

# Neutrino masses and mixing: a flavour symmetry roadmap

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Over the last ten years tri-bimaximal mixing has played an important role in modeling the flavour problem. We give a short review of the status of flavour symmetry models of neutrino mixing. We concentrate on non-Abelian discrete symmetries, which provide a simple way to account for the TBM pattern. We discuss phenomenological implications such as neutrinoless double beta decay, lepton flavour violation as well as theoretical aspects such as the possibility to explain quarks and leptons within a common framework, such as grand unified models.

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## 1 Introduction

The long-standing solar and atmospheric neutrino anomalies suggested the idea of neutrino oscillations, now confirmed by a series of “laboratory” experiments based on reactors and accelerators [1]. Especially puzzling theoretically is the fact that the neutrino mixing angles inferred from experiment follow a pattern rather different from that which characterizes quark mixing [2, 3]. The atmospheric angle  $\theta_{23}$  is close to maximal, with a large value for the solar angle  $\theta_{12}$ , both of which are at odds with their quark sector counterparts. Moreover, following the first indications of nonzero  $\theta_{13}$  reported by accelerator experiments MINOS [4] and T2K [5] three recent measurements of  $\theta_{13}$  have been reported by the reactor experiments Double CHOOZ [6], Daya Bay [7] and RENO [8], as well as by the MINOS collaboration [9]<sup>1</sup>.

While the historic discovery of neutrino oscillations provides strong indications for the need of physics beyond Standard Model (SM), the detailed nature of this physics remains elusive: i) the mechanism responsible for neutrino mass generation, ii) its flavour structure, iii) its characteristic scale, as well as the nature of the associated messenger particle all remain unknown. As a result the nature of neutrinos, their mass and mixing parameters are so far unpredicted [10].

Understanding the pattern of neutrino mixing is part of the flavour problem, one of the deepest in particle physics. Although it may be the result of an accident, the pattern of neutrino mixing angles most likely follows a *rationale*. Indeed there has been a strong effort towards the formulation of symmetry-based approaches to address the flavour problem from first principles, assuming the existence of an underlying flavour symmetry of leptons and/or quarks, separately or jointly.

In 2002 Harrison, Perkins and Scott proposed the tri-bimaximal (TBM) mixing *ansatz* [11] with effective bimaximal mixing of  $\nu_\mu$  and  $\nu_\tau$  at the atmospheric scale and effective trimaximal mixing for  $\nu_e$  with  $\nu_\mu$  and  $\nu_\tau$  at the solar scale (hence ‘tri-bimaximal’ mixing). While large atmospheric mixing was already discussed before 2002, the trimaximal solar angle has represented a milestone for model building. Non-Abelian continuous and discrete flavour symmetries have been extensively used to account for TBM mixing. Here we review the basic features of some of the most interesting models proposed in the last ten years.

To be fair we must say that the global analysis of neutrino oscillation data now indicates a robust measurement of a relatively “large” value of  $\theta_{13}$  [12] which casts some doubt on the validity of the TBM *ansatz* as a good first approximation to the neutrino mixing pattern. Nevertheless it is too early to jump into conclusions, since in concrete theories there may be large corrections to the TBM pattern, so that here we still take it as a useful reference *ansatz*.

Non-Abelian discrete groups have non trivial irreducible representations (*irreps*). Assigning the three known generations of leptons to irreps of a flavour symmetry group one can make predictions for masses and mixings in the lepton sector. In general it is expected that the number of free parameters of models based on Abelian flavour symmetries is typically larger than the corresponding number of free parameters needed to describe non-Abelian flavour symmetry models. Moreover there are non-Abelian discrete groups that contain triplet irreps, exactly as the number of generations in the standard model. Hence there are viable and predictive non-Abelian models to which we dedicate this brief review.

The smallest group<sup>2</sup> that contains triplet irreps is  $A_4$ , the group of the even permutations of four objects, isomorphic to the group of the symmetries of the tetrahedron  $T$ . For a classification of the irreps of different non Abelian discrete groups see for instance [13].  $A_4$  was first used in the lepton sector by Ma and Rajasekaran [14] but the solar angle was not predicted and neutrino masses were degenerate. A realistic model was proposed by Babu, Ma and Valle [15] adopting a supersymmetric context in order to produce the required neutrino mass splittings and mixing angles, predicting maximal atmospheric mixing and vanishing  $\theta_{13}$  to first approximation. Although expected to be sizeable, the solar mixing angle is unpredicted. In order to predict the full tri-bimaximal pattern, the neutrino mass matrix must take the form

<sup>1</sup> The bulk of the data on neutrino oscillations are well described in terms of three active neutrinos.

<sup>2</sup> The order of a finite group is just the number of elements.

$$M_\nu = \begin{pmatrix} y & x & x \\ x & y+z & x-z \\ x & y-z & x+z \end{pmatrix}, \quad (1)$$

where  $x, y, z$  are free parameters. The above matrix has two properties:

- it is  $\mu - \tau$  invariant giving maximal atmospheric and zero reactor angles;
- it satisfies the relation  $(M_\nu)_{11} + (M_\nu)_{12} = (M_\nu)_{22} + (M_\nu)_{23}$  giving trimaximal solar angle.

The neutrino mass matrix of eq. (1) is diagonalized by the TBM mixing matrix

$$U = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix}, \quad (2)$$

independently of the mass eigenvalues. A trimaximal solar angle was first given in a paper of Ma [16] based on type-II seesaw but assuming in an *ad hoc* way that the contribution of two scalar singlets  $\mathbf{1}$  and  $\mathbf{1}'$  were the same. The derivation of TBM mixing from a flavour symmetry was achieved by Altarelli and Feruglio in Refs. [17] and [18] and subsequently by Babu and He [19].

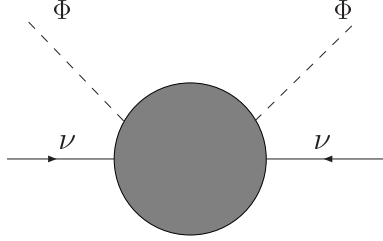
In contrast to the quark sector, neutrino mixing angles are large, possibly TBM, since the flavour group can break into two different subgroups in the charged and neutral lepton sectors, respectively. Consider  $A_4$  as an example.  $A_4$  contains two abelian subgroups, namely  $Z_2$  and  $Z_3$ . When broken into  $Z_3$  in the charged sector, and into  $Z_2$  in the neutrino sector,  $A_4$  leads to a lepton mixing matrix of TBM form, Eq. (2).

Of course  $A_4$  is totally broken, therefore deviations at next to leading order are expected. In general one can not align  $A_4$  in the  $Z_3$  and  $Z_2$  directions in the charged and neutral lepton sectors respectively, this is known as the *alignment* problem. This may be circumvented by using extra dimensions and/or supersymmetry [17, 18] or by assuming a suitably chosen soft breaking sector. Alternatively, using a large discrete group, namely  $Z_2^3 \times U_L(1)^3 \times S_3$ , Grimus and Lavoura have shown [20] how to obtain the TBM form without alignment problem.

## 2 The origin of neutrino mass

Table 1 lists the fifteen fundamental “left-handed” chiral fermions of the Standard Model (SM), sequentially repeated, one set for each generation. In contrast to charged fermions, neutrinos come only in one chiral species, and parity violation in the weak interaction is introduced explicitly by having only “left” fermions transforming as doublets under the  $SU(3) \otimes SU(2) \otimes U(1)$  gauge group.

The simplest and most general way to generate neutrino mass in the Standard  $SU(3) \otimes SU(2) \otimes U(1)$  Model (SM) is by adding an effective dimension-five operator  $\mathcal{O}_{ab} = \lambda_{ab} \ell_a \ell_b \Phi \Phi$ , where  $\ell_a$  denotes any of the three lepton doublets and  $\Phi$  is the SM scalar doublet. [21]. After electroweak symmetry breaking takes place, through the nonzero vacuum expectation value (vev)  $\langle \Phi \rangle$ , Majorana neutrino masses are induced. From such general point of view the emergence of Dirac neutrinos would be an “accident”, justified only in the presence of a fundamental lepton number symmetry, in general absent. The underlying nature of the dimension five operator in Fig. 1 is unknown: little can be said from first principles about the *mechanism* that engenders  $\mathcal{O}_{ab}$ , its associated mass *scale* or its *flavour structure*. The strength of the operator  $\mathcal{O}_{ab}$  can be naturally suppressed if the associated messengers are superheavy, as expected say, in unified scenarios. Alternatively, its strength can be naturally suppressed even in the absence of heavy messengers, due to the fact that  $\mathcal{O}_{ab}$  violates lepton number by two units ( $\Delta L = 2$ ), i.e. in its absence the theory recovers lepton number conservation. This is known as t’Hofft’s naturalness [22]. Correspondingly, one may have high and low-scale neutrino mass models, depending on the mass characterizing the messengers whose exchange induces  $\mathcal{O}_{ab}$ . While the former type are closer to the idea of unification, the latter are closer to experimental testability.



**Fig. 1** Dimension five operator yielding neutrino mass.

	$SU(3) \otimes SU(2) \otimes U(1)$
$\ell_a = (\nu_a, l_a)^T$	$(1, 2, -1)$
$e_a^c$	$(1, 1, 2)$
$Q_a = (u_a, d_a)^T$	$(3, 2, 1/3)$
$u_a^c$	$(\bar{3}, 1, -4/3)$
$d_a^c$	$(\bar{3}, 1, 2/3)$
$\Phi$	$(1, 2, 1)$

**Table 1** Lepton, quark and scalar multiplets of the Standard Model

### 2.1 High scale seesaw mechanisms

The exchange of heavy messenger states, either fermions (type-I or type-III seesaw) or scalars (type-II seesaw) provides a simple way to generate the operator  $\mathcal{O}_{ab}$ . The smallness of its strength is ascribed to the large mass scale characterizing the violation of total lepton number [23, 24]. The simplest and most general description of the seesaw mechanism is in terms of just the  $SU(3) \otimes SU(2) \otimes U(1)$  gauge group with ungauged lepton number broken either explicitly [25] or spontaneously [26]. The latter framework or “1-2-3” scheme is characterized by  $SU(3) \otimes SU(2) \otimes U(1)$  singlet, doublet and triplet mass terms, described by the matrix [25, 26]

$$\mathcal{M}_\nu = \begin{pmatrix} Y_3 v_3 & Y_\nu v_2 \\ Y_\nu^T v_2 & Y_1 v_1 \end{pmatrix} \quad (3)$$

