xy^6 + y^9)$.

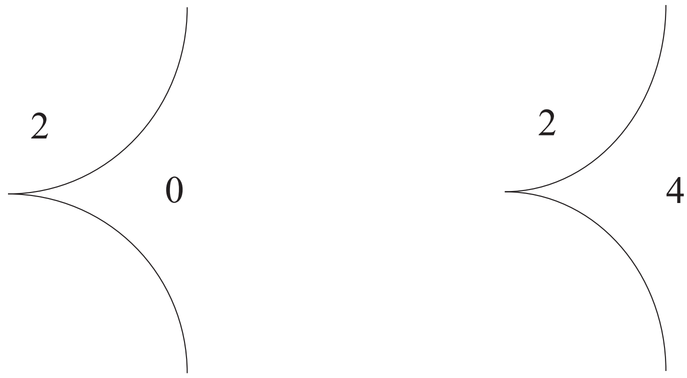


Figure 5.15

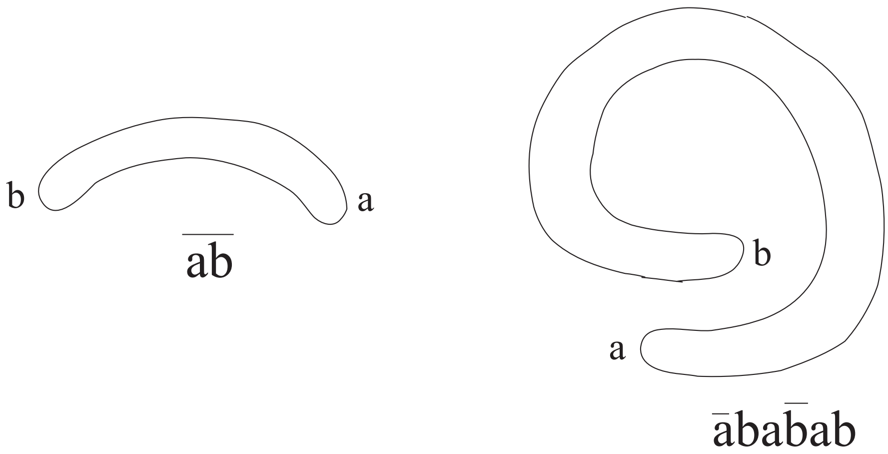


Figure 5.16

- If $B^2 - 4AC < 0$ our singular set, and as a consequence the discriminant, has 3 distinct real branches and the initial number of possible configurations of inverse images in the discriminant is much bigger (see figure 5.17). Let's see that (d) and (e) can't occur.

  If we had the configuration of (d), we would have points of the form $(e, 0) \in \mathbb{R}^2$ with 6 inverse images. Let's suppose that $e > 0$.

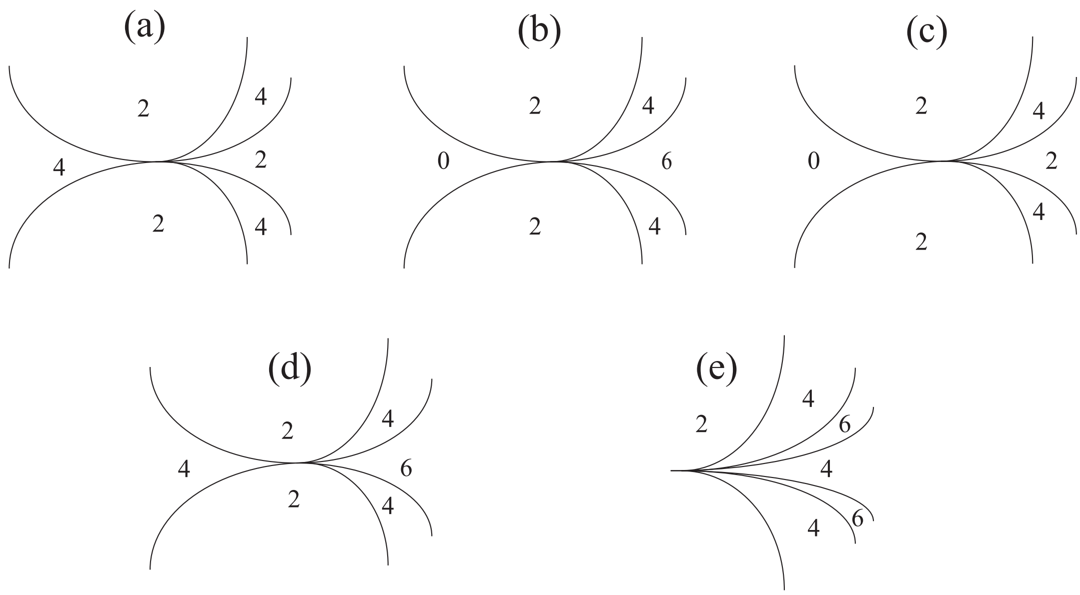


Figure 5.17

We obtain the equation

$$dy^{3(w_1+w_2)} + ce^{w_2}y^{2(w_1+w_2)} + be^{2w_1}y^{w_1+w_2} + ae^{3w_2} = 0.$$

If we apply Descartes method to this polynomial, the only possible signs configuration to get 6 inverse images is $+-+-$ for $y > 0$, obtaining $+-+-$ for $y < 0$. If (d) was possible taking a point of the form $(e, 0)$ with $e < 0$ we should have 4 inverse images. But this is impossible because applying again Descartes method and using the sign of coefficients (a, b, c, d) we have had to choose to obtain 6 inverse images when $e > 0$ we obtain a signs configuration of the form $++++$ for any y. Therefore, we have just arrived to a contradiction and the configuration (d) is not possible.

To prove that (e) is not possible either we will choose a point of the form $(e^{w_1+w_2}, te^{3w_1w_2}) \in \mathbb{R}^2$, that is, a point of a generic cusp and we consider the equations

$$\begin{cases} xy = e^{w_1+w_2} \\ ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1} = te^{3w_1w_2}. \end{cases}$$

If we suppose that $y \neq 0$, we can take $x = \frac{e^{w_1+w_2}}{y}$ and by substituting in the second equation and multiplying both terms by y^{3w_2}

we have

$$a(e^{w_1+w_2})^{3w_2} + b(e^{w_1+w_2})^{2w_2}y^{w_1+w_2} + c(e^{w_1+w_2})^{w_2}y^{2(w_1+w_2)}$$

$$-te^{w_1w_2}y^{3w_2} + dy^{3(w_1+w_2)} = 0,$$

that is, a polynomial constituted by 5 monomials and where, applying Descartes method, we are going to have in the worst of the cases 4 sign changes, and as a consequence, 4 inverse images for $e > 0$ and $e < 0$. Then, (e) is not possible.

Thus, we only have 3 possible configurations $((a), (b)$ and $(c))$ obtaining for each one a single topological class given by its correspondent associated link and Gauss word (see figure 5.18).

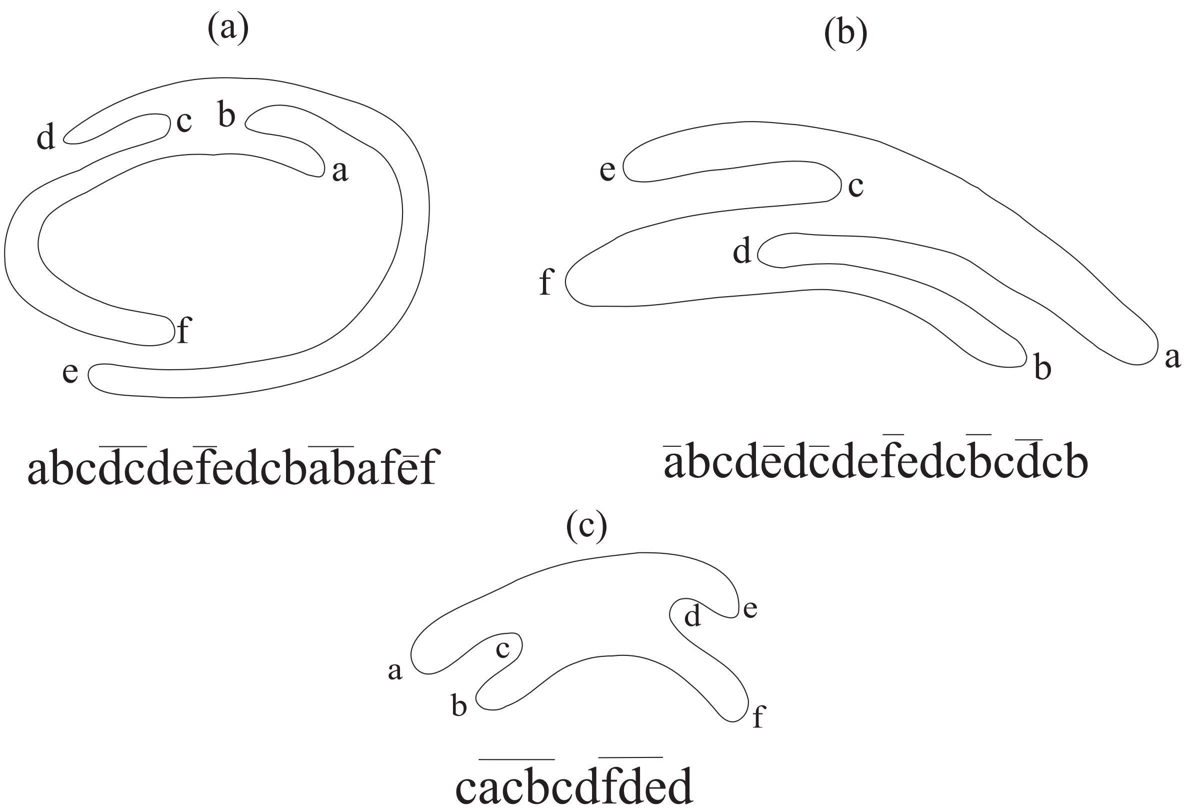


Figure 5.18

To finish we associate to (a) the normal form $(xy, x^3 - 6x^2y^3 + 4xy^6 + y^9)$, to (b) $(xy, x^3 + 6x^2y^3 + 6xy^6 + y^9)$ and to (c) $(xy, x^3 - x^2y^3 + 3xy^6 + y^9)$.

- If we consider the remaining case, $B^2 - 4AC = 0$, following and analogous argument to the case of weights with different parity we conclude that f won't be finitely determined.

□

5.2.2 Germs with prenormal form $(x^2 + y^2, h(x, y))$

As we did with germs with prenormal form $(xy, g(x, y))$, we will suppose that $h(x, y)$ is a weighted homogeneous polynomial, that is,

$$h(x, y) = \sum_{i=0}^{p} b_i (x^{w_2})^i (y^{w_1})^{p-i},$$

although, in general, f won't be weighted homogeneous. We distinguish two different cases, according to the parity of p.

- $\underline{p = 2k}$

  The following theorem will give us the classification of all germs of this type.

  Theorem 5.2.10. *Let f be of type $\Sigma^{2,0}$,*

  $$f(x, y) = (x^2 + y^2, \sum_{i=0}^{p} b_i (x^{w_2})^i (y^{w_1})^{p-i}),$$

  with $p = 2k$. Then, f is not finitely determined.

  Proof. We will prove it for $p = 2$, being analogous for the remaining cases.

  If w_1 or w_2 are greater than 1, when we compute the Jacobian determinant of f we obtain a expression of the form $2yA$ or $2xB$ with A, B depending on w_1, w_2, x, y. In the first case, we have the curve $y = 0$ in the singular set, getting an image (x^2, ax^{2w_2}) that clearly presents double points. If we have $x = 0$ by an analogous procedure we arrive to the same conclusion.

  If $w_1 = w_2 = 1$ we have branches of the form $x = \lambda y$ in the singular set, and as a consequence, each one of the discriminant curves will have the form $((\lambda y)^2 + y^2, a(\lambda y)^2 + b(\lambda y)y + cy^2)$ that present double points of the form $y_1 = -y_2$. Then, f is not finitely determined either. □

- General case Firstly, we will see that if one of the weights is even and the other is different from 1, f won't be finitely determined.

