VNIVERSITAT DE VALÈNCIA

Departamento de Física Teórica, Facultad de Física
Doctorado en Física

Non-supersymmetric black-hole solutions in $N = 2$, $D = 4$ supergravity

PhD dissertation by

Pietro Galli

Thesis director

María Antonia Lledó

Valencia, Spain; July, 2013

Maria Antonia Lledo, Profesor Titular de la Universidad de Valencia,

CERTIFICA:

Que la presente memoria,

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Firma del director de tesis

Maria Antonia Lledo

Dánzame todos los días
como la maldición
más honrada de tu vida:
la verdad.

G.N.

CONTENTS

GENERAL INTRODUCTION

Black holes have always been intriguing objects able, since their conception, to charm the curious mind of people from many different disciplines. Lots of works of unrelated nature took inspiration from these celestial bodies which trap and hide everything behind a black curtain that external observers will never unveil. The presence of a surface of no return, the theoretical prevision of wormholes and the speculative idea of parallel universes made black holes attractive to the public at least as much as their gravitational strength.

From a scientific point of view, black holes are important for several reasons. First of all, it is nowadays widely accepted that they do exist and that they are not a merely theoretical solution of some equations. Indirect observations, like the study of the effects on the surrounding stars or the deflection of light due to strong gravity, indicate their presence in the center of most galaxies (see e.g [1, 2]), including ours [3]. The formation of black holes could be dated back to the Big Bang or, as most cosmologists believe, related with the gravitational collapse of massive stars. In either case, their study may provide useful insights about the origin (or the end in the case of the Big Crunch) of our universe and help in understanding the other giant astronomical objects populating the cosmos. Besides, black holes are interesting by their own since represent a special status of the matter where, supposedly, the Pauli-exclusion principle is violated and where the laws of nature are completely different from anywhere else in our world. In addition, from a more theoretical perspective, black-hole solutions of a theory of quantum gravity can be used as a test of the theory itself. Their thermodynamical description, for example, is an important playground that can be explored, and should be compatible, at both macroscopic and microscopic level. At the same time black-hole geometries appear also in gauge/gravity dualities (AdS/CFT correspondence) relevant in the study of several different fields like condensed matter systems or QCD. At the end, for all these reasons and more, it is not a case if black holes are thought so important to be considered "the harmonic oscillator of the 21st century" [4].

Black-hole history in short

In theoretical physics black holes arise as solutions of the Einstein equations of motion. The first exact black-hole solution was obtained by Schwarzschild [5] in 1915 only one month after the publication of the Einstein theory of gravity. It describes the geometry of the spacetime curved by a spherically symmetric non-rotating mass without any charge. Few months later Reissner solved the Einstein equations for a charged point mass [6] and short after Nordström extended this solution to a spherically symmetric charged body [7]. In 1965 Newman found a metric describing charged rotating black holes [8, 9] that merges the Kerr solution (achieved in 1963 [10]) for a spinning point mass and the Reissner-Nordström one. The Kerr-Newman black-hole solution is characterized by the value of only three parameters: the mass M, the total charge Q and the angular momentum J. This is the statement of the *no-hair theorem* that holds true for Maxwell–Einstein theory in four dimensions and, according to which, all the information concerning the constituent matter, the birth process and conditions at infinity do not affect the externally observable features of a black hole.

The no-hair theorem finds its expression in black-hole thermodynamics, first studied by Bekenstein and Hawking in the seventies. The former worked out a close resemblance between the laws of black-hole mechanics and the laws of thermodynamics [11–13]. His main claim was that, if a black hole admitted a thermodynamical analysis, the entropy would have to be proportional to the area of the event horizon while the temperature to the surface gravity. Hawking, almost simultaneously, demonstrated instead (even if its initial idea was to prove exactly the opposite) that considering quantum effects a black hole emits a perfect black-body radiation spectrum [14]. Merging their results, the two scientists managed to complete the description with the finding of the proportionality coefficients that exactly relate the geometry of the spacetime and the thermodynamical features of a black hole.

The results of Hawking have been one of the first insights into a possible theory of quantum gravity. Nowadays the best candidate for such a theory is string theory. Its low energy limit, known as *supergravity*, accommodates the discussion of the existence of black holes as solutions for the metric. Indeed, besides general relativity, black holes can generally be encountered in theories (such as the bosonic sector of $N = 2$ supergravity in four dimensions) in which gravity is coupled to massless neutral scalar fields and abelian gauge vector fields. Interesting in this framework is the role of the scalars which turn out to affect considerably the description of the other fields. In particular, for *extremal* charged black holes, the spacetime geometry depends exclusively on the values of the scalars on the event horizon which, in turn, are such that minimize a certain effective potential, intuitively understood as the electromagnetic energy of the system. According to the no-hair theorem, this phenomenon, called *attractor mechanism*, prevents conditions at infinity from being relevant in the analysis of the local properties of extremal black holes.

The attractor mechanism was discovered in the middle of the nineties by Ferrara, Kallosh and Strominger for supersymmetric extremal black-hole solutions of $N = 2$ four-dimensional supergravity [15–17]. Afterwards, it has been proven that it is manifested also by non-supersymmetric extremal black holes [18–21]. For supersymmetric attractors, the radial evolution of the scalars is determined by first-order differential equations. The solutions of these, expressed in terms of harmonic functions, properly satisfy the second-order differential equations of motion directly obtainable by varying the Lagrangian of the dynamical system. Subsequent works, starting with [22] and developed in [23–25], have pointed out that this is also true in the non-supersymmetric context and even in the non-extremal case [24, 26, 27].

The achievement of a first-order formalism can be related to the possibility to rewrite the effective Lagrangian describing the theory as a sum of squares. This idea, applied to a more complicated scenario, led to the derivation of *multicenter supersymmetric black holes* [28] and the study of their attractor phenomenon [29,30]. The formulation of non-supersymmetric composites has instead required much more efforts. Many works, by different approaches, have tried to address this issue (see for instance [31–34]) but, yet, their analysis cannot be considered finished.

Still, new advances in the description of the geometry of the scalar manifold and the continuous elaboration of new methods to tackle the equations keep the study of black holes active and interesting, at least as much as it has been since the first appearance of these peculiar objects in the panorama of physics.

A new perspective

Most of the results in the study of black holes are based on a formalism (we will refer to it as the FGK-formalism) introduced in [18] and in its extension to the non-supersymmetric case of [22]. With it, the problem of finding single-center, static, charged, spherically-symmetric black-hole solutions of a generic 4-dimensional Einstein–Maxwell–scalar theory is reduced to the simpler problem of solving the equations of motion of a one-dimensional effective theory whose dynamical variables are just scalar fields. The evolution of the fields is described by second-order differential equations that, even when reduced to first order ones, are generally difficult to solve. The methods implemented for their resolution usually depend on the type of black hole one is interested in. The supersymmetric case is, and has always been, the easier to deal with and, nowadays, it can be considered to be well understood. On the other hand it cannot be said the same for the non-supersymmetric branch. As for all types of symmetry, when supersymmetry is broken the system is less constrained and more complicated configurations become possible. The analysis of non-supersymmetric solutions has been carried out often starting from similarities with the supersymmetric correspondents. Most of the efforts, until short time ago, were concentrated on extremal static solutions. Still a complete classification of all non-supersymmetric black holes is missing.

An important step towards a final understanding of black-hole solutions has been taken with the introduction, by Ortín and collaborators, of the H-FGK formalism in $N = 2$, $D = 4$ (and $D \geq 5$) supergravity [35–37]. By a set of relationships, usually known as stabilisation equations, one can express the physical degrees of freedom in terms of spatial functions $H^M(x)$ and use them as dynamical variables of the system. These relationships remain unchanged for various types of black holes, which means that all black-hole solutions in a given model take the same form in terms of the functions H^M. Only the functions themselves vary. Dealing with the functions H^M in place of the physical fields results in simpler equations of motion and in a better control of the symmetries of the model. Moreover, all types of black holes are put on the same ground and what changes in their description is only the explicit form of the H^M.

The H-FGK formalism has been so far applied to the analysis of static, spherically symmetric, single-center black holes (but its future extension to the stationary case cannot be excluded). For supersymmetric extremal solutions, the H-functions are known to be harmonic with poles in the physical magnetic and electric charges carried by the black hole. By an initial assumption, a harmonic ansatz can be used also for non-supersymmetric black holes, whereas for their non-extremal counterparts a hyperbolic (exponential) ansatz has been shown to solve all the equations [38].

To come in the thesis

The content of this thesis fits the contemporary literature as it aims to contribute to the study and classification of non-supersymmetric black-hole solutions. It is mainly a collection of the papers published during my PhD and is organized in three parts. The first one has just one chapter and is a review dedicated to basic background material about black-hole solutions in general relativity, Eisntein-Maxwell-scalar theories and $N = 2$ four-dimensional supergravity. The second part contains six chapters, each corresponding to a different paper in the same exact form as it has been published. The six articles are ordered according to their submission date to the arXiv and correspond respectively to Ref. [32], [39], [38], [40], [41], [42]. The last part of the manuscript encloses a general discussion briefing the main results and the bibliography. Due to the structure of the thesis, each chapter has its own notation that is every time specified and well defined. For the sake of simplicity, units are chosen so that the four dimensional gravitational constant, the reduced Plank constant and the speed of light are $\mathrm{G}_4 = \hbar = c = 1$.

A more detailed outline of the dissertation is the following:

- In order to accomplish the requirements imposed by the local normative regulating the format of a doctoral thesis, the next chapter is the Spanish translation of this first introductory part.

- Chapter 1 reviews some basic notions taken from the literature that may be useful to the reader to better understand the rest of the thesis. It begins with an introduction about the different kinds of black holes that appear in general relativity. Solutions in the vacuum, with charges, static or stationary are discussed and their macroscopic thermodynamics is commented. It follows a section dedicated to extremal, static, charged black holes in theories where gravity is coupled to abelian gauge vector fields and scalars. The equations of motion are explicitly solved and the attractor mechanism explained in detail. The last part of the chapter regards black holes in $N = 2$, $D = 4$ supergravity. It introduces the FGK-formalism and the first-order description of the evolution of the field. A partial classification of the known solutions is given.

- Chapter 2 corresponds to Ref. [32] and treats non-supersymmetric multicenter black holes with cubic prepotential. The main idea is to rewrite the Lagrangian in a squared form by using the formalism of [28] and the knowledge of the existence of a superpotential driving the flow of the scalars. Multicenter solutions with parallel charge vectors are found and compared with their supersymmetric counterparts.

- Chapter 3 is Ref. [39] and proposes, for static and rotating, non-supersymmetric, extremal black holes, a set of algebraic equations (stabilisation equations) that includes ratios of harmonic functions. The solutions of the algebraic equations satisfy as well the differential equations of motion of the system and are general enough to generate any other solution by applying duality transformations.
- Chapter 4, that is Ref. [38], is about non-extremal black holes. It presents a deformation procedure that allows to obtain non-extremal solutions from supersymmetric extremal ones by replacing, in the expression of the fields, harmonic functions with exponential functions. The recipe is proven successful for several models and the non-extremal black holes obtained interpolate continuously between the supersymmetric and the non-supersymmetric extremal limit. Their thermodynamics and their first-order formulation are analysed.
- Chapter 5 is Ref. [40] and deals with the t^3 model with quantum corrections by using the H-FGK formalism. The first part of the paper explains in detail the formalism while the second part presents the study of the black-hole solutions encountered.
- Chapter 6, corresponding to Ref. [41], confirms the results of [39] and links it to the H-FGK formalism. Moreover, it tries to generalize ratios of harmonic functions to ratios of hyperbolic functions in order to find the seed solution for the non-extremal case.
- Chapter 7, that is Ref. [42], proves the existence of a gauge Freudenthal duality among the variables of the H-FGK formalism. This symmetry allows to express the solutions in terms of “any” type of function. This freedom can be fixed by imposing a constraint that, in the case of harmonic (or hyperbolic) functions, corresponds to absence of NUT charge.
- A general final discussion summarizes the main results achieved and concludes the thesis. Like the introduction, this last chapter is written first in English and then in Spanish.

INTRODUCCIÓN GENERAL

Los agujeros negros, desde su concepción, siempre han sido capaces de encantar la mente curiosa de la gente de diferentes disciplinas. Muchas obras de distinto origen han sido inspiradas por estos misteriosos cuerpos celestes que atrapan y esconden todo detrás de un telón negro donde nadie puede buscar. La presencia de una superficie de no retorno, la previsión teórica de los agujeros de gusano y la idea especulativa de universos paralelos han hecho que los agujeros negros sean conocidos por todos los públicos.

Desde un punto de vista científico, los agujeros negros son importantes por varias razones. En primer lugar, hoy en día, es ampliamente aceptado que existen y que no son sólo una solución de algunas ecuaciones. Observaciones indirectas, como el estudio de sus efectos sobre las estrellas que los rodean o el desvío de la luz debido a su fuerte gravedad, han indicado su presencia en el centro de la mayoría de las galaxias (véase por ejemplo [1, 2]), incluyendo la nuestra [3]. La formación de los agujeros negros podría tener que ver con el Big Bang o, como la mayoría de los cosmólogos creen, estar relacionada con el colapso gravitacional de estrellas masivas. En cualquier caso, su estudio puede aportar ideas útiles acerca del origen (o el final, en el caso del Big Crunch) de nuestro universo y puede ayudar en la comprensión de otros objetos astronómicos gigantes que rellenan el cosmos. Además, los agujeros negros son interesantes por sí mismos, ya que representan un estado especial de la materia en el que, supuestamente, el principio de Pauli es violado y donde las leyes de la naturaleza son completamente diferentes a las que hay en otros lugares del universo. Desde una perspectiva más teórica, las soluciones de agujero negro de una teoría de gravedad cuántica son importantes porque pueden usarse como test para probar la teoría misma. Su descripción termodinámica, por ejemplo, puede ser explorada tanto a nivel macroscópico como a nivel microscópico y ambos análisis deben resultar compatibles. Por otro lado, las geometrías de agujero negro aparecen también en la dualidad “gauge”/gravedad (correspondencia AdS/CFT) que es importante en campos como materia condensada y QCD.

No es por tanto casualidad que los agujeros negros sean considerados como “los osciladores armónicos del siglo XXI” [4].

Breve historia de las soluciones de agujero negro

En física teórica los agujeros negros nacen como soluciones de las ecuaciones de Einstein. La primera solución exacta de agujero negro fue obtenida por Schwarzschild [5] en 1915, sólo un mes después de la publicación de la teoría de la relatividad general. Esta solución describe la geometría de un espacio-tiempo curvado por una masa esférica, neutra, sin momento angular. Pocos meses más tarde, Reissner resolvió las ecuaciones de Einstein para una masa puntual con cargas [6] y poco después Nordström extendió esta solución a un cuerpo esférico [7]. En 1965 Newman encontró una métrica que describe agujeros negros cargados que rotan [8, 9] fusionando así la solución de Kerr (1963 [10]) para masas puntuales rotatorias y la de Reissner-Nordström. La solución de agujero negro de Kerr-Newman se caracteriza por el valor de sólo tres parámetros: la masa M, la carga total Q y el momento angular J. Este resultado es un importante teorema llamado “no-hair theorem” que afirma que toda la información relativa a los constituyentes, al proceso de nacimiento y a las condiciones en el infinito, no afecta las características de un agujero negro que se observan desde el exterior.

Los agujeros negros admiten una descripción termodinámica, que fue estudiada por primera vez en los setenta por Bekenstein y Hawking. El primero elaboró un paralelismo entre las leyes de la mecánica y las de la termodinámica [11–13]. Su argumento principal fue que, si un agujero negro admite un análisis termodinámico, la entropía tendría que ser proporcional al área del horizonte de los eventos y a su vez, la temperatura tendría que ser proporcional a la gravedad en su superficie. Hawking, casi al mismo tiempo, demostró (aunque su idea inicial era demostrar lo opuesto) que, considerando efectos cuánticos, un agujero negro emite una radiación de cuerpo negro perfecta [14]. Juntando sus resultados, los dos científicos lograron completar la descripción expresando en fórmulas la exacta proporcionalidad entre la geometría del espacio-tiempo y las características termodinámicas de un agujero negro.

Los resultados de Hawking han sido una de las primeras indicaciones sobre una posible teoría de gravedad cuántica. Hoy en día el mejor candidato para tal teoría es la teoría de las cuerdas. Es en su límite a baja energía, conocido como supergravedad, que se da el debate sobre la existencia de los agujeros negros. De hecho, además de en relatividad general, este tipo de solución para la métrica se puede encontrar en teorías (como el sector bosonico de supergravedad $N = 2$ en cuatro dimensiones) donde la gravedad se acopla a escalares neutros y a campos vectoriales abelianos. En estos sistemas los escalares juegan un papel muy importante. En particular, para agujeros negros extremos cargados, la geometría del espacio-tiempo depende exclusivamente del valor de los escalares en el horizonte que, a su vez, minimiza un cierto potencial efectivo intuitivamente entendido como la energía electromagnética del sistema. En linea con el "no-hair theorem" este fenómeno, llamado mecanismo del atractor, impide que las condiciones en el infinito sean relevantes en el análisis de las propiedades locales de un agujero negro extremo.

El mecanismo del atractor fue descubierto hacia la mitad de los noventa por Ferrara, Kallosh y Strominger para soluciones supersimétricas de agujero negro extremo en supergravedad $N = 2$, $D = 4$ [15–17]. Poco después se ha encontró también para agujeros negros extremos no supersimétricos [18–21]. La evolución de los escalares, en el caso supersimétrico, se puede describir a través de ecuaciones diferenciales de primer grado que se obtienen por imponer supersimetría y se resuelven en términos de funciones armónicas. Sin embargo, una descripción a primer orden ha sido elaborada también para las demás soluciones de agujeros negros extremos [22, 23] y no extremos [24, 26, 27].

La ventaja del formalismo de orden uno es que se puede reescribir el lagrangiano de la teoría como una suma de términos cuadrados. Esta idea, aplicada a casos más complicados, ha permitido la formulación de agujeros negros supersimétricos de multicentro [28] y el estudio de su mecanismo del atractor [29, 30]. La derivación de sistemas multicentro no supersimétricos ha sido, por contra, mucho más complicada. Muchos esfuerzos han sido dirigidos hacia este tipo de soluciones (véase por ejemplo [31–34]) pero, aún así, su análisis no se puede considerar acabado.

Una vez más, los nuevos avances en la descripción de la geometría de la variedad de los escalares y la continua formulación de nuevos métodos para tratar con las ecuaciones mantienen el estudio de los agujeros negros activo e interesante, como siempre ha sido desde la primera aparición de estas soluciones en el panorama de la física.

Situación presente

En el estudio de los agujeros negros la mayoría de los resultados se basa en un formalismo llamado FGK y introducido en [18]. Este formalismo reduce el problema de la búsqueda de soluciones, estáticas, cargadas y con simetría esférica y un sólo centro, de una teoría Maxwell-Einstein-escalares en cuatro dimensiones al problema más simple de resolver las ecuaciones de movimiento de una teoría efectiva en una dimensión, cuyas variables dinámicas son sólo campos escalares. La evolución de los campos se describe por ecuaciones de segundo orden que, en general, son muy complicadas. Los métodos que se usan para resolverlas dependen del tipo de agujero negro que se quiera investigar. El caso supersimétrico es el más simple y su estudio se puede, hoy en día, considerar completo. No se puede decir lo mismo del caso no supersimétrico. Como ocurre con cualquier simetría, cuando la supersimetría está rota el sistema se encuentra menos vinculado y puede asumir configuraciones más complejas. Hasta hace poco la mayoría de los esfuerzos se habían concentrado principalmente en las soluciones extremas estáticas y todavía falta una clasificación final de todos los agujeros negros no supersimétricos.

Un paso importante en esta dirección ha sido la introducción del formalismo H en supergravedad $N = 2$, $D = 4$ (y $D \geq 5$) por parte de Ortín y colaboradores [35–37]. A través de una serie de relaciones normalmente conocidas como "stabilisations equations", los grados de libertad físicos se pueden expresar en términos de funciones espaciales $H^M(x)$ que devienen así en las nuevas variables del sistema. Estas relaciones se mantienen inalteradas para todos los tipos de agujeros negros de un mismo modelo. Todas las soluciones tienen la misma forma en términos de las funciones H. Lo que diferencia una solución de otra es sólo la forma explícita de las H^M. La ventaja de tratar con las H^M en lugar de con los campos físicos es que las ecuaciones de movimiento resultan más simples y las simetrías del modelo son más fáciles de analizar.

Hasta la fecha, el H-formalismo ha sido aplicado al análisis de agujeros negros estáticos con un solo centro. Para soluciones supersimétricas las funciones H son armónicas con polos en las cargas electromagnéticas del agujero negro. Un ansatz armónico se puede usar también para investigar soluciones extremas no supersimétricas, mientras que para soluciones no extremas se puede demostrar que es suficiente un ansatz hiperbólico.

Esquema de la tesis

Esta tesis se ajusta bien a la bibliografía contemporánea, ya que pretende contribuir al estudio y clasificación de las soluciones no supersimétricas de agujero negro. Es principalmente una colección de los trabajos publicados durante mi doctorado y está organizada en tres partes. La primera tiene un solo capítulo y es una revisión general sobre las soluciones de agujero negro en relatividad general, teorías Eisntein–Maxwell–escalares y supergravedad $N = 2$ en cuatro dimensiones. La segunda parte cuenta con seis capítulos, cada uno correspondiente a un artículo con la forma y contenido en que han sido aceptados para ser publicados. Los seis artículos han sido ordenados de acuerdo a la fecha en que han sido presentados en el arXiv y corresponden a las referencias [32], [39], [38], [40], [41], [42]. Debido a la estructura de la tesis, cada capítulo tiene su propia notación que se explica claramente. La última parte de la disertación contiene una discusión que resume los resultados principales y la bibliografía de todas las citas realizadas.

Una descripción más detallada de la tesis es la siguiente:

- El capítulo 1 examina conceptos básicos que pueden ser útiles al lector para comprender mejor el resto de la tesis. Comienza con una introducción acerca de los diferentes tipos de agujero negro que aparecen en relatividad general y sigue con las soluciones en el vacío, con cargas, estáticas o rotatorias. Para todas ellas se discute su termodinámica. Después hay una sección dedicada a agujeros negros extremos estáticos con cargas en teorías donde la gravedad está acoplada a campos vectoriales abelianos y a escalares. Las ecuaciones de movimiento se resuelven explícitamente y el mecanismo del atractor se analiza con detalle. En la última parte del capítulo se consideran los agujeros negros en supergravedad $N = 2$, $D = 4$, se introduce el formalismo FGK y se discute la descripción de primer orden de la evolución de los campos. También se presenta una clasificación parcial de las soluciones conocidas.

- El capítulo 2 corresponde a la Ref. [32] y trata sobre los agujeros negros no supersimétricos de multicentro en modelos con prepotencial cúbico. La idea principal es reescribir el lagrangiano en forma cuadrática utilizando el formalismo de [28] y el conocimiento de la existencia de un superpotential para la evolución de los escalares.

- El capítulo 3 es la Ref. [39] y en él se propone, para agujeros negros extremos, estáticos y rotatorios, un sistema de ecuaciones algebraicas que incluye fracciones de funciones armónicas. Las soluciones satisfacen las ecuaciones de movimiento y son suficientemente generales como para generar todas las otras a través de transformaciones de dualidad.

- El capítulo 4, que es la Ref. [38], trata de los agujeros negros no extremos. Presenta un procedimiento de deformación que permite obtener soluciones no extremas desde soluciones extremas supersimétricas mediante la sustitución, en la expresión de los campos, de las funciones armónicas por funciones exponenciales. Este método, aplicado con éxito a varios modelos, permite obtener agujeros negros no extremos que interpolan de manera continua entre las soluciones supersimétricas y no supersimétricas en el límite extremo.

- El capítulo 5 es la Ref. [40] y trata sobre el modelo t^3 con correcciones cuánticas, usando el formalismo H. En su primera parte se explica con detalle el formalismo, mientras que en la segunda se analizan las soluciones encontradas.

- El capítulo 6 corresponde a la Ref. [41] y confirma los resultados de [39] mediante el uso del formalismo H. Contiene también un intento de generalización de las fracciones de funciones armónicas a fracciones de funciones hiperbólicas para el caso no extremo.

- En el capítulo 7, que es la Ref. [42], se demuestra la existencia de una dualidad Freudenthal de "gauge" entre las variables del formalismo H. Esta simetría permite expresar una misma solución en términos de diferentes tipos de funciones. Esta libertad puede ser fijada imponiendo un vinculo que, en el caso armónico (o hiperbólico), corresponde a la ausencia de la carga NUT.

- Una discusión final que resume los resultados principales concluye la tesis.

Part I

BACKGROUND

1. BASIC NOTION ABOUT FOUR-DIMENSIONAL BLACK HOLES IN...

1 ...*General Relativity*

One of the most revolutionary ideas introduced in physics at the beginning of the past century is the concept of spacetime. Differently from classical mechanics, the relativistic theory is built from treating time and space on equal footing. Space and time are now coordinates of a manifold describing the four-dimensional framework (the spacetime) where the events take place. Like all types of coordinates, they can be transformed (by Lorentz transformations) without changing the physics of the system (covariant description). In Newtonian mechanics the (three-dimensional) space is generally assumed to be Euclidean but in relativity it is fundamental that the spacetime metric has Minkowskian signature $(-,+,+,+)$. In addition, in general relativity, the spacetime is not even flat. Bodies with their mass curve it as a heavy ball does on a stretched sheet. The bigger is the mass, the bigger is the curvature and the geodesics of the spacetime (the path followed by free objects) will not be straight lines any more.

A black hole in general relativity is usually defined as a region of the spacetime whose curvature becomes so large that the trajectory of all objects, including light, are so bent around the mass that nothing can escape away. The border between the black hole region and the rest of the spacetime is called *event horizon* and nothing behind it can communicate with the outside world. Far from a black hole, in absence of massive bodies, the spacetime is expected to be described by a constant metric with vanishing curvature. That means that any black-hole solution must enjoy the property of being asymptotically flat.

A black hole may originate from the gravitational collapse of a very heavy star. As long as the star "is burning" the gravitational attraction is contrasted by the outward pressure of its incredibly hot constituents. But as the fuel begins to finish the star begins to shrink. When the mass is big enough the gravitational force is so strong to win even the Pauli-exclusion principle, the electrons and the protons will combine in neutrons and the impossibility to contrast the pull of gravity will lead to a black hole.

Already in Newtonian gravity nothing forbids the existence of bodies so massive that they would require, to flee their attraction, an escape velocity bigger than the speed of light. However it is in general relativity that black holes have their proper collocation since they arise as natural solutions of the Einstein field equations:[1]

$$R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} = 8\pi T_{\mu\nu} \,. \tag{1.1}$$

These equations relate the metric (enclosed in the in the Ricci tensor $R_{\mu\nu}$ and in the scalar curvature R) to the energy content (described by the stress-energy tensor $T_{\mu\nu}$) of the spacetime. For this reason, different black-hole solutions exist depending on the fields of the theory that determine the form of $T_{\mu\nu}$.

1.1 Vacuum solution

The simplest case is to consider only the gravitational field and the theory is described by the Einstein–Hilbert action:

$$I = \frac{1}{16\pi} \int \mathrm{d}^4x \, \sqrt{-G} \, R \,. \tag{1.2}$$

Varying the metric $G_{\mu\nu}$ one obtains the Einstein equations in the vacuum that simplify to

$$R_{\mu\nu} = 0 \,. \tag{1.3}$$

The most general spherically symmetric solution to these equations describes the Schwarzschild black hole and reads:

$$\mathrm{d}s^2 = G_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \,, \tag{1.4}$$

where t is the time coordinate, r the radial one, Ω the solid angle on a 2-sphere and M the black hole mass. Notice that as $r \to \infty$ the metric becomes diag$(-1, +1, +1, +1)$ as required by the condition of asymptotic flatness.

[1] Throughout all thesis, for simplicity notation, we choose units so that the four-dimensional gravitational constant, the reduced Planck constant and the speed of light are $\mathrm{G}_4 = \hbar = c = 1$.

It is worth briefly commenting on the concept of mass in general relativity, since it is not generally trivial to define the total energy of a system. This problem stems from the fact that the contribution of the gravitational field to the total energy is part of the Einstein tensor (left hand side of equation (1.1)) and then is not taken into account by the stress-energy tensor. For some case it is possible to rewrite the Einstein equations in a way such that part of the gravitational energy is included in $T_{\mu\nu}$. However such rewriting does not solve the problem since it turns out to be not covariant [43]. What is instead convenient, at least for asymptotically flat spacetimes, is to associate what we call mass to a new quantity: the ADM energy [44]. The ADM energy is defined as a function of the deviation of the metric tensor from the prescribed asymptotic flat form. This quantity in a certain way tries to capture the intuitive notion of "all the mass-energy there is" in some asymptotically flat spacetime model.[2]

The only true singular point of the metric[3] (1.4) turns out to be $r = 0$ as only at this point the curvature invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 48M^2/r^6$ tends to infinity. The singularity in $r = 2M$ is instead a coordinate singularity since a suitable change of coordinates makes it disappear. This fake singularity is interesting because the surface $r = 2M$ is a stationary null surface and defines the event horizon of the Schwarzschild solution. The radial coordinate becomes timelike inside the horizon radius and this is the reason for the impossibility of an infalling object to escape from a black hole once it has crossed the horizon.

The study of the near-horizon geometry reveals many interesting features related to the thermodynamical properties of black holes. If one takes in (1.4) the limit $r \to 2M$ and defines $\zeta^2 = 8M(r - 2M)$, one obtains the

[2] Besides its relation with the ADM energy, it is possible to realize that the parameter M in (1.4) is a mass (understood with the common meaning of gravitational source) if one considers the Newtonian limit of the Einstein equations. In the limit in which gravity is weak, the relative motion of the sources is much slower than the speed of light and the material stresses are much smaller than the mass-energy density, the component tt of the Einstein equations for the perturbation of the metric from the flat form becomes the Poisson's equation of a gravitational field produced by a body of mass M (see e.g. [43,45]).

[3] In general a singularity is a point where some components of the metric vanish or blow up. There is however a difference between coordinate singularities and real singularities. While the former can be remove by an appropriate change of reference frame, a real singularity is something implicit in the nature of the spacetime and independent of the way one describes it. To understand which type of singularity one is dealing with, one should study the behavior of the diffeomorphism invariant quantity $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ (Kretschmann scalar).

near-horizon metric:

$$\mathrm{d}s_{\mathrm{h}}^2 = -\frac{\zeta^2}{16M^2}\mathrm{d}t^2 + \mathrm{d}\zeta^2 + (2M)^2\mathrm{d}\Omega^2\,. \tag{1.5}$$

The geometry factorizes in a (1+1)-dimensional (ζ, t) spacetime times a 2-sphere of radius $2M$. The former, with the identification $\kappa = \frac{1}{4M}$, is the so-called Rindler spacetime

$$\mathrm{d}s_{\mathrm{R}}^2 = -\zeta^2\kappa^2\mathrm{d}t^2 + \mathrm{d}\zeta^2 \tag{1.6}$$

that actually turns out to be a two dimensional Minkowski space $\mathcal{M}^2$ written in "strange" coordinates. For the proof we refer the reader to, for instance, [45]. We want only to point out that the possibility to show the Rindler metric equal to the Minkowski one means that the Schwarzschild spacetime near $r_{\mathrm{h}} = 2M$ is not singular at all since at the end it looks exactly like $\mathcal{M}^2 \times S^2$.

1.2 The Bekenstein-Hawking temperature and entropy

The thermodynamics of a black hole turns out to be governed by some specific geometrical parameters. Their actual values change depending on the black hole that we are considering, but for any solution, they play the same crucial role for defining the thermodynamics. More precisely, in the case of spherical symmetry, there are only two: the *area of the event horizon* $A_{\mathrm{h}} = 4\pi r_{\mathrm{h}}^2$ and the *surface gravity*[4] κ appearing in (1.6).

Indeed there is a deep connection between black-hole geometry and thermodynamics. This is supported by the following observations regarding the possibility to recognize the laws of thermodynamics in the spacetime black-hole geometry (e.g. [46]):

Zeroth law: In thermal physics it states that the temperature T is constant throughout a body at thermal equilibrium. For black holes one can show that the surface gravity κ is constant on the event horizon.

First law: Energy is conserved: $\mathrm{d}E = T\mathrm{d}S + \mu\mathrm{d}Q + \chi\mathrm{d}J$, where E is the energy, S the entropy, Q the charge with electric potential μ and J the angular momentum with angular velocity χ. Correspondingly, for rotating, charged black holes of mass M it holds: $\mathrm{d}M = \frac{\kappa}{8\pi}\mathrm{d}A + \mu\mathrm{d}Q + \chi\mathrm{d}J$.

[4] The surface gravity is usually defined as the acceleration that needs to be exerted at infinity to keep an object at the *Killing horizon*. The Killing horizon is a static, null surface on which there is a null Killing vector field. However, for the black holes we are interested in, the event and the Killing horizon of the definition coincide.

Second law: The total entropy of a physical system never decreases, $\Delta S \geq 0$. Correspondingly for black holes it can be proved that the horizon area never decreases, $\Delta A_{\rm h} \geq 0$ (area theorem).

Third law: The Planck-Nernst form of the third law of thermodynamics states that $S \to 0$ (or a "universal constant") as $T \to 0$. The analog of this law fails in black hole mechanics since there exist black holes, the extremal ones, with surface gravity vanishing but event-horizon area finite. However, also in condensed matter physics there appear several violations of the Planck-Nernst formulation. There are then good reasons to believe that we are not discussing a fundamental law of thermodynamics but rather a property of the density of states near the ground state in the thermodynamic limit [47]. Indeed the weaker formulation of the third law, "the absolute zero cannot be reached in a finite number of processes", turns out to have an analogue for black holes [48].

In spite of the evident analogies $A_{\rm h} \leftrightarrow S$ and $\kappa \leftrightarrow T$, noticed for the first time by Bekenstein [11–13], the above discussion is not sufficient for a thermodynamical description of a black hole. The main problem is that each hot body ($T \neq 0$) must radiate but for a classical black hole, by definition, this is impossible. The missing ingredient of the Bekenstein recipe was added by Hawking. He proved that, after including quantum effects, the spectrum emitted by a black hole is precisely thermal [14]. This phenomenon, known as Hawking radiation, can be intuitively understood by imagining particle-antiparticle pair creation near the horizon. It can happen that one of these two particles falls and then disappears inside the horizon while the other escapes to infinity. A detailed calculation then shows that this effect gives rise to a perfect black body radiation spectrum.

Considering the above discussion, finally, the following identifications for the entropy and the temperature of a black hole come naturally:

$$T = \frac{\kappa}{2\pi}\,, \qquad S = \frac{A_{\rm h}}{4}\,. \tag{1.7}$$

The Bekenstein-Hawking temperature and entropy are given in terms of the surface gravity and the horizon area. For all black holes, they are independent of the carried charges, the spin and the spacetime dimension.

It is important to point out that the thermodynamical description given above is a macroscopic one, that is, it has been carried out without referring to the statistical interpretation of the entropy. Indeed, through the Boltzmann formula,[5] the entropy usually encodes information about the microscopic degrees of freedom of the system. An interpretation in this direction of the black hole entropy has not been yet formulated for Schwarzschild solutions but, thanks to string theory, it is instead possible for a large class of special supersymmetric black holes (see e.g. [49, 50]).

1.3 Charged solutions

As we mentioned at the beginning of the section, the right hand side of the Einstein equations for the metric depends on the field content of the theory. If we add to the Einstein–Hilbert action (1.2) electromagnetic fields we recover the Einstein–Maxwell theory whose action is:

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-G}\left(R - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right)\,, \tag{1.8}$$

where $\mathcal{F}_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ is the electromagnetic field strength. It follows that in (1.1) $T_{\mu\nu} \neq 0$.

Assuming spherical symmetry, the two-form associated with $\mathcal{F}_{\mu\nu}$ must be of the form:[6]

$$\mathcal{F} = p\,\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\psi + \frac{q}{r^2}\,\mathrm{d}t\wedge\mathrm{d}r\,. \tag{1.9}$$

By integrating this expression on a sphere, one obtains the magnetic charge p and the electric charge q:

$$\frac{1}{4\pi}\int_{S^2}\mathcal{F} = p, \qquad \frac{1}{4\pi}\int_{S^2}\star_4\mathcal{F} = q\,. \tag{1.10}$$

The black-hole solution corresponding to a charged, non-rotating, spherically symmetric body is a Reissner–Nordström black hole. The metric for a body carrying total charge $Q = \sqrt{q^2+p^2}$ reads:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \tag{1.11}$$

[5] $S = k\ln N$, where k is the Boltzmann constant and N the total number of microstates of the system.

[6] The Maxwell's equations in the vacuum are invariant under electromagnetic duality. This symmetry exchanges the electric field with the magnetic one without modifying the form of the equations. In presence of sources, electromagnetic duality is preserved only if, besides electric charges, one introduces magnetic charges as well.

The event horizon is located where $g^{rr} = 0$, namely, at those points in space that solve the equation:

$$r^2 - 2Mr + Q^2 = 0\,. \tag{1.12}$$

Provided that $M > Q$, it has two zeros:

$$r_\pm = M \pm \sqrt{M^2 - Q^2}\,. \tag{1.13}$$

There is then in general an outer (true event) horizon, r_+, and an inner (Cauchy[7]) one r_-. The area of the event horizon is $4\pi r_+^2$.

Taking the limit $r \to r_+$ and performing suitable changes of coordinates one can analyze the near-horizon geometry and come out with the following expressions of the Bekenstein-Hawking temperature and entropy:

$$T = \frac{\kappa}{2\pi} = \frac{\sqrt{M^2 - Q^2}}{4\pi M(M + \sqrt{M^2 - Q^2}) - Q^2}\,, \tag{1.14}$$

$$S = \pi r_+^2 = \pi(M + \sqrt{M^2 - Q^2})^2\,. \tag{1.15}$$

Note that the formulae above imply automatically the bound

$$M \geq Q\,. \tag{1.16}$$

Indeed if $M < Q$ we meet several problems: the equations (1.12) would not have real solutions, the corresponding black hole would not have any horizon and we would be in presence of a so-called naked singularity. Furthermore we would be left with troubles in defining the thermodynamics of a such black hole and, most importantly, the previously given definition of the mass itself related to the ADM energy of the system would fail.[8] In order to be able to discard this possibility, Penrose stated the *cosmic censorship conjecture* [52] saying that naked singularities are not physically acceptable and never formed by the gravitational collapse of an celestial body .

[7] A Cauchy horizon can be intuitively understood as a region where causality breaks down. It is the boundary of a region beyond which the initial conditions of the universe, specified on a maximal spacelike hypersurface (Cauchy surface), are no longer sufficient to uniquely define the future. For a more detailed discussion the interested reader can refer to e.g. [43, 45, 51].

[8] As mentioned at the beginning of the section, the ADM energy measures, in a certain way, all the energy of the system. It cannot be that the corresponding mass, the ADM mass, satisfies $M < Q$ since it would mean that not all the electromagnetic energy has been taken in account.

The case $M = Q$ is of special importance and defines extremal black holes. The horizon, which is a double zero of (1.12), is placed at $r = Q$ and the line element becomes:

$$ds^2 = -\left(1 - \frac{Q}{r}\right)^2 dt^2 + \left(1 - \frac{Q}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 \,. \tag{1.17}$$

Concerning the thermodynamics, the entropy (1.15) has a finite non-zero value given by $S = \pi Q^2$ while the temperature (1.14) is zero. The latter result means that an extremal black hole does not radiate and for this reason it is stable against a possible decay through Hawking evaporation. In fact, it can be shown that a radiating black hole looses mass ("evaporates") until it satisfies the extremal condition $M = Q$.

We want now to briefly discuss the near-horizon geometry of extremal black holes. Let us introduce the coordinate $\rho = r - Q$ in which the line element (1.17) becomes:

$$\begin{aligned} ds^2 &= -e^{2U(\vec{x})} dt^2 + e^{-2U(\vec{x})} d\vec{x}^{\,2} \\ &= -\left(1 + \frac{Q}{\rho}\right)^{-2} dt^2 + \left(1 + \frac{Q}{\rho}\right)^2 (d\rho^2 + \rho^2 d\Omega^2) \,, \end{aligned} \tag{1.18}$$

In these new coordinates the horizon is located at $\rho = 0$ ($\leftrightarrow r = Q$) and the physical singularity at $\rho = -Q$ ($\leftrightarrow r = 0$). The asymptotic regime $\rho \to \infty$ remains the same as in the previous coordinates. The main feature of the geometry of an extremal black hole is that its near-horizon limit looks different from the usual Rindler one (1.6). In fact it turns to be:

$$ds^2 = -\frac{\rho^2}{Q^2} dt^2 + \frac{Q^2}{\rho^2} d\rho^2 + Q^2 d\Omega^2 \,. \tag{1.19}$$

The geometry still factorizes but now it is $S^2 \times \mathrm{AdS}_2$, i.e. the product of a two sphere times a two dimensional Anti-de Sitter spacetime (see for instance section 3.4.1 of [51]). An Anti-de Sitter spacetime is defined as a space with constant negative scalar curvature. It is indicated with AdS_n, where n is its dimension. The geodesic radial distance (that is, the physical distance) in AdS is $\ln \rho$ and thus near $\rho = 0$, the geometry looks like an infinite throat whose mouth has radius Q.

Finally, it is worth pointing out that the metric (1.18) for an extremal single-center[9] black hole can be easily generalized to the multicenter case in

[9] When we speak about centers we mean physical singularities.

which there are N charges Q_i at positions $\vec{x}_i$. It is enough to redefine e^{-U} as:

$$e^{-U} = 1 + \sum_{i=1}^{N} \frac{Q_i}{|\vec{x} - \vec{x}_i|} \,. \tag{1.20}$$

The total (ADM) mass, according to the definition of ADM energy as a function of the deviation of the metric from its prescribed asymptotic flat form, is given by the formula (where $\tau = |\vec{x}|^{-1}$):

$$M_{\mathrm{ADM}} = -\lim_{\tau \to 0} \frac{dU}{d\tau} \tag{1.21}$$

and turns out to be equal to the total charge $Q = \sum_i Q_i$. This class of black holes are Papapetrou–Majumdar metrics [53, 54].

1.4 The most general solution

The black holes presented so far are static and spherically symmetric. If we give up these conditions, the most general stationary, non-singular on and outside the event horizon, charged, black-hole solution turns out to be the Kerr-Newman metric. In Boyer–Lindquist coordinates, it results to be:

$$\begin{aligned} \mathrm{d}s^2 = &- \frac{(\Delta - a^2 \sin^2\theta)}{\xi} \, \mathrm{d}t^2 - 2a \frac{2Mr - Q^2}{\xi} \sin^2\theta \, \mathrm{d}t \, \mathrm{d}\phi \\ &+ \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\xi} \right) \sin^2\theta \, \mathrm{d}\phi^2 - \frac{\xi}{\Delta} \, \mathrm{d}r^2 + \xi \, \mathrm{d}\theta^2 \,, \end{aligned} \tag{1.22}$$

where

$$\Delta(r) = r^2 - 2Mr + a^2 + Q^2 \,, \qquad \xi = r^2 + a^2 \cos\theta \,, \qquad a = J/M \,. \tag{1.23}$$

It is a generalization of the Reissner–Nordström black hole describing a rotating charged mass with angular momentum J. The components of the metric are independent of t (stationarity) and ϕ (axisymmetry) that means that the spacetime is left invariant by translations along these coordinates. Like above, depending on the zeros of the equation $\Delta = 0$, extremal or non-extremal solutions are possible. A characteristic peculiar feature related with the rotation is the ergosphere. It is a region outside the event horizon of the form of an oblate spheroid where the component g_{tt} acts like a spatial component. This implies that particles inside the ergosphere co-rotate with the inner mass. Still they can escape the gravitational pull and, if they do, they can extract energy from the black hole (*Penrose process* [55]). For a general overview the interested reader can refer e.g. to the books [43, 45, 56]

Finally it is worth remarking that a stationary black hole is fully determined by only three local parameter: M, Q and J. This is the essence of the *no-hair theorem* [57] according to which all the information concerning the constituent matter, the birth process and objects at infinity do not affect the externally observable features of a black hole.

2 ...Einstein–Maxwell–scalar theories

There are both, extremal and non-extremal, charged black holes also in theories with scalars. In particular, following the treatment carried out in [20], we will consider the coupling between gravity, n_A abelian gauge fields A^Σ_μ ($\Sigma = 0, \ldots, n_A$) and n_ϕ massless, neutral, real, scalar fields ϕ^a ($a = 0, \ldots, n_\phi$). The action we will deal with has the form:

$$I = \int d^4x\sqrt{-G}\left(R - 2g_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b - f_{\Sigma\Lambda}(\phi)\mathcal{F}^\Sigma_{\ \mu\nu}\mathcal{F}^{\Lambda\mu\nu}\right.$$
$$\left. - \frac{1}{2}h_{\Sigma\Lambda}(\phi)\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^\Sigma_{\ \mu\nu}\mathcal{F}^\Lambda_{\ \rho\sigma}\right), \tag{1.24}$$

where g_{ab} is the metric for the real target space of the scalar fields called scalar manifold. In the action no explicit potential appears and, for this reason, the scalars are also named *moduli fields*. The functions $f_{\Sigma\Lambda}(\phi)$ and $h_{\Sigma\Lambda}(\phi)$ determine the gauge couplings, are independent of each other and are symmetric in the indices Σ, Λ. Note that if $h_{\Sigma\Lambda}(\phi)$ did not depend on the scalars, the last term in the Lagrangian would be integrated out, since $\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^\Sigma_{\ \mu\nu}\mathcal{F}^\Lambda_{\ \rho\sigma}$ is a total derivative.[10]

If we could leave the scalars out, a possible simple black-hole solution would be the direct generalization of the Reissner–Nordström one. It would have mass M and n_A electromagnetic charges (p^Σ, q_Σ). However from the structure of (1.24) one can see that the scalars do not decouple and more complicated solutions are expected.

In this framework, and for all types of Einstein–Maxwell–scalar theories, when considering extremal, static, black-hole solutions, an interesting phenomenon called *attractor mechanism* occurs and makes the behavior of the

[10] In the last term of the action (1.24) the symbol $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol of the curved four-dimensional spacetime with metric $G_{\mu\nu}$ defined as: $\epsilon^{\mu\nu\rho\sigma} = |G|^{-1/2}\varepsilon^{\mu\nu\rho\sigma}$. Here $\varepsilon^{\mu\nu\rho\sigma}$ is the usual "flat" anti-symmetric Levi-Civita symbol with the convention $\varepsilon^{0123} = -1$.

scalars so peculiar that can be a subject of study in its own right [15–17]. For this reason, in this section we are going to focus only on these black holes even if they are not the only possible solutions of the equations of motion.

2.1 Equations of motion

From (1.24) one can derive the equations of motion for the metric [20]:[11]

$$R_{\mu\nu} - 2g_{ab}\partial_\mu\phi^a\partial_\nu\phi^b = f_{\Sigma\Lambda}(\phi)\left(2\mathcal{F}^{\Sigma}_{\mu\lambda}\mathcal{F}^{\Lambda\lambda}_{\nu} - \frac{1}{2}G_{\mu\nu}\mathcal{F}^{\Sigma}_{\sigma\lambda}\mathcal{F}^{\Lambda\sigma\lambda}\right), \qquad (1.25)$$

for the scalars (with the notation $\partial_a = \frac{\partial}{\partial\phi^a}$, $\phi_a = g_{ab}\phi^b$):

$$\frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G}\partial^\mu\phi_a) = \frac{1}{4}\partial_a(f_{\Sigma\Lambda})\mathcal{F}^{\Sigma}_{\mu\nu}\mathcal{F}^{\Lambda\mu\nu} + \frac{1}{8}\partial_a(h_{\Sigma\Lambda})\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^{\Sigma}_{\mu\nu}\mathcal{F}^{\Lambda}_{\rho\sigma}, \qquad (1.26)$$

and for the gauge fields:

$$\partial_\mu\left[\sqrt{-G}\left(f_{\Sigma\Lambda}\mathcal{F}^{\Lambda\mu\nu} + \frac{1}{2}h_{\Sigma\Lambda}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda}_{\rho\sigma}\right)\right] = 0\,. \qquad (1.27)$$

The latter have to satisfy also the Bianchi identity:

$$\partial_\mu\mathcal{F}_{\nu\rho} + \partial_\nu\mathcal{F}_{\rho\mu} + \partial_\rho\mathcal{F}_{\mu\nu} = 0\,. \qquad (1.28)$$

In order to search for extremal charged solutions one assumes spherical symmetry. All the quantities will then depend only on r. The static metric ansatz and the field strength satisfying (1.27) and (1.28) are:

$$ds^2 = -A(r)^2\mathrm{d}t^2 + A(r)^{-2}\mathrm{d}r^2 + B(r)^2\mathrm{d}\Omega^2\,, \qquad (1.29)$$

$$\mathcal{F}^{\Sigma} = f^{\Sigma\Lambda}\left(q_\Lambda - h_{\Lambda K}\,p^K\right)\frac{1}{B^2}\mathrm{d}t\wedge\mathrm{d}r + p^{\Sigma}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\psi\,. \qquad (1.30)$$

$A(r)$ and $B(r)$ are the components of the metric to be determined through the equations of motion, while q_Σ and p^Σ are, respectively, the electric and

[11] Varying the Lagrangian in (1.24) with respect to the metric gives:

$$R_{\mu\nu} - 2g_{ab}\partial_\mu\phi^a\partial_\nu\phi^b = 2f_{\Sigma\Lambda}(\phi)\mathcal{F}^{\Sigma}_{\mu\lambda}\mathcal{F}^{\Lambda\lambda}_{\nu} + \frac{1}{2}G_{\mu\nu}\mathcal{L}\,.$$

The trace of this equations yields $R - 2g_{ab}\partial_\mu\phi^a\partial^\mu\phi^b = 0$. Using this result one obtains (1.25).

the magnetic (constant) charges carried by the electromagnetic gauge fields. In our notation $f^{\Sigma\Lambda}$ stands for the inverse of the matrix $f_{\Sigma\Lambda}$.

Using the equation of motion (1.25) one obtains [20]:

$$\begin{aligned} R_{tt} &= \frac{A^2}{B^4} V_{\text{eff}}(\phi)\,, \\ R_{\theta\theta} &= \frac{1}{B^2} V_{\text{eff}}(\phi)\,, \end{aligned} \tag{1.31}$$

where V_{eff} is the effective (or black-hole) potential defined as:

$$V_{\text{eff}}(\phi) = f^{\Sigma\Lambda}(\phi)\big(q_\Sigma - h_{\Sigma\Gamma}(\phi)\, p^\Gamma\big)\big(q_\Lambda - h_{\Lambda\Gamma}(\phi)\, p^\Gamma\big) + f_{\Sigma\Lambda}(\phi) p^\Sigma p^\Lambda\,. \tag{1.32}$$

This last quantity is a function only of the charges and the scalars and, as we will see, it plays a central role in discussing the attractor mechanism. It is proportional to the energy density of the electromagnetic field and invariant under duality transformations exchanging electric with magnetic fields.

After calculating explicitly the components of the Ricci tensor using the above metric ansatz, one can write down the relation $R_{tt} = \frac{A^2}{B^2} R_{\theta\theta}$ (read from (1.31)) as:

$$\big(A^2(r)B^2(r)\big)'' = 2\,, \tag{1.33}$$

where the prime denotes differentiation with respect to r.

Analogously, by (1.25) and the explicit expression of the components of the Ricci tensor, one can write down the combination $R_{rr} - \frac{G^{tt}}{G^{rr}} R_{tt}$ as:

$$\frac{B''}{B} = -g_{ab}(\phi^a)'(\phi^b)'\,. \tag{1.34}$$

In the same way, for the component R_{rr} itself one obtains:

$$-1 + A^2 B'^2 + \frac{A^{2'} B^{2'}}{2} = -\frac{1}{B^2} V_{\text{eff}} + A^2 B^2 g_{ab}(\phi^a)'(\phi^b)'\,. \tag{1.35}$$

Finally, with the use of the ansatz for the metric, the expression of the field strength and the formula for V_{eff}, the equation of motion for the scalars becomes:

$$\partial_r(A^2 B^2 \partial_r \phi_a) = \frac{\partial_a V_{\text{eff}}(\phi)}{2B^2}\,. \tag{1.36}$$

From here it is clear why V_{eff} is called “effective potential”. Indeed it has to be mentioned that the equations of motion above for the scalars and the A

and B components of the metric can be derived from the one-dimensional effective action [20]

$$I_{\text{eff}} = \frac{2}{k^2}\int \mathrm{d}r \left((A^2B)'B' - A^2B^2(\phi')^2 - \frac{V_{\text{eff}}(\phi)}{B^2} \right) \tag{1.37}$$

constrained by equation (1.35).

2.2 Stability of attractors

Looking at the final equation of motion for the scalars (1.36), one can easily realize that a possible solution is given by $\phi^a = \phi_0^a$ (constants) where $\phi_0 = (\phi_1, \ldots, \phi_{n_\phi})$ extremizes the potential:

$$\partial_a V_{\text{eff}}(\phi)|_{\phi_0} = 0\,. \tag{1.38}$$

The possibility to satisfy such condition depends on the charges appearing in the potential. If ϕ_0 exists, it will be a particular combination of the electromagnetic charges $Q = (p^\Sigma, q_\Sigma)$ (we leave for later discussion the eventuality of flat directions in the potential). The scalar solution will be achieved simply by putting $\phi^a(r) = \phi_0^a(Q)$ everywhere and we will speak of *double-extremal black hole* [58–60].

Now, if ϕ_0 is not only an extremum but also satisfies

$$\Pi_{ab} = \frac{1}{2}\partial_a\partial_b V_{\text{eff}}(\phi)|_{\phi_0} > 0 \qquad \forall\, a, b \tag{1.39}$$

(i.e. ϕ_0 is a minimum of V_{eff}), the attractor mechanism takes place: there exists an extremal, charged black-hole solution for the metric which depends on the scalars only through their value on the event horizon. The scalars evolve radially from spatial infinity until reaching the event horizon and, independently of their asymptotic conditions (in the limit of small deviations), they assume on it fixed values equal to $\phi_0^a(Q)$. Whenever this happens we talk about a *stable attractor*.

It is important to stress that the attractor mechanics is characteristic of extremal black holes only. It is due to the special structure of the extremal near-horizon geometry that the scalars take an infinite amount of the “evolution parameter” to reach the horizon and have then enough “time” to “forget” about their asymptotic values. To better figure it out one can make an analogy with systems in classical mechanics exhibiting attractor behavior like for instance the under-damped harmonic oscillator. Its motion is described by $x(t) = ke^{-\gamma t}\cos(ft + \psi_0)$ and the attractor behavior is always manifested as, independently from the starting position, the fixed point $x_{\mathrm{fix}} = 0$ is reached when $t \to \infty$. In our gravitational context the role of the evolution parameter t is played by the physical distance from the horizon and scalars acquire on it values independent of their initial conditions.

From (1.33) and (1.34), by putting $\phi^a(r) = \phi_0^a$ one obtains the following solutions for the components of the metric:

$$A_0(r) = \left(1 - \frac{r_{\mathrm{h}}}{r}\right) , \quad B_0(r) = r \, . \tag{1.40}$$

Inserting this results in equation (1.35) yields [20]:

$$r_{\mathrm{h}}^2 = V_{\mathrm{eff}}(\phi_0) \, . \tag{1.41}$$

The horizon radius is given by the minimum value of the effective potential. It follows that the entropy of the black hole is

$$S = \frac{A_{\mathrm{h}}}{4} = \pi V_{\mathrm{eff}}(\phi_0(Q)) \, . \tag{1.42}$$

This is a very important result: at the end, in agreement with the no-hair theorem, the thermodynamics of an extremal, charged black hole is completely determined by its charges. The attractor mechanism implies that the entropy does not depend on variable parameters at infinity. Instead, it is expressed in terms of the event-horizon values of the scalars that, in turn, are function only of the charges of the black hole. The solution (1.40) for the components of the metric and equation (1.41) defining the horizon radius describe an extremal black hole with mass parameter $M^2 = V_{\mathrm{eff}}(\phi_0)$. Differently from what happened in absence of scalars, here the mass of an extremal black hole is no longer simply equal to the total charge: it is the specific combination of q_Σ and p^Σ that minimizes the black-hole potential.

In [20] it was shown that an extremal black-hole solution continues to exist and behaves like an attractor for small deviations of the scalars from ϕ_0^a at infinity. This would not work if condition (1.39) was not met. In fact if the critical point is a maximum, $\phi^a(r) = \phi_0^a$ is still a solution of the equation of motion but even a tiny deviation at infinity from this value will prevent the scalars from being equal to ϕ_0^a on the horizon. Depending on the shape of the effective potential and from their initial value, the scalars will evolve till they reach the nearest local minimum (of V_{eff}) in the scalar manifold. In this case we say that we are in presence of an *unstable attractor*.

A third possible case is when it holds that $\Pi_{ab} = 0$ for some a, b. To study the stability of the attractor, one needs to understand whether one is dealing with a maximum or a minimum. In view of that it is necessary to look at the first non-vanishing derivative of the effective potential evaluated at the attractor point: only if it is positive the critical point is a minimum and the attractor is stable [21, 61]. However it might also happen that the derivatives vanish at all orders. Then flat directions in V_{eff} appear and it means that there is a continuum of critical points which can transform to one another depending on the symmetry group of the potential. The stability of the attractor depends on the behavior of the potential along the directions orthogonal to the flat ones. The constant values ϕ_0^a that extremize V_{eff} are now dependent on the charges and on the asymptotic values of the scalars as well. However, the dependence on the values of the moduli at infinity cancels out in the expression of the thermodynamical properties of the black hole. Indeed, in the case of stable attractors, the possibility to choose the attractor point among a continuum of values does not affect the entropy (1.42) as, by definition, along a flat direction the potential is constant.

2.3 Symmetries and flat directions in moduli space

Flat directions can be analyzed through geometrical considerations concerning the structure and the symmetry of the scalar manifold [62, 63]. This is a quite large and non-simple argument and here we want to opt for a more practical approach just to give a general idea for understanding the origin of flat directions [64]. We focus on the scalar manifolds of models of $\mathcal{N} = 2$ supergravity in 4 dimensions whose moduli spaces are special Kähler, homogeneous symmetric manifolds.

A homogeneous symmetric scalar manifold X is isomorphic to the coset space G/H, where G is a continuous symmetry group (called *duality group*) which acts on X in a transitive way (i.e. there is a unique orbit of G in X) and H is its maximal compact subgroup. The symmetry transformations defined by G are invertible everywhere in X and the action of a group isomorphic to H leaves invariant each point of the manifold. The duality group acts on the charge vector $Q = (p^\Sigma, q_\Sigma)$ as a linear symplectic transformation.

The crucial point is that the effective potential is left invariant by the action of G on both the scalars and the charges. This is important because one can then retrieve any scalar solution, corresponding to a certain charge configuration, starting from any other, corresponding to a different charge configuration, and by acting on it with a suitable element of G. Practically, to obtain the scalar solution ϕ_{C_2} corresponding to the charge configuration C_2 given C_1 and ϕ_{C_1}, one has first to find the element of G whose action transforms C_1 in C_2. Then acting with this same element on ϕ_{C_1} one directly obtains ϕ_{C_2}. This procedure is very useful since it may happen that the scalar equations of motion are simple in a certain charge configuration but very complicated in the others. Thanks to the symmetry of the system, one has to solve only for the easiest case and apply the suitable transformations to obtain all the other solutions.

There can be elements of G which act trivially on the charges or on the scalars. When they exist, flat directions may arise. We have already said that the symmetry group that leaves the scalars invariant (but not generally the charges) is $H \subset G$. Let $\hat{H} \subset G$ be the subgroup which acts trivially on the charges (but not generally on the scalars). With the definition $h_0 = \hat{H} \cap H$, the coset $\hat{H}/h_0$ is the non-trivial scalar submanifold generated by duality transformations that leave the charges invariant. The physical meaning of this coset is that it corresponds to different scalar solutions with the same charges. If $\hat{H}/h_0$ is not trivial at the horizon, the attractor value of the scalars is not uniquely determined by the charges.

Since H is the maximal compact subgroup of G, $h_0 = H \cap \hat{H}$ is compact. Furthermore on the horizon, h_0 is enhanced to the maximal compact subgroup H_0 of $\hat{H}$. There are then $\dim \hat{H} - \dim H_0$ generators of transformations which leave the charges invariant but not the attractor value of the scalars and at the end this means that there are $\dim \hat{H} - \dim H_0$ flat directions in the black-hole potential. In a suitable coordinate system each basis element of the coset $\hat{H}/H_0$ generates a transformation which simply translates points in moduli space along a certain flat direction of the potential.

3 ...$N = 2$ ungauged Supergravity

Supergravity was proposed in the seventies as a quantum theory of gravity with the hope to unify general relativity with the standard model. Its origin is related with the gauging of global supersymmetry, which is the unique framework where fields of different spins are organized in representations (super-multiplets) of an algebraic system called Poincaré superalgebra. Bosons and fermions are mapped one into another by symmetry transformations with constant spinor parameters and spin-1/2 odd generators called supercharges (for a review see e.g. [65]). Making supersymmetry local, because of Poincaré superalgebra, implies the introduction of gravity and from here the elaboration of a gauge theory invariant under local spacetime symmetries as well [66, 67].

After an initial enthusiasm followed from the first positive results, supergravity was left aside since encountered, among other problems, to be a non-renormalizable theory. By its own it should be then discarded but, seen as the low energy limit of string theory, it acquires again interest and reasons to be explored. String theory is in fact thought nowadays to be the most promising candidate of a theory of quantum gravity, and supergravity provides its low energy effective description (black holes find here their proper collocation).

There are five consistent superstring theories in ten dimensions. They are type I, type IIA, type IIB and heterotic $SO(32)$ and $E_8 \times E_8$. They are connected by dualities[12] which prove them physical equivalent. Indeed it seems that they can be obtained by compactifying on a circle an underlying, more fundamental, 11-dimensional description named M-theory. Its low energy limit is the unique $N = 1$, $D = 11$ supergravity that, properly compactified on a seven-dimensional manifold, gives rise to four-dimensional supergravities with up to 8 supercharges. The number of supersymmetries is indicated with N and the higher it is, the more constrained is the theory. We are going to focus on $N = 2$, since it leaves us sufficient freedom, without being too difficult to deal with (didactic references could be for instance [68–71]).

3.1 The theory

The general multiplet content of $N = 2$, $D = 4$ supergravity counts one supergravity multiplet $(e^i_\mu, \Psi^I_\mu, A^0_\mu)$, $I = 1, 2$, with spins (2, 3/2, 1), n_v vector multiplets $(A^a_\mu, \lambda^{a\,I}, z^a)$, $a = 1, \ldots, n_v$, with spins (1, 1/2, 0) and n_h hypermultiplets (χ^α, ϕ^u), $\alpha = 1, \ldots, 2n_h$, $u = 1, \ldots, 4n_h$, with spins (1/2, 0). However, if we look for classical black-hole solutions, we can set to zero the fermionic fields and we can neglect the hypermultiplets since they do not affect our discussion. The remaining part of the action contains the familiar Einstein–Hilbert term for gravity, n_v neutral complex scalars z^a (we will use an over-bar to indicate complex conjugation) and $n_v + 1$ abelian

[12] There are two types of dualities: T-dualities that link different string theories on different spacetime geometry and S-dualities that relate weakly coupled string theories to strong coupled ones.

gauge fields A_μ^Σ, $\Sigma = 1\ldots, n_v + 1$ (the extra gauge field is the graviphoton of the gravity multiplet) [72, 73]:

$$I_{4D} = \frac{1}{16\pi}\int\Big(R\star 1 - 2\,g_{a\bar{b}}(z,\bar{z})\,\mathrm{d}z^a\wedge\star\mathrm{d}\bar{z}^{\bar{b}} \\ - \mathrm{Im}\,\mathcal{N}_{\Sigma\Lambda}(z,\bar{z})\,F^\Sigma\wedge\star F^\Lambda - \mathrm{Re}\,\mathcal{N}_{\Sigma\Lambda}(z,\bar{z})\,F^\Sigma\wedge F^\Lambda\Big). \quad (1.43)$$

By comparing with (1.24), it is evident that this theory is an Einstein–Maxwell–scalar theory. It follows that the results previously discussed apply here as well.

What makes this setting more interesting is that the scalar manifold is (projective) *special Kähler*, which means that it can be found a suitable symplectic frame [74] where its metric and the gauge couplings $\mathcal{N}_{\Sigma\Lambda}$ follow from a single holomorphic function $F = F(X)$.[13] It is called the prepotential and usually is expressed in terms of the homogeneous coordinates of the scalar manifold related with the affine ones by $z^a = \frac{X^a}{X^0}$. One can then define the covariantly holomorphic symplectic section (or period vector) with Kähler weight $(1, -1)$:

$$\mathcal{V} = \mathrm{e}^{\frac{\mathcal{K}}{2}}\begin{pmatrix} X^\Sigma \\ \partial_\Sigma F\end{pmatrix} \quad (1.44)$$

and the Kähler potential[14] (with the notation $\partial_\Sigma = \frac{\partial}{\partial X^\Sigma}$)

$$\mathcal{K} = -\ln\left[\mathrm{i}\left(\bar{X}^\Sigma\partial_\Sigma F - X^\Sigma\overline{\partial_\Sigma F}\right)\right] =: -\ln\left(\mathrm{i}\langle\bar{\mathcal{V}},\mathcal{V}\rangle\right)\,. \quad (1.47)$$

[13] According to [74], a duality invariant definition of "special Kähler manifold" consists in the requirement of the existence of a holomorphic symplectic section $v(z)$ such that the Kähler potential can be written as $\mathrm{e}^{-\mathcal{K}(z,\bar{z})} = -\mathrm{i}\langle v,\bar{v}\rangle$ and such that it holds the condition $\langle\mathcal{D}_a v,\mathcal{D}_b v\rangle = 0$ (where $\mathcal{D}_a v = \partial_{z^a} v + \partial_{z^a}\mathcal{K}\,v$ is the kähler covariant derivative and $\langle\,,\,\rangle$ the symplectic inner product defined implicitly in (1.47)).

[14] $\mathcal{K}$ is determined up to Kähler transformations

$$\mathcal{K}(z,\bar{z}) \to \mathcal{K}(z,\bar{z}) + f(z) + \bar{f}(\bar{z})\,, \quad (1.45)$$

where f is an arbitrary holomorphic function. Such a transformation does not affect the properties of the Kähler manifold. In general a generic vector v transforms under Kähler transformations as

$$v(z,\bar{z}) \to \exp\Big\{-\tfrac{1}{2}\big[pf(z) + \bar{p}\bar{f}(\bar{z})\big]\Big\}\,v(z,\bar{z})\,, \qquad p,\bar{p}\in\mathbb{R} \quad (1.46)$$

and is said to have Kähler weights $(p,\bar{p})$.

The charges are now encoded in a symplectic vector

$$\mathcal{Q} = \begin{pmatrix} p^{\Sigma} \\ q_{\Sigma} \end{pmatrix}, \tag{1.48}$$

and the metric of the moduli space and the gauge couplings are ($\partial_a = \frac{\partial}{\partial z^a}$):

$$g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} \mathcal{K}\,, \tag{1.49}$$

$$\mathcal{N}_{\Sigma\Lambda} = \overline{\partial_{\Sigma}\partial_{\Lambda} F} + 2\mathrm{i} \frac{\mathrm{Im}(\partial_{\Sigma}\partial_{\Gamma} F)\, \mathrm{Im}(\partial_{X^{\Lambda}}\partial_{X^{\Xi}} F) X^{\Xi} X^{\Gamma}}{\mathrm{Im}(\partial_{\Xi}\partial_{\Gamma} F) X^{\Xi} X^{\Gamma}}\,. \tag{1.50}$$

3.2 BPS and non-BPS solutions

Black-hole solutions in supergravity have the additional property to preserve supersymmetry or not. In order to discuss it, we begin by displaying the anti-commutation rule of two supersymmetry generators Q_i, Q_j

$$\{Q_i,\, Q_j\} = (\delta_{ij}\gamma^{\nu}\mathcal{P}_{\nu} + \varepsilon_{ij}\mathcal{Z})\,\mathcal{C}^{-1}\,, \qquad i,j = 1,2\,, \tag{1.51}$$

where $\mathcal{P}_{\nu}$ is the generator of spacetime translations, $\mathcal{C}$ the charge conjugation matrix and $\mathcal{Z}$ the central charge of the $N = 2$ supersymmetry algebra (i.e. the element that commutes with all the others). One can find a combination $\hat{Q}$ of the supersymmetry generators such that, when anti-commuted with its complex conjugate, gives an expression in terms of the mass M and the central charge $\mathcal{Z}$. Requiring $\{\hat{Q}, \bar{\hat{Q}}\} \geq 0$ leads to a relation between the mass and the central charge of the form [75]

$$M \geq |\mathcal{Z}| \tag{1.52}$$

known as *BPS-bound*.[15] The BPS-bound could be seen as the supergravity correspondent of (1.16) for Reissner–Nordström black holes. From the expression of $\{\hat{Q}, \bar{\hat{Q}}\}$, one can see that when a solution is supersymmetric the bound is saturated (from here the nomenclature BPS black holes for supersymmetric solutions). As discussed in subsection 3.3, BPS black holes are always extremal, while non-supersymmetric (non-BPS) solutions (with $M > |\mathcal{Z}|$) can be extremal or non-extremal.

