

A Comprehensive Mechanism Reproducing the Mass and Mixing Parameters of Quarks and Leptons

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Abstract

It is shown that if, from the starting point of a universal rank-one mass matrix long favoured by phenomenologists, one adds the assumption that it rotates (changes its orientation in generation space) with changing scale, one can reproduce, in terms of only 6 real parameters, all the 16 mass ratios and mixing parameters of quarks and leptons. Of these 16 quantities so reproduced, 10 for which data exist for direct comparison (i.e. the CKM elements including the CP-violating phase, the angles $\theta_{12}, \theta_{13}, \theta_{23}$ in ν -oscillation, and the masses m_c, m_μ, m_e) agree well with experiment, mostly to within experimental errors; 4 others (m_s, m_u, m_d, m_{ν_2}), the experimental values for which can only be inferred, agree reasonably well; while 2 others (m_{ν_1}, δ_{CP} for leptons), not yet measured experimentally, remain as predictions. In addition, one gets as bonuses, estimates for (i) the right-handed neutrino mass m_{ν_R} and (ii) the strong CP angle θ inherent in QCD. One notes in particular that the output value for $\sin^2 2\theta_{13}$ from the fit

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agrees very well with recent experiments. By inputting the current experimental value with its error, one obtains further from the fit 2 new testable constraints: (i) that θ_{23} must depart from its “maximal” value: $\sin^2 2\theta_{23} \sim 0.935 \pm 0.021$, (ii) that the CP-violating (Dirac) phase in the PMNS would be smaller than in the CKM matrix: of order only $|\sin \delta_{CP}| \leq 0.31$ if not vanishing altogether.

1 Introduction and Summary of Results

The mass values and mixing matrices of quarks and leptons together represent 20 independent quantities, accounting for some three-quarters of the total number of empirical parameters appearing in the otherwise very successful standard model. They fall, besides, into a quite bewildering hierarchical pattern which cries out for a theoretical explanation. But while their experimental colleagues have now measured some of these parameters to an impressive accuracy and are now giving shape even to the PMNS matrix so incredibly difficult to access, theoreticians, embarrassingly, have failed so far to produce a commonly accepted explanation even for their qualitative features, let alone their quantitative values to the accuracy that experiments have achieved.

Admittedly, the theoretician has no easy task here. The explanation for these parameters cannot apparently be found in the standard model itself, since this logically seems to admit any chosen values for these parameters. On the other hand, most of the current models or theories which attempt to go beyond the standard model tend rather to increase the number of independent parameters than to place restrictions on the existing ones. There are models giving predictions for some of the quantities, but not all. For this reason, the mass and mixing pattern of quarks and leptons as a whole still remains largely a mystery in particle physics which is still to be understood.

The aim of this paper is to attempt a first understanding of this great mystery via what has become known as the R2M2 (rank-one rotating mass matrix) hypothesis. This starts from the assumption of a “universal” rank-one mass matrix for all quarks and leptons long favoured by phenomenologists [1, 2] as a first approximation and just adds to it the assumption that this matrix rotates (i.e. changes its orientation) with changing scale in generation space. A fermion mass matrix of rank one can always be rewritten by a suitable relabelling of the $su(2)$ -singlet right-handed fields [3] in the following form:

$$m = m_T \boldsymbol{\alpha} \boldsymbol{\alpha}^\dagger, \quad (1)$$

and by “universal” one means that the vector $\boldsymbol{\alpha}$ is the same for all fermion species, that is, whether up-type (U) or down-type (D) quarks, or whether charged leptons (L) or neutrinos (N). The mass matrix (1) has only one massive state for each fermion species, not a bad starting point for the mass hierarchy, and the unit matrix for the mixing matrix (i.e. no mixing), also

not a bad first approximation at least for quarks, hence the original phenomenological interest. The rotation hypothesis in R2M2 is added to give the needed deviations from this first approximation, such as nonzero masses to the two lower generations and nontrivial mixings as observed in experiment. Notice that the fermion mass matrix should rotate with changing scale occurs in the standard model [4]. Thus, the only thing new in R2M2 is the assumption that it is the rotation which gives rise to the mass hierarchy and to mixing rather than the mixing which gives rise to the rotation as in the standard model. Practically, this means that the mass matrix has to rotate faster with respect to scale than it does in the standard model.

That the mass matrix rotates means that not only the masses of the particles but also their state vectors in generation space have to be defined each at its own mass scale. That this will then lead immediately to nontrivial mixing and to nonzero masses for the two lower generations is easily seen as follows. The state vector of the heaviest state in each fermion species is clearly to be taken as the vector $\boldsymbol{\alpha}$ at its own mass scale. Thus for the t quark, the state vector is $\mathbf{t} = \boldsymbol{\alpha}(\mu = m_t)$, and for the b quark, $\mathbf{b} = \boldsymbol{\alpha}(\mu = m_b)$. The state vectors \mathbf{c} of c and \mathbf{u} of u are then of necessity orthogonal to each other and both orthogonal to \mathbf{t} forming together the U -triad. Similarly, we have $(\mathbf{b}, \mathbf{s}, \mathbf{d})$ the D -triad. These two triads are not aligned because the vector $\boldsymbol{\alpha}$ by assumption would have rotated from the scale $\mu = m_t$ to the scale $\mu = m_b$, hence giving immediately nontrivial mixing. Next, the vectors \mathbf{c} and \mathbf{u} , being both by definition orthogonal to $\mathbf{t} = \boldsymbol{\alpha}(\mu = m_t)$, are both null states of the mass matrix (1) at $\mu = m_t$. But it is not to be inferred that c and u have then zero masses for their masses have to be evaluated each at its own mass scale, not at $\mu = m_t$. Thus, the mass m_c has to be evaluated at $\mu = m_c < m_t$, at which scale, the vector $\boldsymbol{\alpha}$ would have rotated to a different direction and acquired a component in the \mathbf{c} direction thus giving the c quark a nonzero mass. A similar conclusion applies to the u quark, and indeed to any quark or lepton of the two lower generations. The nonzero masses that they thus acquire by this so-called “leakage mechanism” are small if the rotation is not too fast, hence giving rise to mass hierarchy, qualitatively as that seen in experiment.

A more detailed analysis in e.g. [5] along these lines yields the following

formulae for the state vectors:

$$\begin{aligned}
\mathbf{t} &= \boldsymbol{\alpha}(m_t); \\
\mathbf{c} &= \mathbf{u} \times \mathbf{t}; \\
\mathbf{u} &= \frac{\boldsymbol{\alpha}(m_c) \times \boldsymbol{\alpha}(m_t)}{|\boldsymbol{\alpha}(m_c) \times \boldsymbol{\alpha}(m_t)|},
\end{aligned}
\tag{2}$$

and for the masses:

$$\begin{aligned}
m_t &= m_U, \\
m_c &= m_U |\boldsymbol{\alpha}(m_c) \cdot \mathbf{c}|^2, \\
m_u &= m_U |\boldsymbol{\alpha}(m_u) \cdot \mathbf{u}|^2.
\end{aligned}
\tag{3}$$

Solving these equations together gives then the masses and state vectors of each individual state. Analogous results hold for the D -type quark and leptons. The mixing matrix then follows for quarks as:

$$V_{UD} = \begin{pmatrix} \mathbf{u} \cdot \mathbf{d} & \mathbf{u} \cdot \mathbf{s} & \mathbf{u} \cdot \mathbf{b} \\ \mathbf{c} \cdot \mathbf{d} & \mathbf{c} \cdot \mathbf{s} & \mathbf{c} \cdot \mathbf{b} \\ \mathbf{t} \cdot \mathbf{d} & \mathbf{t} \cdot \mathbf{s} & \mathbf{t} \cdot \mathbf{b} \end{pmatrix},
\tag{4}$$

with a similar mixing matrix U_{LN} for leptons.

There are two points in the above result which need to be clarified, which the reader might have already noted. The first is that although the mass matrix, by assumption in R2M2, is of rank one throughout with two zero eigenvalues, and therefore invariant under chiral transformations in those eigenstates at every scale μ , yet as a result of the rotation according to (3), all quarks and leptons have acquired a nonvanishing mass. At first sight this may seem surprising, or perhaps even counter-intuitive, it having been ingrained into us by experience working with nonrotating mass matrices, that the eigenvalues of a mass matrix are the mass values and that chiral invariance is synonymous with the existence of zero mass states. A rotating mass matrix, however, of which not only the eigenvalues but also the eigenvectors are scale-dependent, is to most of us a new situation, and it stands to reason that for dealing with this we shall have to start afresh and check how many of those notions gained from experience before will still survive. True, the mass matrix is already known to rotate in the standard model [4] and the question of how to deal with this situation should have arisen there. However, the rotation there being slow, and its effects therefore negligible, practically the

question can be ignored, and has largely been ignored for this reason. But it cannot now be ignored in R2M2. Thus approaching the question anew with an open mind, it will become clear that for a mass matrix that rotates with scale, neither its eigenvalues nor its eigenvectors can in general be taken directly as the masses and state vectors of particles. Instead, the masses and state vectors are to appear as the eigenvalues and eigenvectors of a “physical mass matrix” to be constructed in a specific manner from the rotating matrix. A detailed analysis of the situation is what gives the result (2)—(4) above [5]. That being so, what might before have appeared as surprising or counter-intuitive will now no longer be the case.