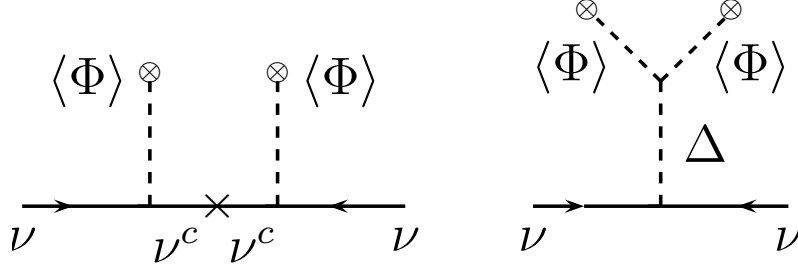
where  $v_2 \equiv \langle \Phi \rangle$  denotes the SM Higgs doublet vev and the basis is  $\nu_L, \nu_L^c$ , corresponding to the three “left” and three “right” neutrinos, respectively. Note that, though symmetric, by the Pauli principle,  $\mathcal{M}_\nu$  is complex, so that its Yukawa coupling sub-matrices  $Y_\nu$  as well as  $Y_3$  and  $Y_1$  are complex matrices denoting the relevant Yukawa couplings, the last two symmetric. Such  $SU(3) \otimes SU(2) \otimes U(1)$  seesaw contains singlet, doublet and triplet scalar multiplets, obeying a simple “1-2-3” vev–seesaw relation of the type

$$v_3 v_1 \sim v_2^2 \quad \text{with} \quad v_1 \gg v_2 \gg v_3 \quad (4)$$

This vev–seesaw is consistent with the minimization condition of the  $SU(3) \otimes SU(2) \otimes U(1)$  invariant scalar potential, and implies that the triplet vev  $v_3 \rightarrow 0$  as the singlet vev  $v_1$  grows. Neutrino masses are suppressed either by heavy  $SU(3) \otimes SU(2) \otimes U(1)$  singlet “right-handed” neutrino exchange (type I) or by the smallness of the induced triplet vev that follows from heavy scalar exchange (type II), as illustrated in Fig. 2. The matrix  $\mathcal{M}_\nu$  is diagonalized by a unitary mixing matrix  $U_\nu$ ,

$$U_\nu^T \mathcal{M}_\nu U_\nu = \text{diag}(m_i, M_i), \quad (5)$$

yielding 6 mass eigenstates: the three light neutrinos with masses  $m_i$ , and the three heavy two-component leptons. The light neutrino mass states  $\nu_i$  are given in terms of the flavour eigenstates via the unitary matrix



**Fig. 2** Left: type-I seesaw (right-handed neutrino exchange). Right: type-II seesaw (scalar triplet exchange).

$U_\nu$  [25]

$$\nu_i = \sum_{a=1}^6 (U_\nu)_{ia} n_a. \quad (6)$$

where the diagonalization matrices are given as a perturbation series, see Ref. [26]. The effective light neutrino mass, obtained this way is of the form

$$m_\nu \simeq Y_3 v_3 - Y_\nu Y_1^{-1} Y_\nu^T \frac{v_2^2}{v_1}. \quad (7)$$

Since in such “1-2-3” seesaw lepton number is ungauged, there is a physical Goldstone boson resulting from its spontaneous breakdown, namely the Majoron [26, 27]. It is often argued that, due to quantum gravity effects the associated Majoron will pick up a mass. It has been shown that, a keV range Majoron can provide the observed dark matter of the Universe [28] and be detected through its X-ray gamma line searches [29]. If B-L is gauged [30] the Majoron is absorbed as the longitudinal mode of a new neutral gauge boson.

## 2.2 Low-scale seesaw mechanisms

A distinguishing feature of the seesaw mechanism as proposed in Ref. [23, 25, 26] and other presentations [24] is that it is formulated in terms of the standard  $SU(3) \otimes SU(2) \otimes U(1)$  SM gauge group. The higher generality implies that the number of “right-handed” neutrinos is totally arbitrary since, being gauge singlets, they carry no anomaly. New important features may emerge when the seesaw is realized with non-minimal lepton content, opening the door to the possibility of low-scale seesaw mechanisms, such as the inverse seesaw [31].

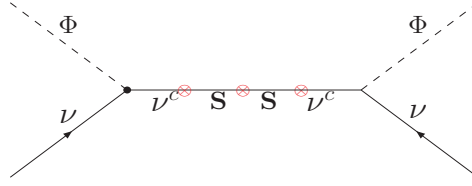
### 2.2.1 Inverse seesaw mechanisms

The model adds a pair of two-component  $SU(3) \otimes SU(2) \otimes U(1)$  singlet leptons,  $\nu_i^c, S_i$ , to each SM generation  $i$  running over 1, 2, 3. In the basis  $\nu, \nu^c, S$ , the neutral leptons mass matrix  $\mathcal{M}_\nu$  is  $9 \times 9$ , i.e.

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & Y_\nu^T v_2 & 0 \\ Y_\nu v_2 & 0 & M^T \\ 0 & M & \mu \end{pmatrix}, \quad (8)$$

with  $\mu \ll Y_\nu v_2 \ll M$ , where  $Y_\nu$  and  $M$  are arbitrary  $3 \times 3$  complex Yukawa matrices, while  $\mu$  is complex symmetric, due to the Pauli principle. In such a scheme the three light neutrino masses are determined from the effective  $3 \times 3$  neutrino mass matrix

$$m_\nu \approx v_2^2 Y_\nu^T M^{T-1} \mu M^{-1} Y_\nu. \quad (9)$$



**Fig. 3** Inverse seesaw mechanism.

The mass generation is illustrated in Fig. 3. Notice that as  $\mu \rightarrow 0$  all neutrinos become massless and lepton number symmetry is restored. The entry  $\mu$  may be proportional to the vev of an  $SU(3) \otimes SU(2) \otimes U(1)$  singlet scalar, in which case the model contains a singlet Majoron [32] which may provide an invisible Higgs boson decay channel [33]. In such schemes one must take into account the existence of sizeable invisible Higgs boson decay channels in the analysis of experimental data on Higgs searches [34, 35].

### 2.2.2 Linear seesaw mechanism

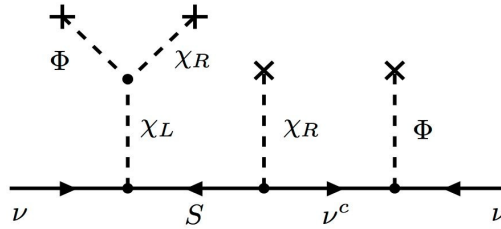
We now turn to a low-energy seesaw mechanism with gauged B-L, originally suggested in the framework of dynamical left-right symmetry [36, 37] and more recently in an  $SO(10)$  model with broken D-parity in which only an Abelian factor survives at low energies [38]. In addition to the three left- and right-handed neutrinos the model contains three sequential gauge singlets  $S_{iL}$  with the following mass matrix

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & Y_\nu \langle \Phi \rangle & F \langle \chi_L \rangle \\ Y_\nu^T \langle \Phi \rangle & 0 & \tilde{F} \langle \chi_R \rangle \\ F^T \langle \chi_L \rangle & \tilde{F}^T \langle \chi_R \rangle & 0 \end{pmatrix} \quad (10)$$

in the basis  $\nu_L, \nu_L^c, S_L$ . The zeros along the diagonal, in the  $\nu_L$ - $\nu_L$  and  $\nu_L^c$ - $\nu_L^c$  entries, are due to the fact that there is no **126**. The resulting neutrino mass is

$$m_\nu \simeq \frac{\langle \Phi \rangle^2}{M_{\text{unif}}} \left[ Y_\nu (F \tilde{F}^{-1})^T + (F \tilde{F}^{-1}) Y_\nu^T \right], \quad (11)$$

where  $M_{\text{unif}}$  is the unification scale,  $F$  and  $\tilde{F}$  denote independent Yukawa coupling combinations of the  $S_{iL}$ . One can see that the neutrino mass is suppressed by the unification scale  $M_{\text{unif}}$  *instead of the B-L breaking scale*. Note that, in contrast to other seesaw schemes, this one is *linear* in the Dirac Yukawa couplings  $Y_\nu$ , as illustrated in Fig. 4. It is rather remarkable that one can indeed take the B-L scale as



**Fig. 4** Linear seesaw mechanism.

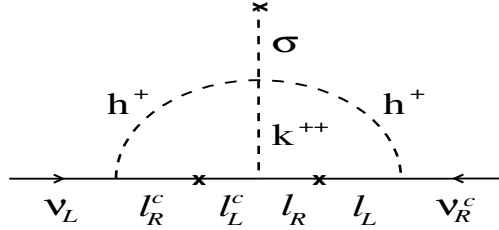
low as TeV without generating inconsistencies with gauge coupling unification [38]. The light neutral gauge boson may be searched directly at the Large Hadron Collider (LHC) or through precision studies of low-energy neutrino-electron scattering [39].

### 2.3 Radiative models of neutrino mass

In addition to the above low-scale seesaw schemes there is a variety of other models of neutrino mass where the operator  $\mathcal{O}_{ab}$  is induced from physics at accessible scales, TeV or less. The first possibility is that neutrino masses are induced by calculable radiative corrections [40, 41], for instance, as illustrated in Fig. 5. Up to a logarithmic factor one has, schematically,

$$\mathcal{M}_\nu \sim \lambda_0 \left( \frac{1}{16\pi^2} \right)^2 f Y_l h Y_l f^T \frac{v_2^2}{(m_k)^2} \langle \sigma \rangle \quad (12)$$

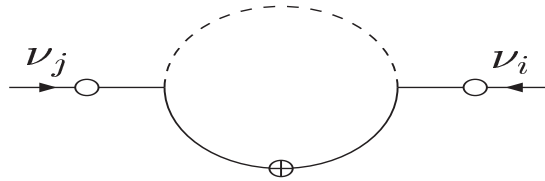
in the limit where the doubly-charged scalar  $k$  is much heavier than the singly charged one. Here  $l$  denotes a charged lepton,  $f$  and  $h$  are their Yukawa coupling matrices and  $Y_l$  denotes the SM Higgs Yukawa couplings to charged leptons and  $\langle \sigma \rangle$  is an  $SU(3) \otimes SU(2) \otimes U(1)$  singlet vev introduced in Ref. [42]. The smallness of the neutrino mass arises from the presence of a product of five small Yukawas and the appearance of the two-loop factor. A special feature of the model is that, thanks to the anti-symmetry of the  $f$  Yukawa coupling matrix, one of the neutrinos is massless.



**Fig. 5** Two-loop origin for neutrino mass.

### 2.4 Supersymmetric neutrino masses

An interesting alternative are models where low energy supersymmetry is the origin of neutrino mass [43] through the breaking of the so-called R parity. This could arise spontaneously, driven by a nonzero vev of an  $SU(3) \otimes SU(2) \otimes U(1)$  singlet sneutrino [44–46]. This way we are led to the minimal way to include neutrino masses into the MSSM, which we take as reference model, with effective bilinear R parity violation [47]. The neutrino mass generation scenario is hybrid, with one scale generated at tree level by the mixing of neutralinos and neutrinos, and the other induced by “calculable” loop corrections [48, 49]. The neutrino mass spectrum naturally follows a normal hierarchy, with the atmospheric scale generated at



**Fig. 6** Loop origin of solar mass scale. Atmospheric scale arises from tree-level neutralino exchange.

the tree level with the solar mass scale arising from calculable loops, as indicated in Fig. 6.