[15] The name BPS is due to the works of Bogomol'nyi, Prasad and Sommerfield [76,77]. It usually refers to a more general concept related to the rewriting of the energy (functional of the fields describing the solutions) as a positive integral plus a boundary term.

The central element of the supersymmetry algebra, in $N = 2$ four-dimensional supergravity, can be associated with a function of the electromagnetic charges and the moduli defined as:

$$Z(\mathcal{Q}, z, \bar{z}) = \mathrm{e}^{\mathcal{Z}/2} \left(p^\Sigma \partial_\Sigma F - q_\Sigma X^\Sigma \right) . \tag{1.53}$$

This object, since at spatial infinity it holds $|\mathcal{Z}| = |Z(\mathcal{Q}, z_\infty, \bar{z}_\infty)|$, inherits the name central charge and, as it will be clearer later, plays an important role in the physics of black-hole solutions.

Generally, a solution is symmetric if it is left invariant by a symmetry transformation. Accordingly, in the present context, a solution is supersymmetric if its infinitesimal variation under (part of) the supersymmetry transformations is vanishing. Since we have set the fermions to zero, supersymmetry variations of bosons are trivial and do not provide any additional information. One has then to consider only the transformations of the fermionic fields. When equated to zero, they become the so-called Killing spinor equations. They are a set of first-order differential equations for the bosonic fields and a set of differential equations and projection conditions for the spinor parameter of the supersymmetry. It can be shown that solving them for black-hole-type solutions implies extremality and absence of NUT charge, local angular momentum and scalar hairs [78, 79]. Moreover it turns out that only one of the two initial supersymmetries is conserved and for this reason supersymmetric solutions are said to be 1/2-BPS. Also, it can be proved that for all models the scalars on the horizon always assume values independent of their asymptotic conditions, meaning that no flat direction exists for supersymmetric solutions [15, 18, 80].

Non-supersymmetric solutions are much less constrained and for this reason they are more difficult to be explored. A complete analysis is still missing but, thanks to the efforts in the last years, several types of non-BPS black holes are now known (see table 1.1 for a partial classification). Like for the supersymmetric case, there exists a first-order formalism reproducing the motion of the scalars (see subsection 3.3) and the extremal solutions exhibit the attractor mechanism (eventually with flat directions). Extremal, rotating, non-BPS black holes have been found and classified depending on the property of having a continuous limit to static solutions [33, 34, 81–87]. Static, spherically symmetric, non-extremal black holes have been shown to interpolate between extremal non-BPS and BPS black holes [38, 88] and very recently a similar conclusion has been discussed also for rotating non-extremal solutions [89].

	Supersymmetric	**Non-Supersymmetric**	
	extremal	extremal	non-extremal
BPS ($M = \|Z\|$)	✓	＼	＼
attractor mechanism	✓	✓	＼
first-order description	✓	✓	✓
rotating	＼	✓	✓
multicenter	✓	✓	?

Tab. 1.1: Classification of the known black-hole solutions of $N = 2$, $D = 4$ supergravity. The checkmark and the diagonal bar respectively confirms or excludes the existence of black holes with the considered property. The question mark means that yet nothing definitive can be said about that kind of solution.

Besides single-center solutions, also extremal, multicenter, BPS and non-BPS black holes have been studied. They can be bound systems with global angular momentum (the centers can move one respect to the other but their relative positions are constrained) or composite with vanishing bounding energy whose constituents can freely move in the space. The flow of the scalars can still be described by first-order differential equations and it can split in moduli space while crossing surfaces called walls of marginal stability. In general, supersymmetric, multicenter black holes have only supersymmetric components while non-supersymmetric compounds can have in principle BPS or non-BPS centers. The mathematical descriptions of these solutions is quite technical and we invite the interested reader to check [28, 29, 31–34] and references therein.

3.3 Spherical symmetry and the FGK formalism

Assuming spherical symmetry and staticity simplifies considerably the study of black-hole solutions in the theory (1.43). These assumptions are sufficient to explore the entire single-center BPS branch and enough to review most of the known non-BPS landscape. The ansatz one should make for the metric is:

$$\mathrm{d}s^2 = -\mathrm{e}^{2U(\tau)}\mathrm{d}t^2 + \mathrm{e}^{-2U(\tau)}\left(r_0^4 \sinh^{-4}(r_0\tau)\mathrm{d}\tau^2 + r_0^2 \sinh^{-2}(r_0\tau)\mathrm{d}\Omega^2\right) , \tag{1.54}$$

with the condition that $\mathrm{e}^{2U}|_{r=\infty} = 1$ in order to guarantee asymptotic flatness. With this ansatz, both spherically symmetric extremal and non-extremal black holes can be discussed depending on the value of the extremality parameter $r_0^2 = 2ST$ [90] that is zero only for the former class of solutions. The new radial coordinate τ is related to the usual one r by

$$\frac{r_0^2}{\sinh^2(r_0\tau)} = (r - r_\mathrm{h})^2 - r_0^2 = (r - r_-)(r - r_+) , \tag{1.55}$$

so that the horizon is placed at $\tau \to \infty$ while spatial infinity corresponds to $\tau \to 0$. The only degree of freedom of the metric to be determined by the equations of motion is the real function $U(\tau)$ usually called warp factor.

Ferrara, Gibbons and Kallosh (from here the name FGK formalism) showed in [18] that, under the assumptions made, the dynamics of the four-dimensional action (1.43) is correctly reproduced by the one-dimensional effective theory (the dot meaning differentiating with respect to τ):

$$I_\mathrm{FGK} = -\frac{1}{2}\int_0^\infty d\tau \left(\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} + e^{2U}V_\mathrm{BH}\right) , \tag{1.56}$$

subject to the constraint (associated with the conservation of energy)

$$\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} - e^{2U}V_\mathrm{BH} = r_0^2 . \tag{1.57}$$

This derivation can be extended to any Einstein–Maxwell–scalar theory as already suggested at the end of subsection 2.1 by formulae (1.35) (here corresponding to (1.57) and coming from the Einstein equations) and (1.37) (here (1.56)). The potential term comes from the gauge fields part of the original action and, thanks to special geometry, it can be written by the double contraction of a scalar-dependent, real, symplectic, symmetric matrix $\mathcal{M}(\mathcal{N}(z,\bar{z}))$ with the charge vector $\mathcal{Q}$:

$$\begin{aligned} V_\mathrm{BH}(\mathcal{Q}, z, \bar{z}) &= -\tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\mathcal{M}_{MN}(\mathcal{N}(z,\bar{z})) \\ &= -\tfrac{1}{2}(p^\Lambda \; q_\Lambda)\begin{pmatrix} (\mathfrak{I} + \mathfrak{R}\mathfrak{I}^{-1}\mathfrak{R})_{\Lambda\Sigma} & -(\mathfrak{R}\mathfrak{I}^{-1})_\Lambda{}^\Sigma \\ -(\mathfrak{I}^{-1}\mathfrak{R})^\Lambda{}_\Sigma & (\mathfrak{I}^{-1})^{\Lambda\Sigma} \end{pmatrix}\begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix} , \end{aligned} \tag{1.58}$$

where $\mathfrak{R}_{\Sigma\Lambda} = \mathrm{Re}\,\mathcal{N}_{\Sigma\Lambda}$ and $\mathfrak{I}_{\Sigma\Lambda} = \mathrm{Im}\,\mathcal{N}_{\Sigma\Lambda}$ (comparing with what in subsection 2.1 one can easily realize that V_BH is the equivalent of effective potential

V_{eff} defined in (1.32) and that $\mathfrak{R}_{\Sigma\Lambda}$, $\mathfrak{I}_{\Sigma\Lambda}$ correspond to the functions $f_{\Sigma\Lambda}$, $h_{\Sigma\Lambda}$ introduced in (1.24)).

The equations of motion for the scalars and the warp factor are a set of coupled second-order differential equations of the form:

$$\ddot{U} - \mathrm{e}^{2U} V_{\text{BH}} = 0\,, \tag{1.59}$$

$$\ddot{z}^a + g^{a\bar{b}} \partial_c g_{d\bar{b}} \dot{z}^c \dot{z}^d - \mathrm{e}^{2U} g^{a\bar{b}} \bar{\partial}_{\bar{b}} V_{\text{BH}} = 0\,. \tag{1.60}$$

In general they are difficult to solve and usually they are not simple enough to be integrated directly.

First-order equations

As already mentioned in the previous subsection, it is possible to write first-order equations for the fields that imply the second-order ones. Historically this was achieved for the first time by requiring some supersymmetry to be conserved [15, 17]. Such a requirement turns out to be mathematically expressed by the Killing spinor equations, obtained from the supersymmetry transformation rules of the chiral gravitino and gaugino fields of the theory. Since the supersymmetry transformation rules are linear in the first derivatives of the fields, imposing the vanishing of the chiral gravitino and gaugino variations gives actually a system of first-order differential equations. This, by construction, helps only in the supersymmetric case and does not say anything when supersymmetry is not conserved.

In order to implement a first-order description for all types of black hole one has to turn to a more general method. It is based on rewriting the effective Lagrangian as a sum of squares (Bogomol'nyi trick) [18]. If it is possible to write

$$\mathcal{L} = \sum_i (\dots)_i^2\,, \tag{1.61}$$

seeking for the equations of motion through the usual variational method, one ends up with the vanishing of the sum of each term times the corresponding variation:

$$0 = \delta\mathcal{L} = \sum_i (\dots)_i \delta(\dots)_i\,. \tag{1.62}$$

Putting to zero each term of the sum yields a solution for the stationary points of the action. If (1.61) is a strict sum of positive terms, as it happens in the case we are interested in, the stationary points of the action are extrema. This procedure, simple to understand but frequently difficult to carry out, gives the sought first-order equations. The Hamiltonian constraint should be in principle imposed separately since we are dealing with a constrained system. However, the method proposed and discussed in [22] is proved to provide in a single step first-order equations that imply both the Hamiltonian constraint and the original second-order equations of motion.

In the framework we are dealing with, the fundamental observation is that the black-hole potential can be rewritten in the quadratic form

$$\mathrm{e}^{2U} V_{\mathrm{BH}} = (\partial_U Y)^2 + 4 g^{a\bar{b}} \partial_{z^a} Y \partial_{\bar{z}^{\bar{b}}} Y - r_0^2 \tag{1.63}$$

by the introduction of a real positive function $Y = Y(\mathcal{Q}, z, \bar{z}, U, r_0)$ called *generalized superpotential* (see [22] for the extremal case and [26] for the non-extremal one). So, the effective Lagrangian, up to a total derivate, is equivalent to the sum of squares:

$$\mathcal{L}_{\mathrm{FGK}} = \left(\dot{U} \pm \partial_U Y\right)^2 + \left| \dot{z}^a \pm 2 g^{a\bar{b}} \partial_{\bar{z}^{\bar{b}}} Y \right|^2 \tag{1.64}$$

from which, straightforwardly, one obtains the first-order equations:

$$\dot{U} = \pm \partial_U Y \,, \tag{1.65}$$

$$\dot{z}^a = \pm 2 g^{a\bar{b}} \partial_{\bar{z}^{\bar{b}}} Y \,. \tag{1.66}$$

The sign must be chosen in agreement with the definition of the mass and for us, by (1.21), it must be a minus. If $Y = \mathrm{e}^U |Z(\mathcal{Q}, z, \bar{z})|$, the flow equations define the evolution of the fields of a supersymmetric black hole, since they imply the relevant Killing spinor equations [15]. Other possibilities describe instead non-supersymmetric black holes, which are extremal whenever $Y = \mathrm{e}^U \mathcal{W}(\mathcal{Q}, z, \bar{z})$, with $\mathcal{W} \neq |Z|$ being the so-called superpotential whose explicit expression depends on the model [22]. In the non-extremal case the generalized superpotential is a model-dependent complicated function which does not factorize but such that, in the limit $r_0 \to 0$, it allows to obtain both the supersymmetric and non-supersymmetric extremal first-order flow equations [26, 38].[16]

[16] It is worth stressing that, at least in the extremal case, the method of [26,38] does not contain more information than that of [22], which automatically also takes into account the Hamiltonian constraint and thus yields the correct first-order equations to solve the system.

BPS bound and extremal attractor

The first-order equations above can be used to re-derive the attractor mechanism and reproduce, without using the superalgebra, the BPS-bound. For the first issue, it is enough to observe that in the extremal case, the absolute value of the central charge or the non-supersymmetric superpotential, depends on τ only trough the scalars. It then follows (we write W to indicate both $\mathcal{W}$ and $|Z|$):

$$\partial_\tau W = \partial_a W \dot{z}^a + \bar{\partial}_{\bar{a}} W \dot{\bar{z}}^{\bar{a}} \,, \tag{1.67}$$

and, by equation (1.66) $\dot{z}^a = -2\mathrm{e}^U g^{a\bar{b}} \bar{\partial}_{\bar{b}} W$, one achieves:

$$\partial_\tau W = -4e^U g^{a\bar{b}} \partial_a W \bar{\partial}_{\bar{b}} W \leq 0 \,. \tag{1.68}$$

This inequality proves that $W(\tau)$ is a monotonically decreasing function of τ (as τ increases, it becomes smaller and smaller). Moreover, considering as well the flow of the scalars, it is not difficult to see

$$\partial_\tau W = 0 \quad \Leftrightarrow \quad \partial_a W = 0 \quad \Leftrightarrow \quad \dot{z}^a = 0 \,, \tag{1.69}$$

which on the horizon ($\tau = \infty$) implies:

$$\begin{aligned} &\partial_a V_{\mathrm{BH}}|_{\tau=\infty} = \Big[W \partial_a W + 4 g^{b\bar{c}} \big(\partial_a \partial_b W \bar{\partial}_{\bar{c}} W + \partial_b W \partial_a \bar{\partial}_{\bar{c}} W \big) \Big]_{\tau=\infty} = 0 \,, \\ &V_{\mathrm{BH}}|_{\tau=\infty} = W^2|_{\tau=\infty} \,. \end{aligned} \tag{1.70}$$

This is practically the essence of the attractor mechanism: the scalars, independently from their asymptotic conditions, evolve in moduli space and assume on the horizon a value such to minimize the black-hole potential. Concerning the stability of the attractors it can be shown that all supersymmetric extrema are minima and that in general, for all supergravities based on homogeneous scalar manifolds, all the extremal (supersymmetric and non-supersymmetric) black hole solutions are stable attractors, up to possible flat directions for the non-BPS case [91].

Concerning the BPS-bound, its derivation follows (as in [92]) from evaluating the Hamiltonian constraint (1.57) at spatial infinity ($\tau = 0$) and by using (1.63) with $Y = \mathrm{e}^U |Z|$ ($r_0 = 0$) and the definition of the mass $U|_{\tau=0} = -M$:

$$\begin{aligned} M^2 &= r_0^2 + |Z_{\tau=0}|^2 + g_{a\bar{b}} \left[4 g^{a\bar{c}} \bar{\partial}_{\bar{c}} |Z| g^{\bar{b}c} \partial_c |Z| - \dot{z}^a \dot{\bar{z}}^{\bar{b}} \right]_{\tau=0} \\ &\geq |Z_{\tau=0}|^2 + g_{a\bar{b}} \left[4 g^{a\bar{c}} \bar{\partial}_{\bar{c}} |Z| g^{\bar{b}c} \partial_c |Z| - \dot{z}^a \dot{\bar{z}}^{\bar{b}} \right]_{\tau=0} \,. \end{aligned} \tag{1.71}$$

It is easy to realize that the inequality becomes an equal sign only in the extremal case. Furthermore one can see that only for supersymmetric solutions (by (1.66)) it results that $M = |Z_{\tau=0}|$.

Part II

PUBLISHED ARTICLES

2. NON-SUPERSYMMETRIC EXTREMAL MULTICENTER BLACK HOLES WITH SUPERPOTENTIALS

Pietro Galli and Jan Perz[◇]

[◇]*Afdeling Theoretische Fysica, Katholieke Universiteit Leuven*
Celestijnenlaan 200D bus 2415, 3001 Heverlee, Belgium.

Publication reference: *JHEP* **02** (2010) 102, arXiv:0909.5185 [hep-th].

Abstract

Using the superpotential approach, we generalize Denef's method of deriving and solving first-order equations describing multicenter extremal black holes in four-dimensional $\mathcal{N} = 2$ supergravity to allow for non-supersymmetric solutions. We illustrate the general results with an explicit example of the *stu* model.

1 Introduction

Most of the insight that we have gained into the origin of black hole entropy comes from the analysis of supersymmetric solutions in supergravity and string theory, an important class of which are multicenter black holes (the Majumdar–Papapetrou solutions [53,93] in the Einstein–Maxwell theory can be seen as their precursors). In four-dimensional $\mathcal{N} = 2$ supergravity coupled to vector multiplets, the general stationary supersymmetric multi-black holes, also known as black hole composites, were obtained by Denef [28] (extending results of Behrndt, Lüst and Sabra [94]; higher-curvature corrections were taken into account by Cardoso, de Wit, Käppeli and Mohaupt [95]). To better understand black holes that are not supersymmetric, however, it is desirable to look for new solutions, in particular those that would still share certain features (such as extremality) with their supersymmetric counterparts, so that at least some of the tools developed for the latter could be applied to the former.

Recently, two methods have been used to construct non-supersymmetric extremal multicenter solutions: Gaiotto, Li and Padi [31], following the earlier idea of Breitenlohner, Maison and Gibbons [96], through dimensional reduction over the timelike Killing direction mapped a class of four-dimensional static multicenter black holes (which includes both supersymmetric and non-supersymmetric solutions) to geodesics on the scalar manifold, for the case when it is a symmetric coset space. The geodesics are then traced by the nilpotent generators of the coset algebra. A similar study, but carried out for maximal rather than $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet, was later performed by Bossard and Nicolai [97]. The same type of dimensional reduction was also the main tool of the systematic study by Mohaupt and Waite [98] of conditions under which static electric multicenter solutions in theories with Einstein–Maxwell-type Lagrangians in five spacetime dimensions can be expressed by harmonic functions.

Goldstein and Katmadas [84] in turn observed that one could break supersymmetry, but still satisfy the equations of motion of five-dimensional extremal supergravity solutions with a four-dimensional Gibbons–Hawking or Taub-NUT base space, by reversing the orientation of the base.[1] By spacelike dimensional reduction these authors were able to obtain non-supersymmetric multicenter configurations also in four spacetime dimensions. Subsequently

[1] In fact the equations of motion will remain satisfied also after replacing the Euclidean four-dimensional hyper-Kähler base with a more general Ricci-flat space [86].

Bena et al. [33, 85] demonstrated examples of non-supersymmetric multicenter solutions with non-zero angular momentum and non-trivial constraints on the relative positions of the centers.

Meanwhile Gimon, Larsen and Simón [64, 99], motivated by the form of the ADM mass formula, provided an interpretation of a single-center extremal non-supersymmetric black hole in the *stu* model as a threshold bound state (where the binding energy between the components vanishes) of four constituents, each of which is supersymmetric when considered individually.

Here, in the context of four-dimensional $\mathcal{N} = 2$ supergravity with cubic prepotentials, we present another way of obtaining extremal non-supersymmetric multicenter solutions, which directly generalizes Denef and Bates's original supersymmetric derivation [28, 29], and which is an application of the superpotential approach, so far employed for single-center solutions [22, 24–27, 100, 101]. Figuratively speaking, this method consists in replacing the central charge in the equations governing the solution by a different, but typically very closely related quantity, known as the (fake) superpotential. To make the merger with Denef's formalism possible with minimal modification, we restrict ourselves to systems which turn out to have constituents with mutually local charges.

Before explaining this technique in more detail in section 3, we will introduce the necessary concepts and notation in section 2. In section 4 we (re-)derive simple examples of non-supersymmetric solutions in the *stu* model: a single-center solution with non-vanishing central charge, first obtained by Tripathy and Trivedi [21], and a multi-center solution, of the type conjectured by Kallosh, Sivanandam and Soroush [102]. We also mention how the BPS constituent interpretation fits into our framework. The final section summarizes and discusses the results.

2 Differential and special geometry

In this technical section we are going to briefly recall some basic concepts of special Kähler geometry [74, 103]—the target space geometry of $\mathcal{N} = 2$ supergravity [72, 73]—needed for finding single-center and multicenter charged extremal black hole solutions in four spacetime dimensions, following the formalism employed by Denef for the supersymmetric case. For a more exhaustive exposition we refer the reader to, for instance, [104, 105] and [106].

We can look at the four-dimensional theory from a higher-dimensional perspective.[2] By compactifying six of the ten dimensions of type IIA string theory on a Calabi–Yau three-fold X (or, equivalently, type IIB on the mirror of X) one finds [110] an effective $\mathcal{N} = 2$ supergravity theory, whose bosonic sector is described by the action

$$I_{4\mathrm{D}} = \frac{1}{16\pi} \int \Big(R \star 1 - 2 g_{a\bar{b}}(z, \bar{z})\, \mathrm{d}z^a \wedge \star \mathrm{d}\bar{z}^{\bar{b}} + \mathrm{Im}\, \mathcal{N}_{IJ}(z, \bar{z})\, \mathcal{F}^I \wedge \star \mathcal{F}^J + \mathrm{Re}\, \mathcal{N}_{IJ}(z, \bar{z})\, \mathcal{F}^I \wedge \mathcal{F}^J \Big). \tag{2.1}$$

In this action the field strengths are defined as $\mathcal{F}^I = \mathrm{d}A^I$ with the index $I = (0, a)$ labeling the abelian gauge fields $A^I = (A^0, A^a)$ of the gravity multiplet and, respectively, the vector multiplets of the theory. The vector multiplets are enumerated by the Hodge number $h^{1,1} = \dim H^{1,1}(X)$. Each vector multiplet contains two neutral real scalars, combined into a complex scalar: $z^a = \mathcal{X}^a + \mathrm{i}\mathcal{Y}^a$. Hypermultiplets (and the tensor multiplet, which can be dualized to another hypermuliplet) do not play a role in our discussion, hence we have set them to zero.

The compactification manifold X is characterized by its intersection numbers defined as:

$$D_{abc} = \int_X D_a \wedge D_b \wedge D_c \,, \tag{2.2}$$

where the set $\{D_a\}$ comprises a basis of $H^2(X) = H^{1,1}(X)$. Using this quantity, we introduce for any $\xi = \xi^a D_a$ the notation:

$$\begin{aligned} \xi^3 &= \int_X \xi \wedge \xi \wedge \xi = D_{abc}\xi^a \xi^b \xi^c \,, \\ \xi^2_a &= \int_X D_a \wedge \xi \wedge \xi = D_{abc}\xi^b \xi^c \,, \\ \xi_{ab} &= \int_X D_a \wedge D_b \wedge \xi = D_{abc}\xi^c \,. \end{aligned} \tag{2.3}$$

The scalar manifold is special Kähler with the metric:

$$g_{a\bar{b}} = \frac{1}{4\mathcal{Y}^3} \int_X D_a \wedge \star D_{\bar{b}} = -\frac{3}{2} \left(\frac{\mathcal{Y}_{a\bar{b}}}{\mathcal{Y}^3} - \frac{3}{2} \frac{\mathcal{Y}^2_a \mathcal{Y}^2_{\bar{b}}}{(\mathcal{Y}^3)^2} \right) = -\partial_{z^a} \partial_{\bar{z}^{\bar{b}}} \Big(\ln \frac{4}{3} \mathcal{Y}^3 \Big). \tag{2.4}$$

[2] Early papers on the subject of black hole composites, such as [28, 29, 107], predominantly adopted type IIB interpretation; we choose type IIA, common in more recent work, e.g. [108, 109].

This equation shows that $g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K$ is a Kähler metric with the Kähler potential $K(z,\bar{z}) = -\ln \frac{4}{3}\mathcal{Y}^3$. In fact both the Kähler potential and the vector couplings $\mathcal{N}_{IJ}(z,\bar{z})$ appearing in (2.1) can be calulated from a single function, the holomorphic cubic prepotential, homogeneous of second degree in the projective coordinates X^I (such that $z^a = X^a/X^0$):

$$F = -\frac{1}{6}D_{abc}\frac{X^a X^b X^c}{X^0} = (X^0)^2 f(z) \qquad f(z) = -\frac{1}{6}D_{abc}z^a z^b z^c \,. \tag{2.5}$$

Many objects of relevance will be most naturally thought of as taking values in the even cohomology of the internal Calabi–Yau manifold X:

$$H^{2*}(X) = H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X)\,. \tag{2.6}$$

The even cohomology has dimension $2h^{1,1}+2$ and each element $E \in H^{2*}(X)$ can be expanded as:

$$E = E^0 + E^a D_a + E_a D^a + E_0 \mathrm{d}V\,. \tag{2.7}$$

$\mathrm{d}V$ is the normalized volume form on X and $\{D^a\}$ is a dual basis of $H^4(X)$ such that:

$$\int_X D_I \wedge D^J = \delta_I^J\,. \tag{2.8}$$

We will make use of the following antisymmetric topological intersection product of two polyforms belonging to $H^{2*}(X)$:

$$\langle E, \hat{E}\rangle = \int_X E \wedge \hat{E}^*\,, \tag{2.9}$$

where the action of the operator * on E is simply a change of sign of the 2- and 6-form components. The intersection product in terms of components then reads:

$$\langle E, \hat{E}\rangle = -E^0\hat{E}_0 + E^a\hat{E}_a - E_a\hat{E}^a + E_0\hat{E}^0\,. \tag{2.10}$$

We define the *period vector* as an object belonging to $H^{2*}(X)$ that entails the quantities introduced so far:

$$\Omega_{\mathrm{hol}}(z) = -1 - z^a D_a - \frac{z_a^2 D^a}{2} - \frac{z^3}{6}\mathrm{d}V\,. \tag{2.11}$$

A normalized version of Ω_{hol} satisfying $\langle \Omega, \bar{\Omega}\rangle = -\mathrm{i}$ is:

$$\Omega(z,\bar{z}) = \mathrm{e}^{K/2}\Omega_{\mathrm{hol}} = \sqrt{\frac{3}{4\mathcal{Y}^3}}\,\Omega_{\mathrm{hol}}\,. \tag{2.12}$$

The period vector Ω transforms under Kähler transformations with Kähler weight $(1,-1)$ and we want its derivative to transform in the same way. To achieve this, we define its covariant derivatives as:

$$\begin{aligned} \mathcal{D}_a\Omega &= \partial_a\Omega + \tfrac{1}{2}\partial_a K\,\Omega\,,\\ \bar{\mathcal{D}}_{\bar a}\Omega &= \bar\partial_{\bar a}\Omega - \tfrac{1}{2}\bar\partial_{\bar a} K\,\Omega = 0\,. \end{aligned} \tag{2.13}$$

The second relation expresses the covariant holomorphicity of the normalized period vector with respect to the Kähler connection.

Using the normalized period vector one can associate a new quantity, which we decide to call *fake central charge function*, to every element $E \in H^{2*}(X)$:

$$Z(E) = \langle E,\Omega\rangle = \sqrt{\frac{3}{4\mathcal{Y}^3}}\left(\frac{E^0 z^3}{6} - \frac{E^a z_a^2}{2} + E_a z^a - E_0\right). \tag{2.14}$$

The name "fake central charge function" is given because, when E encodes the electromagnetic charges carried by the vector fields, this object becomes the central charge function which, at spatial infinity, is the true central charge of the relevant four dimensional supersymmetry algebra.

With the definitions above, the set $\{\Omega, \mathcal{D}_a\Omega, \bar{\mathcal{D}}_{\bar a}\bar\Omega, \bar\Omega\}$ constitutes an alternative basis of $H^{2*}(X)$. In fact one can prove the validity of the following equalities:

$$\begin{aligned} &\langle\Omega,\bar\Omega\rangle = -\mathrm{i}\,,\\ &\langle\mathcal{D}_a\Omega,\bar{\mathcal{D}}_{\bar b}\bar\Omega\rangle = \mathrm{i}g_{a\bar b}\,,\\ &\langle\mathcal{D}\Omega,\Omega\rangle = 0\,. \end{aligned} \tag{2.15}$$

In this new basis a constant real element $E \in H^{2*}(X)$ can be expanded as:

$$\begin{aligned} E &= \mathrm{i}\bar Z(E)\Omega - \mathrm{i}g^{\bar a b}\bar{\mathcal{D}}_{\bar a}\bar Z(E)\mathcal{D}_b\Omega + \mathrm{i}g^{a\bar b}\mathcal{D}_a Z(E)\bar{\mathcal{D}}_{\bar b}\bar\Omega - \mathrm{i}Z(E)\bar\Omega\\ &= -2\,\mathrm{Im}\big[\bar Z(E)\Omega - g^{\bar a b}\bar{\mathcal{D}}_{\bar a}\bar Z(E)\mathcal{D}_b\Omega\big]\,. \end{aligned} \tag{2.16}$$

Let us finally introduce the operator $\diamond$ acting on the basis elements in the following way:

$$\diamond\,\Omega = -\mathrm{i}\Omega\,,\qquad \diamond\bar\Omega = \mathrm{i}\bar\Omega\,,\qquad \diamond\mathcal{D}_a\Omega = \mathrm{i}\mathcal{D}_a\Omega\,,\qquad \diamond\bar{\mathcal{D}}_{\bar a}\bar\Omega = -\mathrm{i}\bar{\mathcal{D}}_{\bar a}\bar\Omega\,. \tag{2.17}$$

Using this new operator one can define a positive non-degenerate norm on $H^{2*}(X,\mathbb{R})$ as:

$$|E|^2 = \langle E,\diamond E\rangle\,. \tag{2.18}$$

3 Extremal black holes with a superpotential

3.1 Single-center black holes

Before generalizing to multicenter black holes let us consider the case in which all the electromagnetic charges are carried by a single center and let us assume spherical symmetry. All the quantities (scalars as well) depend thus only on the radial coordinate r or equivalently on $\tau = \frac{1}{|r-r_{\rm h}|}$. The ansatz for a static metric is:

$$\mathrm{d}s^2 = -\mathrm{e}^{2U}\mathrm{d}t^2 + \mathrm{e}^{-2U}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{2.19}$$

with $U = U(r)$ called warp factor. Requiring asymptotically flat metric imposes the constraint $U_{r\to\infty} = U_{\tau\to 0} \to 0$.

The electromagnetic field strength $\mathcal{F}$ consistent with symmetries is:

$$\mathcal{F} = \mathcal{F}_{\rm m} + \mathcal{F}_{\rm e} = \sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi\otimes\Gamma + \mathrm{e}^{2U}\mathrm{d}t\wedge\mathrm{d}\tau\otimes\diamond\Gamma\,, \tag{2.20}$$

with the components of the polyform

$$\Gamma = \Gamma(Q) = p^0 + p^a D_a + q_a D^a + q_0\mathrm{d}V \tag{2.21}$$

encoding the charges carried by the black hole (or, in the geometrical interpretation, numbers of D-branes wrapping even cycles of the compactification manifold X), which we could alternatively arrange in a symplectic vector $Q = (p^I, q_J)$.

Under these assumptions, the total action (2.1) in terms of τ and per unit time can be recast in the form [18]:

$$I_{\rm eff} = -\frac{1}{2}\int_0^\infty \mathrm{d}\tau\,\big(\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} + \mathrm{e}^{2U}V_{\rm BH}\big) - (\mathrm{e}^U|Z|)_{\tau=\infty}\,. \tag{2.22}$$

Here we have neglected the boundary term proportional to $\dot{U}$ and used the shorthand notation $Z = Z(\Gamma)$. The dot indicates differentiation with respect to τ and the effective black hole potential is given by:

$$V_{\rm BH} = \tfrac{1}{2}\langle\Gamma,\diamond\Gamma\rangle = |Z|^2 + 4g^{a\bar{b}}\,\partial_a|Z|\bar{\partial}_{\bar{b}}|Z|\,. \tag{2.23}$$

The black hole potential (2.23) is a quadratic polynomial in the charges and can be expressed as $V_{\rm BH} = Q^{\rm T}\mathcal{M}Q$ with a certain matrix $\mathcal{M}$. We have the freedom to perform transformations on the charge vector $Q \to SQ$

without changing the value of V_{BH}. This freedom lies in the possibility to choose the symplectic matrix S among all those that satisfy [22]:

$$V_{\mathrm{BH}} = Q^{\mathrm{T}}\mathcal{M}Q = Q^{\mathrm{T}}S^{\mathrm{T}}\mathcal{M}SQ \quad \Rightarrow \quad S^{\mathrm{T}}\mathcal{M}S = \mathcal{M}\,. \tag{2.24}$$

The sum of squares (2.23) is therefore not unique and one can, more generally, consider the effective action

$$\begin{aligned} I_{\mathrm{eff}} = &-\frac{1}{2}\int_0^\infty \mathrm{d}\tau\, \Big(\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} + \mathrm{e}^{2U}(W^2 + 4g^{a\bar{b}}\partial_a W\bar{\partial}_{\bar{b}}W)\Big) \\ &-(\mathrm{e}^U W)_{\tau=\infty}\,, \end{aligned} \tag{2.25}$$

with W, usually called the fake superpotential, not necessarily equal to $|Z|$.

Varying the action we obtain the following first order equations (that by construction imply the second order equations of motion):

$$\dot{U} = -\mathrm{e}^U W\,, \tag{2.26}$$

$$\dot{z}^a = -2\mathrm{e}^U g^{a\bar{b}}\bar{\partial}_{\bar{b}}W\,. \tag{2.27}$$

When W is equal to $|Z(\Gamma)|$ (2.26) and (2.27) describe a supersymmetric attractor flow [15, 18]. (The name "attractor" stems from the fact that the flow has a fixed point determined by the charges, which is reached by the scalars as they approach the event horizon, i.e. when $\tau \to \infty$.) When $W \neq |Z(\Gamma)|$ the flow is non-supersymmetric.

The form (2.26) and (2.27) of the attractor equations emphasizes the gradient nature of the flow, but to be able to integrate them directly, another form is more suitable. In the supersymmetric case it follows from the rewriting of the action in yet another way [28] (but still as a sum of squares):

$$I_{\mathrm{eff}} = -\frac{1}{4}\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{2U}\Big|2\,\mathrm{Im}\big[(\partial_\tau + \mathrm{i}\mathcal{Q}_\tau + \mathrm{i}\dot{\alpha})(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega)\big] + \Gamma\Big|^2 - (\mathrm{e}^U|Z|)_{\tau=0}\,, \tag{2.28}$$

where $\mathcal{Q}_\tau = \mathrm{Im}(\partial_a K\dot{z}^a)$ and $\alpha = \arg Z(\Gamma)$.

Based on the similarity between supersymmetric and non-supersymmetric equations, we generalize this expression by replacing Γ with a different real element of the even cohomology of X, say $\tilde{\Gamma}$. To retain the same form of the expansion (2.16) as that employed in supersymmetric solutions,

$$\tilde{\Gamma} = \mathrm{i}\bar{Z}(\tilde{\Gamma})\Omega - \mathrm{i}g^{\bar{a}b}\bar{\mathcal{D}}_{\bar{a}}\bar{Z}(\tilde{\Gamma})\mathcal{D}_b\Omega + \mathrm{i}g^{a\bar{b}}\mathcal{D}_a Z(\tilde{\Gamma})\bar{\mathcal{D}}_{\bar{b}}\bar{\Omega} - \mathrm{i}Z(\tilde{\Gamma})\bar{\Omega}\,, \tag{2.29}$$

we limit our analysis to those $\tilde{\Gamma}$ that have constant real components in the basis $\{D_I, D^J\}$. We can arrange them in a symplectic vector $\tilde{Q} = (\tilde{p}^I, \tilde{q}_J)$, where $\tilde{Q} = SQ$, so that $\tilde{\Gamma} = \Gamma(\tilde{Q})$. Consequently, the matrix S is restricted to be real and constant.[3]

With $\tilde{\Gamma}$ chosen in this way, we can identify the superpotential with the fake central charge function defined in (2.14), when evaluated for $\tilde{\Gamma}$ as its argument:

$$W = |Z(\tilde{\Gamma})|\,. \tag{2.30}$$

It follows that

$$|\tilde{\Gamma}|^2 = W^2 + 4g^{a\bar{b}}\partial_a W \bar{\partial}_{\bar{b}} W\,. \tag{2.31}$$

Denoting $\tilde{\alpha} = \arg Z(\tilde{\Gamma})$ we can write the effective action (2.25) as

$$I_{\text{eff}} = -\frac{1}{4}\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{2U}\left|2\,\mathrm{Im}\big[(\partial_\tau + \mathrm{i}\mathcal{Q}_\tau + \mathrm{i}\dot{\tilde{\alpha}})(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\Omega)\big] + \tilde{\Gamma}\right|^2 - (\mathrm{e}^U W)_{\tau=0}\,, \tag{2.32}$$

and the attractor equations become, in complete analogy with Denef's original treatment of the supersymmetric case:

$$2\partial_\tau\,\mathrm{Im}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\Omega) = -\tilde{\Gamma}\,. \tag{2.33}$$

The form (2.33) of the attractor equations is suitable for direct integration and gives:

$$2\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\Omega) = -\tilde{H}(\tau)\,, \tag{2.34}$$

with

$$\tilde{H}(\tau) = \tilde{\Gamma}\tau - 2\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\Omega)_{\tau=0}\,. \tag{2.35}$$

The explicit solution for the scalars is [29]:

$$z^a(\tilde{H}) = \frac{\tilde{H}^a - \mathrm{i}\,\mathrm{d}_{\tilde{H}_a}\Sigma(\tilde{H})}{\tilde{H}^0 + \mathrm{i}\,\mathrm{d}_{\tilde{H}_0}\Sigma(\tilde{H})}\,, \tag{2.36}$$

and

$$\mathrm{e}^{-2U(\tilde{H})} = |Z(\tilde{H})|^2\Big|_{z=z(\tilde{H})} = W^2(\tilde{H})\Big|_{z=z(\tilde{H})} = \Sigma(\tilde{H})\,, \tag{2.37}$$

[3] Already in [22] it was argued that only a constant matrix S would allow the rewriting of V_{BH} in (2.24) as a sum of squares in terms of a superpotential obtained from the central charge by acting with S on Q. In our formalism, if S were moduli-dependent, the coefficient of $\mathcal{D}_b\Omega$ (and $\bar{\mathcal{D}}_{\bar{b}}\bar{\Omega}$) in the expansion (2.29) would have an additional term (namely $-\langle\partial_a SQ, \Omega\rangle$), and expressing the effective action in a manner analogous to (2.28) would not be straightforward.

where the entropy function $\Sigma(\tilde{H})$ can be obtained, as in the supersymmetric case, from the entropy of the black hole

$$\mathcal{S}_{\rm BH} = \pi\Sigma(\tilde{\Gamma}) = \pi W^2(\tilde{\Gamma}) \tag{2.38}$$

by replacing the charges with harmonic functions.

The question we are left with thus concerns the conditions allowing the existence of the constant matrix S. They may be met by truncating the theory to a suitable subset of the scalar fields. In particular for the *stu* model it has been shown to mean setting to zero the axion fields $\mathrm{Re}\, z^a$ and considering magnetic or electric configurations [22]. This assumption is the same for the t^3 and st^2 models[4] and all models with cubic prepotentials. In this setting S turns out to be diagonal and acts on the charge vector without changing its electric or magnetic character.

In what follows we will assume the scalars to be purely imaginary and the charge configuration to be either $(p^0, 0, 0, q_a)$ or $(0, p^a, q_0, 0)$.

3.2 Multicenter black holes

For multicenter configurations the spherical symmetry assumption of the previous derivation is no longer valid and we have to consider more general, stationary spacetimes, by including in the metric an extra one-form $\omega = \omega_i \mathrm{d}x^i$:

$$\mathrm{d}s^2 = -\mathrm{e}^{2U}(\mathrm{d}t + \omega_i \mathrm{d}x^i)^2 + \mathrm{e}^{-2U}\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j \tag{2.39}$$

and taking U and ω_i to be arbitrary functions of the position $\mathbf{x}$. We require asymptotic flatness by imposing $U, \omega \to 0$ when $\tau \to 0$.

Although the idea for obtaining the attractor flow equations remains the same (namely, the rewriting of the Lagrangian as a sum of squares), the formalism becomes more involved. Following, with some alterations, reference [28], we adopt the boldface notation for three-dimensional quantities. The 3D Hodge dual with respect to the flat metric δ_{ij} will be denoted by $\star_{\mathbf{0}}$ and for convenience we define $w = \mathrm{e}^{2U}\omega$. We also need to introduce the following scalar product of spatial 2-forms $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{G}}$:

$$(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{G}}) = \frac{\mathrm{e}^{2U}}{1 - w^2}\int_X \boldsymbol{\mathcal{F}} \wedge \left[\star_{\mathbf{0}}(\diamond\boldsymbol{\mathcal{G}}^*) - \star_{\mathbf{0}}(w \wedge \diamond\boldsymbol{\mathcal{G}}^*)\, w + \star_{\mathbf{0}}(w \wedge \star_{\mathbf{0}}\boldsymbol{\mathcal{G}}^*)\right], \tag{2.40}$$

[4] It suffices to compare the BPS and non-BPS attractor solutions of the t^3 or st^2 model [22, 111]. As for the *stu* model, once we impose $\mathrm{Re}\, z^a = \mathcal{X}^a = 0$ and consider the magnetic or electric configuration, the solutions differ only by a switch of sign of the charges—one that would be effected by the matrix S in our treatment.

where * is an operator acting on the elements of the even cohomology of X as defined below formula (2.9). The product just introduced is commutative and we can assume it to be positive definite taking w small enough.

With this notation the effective action reads (dropping the total derivative ΔU):

$$I_{\text{4D eff}} = -\frac{1}{16\pi}\int \mathrm{d}t \int_{\mathbb{R}^3} \Big[2\mathbf{d}U \wedge \star_{\mathbf{0}}\mathbf{d}U - \tfrac{1}{2}\mathrm{e}^{4U}\mathbf{d}\omega \wedge \star_{\mathbf{0}}\mathbf{d}\omega + 2g_{a\bar{b}}\,\mathbf{d}z^a \wedge \star_{\mathbf{0}}\mathbf{d}\bar{z}^{\bar{b}} + (\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}})\Big]. \tag{2.41}$$

Generalizing Denef's derivation in the way we did for single-center black holes, we introduce the electromagnetic field strength corresponding to the modified charges $\tilde{\Gamma}$:

$$\frac{1}{4\pi}\int \tilde{\boldsymbol{\mathcal{F}}} = \tilde{\Gamma}\,. \tag{2.42}$$

Consequently, we define

$$\tilde{\boldsymbol{\mathcal{G}}} = \tilde{\boldsymbol{\mathcal{F}}} - 2\,\mathrm{Im}\,\star_{\mathbf{0}}\mathbf{D}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\Omega) + 2\,\mathrm{Re}\,\mathbf{D}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\Omega\,\omega) \tag{2.43}$$

and write the Lagrangian of (2.41) in the form[5]

$$\mathcal{L} = (\tilde{\boldsymbol{\mathcal{G}}}, \tilde{\boldsymbol{\mathcal{G}}}) \;-\; 4\,(\mathbf{Q} + \mathbf{d}\tilde{\alpha} + \tfrac{1}{2}\mathrm{e}^{2U}\star_{\mathbf{0}}\mathbf{d}\omega) \wedge \mathrm{Im}\langle\tilde{\boldsymbol{\mathcal{G}}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\Omega\rangle + \mathbf{d}\,[2w \wedge (\mathbf{Q} + \mathbf{d}\tilde{\alpha}) + 4\,\mathrm{Re}\langle\tilde{\boldsymbol{\mathcal{F}}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\Omega\rangle]\,. \tag{2.44}$$

with

$$\mathbf{D} = \mathbf{d} + \mathrm{i}(\mathbf{Q} + \mathbf{d}\alpha + \tfrac{1}{2}\mathrm{e}^{2U}\star_{\mathbf{0}}\mathbf{d}\omega)\,, \tag{2.45}$$

$$\mathbf{Q} = \mathrm{Im}\,(\partial_a K\,\mathbf{d}z^a)\,. \tag{2.46}$$

Imposing the first order equations

$$\tilde{\boldsymbol{\mathcal{G}}} = 0\,, \tag{2.47}$$

$$\mathbf{Q} + \mathbf{d}\tilde{\alpha} + \tfrac{1}{2}\mathrm{e}^{2U}\star_{\mathbf{0}}\mathbf{d}\omega = 0 \tag{2.48}$$

solves the equations of motion. From (2.48) it follows that $\mathbf{D} = \mathbf{d}$ and then, as by definition and our assumption $\mathbf{d}\tilde{\boldsymbol{\mathcal{F}}} = 0$, differentiating (2.47) leads to

$$2\mathbf{d}\star_{\mathbf{0}}\mathbf{d}\,\mathrm{Im}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\Omega) = 0\,. \tag{2.49}$$

[5] We assume that the constraints [112] resulting from the components of Einstein's equation not reproduced by this Lagrangian will remain satisfied as in the supersymmetric case.

This is a Laplacian equation which integrated gives (cf. (2.34)):

$$2\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\Omega) = -\tilde{H}\,, \tag{2.50}$$

where $\tilde{H}$ is a generic $H^{2*}(X)$-valued harmonic function. Since we are looking for non-BPS multicenter configurations considering N sources at position $\mathbf{x}_n$, it seems reasonable to take as $\tilde{H}(\mathbf{x})$ a natural generalization of (2.35), namely:

$$\tilde{H}(\mathbf{x}) = \sum_{n=1}^{N} \tilde{\Gamma}_n \tau_n - 2\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\Omega)_{\tau=0}\,, \tag{2.51}$$

with $\tau_n = |\mathbf{x} - \mathbf{x}_n|^{-1}$ and $\tilde{\Gamma}_n = \Gamma(S_n Q_n)$, S_n being constant matrices.

To be able to speak of black hole composites, certain conditions need to be satisfied:

- A single-center non-BPS black hole of total charge Q and its corresponding attractor flow have to exist and be well defined (i.e. they have to be describable with the above procedure).
- For each center of charge Q_n a single-center attractor flow has to exist as well.
- The charges must obey the constraints:

$$Q = \sum_{n=1}^{N} Q_n\,, \tag{2.52}$$

$$\tilde{\Gamma} = \Gamma(SQ) = \sum_{n=1}^{N} \Gamma(S_n Q_n) = \sum_{n=1}^{N} \tilde{\Gamma}_n\,. \tag{2.53}$$

In addition we need to take into account a particular feature of the central charge Z, stemming from our assumptions regarding the charges: taking $Q = (p^0, 0, 0, q_a)$ or $Q = (0, p^a, q_0, 0)$ and imposing $\mathrm{Re}\, z^a = 0$ reveals that the central charge has a constant phase. For instance, with Q electric the corresponding central charge reads

$$Z(\Gamma) = \sqrt{\frac{3}{4\mathcal{Y}^3}}\left(\frac{p^0 z^3}{6} + q_a z^a\right) \tag{2.54}$$

and with $z^a = \mathrm{i}\mathcal{Y}^a$ (where $\mathcal{Y}^a \in \mathbb{R}$) it holds that $\mathrm{e}^{\mathrm{i}\alpha} = \frac{Z}{|Z|} = \mathrm{i}$.

Note that this is true also for $\tilde{Z} = Z(\tilde{\Gamma})$ whenever the difference between Q and $\tilde{Q}$ amounts to constant factors multiplying their components (as is the case when S is a constant diagonal matrix). Then, as a direct consequence of the constancy of $\tilde{\alpha}$, it follows in our treatment that $\mathbf{d}\tilde{\alpha} = 0$ and (2.48) in particular becomes:

$$\mathbf{Q} = -\frac{1}{2} \mathrm{e}^{2U} \star_{\mathbf{0}} \mathbf{d}\omega \,. \tag{2.55}$$

A different form of flow equations

Let us bring the attractor equations (2.50) to a form more closely resembling the first order flow equations for the scalars and the warp factor (2.26, 2.27). In view of this we define:

$$\tilde{\boldsymbol{\xi}} = \langle \mathbf{d}\tilde{H}, \Omega \rangle = \sum_{n=1}^{N} Z(\tilde{\Gamma}_n)\, \mathbf{d}\tau_n = \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}\tilde{\alpha}_n} W_n \, \mathbf{d}\tau_n \,. \tag{2.56}$$

Let us differentiate (2.50) to obtain:

$$\begin{aligned} \mathbf{d}\tilde{H} &= 2\,\mathrm{Im}\big[(\mathbf{d}U\Omega - \mathbf{d}\Omega)\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\big] \\ &= 2\,\mathrm{Im}\big[(\mathbf{d}U\Omega - \mathcal{D}_a\Omega\, \mathbf{d}z^a + \mathrm{i}\mathbf{Q}\Omega)\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\big] \,. \end{aligned} \tag{2.57}$$

Taking now the intersection product of (2.57) with Ω yields:

$$-\,\tilde{\boldsymbol{\xi}} = (\mathbf{d}U - \mathrm{i}\mathbf{Q})\mathrm{e}^{-U}\mathrm{e}^{\mathrm{i}\tilde{\alpha}} \,, \tag{2.58}$$

and then:

$$\mathbf{Q} = \mathrm{e}^{U}\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\tilde{\boldsymbol{\xi}}) \,, \tag{2.59}$$

$$\mathbf{d}U = -\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\tilde{\boldsymbol{\xi}}) \,. \tag{2.60}$$

Similarly taking the intersection product of (2.57) with $\bar{\mathcal{D}}_{\bar{a}}\bar{\Omega}$ gives:

$$\mathbf{d}z^a = -\mathrm{e}^{U} g^{a\bar{b}} \mathrm{e}^{\mathrm{i}\tilde{\alpha}} \bar{\mathcal{D}}_{\bar{b}} \bar{\tilde{\boldsymbol{\xi}}} \,. \tag{2.61}$$

Equations (2.60)–(2.61) are the multicenter version of (2.26)–(2.27). Recalling our assumptions and in particular using $\mathrm{Re}\, z^a = 0$ we have:

$$\begin{aligned} \mathbf{Q} &= \mathrm{Im}(\partial_a K \mathbf{d}z^a) = -\frac{\mathrm{i}}{2}(\partial_a K \mathbf{d}z^a - \bar{\partial}_{\bar{a}} K \mathbf{d}\bar{z}^{\bar{a}}) \\ &= -\frac{\mathrm{i}}{2}(\partial_a K \mathbf{d}z^a - \partial_a K \mathbf{d}z^a) = 0 \,. \end{aligned} \tag{2.62}$$

Hence, with $\tilde{\alpha}_n = \arg Z(\tilde{\Gamma}_n)$, (2.59) becomes:

$$0 = \mathrm{Im}(\mathrm{e}^{-\mathrm{i}\tilde{\alpha}}\tilde{\boldsymbol{\xi}}) = \sum_{n=1}^{N} \mathrm{Im}\big(\mathrm{e}^{-\mathrm{i}(\tilde{\alpha}-\tilde{\alpha}_n)}\big) W_n \, \mathbf{d}\tau_n \, , \tag{2.63}$$

that is $\tilde{\alpha} = \tilde{\alpha}_n \pmod{\pi}$ for all n.

Angular momentum and positions of the centers

It is worth pointing out that equation (2.62) applied to (2.55) yields $\star_0 \mathbf{d}\omega = 0$, implying that the angular momentum $\mathbf{J}$, read off from the metric components as (see e.g. [113], ch. 19)

$$\omega_i = 2\epsilon_{ijk} J^j \frac{x^k}{r^3} + O(1/r^3) \quad \text{for} \quad r \to \infty \, , \tag{2.64}$$

has to vanish and so the metric is in fact static. This is a remarkable difference with respect to the supersymmetric case, where, instead, the one-form ω enclosing the off-diagonal element of the metric is determined by solving equation [28]

$$\star_0 \, \mathbf{d}\omega = \langle \mathbf{d}H, H \rangle \, . \tag{2.65}$$

According to equation (2.63), the "tilded" central charges $\tilde{Z} = Z(\tilde{\Gamma})$ and $\tilde{Z}_n = Z(\tilde{\Gamma}_n)$ have to be aligned either parallel or antiparallel. These are conditions analogous to those defining marginal or antimarginal stability in the BPS case. If we want to use the same terminology, this means that multicenter non-BPS systems described in this paper are marginally (or antimarginally) stable and can decompose into their constituents everywhere in moduli space. In the supersymmetric sector such a decay is for generic charge configurations possible only on a particular surface of the scalar manifold (the wall of marginal stability).

The relative positions of the sources in space are governed by the analogue of equation (7.23) in [28]:

$$\sum_{n=1}^{N} \frac{\langle \tilde{\Gamma}_m, \tilde{\Gamma}_n \rangle}{|\mathbf{x}_m - \mathbf{x}_n|} = 2 \, \mathrm{Im}\big[\mathrm{e}^{-\mathrm{i}\tilde{\alpha}} Z(\tilde{\Gamma}_m)\big]_{\tau=0} \, . \tag{2.66}$$

In the supersymmetric sector one finds $N-1$ constraints, which may even determine a nontrivial topology of the solution space [30]. Here instead, since $\tilde{\alpha} = \tilde{\alpha}_m \pmod{\pi}$ for all m implies $\mathrm{Im}[\mathrm{e}^{-\mathrm{i}\tilde{\alpha}} Z(\tilde{\Gamma}_m)] = 0$ and then

$$\sum_{n=1}^{N} \frac{\langle \tilde{\Gamma}_m, \tilde{\Gamma}_n \rangle}{|\mathbf{x}_m - \mathbf{x}_n|} = 0\,, \tag{2.67}$$

equation (2.66) gives:

$$\langle \tilde{\Gamma}_m, \tilde{\Gamma}_n \rangle = 0 \quad \forall\, m, n\,. \tag{2.68}$$

This result in our context directly holds also for the charges Γ_n, stating that they have to be mutually local with respect to the product (2.9). Indeed, to satisfy the condition of constancy of S, we chose to work with electric or magnetic configurations, which lead to mutually local electric or magnetic constituents. As a consequence, there are no constraints on the positions and the centers are free.

4 Non-BPS composites in the *stu* model

In this section we are going to apply the general procedure described above to the particular case of the *stu* model, as a concrete example. In this extensively studied model (see eg. [114] and references therein), arising in type IIA compactification on a $T^2 \times T^2 \times T^2$, the scalar manifold is the homogeneous symmetric space $\left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)^3$ parameterized by the complex moduli $z^1 \equiv s$, $z^2 \equiv t$ and $z^3 \equiv u$ (corresponding to the complexified volumes of the tori). The prepotential reads:

$$f = stu\,. \tag{2.69}$$

The Bekenstein-Hawking entropy of a *stu* black hole with charge[6] $Q_\ell = (p^I_\ell, q^\ell_J)$ is related through

$$\mathcal{S} = \frac{A_\mathrm{h}}{4} = \pi V_\mathrm{BH}\Big|_{\partial V_\mathrm{BH}=0} = \pi\sqrt{|\mathcal{I}_4(Q_\ell)|} \tag{2.70}$$

to the unique invariant $\mathcal{I}_4$ of the tri-fundamental representation $(\mathbf{2},\mathbf{2},\mathbf{2})$ of the duality group $(\mathrm{SL}(2,\mathbb{Z}))^3$. Explicitly this invariant has the form:

$$\mathcal{I}_4(Q_\ell) = -(p^I_\ell q^\ell_I)^2 + 4\sum_{a<b} p^a_\ell q^\ell_a p^b_\ell q^\ell_b - 4p^0_\ell q^\ell_1 q^\ell_2 q^\ell_3 + 4q^\ell_0 p^1_\ell p^2_\ell p^3_\ell\,. \tag{2.71}$$

[6] To match conventions used in some *stu* literature, we have introduced the vector Q_ℓ, differing from Q by a sign reversal in the electric charges: $q^\ell_a = -q_a$.

Non-BPS black holes with $Z \neq 0$ satisfy $\mathcal{I}_4(Q_\ell) < 0$.

Once we have chosen to deal with an electric charge configuration,[7] it follows that $\tilde{Q} = SQ = (-p^0, 0, 0, q_a) = (-p^0_\ell, 0, 0, -q^\ell_a)$ and we can derive the non-BPS scalar solutions for single-center and multicenter black holes using the equations of our formulation.

In the single-center case we have to use the harmonic function (written here as a symplectic vector)

$$\tilde{H} = \begin{pmatrix} \tilde{p}^0 \\ \tilde{p}^a \\ \tilde{q}_0 \\ \tilde{q}_a \end{pmatrix} \tau + \tilde{h}_\infty = \begin{pmatrix} -p^0_\ell \\ 0 \\ 0 \\ -q^\ell_a \end{pmatrix} \tau + h^\ell_\infty \,, \tag{2.72}$$

where with h_∞ we have indicated the constant vector which at the end determines the value of the scalars at infinity. From (2.36), using $\Sigma^2(Q) = \mathcal{I}_4(Q_\ell)$, we obtain the scalar solutions:

$$z^a(\tau) = \frac{-\mathrm{i}\, \mathrm{d}_{\tilde{H}_1} \Sigma(\tilde{H})}{\tilde{H}^0} = \frac{-\mathrm{i}\, \mathrm{d}_{\tilde{H}_1} \sqrt{4 \tilde{H}^0 \tilde{H}_1 \tilde{H}_2 \tilde{H}_3}}{\tilde{H}^0} = \frac{-\mathrm{i}\, \mathrm{d}_{H^\ell_1} \sqrt{4 H^0_\ell H^\ell_1 H^\ell_2 H^\ell_3}}{H^0_\ell} \tag{2.73}$$

and then

$$z^1(\tau) = -\mathrm{i} \sqrt{\frac{H^\ell_2 H^\ell_3}{H^0_\ell H^\ell_1}} \,, \qquad z^2(\tau) = -\mathrm{i} \sqrt{\frac{H^\ell_1 H^\ell_3}{H^0_\ell H^\ell_2}} \,, \qquad z^3(\tau) = -\mathrm{i} \sqrt{\frac{H^\ell_1 H^\ell_2}{H^0_\ell H^\ell_3}} \,. \tag{2.74}$$

These expressions correctly reproduce the results known from the existing literature [21, 102].

The multicenter case is slightly more complicated. As we mentioned, a composite with N centers of charge Q_n at positions $\mathbf{x}_n$ has to satisfy the constraints (2.52) and (2.68). In addition, at each $\mathbf{x}_n$, there has to exist a single-center black hole described in terms of a harmonic function $\tilde{H}(\tau)$ of the form (2.35). The charge cofiguration at each of the N centers needs to be either electric or magnetic, as these are the only configurations that allow non-BPS attractors describable with our procedure. However, since for both these configurations the matrix S is diagonal, the constraints (2.52) are satisfied only if all Q_n are of the same kind as Q. The composite is then constituted by N single-center black holes with charge $Q_n = (p^0_n, 0, 0, q^n_a) =$

[7] For a more generic non-BPS charge configuration one can apply an $\mathrm{SL}(2,\mathbb{Z})$ duality transformation, see e.g. [64].

$(p^0_{n\,\ell}, 0, 0, -q_a^{n\,\ell})$ such that $Q = \sum_n Q_n$. The positions of the centers, as we discussed in subsection 3.2, are not constrained. The scalar solutions are as in (2.74) but with the harmonic function of the form:

$$H_\ell = \sum_n \frac{-Q_{n\,\ell}}{|\mathbf{x} - \mathbf{x}_n|} + h^\ell_\infty \,. \tag{2.75}$$

Hence, near the n-th center, z^a reads:

$$z^a = -\mathrm{i}\sqrt{\frac{|\varepsilon^{abc}| q_b^{n\,\ell} q_c^{n\,\ell}}{2 p^0_{n\,\ell} q_a^{n\,\ell}}} \qquad \mathbf{x} \to \mathbf{x}_n \,. \tag{2.76}$$

These expressions have the form conjectured in [102].

We close this section with a remark that our framework admits also the interpretation [99] of a non-supersymmetric *stu* black hole as comprised of supersymmetric constituents. This model follows from the observation [64] that the ADM mass

$$m_{\mathrm{ADM}} = \lim_{\tau \to 0} \frac{\mathrm{d}U}{\mathrm{d}\tau} \,, \tag{2.77}$$

of a non-BPS black hole can be written as the sum of the masses of four primitive BPS centers. A direct computation in our setting ($\mathrm{Re}\, z = 0 \Rightarrow B = 0$) gives for a non-BPS black hole of electric charge $Q_\ell = (p^0_\ell, 0, 0, q_a^\ell)$:

$$m_{\text{non-BPS}} = k \left(p^0_\ell + q^\ell_1 + q^\ell_2 + q^\ell_3 \right) , \tag{2.78}$$

with k a constant factor and $p^0_\ell > 0$. Computing instead the sum of the masses of four BPS black hole carrying a single type of charge we obtain:

$$m_{\mathrm{BPS}} = k \left(|p^0_\ell| + q^\ell_1 + q^\ell_2 + q^\ell_3 \right) . \tag{2.79}$$

Naively, one could try to construct a non-supersymmetric configuration with supersymmetric constituents by taking the matrices S_n to be proportional to the unit matrix. This, however, would not satisfy the condition (2.53). A way to have supersymmetric centers is to relax the condition of existence of a regular black hole at each of the centers and to assign to each of them only one type of charge. The supersymmetry of such singular configurations will be unaffected by the matrices S_n, which we now need to choose equal to the matrix S: their effect will be reduced to multiplication by a constant factor.

5 Conclusions

In this paper we extended Denef's formalism for multicenter black hole solutions in four-dimensional $\mathcal{N} = 2$ supergravity on simple non-supersymmetric cases, using the fake superpotential method.

Our generalization requires the superpotential to be related to the central charge in a particular way (through a constant matrix S), which imposes some constraints on the charge configuration. It turns out to be a limitation, since already some single-centered cases for which the superpotential is known to exist would violate this assumption (cf. [22]). To satisfy it, we worked with electric or magnetic configurations, which lead to mutually local electric or magnetic constituents.

Still, in the example of the *stu* model, for the single-centered case we recover the non-supersymmetric black holes previously derived in a different way by Tripathy and Trivedi [21]. The multicenter non-supersymmetric *stu* solutions that we find, apart from the constraints, correspond to the form conjectured by Kallosh, Sivanandam and Soroush [102]. Our approach allows also to resolve a single non-supersymmetric *stu* black hole into a collection of supersymmetric centers in a way consistent with the BPS-constituent model of Gimon, Larsen and Simón [99].

More generally, the multicenter solutions that can be described by the method presented here are in a sense the simplest analogues of their supersymmetric counterparts,[8] yet exhibit different properties. In particular, similarly to non-supersymmetric solutions obtained by Gaiotto, Li and Padi [31] in the group-theoretical approach, but unlike in the generic supersymmetric case, the charges carried by the centers are mutually local and the angular momentum vanishes, rendering the solution static.

The following picture therefore seems to emerge, at least in the considered class of theories: supersymmetric black holes can be split only into supersymmetric composites and only at particular loci of their moduli space, namely on the walls of marginal stability (except for a decomposition into constituents with aligned charge vectors, which is always possible). Non-supersymmetric black holes, on the contrary, can be resolved everywhere in moduli space into a composite consisting of any number of non-supersymmetric centers at arbitrary positions, but also (as Gimon, Larsen and Simón demonstrated for the *stu* model) into a specific number

[8] For instance, the Hessian of the black hole potential at its critical points is still proportional to the Kähler metric, implying stability and the absence of flat directions.

of threshold-bound supersymmetric constituents (by combining the two descriptions, mixed cases would appear to be also possible).

We are aware, however, that the above summary is not complete. The results of Bena et al. [33, 85] obtained in the Goldstein and Katmadas's almost-BPS framework [84] demonstrate that non-supersymmetric composites may also comprise constituents with constrained positions. It would therefore be natural to see how the restriction of our method (specifically, the constancy of S) could be relaxed, and whether one would then obtain solutions with non-trivial angular momentum. Even more interesting, perhaps, would be to clarify the relationship between the various approaches employed to construct non-supersymmetric multicenter solutions (along the lines of [101], for instance, where the superpotential for single-center black holes was obtained through timelike dimensional reduction) and find out whether any of the techniques or their refinements could eventually exhaust all possible classes of extremal solutions. A further step could be then an attempt to use them for non-extremal composites [115].

3. FIRST-ORDER FLOWS AND STABILISATION EQUATIONS FOR NON-BPS EXTREMAL BLACK HOLES

Pietro Galli, Kevin Goldstein[◇]*, Stefanos Katmadas*[†]*,*
and Jan Perz[♡]

[◇]*National Institute for Theoretical Physics (NITHeP), School of Physics and Centre for Theoretical Physics, University of the Witwatersrand. WITS 2050, Johannesburg, South Africa.*

[†]*Institute for Theoretical Physics and Spinoza Institute, Universiteit Utrecht. Postbus 80.195, 3508 TD Utrecht, The Netherlands.*

[♡]*Instituto de Física Teórica UAM/CSIC*
C/ Nicolás Cabrera, 13–15, C.U. Cantoblanco, 28049 Madrid, Spain.

Publication reference: *JHEP* **06** (2011) 070, arXiv:1012.4020 [hep-th].

Abstract

We derive a generalised form of flow equations for extremal static and rotating non-BPS black holes in four-dimensional ungauged $N = 2$ supergravity coupled to vector multiplets. For particular charge vectors, we give stabilisation equations for the scalars, analogous to the BPS case, describing full known solutions. Based on this, we propose a generic ansatz for the stabilisation equations, which surprisingly includes ratios of harmonic functions.

1 Introduction

The study of black holes in theories with eight or more supercharges, resulting from string theory compactifications, has proved to be a very useful tool in uncovering some of the structure of the underlying statistical systems. For supersymmetric black holes this task is facilitated by the fact that they exhibit the attractor mechanism and full supersymmetry enhancement near the event horizon [15–17]. Using the constraints imposed by supersymmetry, general, stationary, asymptotically flat solutions have been found in ungauged $N = 2$ Einstein–Maxwell supergravity, including higher-derivative corrections, both in four and five dimensions [28, 94, 95, 116–118]. The spatial profile of scalars in these solutions follows a first-order gradient flow, which is integrable to (non-differential) stabilisation equations, expressing the scalars in terms of harmonic functions. On the event horizon (the endpoint of the flow), the values of scalars are dictated by the charges through the attractor equations, independently of the asymptotic boundary conditions (the beginning of the flow).[1]

In contrast, when the requirement that the solutions must preserve some supersymmetry is abandoned, much less is known about the general structure of the supergravity solutions and the microscopic theory behind them. The simplest generalisation of BPS black holes to consider are the extremal black holes which do not preserve any supersymmetry (see [119]). These are known to share some desirable features with the BPS branch, most importantly the attractor phenomenon [18–20].

For theories with 8 supercharges coupled to vector multiplets in four and five dimensions,[2] the general structure of these non-BPS extremal solutions is unclear, since only partial results are known. In the static case, a restricted set of examples can be found by simply changing the sign of a subset of the charges, which breaks supersymmetry [21, 102]. It was found that the non-BPS solutions exhibit flat directions in the scalar sector, in the sense that the scalars are not completely fixed at the horizon once the charges are chosen [21]. However, these examples are not generic enough — they contain one less than the minimum number of parameters required for the most general solution to be derived from them by dualities. A solution that does contain enough parameters is called a seed solution.

[1] Some authors interchange the meaning of the terms "stabilisation equations" and "attractor equations".

[2] Since the two are related by dimensional reduction, we do not make a distinction between them in this introduction.

For cubic prepotentials, an appropriate seed was found in [25, 64] and the full duality orbit for the *stu* model was subsequently derived in [114]. This full example clarifies how the non-supersymmetric solutions differ from their BPS counterparts in more than simply changing the signs of charges. In particular, they have flat directions that are subject to symmetries that act along the full flow, including the horizon [62, 63, 120].

If one allows for angular momentum, there are two types of single-centre extremal solutions which display attractor behaviour [121]. The over-rotating (or ergo) branch is very different from the BPS solutions, as they feature an ergoregion and are continuously connected to the Kerr solution [81–83]. In contrast, the under-rotating (or ergo-free) black holes have a continuous limit to static charged black holes and seem to be tractable using BPS-inspired techniques. Recently, the single-centre under-rotating seed solution and various multi-centred generalisations were found in [33, 34, 84–86]. In these cases, the nontrivial parameter appearing in the static seed solutions can be viewed as the constant part of a harmonic function describing rotation.

Despite the existence of these known solutions, finding an organising principle for their general structure has proven challenging. The best developed approaches are based on four-dimensional supergravity, where electric-magnetic duality limits the possible structures. One such framework is provided by the timelike dimensional reduction of Breitenlohner, Maison and Gibbons [96], which relates black holes, regardless of supersymmetry (or even extremality), to geodesics on the (augmented) scalar manifold. Given sufficient symmetry on the scalar manifold, solutions, including multi-centre black holes, may be generated with powerful group-theoretical methods, cf. [31, 97, 122–125] and references therein. Unfortunately, this comes at the expense of the results being expressed less explicitly.

A more direct perspective has been offered by the fake superpotential approach of Ceresole and Dall'Agata [22]. They noticed that the rewriting of the effective black hole potential for the scalars [18] as a sum of squares is not unique, leading to more than one type of first-order flow for the scalar fields. The flow, which in the supersymmetric case is governed by the absolute value of the central charge, may be more generally controlled by a different function, called the fake superpotential. The derivation of first-order equations based on a superpotential has been subsequently extended to static non-extremal black holes and for a number of models superpotentials have been identified explicitly [24, 26, 100, 101, 126] (see [127] for a synopsis of these developments and [23] for earlier related work).

The superpotential method has been first applied to multi-centre black holes in [32], which directly generalised [28]. However, simplifying assumptions restricted the non-supersymmetric solutions, as in [31], to threshold-bound configurations with mutually local charges and unconstrained relative positions of the centres. In view of the recent results on the integrability of the scalar equations of motion in black hole backgrounds [27,128], one might expect also the more complicated multi-centre solutions mentioned earlier to be derivable from first-order flows integrated to stabilisation equations.