The second point to clarify concerns the Kobayashi-Maskawa CP-violating phase in the mixing matrix. Since the vector α is tacitly taken above to be real², all the state vectors and the mixing matrix V_{UD} in (4) deduced from it will be real as well. It might thus appear that R2M2 would not give us a proper CKM mixing matrix with a CP-violating phase. Surprisingly, however, R2M2 contains in itself a solution to this apparent problem, and in a most intriguing manner, giving at the same time a solution to the old strong CP problem in QCD [6]. As is well known, invariance principles allow in principle in the QCD action a CP-violating term of topological origin with arbitrary coefficient θ , which would give rise to, for example, an electric dipole moment for the neutron. The non-observance of this in experiment [7] means, however, that this parameter θ has to be very small, of order 10^{-9} . It is thus customary to insert by fiat into the standard model an implicit assumption that $\theta = 0$, or to add extra particles called axions [8] which are not yet observed. The need for doing so is known as the strong CP problem. It is also well-known that the theta-angle can be removed by a chiral transformation on the quark fields, but at the cost of making the quark mass complex. Hence, unless some quark happens to have zero mass which is apparently at variance with experiment, the strong CP problem still remains. What is now interesting in R2M2, however, is the fact already noted in the

²There is in principle no need in R2M2 to assume that α is real. Even if α is complex, however, the relative phases between its 3 components will have to change with scale μ before it can give rise to CP-violations in the CKM matrix, otherwise it is equivalent to a real α . We have taken α to be real here because (i) we can find no empirical indication for the μ -dependence of its phases, (ii) we have not succeeded to build a model where α changes its phases with scale in the required manner, but most importantly, (iii) we see here that CP-violations of sufficient size will in any case arise by virtue of the theta-angle term in QCD even for real α .

above paragraph that the mass matrix can remain chirally invariant while keeping all quark masses nonvanishing. This means that the theta-angle term can be removed by a chiral transformation without making the mass matrix complex, while keeping all quark masses nonzero as experiment seems to demand. There is, however, a penalty, namely that the chiral transformation needed above to remove the theta-angle term is automatically transmitted by the rotation to the quark mixing matrix and makes it complex [9]. But this is, of course, a penalty one would be most willing to pay, for this phase is exactly what one was missing in the mixing matrix V_{UD} in (4) above. There is yet more, for it can readily be shown that provided that one starts with a V_{UD} with entries roughly similar in magnitude to those of the CKM matrix observed in experiment, then the CP-violation transmitted by rotation from a theta-angle of order unity will automatically be of the same order as that observed in experiment, i.e. corresponding to a Jarlskog invariant [10, 11] J of order 10^{-5} .

For details on the two points above and their clarification, the reader has to be referred elsewhere [5, 9]. Although the analysis is logically straightforward, it does take some care and patience to sort out, for which a freedom from pre-conceived notions would seem, in our own experience, to be a prerequisite. Some readers may find it convenient first tentatively to accept the R2M2 scheme as outlined above and see what it gives in practice before deciding to verify the details of its justification. For this reason we first summarize the results before giving the details later in Section 2 and Appendix A of how they have been obtained. Indeed, given the quality of the results, some readers may find the scheme a useful means of parametrizing the data, before getting into the theoretical details.

According then to the formulae (2)–(4), one should be able to evaluate the masses and mixing matrices of both quarks and leptons once given α as a function of the scale μ . As μ varies, the vector α traces out a curve on the unit sphere, which for our purpose is best described by the Serret–Frenet–Darboux formulae for curves lying on a surface, as will be spelt out in the next section. In this language then, the trajectory for α is completely prescribed first by giving the arc-length s as a function of the scale μ , and second by giving the geodesic curvature κ_g of the traced-out curve, i.e. how much it is bending sideways, as a function of the arc-length s . Inspired by a model we have been studying, we choose to parametrize s as an exponential in $\ln \mu$ (2 parameters) and κ_g as some Breit–Wigner shape of s (2 parameters). To these 4 real parameters, we have to add θ , the theta-angle from the strong

Quark/Lepton	Mass used in fit (GeV)	Experiment
t	172.9	$172.9 \pm 0.6 \pm 0.9$
b	4.19	$4.19^{+0.18}_{-0.06}$
τ	1.777	1.77682 ± 0.00016
(ν_3)	0.020	fitted parameter

Table 1: Mass values for the heaviest generation used in the calculation compared with data, where for the heaviest neutrino ν_3 the entry in the second column denotes the (unknown) Dirac mass.

CP term to give the CKM matrix a CP-violating phase, as explained above. Lastly, we add the Dirac mass m_3^D of the heaviest neutrino ν_3 as a parameter for in R2M2 the rotation mechanism depends on the Dirac masses, and these for neutrinos are obscured by the Majorana mass term of the right-handed neutrino(s) via a see-saw mechanism or something similar. Altogether then, we have 6 real parameters to our scheme, with θ expected to be of order unity.

Once given a set of values for these parameters, the formulae (2)—(4) then allow one to calculate all the mixing matrix elements and masses of both quarks and leptons in terms of the coefficients m_T in (1), one for each fermion species, which may be identified with the mass of the heaviest state in that species, and given the values listed in Table 1. From these, one is then to reproduce the remaining 16 quantities, namely 4 from the CKM matrix, 4 from the PMNS matrix (neglecting for now any possible Majorana phases), together with the 8 masses of the two lighter generations: $m_c, m_s, m_\mu, m_{\nu_2}, m_u, m_d, m_e, m_{\nu_1}$.

By adjusting the values of the above 6 parameters, as detailed in the next section, one arrives without much difficulty at a set (21) which gives the following results. First, for the absolute values of the CKM matrix elements, one obtains the following:

$$|V_{\text{CKM}}^{\text{th}}| = \begin{pmatrix} 0.97423 & 0.2255 & 0.00414 \\ 0.2252 & 0.97342 & 0.0417 \\ 0.01237 & 0.0400 & 0.999122 \end{pmatrix}, \quad (5)$$

and for the Jarlskog invariant, the value:

$$|J^q| = 2.91 \times 10^{-5}. \quad (6)$$

These are to be compared with experiment, for the absolute values of the CKM matrix elements [12]:

$$\begin{pmatrix} 0.97428 \pm 0.00015 & 0.2253 \pm 0.0007 & 0.00347^{+0.00016}_{-0.00012} \\ 0.2252 \pm 0.0007 & 0.97345^{+0.00015}_{-0.00016} & 0.0410^{+0.0011}_{-0.0007} \\ 0.00862^{+0.00026}_{-0.00020} & 0.0403^{+0.0011}_{-0.0007} & 0.999152^{+0.000030}_{-0.000045} \end{pmatrix}, \quad (7)$$

and for the Jarlskog invariant [12]:

$$|J^q| = (2.91^{+0.19}_{-0.11}) \times 10^{-5}. \quad (8)$$

One notes that apart from the two corner elements, $|V_{ub}|$ and $|V_{td}|$, all other quantities above are reproduced to within experimental errors. The corner elements, being second order effects, as will be explained in the next section, point (a), are rather difficult to reproduce correctly by rotation using a simple parametrization such as equations (12) and (13), but have nevertheless roughly the right size and the right asymmetry about the diagonals, as already anticipated [13].

For the PMNS matrix, a first question to settle is whether the chiral transformation applied on quarks above to remove the theta-angle term from the QCD action should affect leptons as well. A priori, given that leptons carry no colour, they need have nothing to do with the theta-angle and have thus no reason to undergo a similar chiral transformation. In that case, one would obtain for the PMNS matrix just the analogue U_{LN} of (4) above, which for the same values of our 6 parameters as before is found to take the form:

$$|U_{\text{PMNS}}^{\text{th}}| = \begin{pmatrix} 0.818 & 0.556 & 0.147 \\ 0.519 & 0.605 & 0.604 \\ 0.247 & 0.570 & 0.784 \end{pmatrix}, \quad (9)$$

with $J^\ell = \sin \delta_{CP}^\ell = 0$.

The values of the ν -oscillation angles $\sin^2 2\theta_{12}, \sin^2 2\theta_{13}, \sin^2 2\theta_{23}$ corresponding to this PMNS matrix are listed together in Table 2 and compared with experiment. They are all seen to be within the experimental bounds given in [12].

Source	$\sin^2 2\theta_{12}$	$\sin^2 2\theta_{23}$	$\sin^2 2\theta_{13}$
Our Output	0.864	0.935	0.0842
Experiment	$0.861^{+0.026}_{-0.022}$	> 0.92	< 0.15

Table 2: Output values of neutrino oscillation mixing angles compared with data [12].

Since [12] was last updated, however, several major experiments [14, 15, 16, 17, 18] have released new exciting results on θ_{13} which deserve a closer examination. This will be done in section 3 where it will be seen that our output value as given in Table 2 agrees very well with the new data (as summarized in Tables 4 and 5).

That there is no need for a chiral transformation on leptons to remove the theta-angle term, however, does not necessarily mean that one cannot be applied nevertheless to give the PMNS matrix a CP-violating phase. This possibility will be examined in Appendix B where it will be shown that in view of the recent data on θ_{13} , the present R2M2 fit can admit only a small value for $\sin \delta_{CP}^\ell$ arising in this way. In addition, considerations will be given there to a situation when such a CP-violating phase may become a theoretical necessity.

Lastly, the quark and lepton masses we obtained are listed together and compared with experiment in Table 3. The agreement is very good for c and μ . It is not bad even for s, e and ν_2 where some delicate extrapolation is involved. For the others u, d and ν_1 , no ready information is available yet for a meaningful comparison. A more detailed discussion on the last two points will be given in the next section.