## 3 Prototype flavour model with tetrahedral symmetry

We now turn to models incorporating flavour symmetries, starting with the BMV (Babu-Ma-Valle) model [15]. The usual quark, lepton, and Higgs superfields transform under  $A_4$  as follows:

	$\hat{Q}$	$\hat{L}$	$\hat{u}_1^c, \hat{d}_1^c, \hat{e}_1^c$	$\hat{u}_2^c, \hat{d}_2^c, \hat{e}_2^c$	$\hat{u}_3^c, \hat{d}_3^c, \hat{e}_3^c$	$\hat{\phi}_{1,2}$
$A_4$	3	3	1	1'	1''	1
$Z_3$	1	1	$\omega$	$\omega$	$\omega$	1

The following heavy  $SU(2)$  singlet quark, lepton, and Higgs superfields are also added

	$\hat{U}$	$\hat{U}^c$	$\hat{D}$	$\hat{D}^c$	$\hat{E}$	$\hat{E}^c$	$\hat{N}^c$	$\hat{\chi}$
$A_4$	3	3	3	3	3	3	3	3
$Z_3$	1	1	1	1	1	1	1	$\omega^2$

with  $\omega^3 = 1$  and  $1 + \omega + \omega^2 = 0$ . The superpotential of this model is then given by

$$\begin{aligned}
\hat{W} = & M_U \hat{U}_i \hat{U}_i^c + f_u \hat{Q}_i \hat{U}_i^c \hat{\phi}_2 + h_{ijk}^u \hat{U}_i \hat{u}_j^c \hat{\chi}_k + M_D \hat{D}_i \hat{D}_i^c + f_d \hat{Q}_i \hat{D}_i^c \hat{\phi}_1 + h_{ijk}^d \hat{D}_i \hat{d}_j^c \hat{\chi}_k \\
& + M_E \hat{E}_i \hat{E}_i^c + f_e \hat{L}_i \hat{E}_i^c \hat{\phi}_1 + h_{ijk}^e \hat{E}_i \hat{e}_j^c \hat{\chi}_k + \frac{1}{2} M_N \hat{N}_i \hat{N}_i^c + f_N \hat{L}_i \hat{N}_i^c \hat{\phi}_2 + \mu \hat{\phi}_1 \hat{\phi}_2 \\
& + \frac{1}{2} M_\chi \hat{\chi}_i \hat{\chi}_i + h_\chi \hat{\chi}_1 \hat{\chi}_2 \hat{\chi}_3,
\end{aligned} \tag{13}$$

with the usual assignment of  $R$  parity to distinguish between the Higgs superfields, i.e.  $\hat{\phi}_{1,2}$  and  $\hat{\chi}_i$ , from the quark and lepton superfields. The terms  $\hat{\chi}_i \hat{N}_j^c \hat{N}_k^c$ , etc. are forbidden by the  $Z_3$ . However,  $Z_3$  can break explicitly but softly, by  $M_\chi \neq 0$ . The scalar potential involving  $\chi_i$  is given by

$$V = |M_\chi \chi_1 + h_\chi \chi_2 \chi_3|^2 + |M_\chi \chi_2 + h_\chi \chi_3 \chi_1|^2 + |M_\chi \chi_3 + h_\chi \chi_1 \chi_2|^2, \tag{14}$$

which has the supersymmetric solution ( $V = 0$ )

$$\langle \chi_1 \rangle = \langle \chi_2 \rangle = \langle \chi_3 \rangle = u = -M_\chi / h_\chi, \tag{15}$$

so that the breaking of  $A_4$  at the high scale  $M_\chi$  does not break the supersymmetry. Consider now the  $6 \times 6$  Dirac mass matrix linking  $(e_i, E_i)$  to  $(e_j^c, E_j^c)$ .

$$\mathcal{M}_{eE} = \begin{bmatrix} 0 & 0 & 0 & f_e v_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_e v_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & f_e v_1 \\ h_1^e u & h_2^e u & h_3^e u & M_E & 0 & 0 \\ h_1^e u & h_2^e \omega u & h_3^e \omega^2 u & 0 & M_E & 0 \\ h_1^e u & h_2^e \omega^2 u & h_3^e \omega u & 0 & 0 & M_E \end{bmatrix}, \tag{16}$$

where  $v_1 = \langle \phi_1^0 \rangle$  with similar forms for the quark mass matrices. The reduced  $3 \times 3$  charged leptons mass matrix is then

$$\mathcal{M}_e = U_L \begin{bmatrix} h_1^{e'} & 0 & 0 \\ 0 & h_2^{e'} & 0 \\ 0 & 0 & h_3^{e'} \end{bmatrix} \frac{\sqrt{3} f_e v_1 u}{M_E}, \tag{17}$$

where  $h_i^{e'} = h_i^e [1 + (h_i^e u)^2 / M_E^2]^{-1/2}$  and

$$U_L = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}. \tag{18}$$

This shows how charged-lepton masses are allowed to be all different, despite the presence of the  $A_4$  symmetry, because there are three inequivalent one-dimensional representations. Clearly, the *up* and *down* quark mass matrices are obtained in the same way, both are diagonalized by  $U_L$ , so that the charged-current



mixing matrix  $V_{CKM}$  is the identity matrix. CKM angles may be generated from corrections associated to the structure of the soft supersymmetry breaking sector [50, 51]. The  $6 \times 6$  Majorana neutrino mass matrix is given by

$$\mathcal{M}_{\nu N} = \begin{bmatrix} 0 & U_L f_N v_2 \\ U_L^T f_N v_2 & M_N \end{bmatrix}, \quad (19)$$

in the basis  $(\nu_e, \nu_\mu, \nu_\tau, N_1^c, N_2^c, N_3^c)$  and  $v_2 \equiv \langle \phi_2^0 \rangle$ . The effective  $(\nu_e, \nu_\mu, \nu_\tau)$  mass matrix becomes

$$\mathcal{M}_\nu = \frac{f_N^2 v_2^2}{M_N} U_L^T U_L = \frac{f_N^2 v_2^2}{M_N} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \quad (20)$$

showing that neutrino masses are degenerate at this stage. Consider now the above as coming from an effective dimension-five operator

$$\frac{f_N^2}{M_N} \lambda_{ij} \nu_i \nu_j \phi_2^0 \phi_2^0, \quad (21)$$

where  $\lambda_{ee} = \lambda_{\mu\tau} = \lambda_{\tau\mu} = 1$  and all other  $\lambda$ 's are zero, at some high scale. As we come down to the electroweak scale, Eq. (14) is corrected by the wave-function renormalizations of  $\nu_e, \nu_\mu,$  and  $\nu_\tau$ , as well as the corresponding vertex renormalizations, lifting the neutrino degeneracy due to the different charged-lepton masses. In order to obtain a pattern suitable for explaining current neutrino oscillation data we assume the presence of radiative corrections associated to a general slepton mass matrix in softly broken supersymmetry [52]. Given the structure of  $\lambda_{ij}$  at the high scale, its low scale form is fixed to first order as

$$\lambda_{ij} = \begin{bmatrix} 1 + 2\delta_{ee} & \delta_{e\mu} + \delta_{e\tau} & \delta_{e\mu} + \delta_{e\tau} \\ \delta_{e\mu} + \delta_{e\tau} & 2\delta_{\mu\tau} & 1 + \delta_{\mu\mu} + \delta_{\tau\tau} \\ \delta_{e\mu} + \delta_{e\tau} & 1 + \delta_{\mu\mu} + \delta_{\tau\tau} & 2\delta_{\mu\tau} \end{bmatrix}, \quad (22)$$

where we have assumed all parameters to be real as a first approximation. [The above matrix is obtained by multiplying that of Eq. (13) on the left and on the right by all possible  $\nu_i \rightarrow \nu_j$  transitions.] Let us rewrite the above with  $\delta_0 \equiv \delta_{\mu\mu} + \delta_{\tau\tau} - 2\delta_{\mu\tau}$ ,  $\delta \equiv 2\delta_{\mu\tau}$ ,  $\delta' \equiv \delta_{ee} - \delta_{\mu\mu}/2 - \delta_{\tau\tau}/2 - \delta_{\mu\tau}$ , and  $\delta'' \equiv \delta_{e\mu} + \delta_{e\tau}$ .

$$\lambda_{ij} = \begin{bmatrix} 1 + \delta_0 + 2\delta + 2\delta' & \delta'' & \delta'' \\ \delta'' & \delta & 1 + \delta_0 + \delta \\ \delta'' & 1 + \delta_0 + \delta & \delta \end{bmatrix}, \quad (23)$$

so that the *exact* eigenvectors and eigenvalues are easily obtained:

$$\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta/\sqrt{2} & \sin \theta/\sqrt{2} \\ -\sin \theta & \cos \theta/\sqrt{2} & \cos \theta/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix}, \quad (24)$$

and

$$\lambda_1 = 1 + \delta_0 + 2\delta + \delta' - \sqrt{\delta'^2 + 2\delta''^2}, \quad (25)$$

$$\lambda_2 = 1 + \delta_0 + 2\delta + \delta' + \sqrt{\delta'^2 + 2\delta''^2}, \quad (26)$$

$$\lambda_3 = -1 - \delta_0. \quad (27)$$

leading to

$$\sin^2 2\theta_{atm} = 1, \quad \tan^2 \theta_{sol} = \frac{\delta''^2}{\delta'^2 + \delta'^2 - \delta' \sqrt{\delta'^2 + 2\delta''^2}}, \quad (28)$$

if  $\delta' < 0$  and  $|\delta''/\delta'| = 1.7$ . Assuming  $\delta''^2/\delta'^2$  and  $\delta', \delta'' \ll \delta$ , we now have

$$\Delta m_{31}^2 \simeq \Delta m_{32}^2 \simeq 4\delta m_0^2, \quad \Delta m_{12}^2 \simeq 4\sqrt{\delta'^2 + 2\delta''^2} m_0^2, \quad (29)$$

where  $m_0$  is the common mass of all 3 neutrinos. This provides a satisfactory first-order description of present neutrino-oscillation data. With  $U_{e3} = 0$  there is no  $CP$  violation in neutrino oscillations. However, if we assume complex  $\lambda_{ij}$  then one has one  $CP$  phase which cannot be rotated away. Without loss of generality, we now rewrite  $\lambda_{ij}$  as

$$\lambda_{ij} = \begin{bmatrix} 1 + 2\delta + 2\delta' & \delta'' & \delta''^* \\ \delta'' & \delta & 1 + \delta \\ \delta''^* & 1 + \delta & \delta \end{bmatrix}, \quad (30)$$

where we have redefined  $1 + \delta_0$  as 1, and  $\delta, \delta'$  are real. Assuming that  $\delta', Re\delta''$  and  $(Im\delta'')^2/\delta$  are all much smaller than  $\delta$ , one can diagonalize this mass matrix approximately,

$$U_{e3} = \frac{iIm\delta''}{\sqrt{2}\delta}, \quad \delta' \rightarrow \delta' + \frac{(Im\delta'')^2}{2\delta}, \quad \delta'' \rightarrow Re\delta''. \quad (31)$$

obtaining that  $U_{e3}$  is imaginary and  $CP$  violation in neutrino oscillations is predicted to be maximal. There is also an interesting relationship, i. e.

$$\left[ \frac{\Delta m_{12}^2}{\Delta m_{32}^2} \right]^2 \simeq \left[ \frac{\delta'}{\delta} + |U_{e3}|^2 \right]^2 + \left[ \frac{Re\delta''}{\delta} \right]^2. \quad (32)$$

indicating that  $|U_{e3}|$  is naturally of the order  $|\Delta m_{12}^2/\Delta m_{32}^2|^{1/2}$

$$|U_{e3}| = \mathcal{O}(|\Delta m_{12}^2/\Delta m_{32}^2|^{1/2})$$

in the limit  $\delta', \delta'' \ll \delta$ .