As a step in this direction, we study extremal under-rotating (ergo-free) black holes in compactifications of Type IIB string theory on Calabi–Yau manifolds, using the formalism of [28]. In section 3 we relax the additional conditions of [32] to arrive at the general form of first-order flow equations for stationary extremal black holes. Unfortunately, we find that unlike their previously known special cases, to which they correctly reduce under the relevant assumptions, they generically do not lend themselves to explicit integration.

In section 4, therefore, we follow a bottom-up approach, trying to find stabilisation equations by rewriting known solutions (expressed in terms of physical scalars or affine coordinates) in a symplectically covariant way, using the projective (homogeneous) coordinates. We find that this is indeed possible for the under-rotating seed solution of [85], if one adds a ratio of harmonic functions to the standard vector of harmonic functions appearing in the stabilisation equations. Motivated by this, we introduce an ansatz for the general case that can incorporate all known extremal solutions. Our arguments are independent of the considerations in section 3, but the general form of the proposed ansatz is compatible with the generic first-order flow equations. However, it is difficult to fully impose it in the rotating case.

Finally, in section 5 we combine our general flow equations with the ansatz in the static case and connect to the fake superpotential formalism. Section 6 is devoted to concluding remarks, whereas in the Appendix we present a general heuristic argument justifying the presence of ratios of harmonic functions in the stabilisation equations.

2 *Bosonic action and special geometry*

In rewriting of the effective action as a sum of squares and deriving the flow equations for stationary black holes in 4-dimensional $N = 2$ supergravity, we largely follow the method and the notational conventions of the two papers [28, 32] whose results we generalise (and where we also refer the reader for more details and additional references).

Omitting the hypermultiplets, which are immaterial for our discussion, the relevant bosonic action [72, 73]

$$I_{4\mathrm{D}} = \frac{1}{16\pi} \int_{M_4} \Big(R \star 1 - 2\, g_{a\bar{b}}\, \mathrm{d}z^a \wedge \star \mathrm{d}\bar{z}^{\bar{b}} - \tfrac{1}{2} F^I \wedge G_I \Big) , \tag{3.1}$$

contains neutral complex scalars z^a (belonging to the n_v vector multiplets) and abelian gauge fields (from both the gravity multiplet and the vector multiplets), all coupled to gravity.

The scalars z^a are affine coordinates on a special Kähler manifold, whose metric can be calculated from the Kähler potential K: $g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K$, where ∂_a is shorthand for $\partial / \partial z^a$.

The field strengths are defined as $F^I = \mathrm{d}A^I$, where $A^I = (A^0, A^a)$, $a = 1, \ldots, n_\mathrm{v}$. The dual field strengths G_I are given in terms of the field strengths and the kinetic matrix $\mathcal{N}_{IJ}$ by

$$G_I = \mathrm{Im}\, \mathcal{N}_{IJ} \star F^J + \mathrm{Re}\, \mathcal{N}_{IJ}\, F^J . \tag{3.2}$$

We will not need the explicit formulae in what follows, but both the Kähler potential and the kinetic matrix $\mathcal{N}_{IJ}$ for the vector fields are derivable from a prepotential, F, which is a homogeneous function of degree 2. The prepotential itself is typically displayed in homogeneous projective coordinates X^I ($z^a = X^a / X^0$) and we will take it to be of the cubic type:

$$F = -\frac{1}{6} D_{abc} \frac{X^a X^b X^c}{X^0} =: (X^0)^2 f(z) , \qquad f(z) = -\frac{1}{6} D_{abc} z^a z^b z^c . \tag{3.3}$$

Surface integrals surrounding the sources of the field strengths and their duals define physical magnetic and electric charges, p^I and q_I, respectively:

$$p^I = \int_{S^2} F^I , \qquad q_I = \int_{S^2} G_I . \tag{3.4}$$

From a geometrical point of view, the above theory can be regarded as the bosonic massless sector of type IIB superstring theory in 10 dimensions compactified on a Calabi–Yau three-fold M_CY.[3] The scalars of the vector multiplets parametrise the moduli space of complex structure deformations

[3] Thanks to mirror symmetry, one can equivalently use the type IIA picture.

of M_{CY}. The complex dimension of this scalar manifold is given by one of the Hodge numbers of M_{CY}, $n_{\text{v}} = h^{2,1}$, with the Kähler potential $K(z,\bar{z})$ being determined by the unique (up to rescaling), nowhere vanishing holomorphic $(3,0)$-form Ω_{hol}, characterising M_{CY}:

$$K = -\ln\Big(\mathrm{i}\int_{M_{\text{CY}}} \Omega_{\text{hol}} \wedge \bar{\Omega}_{\text{hol}}\Big). \tag{3.5}$$

It will be more convenient later to work with the *covariantly* holomorphic version of the top form

$$\Omega(z,\bar{z}) = \mathrm{e}^{K(z,\bar{z})/2}\Omega_{\text{hol}}\,, \tag{3.6}$$

whose Kähler covariant derivative reads

$$\mathcal{D}\Omega = (\mathrm{d} + \mathrm{i}\,Q)\Omega\,, \tag{3.7}$$

where $Q = \operatorname{Im}(\partial_a K \mathrm{d}z^a)$ plays the role of the connection. In components:

$$\mathcal{D}_a\Omega = \partial_a\Omega + \tfrac{1}{2}\partial_a K\,\Omega\,, \qquad \bar{\mathcal{D}}_{\bar{a}}\Omega = \bar{\partial}_{\bar{a}}\Omega - \tfrac{1}{2}\bar{\partial}_{\bar{a}} K\,\Omega = 0\,. \tag{3.8}$$

In the canonical symplectic basis $\{\alpha_I, \beta^J\}$ for the third integral cohomology $H^3(M_{\text{CY}},\mathbb{Z})$, we can expand Ω as:

$$\Omega = X^I\alpha_I - F_I\beta^I\,, \tag{3.9}$$

where the coefficients are the periods of the Calabi–Yau manifold with respect to the dual homology basis of three-cycles $\{A^I, B_J\}$:

$$X^I = \int_{A^I}\Omega = \int_{M_{\text{CY}}}\Omega\wedge\beta^I\,, \qquad F_I = \int_{B_I}\Omega = \int_{M_{\text{CY}}}\Omega\wedge\alpha_I\,. \tag{3.10}$$

F_I are further identified with the derivatives of the prepotential F with respect to X^I: $F_I = \partial F/\partial X^I$ (we hope that no confusion with the gauge field strength two-form F^I arises).

Similarly, the five-form field strength $\mathcal{F}$ of the IIB theory, which takes values in $\Omega^2(M_4)\otimes H^3(M_{\text{CY}},\mathbb{Z})$, where $\Omega^2(M_4)$ represents the space of two-forms on spacetime, can be written as

$$\mathcal{F} = F^I\otimes\alpha_I - G_I\otimes\beta^I\,. \tag{3.11}$$

By integrating the field strength over an appropriate two-sphere in space, we recover the charges as the coefficients of the three-form $\Gamma\in H^3(M_{\text{CY}},\mathbb{Z})$:

$$\Gamma = \int_{S^2}\mathcal{F} = p^I\alpha_I - q_I\beta^I\,. \tag{3.12}$$

The five-form $\mathcal{F}$ in 10 dimensions is self-dual, $\star_{10}\mathcal{F} = (\star \otimes \diamond)\mathcal{F} = \mathcal{F}$, where $\star$ and $\diamond$ represent the Hodge operators in, respectively, spacetime and the internal CY manifold $M_{\rm CY}$. A representation of the Hodge operator on the basis forms $\{\alpha_I, \beta^J\}$ can be given in terms of a scalar-dependent matrix $\check{\mathcal{M}}(\mathcal{N})$:

$$\begin{pmatrix} \diamond\,\beta^I \\ \diamond\,\alpha_J \end{pmatrix} = \check{\mathcal{M}}^{-1}(\mathcal{N}) \begin{pmatrix} \beta^I \\ \alpha_J \end{pmatrix}, \tag{3.13}$$

so that the self-duality constraint on $\mathcal{F}$ can be expressed in terms of components as

$$\check{\mathcal{M}}(\mathcal{N}) \begin{pmatrix} \star F^I \\ \star G_J \end{pmatrix} = \begin{pmatrix} F^I \\ G_J \end{pmatrix}. \tag{3.14}$$

Instead of the canonical basis, one may use the Dolbeault cohomology basis furnished by $\{\Omega, \mathcal{D}_a\Omega, \bar{\mathcal{D}}_{\bar{a}}\bar{\Omega}, \bar{\Omega}\}$, which diagonalises the Hodge operator $\diamond$ on $M_{\rm CY}$:

$$\diamond\,\Omega = -\mathrm{i}\Omega\,, \qquad \diamond\,\mathcal{D}_a\Omega = \mathrm{i}\mathcal{D}_a\Omega\,, \tag{3.15}$$

and satisfies

$$\langle \Omega, \bar{\Omega}\rangle = -\mathrm{i}\,, \qquad \langle \mathcal{D}_a\Omega, \bar{\mathcal{D}}_{\bar{b}}\bar{\Omega}\rangle = \mathrm{i}g_{a\bar{b}}\,, \qquad \langle \mathcal{D}_a\Omega, \Omega\rangle = 0 \tag{3.16}$$

with respect to the antisymmetric intersection product on $H^3(M_{\rm CY})$

$$\langle E_1, E_2\rangle = \int_{M_{\rm CY}} E_1 \wedge E_2\,. \tag{3.17}$$

In this notation, the central charge Z can be written as

$$Z(\Gamma) = \langle \Gamma, \Omega\rangle = q_I X^I - p^I F_I\,, \tag{3.18}$$

and, conversely, one can prove that

$$\langle \Gamma_1, \Gamma_2\rangle = 2\,\mathrm{Im}[-Z(\Gamma_1)\,\bar{Z}(\Gamma_2) + g^{a\bar{b}}\,\mathcal{D}_a Z(\Gamma_1)\,\bar{\mathcal{D}}_{\bar{b}}\bar{Z}(\Gamma_2)]\,, \tag{3.19}$$

where $g^{a\bar{b}}$ is the inverse matrix of $g_{a\bar{b}}$.

Another useful object is the symmetric Hodge product $\langle E_1, \diamond E_2\rangle$, which introduces a norm on $H^3(M_{\rm CY}, \mathbb{R})$:

$$\|E\|^2 = \langle E, \diamond E\rangle\,. \tag{3.20}$$

An example of its utility in the context of the attractor mechanism is provided by the black hole (or effective) potential

$$V_{\rm BH} = \tfrac{1}{2}\|\Gamma\|^2 = -\tfrac{1}{2}\begin{pmatrix} p & q \end{pmatrix} \mathcal{M}(\mathcal{N}) \begin{pmatrix} p \\ q \end{pmatrix}, \tag{3.21}$$

where we suppressed the indices on the charges. (In what follows we will often identify the elements of $H^3(M_{\rm CY})$ with the associated vectors built out of the components in a symplectic basis, such as Γ and $(p^I, q_J)^{\rm T}$ here.) The matrices $\mathcal{M}(\mathcal{N})$ and $\check{\mathcal{M}}(\mathcal{N})$ are related to each other by the symplectic matrix $\mathcal{I}$,

$$\check{\mathcal{M}} = \mathcal{I}\mathcal{M}\,, \qquad \mathcal{I} = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \qquad \check{\mathcal{M}}^{-1} = -\check{\mathcal{M}}\,, \tag{3.22}$$

and are functions of the kinetic matrix $\mathcal{N}_{IJ}$ (see eg. [129]), although detailed expressions will not be needed in our considerations.

3 Flow equations from the action

In this section we derive generalised flow equations for non-BPS extremal black holes in Type IIB compactifications on Calabi–Yau manifolds, using the formalism of [28]. As these equations are not directly integrable, we describe an algorithm for solving them.

3.1 The action as a sum of squares

Since we are interested in asymptotically flat, stationary extremal black hole configurations, the ansatz that we use for the spacetime metric is:

$$ds^2 = -{\rm e}^{2U}({\rm d}t + \omega_i {\rm d}x^i)^2 + {\rm e}^{-2U}\delta_{ij}{\rm d}x^i{\rm d}x^j\,, \tag{3.23}$$

with the condition $U(x^i)$, $\omega_i(x^j) \to 0$ as $r = \sqrt{\delta_{ij}x^ix^j} \to \infty$. The one-form $\omega = \omega_i {\rm d}x^i$ encodes the angular momentum of the system.

The action (3.1) is not invariant under electromagnetic duality rotations, but as remarked in [28], at the cost of discarding manifest Lorentz invariance, one can have a duality invariant formalism [130, 131]. For this purpose it is convenient to introduce the following symmetric product of spatial 2-forms $\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}} \in \Omega^2(\mathbb{R}^3) \otimes H^3(M_{\rm CY})$:

$$(\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}) = \frac{{\rm e}^{2U}}{1-w^2}\int_{M_{\rm CY}} \boldsymbol{\mathcal{B}} \wedge \left[\star_{\mathbf{0}}(\diamond\boldsymbol{\mathcal{C}}) - \star_{\mathbf{0}}(w \wedge \diamond\boldsymbol{\mathcal{C}})\, w + \star_{\mathbf{0}}(w \wedge \star_{\mathbf{0}}\boldsymbol{\mathcal{C}})\right], \tag{3.24}$$

where by $\star_{\mathbf{0}}$ we denote the Hodge dual with respect to the flat three-dimensional metric δ_{ij} and define $w = {\rm e}^{2U}\omega$. Note that for w not too large $(\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{B}}) \geqslant 0$. In general, boldface symbols will be reserved for quantities in the three spatial dimensions.

With this notation, the effective action (3.1) can be written as

$$I_{4\text{D eff}} = -\frac{1}{16\pi}\int \mathrm{d}t \int_{\mathbb{R}^3} \Big[2\mathbf{d}U \wedge \star_0 \mathbf{d}U - \tfrac{1}{2}\mathrm{e}^{4U}\mathbf{d}\omega \wedge \star_0 \mathbf{d}\omega \\ + 2g_{a\bar{b}}\,\mathbf{d}z^a \wedge \star_0 \mathbf{d}\bar{z}^{\bar{b}} + (\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}})\Big]. \tag{3.25}$$

As shown in [28], this action can be re-expressed as a sum of squares, giving first-order flow equations for stationary supersymmetric black holes, including multi-centre composites:

$$\boldsymbol{\mathcal{F}} - 2\,\mathrm{Im}\star_0 \mathbf{D}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) + 2\,\mathrm{Re}\,\mathbf{D}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\omega) = 0\,, \tag{3.26}$$

$$\mathbf{Q} + \mathbf{d}\alpha + \tfrac{1}{2}\mathrm{e}^{2U} \star_0 \mathbf{d}\omega = 0\,, \tag{3.27}$$

where

$$\mathbf{D} = \mathbf{d} + \mathrm{i}(\mathbf{Q} + \mathbf{d}\alpha + \tfrac{1}{2}\mathrm{e}^{2U} \star_0 \mathbf{d}\omega)\,, \qquad \mathbf{Q} = \mathrm{Im}\,(\partial_a K \mathbf{d}z^a)\,. \tag{3.28}$$

In the light of the considerations in [22], one might expect similar equations, involving a modified field strength $\tilde{\boldsymbol{\mathcal{F}}}$ in the place of the actual field strength $\boldsymbol{\mathcal{F}}$, to exist for non-supersymmetric extremal black holes as well. In [32] this was shown to be true for the special case when the fake field strength is related to the real field strength by a constant symplectic matrix. Building on this, we have found a more general way of writing the action as a sum of squares resulting in non-BPS first-order flow equations, based on a non-closed $\tilde{\boldsymbol{\mathcal{F}}}$ and a new auxiliary one-form $\boldsymbol{\eta}$ related to the non-closure of $\tilde{\boldsymbol{\mathcal{F}}}$. These two objects are constrained by two new equations which need to be satisfied in addition to the flow equations we obtained.

While they will be derived and explained below, for ease of comparison with (3.26, 3.27), we state our non-BPS equations now. The first two equations are very similar to the BPS ones:

$$\tilde{\boldsymbol{\mathcal{F}}} - 2\,\mathrm{Im}\star_0 \mathbf{D}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) + 2\,\mathrm{Re}\,\mathbf{D}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\omega) = 0\,, \tag{3.29}$$

$$\mathbf{Q} + \mathbf{d}\alpha + \boldsymbol{\eta} + \tfrac{1}{2}\mathrm{e}^{2U} \star_0 \mathbf{d}\omega = 0\,, \tag{3.30}$$

with $\tilde{\boldsymbol{\mathcal{F}}}$ replacing $\boldsymbol{\mathcal{F}}$ in the first equation and $\boldsymbol{\eta}$ shifting the second.[4] In addition, the two equations constraining our two auxiliary variables are:

$$\big(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}}\big) = \big(\tilde{\boldsymbol{\mathcal{F}}}, \tilde{\boldsymbol{\mathcal{F}}}\big) - 2\mathbf{d}\boldsymbol{\eta} \wedge w\,, \tag{3.31}$$

$$\boldsymbol{\eta} \wedge \mathrm{Im}\langle \tilde{\boldsymbol{\mathcal{G}}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle = \langle \mathbf{d}\tilde{\boldsymbol{\mathcal{F}}}, \mathrm{Re}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega)\rangle - \tfrac{1}{2}\mathbf{d}\boldsymbol{\eta} \wedge w\,, \tag{3.32}$$

[4] $\mathbf{D}$ and $\mathbf{Q}$ are defined as in (3.28).

where

$$\tilde{\boldsymbol{\mathcal{G}}} = \tilde{\boldsymbol{\mathcal{F}}} - 2\,\mathrm{Im}\,\boldsymbol{\star_0}\mathbf{D}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) + 2\,\mathrm{Re}\,\mathbf{D}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\omega)\,. \tag{3.33}$$

Now onto the derivation. In [28] the crucial step in obtaining the first-order manifestly duality-invariant flow equations (that imply the second-order equations of motion) is to pair appropriately the derivatives of the scalars with the gauge fields and use the scalar product (3.24) to re-express the Lagrangian. It was found in [28] that a good choice is

$$\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{F}} - 2\,\mathrm{Im}\,\boldsymbol{\star_0}\mathbf{D}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) + 2\,\mathrm{Re}\,\mathbf{D}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\omega)\,. \tag{3.34}$$

Similarly to [32], we generalise the above to eq. (3.33) by replacing the actual field strength with a two-form valued 'fake' field strength, $\tilde{\boldsymbol{\mathcal{F}}} \in \Omega^2(\mathbb{R}^3) \otimes H^3(M_{\mathrm{CY}})$, but here we define it by demanding only that it reproduces the original electromagnetic part of the action

$$\big(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}}\big) = \big(\tilde{\boldsymbol{\mathcal{F}}}, \tilde{\boldsymbol{\mathcal{F}}}\big) - \boldsymbol{\Xi}\,, \tag{3.35}$$

up to a possible extra term described by the three-form $\boldsymbol{\Xi}$. The form of $\boldsymbol{\Xi}$ will be determined at the end of this subsection by consistency arguments. Unlike the real field strength, we do not require $\tilde{\boldsymbol{\mathcal{F}}}$ to be closed.

One can then rewrite the Lagrangian in terms of $\tilde{\boldsymbol{\mathcal{G}}}$ as

$$\begin{aligned}\mathcal{L} = (\tilde{\boldsymbol{\mathcal{G}}}, \tilde{\boldsymbol{\mathcal{G}}}) &- 4\,(\mathbf{Q} + \mathbf{d}\alpha + \boldsymbol{\eta} + \tfrac{1}{2}\mathrm{e}^{2U}\,\boldsymbol{\star_0}\,\mathbf{d}\omega) \wedge \mathrm{Im}\langle\tilde{\boldsymbol{\mathcal{G}}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle \\ &+ \mathbf{d}\,[\,2w \wedge (\mathbf{Q} + \mathbf{d}\alpha) + 4\,\mathrm{Re}\langle\tilde{\boldsymbol{\mathcal{F}}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle\,]\,,\end{aligned} \tag{3.36}$$

provided that the new one-form $\boldsymbol{\eta}$, which needs to be introduced due to the possible non-closure of $\tilde{\boldsymbol{\mathcal{F}}}$, satisfies

$$\boldsymbol{\eta} \wedge \mathrm{Im}\langle\tilde{\boldsymbol{\mathcal{G}}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle = \langle\mathbf{d}\tilde{\boldsymbol{\mathcal{F}}}, \mathrm{Re}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega)\rangle\, + \tfrac{1}{4}\boldsymbol{\Xi}\,. \tag{3.37}$$

Because adding a total derivative to the Lagrangian does not change the equations of motion, one finds that equation (3.37) needs to hold only up to a total derivative. Observe that the phase $\alpha = \alpha(\mathbf{x})$ is a priori an arbitrary function.

A sufficient condition for a stationary point of the action (hence, for the equations of motion to be satisfied) is met by neglecting the boundary terms and requiring that the variations of the first two terms in (3.36) vanish separately, which leads to (3.29, 3.30). From (3.30) we obtain $\mathbf{D} = \mathbf{d} - \mathrm{i}\boldsymbol{\eta}$, which substituted into (3.29) gives:

$$\tilde{\boldsymbol{\mathcal{F}}} - 2\,\mathrm{Im}\big[\boldsymbol{\star_0}(\mathbf{d} - \mathrm{i}\boldsymbol{\eta})(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega)\big] + 2\,\mathrm{Re}\big[(\mathbf{d} - \mathrm{i}\boldsymbol{\eta})(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\omega)\big] = 0\,. \tag{3.38}$$

Differentiating one finds:

$$\mathbf{d}\star_{\mathbf{0}}\mathbf{d}\,\mathrm{Im}\,\hat{\Omega} - \mathbf{d}\left(\star_{\mathbf{0}}\boldsymbol{\eta}\,\mathrm{Re}\,\hat{\Omega}\right) - \mathbf{d}\left(\boldsymbol{\eta}\wedge w\,\mathrm{Im}\,\hat{\Omega}\right) = \tfrac{1}{2}\mathbf{d}\tilde{\boldsymbol{\mathcal{F}}}\,, \tag{3.39}$$

where $\hat{\Omega} = \mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega$.[5] We see in particular that, as mentioned earlier, the fake field strength is not necessarily closed.

It is now possible to derive $\boldsymbol{\Xi}$ in terms of the other quantities appearing in the rewriting. The fundamental observation is that in the dynamical system we are considering, the electromagnetic part of the Lagrangian acts as a potential for the remaining fields and, for solutions of the equations of motion with vanishing action, it is expected to equal the kinetic energy. So, by expressing $\tilde{\boldsymbol{\mathcal{F}}}$ through equation (3.38), one can compute $\left(\tilde{\boldsymbol{\mathcal{F}}},\tilde{\boldsymbol{\mathcal{F}}}\right)$ and find:

$$\begin{aligned}\left(\tilde{\boldsymbol{\mathcal{F}}},\tilde{\boldsymbol{\mathcal{F}}}\right) &= 2\mathbf{d}U\wedge\star_{\mathbf{0}}\mathbf{d}U - \tfrac{1}{2}\mathrm{e}^{4U}\mathbf{d}\omega\wedge\star_{\mathbf{0}}\mathbf{d}\omega + 2g_{a\bar{b}}\mathbf{d}z^a\wedge\star_{\mathbf{0}}\mathbf{d}z^{\bar{b}}\\ &\quad + \mathrm{e}^{2U}\mathbf{d}w\wedge\star_{\mathbf{0}}\mathbf{d}\omega + 2\mathbf{d}\mathbf{Q}\wedge w\\ &= \left(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{F}}\right) + \mathrm{e}^{2U}\mathbf{d}w\wedge\star_{\mathbf{0}}\mathbf{d}\omega + 2\mathbf{d}\mathbf{Q}\wedge w\,,\end{aligned} \tag{3.40}$$

from which, using (3.35), (3.30) and integration by parts, one obtains

$$\boldsymbol{\Xi} = -2\mathbf{d}\boldsymbol{\eta}\wedge w\,. \tag{3.41}$$

Finally, substituting (3.41) in (3.35) and (3.37) leads to (3.31, 3.32).

In summary, we have obtained a non-supersymmetric generalisation (3.29, 3.30) of the first-order equations (3.26, 3.27). The generalised equations are expressed in terms of a fake field strength $\tilde{\boldsymbol{\mathcal{F}}}$, constrained by (3.35) to reproduce the original gauge part of the action. The non-closure of $\tilde{\boldsymbol{\mathcal{F}}}$ necessitates the introduction of a new, compensating object, $\boldsymbol{\eta}$, in eq. (3.37). The auxiliary three-form $\boldsymbol{\Xi}$ appearing in (3.35) and (3.37) can be expressed in terms of other quantities through eq. (3.41). In comparison with the supersymmetric case we thus have two more unknowns, $\tilde{\boldsymbol{\mathcal{F}}}$ and $\boldsymbol{\eta}$, constrained by (3.31, 3.32). Since they are mutually related, in any model for which $\tilde{\boldsymbol{\mathcal{F}}}$ can be obtained by other means (as in section 5.2), $\boldsymbol{\eta}$ can be eliminated as well.

3.2 Solving the equations

Whenever $\mathbf{d}\tilde{\boldsymbol{\mathcal{F}}} = 0$ and $\boldsymbol{\eta} = 0$, equation (3.39) reduces to the Laplace equation

$$2\,\mathbf{d}\star_{\mathbf{0}}\mathbf{d}\,\mathrm{Im}\,\hat{\Omega} = 0\,, \tag{3.42}$$

[5] Note that in [28] the hat symbolised what we call $\diamond$ here.

which can be integrated to so-called stabilisation equations:

$$2\,\mathrm{Im}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) = \mathcal{H}\,. \tag{3.43}$$

These express the period vector in terms of (possibly multi-centred) harmonic functions $\mathcal{H}$ throughout the flow.

Using this result back in (3.38) one finds that the fake field strength is given by

$$\tilde{\boldsymbol{\mathcal{F}}} = \star_{\mathbf{0}}\mathbf{d}\mathcal{H} - 2\,\mathbf{d}(\mathrm{e}^{2U}\,\mathrm{Re}\,\hat{\Omega}\omega)\,. \tag{3.44}$$

This is manifestly true for the supersymmetric case [28, 94] and for the non-BPS setting considered in [32], for electric $(p^0, 0; 0, q_a)$ or magnetic $(0, p^a; q_0, 0)$ charge configuration and axions $\mathrm{Re}\,z^a$ set to zero.[6] In the BPS case, one has $\tilde{\boldsymbol{\mathcal{F}}} = \boldsymbol{\mathcal{F}}$ by assumption, so that the vector of electric and magnetic charges is determined through (3.12) to be

$$\Gamma = \frac{1}{4\pi}\int_{S^2} \star_{\mathbf{0}}\mathbf{d}\mathcal{H}\,, \tag{3.45}$$

or, in other words, equal to the poles of the harmonic functions $\mathcal{H}$. For non-supersymmetric solutions, $\tilde{\boldsymbol{\mathcal{F}}}$ is related to $\boldsymbol{\mathcal{F}}$ by charge sign reversals, and the same holds for the poles of the harmonic functions $\mathcal{H}$ in (3.44) compared to the physical charges.

In its general form, however, equation (3.39) cannot be solved directly, since the period vector Ω (scalars), $\boldsymbol{\eta}$, ω and $\tilde{\boldsymbol{\mathcal{F}}}$ are all unknown and constrained by (3.32). A way out of this problem is to first make an ansatz for $\mathrm{Im}\,\hat{\Omega}$, try to solve it for Ω and U, find $\boldsymbol{\eta}$ and ω from (3.30) and then, by using (3.38) as a definition for $\tilde{\boldsymbol{\mathcal{F}}}$, check if (3.31, 3.32) are satisfied.

Let us see explicitly how to do that: we start by making an ansatz of the type

$$2\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{-U}\Omega) = \mathcal{J}\,, \tag{3.46}$$

with $\mathcal{J}$ a vector containing all the parameters in terms of which the solutions will be expressed. The solution for the components of $\hat{\Omega}$ (and hence for scalars) can then be obtained in the same way as the solutions to supersymmetric stabilisation equations ([133], see also [134], section 2). We will indicate all the quantities calculated with the aid of the ansatz by adding the subscript $\mathcal{J}$.

[6] By charge redefinitions one can generate physically equivalent solutions also for other charge configurations [132].

We then proceed differentiating both sides of (3.46) and subsequently taking the intersection product with the real and imaginary part of Ω and with $\mathcal{D}_a\Omega$. With the definitions $\Psi := -\langle \mathbf{d}\mathcal{J}, \Omega\rangle$ and $\mathcal{D}_a\Psi := -\langle \mathbf{d}\mathcal{J}, \mathcal{D}_a\Omega\rangle$ one obtains:

$$\mathbf{d}U = -\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\alpha}\Psi)\,, \tag{3.47}$$

$$\mathbf{d}\alpha + \mathbf{Q} = \mathrm{e}^{U}\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\alpha}\Psi) = -\tfrac{1}{2}\mathrm{e}^{2U}\langle \mathbf{d}\mathcal{J}, \mathcal{J}\rangle\,, \tag{3.48}$$

$$\mathbf{d}\bar{z}^{\bar{b}} = -g^{a\bar{b}}\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{D}_a\Psi\,. \tag{3.49}$$

Note that (3.47) and (3.49) are the flow equations for the warp factor and the scalars, while (3.48) gives an explicit relation between α and the other quantities appearing in the rewriting. More specifically, (3.48) combined with (3.30) eliminates α, giving:

$$\langle \mathbf{d}\mathcal{J}, \mathcal{J}\rangle = 2\mathrm{e}^{-2U}\boldsymbol{\eta} + \star_{\mathbf{0}}\mathbf{d}\omega\,. \tag{3.50}$$

If we make an ansatz also for the angular momentum of the black hole ω_{BH} (which must be expressed in terms of parameters appearing in $\mathcal{J}$), we arrive at an expression for $\boldsymbol{\eta}$:

$$\boldsymbol{\eta}_{\mathcal{J},\omega} = \tfrac{1}{2}\mathrm{e}^{2U_{\mathcal{J}}}\Big(\langle \mathbf{d}\mathcal{J}, \mathcal{J}\rangle - \star_{\mathbf{0}}\mathbf{d}\omega_{\mathrm{BH}}\Big)\,. \tag{3.51}$$

Independently of the ansatz, we can use (3.38) as a definition of $\tilde{\boldsymbol{\mathcal{F}}}$ and then substitute (3.39) and (3.29) in (3.32). The left-hand side is clearly zero, whereas on the right-hand side we have an intersection product that we know how to compute. Neglecting the total derivative results in:

$$\langle \mathbf{d}\tilde{\boldsymbol{\mathcal{F}}}, \mathrm{Re}(\mathrm{e}^{2U}\hat{\Omega})\rangle = \langle 2\,\mathbf{d}\star_{\mathbf{0}}\mathbf{d}\,\mathrm{Im}\,\hat{\Omega}, \mathrm{Re}(\mathrm{e}^{2U}\hat{\Omega})\rangle - \boldsymbol{\eta}\wedge\star_{\mathbf{0}}\boldsymbol{\eta} - \mathrm{e}^{2U}\boldsymbol{\eta}\wedge\mathbf{d}\omega + \tfrac{1}{2}\boldsymbol{\eta}\wedge\mathbf{d}w\,. \tag{3.52}$$

The last term here cancels the last term of (3.32). This means that once all the variables have been expressed in terms of the parameters in the vector $\mathcal{J}$, the consistency of the ansatz with the first-order equations (3.29, 3.30) and the constraint (3.32) can be verified by checking whether the following equation is satisfied:

$$\mathrm{e}^{2U_{\mathcal{J}}}\langle \mathbf{d}\star_{\mathbf{0}}\mathbf{d}\mathcal{J}, \mathrm{Re}\,\hat{\Omega}_{\mathcal{J}}\rangle = \boldsymbol{\eta}_{\mathcal{J},\omega}\wedge\star_{\mathbf{0}}\boldsymbol{\eta}_{\mathcal{J},\omega} + \mathrm{e}^{2U_{\mathcal{J}}}\boldsymbol{\eta}_{\mathcal{J},\omega}\wedge\,\mathbf{d}\omega_{\mathrm{BH}}\,. \tag{3.53}$$

As this is an equation for $\mathcal{J}$, in principle it determines an ansatz that satisfies (3.29), (3.30) and (3.32), even though in practice one would not solve it for $\mathcal{J}$, regarding it instead as a check for the specific form of an ansatz assumed beforehand. One would also still have to ensure that (3.31) is satisfied, which can be a rather non-trivial task. Finally, not all the parameters in the ansatz may be constrained by the equations of motion but should rather be fixed by appropriate boundary conditions.

In the next section we will discuss known black hole solutions which satisfy (3.53) with a nontrivial $\boldsymbol{\eta}$ and propose a generic ansatz.

4 Stabilisation equations from an ansatz

An important result for BPS black holes is the direct integrability of the first order flow equations to stabilisation equations, even for multiple centres. As described in the beginning of section 3.2, this result can be extended to some non-supersymmetric solutions. These examples, however, are not generic, in the sense that applying dualities on them does not lead to the most general non-BPS solution.

Given the flow equations in section 3.1, one expects to find a nontrivial $\boldsymbol{\eta}$ and $\tilde{\boldsymbol{\mathcal{F}}}$ in the general case. The non-closure of $\tilde{\boldsymbol{\mathcal{F}}}$ implies that the corresponding expression for the period vector Ω in terms of charges and integration constants should be an anharmonic extension of (3.43), which must still be consistent with symplectic reparametrisations. To gain intuition about the possible terms, one can follow a bottom-up approach. Therefore, we consider known explicit solutions and rewrite the physical scalars z^a and the metric in terms of Ω, aiming towards a generic ansatz that covers all single-centre solutions.

4.1 Special solutions

In order to be as general as possible and to minimise the ambiguity introduced in the process, we find it illuminating to start with a rotating black hole solution, so that the presence of an extra harmonic function describing angular momentum can provide guidance. Consider the rotating extremal black hole of [85], which can be used as the seed solution for four-dimensional under-rotating black holes [81–83] in theories with cubic prepotentials. This is an almost BPS [84] solution in five-dimensional supergravity described by the harmonic functions:

$$H^0 = h^0 + \frac{p^0}{r}\,, \qquad H_a = h_a + \frac{q_a}{r}\,, \qquad M = b + \frac{J\cos\theta}{r^2}\,, \tag{3.54}$$

where h^0, h_a, b are constants that are related to asymptotic moduli, $-p^0$ is the Kaluza–Klein magnetic charge, q_a are the electric charges and J is the angular momentum of the solution. Therefore, the associated four-dimensional charge vector is defined by the harmonic functions:

$$\mathcal{H}_c = (-H^0,\, 0\,;\, 0,\, H_a)\,, \tag{3.55}$$

whereas M controls the angular momentum and is invariant under symplectic transformations.

Using the 4D/5D dictionary of [134–136], one can rewrite the full solution given in five-dimensional notation [85] in terms of variables natural from the four-dimensional perspective. The metric is as in (3.23), while the resulting expressions for the gauge fields and scalars in our notation are:

$$\boldsymbol{\mathcal{F}} = \star_0 \mathbf{d}\mathcal{H}_c - 2\,\mathbf{d}(\mathrm{e}^{2U}\mathrm{Re}\,\hat{\Omega}\,\omega)\,, \qquad 2\,\mathrm{Im}\,\hat{\Omega} = \mathcal{J} \equiv \mathcal{H} + \mathcal{R}\,, \tag{3.56}$$

where we use again the shorthand $\hat{\Omega} = \mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega$ as in section 3. $\mathcal{J}$ is written in terms of a harmonic part, $\mathcal{H}$, and a part containing ratios of harmonic functions $\mathcal{R}$, which are respectively given by:

$$\mathcal{H} = (H^0,\, 0\,;\, 0,\, H_a)\,, \qquad \mathcal{R} = \left(0,\, 0\,;\, -\frac{M}{H^0},\, 0\right). \tag{3.57}$$

Finally, the metric functions are given by:

$$\star_0\, \mathbf{d}\omega = \mathbf{d}M\,, \qquad \mathrm{e}^{-2U} = \mathrm{i}\langle\hat{\Omega}, \bar{\hat{\Omega}}\rangle = \sqrt{I_4(\mathcal{H}) - M^2}\,. \tag{3.58}$$

Here, I_4 is the quartic invariant that appears in the entropy formula for cubic prepotentials (see [137] for explicit expressions) and the physical scalars are given by $z^a = X^a/X^0$, as usual.

The expression for the gauge fields in (3.56) parallels the form of the BPS solutions (3.44), differing in that the vector of harmonic functions associated to the physical charges (3.55), is related to the one appearing in the scalars by a single sign flip, similar to [102]. The period vector Ω is again determined through stabilisation equations similar to (3.43), so that the scalars are given in terms of harmonic functions describing the flow from asymptotic infinity to the horizon. In particular, the asymptotic values of the scalars are controlled by the constant parts of the harmonic functions $\mathcal{H}$ and M, whereas the attractor equations, obtained in the limit $r \to 0$, are controlled by the charges and the angular momentum [121].

The novel addition to $\mathcal{J}$ is a ratio of harmonic functions that was not present in previous attempts to write non-BPS stabilisation equations and allows for nontrivial axions. Deferring the comparison to the rewriting of section 3 for the next section, we note that this solution leads through (3.51) to a nontrivial $\boldsymbol{\eta}$, given by

$$\boldsymbol{\eta} = \mathrm{e}^{2U} \langle \mathbf{d}\mathcal{R}, \mathcal{H} \rangle = -\mathrm{e}^{2U}\, H^0\, \mathbf{d} \left(\frac{M}{H^0} \right) , \tag{3.59}$$

which demonstrates how $\boldsymbol{\eta}$ is related to the anharmonic part of the solution.

In the static limit, it is possible to show that all constraints and flow equations of the previous section are indeed satisfied, if $\mathcal{J}$ in (3.46) is identified with the one in (3.56). The presence of a nontrivial $\boldsymbol{\eta}$ that follows from the ratio M/H^0 through (3.59) is crucial in this respect. In the rotating case, we have verified the constraint (3.53), but it is more challenging to verify the flow equations and especially the first constraint (3.31).

As this is the seed solution for under-rotating extremal black holes, one can apply duality rotations on the full rotating solution using the stabilisation equations (3.56) to find the most general solution. Imposing that the angular momentum harmonic function M is invariant under duality transformations, the result is that a ratio of harmonic functions is generated in all other cases as well.

For example, in the case of the *stu* model, one can explicitly dualise to the frame with only two charges present, corresponding to a D0-D6 brane system in Type IIA theory. For this model, the prepotential is as in (3.3) with $D_{abc} = |\varepsilon_{abc}|$ and the scalar sector is then described by the choice (no sum on $a = 1, 2, 3$):

$$\mathcal{H} = \Big(H^0,\, \frac{1}{\lambda^a} H^a \,;\, H_0,\, \lambda^a H_a \Big) , \qquad \mathcal{R} = \frac{1}{8} \frac{M}{H_0^+} \Big(1,\, \frac{1}{\lambda^a} \,; -1,\, -\lambda^a \Big) , \tag{3.60}$$

where

$$H_I = h_I + \frac{q_0}{r} , \qquad H_0^+ = \frac{1}{4} \Big(h_0 + \sum_a h_a \Big) + \frac{q_0}{r} \tag{3.61}$$

$$H^I = -\lambda^3 H_I , \qquad H^{0\,+} = -\lambda^3 H_0^+ , \qquad D_{abc} \lambda^a \lambda^b \lambda^c \equiv \lambda^3 , \tag{3.62}$$

$$\mathrm{e}^{-4U} = I_4(\mathcal{H}) - M^2 = (H_0\, H^0)^2 - M^2 , \tag{3.63}$$

and λ^3 must be a constant. Note that the individual constants λ^a appear only as multiplicative factors in $\mathcal{H}$ and $\mathcal{R}$, but not in e^{-U}, which depends only on the physical harmonic functions H_0 and H^0. It follows that the metric and gauge fields depend only on the combination λ^3, so that two of the λ^a correspond to flat directions. The structure in (3.60) is consistent with the results on D0-D6 attractors in [61] and seems to be generic for D0-D6 solutions for all cubic prepotentials.

It is interesting to note that unlike in (3.56), the harmonic part $\mathcal{H}$ is not related to the charges by sign flips, as one might expect. In fact the electric solution is special, in the sense that the flat directions can be described through (3.56) by simply allowing for the missing harmonic functions to be constants, at the cost of making $\mathcal{R}$ more complicated, but still proportional to a single ratio as in the D0-D6 case. On the other hand, for both solutions the angular momentum harmonic function can be invariantly characterised by $M = \langle \mathcal{H}, \mathcal{R} \rangle$. The flat directions described by the λ^a are zero modes of this equation.

4.2 The ansatz

On account of the above observations, it is natural to propose an ansatz for the period vector that contains harmonic functions and ratios of harmonic functions, disregarding the precise relation to the physical charges, which is to be fixed later. In fact, it is simple to see that imposing consistency of any generic ansatz $\operatorname{Im}\Omega \sim \mathcal{H} + \mathcal{R}$, leads to inverse harmonic functions. Since one can compute $\langle \mathcal{H}, \mathcal{R} \rangle$ in two ways:

$$\langle \mathcal{H}, \mathcal{R} \rangle = \operatorname{Im}\langle \hat{\Omega}, \mathcal{R} \rangle = -\operatorname{Im}\langle \hat{\Omega}, \mathcal{H} \rangle \,, \tag{3.64}$$

where $\mathcal{H}$ and $\mathcal{R}$ are a priori independent, it follows that $\langle \mathcal{H}, \mathcal{R} \rangle$ must be a scalar-independent quantity. The only other fields in the system are the scale factor and the rotation form ω in the metric, but since $\langle \mathcal{H}, \mathcal{R} \rangle$ does not carry a scale,[7] it cannot depend on e^U, in accord with the explicit solution above, where $\langle \mathcal{H}, \mathcal{R} \rangle = M$. In the static limit M reduces to a constant, in which case the constraint could be solved even if $\mathcal{R}$ were harmonic, but in the rotating case one has to reproduce the full function M, which depends on the angular coordinates. This implies a structure as in (3.56), with the anharmonic part, which then must be present even when the angular momentum is turned off. A more extensive argument about the kind of zero modes allowed for the scalars, which leads to the same conclusion, is given in the Appendix.

Based on the linearity of symplectic reparametrisations and the fact that (3.57) and (3.60) are seed solutions, we expect the structure seen in the

[7] Here we refer to the symmetry of (3.36) under $e^U \to e^D e^U$, $\Omega \to e^D \Omega$, $g_{ij} \to e^{2D} g_{ij}$ for constant D, inherited from the full conformal formulation of the theory [138].

previous section to be universal for all under-rotating extremal black holes. In other words, we take the point of view that there is no essential difference between static non-supersymmetric and under-rotating black holes, since they are continuously connected by setting to zero the nonconstant part of a single harmonic function, as in (3.54). Therefore, we propose the following form for the stabilisation equations for the scalars and the angular momentum:

$$2\,\mathrm{Im}\,\hat{\Omega} \equiv 2\,\mathrm{Im}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) = \mathcal{H} + \mathcal{R}\,, \tag{3.65}$$

$$\star_0 \mathbf{d}\omega = \langle \mathbf{d}\mathcal{H}, \mathcal{H}\rangle + \mathbf{d}\langle \mathcal{H}, \mathcal{R}\rangle\,, \tag{3.66}$$

where $\mathcal{H}$ is a vector of harmonic functions and $\mathcal{R}$ is a vector of ratios of harmonic functions. The integrability condition of the last equation implies that their symplectic inner product $\langle \mathcal{H}, \mathcal{R}\rangle$ is a harmonic function, while the scale function of the metric is given by:

$$\mathrm{e}^{-2U} = \mathrm{i}\langle \hat{\Omega}, \bar{\hat{\Omega}}\rangle = \sqrt{I_4(\mathcal{H}+\mathcal{R})}\,. \tag{3.67}$$

Note that when $\mathcal{R} = 0$ and the charges carried by $\mathcal{H}$ are identified with the physical charges, one recovers the BPS stabilisation equations, as required. More generally, for a physically reasonable solution the harmonic and inverse harmonic functions in (3.65) are quite restricted due to various consistency constraints, both generic and based on known explicit solutions. The rest of this section is devoted to a discussion of these generic constraints and some of their implications.

A first requirement is that in the near-horizon limit the scale factor e^{-4U} of an under-rotating black hole must reduce to [121]:

$$\mathrm{e}^{-4U} \propto |I_4(\Gamma)| - J^2\cos^2\theta\,, \tag{3.68}$$

where $I_4(\Gamma)$ is the quartic invariant of the model and J is the angular momentum. In the simple case of vanishing angular momentum, $\mathcal{R}$ is proportional to inverse harmonic functions and thus vanishes near the horizon. Therefore, a harmonic piece must always be present in the right hand side of (3.65), to make sense of the static solution in the near-horizon region. Similar comments then apply for the full rotating case, hence it is impossible to have a physical solution for the scalars based purely on inverse harmonic functions.

Going over to the constraints posed by the form of the full solution, observe that in the (necessarily static) BPS case the full scale function is simply $\mathrm{e}^{-4U} = I_4(\mathcal{H})$, where the charges are replaced by their corresponding harmonic functions. Similarly, for the *stu* model, where the most general non-BPS static black hole was explicitly constructed in [114] using the seed solution of [25, 64], it has been shown that the scale factor is shifted as $\mathrm{e}^{-4U} \sim I_4(\mathcal{H}) - b^2$, where b is a constant that does not depend on the charges.

Interestingly, for the known under-rotating seed solution the expression for e^{-U} in (3.58) can again be found from (3.68) by replacing the charges and angular momentum by harmonic functions. Moreover, the additional constant b of [25,64,114] is identified with the constant piece in the harmonic function for the angular momentum in (3.58), as in [84, 85]. Therefore it is reasonable to expect that generically the scale factor is a function of the harmonic functions for the charges and angular momentum, thus allowing for the presence of a possible residual constant in the static solutions, when J is set to zero.

Now, for an ansatz of the type (3.65) to describe the known solutions, the vector $\mathcal{R}$ must be such that (3.67) is consistent with the above comments, in particular with (3.58), so that

$$\mathrm{e}^{-4U} = I_4(\mathcal{H} + \mathcal{R}) = I_4(\mathcal{H}) - \langle \mathcal{H}, \mathcal{R} \rangle^2 \,. \tag{3.69}$$

This equality poses very strong restrictions on $\mathcal{R}$, as it does not appear in linear, cubic or quartic terms. In particular, the components of $\mathcal{R}$ must be such that $I_4(\mathcal{R})$ and its first derivatives vanish, implying that it must have at most as many independent components as a two-charge small black hole. Then, given $\mathcal{H}$ and a model in which I_4 is known, the linear term in $\mathcal{R}$ should vanish, further restricting its independent components. Indeed, $\mathcal{R}$ appears to have only one independent component in the explicit solutions (3.57) and (3.60).

For symmetric cubic models this can be made more precise, by Taylor expanding the left hand side of (3.69) explicitly. In these models, the quartic invariant can be rewritten in terms of the central charge as

$$I_4 = (i_1 - i_2)^2 + 4 i_4 - i_5 \,, \tag{3.70}$$

where

$$i_1 = Z\bar{Z} \,, \qquad i_2 = g^{a\bar{b}} \mathcal{D}_a Z \bar{\mathcal{D}}_{\bar{b}} \bar{Z} \,, \tag{3.71}$$

$$i_3 = \tfrac{1}{3} \operatorname{Re}(Z N_3(\bar{Z})) \,, \qquad i_4 = -\tfrac{1}{3} \operatorname{Im}(Z N_3(\bar{Z})) \,, \tag{3.72}$$

$$i_5 = g^{a\bar{a}} D_{abc} D_{\bar{a}\bar{b}\bar{c}} g^{b\bar{d}} g^{c\bar{e}} g^{d\bar{b}} g^{e\bar{c}} \bar{\mathcal{D}}_{\bar{d}} \bar{Z} \, \bar{\mathcal{D}}_{\bar{a}} \bar{Z} \, \mathcal{D}_d Z \, \mathcal{D}_e Z \,, \tag{3.73}$$

and

$$N_3(\bar{Z}) = D_{abc} g^{a\bar{a}} g^{b\bar{b}} g^{c\bar{c}} \bar{\mathcal{D}}_{\bar{a}} \bar{Z}\, \bar{\mathcal{D}}_{\bar{b}} \bar{Z}\, \bar{\mathcal{D}}_{\bar{c}} \bar{Z}\,. \tag{3.74}$$

Although these five invariants all depend on the scalar fields and the charges, the combination in (3.70) is scalar independent. In this form, it is easy to expand $I_4(\mathcal{H}+\mathcal{R})$ and separately consider the different terms, since Z and its derivatives are linear in the charges. Furthermore, as shown in [139], there are relations between the invariants above when the charge vector is that of a small black hole. The previous discussion suggests that $\mathcal{R}$ should have only one independent component, so we assume that it lies in a doubly critical orbit, in which case

$$\begin{aligned} i_2(\mathcal{R}) &= 3 i_1(\mathcal{R})\,; \qquad & i_3(\mathcal{R}) &= 0\,; \\ i_4(\mathcal{R}) &= 2 i_1^2(\mathcal{R})\,; \qquad & i_5(\mathcal{R}) &= 12 i_1^2(\mathcal{R})\,. \end{aligned} \tag{3.75}$$

A straightforward expansion of (3.70), using (3.75) leads to

$$I_4(\mathcal{H}+\mathcal{R}) = I_4(\mathcal{H}) + \langle \delta I_4(\mathcal{H}), \mathcal{R}\rangle - \langle \mathcal{H}, \mathcal{R}\rangle^2\,, \tag{3.76}$$

where $\delta I_4(\mathcal{H})$ denotes the derivative of $I_4(\mathcal{H})$ and the identity (3.19) was used. Thus, the quadratic term reorganizes itself in the desired form without further assumptions.[8] For a given model, $\mathcal{R}$ can then be determined by demanding that the linear term vanishes.

This requirement is enough to ensure that the ansatz (3.65), together with the above assumptions, automatically satisfies the constraint (3.32), as we now show. First, note that for the ansatz in (3.65), $\boldsymbol{\eta}$ takes the form

$$\boldsymbol{\eta} = \mathrm{e}^{2U} \langle \mathbf{d}\mathcal{R}, \mathcal{H}\rangle\,, \tag{3.77}$$

as in (3.59). In section 3.2 it was shown that the constraint (3.32) is equivalent to (3.53), which in view of the last result reads

$$\langle \mathbf{d} \star_{\mathbf{0}} \mathbf{d}\mathcal{J}, \mathrm{Re}\,\hat{\Omega}\rangle = \mathrm{e}^{2U}\, \langle \mathbf{d}\mathcal{R}, \mathcal{H}\rangle \wedge \langle \star_{\mathbf{0}} \mathbf{d}\mathcal{H}, \mathcal{R}\rangle\,. \tag{3.78}$$

Using (3.67) and (3.69), one can then show that

$$\langle \mathbf{d} \star_{\mathbf{0}} \mathbf{d}\mathcal{J}, \mathrm{Re}\,\hat{\Omega}\rangle = \tfrac{1}{2}\, \mathrm{e}^{2U} \langle \mathcal{R}, \mathcal{H}\rangle\, \langle \mathbf{d} \star_{\mathbf{0}} \mathbf{d}\mathcal{R}, \mathcal{H}\rangle = -\mathrm{e}^{2U} \langle \mathcal{R}, \mathcal{H}\rangle\, \langle \star_{\mathbf{0}} \mathbf{d}\mathcal{R}, \mathbf{d}\mathcal{H}\rangle\,, \tag{3.79}$$

[8] Conversely, the decomposition in [140] can be used to show that the quartic invariant takes this form only if $\mathcal{R}$ lies in a doubly critical orbit [141].

where we used the identity [29]

$$\mathrm{Re}\,\hat{\Omega} = \tfrac{1}{2}\begin{pmatrix}\frac{\partial}{\partial \mathcal{J}_I}\\ \frac{\partial}{\partial \mathcal{J}^J}\end{pmatrix} \mathrm{e}^{-2U}\,, \tag{3.80}$$

and the last step follows from the fact that $\langle \mathcal{R}, \mathcal{H}\rangle$ is a harmonic function. Finally, since $\mathcal{R}$ depends only on a single ratio of the form $\langle \mathcal{H}, \mathcal{R}\rangle/\bar{H}$ (with $\bar{H}$ a harmonic function, cf. (3.60)), it is possible to show that

$$\langle \mathcal{H}, \mathcal{R}\rangle\, \mathbf{d}\mathcal{R} = -\langle \mathbf{d}\mathcal{R}, \mathcal{H}\rangle\, \mathcal{R}\,. \tag{3.81}$$

Combining the last relation with (3.79), the constraint (3.78) is identically satisfied, so that (3.65, 3.66) is a solution of the constraint (3.32). The seed solution (3.56) also satisfies these relations by construction.

This is a rather nontrivial result, as (3.53), due to (3.51), is a quartic equation for $\mathcal{J}$. Assuming this to be the general solution, the only constraint remaining at this stage is (3.31), which generalises the constraint on the fake superpotential for static black holes [22] to the case of under-rotating black holes. However, it is difficult to verify (3.31) and (3.38) explicitly for the seed solution (3.56), or find the general solution. In the next section, we give a more detailed comparison to the ansatz (3.65, 3.66) in the static limit.

It follows that the only object missing for a complete characterisation of the ansatz for extremal solutions is an explicit form for $\mathcal{H}$ given a vector of physical charges. In view of the flow equations in the previous section, that would be equivalent to solving (3.31) which, upon using (3.65, 3.66) to determine the scalars and $\boldsymbol{\eta}$, becomes a quadratic equation for the physical charges in terms of $\mathcal{H}$ and $\mathcal{R}$.

Unfortunately, solving this constraint is not a straightforward task. The only a priori requirement on $\mathcal{H}$ is that it must be "BPS" in the sense that $I_4(\mathcal{H}) > 0$ and that its quartic invariant should be related to the one of the physical charges by a sign flip. In fact the result should not be unique, as one can expect in view of the non-uniqueness in the rewriting (3.31). A manifestation of this ambiguity is seen in (3.60), where the two extra unconstrained parameters in λ^a represent the flat directions of the scalar sector. On the other hand, the relation between $\mathcal{H}$ and the physical charges must be the same throughout the flow, as follows from (3.65), so that an attractor analysis would be sufficient for this purpose. In any case, one can always dualise the stabilisation equations for the seed solutions above to find any other solution and we comment on a possible way to construct $\mathcal{H}$ at the end of the next section.

5 The static limit

In this section we specialise the results of section 3 to the static case, using the ansatz of section 4, and connect to the fake superpotential formalism.

5.1 The static flow equations

The static limit of the results in section 3 leads to several simplifications, since the solutions are necessarily spherically symmetric. This implies that $\omega = 0$ and all quantities depend only on the radial coordinate. Similarly as for the actual field strength, spherical symmetry implies that the modified field strength $\tilde{\boldsymbol{\mathcal{F}}}$ is of the form:

$$\tilde{\boldsymbol{\mathcal{F}}} = \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi \otimes \tilde{\Gamma}\,, \tag{3.82}$$

where now $\tilde{\Gamma} \in H^3(M_{\mathrm{CY}})$ is fibred along r. By (3.31), it must reproduce the same black hole potential V_{BH} as the physical charge Γ:

$$\tfrac{1}{2}\|\tilde{\Gamma}\|^2 = V_{\mathrm{BH}} = \tfrac{1}{2}\|\Gamma\|^2\,. \tag{3.83}$$

In this setting one chooses the arbitrary function $\mathrm{e}^{\mathrm{i}\alpha}$ to be the phase of $\langle\tilde{\Gamma}, \Omega\rangle$. In terms of the inverse radial coordinate $\tau = 1/r$ the first-order equations (3.29, 3.30) with $\boldsymbol{\eta} = \eta\,\mathrm{d}\tau$ reduce to[9]

$$2\,\partial_\tau \operatorname{Im}\!\big(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\big) - 2\,\eta\,\operatorname{Re}\!\big(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\big) = -\tilde{\Gamma}\,, \tag{3.84}$$

$$\eta = -\dot{\alpha} - Q_\tau = -\operatorname{Im}(\langle\dot{\tilde{\Gamma}}, \Omega\rangle / Z(\tilde{\Gamma}))\,, \tag{3.85}$$

where the second relation turns out to be equivalent to the second constraint (3.32). Observe again that the presence of a nontrivial η is essential for the generalisation of Denef's formalism with a fake field strength that is not a closed form.

As described in section 3.2, at least in principle these equations can be further simplified by eliminating fake charges $\tilde{\Gamma}$ from them and by using an ansatz $\mathcal{J}$ for the scalars, whereupon we obtain equations for $\mathcal{J}$. In particular,

[9] The signs depend on the conventions chosen for the Hodge dual.

since we already have an ansatz (3.65) for the stabilisation equations, we can determine U and Ω from it, so that eq. (3.84) becomes:

$$2\,\partial_\tau\,\mathrm{Im}\big(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\big) = \partial_\tau\mathcal{J}\,, \tag{3.86}$$

$$-\tilde{\Gamma} = \partial_\tau\mathcal{J} - 2\,\eta\;\mathrm{Re}\big(\mathrm{e}^{-U_{\mathcal{J}}}\mathrm{e}^{-\mathrm{i}\alpha}\Omega_{\mathcal{J}}\big), \tag{3.87}$$

where η is given by the static limit of (3.77) as

$$\eta = \mathrm{e}^{2U_{\mathcal{J}}}\,\langle\partial_\tau\mathcal{R},\mathcal{H}\rangle\,. \tag{3.88}$$

Recall from the previous section that our ansatz is automatically a solution of the constraint (3.32) (and (3.85)), and the equations of motion are solved if one can find a $\mathcal{J}$, along the lines of section 4.2, such that $\tilde{\Gamma}$ constructed above reproduces the black hole potential in (3.83), which now reads:

$$\tfrac{1}{2}\|\partial_\tau\mathcal{J}\|^2 = \tfrac{1}{2}\|\Gamma\|^2 + \mathrm{e}^{2U_{\mathcal{J}}}\,\langle\mathcal{H},\partial_\tau\mathcal{R}\rangle^2\,. \tag{3.89}$$

This quadratic constraint can be used to relate the physical charges to the harmonic functions $\mathcal{H}$, in addition to the generic requirements of section 4.2. In summary, the static equations of motion are integrable if there exists an $\mathcal{H}$ and its corresponding $\mathcal{R}$, constructed along the lines of section 4.2, satisfying (3.89) .

This is similar in spirit, but different than the approach of [100,101,126], were one seeks to rewrite the black hole potential in (3.83) through a function of the physical charges and moduli z^a directly. In contrast, (3.89) is an equation relating the harmonic functions controlling the physical charges to the ones controlling the scalars through the period vector Ω.

We have checked that $\mathcal{J}$ for the known explicit static solutions are such that they satisfy (3.89) and hence are described by the flow equation (3.84) with η as in (3.77). Since all static non-BPS solutions are related by symplectic rotations to the seed solutions of section 4, it follows that they satisfy the same duality-covariant equations. The nontrivial η is reflected in the anharmonic part of (3.56), controlled by the constant b that remains after setting the angular momentum to zero in (3.54). This observation is in line with [64], where it was stressed that the crucial departure of the static non-BPS seed solution from a BPS-like ansatz is the presence of a parameter related to the asymptotic scalars, identified with this residual constant.

5.2 The fake superpotential

One can adopt the opposite point of view to the one taken above: first find fake charges $\tilde{\Gamma}$ reproducing the black hole potential (3.83) and then solve the differential equations. Taking the intersection product of both sides of (3.84) with the basis elements then leads to the equations for the scale factor and the scalars, which, with appropriate identifications, have the form of non-supersymmetric flow equations generated by a superpotential W [22], analogous to the supersymmetric flow equations governed by the absolute value of the central charge:

$$\dot{U} = -\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\langle\tilde{\Gamma},\Omega\rangle = -\mathrm{e}^{U}W\,, \tag{3.90}$$

$$\dot{z}^{a} = -\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}g^{a\bar{b}}\langle\tilde{\Gamma},\bar{\mathcal{D}}_{\bar{b}}\bar{\Omega}\rangle = -2\mathrm{e}^{U}g^{a\bar{b}}\bar{\partial}_{\bar{b}}W\,. \tag{3.91}$$

Whenever W is explicitly known for a given model and charge configuration, a practical way to connect it with our approach may be to first look for a moduli-independent matrix S that rotates the usual charge vector[10] Γ so that:

$$|\tilde{Z}| := |\langle S\Gamma,\Omega\rangle| = W\,. \tag{3.92}$$

Then, its relation with $\tilde{\Gamma}$ is defined by:

$$\tilde{\Gamma} := \mathrm{i}\bar{\tilde{Z}}\Omega - \mathrm{i}g^{\bar{a}b}\bar{\mathcal{D}}_{\bar{a}}\bar{\tilde{Z}}\mathcal{D}_{b}\Omega + \mathrm{i}g^{a\bar{b}}\mathcal{D}_{a}\tilde{Z}\bar{\mathcal{D}}_{\bar{b}}\bar{\Omega} - \mathrm{i}\tilde{Z}\bar{\Omega}\,, \tag{3.93}$$

where $\mathcal{D}_a\tilde{Z} = \partial_a\tilde{Z} + \frac{1}{2}\partial_a K\tilde{Z}$, with K being the Kähler potential. Note that in general $\tilde{\Gamma} \neq S\Gamma$, if we allow for S to be complex, and in fact this turns out to be the simplest choice.

One can find the matrix S explicitly for the electric configuration, as in (3.56), assuming all physical scalars to have the same phase, say f. The relevant superpotential was given in [114]. Then, a suitable matrix S satisfying (3.92) and defining $\tilde{\Gamma}$ through (3.93) is

$$S = \mathrm{diag}\big(\mathrm{e}^{-2\mathrm{i}f}, 1, 1, 1, \mathrm{e}^{2\mathrm{i}f}, 1, 1, 1\big)\,. \tag{3.94}$$

In terms of the parameters appearing in the solution of section 4 one can identify $\cot f = \mathrm{e}^{2U}M$ and check that the equations of motion (3.84) are satisfied. The one-form $\boldsymbol{\eta}$ is given by (3.77).

In the non-supersymmetric axion-free case M vanishes and $f = \pi/2$, so that S is constant (but not identity), while allowing for a τ-dependent f

[10] By the shorthand $S\Gamma$ we mean rotating the symplectic vector of charges corresponding to the coefficients of Γ: $S\cdot(p^I, q_J)^{\mathrm{T}}$, and arranging the result again as a three-form.

leads to more general non-supersymmetric solutions. It is worth noting that $\eta = 0$ whenever S is constant (cf. eq. (3.85)). In particular, when $S = \mathbb{I}$ we recover the supersymmetric case.

Alternatively, one can rewrite (3.87) and (3.89) in terms of a real matrix T such that:

$$T\Gamma := \Gamma_T = \tilde{\Gamma} - 2\eta \operatorname{Re}\big(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\big)\,, \tag{3.95}$$

$$\tfrac{1}{2}\langle\Gamma_T, \diamond\Gamma_T\rangle = V_{\mathrm{BH}} + \mathrm{e}^{-2U}\eta^2\,, \tag{3.96}$$

$$\mathrm{e}^{-\mathrm{i}\alpha}\langle\Gamma_T, \Omega\rangle = W - \mathrm{i}\mathrm{e}^{-U}\eta\,, \tag{3.97}$$

where W is the superpotential in (3.90)–(3.91). If T is known, it leads to simpler equations of motion for the scalars, that is

$$2\partial_\tau \operatorname{Im}\big(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\big) = -\Gamma_T\,, \tag{3.98}$$

which have the advantage of being directly integrable to (3.46), giving the sought solutions as explained above. For the electric example above, T takes the form:

$$T = \begin{pmatrix} \mathbb{I} & 0 \\ \mathcal{W} & \mathbb{I} \end{pmatrix}, \qquad \mathcal{W} = \begin{pmatrix} -\frac{\mathrm{e}^{-2U}\cot f}{(H^0)^2} & \frac{2q_a}{p^0} \\ \frac{2q_a}{p^0} & 0 \end{pmatrix}, \tag{3.99}$$

so that (3.57) can be written as $\mathcal{J} = -T\mathcal{H}_c$, if the constants in (3.55) are appropriately chosen. Similarly to its complex counterpart S, it reduces to a constant matrix in the axion-free case.

It is interesting to point out that the matrix (3.99) is a (spacetime-dependent) element of the Peccei–Quinn group of transformations, defined as the largest subgroup of the symplectic group leaving the X^I's and the Kähler potential invariant. As was shown recently [142], applying such a transformation on the charges indeed shifts the black hole potential, as in (3.96). For generic charges and phases of the scalars, the corresponding T can be found from the one in (3.99) by conjugation with the appropriate element of the symplectic group. Such a matrix would leave a certain combination of X^I's and F_I's unchanged, e.g. for the magnetic dual of the electric solution in (3.55) it would leave the F_I's invariant. Identifying the combinations that must be invariant for a given set of charges could be a way to determine T from first principles.

6 Conclusions and outlook

In this work, we have extended the formalism of [28, 32], deriving symplectically covariant flow equations for non-BPS extremal black holes in $N = 2$ supergravity, and we have constructed an ansatz for the corresponding stabilisation equations. The main novelty was to rewrite the electromagnetic part of the action in terms of a 'fake' field strength two-form $\tilde{\boldsymbol{\mathcal{F}}}$ that does not have to be closed, where the non-closure turns out to be governed by a single one-form $\boldsymbol{\eta}$. The presence of this one-form is further related to the axions in the full black hole solution, apparently rendering such a deformation essential in a general description of non-BPS extremal black holes. Unfortunately, this complication makes the full equations challenging to solve directly, at least without considerable intuition about the form of the solution.

To obtain that insight, we considered the known seed solution for under-rotating extremal black holes in theories with cubic prepotentials. We showed that it can indeed be written in terms of stabilisation equations for the period vector, just as BPS black holes. The crucial difference is that the scalars are not stabilised in terms of harmonic functions only, but one finds that a ratio of harmonic functions is required. When the angular momentum is set to zero, one simply has the inverse of a harmonic function, which vanishes near the horizon, but mixes with the other asymptotic constants at infinity.

As one might expect, a comparison of these explicit solutions with our flow equations reveals the non-closure of $\tilde{\boldsymbol{\mathcal{F}}}$ to be reflected in exactly these non-harmonic parts of the stabilisation equations. Based on this, we proposed an ansatz for the generic stabilisation equations of under-rotating extremal black holes, satisfying several requirements coming both from general arguments and known explicit solutions. Its practical realisation depends heavily on the model and in particular on invariants constructed from two charge vectors, one of which must correspond to a small black hole. Such invariants have been considered recently in [140, 143] in the context of multi-centre solutions.

In the static case, we showed how to combine this ansatz with the first-order flow equations to identify the structure of $\tilde{\boldsymbol{\mathcal{F}}}$ for known solutions and infer its general form. It turned out that there are two ways of connecting the result to previous work. One involves a complex matrix resembling the matrix S introduced in the superpotential formalism [22]. The other formulation simplifies the equations of motion, using the real matrix in

(3.99) belonging to the group of Peccei–Quinn transformations. This hints towards the possibility of obtaining such matrices systematically.

Irrespectively of the precise description, one can characterise any solution by a vector of harmonic functions such that the quartic invariant computed on their poles is related to the one associated to the physical charges by a sign flip. In principle, it is possible to construct this vector using purely algebraic methods, which is equivalent to solving the attractor equations for static non-supersymmetric black holes. Once this is known for particular charge vectors, one can replace the vector of charges with harmonic functions to find $\mathcal{H}$ in (3.65) and directly construct new solutions using our first-order equations.

We expect this to generally hold also for the single-centre rotating solutions covered by our ansatz, as the only difference with respect to static solutions is in the choice for the angular momentum harmonic function, without modifying the structure of (3.65). In line with this expectation, we have verified that the proposed ansatz does satisfy the constraint (3.53). However, it is more difficult to impose (3.31), so it would be useful to find a generalisation of the arguments in section 5 and/or an extension of the fake superpotential formalism to the rotating case.

It is important to note that our flow equations are by construction fully covariant with respect to electric-magnetic duality and are compatible with the general seed solutions in four dimensions. It then follows that they capture the full orbit of non-BPS extremal solutions for suitable choices of $\tilde{\boldsymbol{\mathcal{F}}}$, regardless of the existence of other stationary points of the action, which should not be part of the standard non-BPS orbit of extremal black holes. It is also interesting to point out the similarity to special cases of static non-extremal black hole solutions, which can be obtained through a deformation of extremal solutions controlled by a ratio of harmonic functions [144], except that there it appears in the line element.

On the microscopic side, it would be very interesting to reproduce the stabilisation equations (3.65). In the rotating case the ratio of harmonic functions survives the near-horizon limit and modifies the attractor equations, similarly to [121], so one generally expects this structure to be accessible from microscopics. Given the model of [99], where the constant part of M in (3.57) is interpreted as the angle between wrapped D3 branes, one expects that the full angular momentum harmonic function might have a similar microscopic analogue.

Finally, it is worthwhile stressing that albeit the explicit solutions that we have discussed are only single centre, we have not made any assumptions on the number of centres in the derivation of flow equations in section 3. Also the ansatz (3.65) is compatible with multi-centre harmonic functions. It would be illuminating to make a detailed comparison with the results of [33, 34, 85], as a test on the robustness of the assumption on the existence of stabilisation equations for generic extremal backgrounds.