Theorem 5.2.11. *Let f be of type $\Sigma^{2,0}$,*

$$f(x,y) = (x^2+y^2, \sum_{i=0}^{p} b_i (x^{w_2})^i (y^{w_1})^{p-i}).$$

Then, if w_1 or w_2 is even, with the other weight being greater than 1, f is not finitely determined.

Proof. Let's suppose that w_1 is even and $w_2 > 1$. If we compute the Jacobian determinant of f we get

$$Jf(x,y) = 2x \sum_{i=0}^{p-1} w_1(p-i) b_i (x^{w_2})^i (y^{w_1})^{p-i-1} - 2y \sum_{i=1}^{p} w_2 i b_i (x^{w_2})^{i-1} (y^{w_1})^{p-i}.$$

Since $w_2 > 1$ we can get one x out of the second summation, obtaining

$$Jf(x,y) = 2x(\sum_{i=0}^{p-1} w_1(p-i) b_i (x^{w_2})^i (y^{w_1})^{p-i-1} - y \sum_{i=1}^{p} w_2 i b_i (x^{w_2})^{i-2} (y^{w_1})^{p-i}.$$

Therefore, one of the branches of $S(f)$ will always be given by the equation $x = 0$ and

$$f_{|x=0}(y) = (y^2, y^{pw_1}),$$

that will always present double points of the form $y_1 = -y_2$. Thus, f is not finitely determined. □

Let's see now what happen when both weights are odd. We will give some particular results about it.

Theorem 5.2.12. *($p = 1$) Let f be of type $\Sigma^{2,0}$, $f(x,y) = (x^2 + y^2, ax^{w_2} + by^{w_1})$, with w_1, w_2 both odd.Then, f is topologically equivalent to the germ (x^2+y^2, x^3+y^5).*

Proof. Let's suppose that w_1, w_2 are both odd and greater than 1 (if one of them was 1, f wouldn't be of type $\Sigma^{2,0}$ anymore). In this case

$$Jf(x,y) = 2xy(w_1 b y^{w_1-2} - w_2 a x^{w_2-2}),$$

obtaining that our singular set $S(f)$ will have 3 branches, $x = 0$, $y = 0$ and $y^{w_1-2} = \frac{w_2 a x^{w_2-2}}{w_1 b}$. In the first two f doesn't present any problem. Let's see that it doesn't present any problem in the third one either. To see this we make the coordinates change $\begin{cases} x = \alpha t^{w_1-2} \\ \\ y = \beta t^{w_2-2} \end{cases}$ with $\beta = \sqrt[w_1-2]{w_2 a} \in \mathbb{C}$ and $\alpha = \sqrt[w_2-2]{w_1 b} \in \mathbb{C}$. We have that $f|_{y^{w_1-2}=\frac{w_2 a x^{w_2-2}}{w_1 b}}(t) = (A(t), B(t))$, with $A(t) = \alpha^2 t^{2(w_1-2)} + \beta^2 t^{2(w_2-2)}$ and $B(t) = a\alpha^{w_2} t^{w_2(w_1-2)} + b\beta^{w_1} t^{w_1(w_2-2)}$. It is clear that although $A(t)$ is going to present double points of the form $t_1 = -t_2$, it isn't going to happen with $B(t)$. Then, f is finitely determined.

Thus, $\Delta(f)$ will have three branches and we can only have two possible configurations:

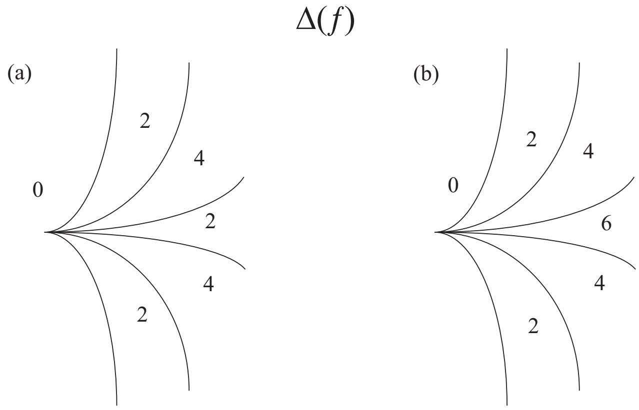


Figure 5.19

Let's see that (b) isn't possible. To prove this we consider a point $(e, 0) \in \mathbb{R}^2$ and we will prove by Descartes method that it will present at most 2 inverse images. We have the equations

$$\begin{cases} x^2 + y^2 = e \\ \\ ax^{w_2} + by^{w_1} = 0 \end{cases},$$

obtaining a single equation of the form $A(y^{\frac{w_1}{w_2}})^2 + y^2 = e$ with $A > 0$. We consider the coordinate change $y = z^{w_2}$ in order to be able to work with integer exponents and we get $Az^{2w_1} + z^{2w_2} - e = 0$ that, applying Descartes method will always present at most 1 root if $z > 0$ and 1 root if $z < 0$, having a total of 2 roots z_1 and z_2 and as a consequence y_1 and y_2. Therefore, the only possible configuration is given by (a) and, since the 3 branches of the singular set are symmetric with respect to the origin of coordinates the only possible topological class is the associated to the link and Gauss word of figure 5.20.

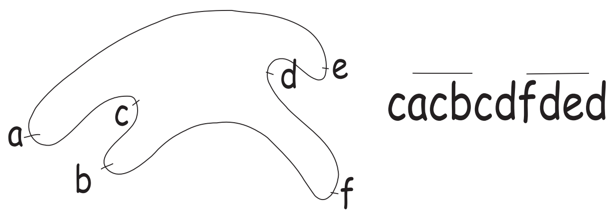


Figure 5.20

Then, f is topologically equivalent to $(x, y^4 - x^2y^2 - \frac{1}{4}x^3y)$ and to the corank 2 normal form $(x^2 + y^2, x^3 + y^5)$.

□

Theorem 5.2.13. *($p = 3$, homogeneous case) Let f be of type of $\Sigma^{2,0}$, $f(x, y) = (x^2 + y^2, ax^3 + bx^2y + cxy^2 + dy^3)$. Then, if we denote by*

$$A = b(\frac{3d - 2b}{3}) - (\frac{2c - 3a}{3})^2,$$

$$B = -bc - \frac{(2c - 3a)(3d - 2b)}{9},$$

$$C = \frac{c(3a - 2c)}{3} - (\frac{3d - 2b}{3})^2$$

we have that

1. *if $B^2 - 4AC > 0$, f is topologically equivalent to the fold,*
2. *if $B^2 - 4AC < 0$, f is topologically equivalent to one of the germs that appear in table 5.4,*
3. *if $B^2 - 4AC = 0$, f is not finitely determined.*

Proof. Applying the result used earlier for map germs of the form $(xy, g(x, y))$ in the case $p = 3$ we obtain coefficients $A = A(a, , b, c, d)$, $B = B(a, b, c, d)$ and $C = C(a, b, c, d)$ such that $Jf(x, y)$ will present a symbolic, elliptical, hyperbolic or parabolic quadratic form if and only if $Ax^2 + Bxy + Cy^2$ presents a symbolic, elliptical, hyperbolic or parabolic quadratic form. Therefore, we have several cases:

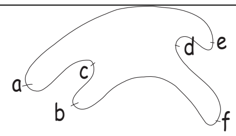
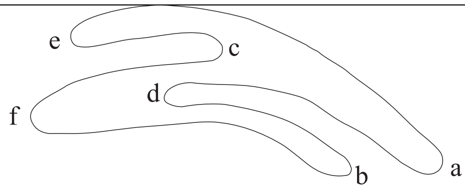
Degree	Germ	Associated link
0	(x^2+y^2, x^3+y^5)	 $\text{c}\overline{\text{acb}}\text{cd}\overline{\text{fde}}\text{d}$
	$(x^2+y^2, x^3+x^2y-3xy^2+y^3)$	 $\text{bcd}\bar{\text{e}}\text{d}\bar{\text{c}}\text{de}\bar{\text{f}}\text{edc}\bar{\text{b}}\text{cdc}\bar{\text{a}}$

Table 5.4

1. If $B^2 - 4AC > 0$, $S(f)$ presents a single branch $x = \lambda y$ whose image will be, as happen with all the germs of this form, symmetric with respect to the x-axis. Since the only possible configuration of inverse images is the one that appears in figure 5.21, f will be topologically equivalent to the fold.

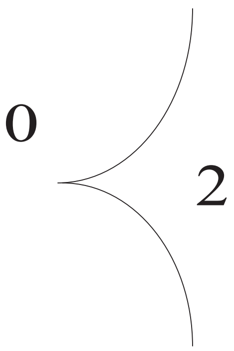


Figure 5.21

2. If $B^2 - 4AC < 0$, $S(f)$ will present three distinct real branches, obtaining in the discriminant the possible configurations of figure 5.22 and from each one of them a single topological class, symmetric with respect to the origin of coordinates. In case (a) we have the associated link and Gauss word of figure 5.23, taking as normal form (x^2+y^2, x^3+y^5) and in case (b) we obtain the link of figure 5.24, taking as normal form $(x^2+y^2, x^3+x^2y-3xy^2+y^3)$.
3. $B^2-4AC = 0$ or $A = B = C = 0$ we obtain, in the best of cases, a

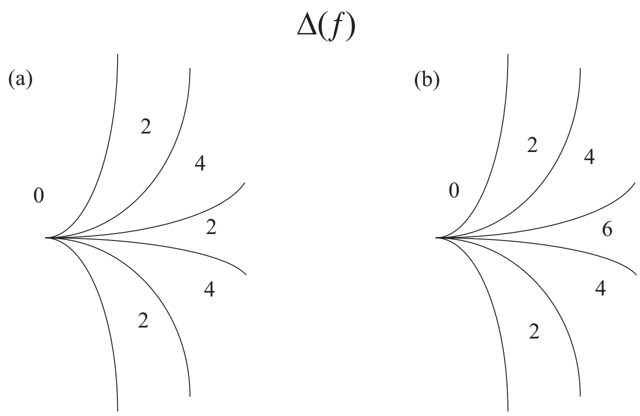


Figure 5.22

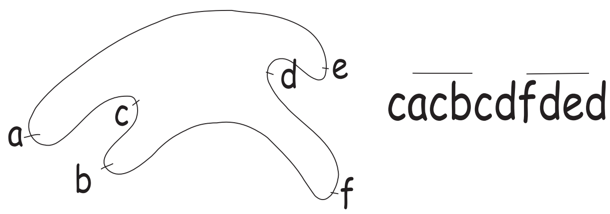


Figure 5.23

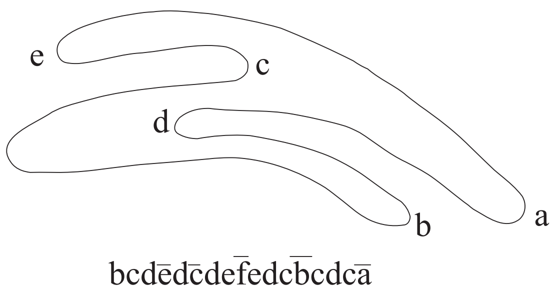


Figure 5.24

curve of double points in $S(f)$. Then, f is not finitely determined.

□

Chapter 6

Topological properties of finitely determined map germs from $\mathbb{R}^3$ to $\mathbb{R}^3$

Here, we want to study the topological classification of finitely determined map germs, $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$, by looking at the topological type of its link. A natural open question is to determine whether given a stable map $\gamma : S^2 \to S^2$, there exists a finitely determined map germ $f : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^3, 0)$ which is topologically equivalent to the cone of γ.

Given a stable map $\gamma : S^2 \to S^2$, then the singular set $S(\gamma)$ is a 1-dimensional closed submanifold of S^2 and its image or discriminant $\Delta(\gamma)$ is a union of curves with only simple cusps or transverse double points. The restriction $\gamma : \gamma^{-1}(\Delta(\gamma)) \to \Delta(\gamma)$ contains all the topological information of γ, although in general we have also to take into account the embedding types of $\gamma^{-1}(\Delta(\gamma))$ and $\Delta(\gamma)$ into S^2. In order to overcome that problem, we restrict ourselves to the case that $S(\gamma)$ is connected. Then, we will use an adapted version of Gauss words to classify such stable maps, demostrating that, with this additional hypothesis, they become a complete topological invariant. In the case that $S(\gamma)$ is not connected, the Gauss words are not enough to classify stable maps and we need to use some other global type invariants (see [14, 38]).