In Fig.7 we plot the maximum achievable value of  $\Delta m_{\text{atm}}^2$  against the overall neutrino mass scale  $m_0$ . The value of  $m_0$  is subject to an upper bound given by  $(\beta\beta)_{0\nu}$  experiments [53,54] and cosmology [55,56]. In order to get large enough neutrino mass splittings to account for current oscillation data we need

$$m_0 \gtrsim 0.4 \text{ eV} \quad (33)$$

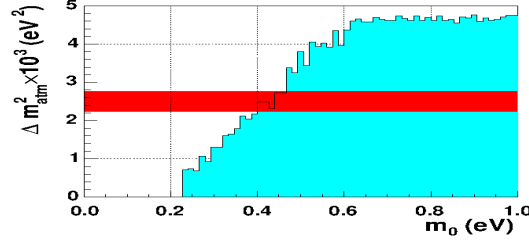
leaving a relatively small room for the value of  $m_0$ . Note also that the mass splittings are related to the parameters  $\delta, \delta'$  and  $\delta''$ , and these are increasing functions of the slepton mixings and also mass splittings. This is potentially in conflict with the restrictions from lepton flavour violation searches which push the spectrum toward mass degeneracy and small mixings. However one can show that viable spectra do exist.

## 4 Quarks, non-abelian discrete flavour symmetries and unification

In the original BMV model [15] as well as in the subsequent Altarelli-Feruglio model [17] the CKM mixing matrix was predicted to be the identity, which provides indeed a good first order approximation. However, realistic quark mixings require either renormalization effects [50,51], or suitable model extensions, e.g [57] and [58].

The largest angle in the CKM matrix is the Cabibbo angle governing the mixing between first and second generations, about  $\lambda_C \sim 0.22$ . Mixing angles between the first/third and second/third families are about  $\lambda_C^2$  and  $\lambda_C^3$ , respectively. This suggests that first and second quark families belong to a doublet of a family symmetry, instead of singlets or triplets.

A non trivial extension of the Altarelli-Feruglio model for quarks was given in [59] where  $A_4$  is extended to its double covering  $T'$ . The main advantage of this group is that it contains doublet irreps besides the



**Fig. 7** Maximum atmospheric mass squared difference versus  $m_0$  (light shaded histogram). The horizontal band is the  $3\sigma$  allowed region for  $\Delta m_{\text{atm}}^2$  by current MINOS/T2K data.

triplet. This feature of  $T'$  is suitable for the quarks since the first two generations can be assigned to doublet irreps, while the third generation belongs to a singlet.

Another group which is interesting in order to include quarks in a TBM pattern for leptons is the permutation group of four objects  $S_4$ . Lam [60] noted that the minimal group for TBM is  $S_4$  and Refs. [61, 62] have studied a model based on  $S_4$ . In Ref. [63] it has been shown that the Cabibbo angle can be predicted using the dihedral flavour symmetry  $D_n$ .

However in general it seems that there is not yet a simple and elegant framework that can explain at the same time neutrinos and quarks. Most of the models considered for quarks contain many flavon fields and extra ad hoc abelian symmetries.

Recently there has been a lot of effort in studying the possibility to embed the TBM *ansatz* within a grand unified (GUT) framework. The most popular unifying groups considered were  $SU(5)$ , Pati-Salam  $SU_c(4) \times SU_L(2) \times SU_R(2)$  and  $SO(10)$ . These can be separated into two classes with respect to TBM. In the  $SU(5)$  framework it is easier to obtain the TBM pattern than in the case of Pati-Salam or  $SO(10)$ , since right-handed neutrinos transform trivially under the gauge group. For some example of extensions of  $SU(5)$  with discrete flavour symmetries see for instance Ref. [64–70], with Pati-Salam [71, 72] and  $SO(10)$  [73–82].

### $SU(5)$ models

As an illustrative example here we consider the model studied in [64]. In typical  $SU(5)$  GUT models, the  $\bar{5}$  contains the lepton doublet  $L$  and the three (colored) right-handed down type quarks  $d^c$  for each family. Whereas the 10 contains the right handed charged leptons, the three right handed up type quarks  $u^c$  and the three quark doublets  $Q$  for each family. Typically in  $A_4$ -based models the three lepton doublets are assigned as triplets of  $A_4$ , while the right-handed charged leptons are assigned to  $A_4$  singlets. This means that we should choose

$$\bar{5} \sim 3, \quad 10 \sim 1, 1', 1'' \quad (34)$$

which implies the following quark assignments

$$d_i^c \sim 3, \quad (u_i, d_i) \sim 1, 1', 1'', \quad u_i^c \sim 1, 1', 1''. \quad (35)$$

One assumes three Higgs doublets  $\bar{5}_H \sim 3$  in the down/lepton sector

$$M_d = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \cdot U_L \cdot \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix} \quad (36)$$

where  $h_i$  are Yukawa couplings and  $v_i \equiv \langle \bar{5}_{H_i}^0 \rangle$ <sup>3</sup>. For the up-quark sector one assumes only two Higgs doublets, namely  $\bar{5}_H \sim 1', 1''$ , so that the up quark mass matrix has the form

$$M_u = \begin{pmatrix} 0 & \mu_2 & \mu_3 \\ \mu_2 & m_2 & 0 \\ \mu_3 & 0 & m_3 \end{pmatrix} \quad (37)$$

where  $m_2$  and  $\mu_3$  arise from the vev of  $\bar{5}_H \sim 1'$  and  $m_3$  and  $\mu_2$  come from the vev of  $\bar{5}_H \sim 1''$ .

In Ref. [64] it has been shown how the CKM mixings  $V_{us}$ ,  $V_{cb}$  and  $V_{ub}$  as well as the CP phase can be obtained in this model. In total the model has 10 free parameters in the quark sector for 10 observable (6 masses, three mixing angles and one phase). Therefore it is clear that in principle such a model can fit realistic quark masses and mixings. Indeed this has been shown, however no light can shed into the structure of quark masses and mixings. Moreover we have the usual  $SU(5)$  relations  $m_\tau = m_b$ ,  $m_\mu = m_s$  and  $m_e = m_d$  at the GUT scale. While the first relation is in good agreement with data, the last two are not. To decouple the charged lepton sector from the down quark sector the usual strategy is to use bigger  $SU(5)$  representations in the scalar sector, like the 45. In the simplest scenario the gauge coupling unification has not been considered.

### $SO(10)$ models

The situation in  $SO(10)$  with type-I seesaw is much more complicated than in  $SU(5)$ . Indeed the main problem is that in  $SO(10)$  TBM requires to distinguish the Dirac neutrino Yukawa coupling from that of the up-quark. In particular the former must be either proportional to the identity or given by eq.(1) in the basis where charged leptons are diagonal. In contrast the up quarks must be strongly hierarchical, namely

$$M_u \sim \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & 1 \end{pmatrix}, \quad m_D \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (38)$$

where  $\times$  indicate small entries much smaller than one. We consider for simplicity the case where the Dirac neutrino mass matrix is proportional to the identity. As already mentioned, in  $SO(10)$  all matter fields (including right-handed neutrinos) belong to only one multiplet, the spinorial 16. In renormalizable  $SO(10)$  models only three types of Yukawa contractions are possible, namely  $16 \cdot 16 = 10 + 120 + 126$ . Two of them  $Y_{10}$  and  $Y_{\bar{126}}$  are symmetric and  $Y_{120}$  is antisymmetric. It is well known that their contributions to up quark and Dirac neutrino mass matrices are given as

$$M^u = M^\nu = Y_{10}v_5, \quad M^u = Y_{\bar{126}}r_5, \quad M^\nu = -3Y_{\bar{126}}r_5, \quad M^u = Y_{120}u_{45}, \quad M^\nu = Y_{120}u_5 \quad (39)$$

where  $v_\alpha, u_\alpha, r_\alpha$  are the vevs of 10, 120 and  $\bar{126}$  respectively, and the lower indexes  $\alpha = 5, 45$  are  $SU(5)$  components. Note that the 10 gives equal contributions to the Dirac and up-quark mass matrices, while the  $\bar{126}$  gives contributions to the Dirac and up-quark mass matrices that are proportional to each other. Therefore if we take hierarchical  $Y_{10}$  and  $Y_{\bar{126}}$  the Dirac neutrino mass matrix will be hierarchical, in contrast with TBM requirement given in eq. (38). The 120 gives different contributions to the up quarks and Dirac neutrino mass matrices, proportional to  $u_{45}$  and  $u_5$ , respectively. Therefore one can in principle distinguish between Dirac neutrino and up quarks by means of the 120. However the coupling  $Y_{120}$  is antisymmetric giving two degenerate eigenvalues and zero determinant. Hence it is not possible to obtain a hierarchical Yukawa coupling with 120. We conclude that it is not possible to obtain the TBM mixing pattern eq.(38) within a simple renormalizable  $SO(10)$  framework with type-I seesaw mechanism as described above. This problem has been circumvented in two ways:

- assuming type-II dominant with respect to type-I seesaw;

<sup>3</sup> Since in minimal  $SU(5)$  we have that  $M_l = M_d^T$  it is clear that one can have TBM mixing in the limit  $v_1 = v_2 = v_3$ .

- introducing non-renormalizable operators.

The idea of type-II dominance was suggested in Ref. [83] in the context of an  $SO(10)$  model with quasi-degenerate neutrinos. A similar scenario has been used in Ref. [79] to accommodate the TBM mixing pattern. The idea is that in  $SO(10)$  the type-II seesaw arises from the coupling with  $\overline{126}$  scalar and is proportional to its 15 component under  $SU(5)$ . It is well known that the 15 does not give contributions to the Dirac fermion masses, see eqs.(39). Therefore assuming type-II dominance neutrino and charged fermion Yukawa couplings become unrelated and we can easily obtain TBM mixing pattern.