A On inverse harmonic functions

It is possible to give a generic heuristic argument for the presence of ratios of harmonic functions of the kind seen in (3.56) in the solution for the scalars. We find it convenient to choose the arbitrary function $\mathrm{e}^{\mathrm{i}\alpha}$ according to (3.30), so that (3.36) reduces to

$$\mathcal{L} = (\tilde{\boldsymbol{\mathcal{G}}}, \tilde{\boldsymbol{\mathcal{G}}})\,. \tag{3.100}$$

Similarly to the gauge part of (3.25), this can be interpreted as an action for the tensor

$$\tilde{\mathcal{G}} = \tilde{\mathcal{F}} - 2\ \mathrm{Im}\,\star_{\mathbf{0}}\mathbf{D}\hat{\Omega} + 2\ \mathrm{Re}\,\mathbf{D}(\mathrm{e}^{2U}\hat{\Omega}) \wedge (\mathrm{d}t + \omega)\,, \tag{3.101}$$

which respects the same pseudo-selfduality condition (3.14) as $\mathcal{F}$, assuming that $\tilde{\mathcal{F}}$ does:

$$\check{\mathcal{M}} \star \tilde{\mathcal{G}} = \tilde{\mathcal{G}}\,. \tag{3.102}$$

The scalar part can be shown to be pseudo-selfdual using (3.13) and (3.15). The matrix $\check{\mathcal{M}}$ is crucial for the existence of such a constraint, since it is not possible to impose selfduality on a four-dimensional field strength unless one complexifies it. However, it can be done in $4n + 2$ dimensions, and (3.102) descends from the ten-dimensional constraint on the five-form.

For the ordinary Einstein–Maxwell theory, gauge field equations in backgrounds of the type (3.23) naturally lead to harmonic functions (cf. eg. appendix B in [138]). Motivated by (3.58), we further assume that the rotational one-form satisfies $\mathbf{d}\omega = \star_{\mathbf{0}}\mathbf{d}M$. Denote spatial directions by $i = 1, 2, 3$ and consider first electric solutions, $F_{ij} = 0$, for which the Bianchi identity implies the existence of an electrostatic potential

$$\partial_i F_{tj} = \partial_j F_{ti} \qquad \Rightarrow \qquad F_{ti} = \partial_i \frac{M}{H}\,. \tag{3.103}$$

The field equations further impose that H is harmonic

$$\mathrm{e}^{-U} = H\,, \qquad \nabla^2 H = 0\,. \tag{3.104}$$

Similarly, for a magnetic solution, $F_{ti} = 0$, the Bianchi identities relate the field strength to a harmonic function

$$\epsilon^{tijk}\partial_i F_{jk} = 0 \qquad \Rightarrow \qquad F_{ij} = \epsilon_{ijk}\partial_k H\,, \tag{3.105}$$

where ϵ_{ijk} is the Levi-Civita permutation symbol and the Einstein equation implies again $\mathrm{e}^{-U} = H$. These solutions are related by an electric-magnetic duality rotation and belong to a class of solutions called the Majumdar–Papapetrou solutions.

Now, consider the case that the field strength is constrained to be pseudo-selfdual, as in (3.102). Then, the distinction between equations of motion and Bianchi identities disappears, and the solutions can no longer be purely electric or purely magnetic. For such a field, the gauge part of the action in (3.25) leads to the equation of motion (see [131] for details)

$$\mathrm{d}(\mathcal{F} - \check{\mathcal{M}} \star \mathcal{F}) = 0\,, \tag{3.106}$$

which is then solved for *both* cases:

$$\boldsymbol{\mathcal{F}} = \star_{\mathbf{0}}\mathbf{d}H\,, \qquad \mathrm{e}^{-U} = I(H)\,, \tag{3.107}$$

$$\boldsymbol{\mathcal{F}} = \star_{\mathbf{0}}\mathbf{d}\frac{M}{H}\,, \qquad \mathrm{e}^{-U} = I\Big(\frac{M}{H}\Big)\,, \tag{3.108}$$

where now H denotes a vector of harmonic functions and I is a model-dependent invariant. Imposing closure of $\boldsymbol{\mathcal{F}}$, which is equivalent to the existence of a gauge potential in four dimensions, one concludes that only the first solution survives, leading to the standard description by harmonic functions.

Going back to the Lagrangian in (3.100), the above discussion of selfdual fields applies including the scalar sector. In view of this, $\tilde{\boldsymbol{\mathcal{G}}}$, unlike $\boldsymbol{\mathcal{F}}$, is not necessarily closed and the two independent solutions in (3.107)–(3.108) are allowed. Based on this, one concludes that a vector of inverse harmonic functions is a zero mode of the equations of motion following from (3.100). Such a vector must be part of the general solution for the scalar sector only, given that the gauge field $\mathcal{F}$ is described by harmonic functions, as above.

Since the equations of motion are nonlinear, it is a nontrivial task to find full solutions with both kinds of zero modes turned on. Nevertheless, this discussion demonstrates that one can consider the scalar sector a priori independently from the vector one. It also lends credibility to the presence of ratios of harmonic functions in the ansatz (3.65) and the different sets of harmonic functions in (3.56).

4. NON-EXTREMAL BLACK HOLES OF $N = 2, D = 4$ SUPERGRAVITY

Pietro Galli, Tomás Ortín[◇], *Jan Perz*[◇],
and Carlos S. Shahbazi[◇]

[◇]*Instituto de Física Teórica UAM/CSIC*
C/ Nicolás Cabrera, 13–15, C.U. Cantoblanco, 28049 Madrid, Spain.

Publication reference: *JHEP* **07** (2011) 041, arXiv:1012.4020 [hep-th].

Abstract

We propose a generic recipe for deforming extremal black holes into non-extremal black holes and we use it to find and study the non-extremal black-hole solutions of several $N = 2, d = 4$ supergravity models ($SL(2,\mathbb{R})/U(1)$, $\overline{\mathbb{CP}}^n$ and STU with four charges). In all the cases considered, the non-extremal family of solutions smoothly interpolates between all the different extremal limits, supersymmetric and not supersymmetric. This fact can be used to find explicitly extremal non-supersymmetric solutions in the cases in which the attractor mechanism does not completely fix the values of the scalars on the event horizon and they still depend on the boundary conditions at spatial infinity.

We compare (supersymmetry) Bogomol'nyi bounds with extremality bounds and we find the first-order flow equations for the non-extremal solutions and the corresponding superpotential, which gives in the different extremal limits different superpotentials for extremal black holes. We also compute the *entropies* (areas) of the inner (Cauchy) and outer (event) horizons, finding in all cases that their product gives the square of the moduli-independent entropy of the extremal solution with the same electric and magnetic charges.

1 Introduction

Black holes are among the most interesting objects that occur in theories of gravity that include or extend general relativity, such as supergravity and superstring theories because their thermal behavior (Hawking radiation and Bekenstein–Hawking entropy) provides a unique window into the quantum-mechanical side of these theories. Their study in the framework of supergravity and superstring theories has generated a huge body of literature, the largest part of which concerns extremal (mostly but not always supersymmetric) black holes.

There are several reasons for having a special interest in extremal black holes: the solutions are simpler to find, they are protected from classical and quantum corrections when they are supersymmetric, there is an attractor mechanism for the scalar fields of most of them [15–17, 145], their entropies are easier to interpret microscopically in the framework of superstring theory [49] etc. Much of the progress in their study has been facilitated by the explicit knowledge of general families of extremal supersymmetric solutions e.g. in $N = 2, d = 4$ supergravity theories, where we know how to find systematically all of them [15, 60, 79, 94, 95, 146–153] (see also the review Ref. [138]).

By contrast, only a few non-extremal black-hole solutions are known (for instance, in $N = 2, d = 4$ theories), partly because they are more difficult to find than their extremal counterparts, and partly because they do not enjoy so many special properties. It is, however, clear that non-extremal black holes are at least as interesting as the extremal ones from a physical point of view, because they are closer to those that we may one day be able to observe. Furthermore, in $d = 4$ dimensions adding any amount of angular momentum to extremal black holes causes the event horizon to disappear [79]. This does not happen in non-extremal black holes, at least as long as the angular momentum does not exceed a certain value.

In order to learn more about them it is necessary to have more examples available for their study. In this paper we are going to propose a procedure to find non-extremal solutions of $N = 2, d = 4$ supergravity theories by deforming in a prescribed way the supersymmetric extremal solutions that we know how to construct systematically.[1] Another prescription has been proposed in the literature, namely the introduction of an additional harmonic function (called *Schwarzschild factor* in Ref. [78] and *non-extremality factor*

[1] For previous work on near-extremal and non-extremal solutions see e.g. Refs. [154–160].

in Ref. [144]), but it is unclear whether this method will work in all cases and for all models.

Our proposal makes crucial use of the formalism of Ferrara, Gibbons and Kallosh in Ref. [18], which turns out to be very convenient for our purposes. This formalism is based on the use of a particular radial coordinate τ that covers the exterior of the event horizon (which is always at $\tau = -\infty$ in these coordinates, a suitable value for the study of attractors). Furthermore, in this formalism the equations of motion have been reduced to a very small number of ordinary differential equations in the variable τ, which should simplify the task of finding solutions. In these equations there is a function of the scalars and the electric charges (the so-called *black-hole potential*), which plays a very important rôle, since its critical points are associated with possible extremal black-hole solutions. Then, using this formalism, we can also relate more easily the non-extremal solutions to the extremal solutions that have the same electric and magnetic charges.

We are going to test our proposal in a number of $N = 2, d = 4$ models and then study the main characteristics of the non-extremal solutions constructed. In this work we consider only regular static black holes.

This paper is organized as follows: in Section 2 we review essential facts concerning extremal and non-extremal black holes in the formalism and coordinates used by Ferrara, Gibbons and Kallosh in Ref. [18]. This will help us to establish our notation and conventions, find an ansatz for the non-extremal black holes based on the expressions for well-known solutions in these coordinates and show that these coordinates also cover the region that is bounded by the inner (Cauchy) horizon. In Section 3 we use the ansatz for the $SL(2, \mathbb{R})/U(1)$ axion-dilaton model to deform the supersymmetric extremal solutions (which we review first in detail) into non-extremal solutions, from which we can obtain in adequate limits supersymmetric and non-supersymmetric extremal black holes. In Section 4 we do the same for the $\overline{\mathbb{CP}}^n$ model. The black hole potential has flat directions and its non-supersymmetric critical points span a hypersurface in the moduli space. In other words: the attractor mechanism does not uniquely fix the values of the scalars on the horizon in terms of the electric and magnetic charges alone. Consequently the prescription of Ref. [102] for constructing full interpolating solutions from the horizon values of scalars by replacing charges with harmonic functions does not work. We will find these extremal non-supersymmetric solutions as limits of the non-extremal ones. In Section 5 we do the same for the well-known 4-charge solutions of the *STU* model. We show that there are 16 possible extremal limits, and discuss which of them are $N = 2$ and/or $N = 8$ supersymmetric. Section 6 contains our conclusions and directions for further work.

2 Extremal and non-extremal black holes

In this section we are going to review some well-known results on static extremal and non-extremal black-hole solutions, of which we will make use later. We will also study some examples of explicit non-extremal solutions in order to gain insight and formulate a general prescription for the deformation of supersymmetric extremal solutions into non-extremal solutions.

2.1 Introductory example: the Schwarzschild black hole

The prime example of a (non-extremal) black-hole is the Schwarzschild solution, which in Schwarzschild coordinates is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2_{(2)}\,, \tag{4.1}$$

where $d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the spherically symmetric metric of the unit 2-sphere. In this case, the "extremal limit" is Minkowski spacetime and the non-extremality parameter that goes to zero in the extremal limit, which we will denote from now on by r_0, is just the mass M:

$$r_0 = M\,. \tag{4.2}$$

The event horizon is located at the Schwarzschild radius $r_\mathrm{h} = 2M$ and there is a curvature singularity at $r = 0$.

The coordinate transformation

$$r = (\rho + r_0/2)^2/\rho\,, \tag{4.3}$$

brings this solution to the spatially isotropic form

$$ds^2 = \left(1 - \frac{r_0/2}{\rho}\right)^2 \left(1 + \frac{r_0/2}{\rho}\right)^{-2} dt^2 - \left(1 + \frac{r_0/2}{\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega^2_{(2)})\,, \tag{4.4}$$

in which the horizon is located at $\rho_\mathrm{h} = r_0/2$.

In order to study the attractor behavior of different quantities on the event horizon of a black hole it is convenient to use a radial coordinate τ

such that $\tau \to -\infty$ on the horizon. In the Schwarzschild black hole there seems to be no attractor behavior, but a coordinate τ with this property can be readily found [161]:

$$\rho = -\frac{r_0}{2\tanh\frac{r_0}{2}\tau} \tag{4.5}$$

and with it the Schwarzschild solution can be put in the form

$$\begin{aligned} ds^2 &= e^{2U}dt^2 - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}}\,, \\ \gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} &= \frac{r_0^4}{\sinh^4 r_0\tau}d\tau^2 + \frac{r_0^2}{\sinh^2 r_0\tau}d\Omega^2_{(2)}\,, \end{aligned} \tag{4.6}$$

which is valid for the exterior of any static non-extremal black hole with different values of the function $U(\tau)$. For the Schwarzschild black hole

$$U = r_0\tau\,, \tag{4.7}$$

and the radial coordinate τ takes values in the interval $(-\infty, 0)$, whose limits correspond to the event horizon and spatial infinity, where the radius of the 2-spheres becomes infinitely large. In the interval $(0, +\infty)$ the above metric describes a Schwarzschild solution with negative mass and a naked singularity at $\tau \to +\infty$ (just transform $\tau \to -\tau$). In more general cases the interval $(0, +\infty)$ will describe different patches of the black-hole spacetime.

Using the above general metric for static, non-extremal black holes, it can be shown [90] that the non-extremality parameter r_0 satisfies

$$r_0^2 = 2ST\,, \tag{4.8}$$

where S is the Bekenstein entropy and T is the Hawking temperature.

In the extremal limit $r_0 \to 0$ all static black holes are described by a metric of the same general form of Eq. (4.6), but the 3-dimensional spatial metric reduces to

$$\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} = \frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}d\Omega^2_{(2)}\,, \tag{4.9}$$

which, as can be seen by setting $\tau = -1/r$, is the Euclidean metric of $\mathbb{R}^3$ in standard spherical coordinates. In the Schwarzschild case, $U = 0$ in the extremal limit and the full metric becomes Minkowski's.

2.2 General results

In Ref. [18], in which the attractor behavior of general, static, $d = 4$ black-hole solutions was first studied, it was assumed that all of them could be written in the general form of Eq. (4.6), U being a function of τ to be determined and r_0 (denoted by c in Ref. [18]) being a general non-extremality parameter whose value as a function of physical constants (mass, electric and magnetic charges and asymptotic values of the scalars) also has to be determined. The action considered in that reference (slightly adapted to our conventions)[2] reads

$$I = \int d^4x\sqrt{|g|}\Big\{R + \mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + 2\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}F^{\Sigma\,\mu\nu} - 2\,\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}\star F^{\Sigma\,\mu\nu}\Big\}\,, \tag{4.10}$$

and can describe the bosonic sectors of all 4-dimensional ungauged supergravities for appropriate σ-model metrics and kinetic matrices $\mathcal{N}_{\Lambda\Sigma}(\phi)$. The indices $i, j, \ldots$ run over the scalar fields and the indices $\Lambda, \Sigma, \ldots$ over the 1-form fields. Their values are related only for $N \geq 2$ supergravity theories.

Using the general form of the metric for a static non-extremal black hole, Eq. (4.6), as well as the conservation of the electric and magnetic charges, the equations of motion of the above generic action can be reduced to those of an effective mechanical system with variables $U(\tau), \phi(\tau)$:

$$U'' + e^{2U}V_{\rm bh} = 0\,, \tag{4.11}$$

$$(U')^2 + \tfrac{1}{2}\mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} + e^{2U}V_{\rm bh} = r_0^2\,, \tag{4.12}$$

$$(\mathcal{G}_{ij}\phi^{j\,\prime})' - \tfrac{1}{2}\partial_i\mathcal{G}_{jk}\phi^{j\,\prime}\phi^{k\,\prime} + e^{2U}\partial_i V_{\rm bh} = 0\,. \tag{4.13}$$

Primes signify differentiation with respect to the inverse radial coordinate τ, which plays the role of the evolution parameter. The so-called *black-hole potential* is given by[3]

$$\begin{aligned} -V_{\rm bh}(\phi,\mathcal{Q}) &\equiv -\tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\mathcal{M}_{MN} \\ &\equiv -\tfrac{1}{2}(p^\Lambda \quad q_\Lambda)\begin{pmatrix} (\mathfrak{I}+\mathfrak{R}\mathfrak{I}^{-1}\mathfrak{R})_{\Lambda\Sigma} & -(\mathfrak{R}\mathfrak{I}^{-1})_\Lambda{}^\Sigma \\ -(\mathfrak{I}^{-1}\mathfrak{R})^\Lambda{}_\Sigma & (\mathfrak{I}^{-1})^{\Lambda\Sigma}\end{pmatrix}\begin{pmatrix} p^\Sigma \\ q_\Sigma\end{pmatrix}, \end{aligned} \tag{4.14}$$

[2] Our conventions are those of Refs. [79, 151].

[3] We adopt the sign of the black-hole potential opposite to most of the literature on black-hole attractors, conforming instead to the conventions of Lagrangian mechanics.

where we replaced each symplectic pair of superscript and subscript indices $\Lambda, \Sigma, \ldots$ with a single Latin letter $M, N, \ldots$, and used the shorthand

$$\mathfrak{R}_{\Lambda\Sigma} \equiv \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\,, \qquad \mathfrak{I}_{\Lambda\Sigma} \equiv \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\,, \qquad (\mathfrak{I}^{-1})^{\Lambda\Sigma}\mathfrak{I}_{\Sigma\Gamma} = \delta^{\Lambda}{}_{\Gamma}\,. \quad (4.15)$$

Eqs. (4.11) and (4.13), but not the constraint Eq. (4.12), can be derived from the effective action[4]

$$I_{\rm eff}[U,\phi^i] = \int d\tau\,\left\{(U')^2 + \tfrac{1}{2}\mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} - e^{2U}V_{\rm bh} + r_0^2\right\}\,. \qquad (4.17)$$

In Ref. [18] it was shown that for regular extremal ($r_0 = 0$) black holes the values of the scalars on the event horizon $\phi^i_{\rm h}$ are critical points of the black hole potential,[5] i.e. they satisfy

$$\partial_i V_{\rm bh}\big|_{\phi_{\rm h}} = 0\,. \qquad (4.18)$$

These equations can be solved in terms of the charges but, if the black hole potential has *flat directions*, the equations will be underdetermined and their solution will have residual dependence on the asymptotic values of the scalars at spatial infinity ($\tau \to 0^-$):

$$\phi_{\rm h} = \phi_{\rm h}(\phi_\infty, \mathcal{Q})\,. \qquad (4.19)$$

Furthermore, it was shown that the value of the black-hole potential at the critical points gives the entropy:

$$S = -\pi V_{\rm bh}(\phi, \mathcal{Q})\big|_{\phi_{\rm h}} \qquad (4.20)$$

and that the near-horizon geometry is that of $AdS_2 \times S^2$ with the AdS_2 and S^2 radii both equal to $(-V_{\rm bh}|_{\phi_{\rm h}})^{1/2}$. Even though the critical loci may not be isolated points, in which case the scalars will vary along the flat

[4] The three equations (4.11)–(4.13) can be derived from a more general effective action, which is reparametrization invariant:

$$I_{\rm eff}[U,\phi^i,e] = \int d\tau\,\left\{e^{-1}\left[(U')^2 + \tfrac{1}{2}\mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime}\right] - e\left[e^{2U}V_{\rm bh} - r_0^2\right]\right\}\,, \qquad (4.16)$$

where $e(\tau)$ is an auxiliary einbein. We can recover the effective action Eq. (4.17) in the gauge $e(\tau) = 1$, in which the equation of motion of e gives precisely the constraint Eq. (4.12). The constant term in Eq. (4.17) is usually ignored, as it is a total derivative.

[5] In the absence of stationary points the scalars would be singular on the horizon. We do not consider such cases.

directions of the potential when one changes ϕ_∞, the stationary value itself will not be affected, hence the entropy depends on the charges only [162]. Each solution to Eq. (4.18) yields a possible set of values of the scalars on the event horizon and of the radii, thus a possible extremal black-hole solution.

In the general case one can prove the following extremality bound [18]:

$$r_0^2 = M^2 + \tfrac{1}{2}\mathcal{G}_{ij}(\phi_\infty)\Sigma^i\Sigma^j + V_{\rm bh}(\phi_\infty, \mathcal{Q}) \geq 0\,, \tag{4.21}$$

where M, Σ^i are the mass and scalar charges defined by the behavior at spatial infinity ($\tau \to 0^-$)

$$\begin{aligned} U &\sim 1 - M\tau\,,\\ \phi^i &\sim \phi^i_\infty - \Sigma^i\tau\,. \end{aligned} \tag{4.22}$$

Flow equations

Whenever the potential term can be represented as a sum of squares of derivatives of a so-called *(generalized) superpotential* function $Y(U, \phi^i, \mathcal{Q}, r_0)$ of the warp factor U and the scalars ϕ^i,

$$-\left[e^{2U}V_{\rm bh} - r_0^2\right] = (\partial_U Y)^2 + 2\,\mathcal{G}^{ij}\partial_i Y \partial_j Y\,, \tag{4.23}$$

the effective action Eq. (4.17) also admits a rewriting as a sum of squares (up to a total derivative)

$$\begin{aligned} I_{\rm eff}[U,\phi^i] = \int d\tau \Big\{ &\tfrac{1}{2}\mathcal{G}_{ij}(\phi^{i\,\prime} \pm 2\,\mathcal{G}^{ik}\partial_k Y)(\phi^{j\,\prime} \pm 2\,\mathcal{G}^{jl}\partial_l Y)\\ &+ (U' \pm \partial_U Y)^2 \mp 2Y' \Big\}\,, \end{aligned} \tag{4.24}$$

whose variation leads to first-order gradient flow equations, solving the second-order equations of motion [26]:[6]

$$U' = \partial_U Y\,, \tag{4.25}$$

$$\phi^{i\,\prime} = 2\,\mathcal{G}^{ij}\partial_j Y\,. \tag{4.26}$$

[6] This generalized the results of Refs. [23, 163]. For first-order equations with a τ-dependent superpotential see Refs. [24, 27].

(Of the two signs in Eq. (4.24), only one, dependent on conventions, is physically admissible.) It is easy to see that

$$\partial_i Y = 0 \;\; \Rightarrow \;\; \partial_i V_{\rm bh} = 0 \,, \tag{4.27}$$

which sometimes simplifies the task of finding critical points of the black-hole potential. Observe also that when there is a generalized superpotential Y, the mass and scalar charges are determined by its derivatives at spatial infinity $\tau \to 0^-$:

$$M = - \lim_{\tau \to 0^-} \partial_U Y \,, \qquad \Sigma^i = - \lim_{\tau \to 0^-} \mathcal{G}^{ij} \partial_j Y \,. \tag{4.28}$$

The generalized superpotential $Y(U, \phi^i, \mathcal{Q}, r_0)$ has been proven [27, 128] to exist in theories whose scalar manifold (after timelike dimensional reduction) is a symmetric coset space, thus in particular for extended supergravities with more than 8 supercharges.

In the extremal cases, when there is a generalized superpotential function $Y(U, \phi, \mathcal{Q})$, it factorizes into

$$Y(U, \phi, \mathcal{Q}) = e^U W(\phi, \mathcal{Q}) \,, \tag{4.29}$$

where W is called the *superpotential*. The flow equations take the form [22]

$$U' = e^U W \,, \tag{4.30}$$

$$\phi^{i\,\prime} = 2\, e^U \mathcal{G}^{ij} \partial_j W \,. \tag{4.31}$$

In supergravities with more than 8 supercharges and in the extremal limit there is always at least one superpotential associated with the skew eigenvalues of the central charge, the above flow equations are related to the Killing spinor identities, and the corresponding extremal black-hole solutions are supersymmetric. However, in general there are extremal black-hole solutions that are not supersymmetric and satisfy the above flow equations for a different superpotential. We will discuss this point in more detail for $N = 2$ supergravity in the next section.

The stationary values of the superpotential

$$\partial_i W|_{\phi_{\rm h}} = 0 \tag{4.32}$$

give the the entropy:

$$S = \pi |W(\phi_{\rm h}, \mathcal{Q})|^2 \,. \tag{4.33}$$

2.3 $N = 2, d = 4$ supergravity

In this paper we will focus on theories of ungauged $N = 2, d = 4$ supergravity coupled to n vector supermultiplets (that is, with $\bar{n} = n + 1$ vector fields $A^{\Lambda}{}_{\mu}$, $\Lambda = 0, 1, \ldots, n$, taking into account the graviphoton).[7] The n scalars of these theories, denoted by Z^i, $i = 1, \ldots, n$ are complex and parametrize a special Kähler manifold with Kähler metric $\mathcal{G}_{ij^*} = \partial_i \partial_{j^*} \mathcal{K}$, where $\mathcal{K}(Z, Z^*)$ is the Kähler potential, and the Eqs. (4.11)–(4.13) can be rewritten in the form

$$U'' + e^{2U} V_{\rm bh} = 0\,, \tag{4.34}$$

$$(U')^2 + \mathcal{G}_{ij^*} Z^{i\,\prime} Z^{*\,j^*\,\prime} + e^{2U} V_{\rm bh} = r_0^2\,, \tag{4.35}$$

$$Z^{i\,\prime\prime} + \mathcal{G}^{ij^*} \partial_k \mathcal{G}_{lj^*} Z^{k\,\prime} Z^{l\,\prime} + e^{2U} \mathcal{G}^{ij^*} \partial_{j^*} V_{\rm bh} = 0\,. \tag{4.36}$$

Furthermore, the black-hole potential takes the simple form

$$-V_{\rm bh}(Z, Z^*, \mathcal{Q}) = |\mathcal{Z}|^2 + \mathcal{G}^{ij^*} \mathcal{D}_i \mathcal{Z} \mathcal{D}_{j^*} \mathcal{Z}^*\,, \tag{4.37}$$

where

$$\mathcal{Z} = \mathcal{Z}(Z, Z^*, \mathcal{Q}) \equiv \langle \mathcal{V} \mid \mathcal{Q} \rangle = -\mathcal{V}^M \mathcal{Q}^N \Omega_{MN} = p^{\Lambda} \mathcal{M}_{\Lambda} - q_{\Lambda} \mathcal{L}^{\Lambda}\,, \tag{4.38}$$

is the *central charge* of the theory, $\mathcal{V}^M = (\mathcal{L}^{\Lambda}, \mathcal{M}_{\Lambda})$ is the covariantly holomorphic symplectic section, $(\Omega_{MN}) = \begin{pmatrix} 0 & \mathbb{I}_{\bar{n}\times\bar{n}} \\ -\mathbb{I}_{\bar{n}\times\bar{n}} & 0 \end{pmatrix}$ is the symplectic metric, and

$$\mathcal{D}_i \mathcal{Z} = e^{-\mathcal{K}/2} \partial_i \left(e^{\mathcal{K}/2} \mathcal{Z} \right)\,, \tag{4.39}$$

is the Kähler covariant derivative.

Since $\mathcal{D}_i \mathcal{Z} = 2(\mathcal{Z}/\mathcal{Z}^*)^{1/2} \partial_i |\mathcal{Z}|$, in $N = 2$ theories there is always at least one superpotential

$$W = |\mathcal{Z}|\,, \tag{4.40}$$

and the associated flow equations (4.30), (4.31) for extremal black holes take the form

$$U' = e^U |\mathcal{Z}|\,, \tag{4.41}$$

$$Z^{i\,\prime} = 2 e^U \mathcal{G}^{ij^*} \partial_{j^*} |\mathcal{Z}|\,. \tag{4.42}$$

[7] See, for instance, Ref. [164], the review [165], and the original works [72, 73] for more information on $N = 2, d = 4$ supergravities.

It can be shown that these flow equations follow from the $N = 2$ Killing spinor identities and the corresponding extremal black-hole solutions are supersymmetric.[8] $|\mathcal{Z}|$ is the only superpotential associated to supersymmetric solutions in $N = 2$ theories, but there can be more non-supersymmetric superpotentials W.

Then, for $N = 2$ theories, the critical points of the black-hole potential (that we will loosely call *attractors* from now on) are of two kinds:

Supersymmetric (or BPS) attractors, for which

$$\mathcal{D}_i\mathcal{Z}\big|_{Z_\mathrm{h}} = 0 \qquad \text{or, equivalently} \qquad \partial_i|\mathcal{Z}|\Big|_{Z_\mathrm{h}} = 0\,. \tag{4.43}$$

As we have mentioned, the extremal black-hole solutions associated to these attractors are supersymmetric and the functions $U(\tau), Z^i(\tau)$ satisfy the above flow equations. Furthermore, according to the general results, the entropy is given by the value of the central charge at the horizon

$$S = \pi|\mathcal{Z}(Z_\mathrm{h}, Z^*_\mathrm{h}, \mathcal{Q})|^2 \tag{4.44}$$

and the mass of the black hole is given by the value of the central charge at infinity (BPS relation)

$$M = |\mathcal{Z}(Z_\infty, Z^*_\infty, \mathcal{Q})|\,. \tag{4.45}$$

In this case, since at supersymmetric critical points the Hessian of the black hole potential $-V_\mathrm{bh}$ is proportional to the (positive definite) metric on the scalar manifold, these points must be minima [18]. As a consequence, the scalars on the horizon take *attractor values* $Z_\mathrm{h} = Z_\mathrm{h}(\mathcal{Q})$, determined only by the electric and magnetic charges and independent of the asymptotic boundary conditions (at least within a single "basin of attraction" [166]). To put it differently: supersymmetric attractors are stable. As already remarked, the attractor mechanism may fail for certain choices of charges for which the horizon is singular (*small black holes*).

Non-supersymmetric attractors [19–21]. They satisfy an equation of the form

$$\partial_i W|_{Z_\mathrm{h}} = 0\,, \tag{4.46}$$

[8] For a rigorous proof, see Ref. [79].

for a superpotential function $W(Z, Z^*, \mathcal{Q}) \neq |\mathcal{Z}|$ [22], and the solution satisfies the corresponding flow equations (4.30), (4.31). The entropy will be given by Eq. (4.33) and the mass and scalar charges by Eqs. (4.28):

$$\begin{aligned} S =& \pi |W(Z_{\rm h}, Z^*_{\rm h}, \mathcal{Q})|^2 \,, \qquad M = |W(Z_\infty, Z^*_\infty, \mathcal{Q})| \,, \\ \Sigma^i =& -\mathcal{G}^{ij} \partial_j W(Z_\infty, Z^*_\infty, \mathcal{Q}) \,. \end{aligned} \tag{4.47}$$

One of the main differences with the supersymmetric case is that the stationary points of the black hole potential do not necessarily need to be minima. For models whose scalar manifold is a homogeneous space (in particular, thus, for all models embeddable in $N > 2$ supergravity) the Hessian at these points (expressed in a real basis [167, 168]), has non-negative eigenvalues, therefore such stationary points are also stable, but only up to possible flat directions [63, 169]. It means that the attractor mechanism is no longer guaranteed to completely fix the values of the scalars on the horizon $Z^i_{\rm h}$, which may still depend on the asymptotic values Z^i_∞ as well as on the charges $\mathcal{Q}$, even though the entropy will only depend on the charges. In this sense one may speak of *moduli spaces of attractors* parametrized by (combinations of) the Z^i_∞, as opposed to the supersymmetric attractors, which are isolated points in the target space of the scalars.

Only in the supersymmetric case $\Sigma^i = \mathcal{D}^i \mathcal{Z}^*\big|_{Z_\infty}$ and, therefore, the general extremality bound Eq. (4.21) does not reduce to just the BPS bound $r_0^2 = M^2 - |\mathcal{Z}(Z_\infty, Z^*_\infty, \mathcal{Q})|^2 \geq 0$ (otherwise, all extremal black holes in $N = 2$ supergravity would automatically be supersymmetric, which is not true). One of our goals is to study the general extremality bound and interpret it in terms of the central charge and other known quantities, explaining why and how it happens that supersymmetry always implies extremality, but not the other way around, as first shown in Ref. [119] (see also [170]).

$N = 2, d = 4$ *black-hole solutions*

How are the complete black-hole solutions (or, equivalently, the variables $U(\tau), Z^i(\tau)$) found? For supersymmetric (and, therefore, extremal) $N = 2$ supergravity solutions there is a well-established method to construct systematically all the possible black-hole solutions [15, 16, 28, 29, 94, 95, 148, 149, 151]. We will follow the prescription given in Ref. [151]:

1. Introduce a complex function $X(Z, Z^*)$ with the same Kähler weight as the canonical symplectic section $\mathcal{V}$ so that the quotient $\mathcal{V}/X$ is invariant under Kähler transformations.[9]

2. Define the real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$ by

$$\mathcal{R} + i\mathcal{I} \equiv \mathcal{V}/X\,. \tag{4.48}$$

   The components of $\mathcal{R}$ can always be expressed in terms of those of $\mathcal{I}$ (by solving the *stabilization equations* of Refs. [94, 149]),[10] although in some cases the relations may be difficult to find explicitly.

3. The $2\bar{n}$ components of imaginary part $\mathcal{I}^M$ are given by as many real harmonic functions in $\mathbb{R}^3$. For single-center, spherically symmetric, black-hole solutions, they must have the form[11]

$$\mathcal{I}^M = \mathcal{I}^M_\infty - \tfrac{1}{\sqrt{2}}\mathcal{Q}^M \tau\,. \tag{4.49}$$

   Furthermore, in order not to have NUT charge (and have staticity) we must require [79]

$$\langle \mathcal{I}_\infty \mid \mathcal{Q} \rangle = -\mathcal{I}^M_\infty \mathcal{Q}^N \Omega_{MN} = 0\,. \tag{4.50}$$

   The choice of $\mathcal{I}^M$ determines the components $\mathcal{R}^M$ according to the pervious discussion.

4. The scalar fields are given by

$$Z^i = \frac{\mathcal{L}^i}{\mathcal{L}^0} = \frac{\mathcal{L}^i/X}{\mathcal{L}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,, \tag{4.51}$$

   and the metric function U is given by

$$e^{-2U} = \frac{1}{2|X|^2} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle = -\mathcal{R}^M \mathcal{I}^N \Omega_{MN}\,. \tag{4.52}$$

[9] This prescription does not depend on the Kähler gauge. A function playing the same role as X, namely $1/\mathcal{Z}^*$, was also introduced in Ref. [60].

[10] Since the relations must remain the same at all points in space, it suffices to infer them on the horizon, where the stabilization equations reduce to the *attractor equations* of Ref. [17].

[11] The factor $1/\sqrt{2}$ is required for the correct normalization of the charges (in particular, to have the same normalization of the charges used in the definition of the black-hole potential) and it was omitted in Ref. [79].

We will not need the explicit form of the vector fields but they can be found in Ref. [79].

Some extremal but non-supersymmetric solutions can be constructed from the attractor values [92, 102] by replacing the electric and magnetic charges in the expressions for the scalars on the horizon by harmonic functions (the metric function is obtained in the same way from the entropy). It is not clear, however, that this is always applicable, in particular when there are moduli spaces of non-supersymmetric attractors, as in the $\overline{\mathbb{CP}}^n$ model (Section 4).

There is no general algorithm to construct non-extremal black-hole solutions either. In some cases, the introduction of an additional harmonic function (called *Schwarzschild factor* in Ref. [78] and *non-extremality factor* in Ref. [144]) appears to be enough, but the explicit non-extremal solution [171] seems to suggest that this prescription may not always work. In order to gain more insight into this problem, which is of our main interest in this paper, we are going to examine in detail more examples of non-extremal solutions. Then we will formulate a prescription to deform any static extremal supersymmetric black-hole solution of $N = 2, d = 4$ supergravity into a non-extremal one and, next, we will apply it to several examples in the following sections.

2.4 Second example: the Reissner–Nordström black hole

Let us consider pure $N = 2, d = 4$ supergravity, with the bosonic action

$$I = \int d^4x \sqrt{|g|}\, \left[R - F^2\right] \,, \tag{4.53}$$

which corresponds to a canonical section and period matrix

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^0 \\ \mathcal{M}_0 \end{pmatrix} = \begin{pmatrix} i \\ \frac{1}{2} \end{pmatrix} \,, \qquad \mathcal{N}_{00} = -\frac{i}{2} \,. \tag{4.54}$$

The central charge and black-hole potential are

$$\mathcal{Z} = \tfrac{1}{2}p - iq \,, \qquad -V_{\rm bh} = |\mathcal{Z}|^2 \,, \tag{4.55}$$

and, since there are no scalars, it has no critical points.

The supersymmetric extremal black-hole solutions can be constructed using the mentioned algorithm of Ref. [151]. First, we introduce the function

X and the two harmonic functions

$$\begin{aligned}\mathcal{I}^0 &= \mathrm{Im}(\mathcal{L}^0/X) = \mathcal{I}^0_\infty - \frac{p^0}{\sqrt{2}}\tau\,,\\ \mathcal{I}_0 &= \mathrm{Im}(\mathcal{M}_0/X) = \mathcal{I}_{0\,\infty} - \frac{q_0}{\sqrt{2}}\tau\,,\end{aligned}\tag{4.56}$$

where $\mathcal{I}^0_\infty, \mathcal{I}_{0\,\infty}$ are constants to be determined later.[12] It is convenient to combine these two real harmonic functions into a single complex harmonic function

$$\mathcal{H} \equiv \tfrac{1}{\sqrt{2}}(\mathcal{I}^0 + 2i\mathcal{I}_0) = \mathcal{H}_\infty - \mathcal{Z}\tau\,. \tag{4.57}$$

Then, it is easy to see that the zero-NUT-charge condition Eq. (4.50) can be written in the form

$$N = \mathrm{Im}(\mathcal{H}_\infty \mathcal{Z}^*) = 0\,. \tag{4.58}$$

The stabilization equations determine the real parts

$$\begin{aligned}\mathcal{R}^0 &= -2\mathcal{I}_0\,,\\ \mathcal{R}_0 &= \frac{1}{2}\mathcal{I}^0\,,\end{aligned}\tag{4.59}$$

and then the metric function is given by

$$e^{-2U} = |\mathcal{H}|^2 = |\mathcal{H}_\infty|^2 - 2\,\mathrm{Re}(\mathcal{H}_\infty\mathcal{Z}^*)\tau + |\mathcal{Z}|^2\tau^2\,. \tag{4.60}$$

Asymptotic flatness requires $|\mathcal{H}_\infty|^2 = 1$ and indicates that $M = \mathrm{Re}(\mathcal{H}_\infty\mathcal{Z}^*)$, and then we get the well-known extremal, dyonic, Reissner–Nordström (RN) solution:

$$\mathcal{H}_\infty = \frac{\mathcal{Z}}{|\mathcal{Z}|}\,,\qquad M = |\mathcal{Z}|\,,\qquad S = \pi|\mathcal{Z}|^2\,.\qquad e^{-2U} = (1 - |\mathcal{Z}|\tau)^2\,. \tag{4.61}$$

Observe that e^{-2U} ends up as the square of a real harmonic function, which we can call H.

The non-extremal RN solutions are, of course, known as well. In our conventions, and Schwarzschild-like coordinates, the metric takes the form

$$ds^2 = \frac{(r-r_+)(r-r_-)}{r^2}dt^2 - \frac{r^2}{(r-r_+)(r-r_-)}dr^2 - r^2 d\Omega^2_{(2)}\,, \tag{4.62}$$

[12] These constants are often set equal to 1 from the beginning, which is in general incorrect, as we are going to show.

where

$$r_{\pm} = M \pm r_0 \,, \tag{4.63}$$

are the values of r at which the outer (event) horizon (+) and inner (Cauchy) horizon (−) are located, and

$$r_0^2 = M^2 - |\mathcal{Z}|^2 \,, \tag{4.64}$$

is the non-extremality parameter.

In order to study this solution using the black-hole potential formalism we first need to reexpress it in terms of the coordinate τ. As an intermediate step we reexpress it in terms of spatially isotropic coordinates

$$r = [\rho^2 + M\rho + r_0^2/4]/\rho \,, \tag{4.65}$$

so it takes the form ($\rho_{\pm} \equiv M \pm |\mathcal{Z}|$)

$$\begin{aligned} ds^2 = {} & \frac{\left(1 - \frac{r_0/2}{\rho}\right)^2 \left(1 + \frac{r_0/2}{\rho}\right)^2}{\left(1 + \frac{\rho_+/2}{\rho}\right)^2 \left(1 + \frac{\rho_-/2}{\rho}\right)^2} dt^2 \\ & - \left(1 + \frac{\rho_+/2}{\rho}\right)^2 \left(1 + \frac{\rho_-/2}{\rho}\right)^2 (d\rho^2 + \rho^2 d\Omega^2_{(2)}) \,. \end{aligned} \tag{4.66}$$

For $M = |\mathcal{Z}|$ ($\rho_- = r_0 = 0$) we recover the extremal solution just studied (with $\rho = -1/\tau$). Next, we change to the coordinate τ as in the Schwarzschild case with M replaced by r_0

$$\rho = -\frac{r_0}{2 \tanh \frac{r_0}{2}\tau} \,, \tag{4.67}$$

to obtain a metric of the standard form Eq. (4.6) with

$$e^{-2U} = e^{-2r_0\tau} \left[\frac{r_+}{2r_0} - \frac{r_-}{2r_0} e^{2r_0\tau}\right]^2 \,. \tag{4.68}$$

This metric function contains a *Schwarzschild factor* $e^{-2r_0\tau}$, which is the only one that remains when the charge vanishes, and the square of a function which is not a harmonic function in $\mathbb{R}^3$ but can be seen as a deformation of the function $H = 1 - |\mathcal{Z}|\tau$:

$$\lim_{r_0 \to 0} \left[\frac{r_+}{2r_0} - \frac{r_-}{2r_0} e^{2r_0\tau}\right] = H \,. \tag{4.69}$$

As in the Schwarzschild case, when the radial coordinate coordinate τ takes values in the interval $(-\infty, 0)$, whose limits correspond to the event horizon and spatial infinity, the metric covers the exterior of the horizon. The explicit relation between the original Schwarzschild-like radial coordinate r and τ in that interval is

$$\tau = \frac{2}{r_0}\text{arctanh}\left\{\frac{-r_0}{(r-M)+\sqrt{(r-M)^2-r_0^2}}\right\}, \qquad r \in (r_+, +\infty)\,. \tag{4.70}$$

In the RN case, however, *the same metric* also covers the interior of the inner horizon when τ takes values in the interval $(\frac{2}{r_0}\text{arctanh}\sqrt{\frac{M-|\mathcal{Z}|}{M+|\mathcal{Z}|}}, +\infty)$, whose limits correspond to the singularity at the origin and the inner horizon. The explicit relation between the original Schwarzschild-like radial coordinate r and τ in that interval is

$$\tau = \frac{2}{r_0}\text{arctanh}\left\{\frac{-r_0}{(r-M)-\sqrt{(r-M)^2-r_0^2}}\right\}, \qquad r \in (0, r_-)\,. \tag{4.71}$$

It is easy to see that e^{2U} tends to zero in the two limits $\tau \to \pm\infty$ and that the coefficient of $d\Omega^2_{(2)}$ in the metric, which can be understood as the square radius of the spatial sections of the horizons

$$\frac{(2r_0)^2 e^{-2U}}{(e^{r_0\tau}-e^{-r_0\tau})^2} = \left(\frac{r_\pm - r_\mp e^{\pm 2r_0\tau}}{e^{\pm 2r_0\tau}-1}\right)^2 \stackrel{\tau\to\mp\infty}{\longrightarrow} r_\pm^2\,. \tag{4.72}$$

This allows us to compute the areas and, therefore, the "entropies" associated with both horizons using the standard metric:

$$S_\pm/\pi = (r_\pm)^2\,, \tag{4.73}$$

and, using the general result Eq. (4.8), the temperatures

$$T_\pm = \frac{r_0^2}{2S_\pm} = \frac{1}{2\pi}(r_0/r_\pm)^2\,. \tag{4.74}$$

2.5 General prescription

The previous result suggests the following prescription for deforming extremal, static, supersymmetric solutions of $N = 2, d = 4$ supergravity into non-extremal solutions: if the supersymmetric solution is given by

$$U(\tau) = U_{\text{e}}[H(\tau)]\,, \qquad Z^i(\tau) = Z^i_{\text{e}}[H(\tau)]\,, \tag{4.75}$$

where $U_{\rm e}$ and $Z^i_{\rm e}$ are the functions of certain harmonic functions $H_\alpha(\tau) = H_{\alpha\,\infty} - Q_\alpha\tau$ (α being some index) that one finds following the standard prescription for supersymmetric black holes, then the non-extremal solution is given by

$$U(\tau) = U_{\rm e}[\hat{H}(\tau)] + r_0\tau\,, \qquad Z^i(\tau) = Z^i_{\rm e}[\hat{H}(\tau)]\,, \tag{4.76}$$

where the harmonic functions H have been replaced by the hatted functions

$$\hat{H}_\alpha = a_\alpha + b_\alpha e^{2r_0\tau}\,. \tag{4.77}$$

This ansatz has to be used in the three equations (4.34), (4.35) and (4.36) to determine the actual values of the integration constants a_α, b_α. In the following sections we are going to see how this ansatz works in particular models, showing that the original differential equations are solved by the ansatz if the integration constants satisfy certain algebraic equations that relate them to the charges $\mathcal{Q}^M$ and the non-extremality parameter r_0, and we will argue that it should always work, even if the algebraic equations for the integration constants are in general difficult to solve.

Observe that, since in most cases $e^{-2U_{\rm e}}(H)$ is homogenous of second degree in the harmonic functions, following the same steps as in the RN example, we expect to find the event horizon in the $\tau\to-\infty$ limit and the inner horizon $\tau\to+\infty$ limit, which will allow us to find the entropies and temperatures using Eq. (4.8).

3 *Axion-dilaton black holes*

The so-called axion-dilaton black holes[13] are solutions of the $\bar{n}=2$ theory with prepotential

$$\mathcal{F} = -i\mathcal{X}^0\mathcal{X}^1\,. \tag{4.78}$$

This theory has only one complex scalar that it is usually called τ but we are going to call λ to distinguish it from the radial coordinate. This scalar is given by

$$\lambda \equiv i\mathcal{X}^1/\mathcal{X}^0\,. \tag{4.79}$$

In terms of λ the period matrix is given by

$$(\mathcal{N}_{\Lambda\Sigma}) = \begin{pmatrix} -\lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \tag{4.80}$$

[13] For references on these black-hole solutions see Refs. [172].

and, in the $\mathcal{X}^0 = i/2$ gauge, the Kähler potential and metric are

$$\mathcal{K} = -\ln \operatorname{Im}\lambda\,, \qquad \mathcal{G}_{\lambda\lambda^*} = (2\operatorname{Im}\lambda)^{-2}\,. \tag{4.81}$$

The reality of the Kähler potential requires the positivity of $\operatorname{Im}\lambda$. Therefore, λ parametrizes the coset $SL(2,\mathbb{R})/SO(2)$ and the action for the bosonic fields is

$$\begin{aligned} I = \int d^4x\sqrt{|g|}\Big\{ R + \frac{\partial_\mu\lambda\partial^\mu\lambda^*}{2(\operatorname{Im}\lambda)^2} - 2\operatorname{Im}\lambda\left[(F^0)^2 + |\lambda|^{-2}(F^1)^2\right] \\ + 2\operatorname{Re}\lambda\left[F^0\star F^0 - |\lambda|^{-2}F^1\star F^1\right]\Big\}\,. \end{aligned} \tag{4.82}$$

This theory is a truncation of $N=4, d=4$ supergravity. After replacing the matter vector field A^1 by its dual ($F_1 = \operatorname{Im}\lambda\star F^1 + \operatorname{Re}\lambda F^1$) the action takes the more (manifestly) symmetric form

$$\begin{aligned} I = \int d^4x\sqrt{|g|}\Big\{ R + \frac{\partial_\mu\lambda\partial^\mu\lambda}{(2\operatorname{Im}\lambda)^2} - 2\operatorname{Im}\lambda\left[(F^0)^2 + (F_1)^2\right] \\ + 2\operatorname{Re}\lambda\left[F^0\star F^0 + F_1\star F_1\right]\Big\}\,, \end{aligned} \tag{4.83}$$

in which it has been exhaustively studied [161]– [171]. In particular, the most general (non-extremal and rotating) black-holes of this theory were presented in Ref. [171]. A preliminary check shows that in the static case the metric and scalars are, in the coordinate τ, of the form of our deformation ansatz, but we want to reobtain the non-extremal solutions using the ansatz and the language and notation of $N=2, d=4$ supergravities.

In order to apply the formalism reviewed in the previous section, let us start by constructing the black-hole potential.

The canonically normalized symplectic section $\mathcal{V}$ is, in a certain gauge,

$$\mathcal{V} = \frac{1}{2(\operatorname{Im}\lambda)^{1/2}}\begin{pmatrix} i \\ \lambda \\ -i\lambda \\ 1 \end{pmatrix}\,, \tag{4.84}$$

and, in terms of the complex combinations

$$\Gamma_1 \equiv p^1 + iq_0\,, \qquad \Gamma_0 \equiv q_1 - ip^0\,, \tag{4.85}$$

the central charge and its holomorphic covariant derivative and the black-hole potential are

$$
\begin{aligned}
\mathcal{Z} &= \frac{1}{2\sqrt{\operatorname{Im}\lambda}}\,[\,\Gamma_1^* - \Gamma_0^*\lambda\,]\;, \\
\mathcal{D}_\lambda \mathcal{Z} &= \frac{i}{4(\operatorname{Im}\lambda)^{3/2}}\,[\,\Gamma_1^* - \Gamma_0^*\lambda^*\,]\;, \\
-V_{\rm bh} &= \frac{1}{2\operatorname{Im}\lambda}\left[\,|\Gamma_1|^2 - 2\operatorname{Re}(\Gamma_1\Gamma_0^*)\operatorname{Re}\lambda + |\Gamma_0|^2|\lambda|^2\,\right]\;.
\end{aligned}
\tag{4.86}
$$

It is convenient to define the charge

$$
\tilde{\mathcal{Z}} \equiv \frac{1}{2\sqrt{\operatorname{Im}\lambda}}\,[\,\Gamma_1^* - \Gamma_0^*\lambda^*\,]\;, \tag{4.87}
$$

in terms of which

$$
\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^* = |\tilde{\mathcal{Z}}|^2\,, \tag{4.88}
$$

so we can write

$$
-\,V_{\rm bh} = |\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2\,, \qquad -\partial_\lambda V_{\rm bh} = 2\mathcal{Z}^*\mathcal{D}_\lambda\mathcal{Z} \sim \mathcal{Z}^*\tilde{\mathcal{Z}}\,. \tag{4.89}
$$

3.1 Flow equations

The potential term can be expanded in the following way:

$$
-\left[e^{2U}V_{\rm bh} - r_0^2\right] = \Upsilon^2 + 4\,\mathcal{G}^{\lambda\lambda^*}\Psi\Psi^*\,, \tag{4.90}
$$

where

$$
\Upsilon = \frac{e^U}{\sqrt{2}}\sqrt{e^{-2U}r_0^2 + |\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2 + \sqrt{\left(e^{-2U}r_0^2 + |\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2\right)^2 - 4|\mathcal{Z}|^2|\tilde{\mathcal{Z}}|^2}}\,, \tag{4.91}
$$

$$
\Psi = i\,\frac{e^{2U}\mathcal{Z}^*\tilde{\mathcal{Z}}}{4\ \operatorname{Im}\lambda\,\Upsilon}\,. \tag{4.92}
$$

The vector field generated by (Υ, Ψ, Ψ^*) is conservative or, in other words, can be written as a gradient of a generalized superpotential $Y(U, \lambda, \lambda^*)$

$$
(\Upsilon, \Psi, \Psi^*) = (\partial_U Y, \partial_\lambda Y, \partial_{\lambda^*} Y)\,, \tag{4.93}
$$

if and only if it is irrotational (i.e. its curl vanishes). This is the case here, since

$$\partial_U \Psi - \partial_\lambda \Upsilon = \partial_U \Psi^* - \partial_{\lambda^*} \Upsilon = \partial_\lambda \Psi^* - \partial_{\lambda^*} \Psi = 0\,, \tag{4.94}$$

which could have been expected on the basis of the results mentioned in Section 2.2. The explicit form of the generalized superpotential can be in principle obtained by integrating Eq. (4.91), but in practice this turns out to be very complicated.

The flow equations (4.25, 4.26), in the conventions of Eq. (4.35), now take the form:

$$U' = \Upsilon\,, \tag{4.95}$$

$$\lambda' = 2\,\mathcal{G}^{\lambda\lambda^*} \Psi^*\,. \tag{4.96}$$

In the particular case of the Reissner–Nordström black hole (cf. Section 2.4), the first of these equations reduces to the one derived in [23] (and the second is not applicable, since there are no scalars). For extremal black holes, studied in greater detail below, one recovers Eq. (4.30, 4.31) with either $W = |\mathcal{Z}|$ (the supersymmetric case) or $W = |\tilde{\mathcal{Z}}|$.

3.2 The extremal case

Critical points

The critical points of the black hole potential are those for which $\mathcal{Z} = 0$ or $\tilde{\mathcal{Z}} = 0$. They are two isolated points in moduli space and only the second is supersymmetric. The situation is summarized in Table 4.1.

As already said in Section 2.3, the supersymmetric stationary points of the black hole potential must be a minimum. Indeed, the Hessian evaluated this point in the real basis has the double eigenvalue

$$\frac{|\Gamma_0|^4}{2\,\mathrm{Im}(\Gamma_1 \Gamma_0^*)} = |\Gamma_0|^2 \mathcal{G}_{\lambda\lambda^*}\big|_{\mathrm{h}}^{\mathrm{susy}} = \frac{(p^0)^2 + (q_1)^2}{2(p^0 p^1 + q_0 q_1)}\,. \tag{4.97}$$

Again referring to Section 2.3, one can expect also the non-supersymmetric extremal stationary point of our model to be stable (up to possible flat directions). This is confirmed by the direct calculation of the Hessian, which has the double eigenvalue

$$-\frac{|\Gamma_0|^4}{2\,\mathrm{Im}(\Gamma_1 \Gamma_0^*)} = |\Gamma_0|^2 \mathcal{G}_{\lambda\lambda^*}\big|_{\mathrm{h}}^{\mathrm{nsusy}} = -\frac{(p^0)^2 + (q_1)^2}{2(p^0 p^1 + q_0 q_1)}\,. \tag{4.98}$$

Attractor	$\operatorname{Im}\lambda_{\rm h}$	$\lvert\mathcal{Z}_{\rm h}\rvert^2$	$\lvert\tilde{\mathcal{Z}}_{\rm h}\rvert^2$	$-V_{\rm bh\,h}$	M
$\lambda_{\rm h}^{\rm susy} = \Gamma_1/\Gamma_0$	$\operatorname{Im}(\Gamma_1\Gamma_0^*)$	$\operatorname{Im}(\Gamma_1\Gamma_0^*)$	0	$\operatorname{Im}(\Gamma_1\Gamma_0^*)$	$\lvert\mathcal{Z}_\infty\rvert$
$\lambda_{\rm h}^{\rm nsusy} = \Gamma_1^*/\Gamma_0^*$	$-\operatorname{Im}(\Gamma_1\Gamma_0^*)$	0	$-\operatorname{Im}(\Gamma_1\Gamma_0^*)$	$-\operatorname{Im}(\Gamma_1\Gamma_0^*)$	$\lvert\tilde{\mathcal{Z}}_\infty\rvert$

Tab. 4.1: Critical points of the axidilaton model. Here we are using the notation $\mathcal{Z}_{\rm h} \equiv \mathcal{Z}(\lambda_{\rm h}, \lambda_{\rm h}^*, \mathcal{Q})$ etc. In the supersymmetric case the mass M can be found in the explicit solution or from the saturation of the supersymmetric bound. Then, the scalar charge $\Sigma^\lambda_{\rm susy} = 2ie^{i{\rm Arg}\mathcal{Z}_\infty}\operatorname{Im}\lambda_\infty\,\mathcal{Z}'^*_\infty$ and $\Sigma^\lambda_{\rm nsusy} = 2ie^{i{\rm Arg}\tilde{\mathcal{Z}}_\infty}\operatorname{Im}\lambda_\infty\,\mathcal{Z}_\infty$ follow from the general extremality bound (or from the knowledge of the explicit solution). In the non-supersymmetric case we do not have analogous arguments and we need the explicit solution, given in Section 3.3.

Observe that the supersymmetric stationary point and the non-supersymmetric extremal stationary point exist for mutually exclusive choices of charges and that in this example, given that $\tilde{\mathcal{Z}}$ differs from $\mathcal{Z}$ by complex conjugation in the numerator, one could have also used, with appropriate modifications, the general supersymmetric argument [18] to study the stability of the non-supersymmetric critical point.

Supersymmetric solutions

According to the general procedure, the supersymmetric solutions are built out of the four harmonic functions

$$\mathcal{I}^M = \mathcal{I}^M_\infty - \frac{\mathcal{Q}^M}{\sqrt{2}}\tau\,. \tag{4.99}$$

In this theory the stabilization equations can be easily solved and they lead to

$$\mathcal{R} = \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix}\mathcal{I}\,, \tag{4.100}$$

where σ^1 is the standard Pauli matrix, so

$$\begin{aligned} e^{-2U} &= \langle \mathcal{R} \mid \mathcal{I} \rangle = 2(\mathcal{I}^0\mathcal{I}^1 + \mathcal{I}_0\mathcal{I}_1)\,, \\ \lambda &= i\frac{\mathcal{L}^1/X}{\mathcal{L}^0/X} = \frac{\mathcal{I}^1 + i\mathcal{I}_0}{\mathcal{I}_1 - i\mathcal{I}^0}\,. \end{aligned} \tag{4.101}$$

It is useful to define the complex harmonic functions

$$\begin{aligned} \mathcal{H}_1 &\equiv \mathcal{I}^1 + i\mathcal{I}_0 = \mathcal{H}_{1\,\infty} - \frac{\Gamma_1}{\sqrt{2}}\tau\,, \\ \mathcal{H}_0 &\equiv \mathcal{I}_1 - i\mathcal{I}^0 = \mathcal{H}_{0\,\infty} - \frac{\Gamma_0}{\sqrt{2}}\tau\,, \end{aligned} \tag{4.102}$$

in terms of which we have

$$e^{-2U} = 2\,\mathrm{Im}(\mathcal{H}_1\mathcal{H}_0^*)\,, \qquad \lambda = \frac{\mathcal{H}_1}{\mathcal{H}_0}\,. \tag{4.103}$$

The solution depends on the charges $\mathcal{Q}$ and on the two complex constants $\mathcal{H}_{1\,\infty}$ and $\mathcal{H}_{0\,\infty}$. A combination of them $(\mathcal{H}_{1\,\infty}/\mathcal{H}_{0\,\infty})$ is λ_∞ and the other combination is determined in terms of $\mathcal{Q}$ and λ_∞ by imposing asymptotic flatness

$$2\,\mathrm{Im}(\mathcal{H}_{1\,\infty}\mathcal{H}_{0\,\infty}^*) = 1\,, \tag{4.104}$$

which provides one real condition, and absence of NUT charge

$$\mathrm{Re}(\mathcal{H}_{1\,\infty}\Gamma_0^* - \mathcal{H}_{0\,\infty}\Gamma_1^*) = 0\,, \tag{4.105}$$

which is another real condition. These conditions have two solutions

$$\begin{aligned} \mathcal{H}_{1\,\infty} = \lambda_\infty \mathcal{H}_{0\,\infty}\,, \qquad & \mathcal{H}_{0\,\infty} = \mp\frac{i}{\sqrt{2\,\mathrm{Im}\,\lambda_\infty}}\frac{\mathcal{Z}_\infty^*}{|\mathcal{Z}_\infty|}\,, \\ \mathcal{Z}_\infty \equiv \mathcal{Z}(\lambda_\infty, \lambda_\infty^*, \mathcal{Q})\,, & \end{aligned} \tag{4.106}$$

but, using them in the expression for the mass

$$M = \tfrac{1}{\sqrt{2}}\,\mathrm{Im}(\mathcal{H}_{1\,\infty}\Gamma_0^* - \mathcal{H}_{0\,\infty}\Gamma_1^*)\,, \tag{4.107}$$

one finds that only the upper sign gives a positive mass, which turns out to be equal to $|\mathcal{Z}_\infty|$, as expected.

The complete solution is, therefore, given by the two harmonic functions

$$\begin{aligned} \mathcal{H}_1^{\rm susy} &= -\frac{i\lambda_\infty}{\sqrt{2\,\mathrm{Im}\,\lambda_\infty}}\frac{\mathcal{Z}_\infty^*}{|\mathcal{Z}_\infty|} - \frac{\Gamma_1}{\sqrt{2}}\tau\,, \\ \mathcal{H}_0^{\rm susy} &= -\frac{i}{\sqrt{2\,\mathrm{Im}\,\lambda_\infty}}\frac{\mathcal{Z}_\infty^*}{|\mathcal{Z}_\infty|} - \frac{\Gamma_0}{\sqrt{2}}\tau\,. \end{aligned} \tag{4.108}$$

Extremal non-supersymmetric solutions

According to the proposal made for the STU model in Ref. [102], the metric and scalar fields of the extremal non-supersymmetric, solutions can be constructed by replacing the electric and magnetic charges of their attractor values by the harmonic functions that have those charges as coefficients, that is $\mathcal{Q}^M$ should be replaced by the real harmonic function

$$H^M = H^M_\infty - \tfrac{1}{\sqrt{2}}\mathcal{Q}^M\tau\,. \tag{4.109}$$

The constant parts of the harmonic functions cannot be the same as those of the supersymmetric solution, otherwise the prescription would lead to

$$e^{-2U} = -2(\mathcal{I}^0\mathcal{I}^1 + \mathcal{I}_0\mathcal{I}_1)\,, \qquad \lambda = \frac{\mathcal{I}^1 - i\mathcal{I}_0}{\mathcal{I}_1 + i\mathcal{I}^0}\,, \tag{4.110}$$

or, in terms of the complex harmonic functions defined in Eq. (4.102)

$$e^{-2U} = -2\,\mathrm{Im}(\mathcal{H}_1\mathcal{H}_0^*)\,, \qquad \lambda = \frac{\mathcal{H}_1^*}{\mathcal{H}_0^*}\,. \tag{4.111}$$

If we plug in these expressions the values of the harmonic functions determined before, we get inconsistent results, because the metric function e^{-2U} is that of the supersymmetric case and goes to -1 at spatial infinity. Thus, the prescription given in Ref. [102] should be interpreted as a replacement of the charges by harmonic functions with asymptotic values yet to be determined by imposing asymptotic flatness etc. In Section 3.3 we will determine the form of the extremal non-supersymmetric solutions by taking an appropriate extremal limit of the non-extremal solution.

3.3 Non-extremal solutions

Our ansatz of Section 2.5 for the non-extremal solution is

$$\begin{aligned} e^{-2U} &= e^{-2[U_\mathrm{e}(\hat{\mathcal{H}})+r_0\tau]}\,, \qquad e^{-2U_\mathrm{e}(\hat{\mathcal{H}})} = 2\,\mathrm{Im}(\hat{\mathcal{H}}_1\hat{\mathcal{H}}_0^*)\,,\\ \lambda &= \lambda_\mathrm{e}(\hat{\mathcal{H}}) = \hat{\mathcal{H}}_1/\hat{\mathcal{H}}_0\,, \end{aligned} \tag{4.112}$$

where the deformed harmonic functions are assumed to have the form

$$\hat{\mathcal{H}}_\Lambda \equiv A_\Lambda + B_\Lambda e^{2r_0\tau}\,, \quad \Lambda = 1, 0\,, \tag{4.113}$$

The four complex constants A_Λ, B_Λ need to be determined by imposing on them the equations of motion (4.34)–(4.36), asymptotic flatness, absence of NUT charge plus the definitions of M and λ_∞.

Solving the equations of motion is not as complicated a task as it may look at first sight. First of all, we observe that all the dependence of U and λ on τ is of the form of the Schwarzschild factor $e^{2r_0\tau}$, which we are going to denote by f. Using the chain rule and combining the first two equations, we get

$$\ddot{U}_{\rm e} - (\dot{U}_{\rm e})^2 - \mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*} = 0\,, \tag{4.114}$$

$$(2r_0)^2\left[f\ddot{U}_{\rm e} + \dot{U}_{\rm e}\right] + e^{2U_{\rm e}}V_{\rm bh} = 0\,, \tag{4.115}$$

$$(2r_0)^2\left[f\left(\ddot{Z}^i + \mathcal{G}^{ij^*}\partial_k\mathcal{G}_{lj^*}\dot{Z}^k\dot{Z}^l\right) + \dot{Z}^i\right] + e^{2U_{\rm e}}\mathcal{G}^{ij^*}\partial_{j^*}V_{\rm bh} = 0\,. \tag{4.116}$$

Secondly, $U_{\rm e}$ and λ only depend on f through the deformed harmonic functions and, therefore, by virtue of the chain rule:

$$\begin{aligned}
\dot{U}_{\rm e} &= \partial_\Lambda U_{\rm e}\,B_\Lambda + \text{c.c.}\,,\\
\ddot{U}_{\rm e} &= \partial_\Sigma\partial_\Lambda U_{\rm e}\,B_\Lambda B_\Sigma + \partial^*_\Sigma\partial_\Lambda U_{\rm e}\,B_\Lambda B^*_\Sigma + \text{c.c.}\,,\\
\dot{Z}^i &= \partial_\Lambda Z^i\,B_\Lambda + \partial^*_\Lambda Z^i\,B^*_\Lambda\,,
\end{aligned} \tag{4.117}$$

etc., where $\partial_\Lambda \equiv \partial/\partial\hat{\mathcal{H}}_\Lambda$ and $\partial^*_\Lambda \equiv \partial/\partial\hat{\mathcal{H}}^*_\Lambda$. Then Eq. (4.114) becomes, after multiplication by a convenient global factor, a quadratic polynomial in the deformed harmonic functions with coefficients that are combinations of the integration constants B_Λ. This is true for any $N = 2$ model. For the axidilaton model, the polynomial turns out to be the square of a generalization of the condition of absence of NUT charge:

$$\text{Re}(\mathcal{H}_1 B^*_0 - \mathcal{H}_0 B^*_1) = \text{Re}(A_1 B^*_0 - A_0 B^*_1)\,. \tag{4.118}$$

Setting this quantity to zero yields an algebraic equation for the integration constants, which is enough to solve the first equation. In a similar fashion we find that the other two differential equations are solved by our ansatz if the integration constants satisfy certain algebraic constraints that we summarize here:

$$\mathrm{Re}(A_1 B_0^* - A_0 B_1^*) = 0\,, \tag{4.119}$$

$$|\Gamma_1|^2 A_0 B_0 + |\Gamma_0|^2 A_1 B_1 - \mathrm{Re}(\Gamma_1 \Gamma_0^*)(A_1 B_0 + A_0 B_1) = 0\,, \tag{4.120}$$

$$|\Gamma_1|^2 A_0^2 + |\Gamma_0|^2 A_1^2 - 2\,\mathrm{Re}(\Gamma_1 \Gamma_0^*) A_1 A_0 + 8 i r_0^2\, \mathrm{Im}(A_1 A_0^*)(A_1 B_0 - A_0 B_1) = 0\,, \tag{4.121}$$

$$|\Gamma_1|^2 B_0^2 + |\Gamma_0|^2 B_1^2 - 2\,\mathrm{Re}(\Gamma_1 \Gamma_0^*) B_1 B_0 - 8 i r_0^2\, \mathrm{Im}(B_1 B_0^*)(A_1 B_0 - A_0 B_1) = 0\,, \tag{4.122}$$

$$\mathrm{Re}(A_0 B_0^*) + \frac{1}{8 r_0^2} |\Gamma_0|^2 = 0\,, \tag{4.123}$$

$$\mathrm{Re}(A_1 B_1^*) + \frac{1}{8 r_0^2} |\Gamma_1|^2 = 0\,, \tag{4.124}$$

$$\mathrm{Re}(A_0 B_1^* + A_1 B_0^*) + \frac{1}{4 r_0^2}\, \mathrm{Re}(\Gamma_1 \Gamma_0^*) = 0\,, \tag{4.125}$$

and to which we must add the conditions of asymptotic flatness and the definitions of M and λ_∞:

$$2\,\mathrm{Im}[(A_1 + B_1)(A_0^* + B_0^*)] = 1\,, \tag{4.126}$$

$$2 r_0\, \mathrm{Im}[A_1 A_0^* - B_1 B_0^*] = M\,, \tag{4.127}$$

$$\frac{A_1 + B_1}{A_0 + B_0} = \lambda_\infty\,. \tag{4.128}$$

From these equations we can derive a relation between the non-extremality parameter, mass, charge and moduli, which is convenient to write in this form:

$$M^2 r_0^2 = (M^2 - |\mathcal{Z}_\infty|^2)(M^2 - |\tilde{\mathcal{Z}}_\infty|^2)\,. \tag{4.129}$$

This shows that there are two different extremal limits (supersymmetric and non-supersymmetric) and that the non-extremal family of solutions interpolates between these two limits. This will allow us to obtain the extremal non-supersymmetric solution in a clean way. Observe that in the context of $N = 4$ supergravity both extremal limits are supersymmetric [171, 173].