In summary, of the 16 quantities reproduced, 10 of which can be compared directly with experiment namely: 4 for the CKM matrix, the 3 mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$ from neutrino oscillation, and the masses m_c, m_μ, m_e , agree with experiment, mostly to within present experimental errors, or else to a high accuracy in the case of m_μ . The only exceptions are the corner CKM elements V_{ub}, V_{td} and the electron mass m_e , the agreement for which is only approximate and outside experimental errors, for reasons that will be given later in the next section, points (a) and (b). Of the remaining 6, 4 can be compared only with values inferred from experiment, where m_s, m_{ν_2} agree

Quark/Lepton	Output mass in GeV	Experiment
c	1.234	$1.29^{+0.05}_{-0.11}$
s	0.171^*	$0.100^{*+0.030}_{-0.020}$
μ	0.10564	0.10565837
ν_2	0.0064^\dagger	0.0083^\dagger
u	0.0009^*	0.0017 – 0.0033
d	0.0008^*	0.0041 – 0.0058
e	0.0007	0.0005109989
ν_1	$< 0.0016^\dagger$	unknown

Table 3: Output masses compared with data. Note that those entries marked by a $*$ or a \dagger cannot be compared with data directly. For the light quarks u, d, s marked with $*$ the output values are supposed to represent the mass of the free quark measured at its own mass scale while values given in the PDG review [12], because of quark confinement, are determined at a scale of 2 GeV. For the neutrinos, marked with \dagger , the output masses are the Dirac masses which are not measured directly. The “experimental values” cited are inferred from the measured (physical) mass differences via a see-saw mechanism. For more details, see text, Section 2.

within reason but m_u, m_d only in the rough order of magnitude, again for reasons to be given in the next section, points (d) and (e). This leaves 2 others m_{ν_1} and δ_{CP}^ℓ for leptons as predictions, for which no experimental information is yet available. In addition, one obtains as bonuses estimates for the right-handed neutrino mass m_{ν_R} and the strong CP angle θ , for which no experimental information is likely to be available for some time, and therefore for the moment of theoretical interest only. Although θ was treated above as an input parameter and used to fit the value of the Jarlskog invariant J^q for the CKM matrix, that this was possible is only by virtue of R2M2 so that a value for θ should itself be counted as an output. Altogether then, one has obtained 18 quantities for the price of 6, which is not too bad a bargain. What is perhaps more relevant, however, is that one has obtained, probably for the first time ever, a comprehensive picture of the whole mass and mixing pattern of both quarks and leptons with a single all-embracing mechanism and the same parameters throughout.

The trajectory for α from which these results are obtained is shown in Figure 1. It is seen to possess certain characteristics, e.g. that it accelerates in rotation as the scale μ decreases, while the (geodesic) curvature of the curve it traces out is reduced, in a manner to be detailed in the next section, but is otherwise quite innocuous.

More details of the calculation will be supplied in Appendix A so that, with the help of Figure 1, interested readers can readily verify for themselves the validity of the assertions made as well as the numerical values of the output quantities cited.

2 The Mechanism and How it Works

As the scale changes, the vector α traces out a curve on the unit sphere, for which it is found convenient to adopt the Serret–Frenet–Darboux formalism for curves on a surface [19]. At every point on the curve, one sets up then a Darboux triad, consisting of first the normal \mathbf{n} to the surface, then the tangent vector $\boldsymbol{\tau}$ to the curve, and thirdly the normal $\boldsymbol{\nu}$ to both the above, all normalized and orthogonal to one another. The Serret–Frenet–Darboux

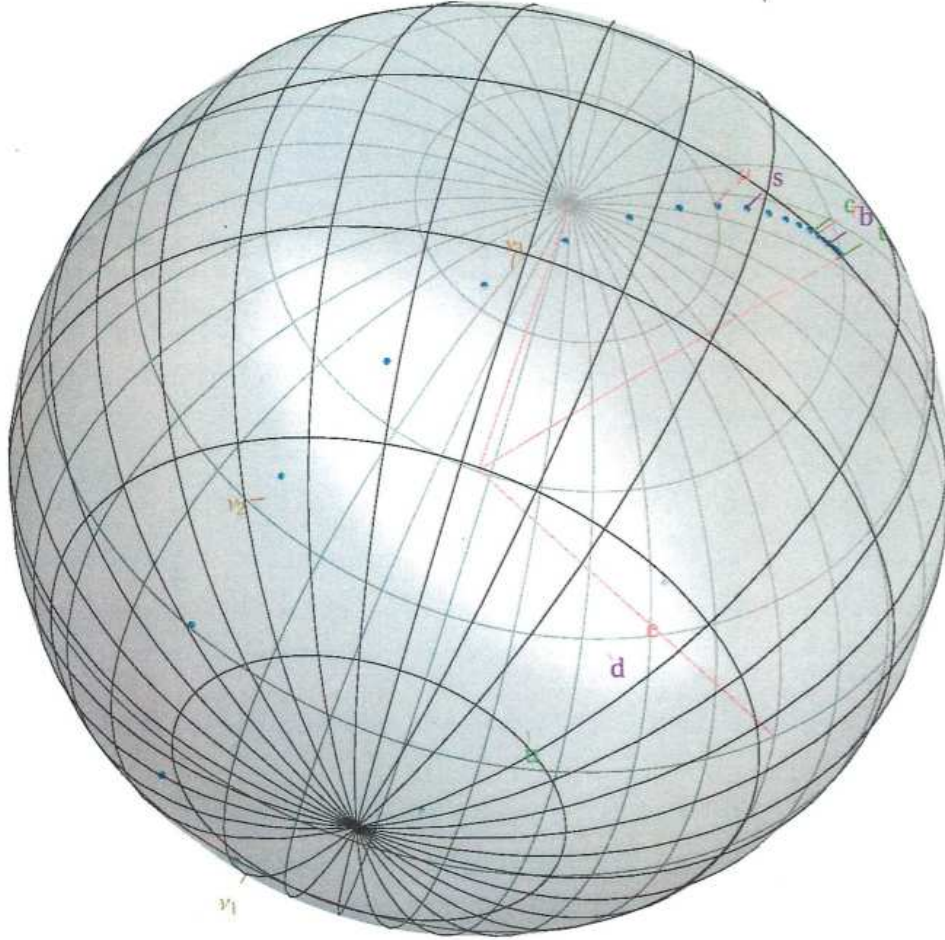


Figure 1: The trajectory traced out by α on the unit sphere as the scale μ decreases over some 6 decades in orders of magnitude from $\mu \sim m_t$ to $\mu \sim m_e$. The points shown on the curve are spaced at equal intervals of $1/5$ of a decade in $\ln \mu$ indicating from their spacing the “local” speed of rotation at any μ . The rotation is seen to accelerate as μ decreases, while the geodesic curvature (i.e. sideways bending) of the curve traced out is reduced. Also, along the trajectory, are marked out the individual mass scales of all the 12 quark and lepton states to show the position they each occupy, where we note that u, d and e are located at the back of the sphere.

formulae then say:

$$\begin{aligned}
\mathbf{n}' &= \mathbf{n}(s + \delta s) = \mathbf{n}(s) - \kappa_n \boldsymbol{\tau}(s) \delta s + \tau_g \boldsymbol{\nu}(s) \delta s, \\
\boldsymbol{\tau}' &= \boldsymbol{\tau}(s + \delta s) = \boldsymbol{\tau}(s) + \kappa_n \mathbf{n}(s) \delta s + \kappa_g \boldsymbol{\nu}(s) \delta s, \\
\boldsymbol{\nu}' &= \boldsymbol{\nu}(s + \delta s) = \boldsymbol{\nu}(s) - \kappa_g \boldsymbol{\tau}(s) \delta s - \tau_g \mathbf{n}(s) \delta s,
\end{aligned} \tag{10}$$

correct to first order in δs , a small increment in the arc-length s , where κ_n is known as the normal curvature (bending along the surface), κ_g the geodesic curvature (bending sideways) and τ_g the geodesic torsion (twist). For the special case here of a curve on the unit sphere, the surface normal \mathbf{n} is the radial vector $\boldsymbol{\alpha}$, $\kappa_n = 1$ and $\tau_g = 0$, so that the formulae reduce to:

$$\begin{aligned}
\boldsymbol{\alpha}' &= \boldsymbol{\alpha}(s + \delta s) = \boldsymbol{\alpha}(s) - \boldsymbol{\tau}(s) \delta s, \\
\boldsymbol{\tau}' &= \boldsymbol{\tau}(s + \delta s) = \boldsymbol{\tau}(s) + \boldsymbol{\alpha}(s) \delta s + \kappa_g \boldsymbol{\nu}(s) \delta s, \\
\boldsymbol{\nu}' &= \boldsymbol{\nu}(s + \delta s) = \boldsymbol{\nu}(s) - \kappa_g \boldsymbol{\tau}(s) \delta s.
\end{aligned} \tag{11}$$

The curve will thus be specified just by giving the geodesic curvature κ_g as a function of the arc-length s .

For the problem here, one is interested not only in the shape of the curve traced out by $\boldsymbol{\alpha}$ but also in the speed with respect to the scale μ at which $\boldsymbol{\alpha}$ moves along that curve. To specify the ‘‘trajectory’’ of $\boldsymbol{\alpha}$, then, one will need to give not only κ_g as a function of s but also s as a function of μ .

These two functions we propose now to parametrize as follows, inspired by a dynamical model [20, 21, 22] we have been considering. For s as a function of $\ln(\mu)$ (μ in GeV) we propose an exponential, thus:

$$s = \exp[-a - b \ln(\mu)] - \exp[-a - b \ln(m_t)] \tag{12}$$

and for κ_g as a function of s , we propose the tail of a square-root Breit-Wigner function, namely:

$$\kappa_g = \frac{g}{\sqrt{(s - s_0)^2 + s_0^2}}, \tag{13}$$

with g being the residue, and s_0 denoting the position of a pole near the physical region. The reason for these proposals is that in the model cited, there is a fixed point for $\boldsymbol{\alpha}$ near $\mu = m_t$, where the speed at which $\boldsymbol{\alpha}$ rotates tends to zero while the geodesic curvature κ_g tends to infinity [22]. What seems to matter is only the distance of t to the pole, not the ratio of its real and imaginary parts, so that we can use just the one parameter s_0 throughout.

However, there is nothing very special about these parametrizations and any similar choice of functions would likely lead to similar results.

With our choice of (12) and of (13) above, the trajectory for $\boldsymbol{\alpha}$ is specified by the 4 parameters a, b, g, s_0 . Once given a set of values for the 4 parameters a, b, g, s_0 , one can reconstruct the trajectory by integrating the equations in (11). And once one has the trajectory, one can then calculate the mass ratios and the mixing parameters for both quarks and leptons.

To calculate the masses of the members of any fermion species, say the U -type quarks t, c, u , one has first to normalize by supplying, say the mass $m_U = m_t$ of the heaviest state t as listed in Table 1. The mass of the c quark is then obtained as per (2) and (3) by solving the equation:

$$m_c = m_U |\mathbf{c} \cdot \boldsymbol{\alpha}(\mu = m_c)|^2. \quad (14)$$

And once we know the masses of t and c , we can then solve (2) for all the state vectors $\mathbf{t}, \mathbf{c}, \mathbf{u}$ of the U -triad. A similar procedure applies for the D -type quarks to give the D -triad. Combining the results, we can then calculate the matrix V_{UD} in (4) above.