In fact we can take the Yukawa coupling  $Y_{126}$  with TBM form given in eq.(1), while Dirac neutrino mass matrix and up quark mass matrix can be taken hierarchical. This is no longer a problem since type-I seesaw contribution is assumed to be negligible with respect to type-II contribution, giving only deviations from the TBM pattern that can generate a sizable  $\theta_{13}$  angle.

The second possibility to address the problem of having a TBM mixing pattern within the  $SO(10)$  type-I seesaw mechanism is to use non-renormalizable operators. For instance one can use the dimension five operator<sup>4</sup>

$$h 16 16 120_H 45_H. \quad (40)$$

This gives a contribution to the up-quark mass matrix but not to the Dirac neutrino [75, 84, 85]. This can be seen in more details as follows. The  $45_H$  can take vev in its singlet  $1_{SU(5)}$  component called  $X$ -direction<sup>5</sup> or along the adjoint  $24_{SU(5)}$  component, that is the hypercharge  $Y$ -direction (see for instance [86]). We indicate their vevs as

$$b_1 = \langle 1_{SU(5)} \rangle, \quad b_{24} = \langle 24_{SU(5)} \rangle. \quad (41)$$

The  $SU(5)$  components of the  $120_H$  of  $SO(10)$  that contain  $SU(2)$  doublet (giving rise to the Dirac masses terms for the fermions) are the  $45_{SU(5)}$ ,  $\overline{45}_{SU(5)}$ ,  $\overline{5}_{SU(5)}$  and  $5_{SU(5)}$  representations. Denoting their vevs as

$$a_5 = \langle 5_{SU(5)} \rangle, \quad a_{\overline{5}} = \langle \overline{5}_{SU(5)} \rangle, \quad a_{45} = \langle 45_{SU(5)} \rangle, \quad a_{\overline{45}} = \langle \overline{45}_{SU(5)} \rangle, \quad (42)$$

we find that

$$M_u = h a_{45} (b_1 - 4b_{24}) - h^T a_{45} (b_1 + b_{24}), \quad (43)$$

$$M_\nu = 5 h a_5 b_1 - h^T a_5 (-3b_1 - 3b_{24}), \quad (44)$$

$$M_d = h (a_{\overline{5}} + a_{\overline{45}}) (-3b_1 + 2b_{24}) - h^T (a_{\overline{5}} + a_{\overline{45}}) (b_1 + b_{24}), \quad (45)$$

$$M_e^T = h (a_{\overline{5}} - 3a_{\overline{45}}) (-3b_1 - 3b_{24}) - h^T (a_{\overline{5}} - 3a_{\overline{45}}) (b_1 + 6b_{24}). \quad (46)$$

where  $h$  is an arbitrary matrix. Note that if we set  $b_{24} = 0$ ,  $M_u$  is proportional to  $h - h^T$  which is antisymmetric. In contrast, if we set  $b_1 = 0$  and  $b_{24} \neq 0$  then  $M_u \propto 4h + h^T$  which is an arbitrary matrix. When  $a_5 = 0$  the operator  $16 16 120_H 45_H$  contributes to  $Y_u, Y_d, Y_e$  but not to  $Y_\nu$ , so we can distinguish between Dirac neutrino and up-quark mass matrices.

Recent observations of a relatively large reactor angle [6–8] pose the questions as to whether the TBM pattern is indeed a good starting point to describe neutrino flavour mixing. The difficulty encountered in  $SO(10)$  based GUT models suggests us to discard the TBM *ansatz* as starting point. However discrete non-Abelian flavour symmetries appear to be better candidates for a family symmetry. Here to give an explicit example of such possibility, briefly presenting a model given in Ref. [87] based on  $SO(10) \times D_3$ <sup>6</sup>. The group  $D_3$  (isomorphic to  $S_3$ —the group of permutations of three objects) contains three irreducible

<sup>4</sup> This operator can be obtained by integrating out a couple  $16_\chi - \overline{16}_\chi$  with renormalizable couplings  $16 16_\chi 120_H, 16 \overline{16}_\chi 45_H$ .

<sup>5</sup> This is the extra  $U(1)$  contained in  $SO(10) \supset SU(5) \times U_X(1)$ .

<sup>6</sup> An extra  $U(1)$  family symmetry is also required.

representations, one symmetric singlet, one antisymmetric singlet and one doublet. The third generation is assigned to the antisymmetric singlet,  $16_3 \sim 1_A$  while the first two families are assigned to a doublet of  $D_3$ . The only renormalizable coupling is for the third generation, with first and second generation masses arising only from next to leading order contributions. Extra messenger fields are introduced in order to make the Lagrangian renormalizable. All charged fermion Yukawa couplings  $Y_u$ ,  $Y_d$ ,  $Y_\ell$  and  $Y_\nu$  have Fritzsch texture [88], namely

$$Y_f \propto \begin{pmatrix} 0 & \times & 0 \\ \times & \times & \times \\ 0 & \times & 1 \end{pmatrix}. \quad (47)$$

Such a model is that it contains only seven real free parameters plus three complex phases for thirteen observable charged fermion masses and mixing angles. While in the neutrino sector there are other four free parameters, giving in total 14 free parameters. Hence this model is highly predictive.

## 5 Dark matter and flavour symmetry

Deciphering the nature of Dark Matter (DM) constitutes one of the main challenges in cosmology since decades. Some recent direct and indirect DM detection experiments have given tantalizing hints in favour of a light WIMP-like DM particle [89–93] feeding the hopes of an imminent detection.

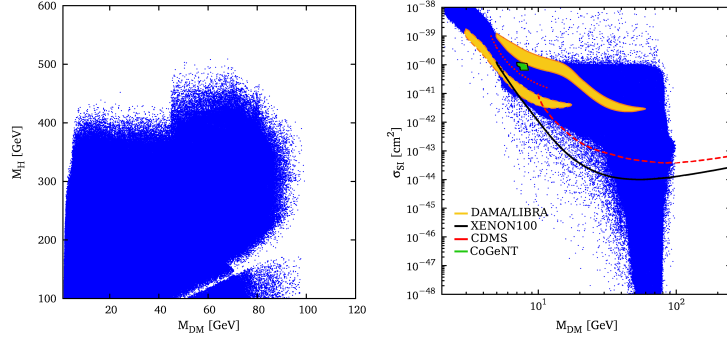
An interesting idea investigated recently has been to link neutrino mass generation to dark matter, two seemingly unrelated problems, into a single framework. This is not only theoretically appealing, but also may bring us new insights on both issues.

Among the requirements a viable DM candidate must obey, stability has traditionally been ensured by imposing of a stabilizing parity in an *ad hoc* way. It would be clearly appealing to obtain stability in a theoretically natural way. This has motivated attempts such as gauged  $U(1)_{B-L}$  [94], gauged discrete symmetries [95] as well as the recently proposed discrete dark matter mechanism (DDM) [96–99], where stability arises as a remnant of a suitable flavour symmetry<sup>7</sup>.

The interplay between decaying dark matter and non-Abelian discrete flavour symmetries has been considered in a number of papers; for instance, in [103–105] non-Abelian discrete symmetries prohibit operators that may induce too fast dark matter decay; in [106] a non-Abelian discrete symmetry (not a flavour symmetry) has been used to stabilize the scalar DM candidate (similar to what has been discussed in the inert scalar models [107]). However, these models differ substantially from what was proposed in [96].

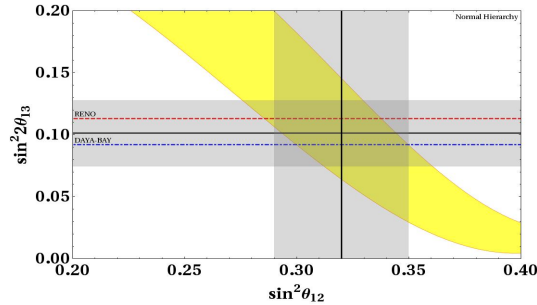
Here we describe the original suggestion. Consider the group of the even permutations of four objects  $A_4$ . It has one triplet and three singlet irreducible representations, denoted  $\mathbf{3}$  and  $\mathbf{1}, \mathbf{1}', \mathbf{1}''$  respectively.  $A_4$  can be broken spontaneously to one of its  $Z_2$  subgroups. Two of the components of any  $A_4$  triplet are odd under such a parity, while the  $A_4$  singlet representation is even. This residual  $Z_2$  parity can be used to stabilize the DM which, in this case, must belong to an  $A_4$  triplet representation, taken as an  $SU(2)_L$  scalar Higgs doublet,  $\eta \sim \mathbf{3}$  [96–99]. Assuming that the lepton doublets  $L_i$  are singlets of  $A_4$  while right-handed neutrinos transform as  $A_4$  triplets  $N \sim \mathbf{3}$ , the contraction rules imply that  $\eta$  couples only to Higgses and heavy right-handed neutrinos  $\bar{L}_i N \tilde{\eta}$ . In this case  $\eta$  and  $N$  have even as well as odd-components while  $L_i$  are even so that  $\bar{L}_i N \tilde{\eta}$  interaction preserves the  $Z_2$  parity. Invariance under  $Z_2$  implies that  $N$  components odd under  $Z_2$  are not mixed with the  $Z_2$ -even light neutrinos  $\nu_i$ . This forbids the decay of the lightest  $Z_2$ -odd component of  $\eta$  to light neutrinos through the heavy right handed neutrinos, ensuring DM stability. In this framework DM has quartic couplings with the SM Higgs doublet as is the Higgs portal DM scenario, and has been shown to have the correct relic density [108], with annihilation and co-annihilation of DM into SM particles (fermions and bosons), see Fig. 8 for the results. Note that assigning the three left-handed leptons to a flavour-triplet implies that the “would-be” DM candidate decays very fast into light leptons, through the contraction of the triplet representations, see general discussion in ref. [100]. This problem

<sup>7</sup> For other flavour models with DM candidates see [97, 100–104]



**Fig. 8** Left plot: Regions in  $(M_{DM} - M_H)$  plane (DM mass–lightest Higgs boson mass) allowed by collider constraints and leading to an adequate DM relic abundance. Right plot: Spin-independent DM scattering cross section off-protons as a function of the dark matter mass. From Ref. [98].

has been considered by Eby and Frampton [109] using a  $T'$  flavour symmetry. While the suggested model has the merit of incorporating quarks non-trivially, it requires an “external”  $Z_2$  asymmetry in order to stabilize dark matter. In fact this observation lead ref. [110] to claim that a successful realization of the DDM scenario requires the lepton doublets to be in three inequivalent singlet representations of the flavour group. Recently Ref. [108] has given an explicit example of a model based on a  $\Delta(54)$  flavour symmetry