Expanding the above expression and comparing with the general result Eq. (4.21) one can find the scalar charge up to a phase. From the complete solution (see later) we obtain the exact result

$$\Sigma^\lambda = \frac{2i\,\mathrm{Im}\,\lambda_\infty\, \mathcal{Z}_\infty \mathcal{Z}'^*_\infty}{M} \,. \tag{4.130}$$

Since the expressions for the metric function and the scalar are invariant if we multiply $\mathcal{H}_1, \mathcal{H}_0$ by the same phase, we can use this freedom to simplify the equations setting $\mathrm{Im}(A_0 + B_0) = 0$. We can later restore the phase by studying the supersymmetric extremal limit.

Under this assumption we find (we use a tilde to stress the fact that these are not the final values of the integration constants):

$$\tilde{A}_1 = \frac{\lambda_\infty}{2\sqrt{2}\,\mathrm{Im}\,\lambda_\infty}\left\{1 + \frac{1}{Mr_0}\left\{M^2 + \tfrac{1}{2}V_{\mathrm{bh}\,\infty} + \tfrac{i}{2}\left[\frac{1}{\lambda_\infty}|\Gamma_1|^2 - \mathrm{Re}(\Gamma_1\Gamma_0^*)\right]\right\}\right\} \,, \tag{4.131}$$

$$\tilde{B}_1 = \frac{\lambda_\infty}{2\sqrt{2}\,\mathrm{Im}\,\lambda_\infty}\left\{1 - \frac{1}{Mr_0}\left\{M^2 + \tfrac{1}{2}V_{\mathrm{bh}\,\infty} + \tfrac{i}{2}\left[\frac{1}{\lambda_\infty}|\Gamma_1|^2 - \mathrm{Re}(\Gamma_1\Gamma_0^*)\right]\right\}\right\} \,, \tag{4.132}$$

$$\tilde{A}_0 = \frac{1}{2\sqrt{2}\,\mathrm{Im}\,\lambda_\infty}\left\{1 + \frac{1}{Mr_0}\left\{M^2 + \tfrac{1}{2}V_{\mathrm{bh}\,\infty} - \tfrac{i}{2}\left[\lambda_\infty|\Gamma_0|^2 - \mathrm{Re}(\Gamma_1\Gamma_0^*)\right]\right\}\right\} \,, \tag{4.133}$$

$$\tilde{B}_0 = \frac{1}{2\sqrt{2}\,\mathrm{Im}\,\lambda_\infty}\left\{1 - \frac{1}{Mr_0}\left\{M^2 + \tfrac{1}{2}V_{\mathrm{bh}\,\infty} - \tfrac{i}{2}\left[\lambda_\infty|\Gamma_0|^2 - \mathrm{Re}(\Gamma_1\Gamma_0^*)\right]\right\}\right\} \,, \tag{4.134}$$

where we are using the shorthand notation $V_{\mathrm{bh}\,\infty} \equiv V_{\mathrm{bh}}(\lambda_\infty, \lambda^*_\infty, \mathcal{Q})$.

Then, the metric function can be put in the two alternative forms

$$e^{-2U} = 1 \pm \frac{M}{r_0}(1 - e^{\pm 2r_0\tau}) + \frac{S_\pm}{\pi}\frac{\sinh^2 r_0\tau}{r_0^2} \,, \tag{4.135}$$

where $S_\pm$ are the entropies associated to the outer $(+)$ and inner $(-)$ horizons, given in Eqs. (4.140)–(4.142). In any of these two forms e^{-2U} is a sum of manifestly positive terms when $r_0^2 > 2$ and $S_\pm > 0$, so all the singularities will be covered by the horizons when they exist. The conditions under which this happens will be studied later.

Supersymmetric and non-supersymmetric extremal limits

The hatted functions have the following extremal limits ($r_0 \to 0$):

1. The supersymmetric extremal limit, when $M \to |\mathcal{Z}_\infty|$

$$\hat{\mathcal{H}}_{1,0} \stackrel{M\to|\mathcal{Z}_\infty|}{\longrightarrow} i\frac{\mathcal{Z}_\infty}{|\mathcal{Z}_\infty|}\,\mathcal{H}^{\rm susy}_{1,0}\,, \tag{4.136}$$

   with $\mathcal{H}^{\rm susy}_{1,2}$ given in Eq. (4.108).

2. The non-supersymmetric extremal limit, when $M \to |\tilde{\mathcal{Z}}_\infty|$

$$\hat{\mathcal{H}}_{1,0} \stackrel{M\to|\tilde{\mathcal{Z}}_\infty|}{\longrightarrow} i\frac{\mathcal{Z}'^{*}_\infty}{|\tilde{\mathcal{Z}}_\infty|}\,\mathcal{H}^{\rm nsusy}_{1,0}\,, \tag{4.137}$$

   with

$$\begin{aligned}\mathcal{H}^{\rm nsusy}_1 &= -\frac{i\lambda_\infty}{\sqrt{2\,{\rm Im}\,\lambda_\infty}}\frac{\tilde{\mathcal{Z}}_\infty}{|\tilde{\mathcal{Z}}_\infty|} - \frac{\Gamma_1^*}{\sqrt{2}}\tau\,,\\ \mathcal{H}^{\rm nsusy}_0 &= -\frac{i}{\sqrt{2\,{\rm Im}\,\lambda_\infty}}\frac{\tilde{\mathcal{Z}}_\infty}{|\tilde{\mathcal{Z}}_\infty|} - \frac{\Gamma_0^*}{\sqrt{2}}\tau\,.\end{aligned} \tag{4.138}$$

   $\mathcal{H}^{\rm nsusy}_{1,0}$ can be obtained by replacing everywhere in $\mathcal{H}^{\rm susy}_{1,0}$ the complex charges $\Gamma_{0,1}$ by their complex conjugates $\Gamma^*_{0,1}$.

   We stress that in this case, the metric function and scalar are still given by

$$e^{-2U} = 2\,{\rm Im}(\mathcal{H}^{\rm nsusy}_1\mathcal{H}^{\rm nsusy\,*}_0)\,, \qquad \lambda = \mathcal{H}^{\rm nsusy}_1/\mathcal{H}^{\rm nsusy}_0\,, \tag{4.139}$$

   and it is immediate to check that they lead to the non-supersymmetric attractor and entropy.

Physical properties of the non-extremal solutions

The "entropies" (one quarter of the areas) of the outer (+) and inner (−) horizon, placed at $\tau = -\infty$ and $\tau = +\infty$, respectively, are given by

$$\frac{S_\pm}{\pi} = (M^2 - |\mathcal{Z}_\infty|^2) \pm (M^2 - |\tilde{\mathcal{Z}}_\infty|^2) \pm 2Mr_0\,. \tag{4.140}$$

They can also be written in the form

$$S_\pm = \pi\left(\sqrt{N_{\rm R}} \pm \sqrt{N_{\rm L}}\right)^2\,, \tag{4.141}$$

with

$$N_{\rm R} \equiv M^2 - |\mathcal{Z}_\infty|^2\,, \qquad N_{\rm L} \equiv M^2 - |\tilde{\mathcal{Z}}_\infty|^2\,, \tag{4.142}$$

so the product of these "entropies" is manifestly moduli-independent:

$$S_+ S_- = \pi^2 (N_{\rm R} - N_{\rm L})^2 = \pi^2 \left[\mathrm{Im}\,(\Gamma_1 \Gamma_0^*)\,\right]^2\,. \tag{4.143}$$

From Ref. [171] we know exactly how these expressions are modified by the introduction of angular momentum $J \equiv \alpha M$: the entropies are given by

$$\frac{S_\pm}{\pi} = (M^2 - |\mathcal{Z}_\infty|^2) \pm (M^2 - |\tilde{\mathcal{Z}}_\infty|^2) \pm 2M\sqrt{r_0^2 - \alpha^2}\,, \tag{4.144}$$

and can be put in the suggestive form of Eq. (4.141) with

$$N_{\rm R,L} \equiv M^2 - \tfrac{1}{2}(|\mathcal{Z}_\infty|^2 + |\tilde{\mathcal{Z}}_\infty|^2) \pm \tfrac{1}{2}\sqrt{(|\mathcal{Z}_\infty|^2 - |\tilde{\mathcal{Z}}_\infty|^2)^2 + 4J^2}\,. \tag{4.145}$$

Again, the product of the two entropies is moduli-independent:

$$S_+ S_- = \pi^2 (N_{\rm R} - N_{\rm L})^2 = \pi^2 \left\{[\mathrm{Im}\,(\Gamma_1 \Gamma_0^*)\,]^2 + 4J^2\right\}\,. \tag{4.146}$$

The temperatures $T_\pm$ can be computed from $S_\pm$ using Eq. (4.8).

In the two extremal cases, the scalar takes attractor values on the horizon, which are independent of its asymptotic value λ_∞. In non-extremal black holes the scalar takes the horizon value

$$\lambda_{\rm h}^{\rm ne} = \frac{\lambda_\infty S_+/\pi + i[|\Gamma_1|^2 - \lambda_\infty\,\mathrm{Re}(\Gamma_1\Gamma_0^*)]}{S_+/\pi - i[\lambda_\infty|\Gamma_0|^2 - \mathrm{Re}(\Gamma_1\Gamma_0^*)]}\,, \tag{4.147}$$

which manifestly depends on λ_∞, from which we conclude that the attractor mechanism does not work in this case.

We observe that if in the general non-extremal case λ_∞ is set equal to one of the two attractor values, then $\lambda(\tau)$ is constant over the space. In other words: the non-extremal deformation of a double-extremal black hole also has constant scalars and, therefore, has the metric of the non-extremal Reissner–Nordström black hole.

In the evaporation of a non-extremal black hole of this theory only M changes, while the charges and λ_∞ remain constant.[14] The value of M will

[14] There are no particles carrying electric or magnetic charges in ungauged $N = 2, d = 4$ supergravity and there is no perturbative physical mechanism that can change the moduli, which are properties characterizing the vacuum.

decrease until it becomes equal to $\max(|\mathcal{Z}_\infty|, |\tilde{\mathcal{Z}}_\infty|)$ This value depends on the values of the charges and moduli in this way:

$$|\mathcal{Z}_\infty| > |\tilde{\mathcal{Z}}_\infty| \quad \Leftrightarrow \quad \cos \mathrm{Arg}(\lambda_\infty/\lambda_{\mathrm{h}}^{\mathrm{susy}}) > \cos \mathrm{Arg}(\lambda_\infty/\lambda_{\mathrm{h}}^{\mathrm{nsusy}})\,. \tag{4.148}$$

Hence, if the phase of λ_∞ is closer to that of the supersymmetric attractor value Γ_1/Γ_0 than to that of the non-supersymmetric one Γ_1^*/Γ_0^*, the central charge $|\mathcal{Z}_\infty|$ will be larger than $|\tilde{\mathcal{Z}}_\infty|$ and the evaporation process will stop at the supersymmetric extremal limit and vice versa. However, in this analysis we must take into account that the imaginary part of λ must be positive at any point, which means that $\mathrm{Im}\,\lambda_\infty > 0$ and only one of $\lambda_{\mathrm{h}}^{\mathrm{susy}}$ and $\lambda_{\mathrm{h}}^{\mathrm{nsusy}}$ will satisfy that condition for a given choice of electric and magnetic charges. Then, it is easy to see that if $\mathrm{Im}\,\lambda_{\mathrm{h}}^{\mathrm{susy}} > 0$, for any λ_∞ satisfying $\mathrm{Im}\,\lambda_\infty > 0$, the above condition is met and the endpoint of the evaporation process should be the supersymmetric one and if $\mathrm{Im}\,\lambda_{\mathrm{h}}^{\mathrm{nsusy}} > 0$, then the opposite will be true for any admissible λ_∞.

We conclude that a family of non-extremal black hole solutions with given electric and magnetic charges $\mathcal{Q}$ and parametrized by r_0 is always attracted to one of the two extremal solutions in the evaporation process, independently of our choice of λ_∞. The same will happen to the non-extremal black holes of the model that we are going to consider next and which can be regarded as an extension of the axidilaton model.

4 Black holes of the $\overline{\mathbb{CP}}^n$ model

This model is characterized by the prepotential

$$\mathcal{F} = -\tfrac{i}{4}\eta_{\Lambda\Sigma}\mathcal{X}^\Lambda\mathcal{X}^\Sigma\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+-\cdots-)\,, \tag{4.149}$$

and has n scalars

$$Z^i \equiv \mathcal{X}^i/\mathcal{X}^0\,, \tag{4.150}$$

to which we add for convenience $Z^0 \equiv 1$, so we have

$$(Z^\Lambda) \equiv \left(\mathcal{X}^\Lambda/\mathcal{X}^0\right) = (1, Z^i)\,, \qquad (Z_\Lambda) \equiv (\eta_{\Lambda\Sigma}Z^\Sigma) = (1, Z_i) = (1, -Z^i)\,. \tag{4.151}$$

This will simplify our notation. Thus, the Kähler potential and metric are given by

$$\begin{aligned} \mathcal{K} = -\log\left(Z^{*\Lambda}Z_\Lambda\right), \quad \mathcal{G}_{ij^*} &= -e^{\mathcal{K}}\left(\eta_{ij^*} - e^{\mathcal{K}}Z_i^*Z_{j^*}\right)\,, \\ \mathcal{G}^{ij^*} &= -e^{-\mathcal{K}}\left(\eta^{ij^*} + Z^iZ^{*\,j^*}\right)\,. \end{aligned} \tag{4.152}$$

The covariantly holomorphic symplectic section reads

$$\mathcal{V} = e^{\mathcal{K}/2} \begin{pmatrix} Z^\Lambda \\ \\ -\frac{i}{2} Z_\Lambda \end{pmatrix} , \tag{4.153}$$

and, in terms of the complex charge combinations

$$\Gamma_\Lambda \equiv q_\Lambda + \tfrac{i}{2} \eta_{\Lambda\Sigma} p^\Sigma , \tag{4.154}$$

the central charge, its holomorphic Kähler-covariant derivative and the black-hole potential are given by

$$\begin{aligned} \mathcal{Z} &= e^{\mathcal{K}/2} Z^\Lambda \Gamma_\Lambda , \\ \mathcal{D}_i \mathcal{Z} &= e^{3\mathcal{K}/2} Z_i^* Z^\Lambda \Gamma_\Lambda - e^{\mathcal{K}/2} \Gamma_i , \\ |\tilde{\mathcal{Z}}|^2 &\equiv \mathcal{G}^{ij^*} \mathcal{D}_i \mathcal{Z} \mathcal{D}_{j^*} \mathcal{Z}^* = e^{\mathcal{K}} |Z^\Lambda \Gamma_\Lambda|^2 - \Gamma^{*\,\Lambda} \Gamma_\Lambda , \\ -V_{\rm bh} &= 2 e^{\mathcal{K}} |Z^\Lambda \Gamma_\Lambda|^2 - \Gamma^{*\,\Lambda} \Gamma_\Lambda . \end{aligned} \tag{4.155}$$

4.1 Flow equations

Similarly as in the axion-dilaton model, the potential term can be expanded into

$$- \left[e^{2U} V_{\rm bh} - r_0^2 \right] = \Upsilon^2 + 4\, \mathcal{G}^{ij^*} \Psi_i \Psi^*_{j^*} , \tag{4.156}$$

where

$$\Upsilon = \frac{e^U}{\sqrt{2}} \sqrt{ |\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2 + e^{-2U} r_0^2 + \sqrt{ \left(|\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2 + e^{-2U} r_0^2 \right)^2 - 4 |\mathcal{Z}|^2 |\tilde{\mathcal{Z}}|^2 } } , \tag{4.157}$$

$$\Psi_i = e^{2U} \frac{\mathcal{Z}^* \, \mathcal{D}_i \mathcal{Z}}{\Upsilon} , \tag{4.158}$$

with the definitions Eqs. (4.155).

Since

$$\partial_U \Psi_i - \partial_i \Upsilon = \partial_i \Psi_j - \partial_j \Psi_i = \partial_{i*} \Psi_j - \partial_j \Psi^*_{i^*} = 0 , \tag{4.159}$$

there exists a superpotential, whose gradient generates the vector field $(\Upsilon, \Psi_i, \Psi^*_{j^*})$ and the first-order equations

$$U' = \Upsilon , \tag{4.160}$$

$$Z^{i\,\prime} = 2\, \mathcal{G}^{ij^*} \Psi^*_{j^*} \tag{4.161}$$

Attractor	$e^{-\mathcal{K}_\mathrm{h}}$	$\lvert\mathcal{Z}_\mathrm{h}\rvert^2$	$\lvert\tilde{\mathcal{Z}}_\mathrm{h}\rvert^2$	$-V_\mathrm{bhh}$	M
$Z_\mathrm{h}^{i\,\mathrm{susy}} = \Gamma^{*\,i}/\Gamma^{*\,0}$	$\Gamma^{*\,\Lambda}\Gamma_\Lambda$	$\Gamma^{*\,\Lambda}\Gamma_\Lambda$	0	$\Gamma^{*\,\Lambda}\Gamma_\Lambda$	$\lvert\mathcal{Z}_\infty\rvert$
$Z_\mathrm{h}^{\Lambda\,\mathrm{nsusy}}\Gamma_\Lambda = 0$	$-\Gamma^{*\,\Lambda}\Gamma_\Lambda$	0	$-\Gamma^{*\,\Lambda}\Gamma_\Lambda$	$-\Gamma^{*\,\Lambda}\Gamma_\Lambda$	$\lvert\tilde{\mathcal{Z}}_\infty\rvert$

Tab. 4.2: Critical points of the $\overline{\mathbb{CP}}^n$ model.

solve the second-order equations of motion.

4.2 The extremal case

Critical points

To find the critical points of the black-hole potential it is simpler to search for the zeros of

$$\mathcal{G}^{ij^*}\partial_{j^*}V_\mathrm{bh} = 2Z^\Lambda\Gamma_\Lambda\left(\Gamma^{*\,i} - \Gamma^{*\,0}Z^i\right) , \tag{4.162}$$

which has two factors that can vanish separately. The second factor vanishes only for the isolated point in moduli space

$$Z_\mathrm{h}^i = \Gamma^{*\,i}/\Gamma^{*\,0} , \tag{4.163}$$

and corresponds to the supersymmetric attractor, whereas the first factor vanishes for the complex hypersurface of the moduli space defined by the condition

$$Z_\mathrm{h}^\Lambda\Gamma_\Lambda = 0 . \tag{4.164}$$

These points are associated with non-supersymmetric black holes (the central charge vanishes). The attractor behavior fixes only a combination of scalars on the horizon, but each of them individually still depends on the asymptotic values Z_∞^i. The situation is summarized in Table 4.2.

As we mentioned earlier, the supersymmetric stationary point must be stable and, since the $\overline{\mathbb{CP}}^n$ model is also based on a homogeneous manifold, the non-supersymmetric stationary points must be stable as well, even though, because the stationary locus is a submanifold of complex codimension 1, rather than an isolated point, one expects $n - 1$ complex flat directions. In fact the Hessian in the real basis has one double eigenvalue

$$4\frac{\delta^{ij}\Gamma_i^*\Gamma_j}{1 - \delta_{kl}Z_{\rm h}^{*\,k}Z_{\rm h}^{l}}\,. \tag{4.165}$$

At first it may seem that for sufficiently large values of the scalars on the horizon the eigenvalue could become negative. The above expression, however, is proportional to a (multiple) eigenvalue of the scalar metric, hence the values for which the Hessian becomes negative semi-definite would also render the scalar metric negative definite and are consequently not physically admissible.

Supersymmetric solutions

The stabilization equations are solved by

$$\mathcal{R}_\Lambda = \tfrac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Sigma\,, \qquad \mathcal{R}^\Lambda = -2\eta^{\Lambda\Sigma}\mathcal{I}_\Sigma\,, \tag{4.166}$$

so

$$\mathcal{L}^\Lambda/X = \mathcal{R}^\Lambda + i\mathcal{I}^\Lambda = -2\eta^{\Lambda\Sigma}(\mathcal{I}_\Sigma - \tfrac{i}{2}\eta_{\Sigma\Omega}\mathcal{I}^\Omega)\,. \tag{4.167}$$

Defining the complex combinations of harmonic functions

$$\mathcal{H}_\Lambda \equiv \mathcal{I}_\Lambda + \tfrac{i}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Sigma \equiv \mathcal{H}_{\Lambda\,\infty} - \tfrac{1}{\sqrt{2}}\Gamma_\Lambda\tau\,, \tag{4.168}$$

where $\mathcal{H}_{\Lambda\,\infty}$ are the values at spatial infinity, we find that the metric function and scalar fields are given by

$$e^{-2U} = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_\Lambda\,, \qquad Z^i = \frac{\mathcal{L}^i/X}{\mathcal{L}^0/X} = \frac{\mathcal{H}^{*i}}{\mathcal{H}^{*0}}\,, \tag{4.169}$$

where we are using η to raise and lower the indices of the complex harmonic functions.

The solution depends on the $\bar{n}$ complex charges Γ_Λ and on the $n + 1$ complex constants $\mathcal{H}_{\Lambda\,\infty}$. n combinations of them are determined by the asymptotic values of the n scalars

$$Z^i_\infty = \mathcal{H}^{*\,i}_\infty/\mathcal{H}^{*\,0}_\infty\,, \tag{4.170}$$

and the remaining one is determined by the two real conditions of asymptotic flatness

$$2\mathcal{H}^{*\,\Lambda}_\infty\mathcal{H}_{\Lambda\,\infty} = 1\,, \tag{4.171}$$

and absence of NUT charge

$$\mathrm{Im}\left(\mathcal{H}^{*\Lambda}_{\infty}\Gamma_{\Lambda}\right) = 0\,. \tag{4.172}$$

The result is

$$\mathcal{H}^{\Lambda}_{\infty} = \pm e^{\mathcal{K}_{\infty}/2}\frac{\mathcal{Z}_{\infty}}{|\mathcal{Z}_{\infty}|}Z^{*\Lambda}_{\infty}\,, \tag{4.173}$$

where $\mathcal{K}_{\infty}$ and $\mathcal{Z}_{\infty}$ are the asymptotic values of the Kähler potential and central charge, although the positivity of the mass, which is given, as expected, by $M = |\mathcal{Z}_{\infty}|$, allows only for the upper sign.

The complete supersymmetric solution is, therefore, given by the $\bar{n}$ complex harmonic functions

$$\mathcal{H}^{\rm susy}_{\Lambda} = e^{\mathcal{K}_{\infty}/2}\frac{\mathcal{Z}_{\infty}}{|\mathcal{Z}_{\infty}|}Z^{*}_{\Lambda\,\infty} - \tfrac{1}{\sqrt{2}}\Gamma_{\Lambda}\tau\,, \tag{4.174}$$

that depend only on the $2n+1$ physical complex parameters $Z^{i}_{\infty}, \Gamma_{\Lambda}$.

In order to find the extremal non-supersymmetric solutions we will first obtain the general non-extremal ones and then we will take the extremal non-supersymmetric limit. We will see that this procedure works as in the axidilaton case because the non-extremal solutions interpolate between the different extremal limits.

4.3 Non-extremal solutions

Our ansatz for the non-extremal solution is again

$$\begin{aligned} e^{-2U} =& e^{-2[U_{\rm e}(\hat{\mathcal{H}})+r_0\tau]}\,, \qquad e^{-2U_{\rm e}(\hat{\mathcal{H}})} = 2\hat{\mathcal{H}}^{*\,\Lambda}\hat{\mathcal{H}}_{\Lambda}\,, \\ & Z^{i} = Z^{i}_{\rm e}(\hat{\mathcal{H}}) = \hat{\mathcal{H}}^{*\,i}/\hat{\mathcal{H}}^{*\,0}\,, \end{aligned} \tag{4.175}$$

where the hatted functions are assumed to have the form

$$\hat{\mathcal{H}}^{\Lambda} \equiv A^{\Lambda} + B^{\Lambda}e^{2r_0\tau}\,, \quad \Lambda = 0,\ldots,n\,. \tag{4.176}$$

As in the axidilaton model, we have to find the $2\bar{n}$ complex constants A_{Λ}, B_{Λ} by requiring that we have a solution to the equations of motion (4.114)–(4.116). It is not difficult to see that this happens if the following algebraic conditions are satisfied:

$$\mathrm{Im}(B^{*\,\Lambda}A_\Lambda) = 0\,, \tag{4.177}$$

$$A^{*\,\Lambda}A^\Sigma\xi_{\Lambda\Sigma} = 0\,, \tag{4.178}$$

$$(A^{*\,\Lambda}B^\Sigma + B^{*\,\Lambda}A^\Sigma)\xi_{\Lambda\Sigma} = 0\,, \tag{4.179}$$

$$B^{*\,\Lambda}B^\Sigma\xi_{\Lambda\Sigma} = 0\,, \tag{4.180}$$

$$(2r_0)^2(B_i^*A_0^* - B_0^*A_i^*)A^{*\,\Lambda}A_\Lambda + (\Gamma_i^*A_0^* - \Gamma_0^*A_i^*)A^{*\,\Lambda}\Gamma_\Lambda = 0\,, \tag{4.181}$$

$$-(2r_0)^2(B_i^*A_0^* - B_0^*A_i^*)B^{*\,\Lambda}B_\Lambda + (\Gamma_i^*B_0^* - \Gamma_0^*B_i^*)B^{*\,\Lambda}\Gamma_\Lambda = 0\,, \tag{4.182}$$

$$(\Gamma_i^*A_0^* - \Gamma_0^*A_i^*)A^{*\,\Lambda}\Gamma_\Lambda + (\Gamma_i^*B_0^* - \Gamma_0^*B_i^*)B^{*\,\Lambda}\Gamma_\Lambda = 0\,, \tag{4.183}$$

where we have defined

$$\xi_{\Lambda\Sigma} \equiv 2\left(\Gamma_\Lambda\Gamma_\Sigma^* + 8r_0^2A_\Lambda B_\Sigma^*\right) - \eta_{\Lambda\Sigma}\left(\Gamma^\Omega\Gamma_\Omega^* + 8r_0^2A^\Omega B_\Omega^*\right)\,. \tag{4.184}$$

In order to fully identify the constants A_Λ, B_Λ in terms of the physical parameters, we must add to the above conditions the requirement of asymptotic flatness and the definitions of mass M and of the asymptotic values of the scalars Z^i_∞:

$$2(A^{*\,\Lambda} + B^{*\,\Lambda})(A_\Lambda + B_\Lambda) = 1\,, \tag{4.185}$$

$$4\,\mathrm{Re}[B^{*\,\Lambda}(A_\Lambda + B_\Lambda)] = 1 - M/r_0\,, \tag{4.186}$$

$$\frac{A^{*\,i} + B^{*\,i}}{A^{*\,0} + B^{*\,0}} = Z^i_\infty\,. \tag{4.187}$$

The condition of absence of NUT charge arises naturally as a consequence of the equations of motion (it is Eq. (4.177)).

To solve these equations we choose $A_0 + B_0$ to be real, as we did in the axidilaton case. Then, we find the following result:

$$A_\Lambda = \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1 + \frac{(M^2 - e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right] + \frac{\Gamma_\Lambda Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma}{Mr_0}\right\}\,, \tag{4.188}$$

$$B_\Lambda = \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1 - \frac{(M^2 - e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right] - \frac{\Gamma_\Lambda Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma}{Mr_0}\right\}\,, \tag{4.189}$$

$$M^2r_0^2 = (M^2 - |\mathcal{Z}_\infty|^2)(M^2 - |\tilde{\mathcal{Z}}_\infty|^2)\,, \tag{4.190}$$

where $|\tilde{\mathcal{Z}}|^2$ is defined in Eq. (4.155) and we remind the reader that $-V_{\rm bh} = |\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2$.

With these values it is easy to see that the metric function e^{-2U} can be put in exactly the same form as in the axidilaton case, given in Eq. (4.135) where r_0 and $S_\pm$ are now those of the present case. This means that the metric is regular in all the $r_0^2 > 0$ cases.

Supersymmetric and non-supersymmetric extremal limits

Again, there are two possible extremal limits in which $r_0 \to 0$:

1. Supersymmetric, when $M^2 \to |\mathcal{Z}|^2 = e^{\mathcal{K}_\infty}|Z^\Sigma_\infty \Gamma_\Sigma|^2$. In this limit we get

$$\hat{\mathcal{H}}_\Lambda \overset{M\to|\mathcal{Z}_\infty|}{\longrightarrow} \pm \frac{\mathcal{Z}^*_\infty}{|\mathcal{Z}_\infty|}\mathcal{H}^{\rm susy}_\Lambda\,, \tag{4.191}$$

where $\mathcal{H}^{\rm susy}_\Lambda$ is given by Eq. (4.174). This determines the phase of $A_0 + B_0$, which we set to zero at the beginning for simplicity, making use of the formal phase invariance of the solution.

2. Non-supersymmetric, when $M^2 \to |\tilde{\mathcal{Z}}|^2 = e^{\mathcal{K}_\infty}|Z^\Sigma_\infty \Gamma_\Sigma|^2 - \Gamma^{*\,\Sigma}\Gamma_\Sigma$. In this limit we get

$$\hat{\mathcal{H}}_\Lambda \overset{M\to|\tilde{\mathcal{Z}}_\infty|}{\longrightarrow} \pm \frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty} - \frac{1}{|\tilde{\mathcal{Z}}_\infty|}\left[-Z^*_{\Lambda\,\infty}\Gamma^{*\,\Sigma}\Gamma_\Sigma + \Gamma_\Lambda Z^{*\,\Sigma}_\infty \Gamma^*_\Sigma\right]\tau\right\}. \tag{4.192}$$

In this case we do not have an explicit solution to compare with and we cannot determine the phase of $A_0 + B_0$. However, the metric and scalar fields do not depend on that phase and the above harmonic functions determine them completely.

It takes little time to see that in this case the entropy is

$$S = -\pi\Gamma^{*\,\Sigma}\Gamma_\Sigma\,, \tag{4.193}$$

as expected, and that on the event horizon the scalars take the values

$$Z^{*\,i}_{\rm h} = \frac{\Gamma^i Z^{*\,\Lambda}_\infty \Gamma^*_\Lambda - Z^{*\,i}_\infty \Gamma^{*\,\Sigma}\Gamma_\Sigma}{\Gamma^0 Z^{*\,\Gamma}_\infty \Gamma^*_\Gamma - \Gamma^{*\,\Omega}\Gamma_\Omega}\,, \tag{4.194}$$

which depend manifestly on the asymptotic values. It is easy to check that the horizon values satisfy the condition $Z^\Lambda_{\rm h}\Gamma_\Lambda = 0$.

Had we tried to implement the prescription of replacement of charges by harmonic functions in the extremal non-supersymmetric horizon values, it is difficult to see how the full solution with the above coefficients in the harmonic functions could have been recovered.

Physical properties of the non-extremal solutions

The entropies of the black-hole solutions of this model can also be put in the form Eqs. (4.140)–(4.142), where now $\mathcal{Z}_\infty$ and $\tilde{\mathcal{Z}}_\infty$ take the form corresponding to the present model. In both extremal limits we obtain finite entropies which are moduli-independent, even though in the extremal non-supersymmetric limit the values of the scalars on the horizon depend on the asymptotic boundary conditions according to Eq. (4.194). In the non-extremal case, the product of the entropies of the inner and outer horizon gives the square of the extremal entropy and, consequently, is moduli-independent.

Also in this case the non-extremal deformation of the double-extremal solutions have constant scalars: if the asymptotic values of the scalars in the general case coincide with their horizon attractor values in the extremal case, then the scalars are constant and the metric is that of the Reissner–Nordström solution.

The endpoint of the evaporation process of the non-extremal black holes of this model is completely determined by their electric and magnetic charges and is independent of the choice of asymptotic values Z^i_∞ for the scalars. Thus, if $\Gamma^{*\Lambda}\Gamma_\Lambda > 0$, which is the property that characterizes the supersymmetric attractor, then $|\mathcal{Z}_\infty| > |\tilde{\mathcal{Z}}_\infty|$ and the evaporation process will stop when $M = |\mathcal{Z}_\infty|$, the supersymmetric case. The opposite will be true if $\Gamma^{*\Lambda}\Gamma_\Lambda < 0$. Again, we can speak of an attractive behavior in the evaporation process.

5 *D0-D4 black holes*

In this section we are going to obtain, following the procedure outlined in Section 2.5, the non-extremal deformation of the well-known supersymmetric D0-D4 black hole embedded in the *STU* model [59, 114, 174].

We have chosen this particular solution because the non-extremal case is manageable, yet general enough to be interesting. Furthermore, the well-known supersymmetric limit has a straightforward microscopic interpretation. This fact could be useful for obtaining a microscopic interpretation in the non-extremal case, although this interpretation may be difficult to find, since for non-extremal black holes we have neither supersymmetry nor attractor mechanism to protect the solution from the effects of a strong-weak change of the coupling.

The STU model is defined through the following prepotential:[15]

$$\mathcal{F} = \frac{\mathcal{X}^1\mathcal{X}^2\mathcal{X}^3}{\mathcal{X}^0}\,, \tag{4.195}$$

and has three scalars customarily defined as

$$Z^1 \equiv \frac{\mathcal{X}^1}{\mathcal{X}^0} \equiv S\,, \qquad Z^2 \equiv \frac{\mathcal{X}^2}{\mathcal{X}^0} \equiv T\,, \qquad Z^3 \equiv \frac{\mathcal{X}^3}{\mathcal{X}^0} \equiv U\,, \tag{4.196}$$

with Kähler potential (in the $\mathcal{X}^0 = 1$ gauge) and metric given by

$$e^{-\mathcal{K}} = -8\,\mathrm{Im}\,S\;\mathrm{Im}\,T\;\mathrm{Im}\,U\,, \qquad \mathcal{G}_{ij^*} = \frac{\delta_{(i)j^*}}{4(\mathrm{Im}\;Z^{(i)})^2}\,. \tag{4.197}$$

The covariantly holomorphic symplectic section is given by

$$\mathcal{V} = \begin{pmatrix}\mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda\end{pmatrix} = e^{\mathcal{K}/2}\begin{pmatrix}1 \\ Z^i \\ -\mathcal{F} \\ 3d_{ijk}Z^jZ^k\end{pmatrix} = \frac{1}{\sqrt{-8\,\mathrm{Im}\,S\;\mathrm{Im}\,T\;\mathrm{Im}\,U}}\begin{pmatrix}1 \\ S \\ T \\ U \\ -STU \\ TU \\ SU \\ ST\end{pmatrix}, \tag{4.198}$$

and therefore, we have

$$\begin{aligned} \mathcal{Z} &= e^{\mathcal{K}/2}W\,, \\ \mathcal{D}_i\mathcal{Z} &= \frac{ie^{\mathcal{K}/2}}{2\,\mathrm{Im}\,Z^{(i)}}W_{(i)}\,, \\ -V_{\rm bh} &= e^{\mathcal{K}}\left\{|W|^2 + \sum_{i=1}^{3}|W_i|^2\right\}\,, \end{aligned} \tag{4.199}$$

[15] Sometimes it is convenient to use the symmetric tensor $d_{ijk} = |\epsilon_{ijk}|$ so $\mathcal{F} = \frac{1}{6}d_{ijk}\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k/\mathcal{X}^0$.

where

$$W = W(S,T,U,\mathcal{Q}) \equiv -p^0\mathcal{F} - q_0 + \sum_{i=1}^{3}\left(3d_{ijk}p^iZ^jZ^k - q_iZ^i\right) , \quad (4.200)$$

$$W_1 \equiv W(S^*,T,U,\mathcal{Q}) , \quad (4.201)$$

$$W_2 \equiv W(S,T^*,U,\mathcal{Q}) , \quad (4.202)$$

$$W_3 \equiv W(S,T,U^*,\mathcal{Q}) . \quad (4.203)$$

The D0-D4 black holes that we are going to consider only have four non-vanishing charges which, when embedded in the STU model, correspond to three magnetic charges p^i, $i = 1,\ldots,3$ from the vector fields in the three vector multiplets, and the electric charge q_0 of the graviphoton. In this case the function W reduces to just

$$W = W(S,T,U,\mathcal{Q}) = 3d_{ijk}p^iZ^jZ^k - q_0 . \quad (4.204)$$

Before we analyze the supersymmetric solution, which eventually is going to be deformed, we discuss the flow equations.

5.1 Flow equations

As in Eq. (4.156), also here it is possible to expand the potential term into squares of

$$\Upsilon = \tfrac{1}{4}e^U\left(\sqrt{e^{-2U}r_0^2 + (\hat{q}_0)^2} + \sum_{j=1}^{3}\sqrt{e^{-2U}r_0^2 + (\hat{p}^j)^2}\right) , \quad (4.205)$$

$$\Psi_i = \frac{ie^U}{16\,\mathrm{Im}\,Z^{(i)}}\left(\sqrt{e^{-2U}r_0^2 + (\hat{q}_0)^2} - \sum_{j=1}^{3}(-1)^{\delta^j_{(i)}}\sqrt{e^{-2U}r_0^2 + (\hat{p}^j)^2}\right) , \quad (4.206)$$

where the (hatted) dressed charges are defined as

$$\begin{aligned} \hat{p}^i &= -4|p^{(i)}|\mathcal{M}_{(i)} = \sqrt{2}e^{\mathcal{K}/2}d_{(i)jk}|p^{(i)}|\,\mathrm{Im}\,Z^j\,\mathrm{Im}\,Z^k , \\ \hat{q}_0 &= 4|q_0|\mathcal{L}^0 = \sqrt{2}|q_0|e^{\mathcal{K}/2} . \end{aligned} \quad (4.207)$$

The superpotential can be obtained explicitly by integrating Eq. (4.205) with respect to U:

$$Y = \Upsilon - \frac{r_0}{4}\Bigg[\ln\left(e^{-U} r_0^2 + r_0\sqrt{e^{-2U} r_0^2 + \hat{q}_0^2}\right) + \sum_{j=1}^{3} \ln\left(e^{-U} r_0^2 + r_0\sqrt{e^{-2U} r_0^2 + (\hat{p}^i)^2}\right)\Bigg], \tag{4.208}$$

and the first-order flow equations take the form:

$$U' = \Upsilon = \partial_U Y\,, \tag{4.209}$$

$$Z^{i\,\prime} = 2\,\mathcal{G}^{ij^*}\Psi^*_{j^*} = 2\,\mathcal{G}^{ij^*}\partial_{j^*} Y\,. \tag{4.210}$$

5.2 The extremal case

Critical points

We start by computing the derivatives of the black-hole potential:

$$-\,\partial_{Z^1} V_{\rm bh} = \frac{ie^{\mathcal{K}}}{\operatorname{Im} Z^1}\left\{W_1 W^* + W_2^* W_3^*\right\} = 0\,. \tag{4.211}$$

This equation and the other two that can be obtained by permuting S with T and U we get the system

$$\begin{aligned} W_1 W^* + W_2^* W_3^* &= 0\,,\\ W_2 W^* + W_1^* W_3^* &= 0\,,\\ W_3 W^* + W_1^* W_2^* &= 0\,, \end{aligned} \tag{4.212}$$

that admits three kinds of solutions:

1. $W \neq 0$ and $W_i = 0\ \forall i$. This is the $N = 2$ supersymmetric solution because $W_i = 0$ implies $\mathcal{D}_i\mathcal{Z} = 0$. It corresponds to an isolated point in moduli space.

2. $W_1 \neq 0$, $W = W_2 = W_3 = 0$ and the other two permutations of this solution. These three isolated points in moduli space are not $N = 2$ supersymmetric but correspond to $N = 8$ supersymmetric critical points since W and the W_i's are associated to the four skew eigenvalues of the central charge matrix of $N = 8$ supergravity [175]. Formally they can be obtained from the supersymmetric critical point by taking the complex conjugate of one of the complex scalars.

3. $|W| = |W_i|$ $\forall i$ and $\operatorname{Arg} W = \sum_{i=1}^{3} \operatorname{Arg} W_i - \pi$. These are only 4 real equations for the 3 complex scalars and admit a 2-parameter space of solutions which are not supersymmetric in either $N = 2$ or $N = 8$ supergravity [175]. The values of the scalars on the horizon will depend on two real combinations of their asymptotic values.

Supersymmetric solutions

Solving directly the equations $W_i = 0$ $\forall i$ is complicated, but we can find the supersymmetric attractor values if we can construct the supersymmetric solutions by the standard method. This requires solving the stabilization equations or the attractor equations on the horizon, which is not straightforward either, but has already been done in Ref. [133].

If $\mathcal{I}^0 \neq 0$, the scalars and metric function of the supersymmetric extremal solutions are given in terms of the real harmonic functions $\mathcal{I}^M$ by

$$Z^i = \frac{\mathcal{I}^\Lambda \mathcal{I}_\Lambda - 2\mathcal{I}^{(i)}\mathcal{I}_{(i)}}{2\mathcal{J}_i} - i\frac{e^{-2U}}{4\mathcal{J}_{(i)}} \tag{4.213}$$

$$e^{-2U} = 2\sqrt{4\mathcal{I}_0\mathcal{I}^1\mathcal{I}^2\mathcal{I}^3 - 4\mathcal{I}^0\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3 + 4\sum_{i<j}\mathcal{I}^i\mathcal{I}_i\mathcal{I}^j\mathcal{I}_j - (\mathcal{I}^\Lambda\mathcal{I}_\Lambda)^2}\,, \tag{4.214}$$

where

$$\mathcal{J}_i \equiv 3d_{ijk}\mathcal{I}^j\mathcal{I}^k - \mathcal{I}_i\mathcal{I}^0\,. \tag{4.215}$$

If $\mathcal{I}^0 = 0$, the metric function e^{-2U} and the scalars Z^i are the restriction to $\mathcal{I}^0 = 0$ of the above expressions.

The harmonic functions have the general form Eq. (4.49) but, as usual, given the charges $\mathcal{Q}^M$, the asymptotic constants $\mathcal{I}^M_\infty$ are restricted by the condition of absence of NUT charge Eq. (4.50).

The simplest supersymmetric extremal D0-D4 black holes, the ones we are going to consider, have $\mathcal{I}^0 = \mathcal{I}_i = 0$ ($\mathcal{I}^0 = \mathcal{I}_i = 0$ implies $p^0 = q_i = 0$ but not the other way around). The scalars and metric function take the simple forms

$$Z^i = -4ie^{2U}\mathcal{I}_0\mathcal{I}^i\,, \tag{4.216}$$

$$e^{-2U} = 4\sqrt{\mathcal{I}_0\mathcal{I}^1\mathcal{I}^2\mathcal{I}^3}\,, \tag{4.217}$$

and the condition of absence of NUT charge Eq. (4.50) is automatically satisfied for arbitrary values of the constants $\mathcal{I}^M_\infty$.

The regularity of the metric and scalar fields (whose imaginary part must be strictly positive in these conventions) for all $\tau \in (-\infty, 0)$ implies

$$\operatorname{sign} \mathcal{I}_{0\,\infty} = \operatorname{sign} q_0 \,, \qquad \operatorname{sign} \mathcal{I}^i_\infty = \operatorname{sign} p^i \,,\ \forall i \,, \tag{4.218}$$

and the reality of the metric function and negative definiteness of the imaginary parts of the scalars imply

$$\mathcal{I}_0 \mathcal{I}^i > 0 \,, \qquad \mathcal{I}_0 \mathcal{I}^1 \mathcal{I}^2 \mathcal{I}^3 > 0 \,, \tag{4.219}$$

which leave us with just two options

$$\begin{aligned} &\mathcal{I}_0, \mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^3 > 0 \,, \quad q_0, p^1, p^2, p^3 > 0 \,, \quad \mathcal{I}_{0\,\infty}, \mathcal{I}^1_\infty, \mathcal{I}^2_\infty, \mathcal{I}^3_\infty > 0 \,, \\ &\mathcal{I}_0, \mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^3 < 0 \,, \quad q_0, p^1, p^2, p^3 < 0 \,, \quad \mathcal{I}_{0\,\infty}, \mathcal{I}^1_\infty, \mathcal{I}^2_\infty, \mathcal{I}^3_\infty < 0 \,. \end{aligned} \tag{4.220}$$

Therefore, in the supersymmetric solution we have two disconnected possibilities (in the sense that it is not possible to go from one to the other continuously without making the metric functions or the scalars singular).

Imposing asymptotic flatness and absence of NUT charge we find that the four harmonic functions can be written in terms of the physical parameters in the form

$$\begin{aligned} \mathcal{I}_0 &= s_0 \left\{ \frac{1}{4\sqrt{2}\mathcal{L}^0_\infty} - \frac{1}{\sqrt{2}} |q_0| \tau \right\} = \frac{s_0}{4\sqrt{2}\mathcal{L}^0_\infty} \left(1 - \hat{q}_{0\,\infty} \tau\right) \,, \\ \mathcal{I}^i &= s^{(i)} \left\{ -\frac{1}{4\sqrt{2}\mathcal{M}_{(i)\,\infty}} - \frac{1}{\sqrt{2}} |p^{(i)}| \tau \right\} = -\frac{s^{(i)}}{4\sqrt{2}\mathcal{M}_{(i)\,\infty}} \left(1 - \hat{p}^{(i)}_\infty \tau\right) \,, \end{aligned} \tag{4.221}$$

where $s_0, s^{(i)}$ are the signs of the charges $q_0, p^{(i)}$ and $\hat{q}_{0\,\infty}, \hat{p}^i_\infty$ are the asymptotic values of the dressed charges defined in Eq. (4.207). These are positive by definition. On the other hand, as previously discussed, the signs s_0, s^i must be either all positive or all negative in the supersymmetric case.

Plugging these expressions into the metric function we can compute the entropy and the mass of the black hole, finding

$$S/\pi = |\mathcal{Z}(Z_{\rm h}, Z^*_{\rm h}, \mathcal{Q})|^2 = 2\sqrt{q_0 p^1 p^2 p^3} \,, \tag{4.222}$$

$$M = |\mathcal{Z}(Z_\infty, Z^*_\infty, \mathcal{Q})| = \tfrac{1}{4} \left(\hat{q}_{0\,\infty} + \hat{p}^1_\infty + \hat{p}^2_\infty + \hat{p}^3_\infty\right) \,. \tag{4.223}$$

s_0	s^1	s^2	s^3
$\pm$	$\pm$	$\pm$	$\pm$
$\pm$	$\mp$	$\pm$	$\pm$
$\pm$	$\pm$	$\mp$	$\pm$
$\pm$	$\pm$	$\pm$	$\mp$
$\mp$	$\pm$	$\pm$	$\pm$
$\mp$	$\mp$	$\pm$	$\pm$
$\mp$	$\pm$	$\mp$	$\pm$
$\mp$	$\pm$	$\pm$	$\mp$

Tab. 4.3: Possible sign choices for extremal black holes of the D0-D4 model. The first two possibilities (first row of the table) correspond to the $N = 2$ supersymmetric black holes. The six choices in the 2nd, 3rd and 4th rows correspond to the extremal black holes that are not supersymmetric in $N = 2$ supergravity in general but that are supersymmetric when the theory is embedded in the $N = 8$ supergravity. The last 8 choices (4 rows) correspond to extremal black holes which are not supersymmetric in any theory.

Extremal, non-supersymmetric solutions

According to the discussion of the critical points of the black-hole potential, we can obtain 3 non-supersymmetric extremal black holes by formally replacing one of the scalars by its complex conjugate. If we do it for Z^1, for instance, we get

$$\begin{aligned} \operatorname{Im} Z^1 &= -4e^{2U}\mathcal{I}_0\mathcal{I}^1 \quad \longrightarrow \quad +4e^{2U}\mathcal{I}_0\mathcal{I}^1\,, \\ e^{-2U} &= 4\sqrt{\mathcal{I}_0\mathcal{I}^1\mathcal{I}^2\mathcal{I}^3} \quad \longrightarrow \quad 4\sqrt{\mathcal{I}_0\mathcal{I}^1\mathcal{I}^2\mathcal{I}^3}\,, \end{aligned} \tag{4.224}$$

with $\operatorname{Im} Z^1$ strictly negative. This transformation is equivalent to the replacement of $\mathcal{I}^1$ by $-\mathcal{I}^1$ everywhere. To take into account these and also further possibilities, we write the extremal solutions in the form

$$Z^{(i)} = -4s_0 s^{(i)} e^{2U}\mathcal{I}_0\mathcal{I}^{(i)}\,, \qquad e^{-2U} = 4\sqrt{s_0 s^1 s^2 s^3 \mathcal{I}_0\mathcal{I}^1\mathcal{I}^2\mathcal{I}^3}\,, \tag{4.225}$$

where s_0, s^i are the signs of the respective harmonic functions (which coincide with those of the charges and those of the asymptotic constants). The possible choices and their relation to supersymmetry are given in Table 4.3.

The entropy of these solutions is given by

$$S/\pi = 2\sqrt{s_0 s^1 s^2 s^3 q_0 p^1 p^2 p^3} = 2\sqrt{|q_0 p^1 p^2 p^3|}\,, \tag{4.226}$$

and the mass is still given by Eq. (4.223)

$$M = \tfrac{1}{4}\left(\hat{q}_{0\,\infty} + \hat{p}^1_\infty + \hat{p}^2_\infty + \hat{p}^3_\infty\right)\,, \tag{4.227}$$

but it coincides with $|\mathcal{Z}(Z_\infty, Z^*_\infty, \mathcal{Q})|$ only for the first two choices of signs in Table 4.3. For the choices in the rows $i+1 = 2, 3, 4$ of the table, the mass equals $e^{\mathcal{K}/2}|W_i|$ (for them $|\mathcal{Z}(Z_\infty, Z^*_\infty, \mathcal{Q})| = 0$) and for the other eight combinations of signs the mass is numerically equal to $4|\mathcal{Z}(Z_\infty, Z^*_\infty, \mathcal{Q})|$. Thus, for all these extremal black holes $M > |\mathcal{Z}(Z_\infty, Z^*_\infty, \mathcal{Q})|$.

5.3 Non-extremal D0-D4 black hole

According to the general prescription we describe the non-extremal solution with four functions $\hat{\mathcal{I}}_0, \hat{\mathcal{I}}^1, \hat{\mathcal{I}}^2, \hat{\mathcal{I}}^3$ of τ, which we will denote collectively by $\mathcal{I}^\Lambda$ in this section and which we assume to be of the general form

$$\hat{\mathcal{I}}^\Lambda = a^\Lambda + b^\Lambda e^{2r_0\tau}\,, \tag{4.228}$$

The metric factor and scalar fields are assumed to take the form

$$e^{-2U} = e^{-2[U_{\rm e} + r_0\tau]}\,, \tag{4.229}$$

$$Z^i = -4ie^{2U_{\rm e}}\hat{\mathcal{I}}_0\hat{\mathcal{I}}^i\,, \tag{4.230}$$

where

$$e^{-2U_{\rm e}} = 4\sqrt{\hat{\mathcal{I}}_0\hat{\mathcal{I}}^1\hat{\mathcal{I}}^2\hat{\mathcal{I}}^3}\,. \tag{4.231}$$

Observe that the consistency of this ansatz requires that all the functions $\hat{\mathcal{I}}^\Lambda$ must be simultaneously positive or negative. Furthermore, they must be finite in the interval $\tau \in (-\infty, 0)$, which implies that

$$\text{sign}\, a^\Lambda \neq \text{sign}\, b^\Lambda\,, \qquad |a^\Lambda| > |b^\Lambda| \quad \forall\,\Lambda\,. \tag{4.232}$$

Plugging this ansatz into the Eqs. (4.114)–(4.116) we find that they are solved if the constants a^Λ, b^Λ satisfy for each value of Λ

$$a^{(\Lambda)}b^{(\Lambda)} = -\frac{(p^\Lambda)^2}{8r_0^2}\,. \tag{4.233}$$

In order to determine all the constants in terms of the physical parameters we impose asymptotic flatness and use the definitions of mass and the asymptotic values of the scalars, which yield the additional relations (the condition of absence of NUT charge is automatically satisfied)

$$\prod_{\Lambda}(a^{\Lambda} + b^{\Lambda}) = \frac{1}{16}\,, \tag{4.234}$$

$$\sum_{\Lambda} \frac{b^{\Lambda}}{a^{\Lambda} + b^{\Lambda}} = 2\left(1 - \frac{M}{r_0}\right)\,, \tag{4.235}$$

$$\operatorname{Im} Z^{i}_{\infty} = -4(a_0 + b_0)(a^{i} + b^{i})\,. \tag{4.236}$$

The solution to these equations that satisfies the finiteness condition Eq. (4.232) is

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \frac{\varepsilon}{8\sqrt{2}\mathcal{L}^{0}_{\infty}}\left\{1 \pm \frac{1}{r_0}\sqrt{r_0^2 + (\hat{q}_{0\,\infty})^2}\right\}\,, \tag{4.237}$$

$$\begin{pmatrix} a^{i} \\ b^{i} \end{pmatrix} = -\frac{\varepsilon}{8\sqrt{2}\mathcal{M}_{i\,,\infty}}\left\{1 \pm \frac{1}{r_0}\sqrt{r_0^2 + (\hat{p}^{i}_{\infty})^2}\right\}\,, \tag{4.238}$$

where the upper sign corresponds to the constant a and the lower to b and ε is the global sign of the functions $\hat{\mathcal{I}}^{\Lambda}$. We must stress that, unlike in the extremal case, this sign is not related to that of the charges.

Physical properties of the non-extremal solutions

The mass is given by

$$M = \tfrac{1}{4}\sum_{\Lambda}\sqrt{r_0^2 + (\hat{p}^{\Lambda}_{\infty})^2}\,, \tag{4.239}$$

and it is evident that in the extremal limit it takes the value Eq. (4.227), while the entropies are given by

$$\frac{S_{\pm}}{\pi} = \frac{A_{\pm}}{4\pi} = \sqrt{\prod_{\Lambda}\left(r_0^2 \pm \sqrt{r_0^2 + (\hat{p}^{\Lambda}_{\infty})^2}\right)}\,, \tag{4.240}$$

and take the value Eq. (4.226), since $\prod_{\Lambda}|\hat{p}^{\Lambda}_{\infty}| = \prod_{\Lambda}|p^{\Lambda}|$. Observe that

$$\frac{S_{+}}{\pi}\frac{S_{-}}{\pi} = 4|q_0 p^1 p^2 p^3|\,, \tag{4.241}$$

which is the square of the moduli-independent entropy of all the extremal black holes.

It is highly desirable to have an explicit expression of the non-extremality parameter r_0 in terms of the physical parameters M, p^Λ, Z^i_∞, which, in turn, would allow us to express mass and entropy as functions of p^Λ, Z^i_∞ alone. Furthermore, such an expression would allow us to study the different extremal limits or relations between M and p^Λ and Z^i_∞ that make r_0 vanish. In the general case, solving Eq. (4.239) explicitly is impossible, though. We can, nevertheless, consider some particular examples, obtained by fixing the relative values of the dressed charges $\hat{p}^i$ and $\hat{q}_0$:

1. If $\hat{p}^1_\infty = \hat{p}^2_\infty = \hat{p}^3_\infty = \hat{q}_{0\,\infty}$, then Eq. (4.239) simplifies to:

   $$M = \sqrt{r_0^2 + (\hat{q}_{0\,\infty})^2}\,, \tag{4.242}$$

   so

   $$r_0^2 = (M - \hat{q}_{0\,\infty})\,(M + \hat{q}_{0\,\infty})\ , \tag{4.243}$$

   from which we conclude that we can reach the extremal limit $M = \hat{q}_{0\,\infty}$ in two different ways:[16] $M = s_0\hat{q}_{0\,\infty}$ and $M = -s_0\hat{q}_{0\,\infty}$. Which one is reached depends on $s_0 = \operatorname{sign} q_0$. Whether this limit is supersymmetric or not will depend on the signs of the charges, as discussed in Table 4.3.

   We can use Eq. (4.243) to express the entropy in terms of the mass, the charges, and the asymptotic values of the scalars at infinity in the familiar form:

   $$\frac{S_\pm}{\pi} = \left(\sqrt{N_{\rm R}^{(1)}} \pm \sqrt{N_{\rm L}^{(1)}}\right)^2\ , \tag{4.244}$$

   where

   $$N_{\rm R}^{(1)} = M^2\,, \qquad N_{\rm L}^{(2)} = M^2 - \hat{q}^2_{0\,\infty}\,. \tag{4.245}$$

2. If $\hat{p}^1_\infty = \hat{p}^2_\infty$ and $\hat{p}^3_\infty = \hat{q}_{0\,\infty}$, then the mass of the black hole is given by

   $$M = \tfrac{1}{2}\left[\sqrt{r_0^2 + (\hat{p}^1_\infty)^2} + \sqrt{r_0^2 + (\hat{q}_{0\,\infty})^2}\right]\ , \tag{4.246}$$

   and Eq. (4.246) can be inverted to obtain

   $$M^2 r_0^2 = \left(M^2 - \frac{(\hat{p}^1_\infty + \hat{q}_{0\,\infty})^2}{4}\right)\left(M^2 - \frac{(\hat{p}^1_\infty - \hat{q}_{0\,\infty})^2}{4}\right)\ , \tag{4.247}$$

[16] We remind the reader that we have defined the dressed charges to always be positive.

from which we find four possible extremal limits:

$$M = \begin{cases} \frac{1}{2}(s^1\hat{p}^1_\infty + s_0\hat{q}_{0\,\infty})\,, \\ \frac{1}{2}(s^1\hat{p}^1_\infty - s_0\hat{q}_{0\,\infty})\,, \\ -\frac{1}{2}(s^1\hat{p}^1_\infty + s_0\hat{q}_{0\,\infty})\,, \\ -\frac{1}{2}(s^1\hat{p}^1_\infty - s_0\hat{q}_{0\,\infty})\,. \end{cases} \tag{4.248}$$

Which extremal limit will be attained if the mass diminishes in the process of evaporation depends on the signs of the charges s^1, s_0 but it will always be the largest possible value so that

$$M = \tfrac{1}{2}(\hat{p}^1_\infty + \hat{q}_{0\,\infty})\,. \tag{4.249}$$

In terms of the mass, the charges, and the asymptotic values of the scalars at infinity, the entropies are again given by

$$\frac{S_\pm}{\pi} = \left(\sqrt{N_{\rm R}^{(2)}} \pm \sqrt{N_{\rm L}^{(2)}}\right)^2\,, \tag{4.250}$$

where

$$N_{\rm R}^{(2)} = M^2 - \frac{\left(\hat{p}^1_\infty + \hat{q}_{0\,\infty}\right)^2}{4}\,, \qquad N_{\rm L}^{(2)} = M^2 - \frac{\left(\hat{p}^1_\infty - \hat{q}_{0\,\infty}\right)^2}{4}\,, \tag{4.251}$$

and the product of the two entropies gives the moduli-independent entropy of the extremal black hole with the same charges, squared.

3. If $\hat{p}^1_\infty = \hat{p}^2_\infty = \hat{p}^3_\infty$, then the mass is given by

$$M = \tfrac{1}{4}\left[3\sqrt{r_0^2 + (\hat{p}^1_\infty)^2} + \sqrt{r_0^2 + (\hat{q}_{0\,\infty})^2}\right]\,, \tag{4.252}$$

This equation can be written in a polynomial form by squaring it several times, and then it can be solved for r_0^2

$$\begin{aligned} r_0^2 = \tfrac{1}{8}\,\Big[&(\hat{q}_{0\,\infty})^2 - 9(\hat{p}^1_\infty)^2 + 20M^2 \\ &-6\sqrt{2}\sqrt{(\hat{q}_{0\,\infty})^2M^2 - (\hat{p}^1_\infty)^2M^2 + 2M^4}\Big]\,. \end{aligned} \tag{4.253}$$

From this equation we can obtain the extremal values of M:

$$M = \begin{cases} \frac{1}{4}(3s^1\hat{p}^1_\infty + s_0\hat{q}_{0\,\infty})\,, \\ \frac{1}{4}(3s^1\hat{p}^1_\infty - s_0\hat{q}_{0\,\infty})\,, \\ -\frac{1}{4}(3s^1\hat{p}^1_\infty + s_0\hat{q}_{0\,\infty})\,, \\ -\frac{1}{4}(3s^1\hat{p}^1_\infty - s_0\hat{q}_{0\,\infty})\,. \end{cases} \tag{4.254}$$

The extremal limit that will be reached first in the evaporation process will be that with the largest value of the mass

$$M = \tfrac{1}{4}(3\hat{p}^1_\infty + \hat{q}_{0\,\infty})\,, \tag{4.255}$$

and the supersymmetry will depend on the signs of the charges.

As in the previous examples, we can write the entropy in terms of $N^{(3)}_{\mathrm{R}}$ and $N^{(3)}_{\mathrm{L}}$, although in this case the expression for them is not very manageable. However, we can compute

$$\frac{\sqrt{S_+S_-}}{\pi} = N^{(3)}_{\mathrm{R}} - N^{(3)}_{\mathrm{L}} = \left(\hat{p}^1_\infty\right)^{3/2}\sqrt{\hat{q}_{0\,\infty}} = 2\sqrt{|q_0p^1p^2p^3|}\,. \tag{4.256}$$

Eq. (4.256) depends only on the charges and it is indeed the supersymmetric entropy, as already demonstrated in the general case (4.240) and (4.241).

In the general case, even though finding a closed-form explicit expression for $r_0\left(Z^i_\infty, \mathcal{Q}, M\right)$ is unfeasible, it is still possible to obtain the values $M_{\mathrm{e}} = M(Z^i_\infty, \mathcal{Q})$ at which extramality is reached by setting $r_0 = 0$ in Eq. (4.239). There are $2^4 = 16$ possible extremal limits given by

$$M = \begin{cases} \pm\frac{1}{4}\left(s_0\hat{q}_0 + s^1\hat{p}^1 + s^2\hat{p}^2 + s^3\hat{p}^3\right)\,, \\ \pm\frac{1}{4}\left(s_0\hat{q}_0 - s^1\hat{p}^1 + s^2\hat{p}^2 + s^3\hat{p}^3\right)\,, \\ \pm\frac{1}{4}\left(s_0\hat{q}_0 + s^1\hat{p}^1 - s^2\hat{p}^2 + s^3\hat{p}^3\right)\,, \\ \pm\frac{1}{4}\left(s_0\hat{q}_0 + s^1\hat{p}^1 + s^2\hat{p}^2 - s^3\hat{p}^3\right)\,, \\ \pm\frac{1}{4}\left(-s_0\hat{q}_0 + s^1\hat{p}^1 + s^2\hat{p}^2 + s^3\hat{p}^3\right)\,, \\ \pm\frac{1}{4}\left(-s_0\hat{q}_0 - s^1\hat{p}^1 + s^2\hat{p}^2 + s^3\hat{p}^3\right)\,, \\ \pm\frac{1}{4}\left(-s_0\hat{q}_0 + s^1\hat{p}^1 - s^2\hat{p}^2 + s^3\hat{p}^3\right)\,, \\ \pm\frac{1}{4}\left(-s_0\hat{q}_0 + s^1\hat{p}^1 + s^2\hat{p}^2 - s^3\hat{p}^3\right)\,, \end{cases} \tag{4.257}$$

for the 2^4 possible choices of s_0, s^1, s^2, s^3 of the charges in Table 4.3. The first limit is $N = 2$ supersymmetric etc. In all cases, the extremal mass will be given by the same expression Eq. (4.227).

It is important to observe that the non-extremal solution has no constraints on the signs (or the absolute values) of the charges, hence it interpolates between the 16 discrete extremal limits.

6 Conclusions

In this paper we have constructed static non-extremal black-hole solutions of three $N = 2, d = 4$ supergravity models using a general prescription based on several well-known examples of non-extremal black holes. While we have given some arguments to justify why this prescription may always work for all models, we are far from having a general proof and more examples need to be considered.

On the other hand, the non-extremal solutions we have found are interesting per se. They seem to share some important properties:

1. Even though in all the models considered there are several disconnected branches of extremal solutions, there is only one non-extremal solution that interpolates between all of them. All the extremal solutions are reachable by taking the appropriate extremal ($r_0 \to 0$) limit. Furthermore, if we let M diminish while leaving the charges and asymptotic values of the scalars constant (as happens in the evaporation process in these theories), which extremal limit is attained depends on the charges alone.

2. There seems to be a unique non-extremal superpotential in each theory and, in the different extremal limits, it gives the different superpotentials associated to the different branches of extremal solutions.

3. The non-extremality parameter r_0, expressed in terms of the mass, charges and asymptotic values of the scalars, holds a great deal of information about the theory because r_0 vanishes whenever the value of the mass equals the value of any of the possible extremal superpotentials (some of which are the skew eigenvalues of the central charge matrix). Therefore, knowing this function $r_0(Z^i_\infty, \mathcal{Q}, M)$ we would know all the possible superpotentials. Unfortunately, there seems to

be no a priori formula to determine it[17] and sometimes (e.g. for the *STU* model) it is not possible to find it explicitly even when the full solution is known.

4. The metrics have generically two horizons at the values $\tau = -\infty$ (the outer, event horizon) and $\tau = +\infty$ (the inner, Cauchy horizon) whose areas and associated entropies are easily calculable and turn out to depend on the values of the scalars at infinity. The product of these two entropies is, in the three cases considered here, the square of the moduli-independent entropy of the extremal black hole that has the same charges.

5. The non-extremal solutions can be used to find some non-supersymmetric extremal solutions that cannot be constructed by the standard methods, as we have shown in the $\overline{\mathbb{CP}}^n$ model case.

If this prescription works also in more complicated cases, it will give us the opportunity to study how non-extremal black holes are affected by quantum corrections and perhaps will give us new insights into the micrscopic interpretation of the black-hole entropy in non-extremal cases. Work in this direction is in progress.

[17] Eq. (4.21) requires the knowledge of the scalar charges $\Sigma^i(Z^i_\infty, \mathcal{Q}, M)$, which we know how to compute only *after* we have the complete black-hole solution.

5. BLACK-HOLE SOLUTIONS OF $N = 2, D = 4$ SUPERGRAVITY WITH A QUANTUM CORRECTION, IN THE H-FGK FORMALISM

Pietro Galli, Tomás Ortín[◇]*, Jan Perz*[*]*,*
and Carlos S. Shahbazi[◇]

[◇]*Instituto de Física Teórica UAM/CSIC*
C/ Nicolás Cabrera, 13–15, C.U. Cantoblanco, 28049 Madrid, Spain.

[*]*Istituto Nazionale di Fisica Nucleare, Sezione di Padova*
Via Marzolo, 8, 35131 Padova, Italy

Publication reference: *JHEP* **04** (2013) 157, arXiv:1212.0303 [hep-th].

Abstract

We apply the H-FGK formalism to the study of some properties of the general class of black holes in $N = 2$ supergravity in four dimensions that correspond to the harmonic and hyperbolic ansätze. We obtain explicit extremal and non-extremal solutions for the t^3 model with and without a quantum correction. Not all solutions of the corrected model (quantum black holes), including in particular a solution with a single q_1 charge, have a regular classical limit.

1 Introduction

In [35,176] a new formalism for constructing single-center, static, spherically-symmetric black-hole solutions of $N = 2, d = 4$ supergravity coupled to vector multiplets was proposed.[1] It is based on rewriting the effective FGK action [18] in terms of a set of functions ("H-variables") of the original dynamical fields, chosen in such a way that they are real and transform linearly under duality. The appropriate choice, which significantly simplifies the equations of motion, can be made with the same algorithm for all supergravity prepotentials and for both extremal and non-extremal black holes. Substituting an ansatz for the H-variables (in [38] taken to be harmonic and hyperbolic for, respectively, extremal and non-extremal solutions[2]) transforms the equations of motion into a system of ordinary equations for the parameters of the ansatz.

The new formalism, as has been shown in the $N = 2, d = 5$ case [36,98,144], should considerably facilitate the construction of new black-hole solutions and their systematic study. This task so far demanded the use of a specific ansatz for each type of solution, which had to be plugged into the equations of motion and checked case by case, although ultimately very general ansätze were proposed. The supersymmetric solutions of ungauged $N = 2, d = 4$ supergravity coupled to vector supermultiplets, which are the only theories we are going to study and discuss here, were constructed in this way in a long series of papers [15, 16, 60, 94, 148, 149]. The effect of the inclusion of R^2 corrections was studied in ref. [95]. The outcome of all this work was a recipe that allows the systematic construction of supersymmetric black-hole solutions from harmonic functions. The same class of solutions was eventually shown to contain regular stationary multicenter black holes [28, 29] and the generality of the construction has been proven by the use of supersymmetry methods in [151]. For extremal non-supersymmetric black holes [119] no completely general procedure is known, although several families of solutions have been found, such as the almost-BPS ones [78, 84], those of the cubic models [178], which originate from more particular examples [25, 39, 64, 85], and the interacting non-BPS solutions of ref. [179]. In the non-extremal case the situation is even worse, as only a few examples of non-extremal black-hole solutions have been obtained [38]. The H-FGK formalism can improve it.

[1] An analogous formalism exists for $N = 2, d = 5$ supergravity theories [35, 98, 144, 177] and can be extended to black-string solutions as well [36, 37].

[2] The same ansatz has been exploited also in five dimensions [35, 88, 98, 144].

Our goals in this paper are similar to those of ref. [36] in the 5-dimensional context: firstly to derive useful model-independent relationships between the quantities appearing in the H-FGK formalism and the physical characteristics of the solutions, in sec. 2, and secondly to use them in sec. 3 for finding explicit examples of black holes in the t^3 model with a quadratic correction to the prepotential, whose string-theoretical origin we recall in appendix A. We restrict ourselves to solutions described by harmonic and hyperbolic functions (for the discussion of generality of these ansätze see refs. [41, 42]). Sec. 4 contains our conclusions.

2 The H-FGK formalism for $N = 2$, $D = 4$ supergravity

In this section we briefly review the H-FGK formalism for theories of $N = 2$, $d = 4$ supergravity coupled to n vector multiplets, following [35], whose conventions we use.