As noted before, however, this real mixing matrix V_{UD} is not yet the CKM matrix we seek. If there is a nonvanishing theta-angle term in the QCD action, as there is in general no reason not to have in theory, then strong interaction will be CP-violating, which is a situation experiment will find hard to accommodate. However, as was mentioned above and worked out in detail in [9], this unwanted theta-angle term can be eliminated in the present R2M2 scheme, hence making the strong sector CP-conserving as desired, by a judicious chiral transformation on the quark state vectors, thus making them complex. The easiest way to specify the effect of this is as follows [5]. At $\mu = m_t$, we recall that $\boldsymbol{\alpha}$ coincides with the state vector \mathbf{t} for the t quark. The state vectors \mathbf{c} and \mathbf{u} must then be orthogonal to $\boldsymbol{\alpha}$ there and to each other, the three quarks t, c and u being by definition independent quantum states. They must therefore be related to $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$, the other two members of the Darboux triad at $\mu = m_t$, which are also orthogonal to $\boldsymbol{\alpha}$, just by a rotation about $\boldsymbol{\alpha}$, say by an angle ω_U , thus:

$$(\mathbf{t}, \mathbf{c}, \mathbf{u}) = (\boldsymbol{\alpha}, \cos \omega_U \boldsymbol{\tau} + \sin \omega_U \boldsymbol{\nu}, \cos \omega_U \boldsymbol{\nu} - \sin \omega_U \boldsymbol{\tau}). \quad (15)$$

Similarly, for the D -type quarks, we have:

$$(\mathbf{b}, \mathbf{s}, \mathbf{d}) = (\boldsymbol{\alpha}', \cos \omega_D \boldsymbol{\tau}' + \sin \omega_D \boldsymbol{\nu}', \cos \omega_D \boldsymbol{\nu}' - \sin \omega_D \boldsymbol{\tau}'), \quad (16)$$

where $(\boldsymbol{\alpha}', \boldsymbol{\tau}', \boldsymbol{\nu}')$ is now the Darboux triad taken at $\mu = m_b$. What the chiral transformation which eliminates the theta-angle term does is to give the $\boldsymbol{\nu}$ component of each of the above vectors a phase $\exp(-i\theta/2)$, these becoming thus:

$$(\tilde{\mathbf{t}}, \tilde{\mathbf{c}}, \tilde{\mathbf{u}}) = (\boldsymbol{\alpha}, \cos \omega_U \boldsymbol{\tau} + \sin \omega_U \boldsymbol{\nu} e^{-i\theta/2}, \cos \omega_U \boldsymbol{\nu} e^{-i\theta/2} - \sin \omega_U \boldsymbol{\tau}) \quad (17)$$

$$(\tilde{\mathbf{b}}, \tilde{\mathbf{s}}, \tilde{\mathbf{d}}) = (\boldsymbol{\alpha}', \cos \omega_D \boldsymbol{\tau}' + \sin \omega_D \boldsymbol{\nu}' e^{-i\theta/2}, \cos \omega_D \boldsymbol{\nu}' e^{-i\theta/2} - \sin \omega_D \boldsymbol{\tau}'). \quad (18)$$

The CKM matrix can then be evaluated with these two new triads of state vectors as before by (4), but this will now be the true complex CKM matrix containing a CP-violating phase. For instance, if one calculates the Jarlskog invariant³ of this new matrix, one will obtain a nonzero answer. On the other hand, the formulae for the masses of the various quark states are not affected by this chiral transformation [5].

The same procedure as the above for calculating the quark masses applies to the charged lepton states. For the neutrinos, there is a difference in that the R2M2 mechanism applies to the Dirac masses, which for neutrinos are unknown, being obscured via a possible see-saw mechanism [23] by the Majorana mass term for the right-handed neutrino. We have therefore to supply the Dirac mass $m_{\nu_3}^D$ of the heaviest neutrino ν_3 as a parameter, as listed in Table 1 above. Once given some value for $m_{\nu_3}^D$, the same procedure as before will yield the Dirac masses of the 2 lighter neutrinos. To compare these output values with experiment, the Dirac masses have to be inferred from the physical masses of the neutrinos measured in experiment by assuming some specific see-saw mechanism. Here, we have taken the simplest model: Type I quadratic see-saw mechanism [23] in which the physical masses of the neutrinos are given as:

$$m_{\nu_i} = (m_{\nu_i}^D)^2 / m_{\nu_R}. \quad (19)$$

It is the values inferred in this way which are entered in the Table 3 for comparison with the output values, and gives incidentally also an estimate for the right-handed neutrino mass m_{ν_R} .

³Note however that in R2M2, where the orientation of the mass matrix is scale dependent, CP-violation in the CKM matrix cannot be directly related to commutators of (hermitian) mass matrices as proposed by Jarlskog in [10]. Instead, one has to rely solely on the unitarity properties of the mixing matrix and work with the quartic rephasing invariants which are scale independent. These invariants appeared in earlier work on CP-nonconservation [11] without use being made of the mass matrix commutators.

The same procedure as the above for calculating the CKM matrix applies in principle also to the PMNS matrix except for the question whether the chiral transformation for eliminating the theta-angle term in the QCD action should affect the leptons as well. Here, in the text, as was explained, we shall consider only the case when it does not, leaving the other case to be considered in Appendix B. The PMNS matrix for leptons is then real and can be calculated in the same way as the UD -matrix for quarks in (4) above. As already mentioned, we neglect for now any possible Majorana phases, the detection of which appears to be only possible in the remote future [24].

Having now completed the specification of how the 16 mass ratios and mixing parameters are calculated given any set of values for the 6 parameters $a, b, g, s_0, \theta, m_{\nu_3}^D$, we can now start varying the input values of the 6 parameters to see whether any choice of those can give a decent description of the data. Before doing so, it is worth gaining first some qualitative understanding of how the value of each parameter will affect the agreement of the output with experiment.

Let us start with the parameters a and b , which govern the dependence of s on μ . Some quantities, such as the mass of the muon m_μ and the CKM element V_{tb} depend rather simply each on the rotation angle between two states, namely between $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ for m_μ , and between \mathbf{t} and \mathbf{b} for V_{tb} . They would thus be sensitive to how the arc-length s , or the angle subtended at the centre of the sphere, depends on the scale μ , or in other words to the parameters a and b . The two quantities m_μ, V_{tb} have also both been determined accurately by experiment, and so can be used as anchors for these two parameters. Indeed, based more or less just on this observation, approximate values for a, b have already been obtained some 10 years ago [25], and the new values we shall suggest will not deviate much from those.

Next, for the two parameters g and s_0 governing the dependence of κ_g on s , it is instructive first to examine the situation when θ_{tb} , the angle between $\mathbf{t} = \boldsymbol{\alpha}(\mu = m_t)$ and $\mathbf{b} = \boldsymbol{\alpha}(\mu = m_b)$, is small, which indeed it is experimentally, its value estimated from the experimental value of V_{tb} by the consideration in the above paragraph being only ~ 0.041 . Substituting θ_{tb} for δs in (11),(15), (16) then easily yields the following approximate formula for the mixing matrix (4) [13]:

$$V_{UD} = \begin{pmatrix} \cos \omega - \kappa_g \sin \omega \theta_{tb} & \sin \omega + \kappa_g \cos \omega \theta_{tb} & \sin \omega_U \theta_{tb} \\ -\sin \omega - \kappa_g \cos \omega \theta_{tb} & \cos \omega - \kappa_g \sin \omega \theta_{tb} & -\cos \omega_U \theta_{tb} \\ -\sin \omega_D \theta_{tb} & \cos \omega_D \theta_{tb} & 1 \end{pmatrix} \quad (20)$$

with $\omega = \omega_D - \omega_U$. We notice there that the geodesic curvature κ_g appears prominently in the Cabibbo angle V_{us}, V_{cd} . Again, the Cabibbo angle, being well measured experimentally, can be used as an anchor for the parameters g and s_0 appearing in (13). The same observation applies qualitatively to the solar neutrino angle U_{e2} in the leptonic equivalent matrix U_{LN} , although the angles involved in this case are no longer small. Thus, by adjusting the parameter g , one can fit roughly the output value of the Cabibbo angle to its experimental value, and by adjusting the position of the pole, namely s_0 in (13), the relative magnitude of κ_g between the quark and lepton sectors and hence the relative magnitude of V_{us} to U_{e2} .

There are two things still missing in the preceding arguments, namely the effect of the theta-angle on the phases of the mixing matrices, and that of the Dirac mass of ν_3 . The former affects mostly the Jarlskog invariant J^q for quarks, while the latter, by specifying where the state vector ν_3 will lie on the trajectory, mostly the atmospheric angle $U_{\mu 3}$, and can each be used to anchor the corresponding quantity. However, they will both also affect the quantities already considered, as will also the other 4 parameters affect the present 2 quantities, though all to a smaller extent. Nevertheless, armed with these preliminary understanding of how the values of our 6 parameters will affect mainly which of the output quantities, it is not difficult to jiggle them around to achieve a tolerably decent fit of the quantities concerned. Working in this way, one arrives at the following set of values for our parameters, which is what gives the result summarized in the preceding section:

$$a = 2.27635; b = 0.512; g = 0.42214; s_0 = -0.069; \theta = 1.28; m_{\nu_3}^D = 20 \text{ MeV}. \quad (21)$$

There is of course nothing immutable about these parameters, nor indeed even about the parametrizations (12) and (13). They are meant to represent just a sample fit by R2M2 to data, and any parametrization which gives a trajectory for α with a similar shape to that shown in Figure 1 is likely to do quite as well. For instance, there has been no serious attempt to optimize the parameters in (21) on our part, for indeed, given the great disparity in accuracy achieved in experiment for the 16 quantities we are after, it is not clear what optimization of the parameters should mean in fitting them. What we wish to assert is that with the above sample choice, one seems able to obtain an adequate description of the data.