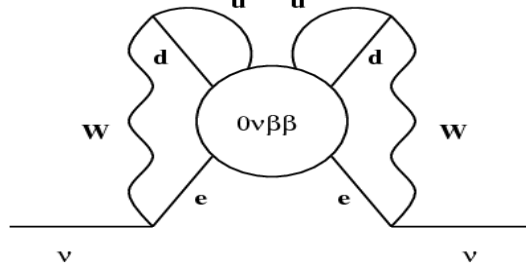
**Fig. 9** The shaded (yellow) curved band gives the predicted correlation between solar and reactor angles at two-sigma for normal hierarchy. The solid (black) lines give the global best fit from Ref. [12]. The dashed lines correspond to the central values of the recent reactor measurements [6–8], along with the corresponding two-sigma bands.

in which left-handed leptons are assigned to non-trivial representations of the flavour group, with a viable stable dark matter particle and a nontrivial inclusion of quarks. In contrast to the simplest “flavour-blind” inert dark matter scheme [111], such a model implies non-trivial restrictions and/or correlations amongst the neutrino oscillation parameters, consistent with the recent reactor angle, see Fig. 9. Although neutrino mixing parameters in the lepton mixing matrix are not strictly predicted, as seen in Fig. (9), there is a non-trivial correlation between the reactor and the atmospheric angle. While the solar angle is clearly unconstrained and can take all the values within in the experimental limits, a nontrivial correlation exists with the reactor mixing angle, indicated by the bands in Fig. 9. These correspond to two and  $3\sigma$  regions corresponding to the global oscillation fit in Ref. [12]. The horizontal lines give the best global fit value and the recent best fit values obtained in Daya–Bay and RENO reactors [7, 8].

## 6 Neutrinoless double beta decay

This is the process *par excellence* which allows us ways to test the fundamental nature - Dirac or Majorana - of neutrinos in a model-independent way, i. e. irrespective of which mechanism generates neutrino

masses and irrespective of which mechanism induces  $0\nu\beta\beta$  (neutrinoless double beta decay)<sup>8</sup>. The basic connection is given by the *black box theorem* [116, 117] illustrated in Fig. 10. Moreover,  $0\nu\beta\beta$  receives a



**Fig. 10** The observation of  $0\nu\beta\beta$  implies the Majorana nature of a neutrino [116].

contribution from the tree-level exchange of Majorana neutrinos, whose amplitude, illustrated in Fig. 11, is proportional to an effective mass parameter which, in contrast to neutrino oscillations, is sensitive also to the absolute scale of neutrino masses, which is independently tested also in searches for tritium beta decay [118, 119] and cosmology [55, 56].

In addition, this amplitude can bring complementary information on the underlying flavour structure as revealed, say, in neutrino oscillation searches<sup>9</sup>.

As we saw in Sec. 3 the BMV model [15] implies a lower bound on the absolute neutrino mass  $m_\nu \gtrsim 0.4$  eV and therefore will be tested fairly soon in  $0\nu\beta\beta$  searches.

On the other hand, even if neutrinos are non-degenerate, many of the models based on non-Abelian discrete flavour symmetries are characterized by a specific (complex) relation between neutrino mass eigenvalues, leading to mass sum rules (MSR). Such MSR lead in most of the cases to lower bounds for the neutrinoless double beta amplitude parameter, depending of the type of the specific MSR. The following types of mass relations hold:

$$A) \quad \chi m_2^\nu + \xi m_3^\nu = m_1^\nu, \quad (48)$$

$$B) \quad \frac{\chi}{m_2^\nu} + \frac{\xi}{m_3^\nu} = \frac{1}{m_1^\nu}, \quad (49)$$

$$C) \quad \chi \sqrt{m_2^\nu} + \xi \sqrt{m_3^\nu} = \sqrt{m_1^\nu}. \quad (50)$$

$$D) \quad \frac{\chi}{\sqrt{m_2^\nu}} + \frac{\xi}{\sqrt{m_3^\nu}} = \frac{1}{\sqrt{m_1^\nu}}. \quad (51)$$

Here  $m_i^\nu = m_i^0$  denote neutrino mass eigenvalues, up to a Majorana phase factor, while  $\chi$  and  $\xi$  are free parameters which specify the model, taken to be positive without loss of generality.

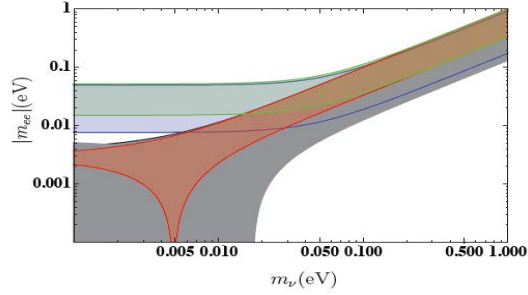
We first consider the amplitude for neutrinoless double- $\beta$  decay within a flavour-generic scheme. The effective neutrino mass parameter  $|m_{ee}|$  determining the  $0\nu\beta\beta$  decay amplitude, as a function of the lightest neutrino mass is given in Fig. 11. As is well-known, by varying the neutrino oscillation parameters in their allowed ranges [12] one obtains two types of relatively broad bands in the  $(|m_{ee}|, m_{\text{light}}^\nu)$  plane corresponding to normal and inverse hierarchy spectra, as shown in Fig. 11. For such “flavour generic” case one finds a lower bound for the neutrinoless double- $\beta$  decay effective mass parameter  $|m_{ee}|$  only in the case of inverse mass hierarchy. Indeed, thanks to the possibility of destructive interference among the light neutrinos no lower bound holds for the case of normal neutrino mass hierarchy [112, 121, 122].

Let us now turn to the case where “flavoured” case where MSR relations like (A), (B), (C) and (D) hold. As mentioned above these can be obtained in flavour models where the neutrino mass matrix depends only on two independent free parameters, so that the resulting mixing angles are fixed, as in the case of tri-bimaximal or bimaximal mixing patterns.

<sup>8</sup> CP and electromagnetic properties of neutrinos [112–115] are also sensitive to the fundamental nature of neutrinos, though experimental prospects are far less clear than those for  $0\nu\beta\beta$ .

<sup>9</sup> Subject, of course, to nuclear matrix element uncertainties [120].



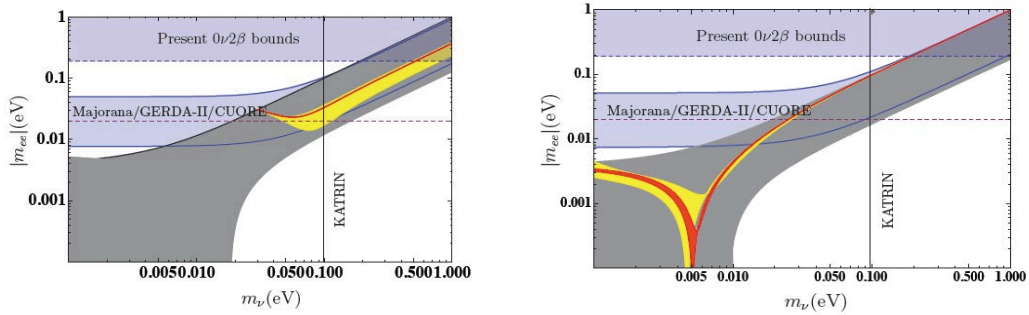


**Fig. 11** Allowed range of  $\langle |m_{ee}| \rangle$  as a function of the lightest neutrino mass for the TBM mixing pattern (red and green bands for NH and IH respectively) and for the full  $3\sigma$  C.L. ranges of oscillation parameters from [12] (gray and blue bands for NH and IH respectively).

For definiteness here we focus on the case of tri-bimaximal neutrino mixing pattern. Taking into account that corrections from higher dimensional operators and/or from the charged lepton sector can yield  $\theta_{13} \neq 0$ , here we retain the TBM approximation as a useful starting point to obtain our MSR relations. However, when evaluating a lower bound on the effective neutrino mass parameter  $|m_{ee}|$  determining the  $0\nu\beta\beta$  decay amplitude, we include explicitly the effects of non-vanishing  $\theta_{13}$ , by taking the  $3\sigma$  oscillation parameter values determined in Ref. [12].

This way one obtains lower limits of  $|m_{ee}|$  corresponding to different integer choices of  $(\chi, \xi)$  between 1 and 3 and for each of the four MSR considered in eqs. (48)-(51), for both normal and inverted hierarchies. These results are summarized in Tab. I of Ref. [123]. A large class of non-Abelian flavour symmetry models discussed in the literature is covered, see for example, references [17–19, 57, 58, 61, 62, 65–67, 75, 124–138, 138–148].

Some comments are in order. First let us consider the effect of a possible non-zero effect of  $\theta_{13}$  as indicated by recent experiments [5, 6] as well as global neutrino oscillation fits [12, 149]. In Fig 12 we show the prediction for  $|m_{ee}|$  as function of  $m_{light}$  obtained from the MSR  $3\sqrt{m_2} + 3\sqrt{m_3} = \sqrt{m_1}$  (right panel) and  $2\sqrt{m_2} + \sqrt{m_3} = \sqrt{m_1}$  (left panel). For the red bands we assumed the TBM values of the oscillation parameters (implying  $\theta_{13} = 0$ ) while the yellow bands corresponds to the same MSR, but now varying the values of  $\theta_{13}$ ,  $\theta_{23}$  and  $\theta_{12}$  within their allowed  $3\sigma$  C.L. interval. By looking at the left panel in



**Fig. 12**  $\langle |m_{ee}| \rangle$  as a function of the lightest neutrino mass corresponding to the mass sum-rule  $2\sqrt{m_2} + \sqrt{m_3} = \sqrt{m_1}$  [150] (left) and  $3\sqrt{m_2} + 3\sqrt{m_3} = \sqrt{m_1}$  (right). The red bands correspond to the TBM mixing pattern, while the yellow bands correspond to the same MSR, but now varying the values of  $\theta_{13}$ ,  $\theta_{23}$  and  $\theta_{12}$  to  $3\sigma$  C.L. range.

Fig. 12 one sees that, indeed, the  $0\nu\beta\beta$  lower bound is sensitive to the value of  $\theta_{13}$ .

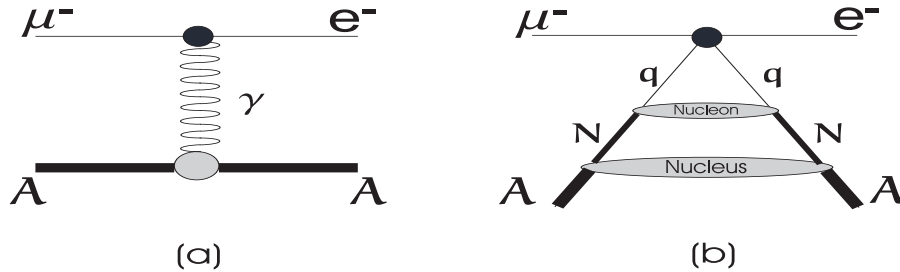
One also finds that, as expected on general grounds, all inverse hierarchy schemes corresponding to various choices of  $(\chi, \xi)$  within sum-rules A-D have a lower bound for the parameter  $|m_{ee}|$ . However, the numerical value obtained depends on the MSR scheme, signaling that not all values within the corresponding band in Fig. 11 are covered for a given flavour symmetry structure.

