Relativistic Wigner Function Approach to Neutrino Propagation in Matter.

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In this work we study the propagation of massive Dirac neutrinos in matter with flavor mixing, using statistical techniques based on Relativistic Wigner Functions. First, we consider neutrinos in equilibrium within the Hartree approximation, and obtain the corresponding relativistic dispersion relations and effective masses. After this, we analyze the same system out of equilibrium. We verify that, under the appropiate physical conditions, the well known equations for the MSW effect are recovered. The techniques we used here appear as an alternative to describe neutrino properties and transport equations in a consistent way.

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I. INTRODUCTION

Neutrino propagation in dense media becomes an important issue in some astrophysical scenarios, such as supernovae, neutron stars during the Kelvin-Helmholtz epoch and the solar neutrino problem [1], [2]. The first two cases correspond to compact stars, where densities a few times the nuclear saturation density are reached. To describe neutrino propagation in the dense core of such compact objects, aside of production and absorption rates[1], one is mostly interested in the neutrino cross section with the surrounding matter, which in turn is related to the imaginary part of the forward amplitude, due to the optical theorem. In the case of solar neutrinos (and also for the 'atmosphere' of compact stars), the possibility of neutrino oscillations and flavor conversion appears. In the standard picture, these oscillations are associated to the real part of the forward scattering amplitude of massive neutrinos with the background.

In this paper we will be concerned about the latter topic, i. e. matter effects on neutrino masses and neutrino oscillations. Since the seminal papers by Wolfenstein [4], [5] and by Mikheyev and Smirnov [6], [7], [8], there have been many papers which have explored the physics of in-medium neutrinos, specially in connection with the solar neutrino problem (for a review, see for example [9] and references [1], [2]). Different techniques have been used to approach this problem. The simplest one consists in describing the matter effect by an *effective potential* (as in the Wolfenstein's papers) which will be added to the mass matrix to give an *effective Hamiltonian* in the Schrödinger's equation. Although this approach will suffice for most purposes, it is clear that it does not describe matter effects in a covariant way. In order to obtain covariant equations, one has to obtain the corresponding dispersion relations, which are the in-medium analogous to the simple mass-shell condition $p^2 = m^2$ of free particles. Dispersion relations of neutrinos in different backgrounds, taking into account the mixing among generations, have been investigated by Nötzold and Raffelt [10]. In that work, dispersion relations appear as the poles of the neutrino propagator (Green function), evaluated at the one-loop approximation.

On the other hand, one has astrophysical situations in which contribution of neutrinos to macroscopic magnitudes, such as the energy, pressure, etc, ... becomes important (this is the case in a supernova collapse [3], or in the Early Universe). In this case, one has to introduce a *distribution function* for neutrinos to describe the number of neutrinos having a given momentum. It is then desirable to develop evolution equations for these functions in the case where neutrino oscillations are present, which implies that distribution functions become non-diagonal matrices in flavor space. Several works have implemented this in different ways. In Ref. [11], use is made of the techniques described in [12] to obtain the kinetic equations for non-relativistic Wigner distribution functions of neutrinos. Alternatively, in references [13], [14] it is derived the time evolution of a neutrino density matrix ρ constructed as macroscopic averages of generalized occupation numbers, in the way $\rho_{ij}(\vec{p}) =< \hat{\rho}_{ij}(\vec{p}) >$, where $\hat{\rho}_{ij}(\vec{p}) = a^+_j(\vec{p})a_i(\vec{p})$ and $a^+_j(\vec{p})$ $(a_i(\vec{p}))$ is the creation (destruction) operator of neutrinos with flavor j (i) and momentum $\vec{p}$. Here, one starts directly from Heisenberg's equation

$$i\partial_t \hat{\rho} = [\hat{\rho}, H] \tag{1}$$

[1]In this work we will concentrate on medium effects in neutrino propagation, and will not discuss production and absortion mechanisms. For these processes, the reader may want to consult [3].

with a Hamiltonian $H = H_0 + H_{int}$, where H_0 is the free Hamiltonian and H_{int} is the interaction piece. Equation (1) is then expanded perturbatively (after macroscopic averaging). With this at hand, the authors have studied the possibility of flavor conversion in a supernova core (see the references above for more details).

Both procedures give rise to an expansion in powers of the Fermi's coupling constant G_F. The first term in this expansion (proportional to G_F) contains the modifications to the mass matrix due to the interaction, while the second, G_F^2 term, is the generalization of the Boltzmann collision integral to the case of flavor mixing. However, because both methods are based on non-covariant techniques, one does not obtain relativistic dispersion relations for the neutrinos.