Notice that with the 6 parameters we have permitted ourselves, we can anchor but 6 of our output quantities by experiment, and these 6 quantities,

as seen, can all be fitted without trouble to within experimental errors or to a high accuracy as in the case of the μ mass. But the other 10 quantities we also want will then have to fend for themselves. That they all have managed to do so reasonably well should thus be regarded as a test of the R2M2 hypothesis.

Before moving on further, the following notes on some points of detail would be in order:

(a) First, one notes that the corner elements in (20) are proportional to $\sin\omega_{U,D}\theta_{tb}$, where each of the angles is itself a consequence of the rotation. These corner elements V_{ub} and V_{td} , therefore, are second order effects of the rotation [13]. This comes about in R2M2 just from the geometrical fact that for a curve on the unit sphere, the geodesic torsion vanishes so that any twist that these elements represent will have to be of at least the second order in the change in arc length δs . It thus follows immediately that they are small, as experiment shows. It also follows that they have the asymmetry $V_{td} > V_{ub}$, again as experiment wants, since by virtue of the behaviour (12) of s as a function of μ , one has $\omega_D > \omega_U$. For more details on these points, see [13]. However, the down side is that, being second order effects in rotation, these corner elements are rather difficult to reproduce accurately in the R2M2 scheme, hence their output values in (5) lying outside the experimental errors in (7) above.

(b) Secondly, one notes that of the parameters in (21), 4 (a, b, g, θ) are anchored to experiment in the high scale region with $\mu > 100$ MeV, using data there of high accuracy, namely $m_\mu, V_{tb}, V_{us} \sim V_{cd}$ and J^q for quarks. The other 2, namely s_0 and $m_{\nu_3}^D$, are anchored on neutrino oscillation data of understandably lower accuracy. This means that the rotation trajectory is quite well determined in the region $\mu > 100$ MeV, less well but still adequately down to about $m_{\nu_2}^D \sim 6$ MeV, below which, however, it is obtained just by extrapolation from the high scale region. This extrapolation does not affect, of course, the masses of the two heavier generations, nor the two mixing matrices since these depend only on the state vectors which, including those of the lightest generation, are already determined by (2) at the scale of the middle generation. The masses of the lightest generation, however, depend on the extrapolation, and given the assumed exponential shape of (12), this can cover quite a considerable range in arc-length s , as is seen in Figure 1. In view of this, that one is able to get a roughly correct value (i.e. 0.7 as compared to 0.5 MeV) for the e mass is already better than can reasonably be expected. In any case, reproducing the mass of the electron

being such an audacious undertaking in itself, an error of this sort probably needs no apology.

(c) Thirdly, again with reference to the e mass, and those of u and d in the lightest generation, one notes that there are in fact more than one solution each to the equations (3). We recall that at the mass scale of the middle generation, say c to be specific, the state \mathbf{u} has zero eigenvalue, but as the scale decreases from m_c , $\boldsymbol{\alpha}(\mu)$ will acquire a component $\langle \mathbf{u} | \boldsymbol{\alpha}(\mu) \rangle$ in the direction of \mathbf{u} . This will gradually increase in size until at some point $\langle \mathbf{u} | \boldsymbol{\alpha}(\mu) \rangle$ matches the value $\sqrt{\mu/m_t}$ and we have, according to (3), a solution, say the first (higher) solution, to the u mass. However, as μ decreases further, $\langle \mathbf{u} | \boldsymbol{\alpha}(\mu) \rangle$ may vanish again, in which case one may find in the neighbourhood again a solution for $\langle \mathbf{u} | \boldsymbol{\alpha}(\mu) \rangle = \sqrt{\mu/m_t}$, namely a second (lower) solution to (3) for the u mass. Indeed, this is what happens to the trajectory here and the mass values cited in Table 3 for u , d , and e correspond to this lower, likely more stable solution. The limit cited in Table 3 for the Dirac mass of the lightest neutrino ν_1 , however, corresponds to the higher solution since the lower solution for $m_{\nu_1}^D$ has not yet been found in the range of scales explored and we are reluctant to trust the extrapolation further. An interesting phenomenological point to note here is that given the great disparity in value between the normalization factors m_T in (1) for the 3 species U, D, L , namely the masses m_t, m_b, m_τ , one would have expected a similar disparity between the output masses of the lower generations obtained by leakage from the heaviest. This is indeed borne out for the middle generation, as seen in Table 3, as well as for the higher solution to u , d and e (not shown), which latter would be in stark contradiction to experiment. However, as seen in Table 3, one gets from the lower solution similar masses of order MeV for all 3 species u, d, e , which is apparently what experiment wants. The question why the lightest generation, in particular, is prone to having such lower solutions to their mass in the R2M2 scheme is an interesting one deserving more clarification, which will be supplied later, with other details, in Appendix C.

(d) Fourthly, quarks being confined, experiment cannot measure the masses of the lighter quarks s, u, d directly at their own mass scales but only at a higher scale, usually taken to be around 1 or 2 GeV. Since the perturbative method currently available for QCD does not allow for running to such low scales, there is no way at present to compare the output masses of the light quarks determined at their own mass scales with their masses given in experiment, e.g. in [12]. For the s quark, the mass scale being of order ~ 100 MeV, not that much below the scale 1–2 GeV at which it is measured,

one can perhaps claim that our output value of 170 MeV at its own mass scale is in reasonable agreement with the value of about 100 MeV given by experiment at a scale of 1–2 GeV. For the u and d quarks, however, one can go no further than saying that the output values are roughly of the same MeV order as the experimentally given values. We have thus no explanation whatsoever so far of the crucial fact why the u quark should be lighter than the d .

(e) Lastly, based on the see-saw mechanism outlined above, one can estimate also the physical masses of the 3 generations of neutrinos, as respectively 0.05, 0.005, < 0.0003 eV, normalized at $m_{\nu_3} \sim 0.05$ eV, as estimated from $\delta m_{23}^2 \sim 2.43 \times 10^{-3}$ eV². Here we have normal hierarchy, as is natural from the leakage mechanism and universality that R2M2 implies. Notice that the above output values for the physical masses of ν_2 and ν_1 would correspond to a value of δm_{12}^2 of only $\sim 2.5 \times 10^{-5}$ eV², i.e., a factor of 3 smaller than the experimental value of $7.59 \pm 0.21 \times 10^{-5}$ eV² [12]. However, since δm_{12}^2 is proportional to the 8th power of the rotation angle calculated by the R2M2 scheme, this factor of 3 translates only to an error of about 15 percent in the rotation angle, which is not bad in view of the extrapolation involved as detailed in (b). These masses correspond to a right-handed neutrino mass of about 8000 TeV, and, as far we know, there are no stringent empirical bounds on this [26].

3 A Closer Look at Lepton Mixing and the PMNS Matrix

Since the PDG website [12] giving the bounds quoted in Table 2 was last updated, 5 experiments [14, 15, 16, 17, 18] have given exciting new results on the mixing angle θ_{13} which now need to be taken into account. We shall do so in the order in which these results appeared.

As regards the two appearance experiments T2K and MINOS, the quantity they quote is $2 \sin^2 \theta_{23} \sin^2 2\theta_{13}$, and the result depends on the CP phase δ_{CP}^ℓ . It is thus the quoted value of this quantity at $\delta_{CP}^\ell = 0$ that should be compared with the value of $2 \sin^2 \theta_{23} \sin^2 2\theta_{13}$ from our output above in Table 2. The result of this comparison is given in Table 4 where it is seen that our output value agrees with the experiments to the extent that the two experiments agree between themselves.

Source		$2 \sin^2 \theta_{23} \sin^2 2\theta_{13}$
Our Output		0.0628
T2K	68%C.L.	$0.12^{+0.09}_{-0.06}$
	90%C.L.	$0.12^{+0.16}_{-0.07}$
MINOS	68%C.L.	$0.04^{+0.05}_{-0.03}$
	90%C.L.	$0.04^{+0.08}_{-0.04}$

Table 4: Output compared with the values of $2 \sin^2 \theta_{23} \sin^2 2\theta_{13}$ at $\delta_{CP}^\ell = 0$ read from [14, 15].

Source	$\sin^2 2\theta_{13}$
Our Output	0.0842
Double Chooz	$0.086 \pm 0.041(\text{stat}) \pm 0.030(\text{syst})$
Daya Bay	$0.092 \pm 0.016(\text{stat}) \pm 0.005(\text{syst})$
RENO	$0.113 \pm 0.013(\text{stat}) \pm 0.019(\text{syst})$
Combined Fit (Blondel)	0.084 ± 0.014

Table 5: Output value for $\sin^2 2\theta_{13}$ compared with Double Chooz [16], Daya Bay [17], RENO [18] and the combined fit of the first two experiments together with T2K [14] and MINOS [15] by A. Blondel [27].

The other three disappearance experiments, Double Chooz, Daya Bay, and RENO measured simply $\sin^2 2\theta_{13}$ and the result does not depend on the CP phase δ_{CP}^ℓ so their result can be directly compared with our output value, as is done in Table 5. The agreement is seen to be very good.

Finally one can compare our output value with the combined fit for $\sin^2 2\theta_{13}$ of the results from the first four experiments given by A. Blondel in his summary talk at the recent Moriond meeting [27]. As seen also in Table 5, our output value falls right in the middle of the allowed experimental range.