In the following section, we consider finitely determined map germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ of corank 1 and whose 2-jet is equivalent to (x, y, xz). This condition guarantees that the singular set is smooth and hence, the singular set of its link is connected. We give a topological classification of weighted homogeneous map germs of this type in a particular form, using this theorem to finish this chapter with the full classification of ruled map germs from $\mathbb{R}^3$ to $\mathbb{R}^3$.

6.1 The link of a germ from $\mathbb{R}^3$ to $\mathbb{R}^3$

Definition 6.1.1. Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ. We say that the stable map $f|_{\widetilde{S}^2_\epsilon} : \widetilde{S}^2_\epsilon \to S^2_\epsilon$ is the *link* of f, where f is a representative such that (1), (2) and (3) of corollary 2.3.3 hold for any ϵ with $0 < \epsilon \leq \epsilon_0$. This link is well defined, up to $\mathcal{A}$-equivalence.

Since any finitely determined map germ is topologically equivalent to the cone of its link, we have the following immediate consequence.

Corollary 6.1.2. *Let $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be two finitely determined map germs such that their associated links are topologically equivalent. Then f and g are topologically equivalent.*

We will see that the converse of this corollary is also true at the end of this section, if we assume that the singular sets $S(f), S(g)$ are smooth. Now we introduce the Gauss paragraph of a stable map with connected singular set.

6.2 Gauss words

We recall that a Gauss word is a word which contains each letter exactly twice, one with exponent $+1$ and another one with exponent -1. They were introduced originally by Gauss to describe the topology of closed curves in the plane $\mathbb{R}^2$ or in the sphere S^2 (see for instance [35]). Here, we use the same terminology of Gauss word to represent a different type of word, adapted to our particular case of stable maps $S^2 \to S^2$.

Along this section, we assume that $\gamma : S^2 \to S^2$ is a stable map, that is, such that all its singularities are folds and cusp points and that $\gamma|_{S(\gamma)}$ only presents simple cusps and double transverse points. Moreover, we assume that $S(\gamma)$ and hence its image $\Delta(\gamma)$ are connected.

Lemma 6.2.1. *Let $\gamma : S^2 \to S^2$ be a stable map such that $S(\gamma)$ is connected. Then:*

1. *$\gamma^{-1}(\Delta(\gamma))$ is also connected,*
2. *the restriction of γ to each connected component of $S^2 \setminus \gamma^{-1}(\Delta(\gamma))$ is a diffeomorphism.*

Proof. If $S(\gamma)$ is empty, then $\gamma^{-1}(\Delta(\gamma))$ is also empty. Moreover, γ is a local diffeomorphism and hence a d-fold covering, for some $d \geq 1$. Then,

$$2 = \chi(S^2) = d\chi(S^2) = 2d,$$

and we have $d = 1$ and γ is a diffeomorphism.

Assume that $S(\gamma)$ is non empty, then $S(\gamma)$ and $\gamma^{-1}(\Delta(\gamma))$ are both non-empty graphs in S^2. Since $S(\gamma)$ is connected, $\Delta(\gamma)$ is also connected and hence, $S^2 \setminus \Delta(\gamma)$ is a disjoint union of open discs. We show that $\gamma^{-1}(\Delta(\gamma))$ is connected by showing that $S^2 \setminus \gamma^{-1}(\Delta(\gamma))$ is also a disjoint union of open discs.

Let C be a connected component of $S^2 \setminus \gamma^{-1}(\Delta(\gamma))$ and let $D = \gamma(C)$ be the connected component of $S^2 \setminus \Delta(\gamma)$. The restriction $\gamma|_C : C \to D$ is again a d-fold covering, for some $d \geq 1$. Therefore,

$$1 - \beta_1(C) = \chi(C) = d\chi(D) = d \geq 1,$$

where $\beta_1(C)$ is the first Betti number of C. Hence, $\beta_1(C) = 0$ and $d = 1$. We deduce that C is an open disc and $\gamma|_C : C \to D$ is a diffeomorphism. □

Now we look at the structure of the singular curves. We split $\gamma^{-1}(\Delta(\gamma))$ into $\gamma^{-1}(\Delta(\gamma)) = S(\gamma) \cup X(\gamma)$ where

$$X(\gamma) = \overline{\gamma^{-1}(\Delta(\gamma)) \setminus S(\gamma)}.$$

The local structure of these curves at a cusp or at a transverse double point is shown in figure 6.1. In general $X(\gamma)$ may have several components, that is, it is equal to a finite union of closed curves with cusps or transverse double points. We denote such components by $X_1(\gamma), \ldots, X_k(\gamma)$.

We now choose orientations on the spheres S^2 (we may take different orientations on each S^2). Then there are natural orientations induced on the singular curves:

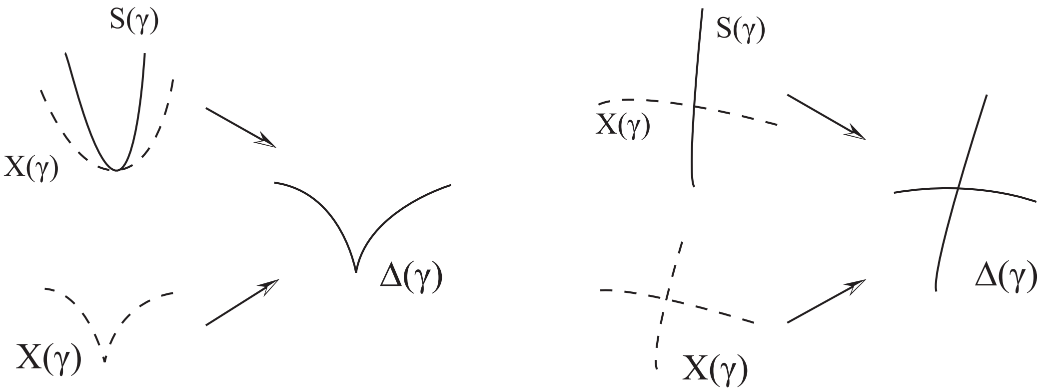


Figure 6.1: Local structure of $X(\gamma), S(\gamma)$ and $\Delta(\gamma)$

- $S(\gamma)$: we have on the left the positive region (where γ preserves the orientation).
- $\Delta(\gamma)$: we have on the left the region of bigger multiplicity (the number of inverse images of a value).
- $X_j(\gamma)$: we have on the left the region of bigger multiplicity (the multiplicity of a point here is the multiplicity of its image).

At a transverse double point we have two oriented branches. One branch is called positive if the other branch crosses from right to left at the double point, otherwise we call it negative. We always have a positive and a negative branch meeting at a double point (see figure 6.2).

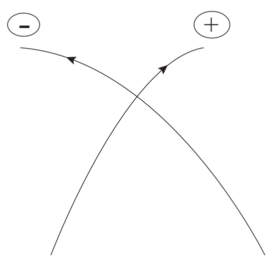


Figure 6.2: Sign of the branches at a double point

The next step is to choose a base point on each curve $S(\gamma)$, $\Delta(\gamma)$ and $X_j(\gamma)$. But we only need to choose a point in $S(\gamma)$ and this point determines in a unique way a base point in all the other curves.

In fact, we put, for simplicity, $X_0(\gamma) = S(\gamma)$ and fix a point $z_0 \in X_0(\gamma)$ which determines a point $\gamma(z_0) \in \Delta(\gamma)$. Since $S(\gamma) \cup X(\gamma)$ is connected, we can reorder the curves $X_1(\gamma), \dots, X_k(\gamma)$ in such a way that $X_j(\gamma)$ has at least one point in common with $X_0(\gamma) \cup \cdots \cup X_{j-1}(\gamma)$. Then we take $z_j \in X_j(\gamma)$ as the first point appearing in the first curve $X_\ell(\gamma)$ such that $X_\ell(\gamma) \cap X_j(\gamma) \neq \emptyset$.

Definition 6.2.2. Assume that $\Delta(\gamma)$ presents r double points and s simple cusps, which are labeled by $r + s$ letters $\{a_1, a_2, \dots, a_{r+s}\}$. The *Gauss word* of $\Delta(\gamma)$ is denoted by W_0 and it is the sequence of cusps and double points that appear when traveling around $\Delta(\gamma)$ starting from the base point and following the orientation. If we arrive to a point a_i, then we put a_i^2 if it is a cusp, a_i if it corresponds to the positive branch of a double point or a_i^{-1} if it corresponds to the negative branch.

For each $j = 1, \dots, k$, the *Gauss word* of $X_j(\gamma)$ is denoted by W_j and it is defined in an analogous way, but we have now more possibilities. Given a point which is an inverse image of a_i, if it belongs to $S(f)$ we use the same letter a_i to label the point; otherwise we put $\overline{a_i}, \overline{\overline{a_i}}, \dots$ (we use multiple bars in order to distinguish between different inverse images). We also use the same convention with the exponents: $a_i^2, \overline{a_i}^2, \overline{\overline{a_i}}^2, \dots$ for a cusp, $a_i, \overline{a_i}, \overline{\overline{a_i}}, \dots$ for a positive branch of double point or $a_i^{-1}, \overline{a_i}^{-1}, \overline{\overline{a_i}}^{-1}, \dots$ for a negative branch of double point.

We call *Gauss paragraph* to the list of Gauss words $\{W_0, W_1, \dots, W_k\}$.

Example 6.2.3. Let's see what form has the link of the three stable singularities.

1. Let $\gamma : S^2 \to S^2$ be the link of the fold $f(x, y, z) = (x, y, z^2)$. Then $\Delta(\gamma)$ doesn't present any simple cusp or double point. The Gauss paragraph is just $\{\emptyset\}$ (figure 6.3).

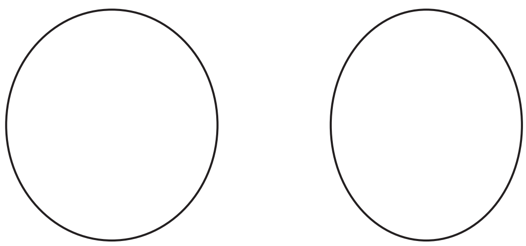


Figure 6.3

2. Let $\gamma : S^2 \to S^2$ be the link of the cuspidal edge $f(x, y, z) = (x, y, xz + z^3)$. Then $\Delta(\gamma)$ presents 2 simple cusps, each one with a single inverse image. The Gauss paragraph in this case is $\{a^2b^2, a^2b^2\}$ (figure 6.4).

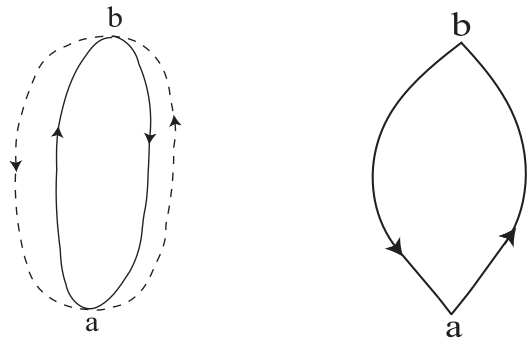


Figure 6.4

3. Let $\gamma : S^2 \to S^2$ be the link of the swallowtail $f(x, y, z) = (x, y, z^4 + xz + yz^2)$. Then $\Delta(\gamma)$ present 2 simple cusps, each one with 2 inverse images, and a double fold point, with 2 inverse images. The Gauss paragraph is $\{a^{-1}b^2c^2a, a^{-1}\overline{b}^2c^2a\overline{c}^2b^2\}$ (figure 6.5).