In this paper, we make use of *relativistic* Wigner functions [15], [16] to describe propagation of neutrinos with flavor mixing in dense media. Relativistic Wigner functions have been successfully used to account for finite density and temperature effects in nuclear matter [17], [18], [19]. They provide an alternative to Green functions in a way which is well adopted to the development of kinetic equations. In addition to this, temperature effects are incorporated in a unique way, contrarily to the situation of Green functions. We also need to introduce correlation functions among neutrinos and the background (which, for simplicity, is considered to consist only on electrons). These correlations, as will be discussed later, take into account for the residual interaction of neutrinos and electrons beyond the mean-field approximation. We will obtain, in the next section, the equations of motion for these functions, under the assumption that the interacting background is in an equilibrium state.

Next, we will consider some particular situations. In section 3, we examine the case when neutrinos are in equilibrium and correlations are neglected. The dispersion relations reproduce, in this case, the expected effective masses and in-matter mixing angles. This result supports our statement that neglecting correlations is equivalent to a mean-field treatment of the surrounding matter.

In section 4 we examine some departure from the above simplest situation, by keeping spatial and/or time variations in the kinetic equations (with correlations still neglected), and we consider propagation on a density-varying medium. In section 5 we examine with more detail a particular case, corresponding to a small effective potential and macroscopic inhomogeneities which are large enough. This is the situation encountered when dealing with solar neutrinos. By making the appropriate approximations, we recover the well-known formulae for the MSW effect. This shows the ability of Wigner function techniques to correctly reproduce both relativistic dispersion equations and transport equations on the same foot, at least in the cases we have studied. We end in section 6 by summarizing our main results and making some remarks. Some auxiliary results will be given in the appendixes. The construction of the Wigner function in the case of free neutrinos is showed in Appendix A. In Appendix B we analyze the neutrino dispersion relations and effective masses which appear within the Hartree approximation, and we extend the results of Appendix A to the construction of the corresponding Wigner function.

In this work the metric is $g^{\mu\nu} = diag(1, -1, -1, -1)$. We adopt the chiral representation for gamma matrices, and natural units ($\hbar = c = 1$) are used.

II. EQUATIONS FOR WIGNER FUNCTIONS

In this section we derive the equations of motion for neutrino Wigner functions. Neutrinos are assumed to propagate on a matter background in equilibrium, an hypothesis which holds in all the astrophysical scenarios mentioned in the introduction.

In order so simplify the equations as much as possible, we consider only two neutrino flavors (namely electron and muon neutrinos) . Our treatment can be generalized in a straightforward way to include more neutrino flavors. Also, our main concern in neutrino propagation is neutrino oscillations and flavor conversion, which are given by charged-current interactions with electrons. Therefore, we consider a background consisting on electrons, although we take into account both neutral and charged currents. As before, the formalism can be trivially extended to account for a more general situation, such as neutral-current interactions on protons and neutrons. In this paper, we treat neutrinos as massive Dirac particles in the most simple model for massive neutrinos, i.e. we treat them the same way as the all other fermions (leptons and quarks). Within this minimal extension of The Standard Electroweak Theory, the charge of the neutrino field which is conserved is the total lepton number $L = L_e + L_\mu$. On the other hand, as we deal with low-energy neutrinos (with energies of the order a few MeV), we adopt an effective contact interaction of neutrinos with the matter background.

In what follows, neutrino magnitudes without a prime will indicate flavor states, and primes will be used for *free mass eigenstates.* In the next sections and in the appendixes, a tilde will be used for *interacting eigenstates.* Flavors, as well as mass eigenstates, will be labeled by lattin superindexes such as a and b. Spin subindexes will be generally omitted ; when needed, we use indices such as $i, j, k, ...$ to label them. Lorentz indices will be labeled by Greek letters.