## 7 Lepton flavour violation and flavour symmetry

Flavour violation is required to account for the current neutrino oscillation data. It should, however, make its appearance also in other sectors, inducing rare processes involving charged leptons, whose strength is not suppressed by the smallness of neutrino masses [151, 152]<sup>10</sup>. For example, in the presence of supersymmetry, the lepton flavour violation required to account for oscillation data in high-scale seesaw schemes induces decays such as  $\mu^- \rightarrow e^- \gamma$  and flavour violating tau decays (Fig. 13) as well as nuclear  $\mu^- - e^-$  conversion (Fig. 14) as a result of the exchange of supersymmetric leptons. The existence of such loop effects, illustrated in Fig. 13, has been known for a while [51, 151]. These lepton flavour



**Fig. 13** Feynman diagrams for  $l_i^- \rightarrow l_j^- \gamma$  involving chargino/neutralinos and sneutrino/charged slepton exchange.



**Fig. 14** Contributions to the nuclear  $\mu^- - e^-$  conversion: (a) long-distance and (b) short-distance.

violation processes can have interesting rates, which depend not only on the seesaw mechanism, but also on the details of supersymmetry breaking and on a possible theory of flavour. The resulting lepton flavour violation rates will be accessible to the upcoming generation of experiments [155–157].

### 7.1 Non-supersymmetric low-scale seesaw models and flavour

Low energy seesaw schemes such as the inverse or linear seesaw mechanisms generate neutrino masses from fermion messengers (right-handed neutrinos) at the TeV scale [31, 36–38]. These are potentially accessible to the LHC, especially in the presence of a new gauge boson “portal” associated, for example, to left-right symmetry [158, 159].

Within low-scale seesaw mechanisms lepton flavour violation and/or CP violating effects arise at the one-loop level from the exchange of relatively light neutral heavy leptons. Their strength is not suppressed by the smallness of neutrino masses [151, 152] so the resulting lepton flavour violation effects

<sup>10</sup> Similar results hold also for leptonic CP violation [153, 154].

are potentially large even in the absence of supersymmetry [152–154] [160, 161] and/or extended gauge structure [158, 159]<sup>11</sup>.

In the inverse and linear seesaw models proposed in [162], the neutrino mass matrix is a  $9 \times 9$  symmetric matrix. It is diagonalized by a unitary matrix  $U_{\alpha\beta}$ ,  $\alpha, \beta = 1 \dots 9$ , leading to three light Majorana eigenstates  $\nu_i$  with  $i = 1, 2, 3$  and six heavy ones  $N_j$  with  $j = 4, \dots, 9$ . The effective charged current weak interaction is characterized by a rectangular lepton mixing matrix  $K_{i\alpha}$  [25],

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} K_{i\alpha} \bar{L}_i \gamma_\mu (1 + \gamma_5) N_\alpha W^\mu, \quad (52)$$

where  $i = 1, 2, 3$  denote the left-handed charged leptons and  $\alpha$  the neutrals. The contribution to the decay  $l_i \rightarrow l_j \gamma$  arises at one loop from the exchanges of the six heavy right-handed Majorana neutrinos  $N_j$  which couple sub-dominantly to the charged leptons. The well-known one-loop contribution to this branching ratio is given by [161]

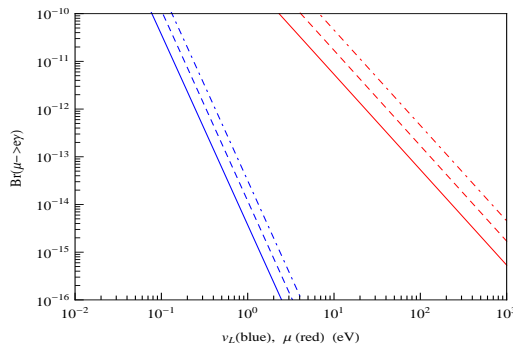
$$Br(l_i \rightarrow l_j \gamma) = \frac{\alpha^3 s_W^2}{256\pi^2} \frac{m_{l_i}^5}{M_W^4} \frac{1}{\Gamma_{l_i}} |G_{ij}|^2 \quad (53)$$

where

$$G_{ij} = \sum_{k=4}^9 K_{ik}^* K_{jk} G_\gamma \left( \frac{m_{N_k}^2}{M_W^2} \right) \quad (54)$$

$$G_\gamma(x) = -\frac{2x^3 + 5x^2 - x}{4(1-x^3)} - \frac{3x^3}{2(1-x)^4} \ln x$$

We note that, thanks to the admixture of the TeV neutral leptons in the charged current weak interaction, this branching ratio can be sizeable [152]. Similar results hold for a class of LFV processes, including nuclear mu-e conversion [163] whose expected rates are strongly correlated to those of  $\mu \rightarrow e \gamma$ , see Fig. 18. As an example we now consider the low-scale seesaw TBM model given in [162]. The simplicity of their mass matrices, which are expressed in terms of very few parameters, makes such a models especially restrictive and this has an impact in the expected pattern of LFV decays. In contrast to the general case considered in [163, 164], in [162] one can easily display the dependence of the  $\mu \rightarrow e \gamma$  branching ratio on the new physics scale represented by the parameters  $M \sim \text{TeV}$  and the parameters  $\mu$  or  $v_L$  characterizing the low-scale violation of lepton number. This is illustrated in Fig. 15.



**Fig. 15**  $Br(\mu \rightarrow e \gamma)$  versus the lepton number violation scale:  $\mu$  for the inverse seesaw (red color), and  $v_L$  for the linear seesaw (blue color). Here  $M$  is fixed as  $M = 100 \text{ GeV}$  (continuous line),  $M = 200 \text{ GeV}$  (dashed line) and  $M = 1000 \text{ GeV}$  (dot-dashed line).

<sup>11</sup> In type-I seesaw schemes the processes  $l_i^- \rightarrow l_j^- \gamma$  are enhanced due to a breakdown of the GIM mechanism. However this is not enough to give large rates since the messenger scale  $M_R$  characterizing lepton number violation is too high.

Note also that, in contrast to a generic inverse or linear seesaw model, such a  $A_4$ -based models the structure of the matrix  $G_{ij}$  is completely fixed, and this leads to predictions for ratios of lepton flavour violation branching ratios such as

$$\frac{Br(\tau \rightarrow e\gamma)}{Br(\mu \rightarrow e\gamma)} = \left(\frac{m_\tau}{m_\mu}\right)^5 \frac{\Gamma_\mu}{\Gamma_\tau} \approx 0.18, \quad (55)$$

for both linear and inverse seesaw and  $Br(\tau \rightarrow \mu\gamma)/Br(\tau \rightarrow e\gamma)$ .

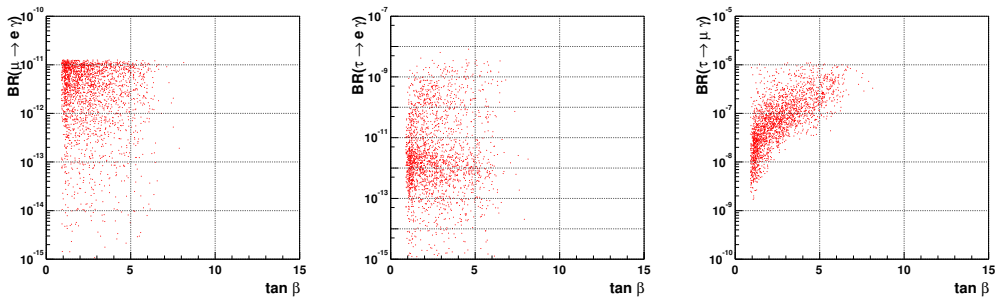
## 7.2 Supersymmetric high-scale seesaw and flavour symmetry

In the presence of supersymmetry, the lepton flavour violation observed in neutrino oscillations induces decays like  $\mu^- \rightarrow e^- \gamma$ , flavour violating tau decays as well as nuclear  $\mu^- - e^-$  conversion (Fig. 14) through the exchange of supersymmetric leptons, as discussed for example, in [165–167] and [168, 169].

Instead of considering a generic “flavour-blind” supersymmetric high-scale seesaw scheme here we consider, as an example, the BMV model already introduced in Sec. 3. A detailed numerical analysis of lepton flavour violation rates has been performed in Ref. [170], using constraints from neutrino oscillation data and confronting with lepton flavour violation searches [171]. The allowed parameter space is determined by a random search through the multi-dimensional parameter space, keeping all supersymmetric masses real and in the range 100 GeV to 1000 GeV. The strongest bounds on lepton flavour violation come from  $\ell_j \rightarrow \ell_i \gamma$ .

The allowed range for the charged slepton parameters is quite restricted. The spectra fall into two different groups. The normal hierarchy having two low mass sleptons ( $\sim 150$  GeV) and one heavy (above  $\sim 500$  GeV), and the inverted hierarchy case having two heavy sleptons and one light. In both cases at least one slepton mass lies below about 200 GeV, detectable at the LHC. Most points fall into the case of normal hierarchy, which often corresponds to a normal hierarchy for the neutrinos as well. The typical case has one small and two large mixing angles. Evidently the small mixing angle is needed to suppress the decay  $\mu \rightarrow e\gamma$ . Also the degeneracy of two of the sleptons helps to minimize the LFV. As a rule of thumb there is at least one pair of sleptons with a mass splitting of less than 40 GeV.

An important outcome of this study is the prediction for the charged lepton decays  $\ell_i \rightarrow \ell_j \gamma$ . As seen in Fig.16 a lower bound of  $10^{-9}$  for  $BR(\tau \rightarrow \mu\gamma)$  is found. This is within reach of BaBar and Belle searches. Also,  $BR(\mu \rightarrow e\gamma)$  is constrained to be larger than about  $10^{-15}$  and therefore stands good chance of being observed in the future [156]. The value of  $\tan(\beta)$ , also plotted in Fig.16, is constrained to be small. For

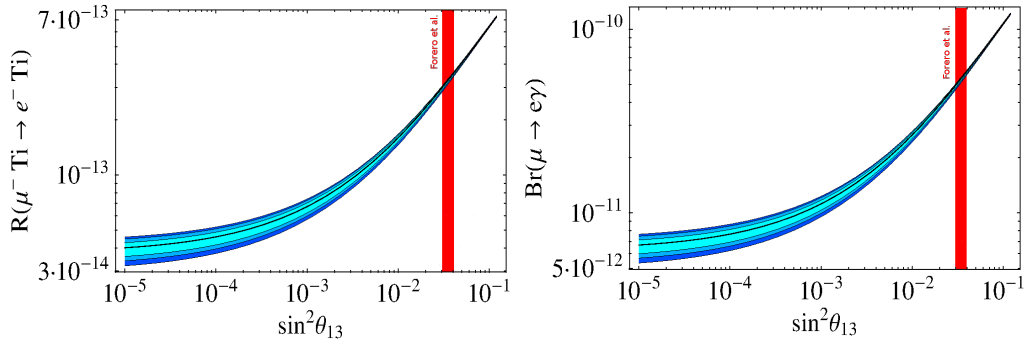


**Fig. 16** The predictions for the branching ratios for the processes  $\ell_i \rightarrow \ell_j \gamma$  as a function of  $\tan(\beta)$ .