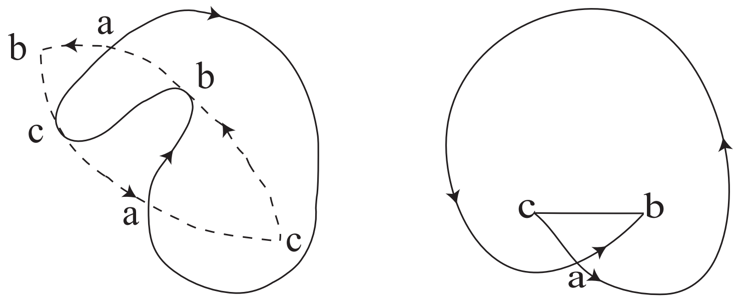


Figure 6.5

4. Let $\gamma : S^2 \to S^2$ be the link of the germ $f(x, y, z) = (x, y, z^4 + xz - y^2z^2)$. Then $\Delta(\gamma)$ presents 4 simple cusps, each one with 2 inverse images, and 2 double fold point, each one with 2 inverse images. The Gauss paragraph in this case is:

$$\{a^{-1}b^2c^2ad^{-1}e^2f^2d, a^{-1}\overline{b}^2c^2a\overline{c}^2b^2, d^{-1}\overline{e}^2f^2d\overline{f}^2e^2\}$$

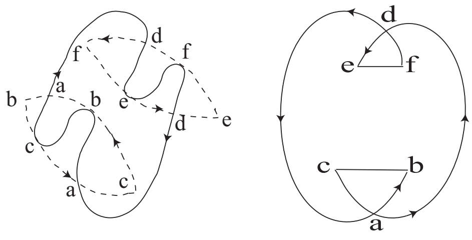


Figure 6.6

(figure 6.6).

5. Let $\gamma : S^2 \to S^2$ be the link of $f(x, y, z) = (x, y, z^5 + xz + yz^3)$. Then $\Delta(\gamma)$ presents 4 simple cusps, one with 3 inverse images and the others with 1, and 3 double points, with 3 inverse images each one. The Gauss paragraph is:

$$\{a^{-1}bc^2d^{-1}e^2af^2b^{-1}dg^2, ae^2d^{-1}\overline{e}^2\overline{a}f^2\overline{b}^{-1}db^{-1}\overline{d}g^2\overline{a}^{-1}ba^{-1}\overline{b}c^2\overline{d}^{-1}\overline{\overline{e}}^2\}$$

(figure 6.7).

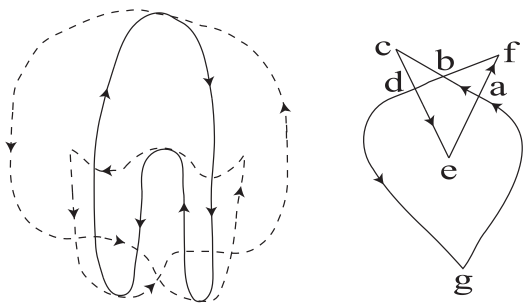


Figure 6.7

6. Let $\gamma : S^2 \to S^2$ be the link of $f(x, y, z) = (x, y, z^5 + xz - y^2z^3)$. Then $\Delta(\gamma)$ presents 6 simple cusps, 2 with 3 inverse images and the others with 1, and 6 double points, with 3 inverse images each one. The Gauss paragraph is:

$$\{a^{-1}bc^2d^{-1}e^2af^2b^{-1}dg^{-1}hi^2j^{-1}k^2gl^2h^{-1}j,$$
$$ae^2d^{-1}\overline{e}^2\overline{a}f^2\overline{b}^{-1}db^{-1}\overline{d}\overline{g}^{-1}hg^{-1}\overline{h}i^2\overline{j}^{-1}\overline{k}^2g^{-1}k^2j^{-1}\overline{\overline{k}}^2\overline{g}l^2\overline{h}^{-1}jh^{-1}\overline{j}\overline{a}^{-1}ba^{-1}\overline{b}c^2\overline{d}^{-1}\overline{\overline{e}}^2\}$$

(figure 6.8).

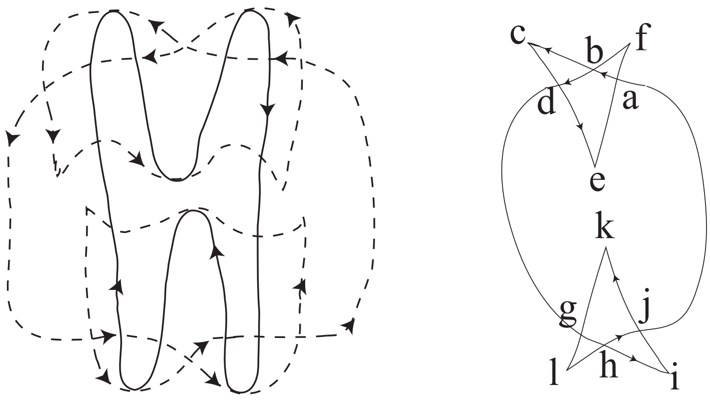


Figure 6.8

It is obvious that the Gauss paragraph is not uniquely determined, since it depends on the labels $a_1, \ldots, a_{r+s}$, the chosen orientations in each S^2 and the base point $z_0 \in S(\gamma)$. Different choices will produce the following changes in the Gauss paragraph:

1. a permutation in the set of the letters $a_1, \ldots, a_{r+s}$,
2. a reversion in the Gauss words together with a change in the exponents $+1$ to -1 and viceversa,
3. a cyclic permutation in the Gauss words.

We say that two Gauss paragraphs are equivalent if they are related through these three operations. Under this equivalence, the Gauss paragraph is now well defined.

In order to simplify the notation, given a stable map $\gamma : S^2 \to S^2$, we denote by $w(\gamma)$ the associated Gauss paragraph and by $\simeq$ the equivalence relation between Gauss paragraphs.

As a consequence of this definition and previous remarks we have the following important result:

Theorem 6.2.4. *Let $\gamma, \delta : S^2 \to S^2$ be two stable maps such that $S(\gamma)$ and $S(\delta)$ are connected and non empty. Then γ, δ are topologically equivalent if and only if $w(\gamma) \simeq w(\delta)$.*

Proof. Let us denote by $w(\gamma) = \{W_0, W_1, \ldots, W_k\}$ the Gauss paragraph of γ with respect to some labels $\{a_1, a_2, \ldots, a_{r+s}\}$, some orientations in the source and the target S^2 and some base point $z_0 \in S(\gamma)$.

Suppose that δ is topologically equivalent to γ. Then, there are homeomorphisms $\phi, \psi : S^2 \to S^2$ such that $\delta = \psi \circ \gamma \circ \phi^{-1}$. We use the same labels $\{a_1, a_2, \dots, a_{r+s}\}$ in such a way that if a_i is the label of a cusp or double point of γ, then it is also the label of its image through ψ and if $a_i, \overline{a_i}, \overline{\overline{a_i}}, \dots$ is the label of an inverse image in γ, then we take the same label for its image through ϕ. We choose the orientations in the source and the target S^2 induced by the orientations of γ and the homeomorphisms ϕ, ψ. Finally, we set $\phi(z_0) \in S(\delta)$ as the base point. With these choices, we have that $w(\delta) = \{W_0, W_1, \dots, W_k\} = w(\gamma)$.

We show now the converse. We divide the proof into several cases.

Case 1: $w(\gamma) = w(\delta)$. We can assume that $w(\gamma) = w(\delta) \neq \emptyset$, since otherwise both maps should be topologically equivalent to the link of the fold.

We first observe that each stable map γ with $w(\gamma) \neq \emptyset$ has a unique cellular structure compatible with the stratification by stable types and such that γ restricted to each cell is a homeomorphism. In the target, the 0-cells are the cusps and double folds and the 1-skeleton is $\Delta(\gamma)$; in the source, the 0-cells are the inverse images of the cusps and double folds and the 1-skeleton is $S(\gamma) \cup X(\gamma)$.

The second fact is that such cellular structure can be deduced in a unique way from the Gauss paragraph of γ. In the target, the 0-cells are labelled by the letters $a_1, \dots, a_{r+s}$, each 1-cell is an oriented edge given by two consecutive letters $a_i^\epsilon a_j^\eta$ in W_0 (including also the edge joining the last to the first letter) and each 2-cell is a face which is determined by a closed sequence of oriented edges or their inverses. In the source, we proceed analogously but this time we take into account all the Gauss words $W_0, \dots, W_k$.

If $w(\gamma) = w(\delta)$, we write $\gamma : M_1 \to P_1$ and $\delta : M_2 \to P_2$ where M_i, P_i denote S^2 with the associated cellular structure in the source or the target respectively. Since the Gauss word of $\Delta(\gamma)$ is equal to the Gauss word of $\Delta(\delta)$, we have that P_1, P_2 are isomorphic as CW-complexes. We choose a cellular homeomorphism $\beta : P_1 \to P_2$. Then we construct another cellular homeomorphism $\alpha : M_1 \to M_2$ such that $\delta \circ \alpha = \beta \circ \gamma$. Given a cell E in M_1, then there is a unique cell E' in M_2 corresponding to the same label in the Gauss word and such that $\beta(\gamma(E)) = E'$. We define $\alpha|_E : E \to E'$ as $\alpha|_E = (\delta|_{E'})^{-1} \circ \beta|_{\gamma(E)} \circ \gamma|_E$.

Case 2: $w(\gamma) \simeq w(\delta)$.

1. Suppose that $w(\gamma), w(\delta)$ are related through a permutation τ in

the set of the letters $a_1, a_2, \ldots, a_{r+s}$. The proof is essentially the same as in case 1, but we construct the homeomorphisms α, β in such a way that a vertex with label a_i is mapped into a vertex with label $a_{\tau(i)}$, and so on.

2. Assume that $w(\gamma), w(\delta)$ are related through a reversion in the Gauss words together with a change in the exponents. We take $J : S^2 \longrightarrow S^2$, with $J(x_1, x_2, x_3) = (x_1, x_2, -x_3)$ such that either $w(\gamma) = w(\delta \circ J)$, $w(\gamma) = w(J \circ \delta)$ or $w(\gamma) = w(J \circ \delta \circ J)$. Then the result follows from case 1.
3. Assume that $w(\gamma), w(\delta)$ are related through cyclic permutations in the Gauss words. Then we can choose again a homeomorphism $T : S^2 \to S^2$ such that $w(\gamma) = w(\delta \circ T)$ and apply case 1.

□

Remark 6.2.5. The equivalence between the Gauss words of $\Delta(\gamma)$ and $\Delta(\delta)$ is not enough to guarantee that γ and δ are topologically equivalent. In fact, if γ, δ have isomorphic discriminants $\Delta(\gamma), \Delta(\delta)$, then they are not topologically equivalent in general (see [2]).

Remark 6.2.6. Note that theorem is not true if $S(\gamma)$ is not connected. We find in [14, Figure 6] an example of two stable maps from S^2 to S^2, both with empty Gauss words, which are not topologically equivalent. In that paper, the authors consider other global type invariants, for instance, the graph associated to the connected components of the complementary of $S(\gamma)$, but again this is far from being a complete invariant.

Now, we are in conditions of stating and proving the converse of corollary 6.1.2 in the case that the singular sets are smooth. In fact, if $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ is a finitely determined map germ such that $S(f)$ is smooth and non empty outside of the origin, then the singular set of its link $S(f|_{\widetilde{S}^2_\epsilon})$ is connected and non empty and hence, we can use theorem 6.2.6.

Theorem 6.2.7. *Let $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be two finitely determined map germs such that $S(f)$ and $S(g)$ are smooth and non empty outside of the origin. Then, if f and g are topologically equivalent, their respective links are topologically equivalent.*

Proof. Since f and g are topologically equivalent, there are homeomorphisms $\phi, \psi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $\psi \circ f = g \circ \phi$. We take small enough representatives and $\epsilon > 0$ such that $f|_{\widetilde{S}^2_\epsilon}$ is the link of f. We denote $M = \phi(\widetilde{S}^2_\epsilon)$ and $P = \psi(S^2_\epsilon)$, then we have a commutative diagram:

$$\begin{array}{ccc} \widetilde{S}^2_\epsilon & \xrightarrow{f|_{\widetilde{S}^2_\epsilon}} & S^2_\epsilon \\ \Big\downarrow \phi & & \Big\downarrow \psi \\ M & \xrightarrow{g|_M} & P \end{array}$$

Let us denote by $w(f|_{\widetilde{S}^2_\epsilon}) = \{W_0, W_1, \ldots, W_k\}$ the Gauss paragraph with respect to some labels $\{a_1, a_2, \ldots, a_{r+s}\}$, some orientations in $\widetilde{S}^2_\epsilon$ and S^2_ϵ and a cusp $z_0 \in S(f|_{\widetilde{S}^2_\epsilon})$ as a base point.