Since we deal with two neutrino species, it is convenient to introduce vectors and matrices in flavor (or mass) space. We therefore define the neutrino and antineutrino vector fields[2] :

$$\widehat{\nu}(x) \equiv \begin{pmatrix} \widehat{\nu}^e(x) \\ \widehat{\nu}^\mu(x) \end{pmatrix}, \quad \widehat{\overline{\nu}}(x) \equiv \begin{pmatrix} \widehat{\overline{\nu}}^e(x) & \widehat{\overline{\nu}}^\mu(x) \end{pmatrix} \tag{2}$$

We also introduce the following matrices in flavor space

$$\Lambda^{ab,\mu} \equiv \begin{pmatrix} 0 & 0 \\ 0 & \lambda^\mu \end{pmatrix} \tag{3}$$

$$\Omega^{ab,\mu} \equiv \begin{pmatrix} \omega^\mu & 0 \\ 0 & 0 \end{pmatrix} \tag{4}$$

With the notations $\lambda^\mu \equiv \gamma^\mu(g_V - g_A\gamma^5)$ and $\omega^\mu \equiv \gamma^\mu(\widetilde{g}_V - \widetilde{g}_A\gamma^5)$. Here, the constant $g_V = -\frac{1}{2} + 2\sin^2\theta_W$ ($g_A = -\frac{1}{2}$) corresponds to the vector (axial) contribution of weak neutral currents, θ_W is the Weinberg's angle, while the constants $\widetilde{g}_V = \frac{1}{2} + 2\sin^2\theta_W$ and $\widetilde{g}_A = \frac{1}{2}$ arise from neutral plus charged currents.

With these notations, the Lagrangian density is written as :

$$\widehat{\mathcal{L}}(x) = \widehat{\mathcal{L}}_e(x) + \widehat{\mathcal{L}}_\nu(x) + \widehat{\mathcal{L}}_I(x)$$

with $\widehat{\mathcal{L}}_e(x)$ the Lagrangian of free electrons,

$$\widehat{\mathcal{L}}_\nu(x) = \widehat{\overline{\nu}}(x) i\gamma^\mu\partial_\mu\widehat{\nu}(x) - \widehat{\overline{\nu}}(x) M \widehat{\nu}(x) \tag{5}$$

the corresponding Lagrangian of free neutrinos, and

$$\widehat{\mathcal{L}}_I(x) = -\frac{G_F}{\sqrt{2}}\widehat{\overline{\nu}}(x)\Omega^\mu\widehat{\nu}(x)\widehat{\overline{e}}(x)\omega_\mu\widehat{e}(x) - \frac{G_F}{\sqrt{2}}\widehat{\overline{\nu}}(x)\Lambda^\mu\widehat{\nu}(x)\widehat{\overline{e}}(x)\lambda_\mu\widehat{e}(x) \tag{6}$$

the interaction piece. In Eq. (5) M is the free mass matrix *in flavor space* [3]. From the above Lagrangian one readily obtains the equations of motion for the neutrinos :

$$i\gamma^\mu\partial_\mu\widehat{\nu}(x) - M\widehat{\nu}(x) - \frac{G_F}{\sqrt{2}}\Omega^\mu\widehat{\nu}(x)\widehat{\overline{e}}(x)\omega_\mu\widehat{e}(x) - \frac{G_F}{\sqrt{2}}\Lambda^\mu\widehat{\nu}(x)\widehat{\overline{e}}(x)\lambda_\mu\widehat{e}(x) = 0 \tag{7}$$

$$\partial_\mu\widehat{\overline{\nu}}(x) i\gamma^\mu + \widehat{\overline{\nu}}(x) M + \frac{G_F}{\sqrt{2}}\widehat{\overline{\nu}}(x)\Omega^\mu\widehat{\overline{e}}(x)\omega_\mu\widehat{e}(x) + \frac{G_F}{\sqrt{2}}\widehat{\overline{\nu}}(x)\Lambda^\mu\widehat{\overline{e}}(x)\lambda_\mu\widehat{e}(x) = 0 \tag{8}$$

We now introduce the neutrino Wigner *operator*

$$\widehat{F}^{ab(\nu)}_{ij}(x,p) = (2\pi)^{-4}\int d^4y\, e^{-ipy}\,\widehat{\overline{\nu}}^b_j(x+y/2)\widehat{\nu}^a_i(x-y/2) \tag{9}$$

and the electron Wigner operator

$$\widehat{F}^{(e)}_{ij}(x,p) \equiv (2\pi)^{-4}\int d^4y\, e^{-ipy}\,\widehat{\overline{e}}_j(x+y/2)\widehat{e}_i(x-y/2) \tag{10}$$

From Eq. (9) one readily gets that the Hermitian conjugate is given by

$$\widehat{F}^{ab(\nu)\dagger}_{ij}(x,p) = \gamma^0_{jk}\widehat{F}^{ba(\nu)}_{kq}(x,p)\gamma^0_{qi} \tag{11}$$

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[2]The symbol $\hat{}$ on top of a magnitude means that we are dealing with a quantum operator. This will be used to distinguish this magnitudes from statisical averages.