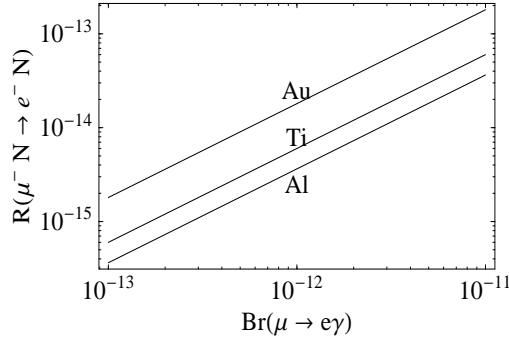
large  $\tan(\beta)$  the RGE effect destroys the agreement with the solar data. The numerical value of  $\delta_\tau$  can not be much bigger than the solar mass scale. A rough estimate gives  $\delta_\tau \lesssim 5 \times 10^{-4}$ , corresponding to the bound  $\tan(\beta) < 10$ . This agrees with the precise bound found in Fig.16. For small values of  $\tan(\beta)$  the threshold corrections dominate and the strongest constraint comes from the bound on  $BR(\mu \rightarrow e\gamma)$ .

### 7.3 Low-scale seesaw and lepton flavour violation

Here we consider the rates for the  $\mu^- \rightarrow e^- \gamma$  decay in the framework of the supersymmetric inverse seesaw model [163, 164]. Fig. 17 displays the dependence of the branching ratios for  $\mu^- - e^-$  conversion in Ti (left) and  $\mu^- \rightarrow e^- \gamma$  (right) with the small neutrino mixing angle  $\theta_{13}$ , for different values of  $\theta_{12}$  (black curve:  $\theta_{12}$  best fit value, blue bands denote  $2\sigma$ ,  $3\sigma$ ,  $4\sigma$  confidence intervals for the solar mixing angle  $\theta_{12}$ ). The inverse seesaw parameters are given by:  $M = 1$  TeV and  $\mu = 30$  eV. The light neutrino parameters used are from [2], except for  $\theta_{13}$  which is varied as shown in the plots. The vertical lines indicate the  $\sin^2 \theta_{13}$  values indicated by recent experiments, as well as by the global fit in Ref. [12]. These lepton flavour violation rates may be testable in the new generation of upcoming experiments [155, 156]. For large  $M$  the estimates recover those of the standard supersymmetric seesaw.



**Fig. 17** LFV branching ratios in the supersymmetric inverse seesaw model of neutrino mass (see text).



**Fig. 18** Correlation between  $Br(\mu \rightarrow e \gamma)$  and muon-electron conversion in nuclei from Ref. [163].

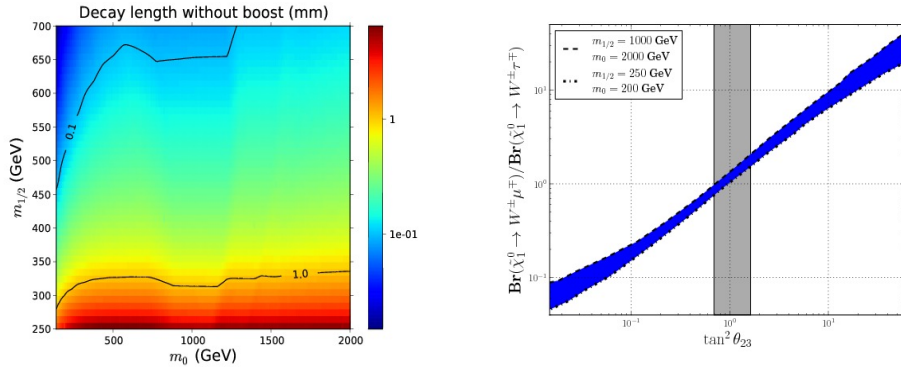
### 7.4 Neutral heavy lepton versus supersymmetric exchange

The novel feature present in low-scale models and not in the minimal seesaw is the possibility of enhancing  $Br(\mu \rightarrow e \gamma)$  and other tau decays with lepton flavour violation due to neutral heavy lepton (right-handed neutrino) versus supersymmetric lepton exchange. In the case where  $M$  is low, around TeV or so this happens even in the *absence* of supersymmetry. In this region of parameters the model also gives rise to large estimates for the nuclear  $\mu^- - e^-$  conversion, depicted in Fig. 14. The latter fall within the sensitivity of future experiments [155]. Note that large lepton flavour violation rates are possible even in the massless neutrino limit. The allowed lepton flavour and CP violation rates are, in fact, unsuppressed by the smallness of neutrino masses [152–154, 160, 161]. Finally, for low enough  $M$  the corresponding heavy leptons could be searched directly at particle accelerators such as LEP already did [172, 173]. Prospects of

LHC detection are less clear, though they are good in the presence of an extended gauge boson “portal”, such as right-handed gauge bosons [158, 159].

## 8 Collider tests: probing neutrino flavour mixing at the LHC

We now turn to the case of low-scale models of neutrino mass. As an example we consider the case of models where supersymmetry is the origin of neutrino mass [43, 48], considered in Sec. 2.4. A general feature of these models is that the lightest supersymmetric particle (LSP) is unstable, since it is not protected by any symmetry. In order to reproduce the masses required by current neutrino oscillation data, the LSP is typically expected to decay inside the detector, leaving a displaced vertex [174–177] as seen in the left panel in Fig. 19. More strikingly, its decay properties correlate with neutrino mixing angles, as seen in the right panel. For example, if the LSP is the lightest neutralino, it is expected to have the same decay rate into muons and taus, since the observed atmospheric angle, is relatively close to  $\pi/4$  [178–180]. This



**Fig. 19** LSP decays typically produce displaced vertices inside the detector, their semi-leptonic decay branching ratios correlate well with the atmospheric mixing angle [176, 178], whose low energy determination is illustrated by the vertical band [12].

opens the tantalizing possibility of testing neutrino mixing at high energy accelerators, like the LHC and the "International Linear Collider" (ILC) and constitutes a smoking gun signature of this proposal that for sure will be tested. This possibility also illustrates the complementarity of accelerator and non-accelerator approaches in elementary particle physics. Before closing this discussion we mention a recent attempt to introduce a flavour symmetry to the bilinear R-parity violation scheme [181].

## 9 Implications of a “large” reactor angle

Recently reactor experiments Double CHOOZ [6], Daya Bay [7] and RENO [8] have published

$$\sin^2 2\theta_{13} = 0.092 \pm 0.016(\text{stat}) \pm 0.005(\text{syst}) \quad \text{at } 5.2\sigma \quad (\text{DayaBay}) \quad (56)$$

$$\sin^2 2\theta_{13} = 0.113 \pm 0.013(\text{stat}) \pm 0.019(\text{syst}) \quad \text{at } 4.9\sigma \quad (\text{RENO}) \quad (57)$$

with similar results recently presented at the Neutrino 2012 conference in Kyoto.

Here we argue that the TBM *ansatz* can still be taken as a good first order approximation. As we discussed in the introduction, in order to have the TBM mixing pattern we need to break separately our flavour group (for instance  $A_4$ ) into  $Z_3$  in the charged sector and into  $Z_2$  in the neutrino sector. Therefore the flavour group is completely broken. Since the flavour symmetry leading to the TBM *ansatz* is in general broken we expect deviations from TBM which, in particular, could generate a nonzero reactor angle.

However many models having TBM at leading order are ruled out because of the recent results which indicate that  $\sin \theta_{13} \sim \lambda_C$  where  $\lambda_C \approx 0.2$ . In fact in general we expect that next-to-leading order terms

give corrections of the same order  $\delta_\theta$  to the three angles  $\theta_{13}$ ,  $\theta_{12}$  and  $\theta_{23}$ . Assuming  $\delta_\theta \sim \lambda_C$  we have

$$\sin^2 2\theta_{13} = 0.087 \quad \text{for } \delta_\theta = 0.15 \quad (58)$$

$$\sin^2 2\theta_{13} = 0.152 \quad \text{for } \delta_\theta = 0.20 \quad (59)$$

close to the best fits in (56) and (57). However the deviations of the solar mixing from its trimaximal values  $\sin^2 \theta_{12}^{TBM} \equiv 1/3$  will be too large if we take  $\delta_\theta = 0.15 \sim \lambda_C$ , namely

$$\sin^2(\theta_{12}^{TBM} + \delta_\theta) = 0.48 \quad (0.38 @ 3\sigma) \quad (60)$$

$$\sin^2(\theta_{12}^{TBM} - \delta_\theta) = 0.20 \quad (0.27 @ 3\sigma). \quad (61)$$

While this poses no problem for the BMV model, which does not predict the solar angle [15], it in principle rules out most TBM schemes. Indeed, most extensions of TBM models which allow for a large reactor angle also predict a deviation of the atmospheric, see for instance [182, 183], and/or the, by now well-measured, solar mixing angle from their TBM values. Therefore one of the most relevant theoretical and experimental questions is to evaluate the extent to which solar and atmospheric mixing angles *deviate* from their TBM values.

Still, not all TBM models proposed in the past are excluded, for example in the model of Ref. [143], based on  $A_4$ , large reactor angle  $\theta_{13} \sim \lambda_C$  has been obtained with deviation of  $\theta_{12}^{TBM}$  of order of  $\lambda_C^2$  in agreement with data. There are other examples in the literature of models where such deviations are negligible, despite the relatively large reactor angle value, see for instance [184–188].

Many alternative *ansatze* have been considered in order to circumvent this problem. An interesting possibility is that the leading order neutrino mass matrix is not diagonalized by the TBM *ansatz*, but rather by the *bi-maximal* one (where both solar and atmospheric mixing angles are maximal from the start) [189] or simply *bi-large* [190], or by the *golden ratio* scheme [188, 191]. Clearly a “large” reactor angle will not only act as a “portal” to a new world of CP violation in the lepton sector, but may also shed light into the flavour problem, one of the deepest puzzles to our current theories of matter.

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