That our output value for θ_{13} should agree so well with experiment is in a sense a little fortuitous for the following reason. As explained in the preceding section, of the 6 parameters for the present R2M2 fit, 5 are anchored on quantities which have been measured with fair accuracy, leaving only $m_{\nu_3}^D$,

the Dirac mass of the heaviest neutrino poorly determined, on which the value of θ_{13} mostly depends. The value of $m_{\nu_3}^D$ does not affect much the other quantities under study apart from those associated with neutrinos. And of the elements of the PMNS matrix, it is the entries of the last column which it will affect the most, as can be seen as follows. The 3 elements of the last column may be thought of as the direction cosines subtended by the state vector for ν_3 on the charged lepton triad, the squares of which add up of course to unity. If the trajectory of α were to lie on a plane, then θ_{13} (i.e. θ_{e3}) would be zero, so that the departure of θ_{13} from zero can be thought of as a measure of the nonplanarity of the trajectory. The further the state vector for ν_3 deviates from the plane containing $\alpha(\mu = m_\tau)$ and $\alpha(\mu = m_\mu)$, the more $\sin \theta_{13} = |U_{e3}| = \sqrt{1 - |U_{\mu 3}|^2 - |U_{\tau 3}|^2}$ will differ from zero. Now we have seen already in the preceding section that in view of the largish values of both the Cabbibo angle $|V_{us}| \sim |V_{cd}|$ and the solar neutrino angle $\sin \theta_{12} \sim |U_{e2}|$, the trajectory must be nonplanar to some fixed extent, in which case, the further that the vector for ν_3 is from that of τ , the greater will be the nonplanarity and the larger the value of θ_{13} as a result. Thus, by adjusting the value of the parameter $m_{\nu_3}^D$, namely the Dirac mass of the heaviest neutrino, and hence the proximity of the state vector of ν_3 to that of τ , one can change the output value of θ_{13} . It was therefore only by a bit of luck that by putting in (21) for $m_{\nu_3}^D$ the value 20 MeV, a round number, that one has ended up with a value for $\sin^2 2\theta_{13}$ so close to the experimental value. The very good agreement obtained in Table 4 and 5 should thus be regarded not so much as a successful prediction, although it was indeed obtained before the data in Table 5 appeared, but as a test of consistency with experiment, first for R2M2, and secondarily for the trajectory as parametrized in (21), even which, of course, is already of significance.

As noted, the output value of θ_{13} can be changed by changing the input value of the parameter $m_{\nu_3}^D$ without affecting the quality of the rest of the fit except that as regards the neutrinos. The changes induced on the output values of the various neutrino quantities, however, will be correlated so that the output value for θ_{13} , for example, will be constrained by the experimental bounds on θ_{12} and θ_{23} , and vice versa. Keeping then the rest of (21) the same and only varying $m_{\nu_3}^D$, one readily comes to the following interesting conclusions:

- $\sin^2 2\theta_{13} > 0.077$ if $\sin^2 2\theta_{12}$ and $\sin^2 2\theta_{23}$ are to remain within their experimental bounds quoted in Table 2 from [12].

- $\sin^2 2\theta_{13} \sim 0.162 > 0.15$ if θ_{23} is to be maximal, and $\sin^2 2\theta_{12}$ is to remain within its experimental bounds given in Table 2.

In other words, both of the two conditions: (i) θ_{13} is vanishing, and (ii) θ_{23} is maximal, much favoured by theoretical models previously [28], would have been excluded by the present R2M2 fit, given just the earlier bounds cited in Table 2 from [12]. But given now the new data summarized in Tables 4 and 5 confirming a nonzero value for θ_{13} , the R2M2 fit would then give predictions for the amount that θ_{23} must depart from maximal. Indeed, if one were to force θ_{23} to be maximal, then the output value of $\sin^2 2\theta_{13}$ will have to miss the Blondel value given in Table 5 by more than 5.5σ . Conversely, inputting the Blondel value for $\sin^2 2\theta_{13}$ with his errors, then the present fit would predict the following value for θ_{23} :

$$\sin^2 2\theta_{23} \sim 0.935 \pm 0.021, \quad (22)$$

a predicted departure from maximal mixing accessible to future tests by experiment.

4 Concluding Remarks

From the results summarized in sections 1 and 3, it would seem that the R2M2 scheme is capable of a comprehensive description of the mass and mixing pattern of both quarks and leptons, which is otherwise so bewildering. Indeed, when supplemented by a parametrization of the rotation trajectory, it seems to have succeeded even in reproducing satisfactorily the mass and mixing parameters. To our knowledge, this is the first time that such has been achieved by any scheme. Given that the generations puzzle, of which these patterns are a part, has long been regarded as one of the great mysteries of particle physics, its explanation would seem a nontrivial achievement. That is, of course, assuming that the results and the method by which they are derived manage to survive the community's close scrutiny.

It is anticipated that much of the scrutiny will be focussed on the unusual property of R2M2 mentioned in the introduction of being able to generate nonzero masses for all quarks and leptons from a universal fermion mass matrix of rank one. As pointed out there, this may seem counter-intuitive in the context of a non-rotating mass matrix and has been the cause of some doubts and criticisms we have encountered in private discussions. As far

as we are aware, these have not appeared in the public domain, although the R2M2 scheme has existed for more than a decade and been developed in numerous publications since it was proposed. But now, in view of its quite alluring consequences as detailed above, it would be profitable for any criticism to be aired and debated in public, for only when they are will the point be properly settled. It is after all just a matter of logic, or so at least it seems to us, which is capable of being sorted out.

However, even if the R2M2 scheme manages to pass all scrutiny on its logical validity and all its attractive results as exhibited in this paper are confirmed, it still leaves the question, of course, why the fermion mass matrix should be of rank one, and why it should rotate. In other words, even if it is successful in explaining mixing and mass hierarchy, the R2M2 as it stands is at best an empirical rule, playing the role for quarks and leptons, perhaps, as the Mendeleev periodic table played for atoms or the Rydberg formula for the hydrogen spectrum. There is still missing the equivalent here of the quantum theory in atomic physics which underlies it all.

For this, of course, one will have to go beyond the standard model, which leaves therefore the field wide open. We have ourselves suggested a possible approach based on what we call framed gauge theory (FGT) [21], namely a gauge theory in which are introduced as dynamical variables not only the usual bosonic gauge and fermionic matter fields, but also, in analogy to the vierbeins in gravity, the frame vectors in internal symmetry space as (Lorentz) scalar fields. This idea has the attractive feature of giving a geometric meaning not only to exactly 3 fermion generations but also to the standard Higgs field needed to break the electroweak symmetry. It has been shown [21, 22] also to lead to a fermion mass matrix which is universal, of rank one, and rotates with changing scales, quite as needed here, although it is not known yet whether it will give a trajectory of the appropriate shape. Whatever its future, however, this framed standard model (FSM) is but one possible implementation of the R2M2 idea which has led to the above results. It is not hard to imagine many other possibilities, and some, we believe, are already on the way [29].

Optimistically then, the R2M2 scheme may play the role of, say, the Mendeleev periodic table to particle physics. And, as the periodic table has led to the quantum theory of atoms, so may, let us hope, the R2M2 scheme, especially now that one has gained here a rough idea of what the rotation trajectory should look like, give us a hint at last to solving the great particle physics mystery of fermion generations.

Postscriptum

After this paper was completed, an article appeared on the preprint arXiv by G.L. Fogli et al. [32] reporting on a global analysis of neutrino mixing data. The values they give on the mixing angles, corresponding at 1σ to: (1) $\sin^2 2\theta_{12} = 0.851 \pm 0.026$, (2) $\sin^2 2\theta_{13} = 0.096_{-0.012}^{+0.013}$, (3) $\sin^2 2\theta_{23} = 0.958_{-0.024}^{+0.022}$, are all consistent with our output values listed in Tables 2 and 5. They note in particular the indication of a departure of θ_{23} from its maximal value of $\pi/4$, as was predicted by our considerations in Section 3. The predicted value in equation (22) obtained by inputting the Blondel value for θ_{13} [27] is within 1σ of their fitted value. However, had we input instead their value of θ_{13} from (2) above, we would have obtained the prediction $\sin^2 2\theta_{23} = 0.953_{-0.018}^{+0.016}$ almost right on top of their value. They also note an indication for a value of $\delta_{CP}^\ell = (0.89_{-0.44}^{+0.29})\pi$ corresponding to $\sin \delta_{CP}^\ell = 0.34_{+0.65}^{-0.88}$ which is again consistent with the bound predicted in equation (27).

Appendix A: Some computational details to facilitate the verification of the results obtained above.

The results of the present R2M2 fit, as summarized in Sections 1 and 3, are obtained from numerical calculations. In this appendix we aim to supply interested readers with sufficient computational details so that they can readily verify the results for themselves. In the numerical results below we provide more significant figures than necessary to avoid cumulative rounding errors.

First, we give the values of the vector $\boldsymbol{\alpha}$ at the mass scales of the various

quarks and lepton states:

$$\begin{aligned}
\boldsymbol{\alpha}^\dagger(\mu = m_t) &= (1.00000, 0.00000, 0.00000) \\
\boldsymbol{\alpha}^\dagger(\mu = m_b) &= (0.99912, -0.04176, -0.00345) \\
\boldsymbol{\alpha}^\dagger(\mu = m_\tau) &= (0.99762, -0.06836, -0.00880) \\
\boldsymbol{\alpha}^\dagger(\mu = m_{\nu_3}) &= (0.74599, -0.52127, -0.41446) \\
\boldsymbol{\alpha}^\dagger(\mu = m_c) &= (0.99643, -0.08350, -0.01281) \\
\boldsymbol{\alpha}^\dagger(\mu = m_s) &= (0.97052, -0.22709, -0.08076) \\
\boldsymbol{\alpha}^\dagger(\mu = m_\mu) &= (0.95172, -0.28232, -0.12048) \\
\boldsymbol{\alpha}^\dagger(\mu = m_{\nu_2}) &= (0.27858, -0.56685, -0.77529) \\
\boldsymbol{\alpha}^\dagger(\mu = m_u) &= (-0.63101, 0.76714, 0.11543) \\
\boldsymbol{\alpha}^\dagger(\mu = m_d) &= (-0.45968, 0.81729, 0.34748) \\
\boldsymbol{\alpha}^\dagger(\mu = m_e) &= (-0.37336, 0.81904, 0.43563) \\
\boldsymbol{\alpha}^\dagger(\mu = m_{\nu_1}) &= (-0.71656, 0.21263, -0.66433), \tag{23}
\end{aligned}$$

where we note that for consistency, the mass values of the various states are taken as either the values listed in Table 1 or the output values listed in Table 3. From (23), it can readily be checked, for example by evaluating their polar co-ordinates, that these vectors do indeed lie on the trajectory shown in Figure 1.