As shown in [35, 176], searching for single-center, static, spherically symmetric black-hole solutions of an $N = 2$, $d = 4$ supergravity coupled to n vector multiplets (and, correspondingly, including n complex scalars Z^i and $n+1$ Abelian vector fields $A^{\Lambda}{}_{\mu}$) with electric (q_Λ) and magnetic (p^Λ) charges described by the $2(n+1)$-dimensional symplectic vector $(\mathcal{Q}^M) \equiv (p^\Lambda, q_\Lambda)^{\mathrm{T}}$ is equivalent to solving the following equations of motion for $2(n+1)$ dynamical variables that we denote by $H^M(\tau)$ and identify below with a certain combination of physical fields:

$$\left(\partial_M\partial_N \log \mathsf{W} - 2\frac{H_M H_N}{\mathsf{W}^2}\right)\ddot{H}^N + \tfrac{1}{2}\partial_M\partial_N\partial_P \log \mathsf{W}\left(\dot{H}^N\dot{H}^P - \tfrac{1}{2}\mathcal{Q}^N\mathcal{Q}^P\right)$$
$$-4\dot{H}_M\frac{\dot{H}^N H_N}{\mathsf{W}^2} + 8H_M\frac{\dot{H}^P\tilde{H}_P\,\dot{H}^N H_N}{\mathsf{W}^3} + 2\mathcal{Q}_M\frac{H^N\mathcal{Q}_N}{\mathsf{W}^2}$$
$$-4\tilde{H}_M\frac{(H^N\dot{H}_N)^2}{\mathsf{W}^3} - 4\tilde{H}_M\frac{(H^N\mathcal{Q}_N)^2}{\mathsf{W}^3} = 0\,, \tag{5.1}$$

$$-\tfrac{1}{2}\partial_M\partial_N \log \mathsf{W}\left(\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) + \left(\frac{\dot{H}^M H_M}{\mathsf{W}}\right)^2 - \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}}\right)^2 = r_0^2\,. \tag{5.2}$$

In these equations r_0 is the non-extremality parameter, we use the symplectic form $(\Omega_{MN}) \equiv \left(\begin{smallmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0\end{smallmatrix}\right)$ and $\Omega^{MN} = \Omega_{MN}$ to lower and raise the symplectic indices according to the convention

$$H_M = \Omega_{MN} H^N \,, \qquad H^M = H_N \Omega^{NM} \,, \tag{5.3}$$

and $\mathsf{W}(H)$ is the *Hesse potential*.[3] For a theory defined by the covariantly holomorphic symplectic section $\mathcal{V}^M$, the Hesse potential can be found as follows: introducing a complex variable X with the same Kähler weight as $\mathcal{V}^M$, we can define the Kähler-neutral real symplectic vectors

$$\mathcal{R}^M = \mathrm{Re}\,\mathcal{V}^M/X \,, \qquad \mathcal{I}^M = \mathrm{Im}\,\mathcal{V}^M/X \,. \tag{5.4}$$

The components $\mathcal{R}^M$ can be expressed in terms of the $\mathcal{I}^M$, to which process we refer later as solving Freudenthal duality equations.[4] Then, the Hesse potential, as a function of the components $\mathcal{I}^M$ is given by

$$\mathsf{W}(\mathcal{I}) \equiv \langle\, \mathcal{R}(\mathcal{I}) \mid \mathcal{I}\,\rangle \equiv \mathcal{R}_M(\mathcal{I})\mathcal{I}^M \,, \tag{5.5}$$

and identifying $\mathcal{I}^M = H^M$ we get $\mathsf{W}(H)$. We can use $\mathcal{R}^M$ to define dual variables:

$$\tilde{H}^M(H) \equiv \mathcal{R}^M(H) \,. \tag{5.6}$$

Given a solution $H^M(\tau)$ of the equations (5.1) and (5.2), the warp factor e^{2U} of the spacetime metric

$$ds^2 = e^{2U} dt^2 - e^{-2U} \left(\frac{r_0^4}{\sinh^4 r_0\tau} d\tau^2 + \frac{r_0^2}{\sinh^2 r_0\tau} d\Omega^2_{(2)} \right) , \tag{5.7}$$

takes the form

$$e^{-2U} = \mathsf{W}(H) \tag{5.8}$$

and the scalar fields are given by

$$Z^i = \frac{\tilde{H}^i + iH^i}{\tilde{H}^0 + iH^0} \,. \tag{5.9}$$

The equations of motion (5.1) can be derived from the effective action

$$- I_{\text{eff}}[H] = \int d\tau \left[\tfrac{1}{2}\partial_M \partial_N \log \mathsf{W} \left(\dot{H}^M \dot{H}^N + \tfrac{1}{2}\mathcal{Q}^M \mathcal{Q}^N \right) - \left(\frac{\dot{H}^M H_M}{\mathsf{W}} \right)^2 - \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}} \right)^2 \right] . \tag{5.10}$$

[3] For a historical perspective on the real formulation of special Kähler geometry and the Hesse potential see e.g. [176, 180].

[4] In earlier papers sometimes called “stabilization equations”.

Then, eq. (5.2) is nothing but the Hamiltonian constraint associated with the τ-independence of the action, with a particular value of the integration constant, which we cannot change because it is part of the transverse metric ansatz.

If we contract the equations of motion (5.1) with H^P and use the homogeneity properties of the different terms and the Hamiltonian constraint eq. (5.2), we find a useful equation

$$\tilde{H}_M \left(\ddot{H}^M - r_0^2 H^M \right) + \frac{(\dot{H}^M H_M)^2}{\mathsf{W}} = 0\,, \tag{5.11}$$

which corresponds to that of the variable U minus the Hamiltonian constraint in the standard formulation.[5]

In what follows we shall impose on the variables H^M the constraint

$$\dot{H}^M H_M = 0\,. \tag{5.12}$$

In the supersymmetric (hence, extremal) case it has been shown [79] that this constraint enforces the absence of NUT charge: a non-zero NUT charge would lead to a non-static metric with string-like singularities. Here, this condition is nothing but a possible simplifying assumption which does not imply non-staticity since staticity has been assumed in this formalism form the onset. Here we take it as a convenient ansatz and leave the possibility and implications of violating this constraint to be studied elsewhere [41,42].

The above constraint simplifies eq. (5.11)

$$\tilde{H}_M \left(\ddot{H}^M - r_0^2 H^M \right) = 0\,, \tag{5.13}$$

which can be solved by harmonic (in the extremal $r_0 = 0$ case) or hyperbolic (in the non-extremal $r_0 \neq 0$ case) ansätze for the variables H^M, satisfying

$$\ddot{H}^M - r_0^2 H^M = 0\,. \tag{5.14}$$

These are the ansätze that we will use in the rest of the paper, bearing in mind that they are adapted to the additional constraint (5.12) that we impose by hand. Taking into account this constraint, the equations that

[5] This equation in the extremal limit agrees with the special static case of eq. (3.31) of ref. [39].

need to be solved are:

$$\partial_P\partial_M \log \mathsf{W}\, \ddot{H}^M + \tfrac{1}{2}\partial_P\partial_M\partial_N \log \mathsf{W} \left(\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) + \partial_P\left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}}\right)^2 = 0\,, \tag{5.15}$$

$$-\tfrac{1}{2}\partial_M\partial_N \log \mathsf{W} \left(\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) - \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}}\right)^2 = r_0^2\,, \tag{5.16}$$

$$\dot{H}^M H_M = 0\,. \tag{5.17}$$

It is also useful to have the expression of the black-hole potential as a zeroth-degree homogeneous function of the variables H^M:

$$-V_{\rm bh}(H,\mathcal{Q}) = -\tfrac{1}{4}\mathsf{W}\left(\partial_M\partial_N \log \mathsf{W} - 4\mathsf{W}^{-2}H_M H_N\right)\mathcal{Q}^M\mathcal{Q}^N\,. \tag{5.18}$$

2.1 Extremal black holes

As explained above, for extremal black holes we take $H^M(\tau)$ to be harmonic in Euclidean $\mathbb{R}^3$, i.e. linear in τ:[6]

$$H^M = A^M - \tfrac{1}{\sqrt{2}}B^M\tau\,, \tag{5.19}$$

where A^M and B^M are integration constants to be determined as functions of the independent physical constants (namely, the charges $\mathcal{Q}^M$ and the values of the scalars at spatial infinity Z^i_∞) by using the equations of motion (5.15)–(5.17) and the asymptotic conditions.

The equations of motion for the above ansatz can be written in a simple and suggestive form[7]

$$\partial_P\left[V_{\rm bh}(H,\mathcal{Q}) - V_{\rm bh}(H,B)\right] = 0\,, \tag{5.20}$$

$$V_{\rm bh}(H,\mathcal{Q}) - V_{\rm bh}(H,B) = 0\,, \tag{5.21}$$

$$A^M B_M = 0\,. \tag{5.22}$$

[6] Known non-supersymmetric extremal solutions that do not conform to this ansatz do not satisfy constraint (5.12) either [39, 41]. On the other hand, the representation of a solution in terms of the H^M may not be unique and the harmonicity or the fact that the constraint eq. (5.17) is satisfied may not always be a characteristic feature of a solution [42].

[7] It is worth stressing that, even though the first equation is the derivative of the second with respect to H^P, solving the second for some functions H^M does not imply having solved the first. Only if we find a B^M such that the second equation is satisfied identically for any H^M will the first equation be satisfied as well. The number of B^M with this property and their value depend on the particular theory under consideration, but their existence is quite a general phenomenon.

Observe that the first two equations are automatically solved for $B^M = \mathcal{Q}^M$, which corresponds to the supersymmetric case. The third equation then takes the form $A^M \mathcal{Q}_M$ and still has to be solved, which can be done generically [28, 29] as we are going to show.

Furthermore, observe that the Hamiltonian constraint (5.21) is equivalent to the requirement that the black-hole potential *evaluated on the solutions* has the same form in terms of the *fake central charge* which we can define for any symplectic (*fake* or not *fake*) charge vector B^M by

$$\tilde{\mathcal{Z}}(Z, Z^*, B) \equiv \langle \mathcal{V} \mid B \rangle \tag{5.23}$$

as in terms of the actual central charge $\mathcal{Z}(Z, Z^*, Q) \equiv \langle \mathcal{V} \mid Q \rangle = \tilde{\mathcal{Z}}(Z, Z^*, \mathcal{Q})$, that is

$$- V_{\rm bh}(Z, Z^*, \mathcal{Q}) = |\tilde{\mathcal{Z}}|^2 + \mathcal{G}^{ij^*} \mathcal{D}_i \tilde{\mathcal{Z}} \, \mathcal{D}_{j^*} \tilde{\mathcal{Z}}^* \,. \tag{5.24}$$

The asymptotic conditions take the form

$$\mathsf{W}(A) = 1 \,, \tag{5.25}$$

$$Z^i_\infty = \frac{\tilde{H}^i(A) + iA^i}{\tilde{H}^0(A) + iA^0} \,, \tag{5.26}$$

but can always be solved, together with (5.22), as follows: if we write X as

$$X = \tfrac{1}{\sqrt{2}} e^{U + i\alpha} \,, \tag{5.27}$$

then, from the definition (5.4) of $\mathcal{I}^M$ we get

$$H^M = \sqrt{2} e^{-U} \ \mathrm{Im}(e^{-i\alpha} \mathcal{V}^M) \,, \tag{5.28}$$

and, at spatial infinity $\tau = 0$, using asymptotic flatness (5.25)

$$A^M = \sqrt{2} \ \mathrm{Im}(e^{-i\alpha_\infty} \mathcal{V}^M_\infty) \,. \tag{5.29}$$

Now, to determine α_∞ we can use (5.22) and the definition of fake central charge (5.23). Observe that

$$A_M B^M = \langle H \mid B \rangle = \mathrm{Im} \langle \mathcal{V}/X \mid B \rangle = \mathrm{Im}(\tilde{\mathcal{Z}}/X) = \sqrt{2} e^{-U} \, \mathrm{Im}(e^{-i\alpha} \tilde{\mathcal{Z}}) = 0 \,, \tag{5.30}$$

from which one first obtains the relation

$$e^{i\alpha} = \pm \tilde{\mathcal{Z}} / |\tilde{\mathcal{Z}}| \tag{5.31}$$

and then the general expression for the A^M as a function of the B^M and the Z^i_∞:

$$A^M = \pm\sqrt{2}\ \mathrm{Im}\left(\frac{\tilde{\mathcal{Z}}^*_\infty}{|\tilde{\mathcal{Z}}_\infty|}\mathcal{V}^M_\infty\right). \tag{5.32}$$

The sign of A^M should be chosen to make H^M finite (and, generically, the metric non-singular) in the range $\tau \in (-\infty, 0)$. The positivity of the mass is a physical condition that eliminates some singularities of the metric. As we shall see in eq. (5.40), this requirement singles out the upper sign in the above formula.

Having reduced the problem of finding a complete solution to the determination of the constants B^M that must satisfy equations (5.20) and (5.21) as functions of the physical parameters $\mathcal{Q}^M, Z^i_\infty$, it is useful to analyze the near-horizon and spatial-infinity limits of these two equations. The near-horizon limit of (5.21) plus the definition of the fake central charge lead to the following chain of relations[8]

$$S/\pi = \tfrac{1}{2}\mathsf{W}(B) = -V_{\mathrm{bh}}(B, \mathcal{Q}) = |\tilde{\mathcal{Z}}(B, B)|^2\,, \tag{5.33}$$

where S is the Bekenstein–Hawking black-hole entropy and $\tilde{\mathcal{Z}}(B, B)$ is the near-horizon value of the fake central charge. The last of these relations, together with the condition (5.24) imply that, on the horizon, the fake central charge reaches an extremum

$$\partial_i|\tilde{\mathcal{Z}}(Z_{\mathrm{h}}, Z^*_{\mathrm{h}}, B)| = 0\,. \tag{5.34}$$

The near-horizon limit of (5.20) leads to

$$\partial_M V_{\mathrm{bh}}(B, \mathcal{Q}) = 0\,, \tag{5.35}$$

which says that the B^M extremize the value of the black-hole potential on the horizon. Since the black-hole potential is invariant under a global rescaling of the H^M, the solutions (that we generically call *attractors* B^M) of these equations are determined up to a global rescaling, which can be fixed by imposing eq. (5.21).

The B^M must transform under the duality group of the theory (embedded in $Sp(2n + 2, \mathbb{R})$) in the same representation as the H^M, the charges $\mathcal{Q}^M$ and the constants A^M. In certain cases this poses strong constraints on

[8] In this and other equations, the expression $V_{\mathrm{bh}}(B, \mathcal{Q})$ stands for the standard black-hole potential with the functions $H^M(\tau)$ replaced by the constants B^M.

the possible solutions, since building from $\mathcal{Q}^M$ and Z^i_∞ an object that transforms in the right representation of the duality group and has dimensions of length squared may be far from trivial. A possibility that is always available is the Freudenthal dual defined in ref. [181], generalizing the definition made in ref. [182]. Freudenthal duality in $N = 2, d = 4$ theories can be understood as the transformation from the H^M to the $\tilde{H}_M(H)$ variables. The same transformation can be applied to any symplectic vector, such as the charge vector. Then, in our notation and conventions, the Freudenthal dual of the charge vector, $\tilde{\mathcal{Q}}_M$, is defined by

$$\tilde{\mathcal{Q}}_M = \frac{1}{2}\frac{\partial \mathsf{W}(\mathcal{Q})}{\partial \mathcal{Q}^M} . \tag{5.36}$$

It is not difficult to prove that this duality transformation is an anti-involution

$$\tilde{\tilde{\mathcal{Q}}}_M = -\mathcal{Q}_M , \tag{5.37}$$

and using eq. (5.5) to show that

$$\mathsf{W}(\tilde{\mathcal{Q}}) = \mathsf{W}(\mathcal{Q}) . \tag{5.38}$$

With more effort one can also show that the critical points of the black-hole potential are invariant under Freudenthal duality [181]. Therefore, as $B^M = \mathcal{Q}^M$ is always an attractor (the supersymmetric one),

$$B^M = \tilde{\mathcal{Q}}^M \tag{5.39}$$

will always be another attractor.

Let us now consider the spatial-infinity limit, taking into account the definition of the mass in these spacetimes and the definition of the fake central charge

$$M = \dot{U}(0) = \tfrac{1}{\sqrt{2}}\langle\, \tilde{A} \mid B \,\rangle = \pm|\tilde{\mathcal{Z}}(A, B)| . \tag{5.40}$$

As mentioned before, to have a positive mass we must use exclusively the upper sign in (5.31) and (5.32) and we do so from now onwards. In the supersymmetric case, when $B^M = \mathcal{Q}^M$ and the fake central charge becomes the true one, this is the supersymmetric BPS relation.

The asymptotic limit of (5.21) plus (5.24) and the above relation give

$$M^2 + \left[\mathcal{G}^{ij^*}\mathcal{D}_i\tilde{\mathcal{Z}}\,\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^*\right]_\infty + V_{\text{bh}\,\infty} = 0 , \tag{5.41}$$

which, when compared with the general BPS bound [18], leads to the identification of the scalar charges Σ_i with the values of the covariant derivatives of the fake central charges at spatial infinity

$$\Sigma_i = \mathcal{D}_i \tilde{\mathcal{Z}}\Big|_{\infty} . \tag{5.42}$$

First-order flow equations

First-order flow equations for extremal BPS and non-BPS black holes can be easily found following [183] but using the generic harmonic functions (5.19): let us consider the Kähler-covariant derivative of the inverse of the auxiliary function

$$\begin{aligned}
\mathcal{D}X^{-1} &= i\langle\, \mathcal{V} \mid \mathcal{V}^*\rangle \mathcal{D}X^{-1} = i\langle\, \mathcal{D}(\mathcal{V}/X) \mid \mathcal{V}^*\rangle = i\langle\, d(\mathcal{V}/X) \mid \mathcal{V}^*\rangle \\
&= i\langle\, d(\mathcal{V}/X) - d(\mathcal{V}/X)^* \mid \mathcal{V}^*\rangle = -2\langle\, dH \mid \mathcal{V}^*\rangle \\
&= -\sqrt{2}\, \tilde{\mathcal{Z}}^*(Z, Z^*, B)\, d\tau\, ,
\end{aligned} \tag{5.43}$$

where we have used the normalization of the symplectic section in the first step, the property $\langle\, \mathcal{D}\mathcal{V} \mid \mathcal{V}^*\rangle = 0$ in the second, the Kähler-neutrality of $\mathcal{V}/X$ in the third, $\langle\, \mathcal{D}\mathcal{V}^* \mid \mathcal{V}^*\rangle = \langle\, \mathcal{V}^* \mid \mathcal{V}^*\rangle = 0$ in the fourth, the definition of $\mathcal{I} = H$ in the fifth, and the ansatz (5.19) and the definition of the fake central charge (5.23) in the sixth.

From this equation, eqs. (5.27) and (5.31) and the relation (cf. eqs. (3.8), (3.28) in ref. [39])

$$\dot{\alpha} = -\mathcal{Q}_\star\, , \qquad \text{where} \qquad \mathcal{Q}_\star = \tfrac{1}{2i}\dot{Z}^i \partial_i \mathcal{K} + \text{c.c.} \tag{5.44}$$

is the pullback of the Kähler connection 1-form, we find the standard first-order equation for the metric function U:

$$\frac{de^{-U}}{d\tau} = -|\tilde{\mathcal{Z}}(Z, Z^*, B)|\, . \tag{5.45}$$

Let us now consider the differential of the complex scalar fields:

$$\begin{aligned}
dZ^i &= i\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^* \mid \mathcal{D}_k \mathcal{V}\,\rangle dZ^k = iX\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^* \mid \mathcal{D}_k(\mathcal{V}/X)\,\rangle dZ^k \\
&= iX\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^* \mid \partial_k(\mathcal{V}/X)\,\rangle dZ^k = iX\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^* \mid d(\mathcal{V}/X)\,\rangle \\
&= iX\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^* \mid d(\mathcal{V}/X) - d(\mathcal{V}/X)^*\,\rangle = -2X\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^* \mid dH\,\rangle \\
&= +\sqrt{2}\, X\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^* \mid B\,\rangle\, d\tau = \sqrt{2}\, X\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^*(Z, Z^*, B)\, d\tau\, ,
\end{aligned} \tag{5.46}$$

where we have used the same properties as before. To put this expression in a more conventional form we can use the covariant holomorphicity of $\tilde{\mathcal{Z}}$ writing

$$\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^* = \mathcal{D}_{j^*}\frac{|\tilde{\mathcal{Z}}|^2}{\tilde{\mathcal{Z}}} = \frac{2|\tilde{\mathcal{Z}}|\partial_{j^*}|\tilde{\mathcal{Z}}|}{\tilde{\mathcal{Z}}} = 2e^{-i\alpha}\partial_{j^*}|\tilde{\mathcal{Z}}|\,, \tag{5.47}$$

and plugging this result in the expression above:

$$\frac{dZ^i}{d\tau} = 2e^U\mathcal{G}^{ij^*}\partial_{j^*}|\tilde{\mathcal{Z}}|\,. \tag{5.48}$$

It is easy to check that these first order equations imply the second-order equations of motion

$$\ddot{U} + e^{2U}V_{\rm bh}(Z, Z^*, B) = 0\,, \tag{5.49}$$

$$\ddot{Z}^i + \Gamma_{jk}{}^i\dot{Z}^j\dot{Z}^k + e^{2U}\partial^i V_{\rm bh}(Z, Z^*, B) = 0\,, \tag{5.50}$$

with $\Gamma_{jk}{}^i = \mathcal{G}^{il^*}\partial_j\mathcal{G}_{kl^*}$, which coincide with the original ones if

$$V_{\rm bh}(Z, Z^*, B) = V_{\rm bh}(Z, Z^*, \mathcal{Q}) \tag{5.51}$$

for any Z^i (not just for the solution; see the remark in footnote 7).

2.2 Non-extremal black holes

Previous experience [38] (see also [176] and, further, [36,88] for 5-dimensional examples) suggests that a suitable ansatz for the variables H^M for non-extremal black holes of $N = 2, d = 4$ supergravity, compatible with the constraint (5.12), is

$$H^M(\tau) = A^M\cosh(r_0\tau) + \frac{B^M}{r_0}\sinh(r_0\tau)\,, \tag{5.52}$$

for some integration constants A^M and B^M that, as in the extremal case, have to be determined by solving the equations of motion and by imposing the standard normalization of the physical fields at spatial infinity.

Using this ansatz, the equations of motion (5.15)–(5.17) take the form

$$\tfrac{1}{2}\partial_P\partial_M\partial_N\log\mathsf{W}\left(B^MB^N - r_0^2A^MA^N\right) - \partial_P\left[V_{\rm bh}(Z, Z^*, \mathcal{Q})/\mathsf{W}\right] = 0\,, \tag{5.53}$$

$$-\tfrac{1}{2}\partial_M\partial_N\log\mathsf{W}\left(B^MB^N - r_0^2A^MA^N\right) - V_{\rm bh}(Z, Z^*, \mathcal{Q})/\mathsf{W} = 0\,, \tag{5.54}$$

$$A^MB_M = 0\,, \tag{5.55}$$

where we have used the third equation and the homogeneity properties of the Hesse potential W in order to simplify the first two.

In the non-extremal case we can define several fake central charges:

$$\tilde{\mathcal{Z}}(Z,Z^*,B) \equiv \langle\, \mathcal{V} \mid B\,\rangle\,, \qquad \tilde{\mathcal{Z}}(Z,Z^*,B_\pm) \equiv \langle\, \mathcal{V} \mid B_\pm\,\rangle\,, \tag{5.56}$$

with the shifted coefficients

$$B^M_\pm \equiv \lim_{\tau\to\mp\infty} \frac{r_0 H^M(\tau)}{\sinh(r_0\tau)} = B^M \mp r_0 A^M\,. \tag{5.57}$$

Imposing the same asymptotic conditions on the fields as in the extremal case and the condition (5.55), we arrive again at (5.32). Left to be determined from the equations of motion are then only the constants B^M and the non-extremality parameter r_0.

The mass is given again by eq. (5.40) and the expressions for the event horizon area (+) and the Cauchy horizon area (−) are

$$\frac{A_{\mathrm{h}\pm}}{4\pi} = \mathsf{W}(B_\pm)\,. \tag{5.58}$$

In the near-horizon limit, the equations of motion, upon use of the above formulae for the area of the event horizon, lead to the following relations

$$\frac{A_{\mathrm{h}\pm}}{4\pi} = -V_{\mathrm{bh}}(B_\pm) \pm 2r_0\mathcal{M}_{MN}[\mathcal{F}(B_\pm)]A^M B^N_\pm = \mathsf{W}(B_\pm)\,, \tag{5.59}$$

$$\partial_P V_{\mathrm{bh}}(B_\pm) = \pm 2r_0\partial_P\mathcal{M}_{MN}[\mathcal{F}(B)]A^M B^N_\pm = -2r_0^2\partial_P\mathcal{M}_{MN}[\mathcal{F}(B)]A^M A^N\,, \tag{5.60}$$

which generalize eqs. (5.33) and (5.35) to the non-extremal case. In the last relation we have used the identity

$$H^M\partial_P\mathcal{M}_{MN}(\mathcal{F}) = 0\,. \tag{5.61}$$

The right-hand side of eq. (5.60) vanishes if $A^M \propto B^M$. This is a special case that we study in section 2.2. Another possibility is that $\mathcal{F}_{\Lambda\Sigma}$ and hence also $\mathcal{M}_{MN}(\mathcal{F})$ are constant, as happens in quadratic models. In general, however, $\partial_P V_{\mathrm{bh}}(B_\pm) \neq 0$ and we conclude that the values of the scalars on the horizon of a non-extremal black hole do not necessarily extremize the black-hole potential.

First-order flow equations

The derivation carried out for extremal black holes in section 2.1 can be straightforwardly extended to the non-extremal case. As in the 5-dimensional case studied in ref. [36], one defines a new coordinate ρ and a function $f(\rho)$

$$\rho \equiv \frac{\sinh(r_0\tau)}{r_0\cosh(r_0\tau)}\,, \qquad f(\rho) \equiv \frac{1}{\sqrt{1-r_0^2\rho^2}} = \cosh(r_0\tau)\,, \tag{5.62}$$

so that the hyperbolic ansatz (5.52) for H^M can be rewritten in the "almost extremal form":

$$H^M = f(\rho)(A^M + B^M\rho) \equiv f(\rho)\hat{H}^M\,. \tag{5.63}$$

Then, following the same steps that led to eqs. (5.45) and (5.65), one can obtain the first-order flow equations:

$$\frac{de^{-\hat{U}}}{d\rho} = \sqrt{2}|\tilde{\mathcal{Z}}(Z,Z^*,B)|\,, \tag{5.64}$$

$$\frac{dZ^i}{d\rho} = -2\sqrt{2}\,e^{\hat{U}}\mathcal{G}^{ij^*}\partial_{j^*}|\tilde{\mathcal{Z}}(Z,Z^*,B)|\,, \tag{5.65}$$

where we have introduced the hatted warp factor $\hat{U} = U + \log f$.

Similarly to the extremal case, it is not difficult to show that this first-order flow implies the second-order equations:

$$\frac{d^2\hat{U}}{d\rho^2} + e^{2\hat{U}}V_{\rm bh}(Z,Z^*,\sqrt{2}B) = 0\,, \tag{5.66}$$

$$\frac{d^2Z^i}{d\rho^2} + \Gamma_{kl}{}^i\frac{dZ^k}{d\rho}\frac{dZ^l}{d\rho} + e^{2\hat{U}}\mathcal{G}^{ij^*}\partial_{j^*}V_{\rm bh}(Z,Z^*,\sqrt{2}B) = 0\,, \tag{5.67}$$

plus the constraint[9]

$$\left(\frac{d\hat{U}}{d\rho}\right)^2 + \mathcal{G}_{ij^*}\frac{dZ^i}{d\rho}\frac{dZ^{*\,j^*}}{d\rho} + e^{2\hat{U}}V_{\rm bh}(Z,Z^*,\sqrt{2}B) = 0\,, \tag{5.68}$$

but now with respect to the new variable ρ and the new function $\hat{U}$.

In order to compare these equations with the actual second-order equations for the warp factor and the scalars we have to rewrite them in terms of

[9] Observe that the right-hand side of this equation is not r_0^2.

the variable τ and rescale $\hat{U}$ to U. For the former, by using $d/d\rho = f^2 d/d\tau$ and eq. (5.64), one finds:

$$\ddot{U} - \frac{2\sqrt{2}\rho}{f} e^U |\mathcal{Z}(Z, Z^*, \sqrt{2}B)| + \frac{r_0^2}{f^2} + \frac{e^{2U}}{f^2} V_{\text{bh}}(Z, Z^*, \sqrt{2}B) \,, \tag{5.69}$$

from which follows the relation between the true and the fake black-hole potential that must hold for the above second-order equations to imply the equations of motion:

$$e^{2U} V_{\text{bh}}(Z, Z^*, \mathcal{Q}) = \frac{e^{2U}}{f^2} V_{\text{bh}}(Z, Z^*, \sqrt{2}B) - \frac{2\sqrt{2} r_0^2 \rho}{f} e^U |\mathcal{Z}(Z, Z^*, \sqrt{2}B)| + \frac{r_0^2}{f^2} \,. \tag{5.70}$$

The same condition ensures that the constraint eq. (5.68) implies the standard Hamiltonian constraint. For the scalar equations we find the condition

$$\partial_i \left(e^{2U} V_{\text{bh}}(Z, Z^*, \mathcal{Q}) - \frac{e^{2U}}{f^2} V_{\text{bh}}(Z, Z^*, \sqrt{2}B) + \frac{4\sqrt{2} r_0^2 \rho}{f} e^U |\mathcal{Z}(Z, Z^*, \sqrt{2}B)| \right) = 0 \,. \tag{5.71}$$

No other conditions need to be satisfied for the first-order equations to imply all the second-order equations of motion. Taking the derivative with respect to ρ of eq. (5.70) we find that, if this relation is satisfied for any Z^i (or any H^M), then the last equation is also satisfied, as are all the second-order equations.

Evaluating eq. (5.70) at spatial infinity ($\tau = 0$, which corresponds to $\rho = 0$) we find the following relation between the charges, the fake charges, the asymptotic values of the moduli and the non-extremality parameter:

$$V_{\text{bh}}(Z_\infty, Z^*_\infty, \mathcal{Q}) - V_{\text{bh}}(Z_\infty, Z^*_\infty, \sqrt{2}B) = r_0^2 \,. \tag{5.72}$$

Non-extremal generalization of doubly-extremal black holes

For non-extremal black holes whose scalars are constant over the whole spacetime, it is possible to solve the equations of motion of the H-FGK system with the hyperbolic ansatz (5.52) in a model-independent way, i.e. for any theory of $N = 2, d = 4$ supergravity. Given the constancy of the scalars we assume

$$Z^i_\infty = Z^i_{\text{h}} \,, \tag{5.73}$$

which requires

$$B^M \propto A^M\,, \tag{5.74}$$

where the constants A^M are given by eq. (5.32).

Using the proportionality of the B^M and A^M in the $\tau \to 0^-$ or $\tau \to \pm\infty$ limit of eq. (5.53) we get

$$\partial_K V_{\rm bh}(Z_\infty, Z^*_\infty, \mathcal{Q}) = 0\,, \tag{5.75}$$

which proves that the scalars must assume attractor values $Z^i_\infty = Z^i_{\rm att}$ that are a stationary point of the black-hole potential, just as in the extremal case. We can thus use eq. (5.33), which gives the value of the black-hole potential at the horizons in terms of the fake central charge there $\tilde{\mathcal{Z}}(B,B)$ (not $\tilde{\mathcal{Z}}(Z,Z^*,B_\pm)$):

$$-V_{\rm bh}(Z_\infty, Z^*_\infty, \mathcal{Q}) = |\tilde{\mathcal{Z}}(B,B)|^2\,. \tag{5.76}$$

The proportionality constant between B^M and A^M is easily determined to be $-\mathsf{W}^{1/2}(B)$ by using the normalization at infinity $\mathsf{W}(A) = 1$ and choosing the sign so as to make the functions $H^M \neq 0$ for $\tau \in (-\infty, 0)$. Then we can write

$$H^M(\tau) = A^M\left(\cosh(r_0\tau) - \mathsf{W}^{1/2}(B)\frac{\sinh(r_0\tau)}{r_0}\right)\,. \tag{5.77}$$

The values of $B^M_\pm$ are

$$B^M_\pm = -\left(\mathsf{W}^{1/2}(B) \pm r_0\right)A^M\,, \tag{5.78}$$

and

$$\mathsf{W}(B_\pm) = \left(\mathsf{W}^{1/2}(B) \pm r_0\right)^2\,. \tag{5.79}$$

A relation between the value of $\mathsf{W}^{1/2}(B)$, the physical parameters and r_0 can be found by taking the $\tau \to 0^-$ limit of eq. (5.54):

$$\mathsf{W}(B) = r_0^2 - V_{\rm bh}(Z_\infty, Z^*_\infty, \mathcal{Q})\,. \tag{5.80}$$

Another relation comes from the definition of mass $M = \dot{U}(0)$, which gives $M = -\tilde{H}_M(A)B^M$. Using the proportionality between A^M and B^M we find that

$$M = \mathsf{W}^{1/2}(B)\,. \tag{5.81}$$

The final expression for the functions $H^M(\tau)$ is, regardless of the details of the model:

$$H^M(\tau) = A^M \left(\cosh(r_0\tau) - M \frac{\sinh(r_0\tau)}{r_0} \right), \tag{5.82}$$

$$S_\pm = \pi (M \pm r_0)^2 , \tag{5.83}$$

where the non-extremality parameter, upon use of eq. (5.76), is given by

$$r_0 = \sqrt{M^2 - |\tilde{\mathcal{Z}}(B,B)|^2} . \tag{5.84}$$

3 One-modulus quantum-corrected geometries

We shall now use the formalism developed in the last section to explore the black-hole solutions of one-modulus quantum-corrected models that typically appear as one-modulus Calabi–Yau compactification of type II string theory. For one-modulus models of this kind the perturbative prepotential $\mathcal{F}_{\rm pert}$ can be brought to the form:

$$\mathcal{F}^{\rm pert}_{\rm IIA} = -\frac{\kappa^0_{1,1,1}}{6} \frac{(\hat{\mathcal{X}}^1)^3}{\hat{\mathcal{X}}^0} - \frac{i}{2} c (\hat{\mathcal{X}}^0)^2 , \tag{5.85}$$

where the *correction* is encoded in the model-dependent positive constant c. $\kappa^0_{1,1,1}$ is the triple intersection number and the hat indicates that we are working in a possibly rotated (by a symplectic matrix) frame of the homogeneous coordinates $\{\mathcal{X}^0, \mathcal{X}^i\}$ of the moduli space. In what follows we take the explicit example of the type IIA superstring compactified on the quintic Calabi–Yau manifold ($\kappa^0_{1,1,1} = 5$), which we review in the appendix.

For the sake of simplicity and in order to be able to make a comparison, in the following we first study the *uncorrected* model corresponding to the prepotential $\mathcal{F}^0_{\rm IIA} \equiv \mathcal{F}^{\rm pert}_{\rm IIA}(c = 0)$ and only afterwards the general case of eq. (5.85).

3.1 Uncorrected case: the t^3 model

In this section we consider the tree-level prepotential:

$$\mathcal{F}^0_{\rm pert}(\mathcal{X}) = -\frac{5}{6} \frac{(\mathcal{X}^1)^3}{\mathcal{X}^0} . \tag{5.86}$$

In terms of the coordinate $t = \mathcal{X}^1/\mathcal{X}^0$ the Kähler potential and metric are given by:

$$e^{-\mathcal{K}^0} = \tfrac{20}{3}(\operatorname{Im} t)^3 \,, \qquad \mathcal{G}^0_{tt^*} = \tfrac{3}{4}\,(\operatorname{Im} t)^{-2} \,, \tag{5.87}$$

whereas the covariantly holomorphic symplectic section is

$$\mathcal{V}^0(t,t^*) = e^{\mathcal{K}^0/2} \begin{pmatrix} 1 \\ t \\ \frac{5}{6}t^3 \\ -\frac{5}{2}t^2 \end{pmatrix} \tag{5.88}$$

and the central charge, its covariant derivative, the black-hole potential and its partial derivative read:

$$\mathcal{Z} \equiv e^{\mathcal{K}^0/2}\hat{\mathcal{Z}} \,, \tag{5.89}$$

$$\mathcal{D}_t\mathcal{Z} = \frac{i}{2}\frac{e^{\mathcal{K}^0/2}}{\operatorname{Im} t}\hat{\mathcal{W}} \,, \tag{5.90}$$

$$-V_{\text{bh}} = e^{\mathcal{K}^0}\left(|\hat{\mathcal{Z}}|^2 + \tfrac{1}{3}|\hat{\mathcal{W}}|^2\right) , \tag{5.91}$$

$$-\partial_t V_{\text{bh}} = \tfrac{i}{20}(\operatorname{Im} t)^{-4}\left((\hat{\mathcal{W}}^*)^2 + 3\hat{\mathcal{W}}\hat{\mathcal{Z}}^*\right) . \tag{5.92}$$

In the above:

$$\hat{\mathcal{Z}} = \tfrac{5}{6}p^0t^3 - \tfrac{5}{2}p^1t^2 - q_1 t - q_0 \,, \tag{5.93}$$

$$\hat{\mathcal{W}} = \tfrac{5}{2}p^0t^2t^* - \tfrac{5}{2}p^1t(t+2t^*) - q^1(2t+t^*) - 3q^0 \,. \tag{5.94}$$

Notice all these objects are well defined only for $\operatorname{Im} t > 0$. Furthermore, it must be taken into account that the theory given by the tree-level prepotential is a good approximation to the full theory only when $|t| \gg 1$.

Extremal solutions

Extremal solutions are associated with the critical points of the black-hole potential. Following from eqs. (5.90) and (5.92), there are two kinds of critical points:

1. Supersymmetric, when

$$\hat{\mathcal{W}} = 0 \,. \tag{5.95}$$

   For generic (non-vanishing) values of the charges, there exist three complex solutions for the critical values t_{att}, but at most two can be physical ($\operatorname{Im} t > 0$). Their expressions are complicated and will be recovered below by taking the appropriate limits in the solutions.

2. Non-supersymmetric [21,184], when $\hat{\mathcal{W}} \neq 0$ and
$$3\hat{\mathcal{Z}}\hat{\mathcal{W}}^* + \hat{\mathcal{W}}^2 = 0\,. \tag{5.96}$$

The extremal BPS solutions can be constructed by the procedure explained in section 2.1. The Freudenthal duality equations can be solved in a general way [133] and the metric function and scalar field read:

$$\begin{aligned} e^{-2U} &= \mathsf{W}(H) = \\ &= \tfrac{2}{\sqrt{3}}\sqrt{\tfrac{8}{15}H^0(H_1)^3 + (H^1H_1)^2 - 3(H^0H_0)^2 - 6H^0H_0H^1H_1 - 10(H^1)^3H_0}\,, \\ t &= -\frac{3H^0H_0 + H^1H_1}{5(H^1)^2 + 2H^0H_1} + i\frac{3e^{-2U}}{2\,[5(H^1)^2 + 2H^0H_1]}\,. \end{aligned} \tag{5.97}$$

The harmonic functions $(H^M) = (H^0, H^1, H_0, H_1)$ are given by eq. (5.19) with $B^M = \mathcal{Q}^M$ and the A^M are given by eq. (5.32) (with the upper sign), where now the asymptotic values of the symplectic section (5.88) and the central charge (5.89) have to be used. This guarantees the absence of NUT charge (necessary for the consistency of the solution) and the correct asymptotic behavior of the above fields: $e^{-2U(0)} = 1$, $t(0) = t_\infty$.

On the horizon, the values taken by these fields can be found by replacing the harmonic functions H^M by $-\mathcal{Q}^M/\sqrt{2}$, that is

$$\begin{aligned} \frac{S_{\rm e}}{\pi} &= \tfrac{1}{2}\mathsf{W}(\mathcal{Q}) = \tfrac{1}{\sqrt{3}}\sqrt{\tfrac{8}{15}p^0(q_1)^3 + (p^1q_1)^2 - 3(p^0q_0)^2 - 6p^0q_0p^1q_1 - 10(p^1)^3q_0}\,, \\ t_{\rm att} &= -\frac{3p^0q_0 + p^1q_1}{5(p^1)^2 + 2p^0q_1} + i\frac{3\mathsf{W}(\mathcal{Q})}{2[5(p^1)^2 + 2p^0q_1]}\,. \end{aligned} \tag{5.98}$$

The values of the fields on the horizon are well defined only if the charges are such that the entropy and, hence, $\mathsf{W}(\mathcal{Q})$ is real and non-vanishing and if $\operatorname{Im} t > 0$. Furthermore, in order to be able to write the above expressions we have assumed that $p^0 > 0$. Then, the conditions that the charges must satisfy are

$$p^0 > 0\,, \tag{5.99}$$
$$5(p^1)^2 + 2p^0q_1 > 0\,, \tag{5.100}$$
$$\tfrac{8}{15}p^0(q_1)^3 + (p^1q_1)^2 - 3(p^0q_0)^2 - 6p^0q_0p^1q_1 - 10(p^1)^3q_0 > 0\,. \tag{5.101}$$

The analysis of the possible values of the charges in the most general case is complicated and unilluminating, so we will not attempt it here. The inequalities (5.99)–(5.101) must be extended to the H^M in order to guarantee the regularity of the solution. The first-order flow equations imply that the metric function grows monotonically from spatial infinity to the event horizon, therefore it is enough to give it admissible values there to ensure that it does not vanish for any value of $\tau \in (-\infty, 0)$. A similar argument applies to the scalar field.[10]

Because the general supersymmetric solution turns out to be very difficult to deform into the general non-extremal solution, we consider a simpler three-charge case with $p^0 = 0$. The supersymmetric solution (with $H^0_\infty = 0$ as well) takes the form:

$$\begin{aligned} e^{-2U} &= \frac{2}{\sqrt{3}}|H^1|\sqrt{(H_1)^2 - 10H^1H_0}\,, \\ t &= -\frac{H_1}{5H^1} + i\frac{\sqrt{3}}{5}\frac{\sqrt{(H_1)^2 - 10H^1H_0}}{|H^1|}\,, \end{aligned} \tag{5.102}$$

For this simpler charge configuration it is also possible to directly study the stationary points of the black-hole potential to find a non-supersymmetric critical point given by:

$$t_{\text{att}} = -\frac{q_1}{5p^1} + i\frac{\sqrt{3}}{5}\frac{\sqrt{-[(q_1)^2 - 10p^1q_0]}}{|p^1|} \tag{5.103}$$

and the corresponding entropy:

$$S_{\text{e}}/\pi = \frac{1}{\sqrt{3}}|p^1|\sqrt{-[(q_1)^2 - 10p^1q_0]}\,. \tag{5.104}$$

They differ from the supersymmetric case by the sign of the discriminant

$$\Lambda = -p^1q_0 + \frac{(q_1)^2}{10}\,. \tag{5.105}$$

Rather than trying to construct the corresponding solutions directly, we shall obtain them as a limit of the non-extremal solution that we construct using the general procedure discussed in the previous section.

[10] With more scalar fields and non-diagonal metrics it would be more complicated to argue the same.

Non-extremal solution with $p^0 = 0$

As we showed in section 2.2, by using the ansatz

$$H^M(\tau) = A^M \cosh(r_0\tau) + \frac{B^M}{r_0} \sinh(r_0\tau)\,. \tag{5.106}$$

valid for non-extremal black holes satisfying $H^M \dot{H}_M = 0$, one can reduce the differential equations of motion to the algebraic equations (5.53)–(5.55) and solve them for the coefficients B^M. For a non-extremal black hole in the t^3 model with charges p^1, q_0 and q_1 one finds:

$$B_0 = s^1\left(\sqrt{\frac{\Lambda^2}{2(p^1)^2} + \frac{5r_0^2(\mathrm{Im}\,t_\infty)^3}{24}} - \frac{q_1^2}{10(p^1)^2}\sqrt{\frac{(p^1)^2}{2} + \frac{3r_0^2}{10(\mathrm{Im}\,t_\infty)^3}}\right), \tag{5.107}$$

$$B^1 = -s^1\sqrt{\frac{3r_0^2}{10\,\mathrm{Im}\,t_\infty} + \frac{1}{2}(p^1)^2}\,, \tag{5.108}$$

$$B_1 = -s^1\frac{q_1}{p^1}\sqrt{\frac{3r_0^2}{10\,\mathrm{Im}\,t_\infty} + \frac{1}{2}(p^1)^2}\,, \tag{5.109}$$

where we have defined

$$s^1 \equiv \mathrm{sgn}(p^1)\,. \tag{5.110}$$

The coefficients A^M can be determined by using the general expression (5.32) and in our case they turn out to be:

$$A_0 = s^1\frac{\sqrt{3}}{10\sqrt{10}\sqrt{\mathrm{Im}\,t_\infty}}\left[\left(\frac{q_1}{p^1}\right)^2 - \frac{25}{3}\,(\mathrm{Im}\,t_\infty)^2\right], \tag{5.111}$$

$$A^1 = s^1\sqrt{\frac{3}{10\,\mathrm{Im}\,t_\infty}}\,, \tag{5.112}$$

$$A_1 = s^1\frac{q_1}{p^1}\sqrt{\frac{3}{10\,\mathrm{Im}\,t_\infty}}\,. \tag{5.113}$$

s^1	s_0	s_Λ
+	−	+
−	+	+
+	+	+
−	−	+
+	+	−
−	−	−

Tab. 5.1: The extremal limits depend on s^1 and s_Λ. Here $s_0 = \mathrm{sgn}(q_0)$, $s^1 = \mathrm{sgn}(p^1)$ and $s_\Lambda = \mathrm{sgn}(\Lambda)$ where the discriminant Λ has been defined in eq. (5.105). There are 6 possible cases : the first 4 possibilities ($s_\Lambda = +1$) would produce a supersymmetric extremal black hole while the others ($s_\Lambda = -1$) a non-supersymmetric one.

From the relation $M = \dot{U}(0)$ the mass is found to be

$$M = \frac{1}{4}\left(\sqrt{\frac{-60p^1q_0(q_1)^2 + 3(q_1)^4 + 25(p^1)^2[12(q_0)^2 + 5r_0^2(\mathrm{Im}\,t_\infty)^3]}{125(p^1)^2(\mathrm{Im}\,t_\infty)^3}} + \sqrt{9r_0^2 + 15(p^1)^2(\mathrm{Im}\,t_\infty)^3}\right). \tag{5.114}$$

One can invert this expression to obtain r_0 in terms of the physical parameters M, $\mathrm{Im}\,t_\infty$, p^1, q_0:

$$\begin{aligned} r_0^2 = \frac{1}{1000(p^1)^4\,\mathrm{Im}\,t_\infty^6}\Big(&-60(p^1)^3q_0(q_1)^2\,\mathrm{Im}\,t_\infty^3 + 3(p^1)^2(q_1)^4\,\mathrm{Im}\,t_\infty^3 \\ &- 1875(p^1)^6\,\mathrm{Im}\,t_\infty^7 + 100(p^1)^4\left[3(q_0)^2\,\mathrm{Im}\,t_\infty^3 + 25M^2\,\mathrm{Im}\,t_\infty^6\right] \\ &+ 10\sqrt{30M^2(p^1)^6\,\mathrm{Im}\,t_\infty^9} \\ &\sqrt{9(q_1)^2\left[(q_1)^2 - 20p^1q_0\right] + 25(p^1)^2\left[36(q_0)^2 - 25(p^1)^2\,\mathrm{Im}\,t_\infty^4 + 30M^2\,\mathrm{Im}\,t_\infty^3\right]}\Big). \end{aligned} \tag{5.115}$$

This result allows one to obtain the expression for the mass in the extremal limit $r_0 \to 0$, namely:

$$M = \sqrt{\frac{3}{5}}\,\frac{25(p^1)^2\,\mathrm{Im}\,t_\infty^2 + 10|\Lambda|}{20|p^1|\,\mathrm{Im}\,t_\infty^{3/2}}\,. \tag{5.116}$$

It is easy to check that $M > 0$. As mentioned at the end of the previous section, when $s_\Lambda = \mathrm{sgn}(\Lambda)$ is positive, the solution is supersymmetric (see table 5.1), in which case the anharmonic function $H_0 = A_0 \cosh(r_0\tau) + \frac{B_0}{r_0}\sinh(r_0\tau)$ becomes for $r_0 \to 0$:

$$H_0 = s^1 \frac{\sqrt{3}}{10\sqrt{10}\sqrt{\mathrm{Im}\, t_\infty}} \left[\left(\frac{q_1}{p^1}\right)^2 - \frac{25}{3}\left(\mathrm{Im}\, t_\infty\right)^2 \right] - \frac{1}{\sqrt{2}} q_0 \tau \,, \tag{5.117}$$

whereas in the non-supersymmetric case:

$$H_0 = s^1 \frac{\sqrt{3}}{10\sqrt{10}\sqrt{\mathrm{Im}\, t_\infty}} \left[\left(\frac{q_1}{p^1}\right)^2 - \frac{25}{3}\left(\mathrm{Im}\, t_\infty\right)^2 \right] + \frac{1}{\sqrt{2}} \left(q_0 - 2\frac{q_1^2}{10p^1} \right) \tau \,. \tag{5.118}$$

The extremal limit for $H^1 = \frac{p^1}{q_1} H_1$ is in turn:

$$H^1 = s^1 \sqrt{\frac{3}{10\,\mathrm{Im}\, t_\infty}} - \frac{1}{\sqrt{2}} p^1 \tau \,. \tag{5.119}$$

Accordingly, for the warp factor after some simplification one obtains

$$e^{-2U} = \tfrac{2}{\sqrt{3}} \sqrt{\pm\left[-10(H^1)^3 H_0 + (H^1 H_1)^2\right]} \,, \tag{5.120}$$

where the plus holds for supersymmetric solutions and the minus for non-supersymmetric.

The entropies associated with the outer ($\tau \to -\infty$) and inner ($\tau \to +\infty$) horizon can be computed to be respectively:

$$\begin{aligned} \frac{S_+}{\pi} &= \frac{1}{15^{3/4}} \left[\frac{\left(\sqrt{3}r_0 + \sqrt{3r_0^2 + 5(p^1)^2\,\mathrm{Im}\, t_\infty}\right)^3}{\mathrm{Im}\, t_\infty^2} \right. \\ &\left. \left(5\sqrt{5}r_0 + \sqrt{\frac{300(q_0)^2}{\mathrm{Im}_\infty^3} - \frac{60 q_0 (q_1)^2}{p^1\,\mathrm{Im}\, t_\infty^3} + \frac{3(q_1)^4}{\mathrm{Im}\, t_\infty^3 (p^1)^2} + 125 r_0^2} \right) \right]^{1/2} , \end{aligned} \tag{5.121}$$

$$\begin{aligned} \frac{S_-}{\pi} &= \frac{1}{15^{3/4}} \left[\frac{\left(-\sqrt{3}r_0 + \sqrt{3r_0^2 + 5(p^1)^2\,\mathrm{Im}\, t_\infty}\right)^3}{\mathrm{Im}\, t_\infty^2} \right. \\ &\left. \left(5\sqrt{5}r_0 - \sqrt{\frac{300(q_0)^2}{\mathrm{Im}\, t_\infty^3} - \frac{60 q_0 (q_1)^2}{p^1\,\mathrm{Im}\, t_\infty^3} + \frac{3(q_1)^4}{\mathrm{Im}\, t_\infty^3 (p^1)^2} + 125 r_0^2} \right) \right]^{1/2} . \end{aligned} \tag{5.122}$$

By taking the limit $r_0 \to 0$ the extremal black-hole entropy is recovered from both S_+ and S_- and their product satisfies the geometric mean property $S_+ S_- = \frac{\pi^2}{3}(p^1)^2\left[-10p^1 q_0 + (q_1)^2\right] = S_\mathrm{e}^2$.

3.2 Quantum-corrected case

For the quantum-corrected model of type IIA superstring on the quintic, whose prepotential can be brought to the form (5.85) by a symplectic rotation of the coordinate frame (see the appendix), the covariantly holomorphic period vector reads:

$$\mathcal{V}_{\text{pert}} = e^{\mathcal{K}_{\text{pert}}/2} \begin{pmatrix} 1 \\ t \\ \frac{5}{6}t^3 - ic \\ -\frac{5}{2}t^2 \end{pmatrix} , \tag{5.123}$$

where (in the compactification we are considering) $c = \frac{25}{\pi^3}\zeta(3) \approx 0.969204$. Because the general case is very complicated, we deal only with two-charge and three-charge black holes.

Supersymmetric solution with $\hat{\mathcal{Q}} = (\hat{p}^0, 0, 0, \hat{q}_1)^{\mathrm{T}}$, $\mathcal{Q} = (p^0, 0, 0, q_1)^{\mathrm{T}}$

The relations between the two pairs of charges in the rotated frame and in the original one are:

$$\hat{p}^0 = p^0 , \qquad \hat{q}_1 = q_1 - \frac{25}{12}p^0 . \tag{5.124}$$

By solving the equation for the extremal supersymmetric case one finds:[11]

$$t = s_i i \sqrt{\frac{2}{5}\frac{H_1}{H^0}} , \tag{5.125}$$

$$e^{-2U_{\rm e}} = s_i \frac{4}{3}\sqrt{\frac{2}{5}H^0 (H_1)^3 + c(H^0)^2} , \tag{5.126}$$

with $H^M = A^M - \frac{1}{\sqrt{2}}\hat{\mathcal{Q}}^M \tau$ and $s_i = +1$ when

$$\sqrt{\frac{2}{5}\frac{\hat{q}_1}{\hat{p}^0}} \in \left(\left(\frac{3c}{5}\right)^{1/3}, \infty \right) , \tag{5.127}$$

while $s_i = -1$ for

$$\sqrt{\frac{2}{5}\frac{\hat{q}_1}{\hat{p}^0}} \in \left(0, \left(\frac{3c}{10}\right)^{1/3} \right) , \tag{5.128}$$

[11] As the H^M in the original frame do not appear (and $\hat{H}^M$ has been already used with a different meaning in eq. (5.62)), we suppress the hats on the rotated H^M.

so that $\operatorname{Im} t$ lies in the allowed domain (5.210) (for other values of the charges the supersymmetric solution simply does not exist). By using (5.32) one can determine the constant part of the harmonic functions:

$$A^0 = s_{\mathcal{Q}} \sqrt{\frac{3}{10\, s_i\, \operatorname{Im} t_\infty^3 + 3c}}\,, \qquad A_1 = s_{\mathcal{Q}}\, \frac{5}{2}\, \operatorname{Im} t_\infty^2 \sqrt{\frac{3}{10\, s_i\, \operatorname{Im} t_\infty^3 + 3c}}\,. \tag{5.129}$$

Notice that two disconnected branches of supersymmetric solutions appear and only one of them, the case (5.127), survives when $c = 0$. For both supersymmetric possibilities $\operatorname{sgn}(\hat{p}^0) = s_{\mathcal{Q}} = \operatorname{sgn}(\hat{q}_1)$ and depending on the charges, the scalar at infinity is bound to a certain set of possible values. If the charges, for example, satisfy (5.127) also $\operatorname{Im} t_\infty$ must belong to this interval and all the flow of the scalar in the moduli space takes place inside this confined region. By looking at the explicit form of the solutions it is possible to convince oneself that the two distinct branches of solutions cannot be connected smoothly by changing the value of the charges.

The entropy and the mass, once computed, can be written in the form:

$$\frac{S_{\rm e}}{\pi} = \frac{\frac{45}{4} c^2 (\hat{p}^0)^3 + 8(\hat{q}_1)^3 + s_i\, 6\sqrt{10}\, c\, \sqrt{(\hat{p}^0 \hat{q}_1)^3}}{\frac{45}{2} c\, \hat{p}^0 + s_i\, 6\sqrt{10}\, \hat{q}_1 \sqrt{\frac{\hat{q}_1}{\hat{p}^0}}}\,, \tag{5.130}$$

$$M_{\rm e} = \frac{\left|6c\, \hat{p}^0 + 6\, \hat{q}_1 \operatorname{Im} t_\infty + 5\, \hat{p}^0 \operatorname{Im} t_\infty^3\right|}{4\sqrt{\frac{9}{2} c + 15 \operatorname{Im} t_\infty^3}}\,. \tag{5.131}$$

The positivity of both the entropy and the mass is guaranteed by the fact that the charges are confined to the intervals (5.127), (5.128).

The study of this two-charge configuration in the rotated symplectic frame allows the analysis of the single charge configurations $\mathcal{Q} = (p^0, 0, 0, 0)^{\rm T}$ and $\mathcal{Q} = (0, 0, 0, q_1)^{\rm T}$ in the original frame. For the former one should substitute in the formulae above $\hat{q}_1 = -\frac{25}{12} \hat{p}^0$ but already here an inconsistency occurs due to the requirement $\operatorname{sgn}(\hat{p}^0) = \operatorname{sgn}(\hat{q}_1)$ that would not be respected. Also for the other single-charge configuration, by setting $\hat{p}^0 = p^0 = 0$, it is easy to realize that the expressions become ill-defined. This suggests that no physical BPS solutions exist for the single-charge case at hand.

Before passing to non-extremal black holes, it is worth mentioning that the Freudenthal duality equations also admit a solution that cannot be accepted, namely:

$$t = \frac{3c}{2H_1}\sqrt{\frac{8}{45c^2}\frac{H_1^3}{H^0} - (H^0)^2} + ic\frac{3H^0}{2H_1}\,, \tag{5.132}$$

$$e^{-2U_{\rm e}} = 2(H^0)^2 c + \frac{2}{45c}\frac{(H_1)^3}{H^0}\,. \tag{5.133}$$

These expressions would be well defined only for charges that violate the constraint (5.210), which leads to invalid Kähler metric and Kähler potential.

Supersymmetric solution with $\hat{\mathcal{Q}} = (0, \hat{p}^1, \hat{q}_0, \hat{q}_1)^{\rm T}$, $\mathcal{Q} = (0, p^1, q_0, q_1)^{\rm T}$

This configuration corresponds to a three-charge black hole also in the original frame, according to the relations:

$$\hat{p}^1 = p^1\,, \qquad \hat{q}_0 = q_0 - \frac{25}{12}p^1\,, \qquad \hat{q}_1 = q_1 + \frac{11}{2}p^1\,. \tag{5.134}$$

We solve the Freudenthal duality equations with the harmonic function H^0 set to zero. This yields:

$$\hat{\mathcal{X}}^0 = \frac{\rho^2 + \rho\alpha^{1/3} + \alpha^{2/3}}{30c\, H^1\, \alpha^{1/3}}\,, \tag{5.135}$$

$$\hat{\mathcal{X}}^1 = iH^1 - \frac{H_1}{5H^1}\hat{\mathcal{X}}^0\,, \tag{5.136}$$

$$U = -\frac{1}{2}\log\left(\frac{\alpha^{1/3}\left(\beta + \gamma\alpha^{1/3}\right) + \delta}{100\, c\, (H^1)^2\, \alpha^{2/3}\left(\alpha^{2/3} + \rho\,\alpha^{1/3} + \rho^2\right)}\right), \tag{5.137}$$

where

$$\begin{aligned}
\rho &= -10H^1 H_0 + (H_1)^2\,,\\
\alpha &= \rho^3 - 11250c^2\,(H^1)^6 + 150\sqrt{(H^1)^6 c^2[5625c^2(H^1)^6 - \rho^3]}\,,\\
\beta &= \rho^2\left(\rho^3 - 7500c^2\,(H^1)^6 + 100\sqrt{(H^1)^6 c^2[5625c^2(H^1)^6 - \rho^3]}\right)\,,\\
\gamma &= \rho\left(\rho^3 - 3750c^2\,(H^1)^6 + 50\sqrt{(H^1)^6 c^2[5625c^2(H^1)^6 - \rho^3]}\right)\,,\\
\delta &= \left(\rho^3 + 7500c^2\,(H^1)^6\right)\alpha\,.
\end{aligned} \tag{5.138}$$

The expression for the physical scalar then becomes

$$t = \frac{\hat{\mathcal{X}}^1}{\hat{\mathcal{X}}^0} = -\frac{\hat{q}_1}{5\hat{p}^1} + i\,\frac{30c\,(H^1)^2\alpha^{1/3}}{\alpha^{2/3} + \rho\,\alpha^{1/3} + \rho^2}\,. \tag{5.139}$$

The constant parts of the harmonic functions turn out to be:

$$A^1 = s^1 \frac{\sqrt{3}\,\mathrm{Im}\, t_\infty}{\sqrt{3c + 10\,\mathrm{Im}\, t_\infty^3}}\,, \qquad A_1 = s_1 \frac{\sqrt{3}\hat{q}_1\ \mathrm{Im}\, t_\infty}{\hat{p}^1\sqrt{3c + 10\,\mathrm{Im}\, t_\infty^3}}\,, \tag{5.140}$$

$$A_0 = s_0 \frac{3(\hat{q}_1)^2\,\mathrm{Im}\, t_\infty - 25(\hat{p}^1)^2\,\mathrm{Im}\, t_\infty^3 - 30c\,(\hat{p}^1)^2}{10(\hat{p}^1)^2\sqrt{9c + 30\,\mathrm{Im}\, t_\infty^3}}\,, \tag{5.141}$$

where $s^M = \mathrm{sgn}(\hat{\mathcal{Q}}^M)$.

The solution just displayed is a purely "quantum black hole": it diverges when c is put to zero and it is well defined only for a restricted set of values of the parameters $\{\hat{p}^1, \hat{q}_0, \hat{q}_1, \mathrm{Im}\, t_\infty\}$. By looking at the expressions of the scalar and the warp factor we realize that the problematic part is the square root

$$\sqrt{(H^1)^6 c^2 [5625 c^2 (H^1)^6 - \rho^3]} \tag{5.142}$$

that, in order to be real, needs the radicand to be bigger than or equal to zero. This condition must be considered besides the requirement that the imaginary part of the scalar should belong to the intervals (5.210) and the positivity of the warp factor. Then, the allowed values of the charges can be determined by studying the behavior of the solutions on the horizon, whereas the allowed values for $\mathrm{Im}\, t_\infty$ are given by the limit at infinity ($\tau \to 0^-$). In the end one obtains the following restrictions:

$$\mathrm{Im}\, t_\infty \in \left(-\left(\frac{3c}{10}\right)^{1/3}, 0\right) \approx \Big(-0.662489\,,\,0\Big), \tag{5.143}$$

$$\hat{q}_0 > \frac{(\frac{75}{2}\,c)^{2/3}\,(\hat{p}^1)^2 + (\hat{q}_1)^2}{10\hat{p}^1} \quad \text{if} \quad \hat{p}^1 > 0\,, \tag{5.144}$$

$$\hat{q}_0 < \frac{(\frac{75}{2}\,c)^{2/3}\,(\hat{p}^1)^2 + (\hat{q}_1)^2}{10\hat{p}^1} \quad \text{if} \quad \hat{p}^1 < 0\,. \tag{5.145}$$

It is not difficult to see that the conditions (5.144), (5.145) would be violated by the charge configuration $\hat{\mathcal{Q}} = (0, \hat{p}^1, 0, \hat{q}_1)^{\mathrm{T}}$, which would produce a black hole with singular metric (differently from the uncorrected t^3 model). Similarly one can exclude the existence of black holes with the charge vector $\hat{\mathcal{Q}} = (0, \hat{p}^1, -\frac{25}{12}\hat{p}^1, \frac{11}{2}\hat{p}^1)^{\mathrm{T}}$, corresponding in the original frame to $\mathcal{Q} = (0, p^1, 0, 0)$: when $\hat{p}^1 = p^1 = 0$ the expression for the scalar would diverge. This last observation, together with the discussion in the previous subsection, indicates that this model does not admit regular supersymmetric single-charge black holes.

On the other hand, solutions with $H_1 = 0$ (corresponding to the charge configuration $\hat{\mathcal{Q}} = (0, \hat{p}_1, \hat{q}_0, 0)^{\mathrm{T}}$, $\mathcal{Q} = (0, p^1, q_0, -\frac{11}{2}p^1)^{\mathrm{T}}$) or with $q_1 = 0$ (two-charge in the unrotated frame), are physical. In the former case the scalar becomes purely imaginary

$$
\begin{aligned}
t = & -3i\,\frac{(H^1)^2 c\,\lambda^{1/3}}{(H^1)^2(H_0)^2 + H^1 H_0 \lambda^{1/3} + \lambda^{2/3}}\,, \\
\lambda = & \frac{45}{4}(H^1)^6 c^2 + (H^1)^3 H_0^3 - 3\sqrt{\frac{5}{4}(H^1)^9 c^2 \left(\frac{45}{4}(H^1)^3 c^2 + 2(H_0)^3\right)}
\end{aligned}
\tag{5.146}
$$

and, in line with eqs. (5.144), (5.145), the charges must satisfy $\operatorname{sgn}\hat{p}^1 = \operatorname{sgn}\hat{q}_0$ and $|\hat{q}_0| > (\frac{(75/2\,c)^{2/3}}{10})\,|\hat{p}^1|$. When instead $q_1 = 0$, the real part of the scalar takes a fixed value independent of parameters, namely $\operatorname{Re} t = -\frac{11}{10}$, and the restrictions on the allowed charges become $\operatorname{sgn}\hat{q}_0 = \operatorname{sgn}\hat{p}^1$ and $|\hat{q}_0| > \frac{4(75/2\,c)^{2/3}+121}{40}\,|\hat{p}^1|$.

The entropy and the mass for the black holes in this section can be calculated as usual, but due to the complexity of the expressions, we do not display them.

Non-extremal solutions

The expressions for the scalar and warp factor are in general very involved and this turns out to make the pursuit of non-supersymmetric black holes cumbersome. The difficulty resides in the fact that the equations for the coefficients turn out to be polynomials of a very high degree, which cannot be solved analytically.

The only non-extremal black holes that can be quite straightforwardly studied are those with the scalar assuming a constant value that extremizes the black-hole potential. From the general treatment in 2.2 we know that for such non-extremal solutions

$$
B^M = -A^M M = -A^M \sqrt{|\mathcal{Z}(Z_\infty, Z^*_\infty, \mathcal{Q})|^2 + r_0^2}\,, \tag{5.147}
$$

and the only quantity to calculate is the absolute value of the central charge in the stationary points of the black-hole potential. In the current case it reads:

$$
|\mathcal{Z}(Z_\infty, Z^*_\infty, \mathcal{Q})| = \frac{|6\hat{q}_0 + 6\hat{q}_1 t_{\mathrm{att}} + 15\hat{p}^1 t_{\mathrm{att}}^2 - 5\hat{p}^0 t_{\mathrm{att}}^3 + 6ic\hat{p}^0|}{\sqrt{6(12c + 5i\,\mathrm{Im}\,t_{\mathrm{att}}^3)}}\,, \tag{5.148}
$$

where $t_{\rm att}$ is the constant value of the scalar all along the flow.

So far no analytic expressions for non-supersymmetric stationary points of $V_{\rm bh}(Z,Z^*,\mathcal{Q})$ have been obtained for a general charge configuration.[12] We study the non-extremal version of (some of) the supersymmetric black holes of the previous subsections and present an example of a constant-scalar non-extremal black hole built from a non-supersymmetric critical point of a system with a particular charge vector.

Configuration $\hat{\mathcal{Q}} = (\hat{p}^0, 0, 0, \hat{q}_1)^{\rm T}$*:* When $\hat{q}_1\hat{p}^0 > 0$, we read off from eq. (5.125) that

$$t_{\rm att} = i\, s_i \sqrt{\frac{2\hat{q}_1}{5\hat{p}^0}} \tag{5.149}$$

and by plugging it in (5.129) and (5.148) we find:

$$B^0 = -s_{\mathcal{Q}} \sqrt{\frac{15\, M^2}{15\, c + s_i\, 4\sqrt{10\left(\frac{\hat{q}_1}{\hat{p}^0}\right)^3}}}\,, \qquad B_1 = \frac{\hat{q}_1}{\hat{p}^0} B^0\,. \tag{5.150}$$

where the mass M is equal to:

$$M = \sqrt{\frac{c}{2}\,(\hat{p}^0)^2 + s_i \sqrt{\frac{8}{45}\,\hat{p}^0(\hat{q}_1)^3} + r_0^2}\,. \tag{5.151}$$

With this last expression, the outer and the inner entropy follow from eq. (5.83). It is worth noticing that all these formulae reduce to the extremal counterparts in the limit $r_0 \to 0$ and for the entropy it holds $S_+S_- = S_{\rm e}^2$.

Configuration $\hat{\mathcal{Q}} = (0, \hat{p}^1, 2\hat{p}^1, 0)^{\rm T}$*:* For the sake of simplicity we take $\hat{q}_0 = 2\hat{p}^1$. The black-hole potential has a charge-independent critical point (corresponding to a supersymmetric attractor) at:

$$\begin{aligned} t_{\rm att} &= -6\, i\, c \frac{\left(64 + 90\, c^2 - 6c\sqrt{5\,(64+45c^2)}\right)^{1/3}}{12 + \left(2 + \left(64 + 90c^2 - 6c\sqrt{5\,(64+45c^2)}\right)^{1/3}\right)^2} \equiv -6\, i\, c\, \xi \\ &\approx -0.447310\, i \end{aligned}$$

[12] An accurate numerical study has been carried out in [185].

and the coefficients of the hyperbolic functions are:

$$B^1 = s^1 \frac{6\, c\, M\, \xi}{\sqrt{c - 720\, c^3\, \xi^3}}\,, \qquad B_0 = \frac{1 - 180\, c^2\, \xi^3}{6\, \xi}\, B^1\,. \tag{5.152}$$

For the mass one finds:

$$M = \sqrt{(p^1)^2\, \frac{2\,(1 - 45\, c^2\, \xi^2)^2}{c\,(1 - 720\, c^2\, \xi^3)} + r_0^2}\,. \tag{5.153}$$

From these expressions it is easy to see by setting $c = 0$ that this black hole does not reduce to a regular solution of the t^3 model.