We also put $R = \phi(\widetilde{D}^3_\epsilon)$ and $Q = \psi(D^3_\epsilon)$ and consider the restriction $g|_R : R \to Q$. We take $\delta > 0$ small enough such that $D^3_\delta \subset Q$ and $g|_{\widetilde{S}^2_\delta}$ is the link of g. Then we consider in R, Q the orientations induced by ϕ, ψ respectively, in $\widetilde{D}^3_\delta, D^3_\delta$ the orientations induced as submanifolds of R, Q respectively and in $\widetilde{S}^2_\delta, S^2_\delta$ the orientations induced as boundaries of $\widetilde{D}^3_\delta, D^3_\delta$ respectively.

For each cusp or double fold in the target of $g|_{\widetilde{S}^2_\delta}$ we can associate a unique letter a_i in the obvious way: consider the curve of cusps or double folds of g joining the origin to this point and take the point of such curve in P, which is the image of a cusp or double fold in the target of $f|_{\widetilde{S}^2_\epsilon}$, labelled by a_i (see figure 6.9). For cusps or double folds in the source of $g|_{\widetilde{S}^2_\delta}$ we proceed analogously.

By using the same procedure, we take as a base point the corresponding cusp $z'_0 \in S(g|_{\widetilde{S}^2_\delta})$ coming from the cusp $z_0 \in S(f|_{\widetilde{S}^2_\epsilon})$.

With these choices it becomes clear that $g|_{\widetilde{S}^2_\delta}$ has the same Gauss paragraph $w(g|_{\widetilde{S}^2_\delta}) = \{W_0, W_1, \ldots, W_k\}$ and therefore, it is topologically equivalent to $f|_{\widetilde{S}^2_\epsilon}$ by theorem 6.2.6.

□

Remark 6.2.8. If $S(f)$ is empty outside the origin its associated link $\gamma : S^2 \to S^2$ becomes a regular map and hence a diffeomorphism by lemma 6.2.1. Hence, in this case we only have one topological class, namely the regular map $f(x, y, z) = (x, y, z)$.

Putting together theorems 6.2.6 and 6.2.7 and corollary 6.1.2, we have the following result.

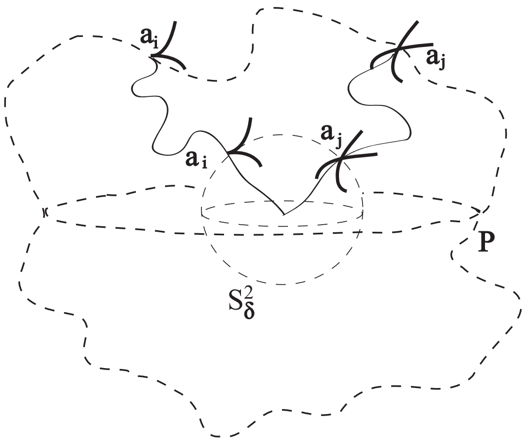


Figure 6.9: Scheme of the relation between S^2_δ and P

Corollary 6.2.9. *Let $f, g : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^3, 0)$ be two finitely determined map germs such that $S(f)$ and $S(g)$ are smooth and non empty outside of the origin. Then f and g are topologically equivalent if and only if their links have equivalent Gauss paragraphs.*

6.3 Topological classification of corank 1 map germs

Given a finitely determined map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$, we want to study its topological type by means of the Gauss paragraph of its link. Since the Gauss paragraph is considered only in the case that the singular set of the link is connected, we have to restrict ourselves to this case. We consider only corank 1 map germs, which can be written in the form $f(x, y, z) = (x, y, g(x, y, z))$. Then $S(f)$ is defined by the equation $g_z(x, y, z) = 0$. We want a condition for the singular set $S(f)$ to be smooth, which guarantees that its transverse intersection with $\widetilde{S}^2_\epsilon$ will be diffeomorphic to S^1.

Lemma 6.3.1. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a corank 1 map germ. Then, the 2-jet $j^2 f(0)$ is $\mathcal{A}$-equivalent to either (x, y, z^2), (x, y, xz) or $(x, y, 0)$.*

Proof. Since f has corank 1, we can assume $j^2 f(0) = (x, y, h(x, y, z))$ where $h(x, y, z)$ is a quadratic form in x, y, z. We eliminate all the terms in x, y by using an appropriate coordinate change in the target. Hence, we arrive to $j^2f(0) \sim_{\mathcal{A}} (x, y, cz^2 + eyz + fxz)$, for some $c, e, f \in \mathbb{R}$. We distinguish several cases.

1. Let $c \neq 0$. We assume, for instance, that $c > 0$ (the case $c < 0$ is analogous). Then, $j^2f(0)$ becomes $\mathcal{A}$-equivalent to (x, y, z^2) by taking the following coordinate changes in the source and target, respectively:
$$\bar{z} = \frac{fx + ey + 2cz}{2\sqrt{c}}, \quad \bar{Z} = Z + \frac{(fX + eY)^2}{4c}.$$
2. Let $c = 0$ and $f \neq 0$. Then, $j^2f(0) \sim_{\mathcal{A}} (x, y, xz)$ by means of the following coordinate changes: $\bar{x} = ey + fx$ and $\bar{X} = eY + fX$.
3. Let $c = f = 0$ and $e \neq 0$. Again, $j^2f(0) \sim_{\mathcal{A}} (x, y, xz)$ by taking $\bar{x} = ey, \bar{y} = x$ and $\bar{X} = eY, \bar{Y} = X$.
4. If $c = e = f = 0$, then $j^2f(0) = (x, y, 0)$.

□

It is well known that the fold $f(x, y, z) = (x, y, z^2)$ is 2-determined. Thus, if a map germ has 2-jet equivalent to (x, y, z^2), then it is in fact $\mathcal{A}$-equivalent to the fold. Hence, we do not need to consider this case.

We center our attention from now on in the case $j^2f(0) \sim_{\mathcal{A}} (x, y, xz)$. Then $S(f)$ is smooth and hence, the singular set of the link is connected.

Lemma 6.3.2. *Let's consider a weighted homogeneous finitely determined map germ whose 2-jet is $\mathcal{A}$-equivalent to (x, y, xz).Then, f will be have the form*

$$(x, y, xz + z^m(a_0(z^{w_2})^r + a_1(z^{w_2})^{r-1}y^{w_3} + \cdots + a_r(y^{w_3})^r))$$

where w_2 and w_3 are the weights of y and z respectively and $m = 2, 3$

Proof. By the fact of being in the $\mathcal{A}^2$-class (x, y, xz) and being weighted homogeneous, f must be have the form

$$(x, y, xz + z^m(a_0(z^{w_2})^r + a_1(z^{w_2})^{r-1}y^{w_3} + \cdots + a_r(y^{w_3})^r)).$$

In addition to this, if $m = 1$ we obtain non isolated swallowtails and the same happen for $m \geq 4$, arriving to a contradiction with the finite determinacy hypothesis. Then, $m = 2, 3$ and the result is proved. □

We are going to work with the simplest case of this germs. Since f is finitely determined a_0, a_r must be distinct from 0, getting $a_0 = 1$ and $a_r = \pm 1$ if we normalize them, but we will take $a_i = 0$, $1 \leq i \leq r-1$. From this point, let's consider the following table (table 6.1):

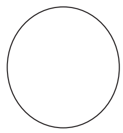
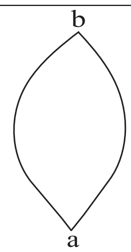
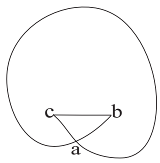
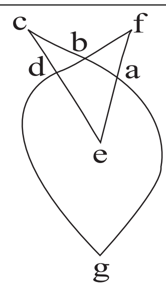
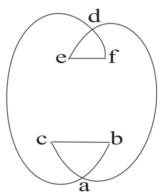
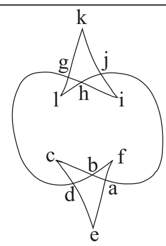
Degree 0		Degree 1	
Germ	Associated link	Germ	Associated link
(x, y, z^2)		$(x, y, z^3 + xz)$	
$(x, y, z^4 + yz^2 + xz)$		$(x, y, z^5 + yz^3 + xz)$	
$(x, y, z^4 - y^2z^2 + xz)$		$(x, y, z^5 - y^2z^3 + xz)$	

Table 6.1

We are going to prove that if we are working with a map germ of the form $f = f_c + \widetilde{f}$, with $f_c(x, y) = (x, y, z^k + xz + y^l z^m)$ with $m = 2, 3$ and $\widetilde{f}$ with only terms of higher weighted degree we are only going to obtain six different topological classes given by the normal forms and discriminant curves that appear in table 6.1. To be in conditions of proving this statement we should remember an important result due to J. R. Quine and stating and proving some previous lemmas and theorems which will help us in the proof of the main result.

Theorem 6.3.3. *([33]) Let M and N be smooth compact oriented con-*

nected 2-manifolds. Suppose $f : M \to N$ is smooth and every point $p \in M$ is either a fold point, cusp point, or regular point of f i.e., f is excellent in the sense of Whitney. Let M^+ be the closure of the set of regular points at which f preserves orientation and M^- the closure of the set of regular points at which f reverses orientation. Let $p_1, \dots, p_n$ be the cusp points and $\mu(p_k)$ the local degree at the cusp point p_k. Then:

$$\chi(M) - 2\chi(M^-) + \Sigma\mu(p_k) = (\deg f)\chi(N)$$

where χ is the Euler characteristic and deg *is the topological degree*

Remark 6.3.4. If we take in Quine's theorem $M = N = S^2$ we have

$$\Sigma\mu(p_k) = 2(\deg f),$$

so if $\deg f = 0$, $\Sigma\mu(p_k) = 0$ and we will have the same number of cusps with positive local degree and cusps with negative local degree in the discriminant curve and if $\deg f = 1$, we have that $\Sigma\mu(p_k) = 2$ and we will have two more cusps with positive local degree or two more cusps with negative local degree in the discriminant curve. We will see the importance of this remark later.

Given a stable map $\gamma : S^2 \to S^2$, we define the multiplicity of γ as $\mathrm{mult}(\gamma) = \min_{p\in S^2} \mathrm{mult}(p)$, where $\mathrm{mult}(p) = \#\gamma^{-1}(p)$.

Theorem 6.3.5. *Let $\gamma : S^2 \to S^2$ be a stable map, with $S(\gamma)$ connected. Then, its multiplicity is determined by $\Delta(\gamma)$.*

Proof. If we denote by c the number of cusps and by d the number of the set of double points $\{z_1, \dots, z_d\}$ of $\Delta(\gamma)$, we have that, applying the Euler formula in the image, $S^2\backslash\Delta(\gamma)$ presents $2+d$ faces $\{F_1, \dots, F_{d+2}\}$. Let's suppose that $\mathrm{mult}(z_i) = n+q_i$, $i = 1, \dots, d$ and $\mathrm{mult}(F_i) = n+p_i$, $i = 1, \dots, d+2$, where $n = \mathrm{mult}(\gamma)$.