Secondly, we give the state vectors of the various quark and lepton states:

$$\begin{aligned}
\mathbf{t}^\dagger &= (1.00000, 0.00000, 0.00000) \\
\mathbf{b}^\dagger &= (0.99912, -0.04176, -0.00345) \\
\boldsymbol{\tau}^\dagger &= (0.99762, -0.06836, -0.00880) \\
\boldsymbol{\nu}_3^\dagger &= (0.74599, -0.52127, -0.41446) \\
\mathbf{c}^\dagger &= (0.00000, 0.98843, 0.15168) \\
\mathbf{s}^\dagger &= (0.03989, 0.92267, 0.38351) \\
\boldsymbol{\mu}^\dagger &= (0.06476, 0.88601, 0.45913) \\
\boldsymbol{\nu}_2^\dagger &= (0.59501, 0.24219, 0.76636) \\
\mathbf{u}^\dagger &= (0.00000, -0.15168, 0.98843) \\
\mathbf{d}^\dagger &= (-0.01284, -0.38331, 0.92353) \\
\mathbf{e}^\dagger &= (-0.02359, -0.45861, 0.88833) \\
\boldsymbol{\nu}_1^\dagger &= (-0.29910, -0.81831, 0.49083) \tag{24}
\end{aligned}$$

It can again be readily checked that these satisfy their defining conditions in (2) with respect to the α 's in (23).

From (24), one can then easily evaluate:

- by taking their inner products with the corresponding α 's in (23), the masses of the lower generation states according to the formulae given in (3),
- by taking the inner products between up and down states the UD matrix (4) for quarks and the PMNS matrix for leptons.

To evaluate the CKM matrix for quarks, one has further to supply, thirdly, the tangent vector $\boldsymbol{\tau}$ and the normal vector $\boldsymbol{\nu}$ to the trajectory at respectively $\mu = m_t$ and $\mu = m_b$, thus:

$$\begin{aligned}
\boldsymbol{\tau}^\dagger(\mu = m_t) &= (0.00000, 1.00000, 0.00000) \\
\boldsymbol{\nu}^\dagger(\mu = m_t) &= (0.00000, 0.00000, 1.00000) \\
\boldsymbol{\tau}^\dagger(\mu = m_b) &= (0.04179, 0.98683, 0.15629) \\
\boldsymbol{\nu}^\dagger(\mu = m_b) &= (-0.00313, -0.15630, 0.98770)
\end{aligned} \tag{25}$$

from which one can easily check that:

$$\sin \omega_U = 0.15168; \quad \sin \omega_D = 0.23446. \tag{26}$$

This then allows one to calculate the individual CKM matrix elements, still just according to (4), although, the vectors being now complex, the matrix elements will also be complex. The Jarlskog invariant will then be nonzero and can be calculated using the formula for it given in [10, 11] (see also footnote 3).

In other words, with the information supplied above in this appendix, the interested reader can easily make spot checks on the cited numerical results, or even, with some patience, verify for themselves all the numerical results cited, and hence to satisfy themselves that the trajectory shown in Figure 1 does indeed lead, via the R2M2 mechanism, to the cited mass and mixing patterns, in close agreement with experiment, as claimed. And this, of course, is the primary question of our main concern.

As to the secondary question of whether the parametrization (12) and (13), with the values of the parameters given in (21), does indeed give rise to the trajectory shown in Figure 1, we know no other way of checking it at

present than by numerically integrating the Serret–Frenet–Darboux formula in (11), which is also straightforward to perform.

Appendix B: The case for a PMNS matrix with a nonvanishing CP-violating phase ($\sin \delta_{CP}^\ell \neq 0$).

As shown in [9], because of the presence of zero modes to the mass matrix at every scale, the R2M2 scheme will always allow a chiral transformation to be made on the state in the ν direction, whether of quarks or of leptons, without making the mass matrix complex. The effect of this transformation will then get transmitted by rotation to the mixing matrix and give it a CP-violating phase. Indeed, in the present set-up where one starts with a real α in (1), this seems the only way to make the otherwise real mixing matrix complex and to give the CP-violating phase a nonzero value. What distinguishes quarks from leptons so far is just that such a chiral transformation on quarks will affect the theta-angle term in the QCD action so that this chiral transformation becomes now not only possible but also necessary and in a sense unique, given that without it, and without having the chiral angle matched to θ , strong interactions will be CP-violating, in contradiction to experiment. For leptons, on the other hand, there seems a priori no reason as yet why such a chiral transformation should be made, and if it is made, no restriction on what value the chiral angle should take. Hence, the suggested simplest solution in the text when no such chiral transformation is made at all, so the mixing matrix remains real, with $\sin \delta_{CP}^\ell = 0$.

There is, however, nothing in principle to stop us making a chiral transformation on the leptons and give the PMNS matrix a CP-violating phase. Indeed, we shall present later in this appendix even some theoretical arguments for a situation where a chiral transformation similar to that for quarks becomes necessary also for leptons. Let us first examine, however, what effect such a chiral transformation will have on our previous results and what sort of values it can give for δ_{CP}^ℓ in the leptonic case.

As was found for the quark mixing matrix earlier, one effect of the chiral transformation in the R2M2 scheme is to enhance the (absolute) value of the top-right corner element in the mixing matrix, namely, for the leptonic case $\sin \theta_{13}$. Given that this quantity is tightly constrained both by our fit and by the new data now available, as explained at the end of section 3, there will not be very much room for this enhancement to operate. Indeed, it is

not hard by playing with the parameters to ascertain that for θ_{13} to remain within the experimental bounds of [27] and θ_{12} and θ_{23} to remain within the bounds cited in Table 2 from [12], the present setup (which neglects possible Majorana phases) can admit only a chiral transformation on leptons with a value for θ of at most 0.64, i.e. about half its previous value required for quarks as given in (21). This corresponds to a CP-violating phase:

$$|\sin \delta_{CP}^\ell| \leq 0.31. \quad (27)$$

The imposition of a chiral transformation on leptons with a θ of the same value (~ 1.28) as suggested in (21) above for quarks would lead to an output value for $\sin^2 2\theta_{13}$ larger than the Blondel value of Table 5 by nearly 4σ . The conclusion (27) is of interest to experiment, but it is not easy to unravel at this stage how much of it is due to the R2M2 scheme and how much to the particular manner by which it is here parametrized.

In the rest of this appendix we shall consider a theoretical situation where a chiral transformation similar to that for quarks becomes necessary also for leptons. This arises from the assumption in R2M2 that the vector α is “universal”, i.e. common not only to the up and down state of quarks and leptons separately but common to quarks and leptons as well. Notice that this last assumption was not made in the original scheme of [1, 2] previously cited, there being then no real initiative for doing so. It was made only in the R2M2 scheme based partly on the observation that it seems to work as seen above, and partly on theoretical prejudices [21, 22]. Nevertheless, once it is accepted, then a new situation can arise, as follows.

The electroweak theory is usually interpreted as one in which the original local gauge symmetry $su(2) \times u(1)$ is broken spontaneously by the Higgs field. However, it could equally be interpreted, according to 't Hooft [30] and to Banks and Rabinovici [31], as a theory in which the local $su(2)$ symmetry confines, what is broken being only a global, say $\widetilde{su}(2)$, symmetry associated or “dual” to it. In present applications of the theory, the two interpretations are mathematically equivalent [30], but some may find one intuitively more appealing than the other.

In the second interpretation where $su(2)$ confines, hereafter referred to as the confinement picture, (left-handed) quarks and leptons, as well as the $W - Z$ and Higgs bosons, appear not as fundamental fields but as bound (confined) states of such under $su(2)$ confinement. For example, a left-handed lepton would appear as a bound state of the fundamental $su(2)$ doublet

scalar field ϕ with a fundamental $su(2)$ doublet left-handed fermion field ψ_L , forming together an $su(2)$ singlet state as confinement requires, thus:

$$\chi_L = \phi^\dagger \psi_L, \quad (28)$$

while a left-handed quark would appear as a bound state of the same ϕ field but now with a fundamental $su(2)$ doublet left-handed fermion field, say ψ_{La} , carrying colour $a = 1, 2, 3$, thus:

$$\chi_{La} = \phi^\dagger \psi_{La}. \quad (29)$$

We notice, however, that the right-handed fundamental fermion fields, say ψ_R and ψ_{Ra} , $a = 1, 2, 3$, being by definition $su(2)$ singlets, need not be so confined, and can thus already function as the right-handed leptons and quarks respectively.

Adopting the above confinement picture for the R2M2 scenario under consideration, we would obtain, for the 3 basis states $\boldsymbol{\alpha}, \boldsymbol{\tau}, \boldsymbol{\nu}$ appearing in (11) above, the following left-handed leptons:

$$\phi^\dagger \boldsymbol{\alpha} \cdot \boldsymbol{\psi}_L, \quad \phi^\dagger \boldsymbol{\tau} \cdot \boldsymbol{\psi}_L, \quad \phi^\dagger \boldsymbol{\nu} \cdot \boldsymbol{\psi}_L, \quad (30)$$

and the following left-handed quarks:

$$\phi^\dagger \boldsymbol{\alpha} \cdot \boldsymbol{\psi}_{La}, \quad \phi^\dagger \boldsymbol{\tau} \cdot \boldsymbol{\psi}_{La}, \quad \phi^\dagger \boldsymbol{\nu} \cdot \boldsymbol{\psi}_{La}, \quad (31)$$

being both bound states, under $su(2)$ confinement, of the fundamental scalar ϕ with the 3-component (in generation space) fundamental fermion fields $\boldsymbol{\psi}$.

The question then arises where the new factors, $\boldsymbol{\alpha}, \boldsymbol{\tau}, \boldsymbol{\nu}$ which distinguish generations, originate, i.e. whether from the bound states' scalar or fermionic constituent. Given that $\boldsymbol{\alpha}$ which appears in the mass matrix (1) is supposed in R2M2 to be “universal”, i.e. the same for (30) and for (31), it seems this factor $\boldsymbol{\alpha}$ can come only from the fundamental scalar field.⁴ And since the other factors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are but constructs from $\boldsymbol{\alpha}$, the same conclusion would seem to apply to them as well.