Configuration $\hat{\mathcal{Q}} = (\hat{p}^0, 0, 0, -\frac{35}{2}(\frac{3}{2}c)^{2/3}\hat{p}^0)^{\mathrm{T}}$*:* Also in this case the stationary point of the black-hole potential does not depend on the value of $\hat{p}^0$ (although this time it corresponds to a non-supersymmetric attractor):

$$t_{\mathrm{att}} = i\left(\tfrac{3}{2}c\right)^{1/3} \approx 1.13284\, i\,. \tag{5.154}$$

The non-extremal solution with a constant scalar is then completely characterized by

$$B^0 = -\, s^0\, \frac{M}{\sqrt{6\, c}}\,, \qquad B_1 = -\, s^0\, \frac{5}{4}\left(\frac{3}{2}\, c\right)^{1/6} M\,, \tag{5.155}$$

with

$$M = \sqrt{48\, c\,(\hat{p}^0)^2 + r_0^2}\,. \tag{5.156}$$

The limit $r_0 \to 0$ gives a doubly-extremal non-supersymmetric black hole. Setting $c = 0$ again does not lead to a regular solution.

Configurations $\hat{\mathcal{Q}} = (0, \hat{p}^1, 0, 0)^{\mathrm{T}}$ *and* $\hat{\mathcal{Q}} = (0, 0, 0, \hat{q}_1)^{\mathrm{T}}$*:* Of these two configurations that are both single-charge in the rotated frame, the second is one-charge also in the original frame, $\mathcal{Q} = (0, 0, 0, \hat{q}_1)^{\mathrm{T}} = (0, 0, 0, q_1)^{\mathrm{T}}$. The admissible critical points of the black-hole potential $-V_{\mathrm{bh}}$ give in each case one non-supersymmetric attractor,

$$t_{\mathrm{att}} = i\sqrt[3]{\left(6 + \sqrt[3]{206 - 6\sqrt{87}} + \frac{17\sqrt[3]{4}}{\sqrt[3]{103 - 3\sqrt{87}}}\right)\frac{3c}{10}} \approx 1.37065\, i \tag{5.157}$$

or

$$t_{\text{att}} = -i\sqrt[3]{(3\sqrt{2}-4)\frac{3c}{10}} \approx -0.327962\,i\,, \tag{5.158}$$

which (by the analysis of eigenvalues of the Hessian matrix of $-V_{\text{bh}}$ with respect to t and t^*, in a real basis [167,168]) is found to be stable.[13] Neither depends on the value of the charge.

The metric function of non-extremal solutions with the constant scalar, fixed to one of the above values,[14]

$$e^{-U} = e^{-r_0\tau}\left(\frac{-V_{\text{bh}}|_{\text{att}}}{-2r_0^2 \pm 2\sqrt{r_0^2(r_0^2 - V_{\text{bh}}|_{\text{att}})}}(e^{2r_0\tau}-1)+1\right), \tag{5.159}$$

has the extremal ($r_0 \to 0$) limit:

$$\lim_{r_0\to 0} e^{-U} = -\sqrt{-V_{\text{bh}}}\Big|_{\text{att}}\tau + 1\,, \tag{5.160}$$

with the minus sign due to the negative τ in our conventions and the constant 1 for asymptotic flatness. The respective stationary values of the black-hole potential read

$$-V_{\text{bh}}\Big|_{\text{att}} = -\frac{5t_{\text{att}}\left(144c^2+30ct_{\text{att}}^3+100t_{\text{att}}^6\right)}{8\left(36c^2+30ct_{\text{att}}^3-50t_{\text{att}}^6\right)}(\hat{p}^1)^2 \approx 2.20225(\hat{p}^1)^2 \tag{5.161}$$

and

$$-V_{\text{bh}}\Big|_{\text{att}} = \frac{\sqrt{2}}{2}\left(\frac{3\sqrt{2}+4}{75c}\right)^{1/3}(\hat{q}_1)^2 \approx 0.431213(\hat{q}_1)^2\,, \tag{5.162}$$

the second of which does not have a finite $c \to 0$ limit.

4 Conclusions

The use of the H-FGK approach has enabled us to study some model-independent properties of black holes in four-dimensional $N=2$ supergravity.

[13] In each case there are in addition multiple stationary points outside of the allowed domain. For q_1 there is also one admissible saddle point of the black-hole potential at $t = i[(3\sqrt{2}+4)\frac{3c}{10}]^{1/3} \approx 1.06216\,i$.

[14] To derive this expression, it is easiest to integrate twice one of the non-trivial equations of motion in the form of eq. (2.37) of [38] and substitute into the other, eq. (2.38) therein.

We have then applied the H-FGK formalism to find extremal and non-extremal solutions of the t^3 model without and, for the first time analytically, with a quadratic quantum correction to the prepotential. We have studied the solutions for the corrected model in a symplectically rotated frame of homogenous coordinates on the scalar manifold, which simplifies the prepotential (and allows one to interpret the results as pairs of solutions for two closely related, but not mutually dual prepotentials with quadratic corrections). The formalism itself can be applied with equal ease to any charge configuration of either model, but the polynomial equations that determine the parameters make the explicit solutions unfeasible except when some charges vanish and, in the non-extremal case, when the scalar is constant.

The correction leads to the appearance of solutions, which one might call quantum black holes [60], that do not possess a regular classical limit. Perhaps surprisingly, we find in particular that the quantum correction is sufficient to render the otherwise divergent solution with only one charge, q_1, regular. (The other solution that is single-charge in the rotated frame, but which is not single-charge in the original frame, without the quantum correction reduces to the empty Minkowski spacetime.) Although this effect resembles the previously known cloaking of the classical naked singularity by higher-curvature corrections [186], there are important differences: first, here the correction pertains to the vector multiplets and not to the Einstein–Hilbert action; second, c is not a quantum parameter *sensu stricto*. It is more properly understood as measuring how the Calabi–Yau manifold on which a type II superstring theory has been compactified deviates from a torus, since the "classical" $N = 2, d = 4$ theory can be obtained as a special case of a toroidal compactification. On the other hand, the existence of these and other "quantum black-hole" solutions with no regular "classical" limit is a clear indication that the "classical" approximations in which c is set to zero are not completely valid. In order to study the black-hole solutions of Calabi-Yau compactifications, one should aim to construct solutions using the complete prepotential, including perturbative and non-perturbative corrections. This does not mean that the "classical" solutions are not correct. It just means that they are correct solutions of a model which is not a good approximation to the one coming form the Calabi-Yau compactification.

In contrast to the solutions in ref. [187], the truncations ($H^M = 0$ for some M) of the H-functions corresponding to our quantum black holes are non-singular in the classical limit. This means that in our case we can construct the classical counterpart to a corrected solution with no regular $c \to 0$ limit by simply considering the theory with $c = 0$, imposing the same constraints on H^M and $\mathcal{Q}^M$, and then solving the Freudhental duality equations.

A Type II Calabi–Yau compactifications

In this appendix we review the compactification of the type IIA theory on the quintic manifold $\mathcal{M}$ and of the type IIB on the mirror quintic manifold $\mathcal{W}$, following refs. [188–196]. It is well known that the low-energy limit of type II superstring theory compactified on a Calabi–Yau manifold is an $N = 2, d = 4$ supergravity with a number of vector multiplets and hypermultiplets that depends on the Hodge numbers of the Calabi–Yau manifold. Only the vector multiplets moduli space is relevant for the construction of black-hole solutions in these theories: black-hole-type solutions with non-trivial hyperscalars in ungauged $N = 2, d = 4$ theories are expected to be generically singular since they would have primary scalar hair [197]. On the other hand, in the unguaged theories, the only bosonic field the hyperscalars couple to in the ungauged theories is the metric, and, therefore, they can always be consistently truncated or, equivalently, set to some constant value.

A.1 Type IIB on the mirror quintic $\mathcal{W}$

Let $\mathcal{M}$ be the family of manifolds associated with the vanishing of a quintic polynomial in $\mathbb{CP}_4$. An element of $\mathcal{M}$ has $h^{(2,1)} = 101$ degrees of freedom describing the complex structure of the manifold, that can be associated with the coefficients of the defining polynomial.[15] Furthermore, $h^{(1,1)} = 1$ and the only independent harmonic $(1,1)$-form can be identified with the Kähler form of the manifold: any other harmonic $(1,1)$-form is the Kähler form multiplied by a real number, which corresponds to the freedom to adjust the overall scale of the manifold. The Euler number of a quintic manifold is $\chi = -200$.

Let us consider the family of quintic polynomials [191, 192]

$$p_\psi = \sum_{k=1}^{5} x_k^5 - 5\psi \prod_{k=1}^{5} x_k, \qquad \psi \in \mathbb{C}\,, \tag{5.163}$$

[15] A quintic polynomial has 126 possible terms and complex coefficients. However, 25 of them can be eliminated by linear transformations of the 5 complex coordinates.

parametrized by the complex modulus ψ, M_ψ the manifold described by $p_\psi = 0$ and $\mathcal{M}_0 \subset \mathcal{M}$ the family of all manifolds M_ψ for $\psi \in \mathbb{C}$ The family of quintic polynomials $(p_\psi,\ \psi \in \mathbb{C})$ is invariant under the group generated by:

$$\begin{aligned} g_0 &= (1,0,0,0,4)\ , \\ g_1 &= (0,1,0,0,4)\ , \\ g_2 &= (0,0,1,0,4)\ , \\ g_3 &= (0,0,0,1,4)\ , \end{aligned} \tag{5.164}$$

where g_i, $i = 0, \ldots, 3$, acts on $(x_1, \ldots, x_5)$ by multiplying the $(i+1)$-th entry by the phase $\alpha = e^{2\pi i/5}$ and the last entry by α^4, so $g_i^5 = 1$ for all i. The transformation $g_0 g_1 g_2 g_3$ leaves each p_ψ invariant because it multiplies the homogeneous coordinates by a common phase, hence only three of the g_i are independent, say g_1, g_2 and g_3. These three elements generate the group $\mathbb{Z}_5^3$.

It turns out that the mirror family $\mathcal{W}$ is $\mathcal{W} = W_\psi \equiv M_\psi/\mathbb{Z}_5^3, \psi \in \mathbb{C}$. It can be shown that the elements of $\mathcal{W}$ have $h^{(2,1)} = 1$, $h^{(1,1)} = 101$ and $\chi = 200$, as they must.

Since the transformation $\psi \to \alpha\psi$ can be undone by a coordinate transformation, we have that $\psi \sim \alpha\psi$, thus it is ψ^5 that plays the role of the modulus that parametrizes the complex-structure moduli space of $\mathcal{W}$ that we denote by $C_{\rm IIB}^{(2,1)}$. This is in agreement with $h^{(2,1)} = 1$. There are two values of ψ^5 for which M_ψ (and, correspondingly, W_ψ) is singular: $\psi^5 = 1$ and $\psi = \infty$.

W_1 has a single singular point given by the equivalence class $[(1,1,1,1)]$ and W_∞ is given by the quotient by $\mathbb{Z}_5^3$ of the singular quintic

$$p_\infty = \prod_{k=1}^{5} x_k = 0\,. \tag{5.165}$$

W_∞ is the large complex structure limit of $\mathcal{W}$: we will see in the following section that it is the mirror of the large-radius limit of $\mathcal{M}$.

We are interested in the compactification of the type IIB theory on $\mathcal{W}$. The low-energy effective field theory is an ungauged $N = 2, d = 4$ supergravity coupled to $h^{(2,1)} = 1$ vector multiplets and $h^{(1,1)} + 1 = 102$ hypermultiplets that can be consistently ignored (set to some constant value). We will thus be dealing with just one complex scalar parametrizing the special Kähler manifold $C_{\rm IIB}^{(2,1)}$.

Following ref. [195], we can describe the complex-structure moduli space $C_{\rm IIB}^{(2,1)}$ by the periods of the holomorphic three-form Ω over a canonical basis of $H_3(W_\psi,\mathbb{Z})$, which in our case, since $b_3=4$, can be taken to be $(\gamma^M)=\left(A^0,A^1,B_0,B_1\right)^{\rm T}$ with the intersections

$$A^\Lambda\cap B_\Gamma=\delta_\Gamma^\Lambda\,,\qquad A^\Lambda\cap A^\Gamma=0\,,\qquad B_\Lambda\cap B_\Gamma=0\,. \tag{5.166}$$

The dual cohomology basis is denoted by $\left(\alpha_\Lambda,\beta^\Gamma\right)$ and obeys

$$\int_{A^\Lambda}\alpha_\Gamma=\delta^\Lambda{}_\Gamma\,,\qquad \int_{B_\Lambda}\beta^\Gamma=-\delta^\Gamma{}_\Lambda\,,\qquad \int_{A^\Lambda}\beta^\Gamma=\int_{B_\Lambda}\alpha_\Gamma=0\,. \tag{5.167}$$

The holomorphic 3-form Ω is given by

$$\Omega=\mathcal{X}^\Lambda\alpha_\Lambda-\mathcal{F}_{{\rm IIB},\Lambda}\beta^\Lambda\,, \tag{5.168}$$

where $\mathcal{X}^\Lambda$ and $\mathcal{F}_{{\rm IIB}\,\Lambda}$, which will be identified as the components of the holomorphic symplectic section

$$\Pi_{\rm IIB}(\psi)=\begin{pmatrix}\mathcal{X}^0\\ \mathcal{X}^1\\ \mathcal{F}_{{\rm IIB}\,0}\\ \mathcal{F}_{{\rm IIB}\,1}\end{pmatrix}\,, \tag{5.169}$$

are the periods of the holomorphic 3-form with respect to the canonical homology basis

$$\mathcal{X}^\Lambda=\int_{A^\Lambda}\Omega\,,\qquad \mathcal{F}_\Lambda=\int_{B_\Lambda}\Omega\,. \tag{5.170}$$

There are 4 periods, but the complex-structure manifold is one-dimensional and hence we can take the $\mathcal{F}_\Lambda$ to be holomorphic functions of the $\mathcal{X}^\Lambda$. Since Ω is defined up to rescalings $\Omega\to g(\psi)\Omega$, where $g(\psi)$ is a holomorphic function of the modulus ψ, we can take the $\mathcal{X}^\Lambda$ to be projective coordinates of the scalar manifold, and hence we end up with one complex coordinate, which is what we need in order to parametrize $C_{\rm IIB}^{(2,1)}$. Different choices of $g(\psi)$ can be understood as different gauge choices. In addition, the periods $\mathcal{F}_{{\rm IIB}\,\Lambda}$ can be expressed as derivatives of a single function $\mathcal{F}_{\rm IIB}$ of the $\mathcal{X}^\Lambda$:

$$\mathcal{F}_{{\rm IIB}\,\Lambda}=\frac{\partial\mathcal{F}_{\rm IIB}}{\partial\mathcal{X}^\Lambda}\,. \tag{5.171}$$

We will find later on that it is more natural to consider $\mathcal{F}_{\text{IIB}\,\Lambda}$ as the projective coordinates and the $\mathcal{X}^\Lambda$ given in terms of them. A good special coordinate in the large complex-structure limit is therefore provided by:

$$Z(\psi) = \frac{\mathcal{F}_{\text{IIB}\,0}(\psi)}{\mathcal{F}_{\text{IIB}\,1}(\psi)} \,. \tag{5.172}$$

It can be shown [198, 199] that the components of the holomorphic symplectic section of an $N=2, d=4$ supergravity theory have to obey a set of differential identities due to the properties of the special Kähler geometry. When the theory originates from a Calabi–Yau compactification, these identities are the Picard–Fuchs equations. In our case, there is only one fourth-order Picard–Fuchs equation associated with $\mathcal{W}$ [198, 200]

$$(1-\psi^5)\omega^{iv} - 10\psi^4\omega''' - 25\psi^3\omega'' - 15\psi^2\omega' - \psi\,\omega = 0\,. \tag{5.173}$$

and its 4 independent solutions $\omega_0, \omega_1, \omega_2, \omega_3$ can be identified with the 4 periods [196].

Eq. (5.173) is an ordinary differential equation with regular singular points at $\psi^5 = 0, 1, \infty$ and, hence, a system of solutions may be obtained following the method of Froebenius for such equations. At $\psi^5 = \infty$ one solution, ω_0, is given as a pure power series and the other three solutions $\omega_1, \omega_2, \omega_3$ contain logarithms, with powers 1, 2 and 3, respectively. At $\psi^5 = 0$ all four solutions are pure power series. We will not need the solutions at $\psi^5 = 1$.

The pure power series solution around $\psi^5 = \infty$ is

$$\omega_0(\psi) = \frac{1}{5\psi}\sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5\,(5\psi)^{5n}}\,, \qquad |\psi|>1\,, \qquad 0 \leq \text{Arg}(\psi) < \frac{2\pi}{5}\,. \tag{5.174}$$

This expression has been obtained with the choice of $g(\psi)$ normally used to study the (mirror) Landau–Ginzburg or Fermat limit $\psi \to 0$. An expression for ω_0 in the large complex structure limit can be obtained from the one above by a gauge transformation with $g(\psi) = 5\psi$ [200] that gets rid of the overall factor $(5\psi)^{-1}$. We will use this new gauge for both limits, since we have found no complications in using it in the Fermat limit $\psi \to 0$. In conclusion, we take ω_0 to be

$$\omega_0(\psi) = \sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5\,(5\psi)^{5n}}\,, \qquad |\psi|>1\,, \qquad 0 \leq \text{Arg}(\psi) < \frac{2\pi}{5}\,. \tag{5.175}$$

The solution around $\psi = 0$ can be obtained by analytical continuation of eq. (5.175):

$$\omega_0(\psi) = -\frac{1}{5}\sum_{m=1}^{\infty}\frac{\alpha^{2m}\Gamma\left(m/5\right)\left(5\psi\right)^m}{\Gamma\left(m\right)\Gamma^4\left(1-m/4\right)}\,, \qquad |\psi| < 1\,. \tag{5.176}$$

The 5 functions

$$\omega_k(\psi) \equiv \omega_0(\alpha^k\psi)\,, \qquad k = 0, \ldots, 4\,, \tag{5.177}$$

are also solutions, but one of them cannot be linearly independent: the ω_k obey a linear relation which turns out to be

$$\sum_{k=0}^{4}\omega_k = 0\,. \tag{5.178}$$

The expressions for the ω_k, $k = 1, \ldots, 4$ for $|\psi| > 1, 0 < \mathrm{Arg}(\psi) < \frac{2\pi}{5}$ are quite involved and can be found in appendix A.3.

To construct the holomorphic symplectic section Π_{IIB} we choose a set of four linearly independent solutions, that we combine into a vector $\hat{\omega}$ (also called the period vector on the Picard–Fuchs basis)

$$\hat{\omega} = -\left(\frac{2\pi i}{5}\right)^3\begin{pmatrix}\omega_2\\ \omega_1\\ \omega_0\\ \omega_4\end{pmatrix}\,, \tag{5.179}$$

and then define $\Pi_{IIB}(\psi)$ by

$$\Pi_{\mathrm{IIB}}(\psi) = M\,\hat{\omega} \qquad M = \begin{pmatrix}-1 & 0 & 8 & 3\\ 0 & 1 & -1 & 0\\ -3/5 & -1/5 & 21/5 & 8/5\\ 0 & 0 & -1 & 0\end{pmatrix}\,. \tag{5.180}$$

The Kähler potential is given by

$$e^{-\mathcal{K}} = i\left(\mathcal{X}^{*\Sigma}\mathcal{F}_{\mathrm{IIB}\,\Sigma} - \mathcal{X}^{\Sigma}\mathcal{F}^{*}_{\mathrm{IIB}\,\Sigma}\right) = \omega^{\dagger}\sigma\,\omega\,, \tag{5.181}$$

where

$$\sigma \equiv \frac{1}{5}\begin{pmatrix}0 & 1 & 3 & 1\\ -1 & 0 & 3 & 3\\ -3 & -3 & 0 & 1\\ -1 & -3 & -1 & 0\end{pmatrix}\,. \tag{5.182}$$

Eq. (5.181) is a very complicated function of ψ, hence some simplification limit is in order. It can be shown that in the large complex-structure limit (given by eq. (5.202)) $\psi \to \infty$ the Kähler potential is given by:

$$e^{-\mathcal{K}} = \left(\frac{2\pi}{5}\right)^3 \left(\frac{20}{3}\log^3|5\psi| + \frac{16}{5}\zeta(3)\right) . \tag{5.183}$$

From (5.183) we can compute the Kähler metric

$$\mathcal{G}_{\psi\psi^*} = \frac{15\left(-24\zeta\,(3)\log|5\psi| + 5\log^4|5\psi|\right)}{|\psi|^2\left(24\zeta\,(3) + 10\log^3|5\psi|\right)} . \tag{5.184}$$

We can expand (5.184) as to obtain:

$$\mathcal{G}_{\psi\psi^*} = \frac{3}{4|\psi|^2\log^2|5\psi|}\left(1 - \frac{48\zeta(3)}{25\log^3|5\psi|} + \cdots\right) . \tag{5.185}$$

We perform the change of variable

$$t \equiv -\frac{5}{2\pi i}\log(5\psi) \tag{5.186}$$

in order to make easier the comparison with the metric of the large-radius limit of type IIA on $\mathcal{M}$, which is obtained in the following section. The leading term of (5.185) becomes

$$\mathcal{G}_{tt^*} = \tfrac{3}{4}(\operatorname{Im} t)^{-2} , \tag{5.187}$$

which is, as we will see, the large-radius limit metric of the Kähler-structure moduli space, the scalar manifold of type IIA on $\mathcal{M}$.

A.2 Type IIA on the quintic $\mathcal{M}$ and mirror map

The low-energy effective theory of of type IIA superstring theory compactified on a Calabi–Yau manifold is $N=2$ supergravity coupled to $h^{(1,1)}$ vector multiplets and $h^{(2,1)}+1$ hypermultiplets. The prepotential in the large compactification radius limit is given by [195]

$$\mathcal{F}^0_{\mathrm{IIA}}(\mathcal{X}) = -\frac{1}{3!}\frac{\kappa^0_{ijk}\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0} , \qquad i,j,k = 1,\ldots,h^{(1,1)} . \tag{5.188}$$

where κ^0_{ijk} are the triple intersection numbers.

We take the compactification manifold to be quintic $\mathcal{M}$, hence $h^{(1,1)} = 1$ and $h^{(2,1)} = 101$. Since, as in the type IIB case, we are only interested in the vector multiplet moduli space, we set the hypermultiplets to zero and deal solely with the complex Kähler-structure moduli space $C_{\text{IIA}}^{(1,1)}$, which is a complex one-dimensional special Kähler manifold.

If we denote by e the generator of $H^2(\mathcal{M},\mathbb{Z})$, the only non-vanishing triple intersection number at tree level is

$$\kappa^0_{1,1,1} = \int_{\mathcal{M}} e \wedge e \wedge e = 5\,. \tag{5.189}$$

Then, in terms of the coordinate

$$t \equiv \mathcal{X}^1/\mathcal{X}^0 \tag{5.190}$$

and in the Kähler gauge $\mathcal{X}^0 = 1$, the Kähler potential is given by

$$\mathcal{K}^0_{\text{IIA}} = -\log\left[\tfrac{20}{3}(\text{Im}\, t)^3\right]. \tag{5.191}$$

The Kähler metric reads

$$\mathcal{G}^0_{tt^*} = \tfrac{3}{4}\,(\text{Im}\, t)^{-2}\ . \tag{5.192}$$

Comparing eqs. (5.187) and (5.192) we can see that the large complex-structure limit of the metric of $C_{\text{IIB}}^{(2,1)}$ agrees with the corresponding bare (uncorrected) quantities for $C_{\text{IIA}}^{(1,1)}$.

We are interested in how the loop corrections and worldsheet instanton corrections (we restrict ourselves to a two-derivative action) to eq. (5.188) affect non-extremal black-hole solutions. One can write the corrected prepotential [196] in the form $\mathcal{F}_{\text{IIA}} = \mathcal{F}^{\text{pert}}_{\text{IIA}} + \mathcal{F}^{\text{npert}}_{\text{IIA}}$, where $\mathcal{F}^{\text{pert}}_{\text{IIA}}$ denotes the perturbatively-corrected prepotential and $\mathcal{F}^{\text{npert}}_{\text{IIA}}$ denotes the exponentially small terms due to instanton corrections. They are given by:

$$\mathcal{F}^{\text{pert}}_{\text{IIA}} = \mathcal{F}^0_{\text{IIA}} + \mathcal{F}^{\text{loop}}_{\text{IIA}} = -\frac{5}{6}\frac{(\mathcal{X}^1)^3}{\mathcal{X}^0} - \frac{11}{4}(\mathcal{X}^1)^2 + \frac{25}{12}\mathcal{X}^0\mathcal{X}^1 - ik(\mathcal{X}^0)^2, \tag{5.193}$$

$$\mathcal{F}^{\text{npert}}_{\text{IIA}} = \sum_l n_l \,\text{Li}_3\left(e^{2\pi i l \mathcal{X}^1/\mathcal{X}^0}\right), \tag{5.194}$$

where

$$\mathrm{Li}_3(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^3}\,, \tag{5.195}$$

and n_k is the number of rational curves of degree k, and where we have defined the real numerical constant

$$k \equiv \frac{c}{2} \equiv \frac{25}{2\pi^3}\zeta(3)\,. \tag{5.196}$$

For large values of the quintic radius $\mathrm{Im}\, t \gg 1$, the non-perturbative contribution to the prepotential are exponentially small and can be ignored.

The type IIB theory compactified on $\mathcal{W}$ is related to the type IIA one compactified on $\mathcal{M}$ through the mirror map, which can be expressed as a symplectic transformation of the holomorphic symplectic section with matrix N given by [196]

$$\Pi_{\mathrm{IIA}} = \frac{\mathcal{F}_{\mathrm{IIB}\,1}}{\mathcal{X}^0} N \Pi_{\mathrm{IIB}}\,, \qquad N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}\,, \tag{5.197}$$

and the coordinate transformation

$$t = \frac{2\,(\omega_1 - \omega_0) + \omega_2 - \omega_4}{5\omega_0}\,, \tag{5.198}$$

where we are denoting the holomorphic symplectic section of the type IIA theory compactified on $\mathcal{W}$ by

$$\Pi_{\mathrm{IIA}}(\psi) = \begin{pmatrix} \mathcal{X}^0 \\ \mathcal{X}^1 \\ \mathcal{F}_{\mathrm{IIA}\,0} \\ \mathcal{F}_{\mathrm{IIA}\,1} \end{pmatrix}\,. \tag{5.199}$$

Consequently, at the supergravity level, both theories are the same theory in different coordinates and symplectic frames.

A.3 Large complex-structure limit

In this section we give the explicit expressions for the periods in region $|\psi| > 1, 0 \leq \mathrm{Arg}(\psi) < \frac{2\pi}{5}$, and we also obtain the large complex-structure limit [196]. The periods are given by:

$$\omega_j(\psi) = \sum_{r=0}^{3} \log^r(5\psi) \sum_{n=0}^{\infty} b_{jrn} \frac{(5\psi)!}{(n!)^5 (5\psi)^{5n}} , \qquad |\psi| > 1 , \tag{5.200}$$

where the coefficients are given by lengthy expressions that can be found in [196]. In the large complex-structure limit $\psi \to \infty$ we keep the first term in the pure power expansion of eq. (5.200). We can then write a vector of coefficients:

$$b_r = -\left(\frac{2\pi i}{5}\right)^3 \begin{pmatrix} b_{2r0} \\ b_{1r0} \\ b_{0r0} \\ b_{4r0} \end{pmatrix} , \tag{5.201}$$

in terms of which the large complex-structure limit of the period vector in Picard–Fuchs basis is written as:

$$\hat{\omega} \sim \sum_{r=0}^{3} b_r \log^r(5\psi) . \tag{5.202}$$

Eq. (5.202) is the starting point for obtaining the relevant quantities of the model in the limit $\psi \to \infty$.

A.4 A simpler prepotential

As already mentioned, for large values of the quintic radius $\mathrm{Im}\, t \gg 1$, the non-perturbative contributions to the prepotential are exponentially small, so $\mathcal{F}_{\mathrm{IIA}}^{\mathrm{npert}}$ of eq. (5.194) can be neglected. Taking into account just eq. (5.193), the holomorphic symplectic section is given by

$$\Pi_{\mathrm{pert}} = \begin{pmatrix} \mathcal{X}^0 \\ \mathcal{X}^1 \\ \frac{5}{6}\frac{(\mathcal{X}^1)^3}{(\mathcal{X}^0)^2} + \frac{25}{12}\mathcal{X}^1 - ic\mathcal{X}^0 \\ -\frac{5}{2}\frac{(\mathcal{X}^1)^2}{\mathcal{X}^0} - \frac{11}{2}\mathcal{X}^1 + \frac{25}{12}\mathcal{X}^0 \end{pmatrix} , \tag{5.203}$$

In the spirit of ref. [142], the symplectic (Peccei–Quinn) transformation

$$\hat{\mathcal{S}} \equiv \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbb{I}} & 0 \\ \hline 0 & -\frac{25}{12} & \\ -\frac{25}{12} & \frac{11}{2} & \mathbb{I} \end{array}\right) , \tag{5.204}$$

brings the section to the simpler form

$$\hat{\Pi}_{\rm pert} = \begin{pmatrix} \hat{\mathcal{X}}^0 \\ \hat{\mathcal{X}}^1 \\ \frac{5}{6}\frac{(\hat{\mathcal{X}}^1)^3}{(\hat{\mathcal{X}}^0)^2} - ic\hat{\mathcal{X}}^0 \\ -\frac{5}{2}\frac{(\hat{\mathcal{X}}^1)^2}{\hat{\mathcal{X}}^0} \end{pmatrix}, \tag{5.205}$$

which can be derived from the prepotential

$$\hat{\mathcal{F}}^{\rm pert}_{\rm quintic} = -\frac{5}{6}\frac{(\hat{\mathcal{X}}^1)^3}{\hat{\mathcal{X}}^0} - \frac{i}{2}c(\hat{\mathcal{X}}^0)^2 \,. \tag{5.206}$$

The geometry of the scalar manifold in the corrected case is quite different from $SL(2,\mathbb{R})/U(1)$ of the pure t^3 model. It is not a homogeneous space and the conditions that $\mathrm{Im}\, t$ has to satisfy are also different: the Kähler potential is given by

$$e^{-\mathcal{K}_{\rm pert}} = \tfrac{20}{3}(\mathrm{Im}\, t)^3 + 2c \tag{5.207}$$

and the fact that $\mathcal{K}$ must be real implies

$$\mathrm{Im}\, t > -\left(\tfrac{3}{10}c\right)^{1/3} \,. \tag{5.208}$$

The Kähler metric is given by

$$\mathcal{G}_{t\bar{t}} = \frac{15\ \mathrm{Im}\, t\left[-3c + 5(\mathrm{Im}\, t)^3\right]}{[3c + 10(\mathrm{Im}\, t)^3]^2} \,. \tag{5.209}$$

For it to be positive definite, we need to demand $\mathrm{Im}\, t\ \left[-3\epsilon c + 5(\mathrm{Im}\, t)^3\right] > 0$. This condition, together with eq. (5.208), gives the domain of definition for $\mathrm{Im}\, t$:

$$\mathrm{Im}\, t \in \left(-\left(\frac{3c}{10}\right)^{1/3}, 0\right) \cup \left(\left(\frac{3c}{5}\right)^{1/3}, \infty\right) \,. \tag{5.210}$$

From the point of view of the supergravity theory, this is the only condition that the scalar needs to satisfy for the solution to be well defined. If, however, this supergravity is to be seen as an effective description of the underlying superstring theory, there are more conditions to be met by t. In particular, the prepotential (5.206) is an expansion around $t \to \infty$, valid only inside the radius of convergence:

$$\operatorname{Im} t > \operatorname{Im} t(1)\,, \tag{5.211}$$

where $t(\psi)$ is the mirror map, ψ is the modulus of the mirror related theory, and the conifold point is assumed to be at $\psi = 1$.

6. ON ANHARMONIC STABILISATION EQUATIONS FOR BLACK HOLES

Pietro Galli, Kevin Goldstein[◇]*, and Jan Perz*[†]

[◇]*National Institute for Theoretical Physics (NITHeP), School of Physics and Centre for Theoretical Physics, University of the Witwatersrand. WITS 2050, Johannesburg, South Africa.*

[†]*Istituto Nazionale di Fisica Nucleare, Sezione di Padova Via Marzolo, 8, 35131 Padova, Italy.*

Publication reference: JHEP **03** (2013) 036 arXiv:1211.7295 [hep-th].

Abstract

We investigate the stabilisation equations for sufficiently general, yet regular, extremal (supersymmetric and non-supersymmetric) and non-extremal black holes in four-dimensional $N = 2$ supergravity using both the H-FGK approach and a generalisation of Denef's formalism. By an explicit calculation we demonstrate that the equations necessarily contain an anharmonic part, even in the static, spherically symmetric and asymptotically flat case.

1 Introduction

Among the efforts to systematise the construction of non-supersymmetric black hole solutions in four-dimensional $N = 2$ supergravity one can discern two intersecting lines of research: on the one hand the generalisation [32,39] of Denef's formalism [28], applicable to stationary extremal black holes, and the H-FGK approach [35,176] for static extremal and non-extremal solutions on the other. In distinct ways each arrives to a set of relationships, which we shall call stabilisation equations, between duality-covariant combinations of physical degrees of freedom and ansätze for spatial functions $H^M(\mathbf{x})$. These relationships remain unchanged for various types of black holes, which means that all black hole solutions (supersymmetric, extremal, non-extremal) in a given model take the same form in terms of the functions H and only the functions themselves vary.

For supersymmetric extremal solutions, functions H are known to be harmonic, with poles corresponding to physical magnetic and electric charges carried by the black hole [94, 148]. In the context of the H-FGK formalism a harmonic ansatz has been used also for non-supersymmetric, static, spherically symmetric extremal black holes, whereas for their non-extremal counterparts a hyperbolic (exponential) ansatz has been employed [36,38,40,144,187].

In this short note we examine the exhaustiveness of these ansätze in the static, spherically symmetric case, i.e. with the metric of the form

$$\mathrm{d}s^2 = -\mathrm{e}^{2U(\tau)}\mathrm{d}t^2 + \mathrm{e}^{-2U(\tau)}\left(\frac{r_0^4}{\sinh^4(r_0\tau)}\mathrm{d}\tau^2 + \frac{r_0^2}{\sinh^2(r_0\tau)}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right), \tag{6.1}$$

providing in the process some portions of a dictionary between the generalised formalism of Denef and the H-FGK formulae.

2 Non-superysmmetric extremal black holes

In [39], to which we refer the reader for the description of the general setup and whose numerical conventions we follow (occasionally adopting some of the notation from the H-FGK literature), the generating single-center underrotating solution [85] for the metric warp factor $U(\mathbf{x})$ and the complex scalars $z^a(\mathbf{x})$ from n_v vector multiplets in models with cubic prepotentials has been recast in the form of stabilisation equations

$$2\,\mathrm{Im}\left(\mathrm{e}^{-U-\mathrm{i}\alpha}\,\Omega^M(z,\bar{z})\right) = H^M(\mathbf{x})\,, \tag{6.2}$$

where $\Omega(z,\bar{z})$ is the covariantly holomorphic symplectic section (period vector) of special geometry, α is a phase and the single superscript M is understood to run over $2(n_{\mathrm{v}}+1)$ components, otherwise indexed with subscripts and superscripts ^{0,a} and $_{0,a}$. H was written in [39] as a sum of harmonic functions and a ratio of harmonic functions (note the minus sign in the zeroth magnetic component):

$$(H^M) = \big(h^0 - p^0\tau, 0; 0, h_a + q_a\tau\big) + \left(0, 0; \frac{b + J\tau^2\cos\theta}{h^0 - p^0\tau}, 0\right), \tag{6.3}$$

where τ is a radial coordinate and the anharmonic part persists also in the absence of rotation ($J=0$), when the solution reduces to that of [25, 64]. Although the quotient form of H was later confirmed by [178], one could nonetheless wonder whether the anharmonic part is necessary (as opposed to being an artifact of the specific rewriting with the particular coefficients used) and whether the solutions that seem to require it do not carry NUT charge (which would render them only locally asymptotically flat).

To answer these questions we solve the spherically symmetric, static case of the t^3 model[1] for the charge configuration $(Q^M) = (0, p^1; q_0, 0)$, dual to that in eq. (6.3) for $n_{\mathrm{v}} = 1$. It is easiest to start with the equation (2.27) of [35],[2] which corresponds to the equation of motion for the warp factor:

$$\frac{1}{2}\frac{\partial \log \mathrm{e}^{-2U}}{\partial H^M}(\ddot{H}^M - r_0^2 H^M) + \left(\frac{\dot{H}^M H_M}{2\mathrm{e}^{-2U}}\right)^2 = 0\,, \tag{6.4}$$

where the dot denotes differentiation with respect to τ, the index M has been lowered with the symplectic form $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and where

$$\mathrm{e}^{-2U} = \sqrt{-\tfrac{10}{3}(H^1)^3 H_0 - (H^0H_0)^2 - 2H^0H^1H_0H_1 + \tfrac{1}{3}(H^1H_1)^2 + \tfrac{8}{45}H^0(H_1)^3}\,. \tag{6.5}$$

As remarked in [35], when $r_0 = 0$ (extremal black holes) and upon assuming,

$$\dot{H}^M H_M = 0\,, \tag{6.6}$$

[1] Normalization: $\Omega_0 = \frac{5}{6}(\Omega^1)^3/(\Omega^0)^2$. (Here, unlike in [39], Ω_0 stands for one of the components of Ω.)

[2] This equation can also be derived from a further generalisation of Denef's formalism to non-extremal solutions. Although this derivation does not appear in the literature, we do not include the rather technical details here since they are not directly relevant to our discussion.

(6.4) reduces to

$$\frac{\partial \log \mathrm{e}^{-2U}}{\partial H^M} \ddot{H}^M = 0\,, \tag{6.7}$$

which can be solved by harmonic functions $\ddot{H}^M = 0$. The harmonic function solution sets each term in (6.7) to zero individually.

One may however relax the assumption (6.6), setting $H_1 = 0$, taking only the two functions corresponding to non-vanishing charges to be harmonic with arbitrary coefficients ($H^1 = A^1 + B^1\tau$, $H_0 = A_0 + B_0\tau$) and leaving H^0 unspecified. Eq. (6.4) then becomes

$$(A_0 + B_0\tau)^2 H^0 \ddot{H}^0 - \frac{1}{2}\left(B_0 H^0 - (A_0 + B_0\tau)\dot{H}^0\right)^2 = 0\,, \tag{6.8}$$

a model-dependent differential equation for $H^0(\tau)$, whose solution reads

$$H^0 = \pm\left(c_1\sqrt{A_0 + B_0\tau} + \frac{c_2}{\sqrt{A_0 + B_0\tau}}\right)^2\,, \tag{6.9}$$

with constants of integration c_1, c_2. The remaining equations of motion fix the coefficients as either

$$c_1 = 0\,, \qquad B_0 = -q_0\,, \qquad B^1 = p^1\,, \tag{6.10}$$

in exact analogy with eq. (6.3), or

$$c_1 = 0\,, \qquad c_2 = 0\,, \qquad B_0 = 0\,, \qquad B^1 = 0\,, \tag{6.11}$$

which leads to a (doubly extremal) solution with constant scalars. The other parameters and the overall sign in (6.9) are determined by the asymptotic boundary conditions. In particular, for the non-constant solution (we suppress the superscript 1 on the single scalar $z = \Omega^1/\Omega^0$):

$$\operatorname{sgn}(H^0) = -\operatorname{sgn}(\operatorname{Re} z_\infty)\,, \qquad c_2^2 = \left|\frac{\operatorname{Re} z_\infty}{\operatorname{Im} z_\infty}\right|. \tag{6.12}$$

3 Non-extremal black holes

For $r_0 \neq 0$ and with the additional assumption $\dot{H}^M H_M = 0$, eq. (6.4) reduces to

$$\frac{\partial \log \mathrm{e}^{-2U}}{\partial H^M} (\ddot{H}^M - r_0^2 H^M) = 0\,, \tag{6.13}$$

which can be solved by hyperbolic functions $\ddot{H}^M = r_0^2 H^M$. Searching for a more general solution we take, similarly to the extremal case above, $H_1 = 0$, $H^1 = A^1 \cosh(r_0\tau) + \frac{B^1}{r_0}\sinh(r_0\tau)$ and $H_0 = A_0\cosh(r_0\tau) + \frac{B_0}{r_0}\sinh(r_0\tau)$. H^0 is then determined from

$$\begin{aligned}
&\Big(A_0\cosh(r_0\tau) + \tfrac{B_0}{r_0}\sinh(r_0\tau)\Big)^2 H^0(\ddot{H}^0 - r_0^2 H^0)\\
&-\frac{1}{2}\Big[\Big(r_0 A_0\sinh(r_0\tau) + B_0\cosh(r_0\tau)\Big)H^0\\
&\qquad\qquad - \Big(A_0\cosh(r_0\tau) + \tfrac{B_0}{r_0}\sinh(r_0\tau)\Big)\dot{H}^0\Big]^2 = 0
\end{aligned} \tag{6.14}$$

and turns out to be

$$H^0 = \pm\frac{\Big(c_1\cosh(r_0\tau) + \frac{c_2}{r_0}\sinh(r_0\tau)\Big)^2}{A_0\cosh(r_0\tau) + \frac{B_0}{r_0}\sinh(r_0\tau)}\,. \tag{6.15}$$

Numerical tests indicate that the analytical solution for the coefficients

$$B_0 = c_2 A_0\,, \qquad B^1 = c_2 A^1\,, \tag{6.16}$$

$$\begin{aligned}
c_2 = &\pm c_1\left(4c_1^4 + 10A_1^3 A_0\right)^{-1}\big[75(A^1)^4(A_0)^2(p^1)^2 - 45c_1^4 A^1 A_0(p^1)^2\\
&+ 45c_1^4(A^1)^2 p^1 q_0 + 25(A^1)^6(q_0)^2 + 9c_1^8 r_0^2 + 60(A^1)^3 A_0 c_1^4 r_0^2\\
&+ 100(A^1)^6(A_0)^2 r_0^2\big]^{\frac{1}{2}}
\end{aligned} \tag{6.17}$$

is the only admissible solution. Such coefficients lead to a constant scalar, which must take the extremal attractor value. It follows that $c_1 = 0$, so ultimately $H^0 = 0$, the solution is given purely in terms of hyperbolic functions (and compatible with the condition $\dot{H}^M H_M = 0$).

4 Discussion and conclusions

In spite of their different origins, the non-supersymmetric extension of Denef's approach and the H-FGK formalism both match the scalar degrees of freedom with the vector part of the action in the same way, one that respects duality covariance. The corresponding non-differential stabilisation equations have consequently (up to the differences in conventions) identical form. The fact that the H-functions differ stems from the specific additional assumptions made in the H-FGK literature, namely that $\dot{H}^M H_M = 0$ and that the rest of eq. (6.4) vanishes term by term.

The condition $\dot{H}^M H_M = 0$ in the BPS context is synonymous with the absence of NUT charge [79]. For the non-supersymmetric extremal solution discussed here this cannot be the case, since all the equations of motion are satisfied with the static metric (6.1), whose NUT charge is 0. Indeed, ref. [39], eq. (3.28) showed that the spatial Hodge dual of the spatial exterior derivative of the one-form ω encoding the relevant part of the metric depends on two terms,

$$\star_0 \, \mathbf{d}\omega = \langle \mathbf{d}H, H \rangle - 2\mathrm{e}^{-2U} \boldsymbol{\eta} \,, \tag{6.18}$$

the first of which directly generalises $\dot{H}^M H_M$. (The second term measures the non-closure of the fake electromagnetic field strength two-form introduced therein.) We see that for the left-hand side to be zero it suffices that, rather than each part vanishes (as happens for BPS solutions), the two terms only cancel each other, as in the extremal example discussed above.[3]

It is worth pointing out that the inverse harmonic part of the functions H is essential for the non-trivial behaviour of the real parts of z^a, usually referred to as axions. We have checked that the constant c_2 (or c in [25], B in [64] and b in eq. (6.3)), originating here from the product $H^0 H_0$, cannot be consistently extracted from the other constants when H^M are purely harmonic (the system equations that one would write does not admit any solution), even if none of them were a priori vanishing.

The non-extremal case remains less lucid. The existence of non-hyperbolic solutions has been postulated in [177], but the non-hyperbolic part of the natural generalization of the extremal anharmonic solution in our example turned out to be zero. Arguably, however, by setting some of the H^M to be harmonic or hyperbolic functions we might not yet have searched for the most general extremal or non-extremal solution.

A *Comparison of conventions*

Some of the original symbols have been replaced with those used here to make the meaning of the expressions clearer. Comparison with the respective papers provides a dictionary. $\hat{\Omega} = \mathrm{e}^{-U-\mathrm{i}\alpha}\Omega(z,\bar{z})$.

[3] Cf. also [42] for the discussion of gauge dependence of the condition $\dot{H}^M H_M = 0$.

	ref. [39]	ref. [35] (H-FGK)	here
metric signature	$(-,+,+,+)$	$(+,-,-,-)$	$(-,+,+,+)$
$\tau \in$	$(0,\infty)$	$(0,-\infty)$	$(0,\infty)$
physical scalars	z^a	Z^i	z^a
vector super- and subscript	$I=0,a$	$\Sigma=0,i$	not used
single index	not used	$M={}^{\Sigma},{}_{\Sigma}$	M
H-functions	$2\,\mathrm{Im}\,\hat{\Omega}=\mathcal{J}$	$\mathrm{Im}\,\hat{\Omega}^M=H^M$	$2\,\mathrm{Im}\,\hat{\Omega}^M=H^M$
symplectic form	$\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$
warp factor	$\mathrm{e}^{-2U}=\mathrm{i}\hat{\Omega}^M\bar{\hat{\Omega}}_M$	$\mathrm{e}^{-2U}=-\frac{\mathrm{i}}{2}\hat{\Omega}^M\bar{\hat{\Omega}}_M$	$\mathrm{e}^{-2U}=\mathrm{i}\hat{\Omega}^M\bar{\hat{\Omega}}_M$
poles of BPS H	$\Gamma\tau$	$-\frac{Q^M}{\sqrt{2}}\tau$	$Q^M\tau$

Note the symplectic form hidden in the expression for the warp factor.

In this paper by "stabilisation equations" we mean $\mathrm{Im}\,\hat{\Omega} \propto H$, whereas the H-FGK papers use that term for the relations between the real and imaginary parts of $\hat{\Omega}$: $\mathrm{Re}\,\hat{\Omega} = \mathrm{Re}\,\hat{\Omega}(\mathrm{Im}\,\hat{\Omega})$.

7. THE FREUDENTHAL GAUGE SYMMETRY OF THE BLACK HOLES OF $N = 2, D = 4$ SUPERGRAVITY

Pietro Galli, Patrick Meessen[†]*,*
and Tomás Ortín[◇]*,*

[†]*HEP Theory Group, Departamento de Física, Universidad de Oviedo*
Avda. Calvo Sotelo s/n, 33007 Oviedo, Spain

[◇]*Instituto de Física Teórica UAM/CSIC*
C/ Nicolás Cabrera, 13–15, C.U. Cantoblanco, 28049 Madrid, Spain.

Publication reference: *JHEP* **05** (2013) 011, arXiv: 1211.7296 [hep-th].

Abstract

We show that the representation of black-hole solutions in terms of the variables H^M, which are harmonic functions in the supersymmetric case, is non-unique due to the existence of a local symmetry in the effective action. This symmetry is a continuous (and local) generalization of the discrete Freudenthal transformations, initially introduced for the black-hole charges, and can be used to rewrite the physical fields of a solution in terms of entirely different-looking functions.

The FGK formalism developed in Ref. [18] reduces the problem of finding single, static, charged, spherically-symmetric black-hole solutions of a generic 4-dimensional theory of gravity coupled to a number of Abelian vectors $A^{\Lambda}{}_{\mu}$ and scalars ϕ^i (without scalar potential) to the simpler problem of finding solutions of a dynamical system whose dynamical variables are just the metric function $U(\tau)$ and the scalar fields $\phi^i(\tau)$; the evolution parameter τ corresponds to a radial coordinate in the black hole spacetime metric. This dramatic simplification allowed the authors of Ref. [18] to derive the very important result, valid for the extremal black-hole solutions of any of these theories (including all the 4-dimensional ungauged supergravity theories), relating the attractor values of the scalars on the event horizon with the entropy through the so-called black-hole potential. We will refer to this famous result as the *FGK theorem*.

Following these results, most of the work in this field has focused on extremal black holes (supersymmetric and non-supersymmetric) since they can be characterized, to a large extent, by the possible attractors and the entropy which, in many supersymmetric theories with large enough duality groups, can be determined by purely algebraic methods.

The FGK formalism was not used for the explicit construction of the extremal solutions, though. The dynamical system is simpler than the original equations but it is still highly non-linear and very complicated. The supersymmetric extremal solutions were constructed by methods based on the study of the consistency conditions of the Killing spinor equations. Even though the form of these solutions is known, showing that they solve the equations of motion of the FGK formalism is not a simple task. Non-supersymmetric extremal solutions have received a lot of attention in the last few years: there are more of these than supersymmetric ones and, furthermore, they have a richer structure. A first-order formalism has been constructed for them starting from the FGK dynamical system and a lot has been learned about the possible attractors, entropies etc., see *e.g.* Refs. [23, 39]. However, not many explicit solutions have been constructed since the first-order equations are not easy to integrate.

Non-extremal black-hole solutions have been left untouched by these developments since the FGK theorem does not apply to them: one needs to construct the explicit solution in order to compute the entropy, the temperature and the dependence of the very important *non-extremality* parameter r_0 on the physical constants, *i.e.* mass, electric and magnetic charges and the values of the scalars at infinity. In Ref. [38] a general ansatz for non-extremal black holes of ungauged $N = 2, d = 4$ supergravity was proposed and it was shown that using this ansatz the equations of motion of the FGK formalism can be solved at least for some simple theories[1]. Non-extremal solutions interpolate between different extremal solutions, supersymmetric and non-supersymmetric alike, that can be recovered by taking the extremal limit. This provides a new method for constructing the extremal non-supersymmetric solutions.

The *hyperbolic ansatz* proposed in Ref. [38] was based on the assumption that all the black-hole solutions of a given theory have exactly the same expression in terms of some functions $H^M(\tau)$, called *seed functions*. Different solutions correspond to different profiles for the seed functions, since they satisfy different equations. For supersymmetric solutions, the functions $H^M(\tau)$ will just be harmonic functions (linear in the coordinate τ). For non-extremal solutions, Ref. [38] proposed that the seed functions $H^M(\tau)$ should be linear combinations of hyperbolic functions. The hyperbolic ansatz was known to be valid in the few non-extremal solutions known to the literature [171, 201]. Furthermore, the expression of the physical fields in terms of the $H^M(\tau)$ was known to remain the same after the gauging of global symmetries [152].

The assumption that the black hole solutions have the same form in terms of the seed functions was proven in the formulation of the H-FGK formalism for $N = 2, d = 4$ supergravity theories, developed in Refs. [35, 176]: this formalism is obtained from the standard FGK one by a change of variables, the new variables being, precisely, the H^Ms mentioned above[2]. The very existence of the change of variables in all $N = 2, d = 4$ theories proves the assumption. However, the new formulation has additional advantages: since the new variables are, somehow, the "right" variables, finding new solutions and general results (attractor theorems, first-order flow equations etc.) becomes much simpler [40][3]. In particular, it is extremely easy to prove that the supersymmetric extremal black-hole solutions with harmonic H^Ms are solutions of the equations of motion; the situation w.r.t. extremal

[1] A generalization of the FGK formalism for higher-dimensional theories as made in Ref. [88], where a similar ansatz was shown to work in a simple $N = 2$, $d = 5$ supergravity theory.

[2] This formulation is clearly related to the real formulation of local special geometry of Ref. [202].

[3] There is also an H-FGK formulation for black holes and black strings of $N = 2, d = 5$ supergravity [35,98,144]. The derivation of the attractor theorem, first-order flow equations etc. has been done in Ref. [36].

non-supersymmetric black hole solutions is more complicated.

There are, however, some loose ends in these developments: in Ref. [25, 99] an extremal non-supersymmetric solution for cubic models was constructed in which one of the $H^M(\tau)$s, rather than being harmonic, has been shown in Ref. [39] to be the inverse of a harmonic function. Ratios of harmonic functions have been later on discussed and confirmed in Ref. [41, 178]. On the other hand, the general study performed in [40] suggests that in extremal black holes, supersymmetric or not, all the H^Ms should be harmonic[4]. Furthermore, the hyperbolic ansatz is used together with a simplifying constraint on the variables H^M which arises quite naturally in the supersymmetric case [79], but which has no justification in the non-supersymmetric cases, both extremal an non-extremal. The non-harmonic solutions of Refs. [25, 39, 41, 99, 178] do not satisfy that constraint.

In this paper we take a first step towards the clarification of the situation by showing how the description of a solution in terms of the variables H^M is not unique. We are going to show the existence of a gauge symmetry in the 4-dimensional H-FGK formalism that acts on the variables H^M in a highly non-trivial and non-linear way, but that preserves the physical fields of the black-hole solution: the metric function $U(\tau)$ and the complex scalar fields $Z^i(\tau)$. This symmetry does not preserve the above-mentioned constraint and, as we are going to see, it can relate a configuration of the H^Ms that does not satisfy it to another configuration that does: both configurations, however, describe the same physical black-hole solution. Whether the transformed H^M that do satisfy the constraint are harmonic is more difficult to prove in general and we will study this problem in another publication.

An interesting aspect of the gauge symmetry that we have discovered is that it is based on a generalization of the Freudenthal duality transformation discovered in Ref. [182] and generalized in the context of $\mathcal{N} = 8, d = 4$ supergravity and generalized to $\mathcal{N} \geq 2, d = 4$ supergravities in Ref. [181]. The original Freudenthal transformation is a discrete transformation that acts on the symplectic vector of magnetic and electric charges of a given theory[5] but one can define the same action on any other symplectic vector of the same theory and, in particular on the variables H^M. As we will show, the discrete transformations are a particular case of a continuous

[4] Observe that the hyperbolic ansatz always gives harmonic functions in the extremal limit.

[5] The transformation depends on the particular theory under consideration.

local symmetry of the H-FGK.

We start by reviewing in depth the H-FGK formalism for $N=2, d=4$ theories in section (1). In section (2) we discuss the discrete Freudenthal transformations and in section (3) we show that the HFGK action has a Freudenthal gauge symmetry. In section (4) we discuss the interplay of the Freudenthal gauge symmetry with the constraint, identifying the latter as a gauge fixing condition. Finally, in Sec. (5) we present our conclusions and discuss, briefly, the implications of the local Freudenthal symmetry for the extremal solutions.

1 *The H-FGK formalism for $N=2$, $d=4$ supergravity revisited*

The action of all ungauged $N=2, d=4$ supergravity theories coupled to n vector multiplets takes the form[6]

$$
\begin{aligned}
I[g_{\mu\nu}, A^{\Lambda}{}_{\mu}, Z^{i}] = \int d^{4}x\sqrt{|g|}\,\Big\{ & R + 2\mathcal{G}_{ij^{*}}\partial_{\mu}Z^{i}\partial^{\mu}Z^{*\,j^{*}} + 2\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}F^{\Sigma\,\mu\nu} \\
& -2\,\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}{}_{\mu\nu}\star F^{\Sigma\,\mu\nu}\Big\}\,,
\end{aligned}
\tag{7.1}
$$

where $i,j=1,\ldots,n$ and $\Lambda,\Sigma=0,1,\ldots,n$. The scalar-dependent Kähler metric $\mathcal{G}_{ij^{*}}$ and period matrix $\mathcal{N}_{\Lambda\Sigma}$ are related by supersymmetry and can be derived, in general, from a holomorphic prepotential function $\mathcal{F}(\mathcal{X})$ homogeneous of degree 2 in the coordinates $\mathcal{X}^{\Lambda}$ or, equivalently, from a canonically normalized, covariantly holomorphic symplectic section $(\mathcal{V}^{M}) = \left(\begin{smallmatrix}\mathcal{L}^{\Lambda}\\ \mathcal{M}_{\Lambda}\end{smallmatrix}\right)$. Here $M,N,\ldots$ are $(2n+2)$-dimensional symplectic indices and we use the symplectic metric $(\Omega_{MN}) \equiv \left(\begin{smallmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{smallmatrix}\right)$ and $\Omega^{MP}\Omega_{NP}=\delta^{M}{}_{N}$ to lower and rise the symplectic indices according to the convention

$$
\mathcal{V}_{M} = \Omega_{MN}\mathcal{V}^{N}\,, \qquad \mathcal{V}^{M} = \mathcal{V}_{N}\Omega^{NM}\,. \tag{7.2}
$$

The metrics of all the single, static, 4-dimensional black-hole solutions to these theories can be put in the form

$$
\begin{aligned}
ds^{2} &= e^{2U}dt^{2} - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}}\,,\\
\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} &= \frac{r_0^{4}}{\sinh^{4} r_0\tau}d\tau^{2} + \frac{r_0^{2}}{\sinh^{2} r_0\tau}d\Omega^{2}_{(2)}\,,
\end{aligned}
\tag{7.3}
$$

where r_0 is the so-called *non-extremality parameter* and $U(\tau)$ the metric function that characterizes a particular solution[7]. Assuming that all the

[6] We will follow the notation and conventions of Ref. [35].

[7] More information about this metric can be found in Ref. [38].

fields are static and spherically symmetric, so that they only depend on the radial coordinate τ, the action (7.1) reduces to the FGK effective action [18]

$$I_{\rm FGK}[U,Z^i] = \int d\tau \left\{(\dot{U})^2 + \mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*} - e^{2U}V_{\rm bh}(Z,Z^*,\mathcal{Q})\right\}\,, \tag{7.4}$$

which has to be supplemented by the Hamiltonian constraint

$$(\dot{U})^2 + \mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*} + e^{2U}V_{\rm bh}(Z,Z^*,\mathcal{Q}) = r_0^2\,. \tag{7.5}$$

In the above formulae $V_{\rm bh}(Z,Z^*,\mathcal{Q})$ is the so-called *black-hole potential* and is given by

$$-V_{\rm bh}(Z,Z^*,\mathcal{Q}) \;=\; -\tfrac{1}{2}\mathcal{M}_{MN}(\mathcal{N})\mathcal{Q}^M\mathcal{Q}^N\,; \tag{7.6}$$

$\mathcal{Q}^M$ is the $(2n+2)$-dimensional symplectic vector of electric q and magnetic p charges $(\mathcal{Q}^M) = \binom{p^\Lambda}{q_\Lambda}$ and $\mathcal{M}_{MN}(\mathcal{N})$ is the symmetric, symplectic matrix defined by

$$\mathcal{M}_{MN}(\mathcal{N}) \equiv \begin{pmatrix} I+RI^{-1}R & -RI^{-1} \\ & \\ -I^{-1}R & I^{-1} \end{pmatrix}\,, \qquad R\equiv {\rm Re}\mathcal{N}\,, \quad I\equiv {\rm Im}\,\mathcal{N}\,. \tag{7.7}$$

Observe that since there is no explicit τ dependence in the effective action (7.4), the corresponding Hamiltonian must take a constant value: the Hamiltonian constraint (7.5) fixes this *a priori* unconstrained value to be r_0^2.

The change of variables that brings us to the H-FGK formalism is inspired in the general form of the timelike supersymmetric solutions of these theories obtained by analyzing the consistency of the Killing spinor equations (see *e.g.* Ref. [151]): given an $\mathcal{N}=2$, $d=4$ theory with canonical symplectic section $\mathcal{V}^M$, introducing a complex variable X with the same Kähler weight as $\mathcal{V}^M$, we can define the real Kähler-neutral symplectic vectors

$$\mathcal{R}^M \;\equiv\; {\rm Re}\left(\mathcal{V}^M/X\right)\,, \qquad \mathcal{I}^M \;\equiv\; {\rm Im}\left(\mathcal{V}^M/X\right)\,. \tag{7.8}$$

The components $\mathcal{R}^M$ can be expressed in terms of the $\mathcal{I}^M$ by solving a set of algebraic equations commonly called the stabilization equations [17] (although this name is used with a different meaning in part of the literature), but to which we shall refer henceforth, for reasons that will become clear in the following and to avoid confusion, as the *Freudenthal duality equations*. The functions $\mathcal{R}^M(\mathcal{I})$ are characteristic of each theory, but they are always homogeneous of first degree in the $\mathcal{I}^M$.

Given the fact that, in supersymmetric solutions, the $\mathcal{I}^M$ are harmonic functions, it is customary to relabel these variables as

$$H^M \equiv \mathcal{I}^M\,, \qquad \tilde{H}^M \equiv \mathcal{R}^M\,. \tag{7.9}$$

Given those functions we can define the *Hesse potential* $\mathsf{W}(H)$ [29,35,176]

$$\mathsf{W}(H) \equiv \langle \tilde{H} \mid H \rangle \equiv \tilde{H}_M H^M\,, \tag{7.10}$$

which is homogeneous of second degree in H^M. The relation between $\tilde{H}^M$ and H^M can be inverted and the Hesse potential can also be written as $\mathsf{W}(\tilde{H})$; from the homogeneity of W one can deduce that

$$\tilde{H}_M = \tfrac{1}{2}\frac{\partial \mathsf{W}}{\partial H^M} \equiv \tfrac{1}{2}\partial_M \mathsf{W}\,, \qquad H^M = \tfrac{1}{2}\frac{\partial \mathsf{W}}{\partial \tilde{H}_M}\,. \tag{7.11}$$

Of special importance to the H-FGK formalism is the symmetric symplectic matrix $\mathcal{M}_{MN}(\mathcal{F})$ which is obtained by replacing in the expression (7.7) the period matrix $\mathcal{N}_{\Lambda\Sigma}$ by

$$\mathcal{F}_{\Lambda\Sigma} \equiv \frac{\partial^2 \mathcal{F}(\mathcal{X})}{\partial \mathcal{X}^\Lambda \partial \mathcal{X}^\Sigma}\,, \tag{7.12}$$

where $\mathcal{F}(\mathcal{X})$ is the prepotential of the theory; the relation between them can be seen to be

$$\mathcal{M}_{MN}(\mathcal{F}) = -\mathcal{M}_{MN}(\mathcal{N}) - 2\mathsf{W}^{-1}\left(H_M H_N + \tilde{H}_M \tilde{H}_N\right). \tag{7.13}$$

From the fundamental properties of the matrix $\mathcal{M}(\mathcal{F})$, namely

$$\begin{aligned} \tilde{H}_M &= -\mathcal{M}_{MN}(\mathcal{F}) H^N\,, \qquad & d\tilde{H}_M &= -\mathcal{M}_{MN}(\mathcal{F}) dH^N\,, \\ H_M &= \mathcal{M}_{MN}(\mathcal{F}) \tilde{H}^N\,, \qquad & dH_M &= \mathcal{M}_{MN}(\mathcal{F}) d\tilde{H}^N\,, \end{aligned} \tag{7.14}$$

one can infer that

$$\mathcal{M}_{MN}(\mathcal{F}) = -\tfrac{1}{2}\frac{\partial^2 \mathsf{W}}{\partial H^M \partial H^N} = \tfrac{1}{2}\frac{\partial^2 \mathsf{W}}{\partial \tilde{H}^M \partial \tilde{H}^N}\,, \tag{7.15}$$

this equation can be rewritten using eqs. (7.11) as

$$\frac{\partial \tilde{H}_N}{\partial H^M} = \Omega_{MP}\Omega_{NQ}\frac{\partial H^Q}{\partial \tilde{H}_P}\,, \tag{7.16}$$

which is equivalent to saying that $\mathcal{M}$ is a symplectic matrix.

Eq. (7.15) tells us that the Hesse potential W is closely related to the prepotential and is to be considered a *real prepotential*.

Observe that the above discovered hessianity implies that $\partial_P\mathcal{M}_{MN}(\mathcal{F}) = \partial_{(P}\mathcal{M}_{MN)}(\mathcal{F})$, whereas the homogeneity implies

$$0 \;=\; H^P\partial_P\mathcal{M}_{MN}(\mathcal{F}) \;=\; \tilde{H}^P\partial_P\mathcal{M}_{MN}(\mathcal{F})\;. \tag{7.17}$$

Now, using general properties of Special Geometry and the above properties one can rewrite the effective action (7.4) and the Hamiltonian constraint (7.5) entirely in terms of the new variables H^M [35]:

$$-I_{\text{H-FGK}}[H] = \int d\tau\,\left\{\tfrac{1}{2}g_{MN}\dot{H}^M\dot{H}^N - V\right\}\,, \tag{7.18}$$

$$-r_0^2 = \tfrac{1}{2}g_{MN}\dot{H}^M\dot{H}^N \;+\; V\;, \tag{7.19}$$

where we have defined the H-dependent metric

$$g_{MN} \;\equiv\; \partial_M\partial_N\log\mathsf{W} - 2\frac{H_MH_N}{\mathsf{W}^2} \;=\; \frac{\partial_M\partial_N\mathsf{W}}{\mathsf{W}} - 2\frac{H_MH_N}{\mathsf{W}^2} - 4\frac{\tilde{H}_M\tilde{H}_N}{\mathsf{W}^2}\,, \tag{7.20}$$

and the potential

$$\begin{aligned} V(H) &= \left\{-\tfrac{1}{4}\partial_M\partial_N\log\mathsf{W} + \frac{H_MH_N}{\mathsf{W}^2}\right\}\mathcal{Q}^M\mathcal{Q}^N \\ &= \left\{-\tfrac{1}{4}g_{MN} + \tfrac{1}{2}\frac{H_MH_N}{\mathsf{W}^2}\right\}\mathcal{Q}^M\mathcal{Q}^N\,. \end{aligned} \tag{7.21}$$

The relation of this potential to the black-hole potential (7.6) is given by

$$V_{\text{bh}} \;=-\,\mathsf{W}\,V\,. \tag{7.22}$$

2 Discrete Freudenthal transformations

The relation between the tilded and untilded variables can be understood as a duality transformation $H^M \to \tilde{H}^M$ which can be iterated if we define $\tilde{\tilde{H}}^M \equiv \tilde{H}^M(\tilde{H})$. Using the properties in Eqs. (7.11–7.17), we find that this duality is an anti-involution, *e.g.*

$$\tilde{\tilde{H}}^M \;=\; -H^M\,. \tag{7.23}$$

It is not difficult to see that the duality transformation is just the generalization to $\mathcal{N}=2$, $d=4$ supergravity theories made in Ref. [181] of the *Freudenthal duality* introduced in Ref. [182] in the context of $\mathcal{N}=8$, $d=4$ supergravity. The same operation can be performed on any symplectic vector of a given theory and, in particular, on the charge vector $\mathcal{Q}$.

In Ref. [181] it was shown that the entropy and the critical points of the black-hole potential are invariant under Freudenthal duality. We will recover this result later as a particular case of the invariance of the H-FGK system under *local Freudenthal rotations*.

The variables that we have just defined are related to the physical variables of the FGK formalism U, Z^i by [35][8]

$$e^{-2U} \equiv \mathsf{W}(H) = \tilde{H}_M H^M \,, \qquad Z^i \equiv \frac{\tilde{H}^i + iH^i}{\tilde{H}^0 + iH^0} \,. \tag{7.24}$$

We can immediately see that the physical variables are invariant under the above Freudenthal duality transformations, *i.e.*

$$e^{-2U}(\tilde{H}) = e^{-2U}(H) \,, \qquad Z^i(\tilde{H}) = Z^i(H) \,, \tag{7.25}$$

It is interesting to study how the central charge changes under Freudenthal duality: first, we rewrite the central charge, whose definition is $\mathcal{Z}(\phi, \mathcal{Q}) \equiv \langle \mathcal{V} \mid \mathcal{Q} \rangle$ in the form

$$\mathcal{Z}(\phi, \mathcal{Q}) = \frac{e^{i\alpha}}{\sqrt{2\mathsf{W}(H)}} (\tilde{H}_M + iH_M)\mathcal{Q}^M \,, \tag{7.26}$$

where $e^{i\alpha}$ is the phase of X and satisfies the equation [151]

$$\dot{\alpha} = \mathsf{W}^{-1} \, \dot{H}^M H_M - \mathcal{Q}_\star \,, \tag{7.27}$$

where $\mathcal{Q}_\star$ is the pullback of the Kähler connection 1-form

$$\mathcal{Q}_\star = \tfrac{1}{2i} \dot{Z}^i \partial_i \mathcal{K} + \text{c.c.} \tag{7.28}$$

Under discrete Freudenthal duality transformations, $\mathsf{W}(H)$, the scalars and the Kähler potential are invariant. α is also invariant and

$$(\tilde{H}_M + iH_M)' = -i(\tilde{H}_M + iH_M) \,, \tag{7.29}$$

[8] The expression for the scalars is not unique (only up to reparametrizations). The expression we give is, however, convenient and simple.

which implies that

$$\mathcal{Z}'(\phi, \mathcal{Q}) = -i\mathcal{Z}(\phi, \mathcal{Q})\,, \tag{7.30}$$

but its absolute value will remain invariant.