On the other hand, if we center now our attention in the inverse image set of the discriminant $\gamma^{-1}(\Delta(\gamma)) \subset S^2$ the Euler formula is also satisfied and, as a consequence, we obtain that the number of faces of $S^2 \setminus \gamma^{-1}(\Delta(\gamma))$ is $2+c+2d+\widetilde{d}$ with $\widetilde{d}$ being the number of non singular inverse images of double points that appear in $X(\gamma)$

So, we have that by one side # of faces of $S^2\backslash\gamma^{-1}(\Delta(\gamma)) = \sum_{i=1}^{d+2} \mathrm{mult}(F_i) = (d+2)n + \sum_{i=1}^{d+2} p_i$. and by the other # of faces of $S^2 \setminus \gamma^{-1}(\Delta(\gamma)) =$

$2 + c + \sum_{i=1}^{d} \text{mult}(z_i) = 2 + c + nd + \sum_{i=1}^{d} q_i$. Simplifying we obtain the formula

$$n = \frac{1}{2}(2 + c - \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} q_i)$$

which is completely determined by the topological information given by the discriminant of γ. □

Lemma 6.3.6. *Let $\gamma, \delta : S^2 \to S^2$ be two stable maps, with $S(\gamma), S(\delta)$ both connected. Assume that $\Delta(\gamma), \Delta(\delta)$ are both diffeomorphic to one of the discriminant curves that appear in table 6.1 and the signs of their cusps (their local degrees) coincide too. Then, γ and δ are topologically equivalent.*

Proof. If both discriminants coincides with the discriminant curve of one of the three stable singularities (fold, cusp and swallowtail), the result is trivial. Let's suppose first that $\Delta(\gamma), \Delta(\delta)$ are equal to the discriminant curve given by the Gauss word $a^{-1}bcad^{-1}efd$. By theorem 6.3.5 we can determine its inverse images distribution uniquely (figure 6.10) and that each one of the simple cusps is going to present 2 inverse

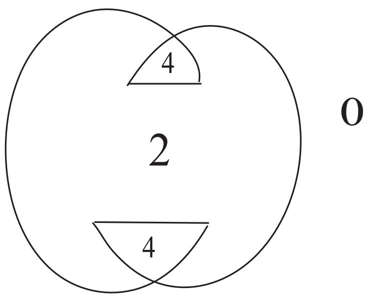


Figure 6.10

images, and by remark 6.3.4, we know that, taking into account the local topological degree of each one of the 4 cusps that appear, we only can have two possible cases:

1 The signs of the cusps are alternating (figure 6.11, left).
The structure of $X(\gamma) \cup S(\gamma)$ will have the form given by the right side of figure 6.11, with Gauss words $a^{-1}\overline{b}^2c^2a\overline{c}^2b^2$ and $d^{-1}\overline{e}^2f^2d\overline{f}^2e^2$. Then, $w(\gamma) = w(\delta)$ and as a consequence, γ and δ are topologically equivalent.

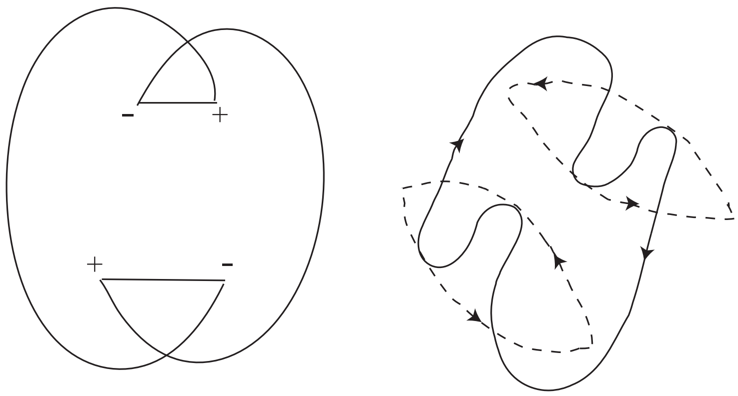


Figure 6.11

2 The first and the last cusp are negative(figure 6.12, left).
In this case the structure of $X(\gamma) \cup S(\gamma)$ will have the form that appears in the right side of figure 6.12,

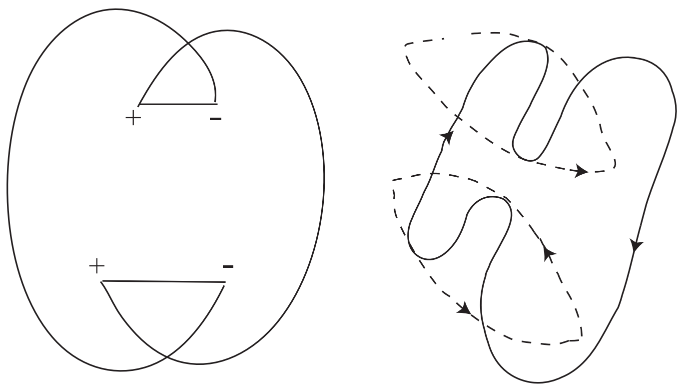


Figure 6.12

with Gauss words $a^{-1}\overline{b}^2c^2a\overline{c}^2b^2$ and $d^{-1}e^2\overline{f}^2df^2\overline{e}^2$. Therefore, both Gauss paragraphs are equal and as a consequence γ and δ are topologically equivalent.

Following an analogous procedure in the discriminant curves given by Gauss words $a^{-1}bc^2d^{-1}e^2af^2b^{-1}dg^2$ (figure 6.13, left) and

$$a^{-1}bc^2d^{-1}e^2af^2b^{-1}dg^{-1}hi^2j^{-1}k^2gl^2h^{-1}j$$

(figure 6.13, right) we also arrive to the conclusion that in both cases $w(\delta) = w(\gamma)$ and both stable maps are topologically equivalent. $\square$

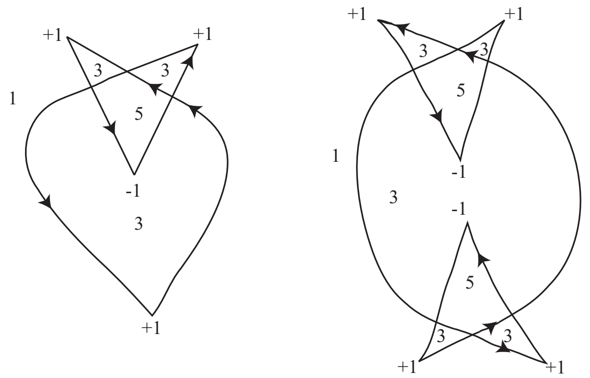


Figure 6.13

Lemma 6.3.7. *The map germ* $f(x,y,z) = (x, y, z^k + xz \pm y^l z^2)$ *is topologically equivalent to one of the germs of the table 6.1.*

Proof. The equation of $S(f)$ is $kz^{k-1} \pm 2y^l z + x = 0$, so that we can put x in terms of y, z and substitute in f to get $f|_{S(f)}$. Then $f|_{S(f)}$ can be seen as an unfolding of a plane curve $p_y(z)$ with parameter y, where

$$p_y(z) = (-kz^{k-1} \mp 2y^l z, (1-k)z^k \mp y^l z^m).$$

If we compute its derivative we obtain

$$p_y'(z) = (-k(k-1)z^{k-2} \mp 2y^l, -k(k-1)z^{k-1} \mp 2y^l z).$$

The equation for the cuspidal edges is $p_y'(z) = 0$, which gives

$$y^l = \frac{\mp k(k-1)}{2} z^{k-2}.$$

To obtain the double point curves we compute the pairs (z, u) such that

$$\frac{p_{y,1}(z) - p_{y,1}(u)}{z-u} = \frac{p_{y,2}(z) - p_{y,2}(u)}{z-u}, \tag{6.1}$$

where $p_y(z) = (p_{y,1}(z), p_{y,2}(z))$. This gives the equation

$$\frac{(z^{k-2} + z^{k-3}u + z^{k-4}u^2 + \cdots + zu^{k-3} + u^{k-2})(z+u)}{z^{k-1} + z^{k-2}u + \cdots + zu^{k-2} + u^{k-1}} = \frac{2(k-1)}{k}. \tag{6.2}$$

In order to simplify our computations we take $u = 1$, obtaining thus the equation

$$\frac{(z^{k-2} + z^{k-3} + \cdots + z + 1)(z + 1)}{z^{k-1} + z^{k-2} + \cdots + z + 1} = \frac{2(k-1)}{k}. \tag{6.3}$$

We know, by [21] that the left hand side of (6.3) has its maximum at $z = 1$, taking the value $\frac{2(k-1)}{k}$. Thus, the only solution of our equation is $z = u$, and as a consequence, if k is odd, f doesn't present any double fold point. If k is even, $z = -1$ is a solution of the polynomial $z^{k-1} + z^{k-2} + \cdots + z + 1$ and as a consequence $z = -u$ is a solution of (6.2). If we suppose that k is even and substitute the equality $z = -u$ in (6.1) we obtain our double point curve

$$y^l = \frac{\mp k z^{k-2}}{2}.$$

Now, let's take into account some important facts:

1. y^l is topologically equivalent to either y^2 when l is even, or y when l is odd.
2. If k is even the map germs $(x, y, z^k + xy + yz^2)$ and $(x, y, z^k + xz - yz^2)$ are topologically equivalent, so we only need to consider one of the topological classes.

As a consequence, we obtain the following configuration of the curves of the singular set as well as the discriminant of the link of f (figures $6.14, 6.15, 6.16, 6.17$) :

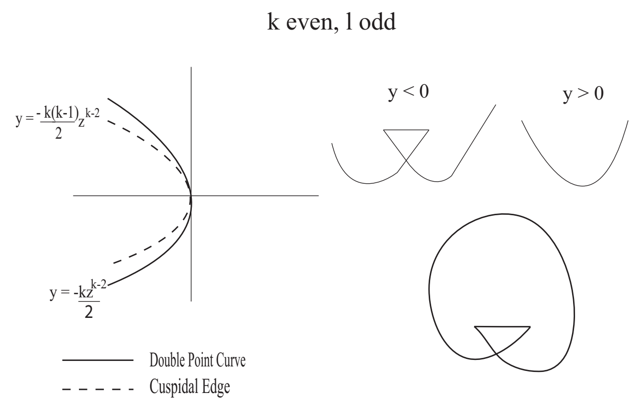


Figure 6.14

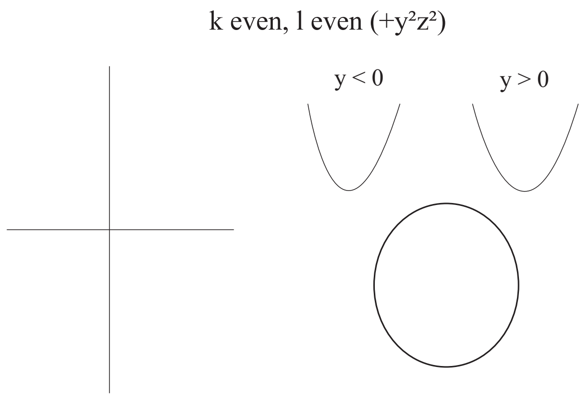


Figure 6.15

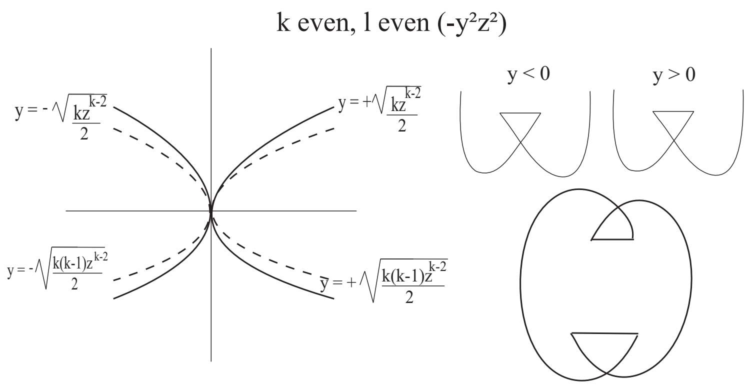


Figure 6.16

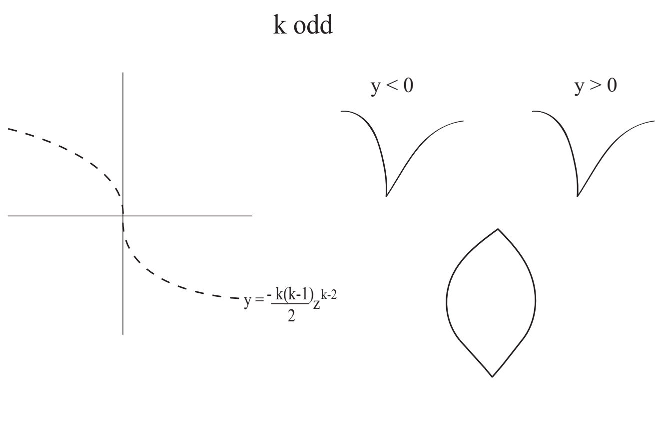


Figure 6.17

The last step to check that each one of these discriminant curves originates a single topological class is to verify that there is a single configuration of the local degrees of the cusps that appear in each one of the discriminant curves. Then, applying lemma 6.3.6 we will have finished. In the case of the fold, the cusp and the swallowtail there is nothing to prove. Let's see what happen with the discriminant curve of figure 6.16. As we saw in the proof of lemma, according to the different configuration of their cusps we can have two cases, but taking into account the normal form we are working with ($(x, y, z^k + xz - y^2z^2)$, with k even) the only possible configuration is the one whose curves are symmetric respect of the origin (figure 6.11). Then, we only need to choose a normal form of each of these classes and the proof is concluded. □