Supposing this is true, then we have the following interesting result. We have noted before that a chiral transformation is to be performed on the $\boldsymbol{\nu}$

⁴We note that in the framed standard model FSM [21, 22] mentioned earlier as providing a possible theoretical basis for R2M2, the factor $\boldsymbol{\alpha}$ in the fermion mass matrix (1) does indeed originate from the scalar constituent, as stipulated. These scalar fields play in FSM a geometric role, namely that of framed vectors or vierbeins in the internal symmetry space.

component of the quark fields so as to eliminate the theta-angle term [9]. We note now that, in the confinement picture of 't Hooft and others as exhibited in (31), the same chiral transform can be obtained just by a phase change in the ν component of the scalar field. That this is so can be seen as follows. A chiral transformation $\exp(i\alpha\gamma_5)$ applied on a quark field ψ changes the left-handed component by the phase $-\alpha$ but the right-handed component by the opposite phase α , thus:

$$\begin{aligned} \exp(i\alpha\gamma_5)\psi &= \exp(i\alpha\gamma_5)\left[\frac{1}{2}(1 - \gamma_5)\psi + \frac{1}{2}(1 + \gamma_5)\psi\right] \\ &= \left[\exp(-i\alpha)\frac{1}{2}(1 - \gamma_5)\psi + \exp(i\alpha)\frac{1}{2}(1 + \gamma_5)\psi\right] \\ &= \exp(i\alpha)\left[\exp(-2i\alpha)\frac{1}{2}(1 - \gamma_5)\psi + \frac{1}{2}(1 + \gamma_5)\psi\right]. \end{aligned} \quad (32)$$

Or equivalently, it changes the phase of the whole of ψ by α , but the left-handed component by an extra phase -2α . And since a change of the overall phase of the quark field ψ has no physical significance, this is again equivalent to just changing the left-handed component by a phase -2α while leaving the right-handed component unchanged. But in the confinement picture, we see that this is exactly what changing the phase of the ν component of ϕ by -2α would do. We recall there that the left-handed quark is a bound state of the scalar field with a fundamental fermion, so that a change in phase in the former will be transmitted to the left-handed quarks. On the other hand, as noted above, the right-handed quarks are not bound states at all and are not affected by the change in phase of the scalar field. In other words, this means that, in the confinement picture we have adopted, just a change in phase of the ν component of ϕ by $-2\alpha = \theta$ will eliminate the theta-angle term from the QCD action, and give a CP-violating phase to the CKM matrix. We note also that a mere change in phase of ν will not affect the orthonormality of the basis Darboux triad.

The really interesting thing now is the following. If the chiral transformation to eliminate the theta-angle term is indeed to be transferred to a change of phase in the ν component of the scalar field as above suggested, then the same chiral transformation would have to be performed also on the lepton fields, for according to (30) above, the left-handed leptons, in the confinement picture, are bound states of the same scalar fields as in the quarks, although now to some different fundamental fermion field. As a result, the same argument that was applied to quarks via the rotation mechanism to deduce a CP-violating phase in the CKM matrix will apply also to the leptons to give a CP-violating phase in the PMNS matrix.

However, that the “same” chiral transformation be applied to both quarks and leptons does not necessarily mean that the value of θ is the same for both, for the two mixing matrices for quarks and leptons refer to very different scales and the angle θ may itself be scale-dependent. The conclusion (27) reached above would thus seem to suggest, that if this theoretical scenario were to hold, then the strong CP angle θ would have to lose about half its value in running from the quark to the leptonic scale.

Appendix C: Solutions for the lightest generation mass

Empirically, the lightest generation is a little anomalous. Whereas for the 2 heavier generations, one has $m_t \gg m_b > m_\tau$ and $m_c \gg m_s \stackrel{?}{>} m_\mu$, for the lightest generation one has $m_d > m_u > m_e$ and all of roughly the same MeV order. Of course, so long as one has no explanation at all for the mass spectrum, the anomaly is just a curiosity. But as soon as one aims to explain why the masses should take the values they do, then this departure of the lightest generation from the pattern of earlier generations would have to be accounted for. At first sight, the R2M2 scheme would seem to have difficulty with this, for in the mass matrix (1) it is the heaviest generation which sets the scale m_T for the masses in each species. And if the heaviest generation follows the above pattern, so should apparently the lower generations by leakage from the heaviest. This holds for the second heaviest generation, so why the anomaly in the lightest generation? As indicated in Section 2, point (c), the occurrence of multiple solutions to the mass equation (3) give us at least a partial answer.

But how do these multiple solutions to the mass equation (3) come about for the lightest generation, and especially why only for it, not for the middle one as well? To see this, we recall first again how solutions to the u mass come about. The state vector \mathbf{u} for u is by definition orthogonal to the \mathbf{tc} -plane. At $\mu = m_c$, $\boldsymbol{\alpha}(\mu)$ lies on the \mathbf{tc} -plane and is therefore orthogonal to \mathbf{u} and $\langle \mathbf{u} | \boldsymbol{\alpha}(\mu) \rangle$ vanishes. As μ decreases, the vector $\boldsymbol{\alpha}(\mu)$ rotates away from the \mathbf{tc} -plane giving then a nonvanishing component $\langle \mathbf{u} | \boldsymbol{\alpha}(\mu) \rangle$ in the \mathbf{u} direction, which gradually increases until at some value of μ it matches the value $\sqrt{\mu/m_t}$ and one has a solution to the equation (3) for the mass of u . This is then our first solution to the u mass, which would occur on the trajectory not so far from c , and has in actual fact a value of about 300MeV, which is quite different from what experiment indicates.

However, as μ lowers further, especially when the arc length s is growing exponentially with μ as suggested by (12), it can easily happen that the trajectory should hit the **tc**-plane again. Then at this point, one has again $\langle \mathbf{u} | \boldsymbol{\alpha}(\mu) \rangle$ vanishing, and the same opportunity as before will present itself for again a solution for the u mass. This is exactly what happens for the trajectory of Figure 1 and accounts for the output mass value for u listed in Table 3. One sees therefore that it is rather easy for the u mass to acquire this lower solution; all it needs is for the trajectory to hit the **tc**-plane again at some lower scale μ , which it can easily do so long as the trajectory is not prematurely terminated, say by a rotational fixed point, for example, before it has gone that far.

The same argument would hold also for d and e if one just replaces the **tc**-plane above by respectively the **bs**- and $\boldsymbol{\tau}\boldsymbol{\mu}$ -plane. These planes are defined at the higher scales of the two heavier generations where the trajectory is moving relatively slowly so that the planes are not too different from each other in terms of the arc length s , nor from the **tc**-plane for u . Hence the lower solutions for u, d, e would occur consecutively and fairly close to one another on the trajectory, as is indeed seen in Figure 1. And since at these lower scales, the trajectory is moving fast, meaning that changing s requires very little change in μ , the solutions for the masses of u, d, e would be rather close in values to one another, as is indeed seen in Table 3. So much then for the properties noted in the text for the output masses for the lightest generation.

A natural question to ask at this point is, of course, what about the middle generation; will there not be other lower solutions to their masses too? Interestingly, the answer is no, there being a rather subtle difference in the R2M2 scheme between the 2 lighter generations which makes the likelihood of lower solution practically zero for the middle generation. This comes about as follows. We recall, as we have done already several times, that all the state vectors are determined in the R2M2 scheme by the scale of the middle generation. This means in particular that by the u mass scale where one looks for solution to the mass equation (3) for m_u , the state vector \mathbf{u} is already known, namely in that direction orthogonal to the **tc**-plane. In contrast, in seeking a solution to the c mass at scale $\mu = m_c$, one has first to ascertain what is the state vector \mathbf{c} lying on the plane orthogonal to the state vector \mathbf{t} of t . In words, what the equations (2) and (3) say is that one should project the vector $\boldsymbol{\alpha}(\mu)$ at $\mu = m_c$ on to the plane orthogonal to \mathbf{t} . Then the direction of the shadow cast by $\boldsymbol{\alpha}(m_c)$ on to that plane is the vector \mathbf{c} ,

and the length of that shadow squared times m_t is the mass m_c . In other words, for a solution to the c mass at a scale μ , one will need the shadow to be of order only $\sqrt{\mu/m_t}$, a small number especially when μ itself is small when looking for a lower solution. This means that for a lower solution, one will need to have $\boldsymbol{\alpha}(\mu)$ nearly parallel or antiparallel to \mathbf{t} itself. In other words, for m_c to have a lower solution, one will need the trajectory to pass again at a lower scale very close to the starting point $\mathbf{t} = (1, 0, 0)$ or to its antipodes $(-1, 0, 0)$, which can happen, of course, but is very unlikely, and it did not happen for our special trajectory in Figure 1. The demand is very much more stringent than for a lower solution for the u mass which needs only that the trajectory hits again a plane, i.e. the \mathbf{tc} -plane, which it will almost always do so long as it is not terminated prematurely. The difference comes about just because there are 3 generations in nature and generation space is 3-dimensional.

We note that the trajectory in Figure 1 actually passes through the \mathbf{tc} -plane at μ around 1 MeV so that there are two solutions very close to each other to the mass equation (3) for the u mass, one lying just above and one just below the \mathbf{tc} -plane. Numerically, they differ by only about 2.5 KeV, and to the accuracy one is interested in at present, may be regarded practically as the same solution. Theoretically, however, we are yet unclear what this double solution means. We note also that if we further extrapolate our trajectory, it might pass through the \mathbf{tc} -plane again and give further and even lower solutions to (3) for the u mass. We are reluctant to do so, however, for we do not believe that the exponential growth in rotation speed as parametrized by (12) can go on forever. We think it is more likely that the trajectory will eventually terminate at some low scale fixed point as it actually does in the theoretical model for R2M2 [21, 22] that we are studying.

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