Observe that when the Freudenthal transformations are non-linear (which is the general case), if we transform a supersymmetric solution, which must have harmonic H^Ms of the form

$$H^M = A^M - \tfrac{1}{\sqrt{2}}\mathcal{Q}^M\tau\,, \tag{7.31}$$

we will obtain non-harmonic H^M and the transformed solution couldn't possibly be supersymmetric. We must remember, however, that all the physical fields are invariant, whence their supersymmetry properties must also remain invariant. This implies that the variables H^M cannot immediately be identified with those appearing in the analysis of the Killing spinor equations: this is possible only up to discrete Freudenthal transformations.

The near-horizon limit of the transformed H^Ms is dominated by the Freudenthal dual of the charges $\mathcal{Q}^M$, defined in Refs. [181,182], namely

$$\tilde{\mathcal{Q}}^M \;\equiv\; -\tfrac{1}{2}\,\Omega^{MN}\,\frac{\partial \mathsf{W}(\mathcal{Q})}{\partial \mathcal{Q}^N}\,. \tag{7.32}$$

3 Local Freudenthal rotations

In the change of variables taking us to the H-FGK formalism, we have gone from a formulation based on $2n+1$ real variables, namely U and the Z^i, to one which is based on $2n+2$ variables, whence we obtained an over-complete formulation. This suggests that there should be a local symmetry in the H-FGK formalism allowing the elimination of one of its degrees of freedom. The variables H^M, on the other hand, transform linearly under the duality group (embedded in $\mathrm{Sp}(n+1;\mathbb{R})$), as follows from its definition.

The looked-for gauge symmetry can be found by observing that the metric g_{MN} is singular: using the properties (7.14–7.17) it is easy to show that it always admits an eigenvector with zero eigenvalue, namely[9]:

$$\tilde{H}^M g_{MN} = 0\,. \tag{7.34}$$

[9] For the sake of completeness we also quote the relation

$$g_{MN}H^N = -2\tilde{H}_M/\mathsf{W} \quad\Rightarrow\quad g_{MN}H^M H^N = -2\,. \tag{7.33}$$

The equations of motion in the H-FGK formalism are

$$\frac{\delta I_{\text{H-FGK}}}{\delta H^M} = g_{MN}\ \ddot{H}^N\ +\ [PQ, M]\ \ \dot{H}^P \dot{H}^Q\ +\ \partial_M V = 0\,, \tag{7.35}$$

where, as g_{MN} is not invertible, we have used the Christoffel symbol of the first kind, *i.e.*

$$[PQ, M]\ \equiv\ \partial_{(P} g_{Q)M}\ -\ \tfrac{1}{2}\partial_M g_{PQ}\,. \tag{7.36}$$

Using the properties (7.14–7.17) it is not difficult to show that

$$\left.\begin{aligned}[PQ, M]\ \tilde{H}^M = 0\\ \tilde{H}^M \partial_M V = 0\end{aligned}\right\} \xrightarrow{\quad\text{so that}\quad} \tilde{H}^M \frac{\delta I_{\text{H-FGK}}}{\delta H^M} = 0\,. \tag{7.37}$$

This is a constraint that relates the equations of motion of the H-FGK formalism. This kind of constraints arise in systems with gauge symmetries, as a consequence of Noether's second theorem and it is a *gauge identity*. Indeed, multiplying the constraint by an arbitrary infinitesimal function $f(\tau)$ and integrating over τ we find that Eq. (7.37) implies

$$\delta_f I_{\text{H-FGK}} = \int d\tau \delta_f H^M \frac{\delta I_{\text{H-FGK}}}{\delta H^M} = 0\,, \tag{7.38}$$

where we have defined the local infinitesimal transformations

$$\delta_f H^M \equiv f(\tau)\tilde{H}^M\,. \tag{7.39}$$

As one can expect from a gauge invariance, this transformation leaves invariant the physical variables of the FGK formalism U, Z^i. To check it, it is enough to use

$$\delta_f \tilde{H}^M \equiv -f(\tau) H^M\,, \tag{7.40}$$

which follows from Eq. (7.23) and Eqs. (7.24).

The finite gauge transformations can be obtained by exponentiating the infinitesimal ones:

$$\delta_f H^M \equiv f(\tau)\pounds_K H^M \quad \longrightarrow \quad H'^M = e^{f(\tau)\pounds_K} H^M \ \text{ where }\ K^M(H) = \tilde{H}^M\,. \tag{7.41}$$

It is not difficult to see that the finite transformations are

$$\left\{\begin{aligned} H'^M &= \cos f\ H^M - \sin f\ \Omega^{MN}\tilde{H}_N\,,\\ \tilde{H}'_M &= -\sin f\ \Omega_{MN} H^N + \cos f\ \tilde{H}_M\,.\end{aligned}\right. \tag{7.42}$$

By defining the complex variables $\mathcal{H}^M \equiv \tilde{H}^M + iH^M$ we can write the transformation as

$$\mathcal{H}'^M = e^{if(\tau)}\mathcal{H}^M \,. \tag{7.43}$$

Using this form of the transformation and expressing the scalars and the metric function in the form

$$e^{-2U} = \mathsf{W}(H) = \tfrac{i}{2}\mathcal{H}_M\mathcal{H}^{*\,M} \,, \qquad Z^i \equiv \mathcal{H}^i/\mathcal{H}^0 \,, \tag{7.44}$$

the invariance of the physical fields under this gauge symmetry is paramount.

A direct proof of the invariance of the H-FGK effective action is also desirable: the invariance of the kinetic term, *i.e.* $\frac{1}{2}g_{MN}\dot{H}^M\dot{H}^N$, follows from the identities

$$(\tilde{H}_M\dot{H}^M)' = \tilde{H}_M\dot{H}^M \,, \qquad \dot{\tilde{H}}^M\mathcal{M}_{MN}(\mathcal{F}) = \dot{H}_N \,, \\ \dot{H}^M\mathcal{M}_{MN}(\mathcal{F}) = -\dot{\tilde{H}}_N \,, \tag{7.45}$$

which can be derived from Eqs. (7.14). The invariance of the potential $V(H)$ follows from Eq. (7.17).

The existence of this symmetry does not help in solving the equations of motion as the Noether charge associated to the invariance under the global Freudenthal rotations vanishes identically:

$$\mathsf{Q} = \delta_f H^M \frac{\partial L}{\partial H^M} \sim f\tilde{H}^M g_{MN}\dot{H}^N = 0 \,. \tag{7.46}$$

We have already said that the origin of this gauge symmetry is the introduction of one additional degree of freedom in the passage from the FGK to the H-FGK formalism. Had the original FGK formulation contained the full complex variable $X = e^{U+i\alpha}$ instead of just U, the change of variables would actually have been much simpler; alas, the phase α is completely absent from the FGK effective action. The local Freudenthal symmetry is associated to this absence, which allows to change α arbitrarily leaving everything else invariant. Indeed, from Eq. (7.27) that defines α, we can easily see that

$$\delta_f\dot{\alpha} = -\dot{f} \,. \tag{7.47}$$

On the other hand, the Freudenthal gauge symmetry can be made manifest as follows: first, observe that the metric

$$G_{MN}(H) \equiv \partial_M\partial_N \log\mathsf{W} - 2(1+\varepsilon)\frac{H_M H_N}{\mathsf{W}} \,, \qquad \varepsilon = \pm 1 \,, \tag{7.48}$$

always admits $K^M(H) = \tilde{H}^M$ as a Killing vector. Then, consider the action

$$- I_{\text{ungauged}}[H] = \int d\tau \left\{ \tfrac{1}{2} G_{MN} \dot{H}^M \dot{H}^N - V \right\} , \tag{7.49}$$

which has a global Freudenthal symmetry generated by $\delta H^M = f \tilde{H}^M$ with $\dot{f} = 0$. To gauge the Freudenthal symmetry, we just have to replace in this action the derivatives with respect to τ by the covariant derivatives

$$\begin{aligned} \dot{H}^M &\to \mathfrak{D} H^M \equiv \dot{H}^M + A \tilde{H}^M , \\ \dot{\tilde{H}}^M &\to \mathfrak{D} \tilde{H}^M \equiv \dot{\tilde{H}}^M - A H^M , \end{aligned} \tag{7.50}$$

which transform covariantly under the infinitesimal transformations Eq. (7.40)

$$\begin{aligned} \delta_f \mathfrak{D} H^M &= f \mathfrak{D} \tilde{H}^M , \\ \delta_f \mathfrak{D} \tilde{H}^M &= - f \mathfrak{D} H^M , \end{aligned} \tag{7.51}$$

if the 1-form A transforms as

$$\delta_f A = - \dot{f}(\tau) . \tag{7.52}$$

The action

$$- I_{\text{gauged}}[H, A] = \int d\tau \left\{ \tfrac{1}{2} G_{MN} \mathfrak{D} H^M \mathfrak{D} H^N - V \right\} , \tag{7.53}$$

is manifestly invariant under local Freudenthal rotations and equivalent to the effective H-FGK action Eq. (7.18) as one can see by integrating out the auxiliary field A: its equation of motion is solved by

$$A = \frac{H_N \dot{H}^N}{\mathsf{W}} , \tag{7.54}$$

and, upon this substitution

$$G_{MN} \mathfrak{D} H^M \mathfrak{D} H^N = \left(G_{MN} + 2\varepsilon \frac{H_M H_N}{\mathsf{W}} \right) \dot{H}^M \dot{H}^N = g_{MN} \dot{H}^M \dot{H}^N . \tag{7.55}$$

The choice $\varepsilon = +1$, which leads to $G_{MN} = 2\mathsf{W}^{-1} \mathcal{M}_{MN}(\mathcal{N})$ is, perhaps, the most natural since the same metric would then occur in the kinetic term and in the potential. It follows that we can rewrite the effective action Eq. (7.18) and the Hamiltonian constraint Eq. (7.19) in the suggestive form

$$I_{\text{H-FGK}}[H] = \int d\tau \left\{ V(H, \sqrt{2}\,\mathfrak{D}H) + V(H, \mathcal{Q}) \right\} , \tag{7.56}$$

$$r_0^2 = V(H, \sqrt{2}\,\mathfrak{D}H) - V(H, \mathcal{Q}) , \tag{7.57}$$

with

$$\mathfrak{D}H^M = \dot{H}^M + \frac{H_N \dot{H}^N}{\mathsf{W}} \tilde{H}^M . \tag{7.58}$$

Finally, it is worth noting that this Freudenthal gauge theory is unrelated to the one constructed in Ref. [203].

4 Unconventional solutions and Freudenthal gauge freedom

If we contract the equations of motion (7.35) with H^P and use the homogeneity properties of the different terms and the Hamiltonian constraint Eq. (7.19), we find a useful equation

$$\tilde{H}_M \left(\ddot{H}^M - r_0^2 H^M \right) + \frac{(\dot{H}^M H_M)^2}{\mathsf{W}} \;=\; 0 , \tag{7.59}$$

which corresponds to that of the variable U in the FGK formulation.

In the supersymmetric (hence, extremal) case, the constraint

$$\dot{H}^M H_M = 0 , \tag{7.60}$$

enforcing the absence of NUT charge must be satisfied, in agreement with the assumption of staticity of the metric [79]. Using this constraint the above equation takes the form

$$\tilde{H}_M \left(\ddot{H}^M - r_0^2 H^M \right) = 0 , \tag{7.61}$$

and can be solved in the extremal case by assuming that the H^M are linear in τ, whence they are harmonic, and in the non-extremal case by assuming that the H^M are linear combinations of hyperbolic functions of $r_0 \tau$ (the hyperbolic ansatz). The solutions that one can get with these assumptions have been intensively studied in Ref. [40].

The constraint Eq. (7.60) is not preserved by the local Freudenthal symmetry: a small calculation gives

$$\delta_f (\dot{H}^M H_M) \;=\; -\dot{f}\mathsf{W} , \tag{7.62}$$

which can be integrated straightforwardly to a finite rotation, namely

$$(\dot{H}^M H_M)' = -\dot{f}\mathsf{W} + \dot{H}^M H_M\,. \tag{7.63}$$

This equation implies that given a configuration H^M with $\dot{H}^M H_M \neq 0$, we can find another configuration H'^M with $\dot{H}'^M H'_M = 0$ *describing exactly the same configuration of physical fields* by performing a finite local Freudenthal transformation with a parameter $f(\tau)$ satisfying

$$\dot{f} = \frac{\dot{H}^M H_M}{\mathsf{W}}\,. \tag{7.64}$$

This shows that it is always possible to impose the constraint Eq. (7.60) without loss of generality because it can be understood just as a good gauge-fixing condition.

5 Conclusions

The extremal static black-hole solutions of $\mathcal{N} = 2, d = 4$ supergravity constructed so far in the literature and written in terms of the variables H^M can be classified using two criteria: the harmonicity of the H^Ms and whether they satisfy the constraint $H_M \dot{H}^M = 0$ or not. Out of the four possible cases, represented in table (7.1), the equation of motion Eq. (7.59) excludes the one corresponding to the upper right corner. The upper left corner corresponds to the supersymmetric black-hole solutions and, as shown in Ref. [38], also to some non-BPS solutions as well. The lower-right corner corresponds to the extremal non-BPS solutions discovered in Refs. [25,39,41,99,178] and the lower-left corner does not correspond to any known solution.

In this paper we have shown that the representation of the solutions in terms of these variables is non-unique due to the presence of the local Freudenthal invariance. Furthermore, we have shown that this symmetry can be used to transform all the solutions in the lower-right corner to solutions in the left column. It is not yet clear whether they will be transformed into solutions in the upper or lower row although preliminary results in simple examples suggest that, typically, they will transformed into solutions in the lower-left corner. The form of the H^Ms in this class is probably quite complicated as they must satisfy the equation

$$\tilde{H}_M \ddot{H}^M = 0\,, \tag{7.65}$$

	$H_M\dot{H}^M = 0$	$H_M\dot{H}^M \neq 0$
$\ddot{H}^M = 0$	BPS and some non-BPS	no solutions
$\ddot{H}^M \neq 0$		some non-BPS

Tab. 7.1: Classification of the extremal static black-hole solutions of $\mathcal{N} = 2, d = 4$ supergravity according to their representation in terms of the variables H^M. It must be taken into account that they satisfy Eq. (7.59) with $r_0 = 0$.

and, at the same time, $\ddot{H}^M \neq 0$. Furthermore, solutions of this kind must be possible only in very special cases and only in some theories, as it happens for the solutions in the lower-right corner. Clearly, more work is needed to arrive at a complete understanding of the situation and to chart the space of extremal black-hole solutions of these theories. The non-extremal case is even more challenging. Work in these directions is in progress.

Part III

CONCLUSIONS

FINAL CONCLUSIONS

In this thesis we have studied black holes in $N = 2$ four-dimensional supergravity. In particular our results can be viewed as a step toward a final classification and understanding of the solutions of the theory. We have mainly concentrated on the non-BPS branch where the literature on explicit solutions is less extended with respect to the supersymmetric case. Such focus is reasonable if one aims to describe our real world that is non-supersymmetric since at low energies supersymmetry is broken.

First, we have discussed extremal, non-supersymmetric, multicenter black holes in models with cubic prepotential. We have adopted the approach of [28], developed for BPS multicenter solutions, and merged it with the first-order formulation in terms of the superpotential given in [22]. We have rewritten the effective action for stationary metrics as a sum of squares, generalizing the already known supersymmetric rewriting. We then derived differential (Laplacian) equations describing a system of multi black holes with vanishing bounding energy. The solutions for the scalars and the warp factor have the same structure as the usual single-center ones, but now they are expressed in terms of a sum of harmonic functions with poles in the fake charges of each center of the composite. In our construction, fake and physical charges must be related by a constant matrix (so that the corresponding fake field strength is a closed two-form). In order to satisfy this requirement, one has to choose electric or magnetic charge configurations and set the axions to zero. As a consequence, the constituents are mutually local, electric or magnetic, their position is unconstrained and the total angular momentum is zero. Our results agree with what was in the literature at the time of their publication but we are aware that they are not general enough to reproduce the bound stationary multicenter solutions of Bena et al. [33, 34, 85] (published later on). The latter have been obtained by a completely different approach based on the underlying 11-dimensional M-theory and a suitable compactification ansatz. It would be nice to find a way to reproduce them by the method we used.

A first attempt in this direction has been the treatment of chapter 3 (Ref. [39]). The rewriting of the action has been extended by allowing the fake field strength to be a two-form not necessarily closed. A new one-form has then appeared and the new degree of freedom associated with it turned out, for cubic models, to reflect the possibility of having non-vanishing axions. Since the equations of motion found look very complicated they have been faced by plugging in an ansatz and checking if they are satisfied. A successful analysis for single-center extremal static and rotating black holes has led to understand that harmonic functions are not enough to describe the entire non-BPS sector. We have discovered that in order to reproduce a generating single-center solution we have to write stabilization equations where ratios of harmonic functions must be considered. This result has been recently confirmed in [178] and proved right also in the H-FGK formalism (see the discussion of chapter 6 Ref. [41]). Due to the complexity of the calculation, when considering multicenter configurations, yet, we do not have a final solution, but we expect ratios of harmonic functions to be involved in its description.

In chapter 4 (Ref. [38]) we have studied non-extremal black holes. We have shown by explicit examples that it is possible to obtain non-extremal black holes by "deforming" known BPS solutions. The deformation procedure consists in shifting U to $U + r_0\tau$ and replacing, in the expression of the fields, the harmonic functions with exponential functions of the form $a_\alpha + b_\alpha\, \mathrm{e}^{2r_0\tau}$. The coefficients a_α and b_α are determined by requiring the equations of motion and the correct asymptotic conditions to be satisfied. The non-extremal black holes constructed in that way have turned out to continuously interpolate (by sending the extremality parameter r_0 to zero) between extremal BPS and non-BPS solutions so they may allow to find non-supersymmetric extremal black holes that cannot be worked out by standard methods. The first-order description of the evolution of the fields has been written down and the existence of a generalized superpotential proved true always. The analysis of the evaporation process has revealed that the final extremal status will be BPS or non-BPS depending only on the sign of the electromagnetic charges. The study of the macroscopic thermodynamics for non-extremal solutions has outlined that the product of the entropies of the inner and outer horizon gives the square of the extremal entropy. This is an interesting result we do not have yet a clear interpretation of but it may be that an analysis from the microscopic point of view may lead to a better understanding.

The non-extremal solutions obtained by the prescription just discussed found confirmation in the subsequent works of Ortín and collaborators [35–37] on the H-FGK formalism. For all models, after the initial condition (6.6) $\dot{H}^M H_M = 0$, a hyperbolic ansatz (equivalent to the exponential one but in a form easier to deal with) for the H-functions is sufficient to find non-extremal black holes. In chapter 5 (Ref. [40]) we have used this H-FGK formulation to find solutions of the t^3 model with quadratic quantum corrections. After working out in detail the formalism, we have presented new supersymmetric black-hole solutions carrying up to three charges and new non-extremal black-hole solutions with constant scalars and one or two charges. The latter are the non-extremal generalization of doubly-extremal black holes and can be completely defined by just knowing the critical points of the black-hole potential (it follows that they have only one possible extremal limit). Analysing our results, we have differentiated between solutions that can be continuously connected with those of the uncorrected t^3 model and "quantum" solutions which do not have a regular classical limit. When setting the quantum parameter to zero only black holes of the former category survive while the others become singular.

As we already mentioned, all the non-extremal black-hole solutions obtained so far are written in terms of hyperbolic functions which, in the extremal limit, always become harmonic. Nevertheless, it is clear from the results of chapter 3 (Ref. [39]) that this is not good enough to retrieve the entire non-BPS extremal scenario where, at least for cubic models, ratios of harmonic functions are necessary. In order to look for a generalization of the hyperbolic ansatz that could reproduce in the extremal limit all the known solutions, we have studied in chapter 6 (Ref. [41]) how ratios of harmonic functions appear in the H-FGK $r_0 = 0$ context. We have considered a magnetic charge configuration in the t^3 model, relaxed condition (6.6) and assumed the Hs corresponding to the non-vanishing physical charges to be harmonic while the one corresponding to the vanishing electric charge to be zero. The H-FGK equation relative to the equation of motion of the warp factor has become, in this way, a differential equation for just one variable and its direct integration has provided the wanted ratio of harmonic functions. In the light of this analysis, we have then tried to look for ratios of hyperbolic functions by the same recipe. Curiously enough, we have found that such functional form for the Hs does appear in a similar way as ratios of harmonic functions do but we have concluded that no non-extremal black-hole solution of this kind can exist since it does not satisfy the remaining equations of motion of the system. The generalization of the hyperbolic ansatz, if any, must be more complicated.

New additional insights to clarify the situation could come from analysing the symmetries of the H-FGK formalism. Aware of this in chapter 7 (Ref. [42]) we have discussed a global Freudenthal duality among the H-variables that exchanges H^M with $\tilde{H}^M$ leaving untouched the field solutions. This discrete symmetry reproduces the results of the literature but we have shown that it is only a particular case of a more general gauge transformation. The action of the theory written in the H-FGK formalism is in fact invariant under a local symmetry, that acts on the Hs like a kind of rotation with angle an arbitrary real function $f(\tau)$. So, applying this transformation to harmonic/hyperbolic Hs will transform them into complicated functions that also satisfy the equations of motion of the H-FGK system and describe the fields without changing their expression. The freedom we have in choosing the functional form of the H-variables is due to the additional variable one has to add passing from the FGK to the H-FGK formulation and can be gauge-fixed by requiring (6.6) to be satisfied. This last condition constrains the Hs to be always harmonic only in the BPS case. Some extremal, non-supersymmetric, black-hole solutions are written in terms of anharmonic functions, but the existence of non-extremal, non-hyperbolic, black-hole solutions is still unclear and should be addressed in future investigations.

The results exposed in this thesis have contributed to the analysis of black-hole solutions in $N = 2$, $D = 4$ supergravity but at the same time have opened new doors on still unexplored possibilities. For example, it could be worth trying to generalize the H-FGK formalism to stationary black holes. It could lead to non-extremal, multicenter solutions possibly interpolating between different extremal, multicenter black holes. A first indication that such configurations could exist may come from the fact that for both, single-center and multicenter solutions with parallel charge vectors (see chapter 2 Ref. [32]), a switch of sign in the poles of the harmonic functions (up to an eventual duality rotation) connects supersymmetric to non-supersymmetric solutions. For the single-center case, the corresponding non-extremal, interpolating black holes are now well known and the switch of sign just mentioned is “hidden” in the non-extremal expressions. Surely, the multicenter case is more complicated but, if something similar can be possible, it may be worth investigating it.

Another interesting question could be the application of the H-FGK formalism in $D > 5$ supergravity, or in extended supergravities. The path to follow is the one traced in [35] and begins with finding the correct change of variables that would rewrite the Lagrangian in terms of the functions H. In an analogous way, also gauged supergravity could be explored. Extremal and non-extremal (at least for simple models) black-hole solutions in Fayet-Iliopoulos, gauged, matter-coupled supergravity have been already constructed in different ways, like by writing first-order flow equations or by using the c-map (among the last publications see e.g. [176, 204]). The implementation of the H-FGK formalism in this context could provide (as in ungauged supergravity) an efficient method to study systematically new types of solutions.

CONCLUSIONES FINALES

En esta tesis hemos estudiado los agujeros negros de supergravedad $N = 2$ en cuatro dimensiones. En particular, nuestros resultados pueden ser considerados un paso importante hacia una comprensión y una clasificación final de las soluciones de la teoría. Nos hemos concentrado principalmente en el estudio del sector no BPS, ya que la literatura puede considerarse incompleta en comparación con la del caso supersimétrico. Este enfoque, además, se justifica porque se pretende describir el mundo real que es no supersimétrico debido a su baja energía. A bajas energías todas las supersimetrías están rotas.

En primer lugar, hemos discutido los agujeros negros extremos no supersimétricos en modelos con prepotencial cúbico. Hemos usado el método de [28], desarrollado para soluciones BPS con multi-centro, junto con el formalismo de primer orden en términos del superpotencial definido en [22]. Hemos reescrito la acción efectiva para una métrica estacionaria en forma de un cuadrado perfecto, generalizando la reescritura conocida para el caso supersimétrico, y hemos obtenido ecuaciones diferenciales de grado dos que describen un sistema de varios agujeros negros con energía de unión cero. Las soluciones para los escalares y el factor "warp" tienen la misma estructura que las de un solo centro, pero estas están escritas en términos de una suma de funciones armónicas, con polos en la carga falsa de cada centro del compuesto. En nuestra construcción, las cargas falsas y las físicas deben estar relacionadas por una matriz constante (de manera que el tensor correspondiente sea una dos-forma cerrada) y por eso se tiene que escoger una configuración de carga eléctrica o magnética y poner a cero los axiones. Como consecuencia, los constituyentes son todos eléctricos o magnéticos (mutuamente locales) y se pueden mover libremente en el espacio, dado que no hay restricciones sobre sus posiciones relativas y el momento angular total es nulo. Nuestros resultados concuerdan con los conocidos en el momento de la publicación, pero sabemos que no son bastante generales para reproducir las posteriores soluciones rotativas a multi-centro de Bena et al. [33, 34, 85]. Sin embargo, estas últimas han sido elaboradas con de un método completamente distinto, basado en la teoría subyacente en 11 dimensiones ("M-theory"). Sería interesante encontrar una manera de describirlas con nuestro método.

Un primer paso en esta dirección ha sido el análisis elaborado en el capítulo 3 (Ref. [39]) donde se ha reescrito la acción, permitiendo esta vez que la intensidad de campo falsa sea una dos-forma no necesariamente cerrada. De esta manera se ha tenido que introducir una nueva uno-forma que, para modelos con prepotencial cúbico, resulta estar asociada a la posibilidad de encontrar axiones no nulos. Siendo las ecuaciones de movimiento muy complicadas, hemos decidido resolverlas haciendo un ansatz para las ecuaciones de estabilización. De este modo hemos descubierto que si queremos reproducir una solución generadora para agujeros negros extremos, no BPS, con un solo centro, es necesario introducir fracciones de funciones armónicas. Este resultado ha sido recientemente confirmado en [178] mientras que en la discusión del capítulo 6 (Ref. [41]) se ha visto cómo aparecen también en el formalismo H. Debido a la complejidad del cálculo, todavía no tenemos una respuesta definitiva sobre los agujeros multi-centro, pero esperamos que algo parecido a fracciones de funciones armónicas también aparezca en este contexto.

En el capítulo 4 (Ref. [38]) hemos estudiado agujeros negros no extremos. Hemos mostrado con ejemplos específicos que es posible obtener agujeros negros no extremos deformando soluciones BPS ya conocidas. El procedimiento de deformación consiste en redefinir U como $U + r_0\tau$ y remplazar, en la expresión de los campos, las funciones armónicas con funciones exponenciales de la forma $a_\alpha + b_\alpha\, \mathrm{e}^{2r_0\tau}$. Los coeficientes a_α y b_α se determinan substituyendo nuestro ansatz en las ecuaciones de movimiento y resolviendo las resultantes ecuaciones algebraicas. Los grados de libertad que quedan se fijan requiriendo la satisfacción de ciertas condiciones asintóticas. Los agujeros negros así construidos interpolan de manera continua (tendiendo a cero el parámetro de extremidad) entre soluciones extremas BPS y no BPS y por lo tanto permiten en algunos casos encontrar nuevas soluciones extremas no supersimétricas. Además de encontrar las soluciones, hemos logrado escribir explícitamente las ecuaciones diferenciales de orden uno que las describen y en todos los casos hemos podido demostrar la existencia de un superpotencial generalizado. El estudio de la termodinámica macroscópica nos ha enseñado que el producto de las entropías asociadas con el horizonte interior y exterior es igual al cuadrado de la entropía en el límite extremo, mientras que el análisis del proceso de evaporación nos ha revelado que el estado extremo final (BPS o no BPS) depende sólo de las cargas electromagnéticas del agujero negro.

Las soluciones no extremas obtenidas por deformación de las BPS han sido confirmadas en los trabajos siguientes de Ortín y colaboradores acerca del formalismo H [35–37]. Para todos los modelos, después de imponer la condición inicial (6.6) $\dot{H}^M H_M = 0$, se ha mostrado que un ansatz hiperbólico (equivalente al exponencial pero en una forma más fácil de tratar) para las funciones H es suficiente para encontrar soluciones de agujero negro no extremos. En el capítulo 5 (Ref. [40]) hemos utilizado el formalismo H para estudiar las soluciones del modelo t^3 con correcciones cuadráticas de origen cuántico. Después de haber elaborado el formalismo en detalle, hemos presentado nuestras nuevas soluciones supersimétricas con hasta tres cargas y no supersimétricas con una o dos cargas y escalares constantes. Estas últimas son la generalización de los agujeros negros doblemente extrémales y basta conocer los puntos críticos del potencial de agujero negro para definirlas completamente (por esta razón tienen un solo límite extremo). Analizando nuestros resultados hemos distinguido las soluciones que se pueden conectar continuamente con las del modelo t^3 sin correcciones y las soluciones cuánticas, que no tienen ningún límite clásico. Sólo las soluciones pertenecientes a la primera categoría sobreviven sin hacerse singulares cuando se manda a cero el parámetro cuántico.

Como ya comentamos, todas las soluciones no extremas de agujero negro obtenidas hasta la fecha se escriben en términos de funciones hiperbólicas que, en el límite extremo, siempre se transforman en funciones armónicas. Sin embargo, de acuerdo con los resultados del capítulo 3 (Ref. [39]), no todas las soluciones extremas no BPS caben en esta descripción ya que hay algunas que necesitan la introducción de fracciones de funciones armónicas. En el capítulo 6 (Ref. [41]) hemos mostrado cómo aparecen las fracciones de funciones armónicas en el contexto del formalismo H cuando se relaja la condición (6.6) y se toma $r_0 = 0$. Si se considera una configuración de carga eléctrica (o magnética) y se toman como armónicas las Hs que le corresponden y como cero aquellas relativa a las cargas magnéticas (eléctricas) nulas, nos quedamos con una sola H indeterminada. Integrando la ecuación diferencial correspondiente a la ecuación de movimiento del factor "warp" se encuentra un cociente de funciones armónicas. De manera análoga, remplazando las Hs armónicas con Hs hiperbólicas, se puede demostrar que en el caso no extremo se obtiene un cociente de funciones hiperbólicas.

Sin embargo, mientras que en el caso extremo es posible resolver también todas las otras ecuaciones, en el caso no extremo, los cocientes de funciones hiperbólicas nos impiden llegar a una solución final. La generalización del ansatz hiperbólico, si existe, debe de ser más complicada.

Con el fin de aclarar la situación, hemos analizado en el capítulo 7 (Ref. [42]) las simetrías del formalismo H. Hemos delineado una dualidad global de tipo Freudenthal que intercambia H^M con $\tilde{H}^M$, dejando intacta la solución de los campos físicos. Posteriormente se ha demostrado que esta simetría discreta es sólo un caso particular de una transformación de “gauge” más general que actúa sobre las variables H como una especie de rotación con ángulo $f(\tau)$. Debido a que podemos escoger una función real cualquiera como $f(\tau)$, la forma funcional del las Hs podría ser de cualquier tipo. Esta libertad deriva de la variable que se tiene que añadir pasando del formalismo FGK al formalismo H para equiparar los grados de libertad en las dos formulaciones. Un buen “gauge-fixing” puede ser la imposición de la condición (6.6) que, para todas las configuraciones BPS, implica que las Hs sean armónicas. En el caso extremo no supersimétrico sabemos que formas funcionales enarmónicas son necesarias, mientras que aún carecemos de una comprensión completa para los agujeros negros no extremos.

Los resultados expuestos en esta tesis han contribuido al análisis de soluciones de agujero negro en supergravedad $N = 2$, $D = 4$, pero al mismo tiempo han indicado nuevas posibilidades que merecen ser exploradas en el futuro. Podría ser interesante intentar generalizar el formalismo H al caso estacionario o a la supergravedad $D > 5$ y/o $N > 2$. La manera de hacerlo sería la misma delineada en [35] y empezaría con encontrar el cambio correcto de variables que permita de reescribir el lagrangiano en términos de las variables H. En modo análogo, se podría probar a estudiar la supergravedad gaugeada, para la cual se han encontrado recientemente soluciones de agujero negro extremo y no extremo a través de ecuaciones de orden uno, o bien usando el mapa c (entre la últimas publicaciones ver [176, 204]). El desarrollo del formalismo H en estos otros contextos podría proporcionar un método sistemático y eficiente para estudiar nuevos tipos de agujeros negros.

BIBLIOGRAPHY

[1] J. Kormendy and D. Richstone, "Inward bound: The Search for supermassive black holes in galactic nuclei," *Ann.Rev.Astron.Astrophys.* **33** (1995) 581.

[2] A. King, "Black holes, galaxy formation, and the MBH - sigma relation," *Astrophys.J.* **596** (2003) L27–L30, `arXiv:astro-ph/0308342 [astro-ph]`.

[3] S. Gillessen, F. Eisenhauer, S. Trippe, T. Alexander, R. Genzel, F. Martins, and T. Ott, "Monitoring Stellar Orbits Around the Massive Black Hole in the Galactic Center," *ApJ* **692** (feb, 2009) 1075–1109, `arXiv:0810.4674`.

[4] A. Strominger, "Black holes: The harmonic oscillators of the 21st century." http://www.physics.mcgill.ca/seminars/online/PSC/strominger/.

[5] K. Schwarzschild, "Über das Gravitationsfeld eines Massenpunktes nach der Einstein'schen Theorie," *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* **1** (1916) 189–196.

[6] H. Reissner, "Über die Eigengravitation des elektrischen Feldes nach der Einstein'schen Theorie," *Annalen der Physik* **50** (1916) 106–120.

[7] G. Nordström, "On the energy of the gravitational field in Einstein's theory," *Verhandl. Koninkl. Ned. Akad. Wetenschap., Afdel. Natuurk., Amsterdam* **26** (1918) 1201–1208.

[8] E. T. Newman and A. I. Janis, "Note on the Kerr spinning particle metric," *J. Math. Phys.* **6** (1965) 915–917.

[9] E. T. Newman *et al.*, "Metric of a rotating, charged mass," *J. Math. Phys.* **6** (1965) 918–919.

[10] R. P. Kerr, "Gravitational field of a spinning mass as an example of algebraically special metrics," *Phys. Rev. Lett.* **11** (1963) 237–238.

[11] J. D. Bekenstein, "Black holes and entropy," *Phys.Rev.* **D7** (1973) 2333–2346.

[12] J. Bekenstein, "Statistical Black Hole Thermodynamics," *Phys.Rev.* **D12** (1975) 3077–3085.

[13] J. D. Bekenstein, "Generalized second law of thermodynamics in black hole physics," *Phys.Rev.* **D9** (1974) 3292–3300.

[14] S. Hawking, "Black hole explosions," *Nature* **248** (1974) 30–31.

[15] S. Ferrara, R. Kallosh, and A. Strominger, "N=2 extremal black holes," *Phys.Rev.* **D52** (1995) 5412–5416, `arXiv:hep-th/9508072 [hep-th]`.

[16] A. Strominger, "Macroscopic entropy of N=2 extremal black holes," *Phys.Lett.* **B383** (1996) 39–43, `arXiv:hep-th/9602111 [hep-th]`.

[17] S. Ferrara and R. Kallosh, "Supersymmetry and attractors," *Phys. Rev.* **D54** (1996) 1514–1524, `arXiv:hep-th/9602136`.

[18] S. Ferrara, G. W. Gibbons, and R. Kallosh, "Black holes and critical points in moduli space," *Nucl.Phys.* **B500** (1997) 75–93, `arXiv:hep-th/9702103 [hep-th]`.

[19] A. Sen, "Black hole entropy function and the attractor mechanism in higher derivative gravity," *JHEP* **09** (2005) 038, `arXiv:hep-th/0506177`.

[20] K. Goldstein, N. Iizuka, R. P. Jena, and S. P. Trivedi, "Non-supersymmetric attractors," *Phys. Rev.* **D72** (2005) 124021, `arXiv:hep-th/0507096`.

[21] P. K. Tripathy and S. P. Trivedi, "Non-supersymmetric attractors in string theory," *JHEP* **0603** (2006) 022, `arXiv:hep-th/0511117 [hep-th]`.

[22] A. Ceresole and G. Dall'Agata, "Flow equations for non-BPS extremal black holes," *JHEP* **03** (2007) 110, `arXiv:hep-th/0702088`.

[23] C. M. Miller, K. Schalm, and E. J. Weinberg, "Nonextremal black holes are BPS," *Phys. Rev.* **D76** (2007) 044001, arXiv:hep-th/0612308.

[24] L. Andrianopoli, R. D'Auria, E. Orazi, and M. Trigiante, "First order description of black holes in moduli space," *JHEP* **0711** (2007) 032, arXiv:0706.0712 [hep-th].

[25] G. L. Cardoso, A. Ceresole, G. Dall'Agata, J. M. Oberreuter, and J. Perz, "First-order flow equations for extremal black holes in very special geometry," *JHEP* **0710** (2007) 063, arXiv:0706.3373 [hep-th].

[26] J. Perz, P. Smyth, T. Van Riet, and B. Vercnocke, "First-order flow equations for extremal and non-extremal black holes," *JHEP* **03** (2009) 150, arXiv:0810.1528 [hep-th].

[27] L. Andrianopoli, R. D'Auria, E. Orazi, and M. Trigiante, "First order description of d=4 static black holes and the Hamilton–Jacobi equation," arXiv:0905.3938 [hep-th].

[28] F. Denef, "Supergravity flows and D-brane stability," *JHEP* **0008** (2000) 050, arXiv:hep-th/0005049 [hep-th].

[29] B. Bates and F. Denef, "Exact solutions for supersymmetric stationary black hole composites," *JHEP* **1111** (2011) 127, arXiv:hep-th/0304094 [hep-th].

[30] J. de Boer, S. El-Showk, I. Messamah, and D. Van den Bleeken, "Quantizing N=2 Multicenter Solutions," *JHEP* **0905** (2009) 002, arXiv:0807.4556 [hep-th].

[31] D. Gaiotto, W. Li, and M. Padi, "Non-supersymmetric attractor flow in symmetric spaces," *JHEP* **12** (2007) 093, arXiv:0710.1638 [hep-th].

[32] P. Galli and J. Perz, "Non-supersymmetric extremal multicenter black holes with superpotentials," *JHEP* **1002** (2010) 102, arXiv:0909.5185 [hep-th].

[33] I. Bena, S. Giusto, C. Ruef, and N. P. Warner, "Multi-center non-BPS black holes—the solution," *JHEP* **11** (2009) 032, arXiv:0908.2121 [hep-th].

[34] I. Bena, S. Giusto, C. Ruef, and N. P. Warner, "Supergravity Solutions from Floating Branes," *JHEP* **1003** (2010) 047, arXiv:0910.1860 [hep-th].

[35] P. Meessen, T. Ortín, J. Perz, and C. Shahbazi, "H-FGK formalism for black-hole solutions of N=2, d=4 and d=5 supergravity," *Phys.Lett.* **B709** (2012) 260–265, arXiv:1112.3332 [hep-th].

[36] P. Meessen, T. Ortín, J. Perz, and C. S. Shahbazi, "Black holes and black strings of N=2, d=5 supergravity in the H-FGK formalism," *JHEP* **1209** (2012) 001, arXiv:1204.0507 [hep-th].

[37] A. de Antonio Martín, T. Ortín, and C. S. Shahbazi, "The FGK formalism for black p-branes in d dimensions," *JHEP* **1205** (2012) 045, arXiv:1203.0260 [hep-th].

[38] P. Galli, T. Ortín, J. Perz, and C. S. Shahbazi, "Non-extremal black holes of N=2, d=4 supergravity," *JHEP* **1107** (2011) 041, arXiv:1105.3311 [hep-th].

[39] P. Galli, K. Goldstein, S. Katmadas, and J. Perz, "First-order flows and stabilisation equations for non-BPS extremal black holes," *JHEP* **1106** (2011) 070, arXiv:1012.4020 [hep-th].

[40] P. Galli, T. Ortin, J. Perz, and C. S. Shahbazi, "Black hole solutions of N=2, d=4 supergravity with a quantum correction, in the H-FGK formalism," arXiv:1212.0303 [hep-th].

[41] P. Galli, K. Goldstein, and J. Perz, "On anharmonic stabilisation equations for black holes," *JHEP* **1303** (2013) 036, arXiv:1211.7295 [hep-th].

[42] P. Galli, P. Meessen, and T. Ortin, "The Freudenthal gauge symmetry of the black holes of N=2,d=4 supergravity," arXiv:1211.7296 [hep-th].

[43] K. S. Thorne and C. W. Misner, *Gravitation*. W. H. Freeman, 1973.

[44] R. L. Arnowitt, S. Deser, and C. W. Misner, "Canonical variables for general relativity," *Phys.Rev.* **117** (1960) 1595–1602.

[45] R. Wald, *General Relativity*. University of Chicago Press, Chicago, 1984.

[46] A. Dabholkar, "Black hole entropy and attractors," *Class.Quant.Grav.* **23** (2006) S957–S980.

[47] R. M. Wald, "The thermodynamics of black holes," *Living Rev.Rel.* **4** (2001) 6, arXiv:gr-qc/9912119 [gr-qc].

[48] W. Israel, "Third law of black-hole dynamics: a formulation and proof," *Phys. Rev. Lett.* **57** (1986) 397–399.

[49] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," *Phys.Lett.* **B379** (1996) 99–104, arXiv:hep-th/9601029 [hep-th].

[50] A. W. Peet, "TASI lectures on black holes in string theory," arXiv:hep-th/0008241 [hep-th].

[51] P. Townsend, "Black holes: Lecture notes," arXiv:gr-qc/9707012 [gr-qc].

[52] R. Penrose, "Gravitational collapse: The role of general relativity," *Riv.Nuovo Cim.* **1** (1969) 252–276.

[53] S. Majumdar, "A class of exact solutions of Einstein's field equations," *Phys.Rev.* **72** (1947) 390–398.

[54] A. Papapetrou, "A Static solution of the equations of the gravitational field for an arbitrary charge distribution," *Proc.Roy.Irish Acad.(Sect.A)* **A51** (1947) 191–204.

[55] M. Bhat, S. Dhurandhar, and N. Dadhich, "Energetics of the kerr-newman black hole by the penrose process," *Journal of Astrophysics and Astronomy* **6** (1985) 85–100.

[56] G. F. R. Ellis and S. W. Hawking, *The Large scale structure of space-time*. Cambridge University Press, 1973.

[57] R. Ruffini and J. A. Wheeler, "Introducing the black hole," *Physics Today* **24** (1971) 30.

[58] R. Kallosh, M. Shmakova, and W. K. Wong, "Freezing of moduli by N=2 dyons," *Phys.Rev.* **D54** (1996) 6284–6292, arXiv:hep-th/9607077 [hep-th].

[59] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova, and W. K. Wong, "STU black holes and string triality," *Phys. Rev.* **D54** (1996) 6293–6301, arXiv:hep-th/9608059.

[60] K. Behrndt, G. L. Cardoso, B. de Wit, R. Kallosh, D. Lüst, *et al.*, "Classical and quantum N=2 supersymmetric black holes," *Nucl.Phys.* **B488** (1997) 236–260, arXiv:hep-th/9610105 [hep-th].

[61] S. Nampuri, P. K. Tripathy, and S. P. Trivedi, "On the stability of non-supersymmetric attractors in string theory," *JHEP* **08** (2007) 054, arXiv:0705.4554 [hep-th].

[62] S. Bellucci, S. Ferrara, M. Günaydin, and A. Marrani, "Charge orbits of symmetric special geometries and attractors," *Int. J. Mod. Phys.* **A21** (2006) 5043–5098, arXiv:hep-th/0606209.

[63] S. Ferrara and A. Marrani, "On the moduli space of non-BPS attractors for n=2 symmetric manifolds," *Phys. Lett.* **B652** (2007) 111–117, arXiv:0706.1667 [hep-th].

[64] E. G. Gimon, F. Larsen, and J. Simón, "Black holes in supergravity: the non-BPS branch," *JHEP* **01** (2008) 040, arXiv:0710.4967 [hep-th].

[65] S. P. Martin, "A Supersymmetry primer," arXiv:hep-ph/9709356 [hep-ph].

[66] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, "Progress Toward a Theory of Supergravity," *Phys.Rev.* **D13** (1976) 3214–3218.

[67] S. Deser and B. Zumino, "Consistent Supergravity," *Phys.Lett.* **B62** (1976) 335.

[68] P. C. West, "Supergravity, brane dynamics and string duality," arXiv:hep-th/9811101 [hep-th].

[69] Y. Tanii, "Introduction to supergravities in diverse dimensions," arXiv:hep-th/9802138 [hep-th].

[70] B. de Wit, "Supergravity," arXiv:hep-th/0212245 [hep-th].

[71] A. Van Proeyen, "Structure of supergravity theories," arXiv:hep-th/0301005 [hep-th].

[72] B. de Wit and A. Van Proeyen, "Potentials and Symmetries of General Gauged N=2 Supergravity: Yang-Mills Models," *Nucl.Phys.* **B245** (1984) 89.

[73] B. de Wit, P. Lauwers, and A. Van Proeyen, "Lagrangians of N=2 Supergravity - Matter Systems," *Nucl.Phys.* **B255** (1985) 569.

[74] B. Craps, F. Roose, W. Troost, and A. Van Proeyen, "What is special Kähler geometry?," *Nucl. Phys.* **B503** (1997) 565–613, `arXiv:hep-th/9703082`.

[75] E. Witten and D. I. Olive, "Supersymmetry Algebras That Include Topological Charges," *Phys.Lett.* **B78** (1978) 97.

[76] E. Bogomolny, "Stability of Classical Solutions," *Sov.J.Nucl.Phys.* **24** (1976) 449.

[77] M. Prasad and C. M. Sommerfield, "An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon," *Phys.Rev.Lett.* **35** (1975) 760–762.

[78] T. Ortín, "Extremality versus supersymmetry in stringy black holes," *Phys.Lett.* **B422** (1998) 93–100, `arXiv:hep-th/9612142 [hep-th]`.

[79] J. Bellorín, P. Meessen, and T. Ortín, "Supersymmetry, attractors and cosmic censorship," *Nucl.Phys.* **B762** (2007) 229–255, `arXiv:hep-th/0606201 [hep-th]`.

[80] G. Gibbons, "Supergravity vacua and solitons," `arXiv:1110.0918 [hep-th]`.

[81] D. Rasheed, "The rotating dyonic black holes of Kaluza–Klein theory," *Nucl. Phys.* **B454** (1995) 379–401, `arXiv:hep-th/9505038`.

[82] T. Matos and C. Mora, "Stationary dilatons with arbitrary electromagnetic field," *Class. Quant. Grav.* **14** (1997) 2331–2340, `arXiv:hep-th/9610013`.

[83] F. Larsen, "Rotating Kaluza–Klein black holes," *Nucl. Phys.* **B575** (2000) 211–230, `arXiv:hep-th/9909102`.

[84] K. Goldstein and S. Katmadas, "Almost BPS black holes," *JHEP* **0905** (2009) 058, `arXiv:0812.4183 [hep-th]`.

[85] I. Bena, G. Dall'Agata, S. Giusto, C. Ruef, and N. P. Warner, "Non-BPS Black Rings and Black Holes in Taub-NUT," *JHEP* **0906** (2009) 015, arXiv:0902.4526 [hep-th].

[86] I. Bena, S. Giusto, C. Ruef, and N. P. Warner, "A (running) bolt for new reasons," *JHEP* **11** (2009) 089, arXiv:0909.2559 [hep-th].

[87] L. Andrianopoli, R. D'Auria, P. Giaccone, and M. Trigiante, "Rotating black holes, global symmetry and first order formalism," *JHEP* **1212** (2012) 078, arXiv:1210.4047 [hep-th].

[88] P. Meessen and T. Ortín, "Non-Extremal Black Holes of N=2,d=5 Supergravity," *Phys.Lett.* **B707** (2012) 178–183, arXiv:1107.5454 [hep-th].

[89] L. Andrianopoli, R. D'Auria, A. Gallerati, and M. Trigiante, "Extremal Limits of Rotating Black Holes," arXiv:1303.1756 [hep-th].

[90] G. W. Gibbons, R. Kallosh, and B. Kol, "Moduli, scalar charges, and the first law of black hole thermodynamics," *Phys.Rev.Lett.* **77** (1996) 4992–4995, arXiv:hep-th/9607108 [hep-th].

[91] S. Ferrara, K. Hayakawa, and A. Marrani, "Lectures on attractors and black holes," *Fortsch. Phys.* **56** (2008) 993–1046, arXiv:0805.2498 [hep-th].

[92] R. Kallosh, N. Sivanandam, and M. Soroush, "The Non-BPS black hole attractor equation," *JHEP* **0603** (2006) 060, arXiv:hep-th/0602005 [hep-th].

[93] A. Papapetrou, "A static solution of the equations of the gravitational field for an arbitrary charge distribution," *Proc. Roy. Irish Acad. (Sect. A)* **51** (1947) 191–204.

[94] K. Behrndt, D. Lüst, and W. A. Sabra, "Stationary solutions of N=2 supergravity," *Nucl.Phys.* **B510** (1998) 264–288, arXiv:hep-th/9705169 [hep-th].

[95] G. L. Cardoso, B. de Wit, J. Käppeli, and T. Mohaupt, "Stationary BPS solutions in N=2 supergravity with R**2 interactions," *JHEP* **0012** (2000) 019, arXiv:hep-th/0009234 [hep-th].

[96] P. Breitenlohner, D. Maison, and G. W. Gibbons, "Four-dimensional black holes from Kaluza–Klein theories," *Commun. Math. Phys.* **120** (1988) 295.

[97] G. Bossard and H. Nicolai, "Multi-black holes from nilpotent Lie algebra orbits," *Gen. Rel. Grav.* **42** (2010) 509–537, `arXiv:0906.1987 [hep-th]`.

[98] T. Mohaupt and K. Waite, "Instantons, black holes and harmonic functions," *JHEP* **0910** (2009) 058, `arXiv:0906.3451 [hep-th]`.

[99] E. G. Gimon, F. Larsen, and J. Simon, "Constituent Model of Extremal non-BPS Black Holes," *JHEP* **0907** (2009) 052, `arXiv:0903.0719 [hep-th]`.

[100] A. Ceresole, G. Dall'Agata, S. Ferrara, and A. Yeranyan, "First order flows for n=2 extremal black holes and duality invariants," *Nucl. Phys.* **B824** (2010) 239–253, `arXiv:0908.1110 [hep-th]`.

[101] G. Bossard, Y. Michel, and B. Pioline, "Extremal black holes, nilpotent orbits and the true fake superpotential," *JHEP* **01** (2010) 038, `arXiv:0908.1742 [hep-th]`.

[102] R. Kallosh, N. Sivanandam, and M. Soroush, "Exact attractive non-BPS STU black holes," *Phys. Rev.* **D74** (2006) 065008, `arXiv:hep-th/0606263`.

[103] A. Strominger, "Special geometry," *Commun. Math. Phys.* **133** (1990) 163–180.

[104] P. Candelas and X. de la Ossa, "Moduli space of Calabi–Yau manifolds," *Nucl. Phys.* **B355** (1991) 455–481.

[105] T. W. Grimm and J. Louis, "The effective action of type IIA Calabi–Yau orientifolds," *Nucl. Phys.* **B718** (2005) 153–202, `arXiv:hep-th/0412277`.

[106] B. Pioline, "Lectures on black holes, topological strings and quantum attractors (2.0)," *Lect. Notes Phys.* **755** (2008) 1–91, `arXiv:hep-th/0607227`.

[107] F. Denef, B. R. Greene, and M. Raugas, "Split attractor flows and the spectrum of BPS D-branes on the quintic," *JHEP* **05** (2001) 012, arXiv:hep-th/0101135.

[108] J. de Boer, F. Denef, S. El-Showk, I. Messamah, and D. Van den Bleeken, "Black hole bound states in AdS(3) x S**2," *JHEP* **0811** (2008) 050, arXiv:0802.2257 [hep-th].

[109] W. Van Herck and T. Wyder, "Black hole meiosis," arXiv:0909.0508 [hep-th].

[110] M. Bodner, A. Cadavid, and S. Ferrara, "(2,2) vacuum configurations for type IIA superstrings: N=2 supergravity Lagrangians and algebraic geometry," *Class.Quant.Grav.* **8** (1991) 789–808.

[111] S. Bellucci, A. Marrani, E. Orazi, and A. Shcherbakov, "Attractors with vanishing central charge," *Phys. Lett.* **B655** (2007) 185–195, arXiv:0707.2730 [hep-th].

[112] A. Van Proeyen and B. Vercnocke, "Effective action for the field equations of charged black holes," *Class. Quant. Grav.* **25** (2008) 035010, arXiv:0708.2829 [hep-th].

[113] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. WH Freeman, San Francisco, 1973.

[114] S. Bellucci, S. Ferrara, A. Marrani, and A. Yeranyan, "stu black holes unveiled," *Entropy* **10** (2008) 507–555, arXiv:0807.3503 [hep-th].

[115] C. Stelea, K. Schleich, and D. Witt, "Charged Kaluza–Klein multi-black holes in five dimensions," arXiv:0909.3835 [hep-th].

[116] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall, "All supersymmetric solutions of minimal supergravity in five dimensions," *Class. Quant. Grav.* **20** (2003) 4587–4634, arXiv:hep-th/0209114.

[117] J. P. Gauntlett and J. B. Gutowski, "General concentric black rings," *Phys. Rev.* **D71** (2005) 045002, arXiv:hep-th/0408122.

[118] A. Castro, J. L. Davis, P. Kraus, and F. Larsen, "String theory effects on five-dimensional black hole physics," *Int. J. Mod. Phys.* **A23** (2008) 613–691, arXiv:0801.1863 [hep-th].

[119] R. R. Khuri and T. Ortín, "A Nonsupersymmetric dyonic extreme Reissner–Nordström black hole," *Phys.Lett.* **B373** (1996) 56–60, arXiv:hep-th/9512178 [hep-th].

[120] S. Ferrara and M. Günaydin, "Orbits of exceptional groups, duality and BPS states in string theory," *Int. J. Mod. Phys.* **A13** (1998) 2075–2088, arXiv:hep-th/9708025.

[121] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, and S. P. Trivedi, "Rotating attractors," *JHEP* **10** (2006) 058, arXiv:hep-th/0606244.

[122] M. Günaydin, A. Neitzke, B. Pioline, and A. Waldron, "BPS black holes, quantum attractor flows and automorphic forms," *Phys. Rev.* **D73** (2006) 084019, arXiv:hep-th/0512296.

[123] E. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante, and T. Van Riet, "Generating geodesic flows and supergravity solutions," *Nucl. Phys.* **B812** (2009) 343–401, arXiv:0806.2310 [hep-th].

[124] G. Bossard, H. Nicolai, and K. S. Stelle, "Universal BPS structure of stationary supergravity solutions," *JHEP* **07** (2009) 003, arXiv:0902.4438 [hep-th].

[125] G. Bossard, "1/8 BPS black hole composites," arXiv:1001.3157 [hep-th].

[126] A. Ceresole, G. Dall'Agata, S. Ferrara, and A. Yeranyan, "Universality of the superpotential for d=4 extremal black holes," arXiv:0910.2697 [hep-th].

[127] A. Ceresole and S. Ferrara, "Black holes and attractors in supergravity," arXiv:1009.4175 [hep-th].

[128] W. Chemissany, P. Fré, J. Rosseel, A. S. Sorin, M. Trigiante, and T. Van Riet, "Black holes in supergravity and integrability," *JHEP* **09** (2010) 080, arXiv:1007.3209 [hep-th].

[129] A. Ceresole, R. D'Auria, and S. Ferrara, "The Symplectic structure of N=2 supergravity and its central extension," *Nucl.Phys.Proc.Suppl.* **46** (1996) 67–74, arXiv:hep-th/9509160 [hep-th].

[130] M. Henneaux and C. Teitelboim, "Dynamics of chiral (self-dual) p-forms," *Phys. Lett.* **B206** (1988) 650.

[131] X. Bekaert and M. Henneaux, "Comments on chiral p-forms," *Int. J. Theor. Phys.* **38** (1999) 1161–1172, arXiv:hep-th/9806062.

[132] S. Nampuri and M. Soroush, "New perspectives on attractor flows and trees from CFT," arXiv:1009.5768 [hep-th].

[133] M. Shmakova, "Calabi-Yau black holes," *Phys.Rev.* **D56** (1997) 540–544, arXiv:hep-th/9612076 [hep-th].

[134] K. Behrndt, G. L. Cardoso, and S. Mahapatra, "Exploring the relation between 4D and 5D BPS solutions," *Nucl. Phys.* **B732** (2006) 200–223, arXiv:hep-th/0506251.

[135] D. Gaiotto, A. Strominger, and X. Yin, "New connections between 4D and 5D black holes," *JHEP* **02** (2006) 024, arXiv:hep-th/0503217.

[136] D. Gaiotto, A. Strominger, and X. Yin, "5D black rings and 4D black holes," *JHEP* **02** (2006) 023, arXiv:hep-th/0504126.

[137] B. L. Cerchiai, S. Ferrara, A. Marrani, and B. Zumino, "Duality, entropy and ADM mass in supergravity," *Phys. Rev.* **D79** (2009) 125010, arXiv:0902.3973 [hep-th].

[138] T. Mohaupt, "Black hole entropy, special geometry and strings," *Fortsch.Phys.* **49** (2001) 3–161, arXiv:hep-th/0007195 [hep-th].

[139] A. Ceresole, S. Ferrara, and A. Marrani, "Small n=2 extremal black holes in special geometry," arXiv:1006.2007 [hep-th].

[140] S. Ferrara, A. Marrani, E. Orazi, R. Stora, and A. Yeranyan, "Two-center black holes duality-invariants for stu model and its lower-rank descendants," arXiv:1011.5864 [hep-th].

[141] A. Marrani (private communication).

[142] S. Bellucci, A. Marrani, and R. Roychowdhury, "Topics in Cubic Special Geometry," *J.Math.Phys.* **52** (2011) 082302, arXiv:1011.0705 [hep-th].

[143] S. Ferrara, A. Marrani, and E. Orazi, "Split attractor flow in n=2 minimally coupled supergravity," arXiv:1010.2280 [hep-th].

[144] T. Mohaupt and O. Vaughan, "Non-extremal Black Holes, Harmonic Functions, and Attractor Equations," *Class.Quant.Grav.* **27** (2010) 235008, arXiv:1006.3439 [hep-th].

[145] S. Ferrara and R. Kallosh, "Universality of supersymmetric attractors," *Phys.Rev.* **D54** (1996) 1525–1534, arXiv:hep-th/9603090 [hep-th].

[146] G. Gibbons and C. Hull, "A Bogomolny Bound for General Relativity and Solitons in N=2 Supergravity," *Phys.Lett.* **B109** (1982) 190.

[147] K. Tod, "All Metrics Admitting Supercovariantly Constant Spinors," *Phys.Lett.* **B121** (1983) 241–244.

[148] W. Sabra, "General static N=2 black holes," *Mod.Phys.Lett.* **A12** (1997) 2585–2590, arXiv:hep-th/9703101 [hep-th].

[149] W. Sabra, "Black holes in N=2 supergravity theories and harmonic functions," *Nucl.Phys.* **B510** (1998) 247–263, arXiv:hep-th/9704147 [hep-th].

[150] M. M. Caldarelli and D. Klemm, "All supersymmetric solutions of N=2, D = 4 gauged supergravity," *JHEP* **0309** (2003) 019, arXiv:hep-th/0307022 [hep-th].

[151] P. Meessen and T. Ortín, "The Supersymmetric configurations of N=2, D=4 supergravity coupled to vector supermultiplets," *Nucl.Phys.* **B749** (2006) 291–324, arXiv:hep-th/0603099 [hep-th].

[152] M. Huebscher, P. Meessen, T. Ortin, and S. Vaula, "Supersymmetric N=2 Einstein-Yang-Mills monopoles and covariant attractors," *Phys.Rev.* **D78** (2008) 065031, arXiv:0712.1530 [hep-th].

[153] S. L. Cacciatori, D. Klemm, D. S. Mansi, and E. Zorzan, "All timelike supersymmetric solutions of N=2, D=4 gauged supergravity coupled to abelian vector multiplets," *JHEP* **0805** (2008) 097, arXiv:0804.0009 [hep-th].

[154] G. T. Horowitz and A. Strominger, "Counting states of near extremal black holes," *Phys.Rev.Lett.* **77** (1996) 2368–2371, arXiv:hep-th/9602051 [hep-th].

[155] M. Cvetic and D. Youm, "Entropy of nonextreme charged rotating black holes in string theory," *Phys.Rev.* **D54** (1996) 2612–2620, arXiv:hep-th/9603147 [hep-th].

[156] D. Kastor and K. Win, "Nonextreme Calabi-Yau black holes," *Phys.Lett.* **B411** (1997) 33–38, arXiv:hep-th/9705090 [hep-th].

[157] K. Behrndt, M. Cvetic, and W. A. Sabra, "The Entropy of near extreme N=2 black holes," *Phys.Rev.* **D58** (1998) 084018, arXiv:hep-th/9712221 [hep-th].

[158] G. Cardoso and V. Grass, "On five-dimensional non-extremal charged black holes and FRW cosmology," *Nucl.Phys.* **B803** (2008) 209–233, arXiv:0803.2819 [hep-th].

[159] E. Gruss and Y. Oz, "Large Charge Four-Dimensional Non-Extremal N = 2 Black Holes with R**2-Terms," *Class.Quant.Grav.* **26** (2009) 245019, arXiv:0902.3799 [hep-th].

[160] E. Gruss and Y. Oz, "Large Charge Four-Dimensional Extremal N=2 Black Holes with R**2 - Terms," *JHEP* **0905** (2009) 041, arXiv:0902.0831 [hep-th].

[161] G. Gibbons, "Antigravitating Black Hole Solitons with Scalar Hair in N=4 Supergravity," *Nucl.Phys.* **B207** (1982) 337–349.

[162] A. Sen, "Entropy function for heterotic black holes," *JHEP* **0603** (2006) 008, arXiv:hep-th/0508042 [hep-th].

[163] B. Janssen, P. Smyth, T. Van Riet, and B. Vercnocke, "A First-order formalism for timelike and spacelike brane solutions," *JHEP* **0804** (2008) 007, arXiv:0712.2808 [hep-th].

[164] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, *et al.*, "N=2 supergravity and N=2 superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map," *J.Geom.Phys.* **23** (1997) 111–189, arXiv:hep-th/9605032 [hep-th].

[165] A. Van Proeyen, *N=2 supergravity in d=4,5,6 and its matter couplings*.

[166] G. W. Moore, "Arithmetic and attractors," arXiv:hep-th/9807087 [hep-th].

[167] S. Bellucci, S. Ferrara, and A. Marrani, "On some properties of the attractor equations," *Phys.Lett.* **B635** (2006) 172–179, arXiv:hep-th/0602161 [hep-th].

[168] G. L. Cardoso, V. Grass, D. Lüst, and J. Perz, "Extremal non-BPS black holes and entropy extremization," *JHEP* **0609** (2006) 078, arXiv:hep-th/0607202 [hep-th].

[169] S. Ferrara and A. Marrani, "N=8 non-BPS Attractors, Fixed Scalars and Magic Supergravities," *Nucl.Phys.* **B788** (2008) 63–88, arXiv:0705.3866 [hep-th].

[170] T. Ortin, "Nonsupersymmetric (but) extreme black holes, scalar hair and other open problems," arXiv:hep-th/9705095 [hep-th].

[171] E. Lozano-Tellechea and T. Ortin, "The General, duality invariant family of nonBPS black hole solutions of N=4, D = 4 supergravity," *Nucl.Phys.* **B569** (2000) 435–450, arXiv:hep-th/9910020 [hep-th].

[172] D. Garfinkle, G. T. Horowitz, and A. Strominger, "Charged black holes in string theory," *Phys.Rev.* **D43** (1991) 3140.

[173] R. Kallosh, A. D. Linde, T. Ortin, A. W. Peet, and A. Van Proeyen, "Supersymmetry as a cosmic censor," *Phys.Rev.* **D46** (1992) 5278–5302, arXiv:hep-th/9205027 [hep-th].

[174] M. Duff, J. T. Liu, and J. Rahmfeld, "Four-dimensional string-string-string triality," *Nucl.Phys.* **B459** (1996) 125–159, arXiv:hep-th/9508094 [hep-th].

[175] S. Ferrara and R. Kallosh, "On N=8 attractors," *Phys.Rev.* **D73** (2006) 125005, arXiv:hep-th/0603247 [hep-th].

[176] T. Mohaupt and O. Vaughan, "The Hesse potential, the c-map and black hole solutions," *JHEP* **1207** (2012) 163, arXiv:1112.2876 [hep-th].

[177] T. Mohaupt and O. Vaughan, "Non-extremal black holes from the generalised r-map," arXiv:1208.4302 [hep-th].

[178] G. Bossard, P. Howe, and K. Stelle, "Anomalies and divergences in N=4 supergravity," *Phys.Lett.* **B719** (2013) 424–429, arXiv:1212.0841 [hep-th].

[179] G. Bossard and C. Ruef, "Interacting non-BPS black holes," *Gen.Rel.Grav.* **44** (2012) 21–66, arXiv:1106.5806 [hep-th].

[180] G. L. Cardoso, B. de Wit, and S. Mahapatra, "Non-holomorphic deformations of special geometry and their applications," arXiv:1206.0577 [hep-th].

[181] S. Ferrara, A. Marrani, and A. Yeranyan, "Freudenthal Duality and Generalized Special Geometry," *Phys.Lett.* **B701** (2011) 640–645, arXiv:1102.4857 [hep-th].

[182] L. Borsten, D. Dahanayake, M. Duff, and W. Rubens, "Black holes admitting a Freudenthal dual," *Phys.Rev.* **D80** (2009) 026003, arXiv:0903.5517 [hep-th].

[183] T. Ortín, "A Simple Derivation of Supersymmetric Extremal Black Hole Attractors," *Phys.Lett.* **B700** (2011) 261–264, arXiv:1103.2738 [hep-th].

[184] K. Saraikin and C. Vafa, "Non-supersymmetric black holes and topological strings," *Class.Quant.Grav.* **25** (2008) 095007, arXiv:hep-th/0703214 [hep-th].

[185] S. Bellucci, S. Ferrara, A. Marrani, and A. Shcherbakov, "Splitting of Attractors in 1-modulus Quantum Corrected Special Geometry," *JHEP* **0802** (2008) 088, arXiv:0710.3559 [hep-th].

[186] A. Dabholkar, R. Kallosh, and A. Maloney, "A Stringy cloak for a classical singularity," *JHEP* **0412** (2004) 059, arXiv:hep-th/0410076 [hep-th].

[187] P. Bueno, R. Davies, and C. S. Shahbazi, "Quantum Black Holes in Type-IIA String Theory," arXiv:1210.2817 [hep-th].

[188] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum Configurations for Superstrings," *Nucl.Phys.* **B258** (1985) 46–74.

[189] A. Strominger and E. Witten, "New Manifolds for Superstring Compactification," *Commun.Math.Phys.* **101** (1985) 341.

[190] P. Candelas, "Yukawa Couplings Between (2,1) Forms," *Nucl.Phys.* **B298** (1988) 458.

[191] B. R. Greene and M. Plesser, "(2,2) and (2,0) superconformal orbifolds,".

[192] B. R. Greene and M. Plesser, "Duality in Calabi–Yau moduli space," *Nucl.Phys.* **B338** (1990) 15–37.

[193] P. Candelas, P. S. Green, and T. Hübsch, "Rolling Among Calabi-Yau Vacua," *Nucl.Phys.* **B330** (1990) 49.

[194] P. Candelas and X. C. de la Ossa, "Comments on Conifolds," *Nucl.Phys.* **B342** (1990) 246–268.

[195] P. Candelas and X. de la Ossa, "Moduli Space of Calabi–Yau Manifolds," *Nucl.Phys.* **B355** (1991) 455–481.

[196] P. Candelas, X. C. De la Ossa, P. S. Green, and L. Parkes, "An Exactly soluble superconformal theory from a mirror pair of Calabi-Yau manifolds," *Phys.Lett.* **B258** (1991) 118–126.

[197] M. Hübscher, P. Meessen, and T. Ortín, "Supersymmetric solutions of N=2 D=4 sugra: The Whole ungauged shebang," *Nucl.Phys.* **B759** (2006) 228–248, `arXiv:hep-th/0606281 [hep-th]`.

[198] S. Ferrara and J. Louis, "Flat holomorphic connections and Picard-Fuchs identities from N=2 supergravity," *Phys.Lett.* **B278** (1992) 240–245, `arXiv:hep-th/9112049 [hep-th]`.

[199] A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche, and J. Louis, "Picard-Fuchs equations and special geometry," *Int.J.Mod.Phys.* **A8** (1993) 79–114, `arXiv:hep-th/9204035 [hep-th]`.

[200] A. Klemm and S. Theisen, "Considerations of one modulus Calabi-Yau compactifications: Picard-Fuchs equations, Kahler potentials and mirror maps," *Nucl.Phys.* **B389** (1993) 153–180, `arXiv:hep-th/9205041 [hep-th]`.

[201] R. Kallosh and T. Ortin, "Charge quantization of axion - dilaton black holes," *Phys.Rev.* **D48** (1993) 742–747, arXiv:hep-th/9302109 [hep-th].

[202] S. Ferrara and O. Macia, "Real symplectic formulation of local special geometry," *Phys.Lett.* **B637** (2006) 102–106, arXiv:hep-th/0603111 [hep-th].

[203] A. Marrani, C.-X. Qiu, S.-Y. D. Shih, A. Tagliaferro, and B. Zumino, "Freudenthal Gauge Theory," arXiv:1208.0013 [hep-th].

[204] G. Dall'Agata and A. Gnecchi, "Flow equations and attractors for black holes in N = 2 U(1) gauged supergravity," *JHEP* **1103** (2011) 037, arXiv:1012.3756 [hep-th].