Lemma 6.3.8. *The map germ $f(x, y, z) = (x, y, z^k + xz \pm y^l z^3)$ is topologically equivalent to one of the germs of the table 6.1.*

Proof. Following the same procedure as in the proof of last lemma, we get the cuspidal edge equation which is given by

$$\begin{cases} z = 0, \\ y^l = \frac{\mp k(k-1)}{6y} z^{k-3}, \end{cases}$$

as well as the equation of the double fold curve:

$$\frac{(z^{k-2} + z^{k-3}u + \cdots + zu^{k-3} + u^{k-2})(z^2 + zu + u^2)}{(z^{k-1} + z^{k-2}u + \cdots + zu^{k-2} + u^{k-1})(z + u)} = \frac{3(k-1)}{2k}, \quad (6.4)$$

Again we take $u = 1$, arriving to the equation

$$\frac{(z^{k-2} + z^{k-3} + \cdots + z + 1)(z^2 + z + 1)}{(z^{k-1} + z^{k-2} + \cdots + z + 1)(z + 1)} = \frac{3(k-1)}{2k}. \quad (6.5)$$

Let's consider the real function

$$h(z) = \frac{(z^{k-2} + z^{k-3} + \cdots + z + 1)(z^2 + z + 1)}{(z^{k-1} + z^{k-2} + \cdots + z + 1)(z + 1)},$$

and let's study its graph. It satisfies the following properties:

- $h(\frac{1}{z}) = h(z)$, $\forall z \neq 0$
- $h(z) > 1$ if $z \neq 0$ and $h(0) = 1$

In order to determine the solutions of $h(z) = c$, with $c = \frac{3(k-1)}{2k}$, we study its positive and negative real roots by applying Descartes rule of signs to the polynomial

$$g(z) = (z-1)^2 H(z) = (1-c)z^{k+2} + cz^k - z^{k-1} - z^3 + cz^2 + (1-c),$$

where $H(z) = (z^{k-2} + z^{k-3} + \cdots + z + 1)(z^2 + z + 1) - c(z^{k-1} + z^{k-2} + \cdots + z + 1)(z+1)$. We find 4 positive roots and 2 negative roots when k is even and 4 positive and 3 negative roots when k is odd for $g(z)$. We have to take into account that

$$\begin{cases} \text{positive roots of } g(z) = \text{ positive roots of } h(z) + 2, \\ \text{negative roots of } g(z) = \text{ negative roots of } h(z). \end{cases}$$

Hence, we have that $h(z)$ presents at most 2 positive and 2 negative roots when k is even and 2 positive and 3 negative roots when k is odd. The graph of $h(z)$ is shown in figure 6.18 in each case:

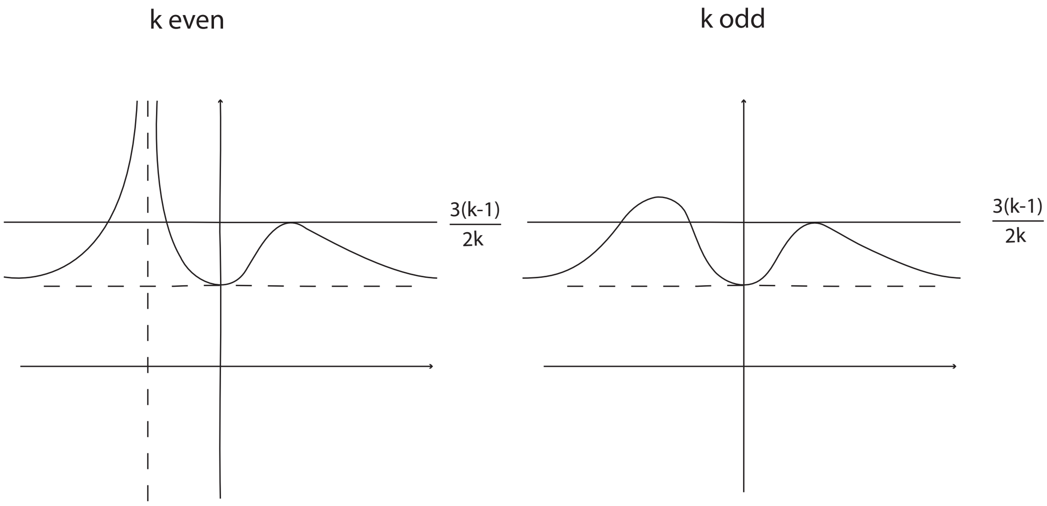


Figure 6.18

By looking at the graph of $h(z)$, we see that in both cases, (6.5) has 3 real roots, namely $z = A_1$, with $-2 < A_1 < -1$, $z = A_2$, with $-1 < A_2 < 0$, and $z = 1$. As a consequence, (6.4) presents the double points $z = A_1 u$, $z = A_2 u$ when k is even, obtaining respectively the double fold curves

$$y^l = \frac{\mp k(A_1^{k-1} - 1)z^{k-3}}{3(A_1^2 - 1}), \quad y^l = \frac{\mp k(A_2^{k-1} - 1)z^{k-3}}{3(A_2^2 - 1)}.$$

If k is odd we obtain this two roots, together with $z = -z'$ (by using an analogous argument as in case 1) getting in this case the equation

$$y^l = \frac{\mp k z^{k-3}}{3}.$$

Now, as we did in the proof of lemma 6.3.7 we must take into account the following facts:

1. y^l is topologically equivalent to either y^2 when l is even, or y when l is odd
2. If k is odd the map germs $(x, y, z^k + xy + yz^3)$ and $(x, y, z^k + xz - yz^3)$ are topologically equivalent, so we only need to consider one of the topological classes.

The configuration of the curves of the singular set, as well as the corresponding links of the discriminant and Gauss words is shown in figures 6.19, 6.20, 6.21, 6.22, where $C_1 = \sqrt{\frac{k(A_2^{k-1}-1)z^{k-3}}{3(A_2^2-1}}$, $C_2 = \sqrt{\frac{k(A_1^{k-1}-1)z^{k-3}}{3(A_1^2-1}}$, $C_3 = \sqrt{\frac{k(k-1)z^{k-3}}{6}}$ and $C_4 = \sqrt{\frac{kz^{k-3}}{3}}$.

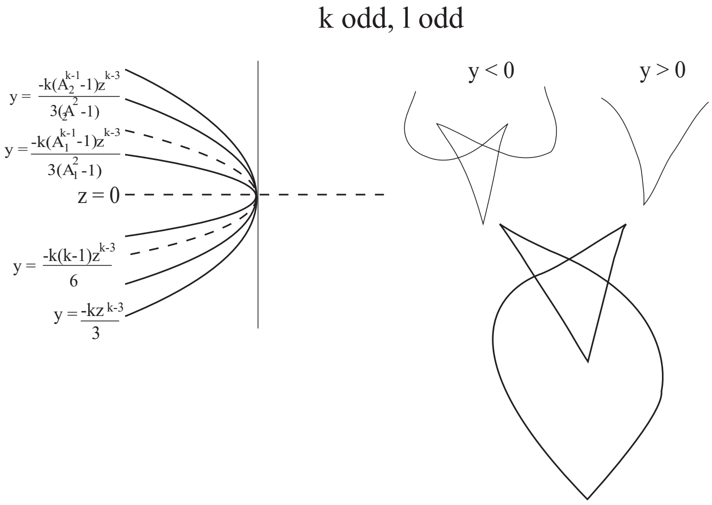


Figure 6.19

By an analogous procedure of the end of the proof of lemma 6.3.7, using lemma 6.3.6 again, we obtain a single topological class for each one of the discriminant curves. Then, we only need to choose again a normal form of each of these classes and the proof is concluded.

□

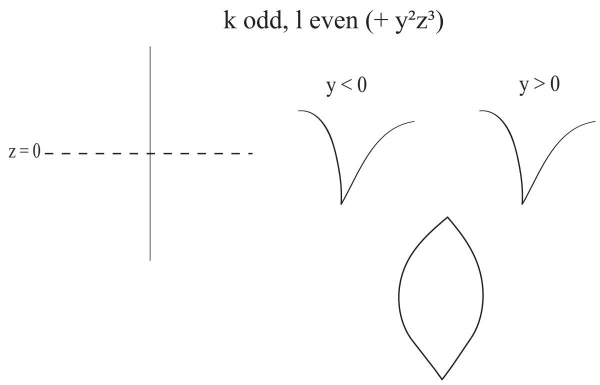


Figure 6.20

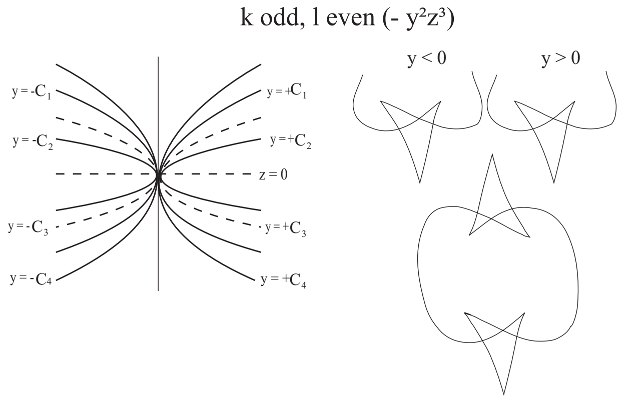


Figure 6.21

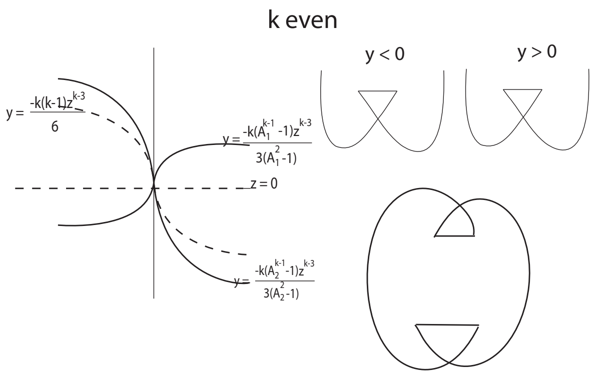


Figure 6.22

Theorem 6.3.9. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ of the form $f = f_c + \widetilde{f}$, with $\widetilde{f}$ with only terms of higher weighted degree. Then, $w(f)$ and $w(f_c)$ are equivalent, and, as a consequence f and f_c are topologically equivalent.*

Proof. Let's check first that the number of branches of the double point curves $D_2(f|_{S(f)})$ and $D_2(f_c|_{S(f)})$ coincides. First of all, let's notice that the number of branches of these curves is equal to the number of branches of the plane curves defined by the equations (2) and (4) of the proof of the lemmas 6.3.7 and 6.3.8. Making use of these equations we will have that

$$k(z^{k-2}+z^{k-3}u+\cdots+u^{k-2})(z+u)+2(k-1)(z^{k-1}+z^{k-2}u+\cdots+u^{k-1})+$$

$$+G(z,u) = G_0(z,u) + G(z,u) = 0$$

in the first case and

$$2k(z^{k-2} + z^{k-3}u + \cdots + zu^{k-3} + u^{k-2})(z^2 + uz + u^2)+$$

$$+3(k-1)(z^{k-1} + z^{k-2}u + \cdots + u^{k-1})(z+u) + H(z,u) =$$

$$= H_0(z,u) + H(z,u) = 0$$

in the second case, where G_0, H_0 are homogeneous and G, H are polynomials of higher order terms.

Since H_0 and G_0 are homogeneous we are able to apply a result due to H. Kuiper ([17]), which tell us that $H_0 + H$ and $G_0 + G$ have the same number of branches than H_0 and G_0 respectively.

If we consider the map germ f_c we know for the proof of lemmas 6.3.7 and 6.3.8 that its double point curves and cuspidal edges are given by equations of the form $y^l = A_i z^{k-m}$ with $A_i \neq A_j$ if $i \neq j$ because f_c is not degenerate. It follows that these curves, in the case of f will have the form $y^l = A_i z^{k-m} + F(z)$, with F being a polynomial of higher order terms. So, if we take values of z small enough, we will have that the relative position of the curves doesn't change with respect to the initial part, it only depends on $A - i$.

Let's finish this proof seing that the sign of the double points in the Gauss word doesn't change either. This sign is given by the determinant formed by the vectors $\{\delta_1 \frac{\partial f}{\partial y}(y,z), \delta_2 \frac{\partial f}{\partial y}(y,u), \delta_3 \frac{\partial f}{\partial z}(y,z)\}$, with $\delta_i = \pm 1$.If we compute this determinant in the case of $g_c = f_c|_{S(f_c)}$ and

and taking into account the relations $y^l = A_i z^{k-m}$ and $u = B_i z$ we obtain the following expression

$$det(\delta_1 \frac{\partial g_c}{\partial y}(y,z), \delta_2 \frac{\partial g_c}{\partial y}(y,u), \delta_3 \frac{\partial g_c}{\partial z}(y,z)) =$$

$$= (z^{m-1} - u^{m-1}) lm(\pm k(k-1) y^{l-1} z^{k-1} + m(m-1) y^{2l-1} z^{m-1}) +$$
$$+ (z^m - u^m) l(1-m)(k(k-1) y^{l-1} z^{k-2} + m(m-1) y^{2l-1} z^{m-2} =$$
$$= C_i D_i l y^{l-1} z^{k+m-2}$$

with

$$C_i = \mp k(k-1) \pm m(m-1) A_i$$

and

$$D_i = m(1 - B_i^{m-1}) + (m-1)(1 - B_i^m)$$

As the terms C_i and D_i never vanish, taking into account the values of A_i and B_i obtained in the proof of lemmas 6.3.7 and 6.3.8 this determinant never vanishes out of the origin and using a continuity argument if we choose $\epsilon > 0$ small enough we will have that for any z, with $|z| < \epsilon$ the sign of the double points of the Gauss word of f coincides with the sign of the double points the Gauss word of f_c and the result is proved. □

Now, putting together lemmas 6.3.7 and 6.3.8 and theorem 6.3.9 we have the following result:

Corollary 6.3.10. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ of the form of theorem 6.3.9. Then, f is topologically equivalent to one of the germs of table 6.1.*

6.4 Ruled map germs from $\mathbb{R}^3$ to $\mathbb{R}^3$

Ruled surfaces are surfaces generated by straight lines or rulings and have been studied for centuries by geometers. We can find examples of this kind of surfaces in the discriminant of the stable maps from $\mathbb{R}^3$ to $\mathbb{R}^3$.

In this last section we will define formally what is a ruled map germ in our case of study, how we can relate them with its discriminant and at the end we will give a complete topological classification of them.

Definition 6.4.1. A ruled map from $\mathbb{R}^n$ to $\mathbb{R}^p$, with $n \leq p$ is a map $f : I \times \mathbb{R}^{n-1} \to \mathbb{R}^p$ given by

$$f(t, u_1, \dots, u_{n-1}) = a_1(t) + u_1 a_2(t) + \dots + u_{n-1} a_n(t),$$

where $I \subset \mathbb{R}$ is an interval, $a_1 : I \to \mathbb{R}^p$ is a curve in $\mathbb{R}^p$ and $a_2, \dots, a_n : I \to \mathbb{R}^p$ are vectors fields along a_1, such that they are linearly independent in each point. If $(t_0, u_{1_0}, \dots, u_{n-1_0}) \in I \times \mathbb{R}^n$ and $f(t_0, u_{1_0}, \dots, u_{n-1_0}) = p$, then the map germ

$$f : (I \times \mathbb{R}^{n-1}, (t_0, u_{1_0}, \dots, u_{n-1_0})) \to (\mathbb{R}^p, p)$$

will be call a *ruled map germ.*

If we take $n = p = 3$ we will have a ruled map $f : I \times \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(t, u, v) = a(t) + ub(t) + vc(t)$$

where $I \subset \mathbb{R}$ is an interval, $a : I \to \mathbb{R}^3$ is a space curve and $b, c : I \to \mathbb{R}^3$ are vectors fields along a, such that they are linearly independent in each point. Analogously as in the general case we obtain from here a ruled map germ from $\mathbb{R}^3$ to $\mathbb{R}^3$.

We have now the following result:

Proposition 6.4.2. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a ruled map germ. Then, up to $\mathcal{A}$ - equivalence, f can be written in the form $f(x, y, z) = (x, y, \alpha(z) + x\beta(z) + y\gamma(z))$.*

Proof. For the fact of being a ruled map germ we have $f(t, u, v) = a(t) + ub(t) + vc(t)$ with b and c linearly independent along a. Let's suppose that $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ is the minor that doesn't vanish. We are looking for obtaining a C^∞ map germ $\phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $f \cdot \phi(x, y, z) = (x, y, \alpha(z) + x\beta(z) + y\gamma(z))$

If we take

$$\phi(x, y, z) = (z, \frac{B(z)c_2(z) - C(z)c_1(z)}{A(z)}, \frac{b_1(z)C(z) - B(z)b_2(z)}{A(z)}),$$

with $A(z) = c_2(z)b_1(z) - c_1(z)b_2(z)$, $B(z) = x - a_1(z)$ and $C(z) = y - a_2(z)$, we get the desired result, with $\alpha(z) = \frac{\det(a(z), b(z), c(z))}{A(z)}$, $\beta(z) = \frac{c_2(z)b_3(z) - b_2(z)c_3(z)}{A(z)}$ and $\gamma(z) = \frac{b_1(z)c_3(z) - c_1(z)b_3(z)}{A(z)}$. □

Let's see now that using this normal form of a ruled map germ we can easily proof a direct relation with its discriminant.

Proposition 6.4.3. *Let* $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ *be a map germ, with* $S(f)$ *smooth. Then,if* f *is a ruled map germ if and only if* $\Delta(f)$ *is a ruled surface in* $\mathbb{R}^3$.

Proof. Let's suppose first that f is a ruled map germ. This directly implies than f can be written in the form $f(x, y, z) = (x, y, \alpha(z)+x\beta(z)+y\gamma(z))$ We have that $S(f)$ is given by the equation $\alpha'(z) + x\beta'(z) + y\gamma'(z) = 0$. So $\Delta(f)$ is parameterized by $(A(z)y + B(z), y, C(z)y + D(z))$, with $A(z) = \frac{-\gamma'(z)}{\beta'(z)}$, $B(z) = \frac{-\alpha'(z)}{\beta'(z)}$, $C(z) = \gamma(z) + \frac{-\gamma'(z)\beta(z)}{\beta'(z)}$ and $D(z) = \alpha(z) + \frac{-\alpha'(z)\beta(z)}{\beta'(z)}$. Therefore, $\Delta(f)$ is a ruled surface in $\mathbb{R}^3$.

Reciprocally, if $\Delta(f)$ is a ruled surface in $\mathbb{R}^3$, $\Delta(f) = f(S(f))$, we will have that, for the smoothness of $S(f)$, $f|_{S(f)}(y, z) = (a(z)y + b(z), y, c(z)y + d(z))$, with $S(f) = \{(x, y, z) : x - a(z)y - b(z) = 0\}$ and f will be given by $f(x, y, z) = (x, y, \int(x - a(z)y - b(z))dz = (x, y, A(z)y + B(z) + xz)$. Then, f is a ruled map germ. □

To finish this section and this article we are going to give a complete topological classification of this kind of germs. We will need some previous lemmas.

Lemma 6.4.4. *Let* $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ *be a finitely determined ruled map germ. Then* $j^2f(0)$ *is* $\mathcal{A}$ *- equivalent to either* (x, y, z^2) *or* (x, y, xz).

Proof. We know from the lemma 6.3.1 that $j^2f(0)$ is $\mathcal{A}$ - equivalent to either $(x, y, z^2), (x, y, xz)$ or $(x, y, 0)$. Let's see than in this case it can't be $\mathcal{A}$ - equivalent to the class $(x, y, 0)$

By the fact of being a ruled map germ we have $f(x, y, z) = (x, y, \alpha(z)+x\beta(z) + y\gamma(z))$, so $j^2f(0)$ will be $\mathcal{A}$ - equivalent to a germ of the form $(x, y, \alpha''(0)z^2 + \beta'(0)xz + \gamma'(0)yz)$.

If $j^2f(0)$ were $\mathcal{A}$ - equivalent to $(x, y, 0)$ we would have $\alpha''(0) = 0$, $\beta'(0) = 0$, $\gamma'(0) = 0$ and as a consequence a map germ f of the form $(x, y, z^k + \cdots + x(z^2 + \cdots) + y(z^3 + \cdots))$. We would get as a singular set's equation $kz^{k-1} + x(z + \cdots) + y(z^2 + \cdots)$.Let's see that it presents a non isolated singularity. If we derive the singular set, obtaining the vector $(z + \cdots, z^2 + \cdots, (k - 1)kz^{k-2} + x(1 + \cdots) + y(2z + \cdots))$ it's obvious that it doesn't present pure terms on y, obtaining points of the form $(0, y, 0)$ where this vector vanishes and arriving to a contradiction with the finite determinacy of f. □

Lemma 6.4.5. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined ruled map germ. Then f is topologically equivalent to a germ of the form $(x, y, z^k + xz + yz^m)$ with $m = 2, 3$.*

Proof. First of all, let's notice that, applying the last lemma, f can be written in the form $f_c + \widetilde{f}$, with $f_c(x, y, z) = (x, y, z^k + xz + yz^m)$, $m = 2, 3$ and $\widetilde{f}(x, y, z) = (0, 0, (a_{k+1}z^{k+1} + \cdots) + x(b_2 z^2 + \cdots) + y(c_{m+1}z^{m+1} + \cdots))$. Now, the proof is a direct consequence of theorem 6.3.9. □

We are in conditions now of stating and proving the main result of this last section.

Theorem 6.4.6. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined ruled map germ. Then f is topologically equivalent to one of the map germs of the table 6.2.*

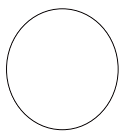
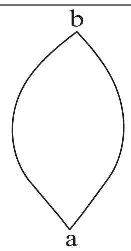
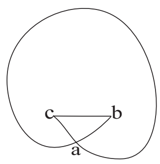
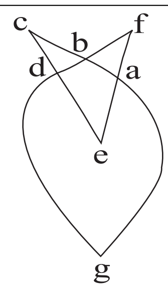
Degree 0		Degree 1	
Germ	Associated link	Germ	Associated link
(x, y, z^2)		$(x, y, z^3 + xz)$	
$(x, y, z^4 + yz^2 + xz)$		$(x, y, z^5 + yz^3 + xz)$	

Table 6.2

Proof. We obtain the desired result by just applying together lemmas 6.3.7, 6.3.8 and 6.4.5. □

Chapter 7

Open problems

During this work we got to reach the following goals:

1. We constructed a complete topological invariant for stable maps from S^1 to S^1 and as a consequence, applying the results of Fukuda, a complete topological invariant for finitely determined map germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.
2. Using this invariant we obtained a wide topological classification of these map germs in the case of corank 1 and we extended it to corank 2 in some particular cases.
3. We proved that any unfolding of a finitely determined map germ $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ that is excellent in the sense of Gaffney and such that $\Delta(F)$ is topologically trivial, is topologically trivial. We putted a bit more our attention in this 1-parameter families, giving a result related with the parity of the number of cusps that appear on them.
4. As a final result of this PhD-Thesis we constructed a complete topological invariant again, in this case to study the topological behavior of stable maps $\gamma : S^2 \to S^2$ and restricting the singular set of γ, $S(\gamma)$, to be connected. With this tool we studied the different topological classes that are contain in the $\mathcal{A}^2$-class (x, y, xz) and we gave a full classification of ruled map germs from $\mathbb{R}^3$ to $\mathbb{R}^3$.

For the results achieved many future lines of research arise:

1. Complete the topological classification of finitely determined map germs from the plane to the plane, specially in the corank 2 case.

2. Departing from the Gauss words try to define a complete topological invariant that control globally the topological behavior of the links associated to multigerms and as a consequence, being in conditions of classifying these multigerms.
3. Taking into account the work achieved for the planar case, find conditions that give us the topological triviality of families in higher dimensions.
4. As in the planar case, construct a complete topological invariant that let us to deal with stable maps $\gamma : S^2 \to S^2$ which singular set is not connected as well as study the topology of multigerms in this case.

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