[3]If flavor mixing exists, M is non-diagonal, its eigenvalues being the masses of free mass eigenstates.

Here and hereafter summation over repeated indices is understood. With the help of Eqs. (7) and (8) one can obtain the equations of motion obeyed by the neutrino Wigner operators. After some algebra, we get :

$$\gamma\left[\partial\widehat{F}^{(\nu)}(x,p)-2ip\widehat{F}^{(\nu)}(x,p)\right]+2iM\widehat{F}^{(\nu)}(x,p)= \tag{12}$$
$$\frac{-2iG_F}{\sqrt{2}}(2\pi)^{-4}\int d^4yd^4k\left[\Omega\widehat{\bar{e}}(y)\omega\widehat{e}(y)+\Lambda\widehat{\bar{e}}(y)\lambda\widehat{e}(y)\right]\widehat{F}^{(\nu)}(x,p-k/2)e^{ik(y-x)}$$

$$\left[\partial\widehat{F}^{(\nu)}(x,p)+2ip\widehat{F}^{(\nu)}(x,p)\right]\gamma-2iM\widehat{F}^{(\nu)}(x,p)= \tag{13}$$
$$\frac{2iG_F}{\sqrt{2}}(2\pi)^{-4}\int d^4yd^4k\widehat{F}^{(\nu)}(x,p-k/2)\left[\Omega\widehat{\bar{e}}(y)\omega\widehat{e}(y)+\Lambda\widehat{\bar{e}}(y)\lambda\widehat{e}(y)\right]e^{-ik(y-x)}$$

One could also derive the corresponding equations for the electron Wigner operator $\widehat{F}^{(e)}_{ij}(x,p)$. However, we will not need these equations under the approximations discussed in this paper, and therefore we will omit them. Of course, if one wants to investigate the next order to this approximation, the whole system of equations has to be taken into account.

We are now interested in introducing statistical averages from the quantum operators introduced above. These statistical averages are called Wigner functions [20], and are the analogous to the distribution functions we need to describe many-particles systems. These are, in general, complex functions, and also contain a Lorentz structure which will be discussed later. The electron and neutrino Wigner functions are defined, respectively, as :

$$F^{(e)}_{ij}(x,p)\equiv<\widehat{F}^{(e)}_{ij}(x,p)>=(2\pi)^{-4}\int d^4y\,e^{-ipy}<\widehat{\bar{e}}_j(x+y/2)\widehat{e}_i(x-y/2)> \tag{14}$$

$$F^{ab(\nu)}_{ij}(x,p)\equiv<\widehat{F}^{ab(\nu)}_{ij}(x,p)>=(2\pi)^{-4}\int d^4y\,e^{-ipy}<\widehat{\bar{\nu}}^b_j(x+y/2)\widehat{\nu}^a_i(x-y/2)> \tag{15}$$

Here, the symbol $<\widehat{A}>$ means the average of a given quantum operator $\widehat{A}$ over a basis of quantum states which are compatible with the macroscopical knowledge of the system. The latter determines a given density matrix operator $\widehat{\rho}$. Thus the averaging is performed according to

$$<\widehat{A}>\equiv Sp\{\widehat{\rho}\widehat{A}\} \tag{16}$$

where Sp means the trace performed over the quantum basis. By taking this average on Eqs. (12) and (13) one arrives to the following equations :

$$\left[\gamma(\partial-2ip)+2iM\right]F^{(\nu)}(x,p)=-\frac{iG_F\sqrt{2}}{(2\pi)^4}\int d^4yd^4kd^4k'e^{ik(y-x)}$$
$$[<\Omega Tr(\omega\widehat{F}^{(e)}(y,k'))\widehat{F}^{(\nu)}(x,p-k/2)>+<\Lambda Tr(\lambda\widehat{F}^{(e)}(y,k'))\widehat{F}^{(\nu)}(x,p-k/2)>] \tag{17}$$
$$F^{(\nu)}(x,p)\left[\gamma(\partial+2ip)-2iM\right]=\frac{iG_F\sqrt{2}}{(2\pi)^4}\int d^4yd^4kd^4k'e^{-ik(y-x)}$$
$$[<\widehat{F}^{(\nu)}(x,p-k/2)\Omega Tr(\omega\widehat{F}^{(e)}(y,k'))>+<\widehat{F}^{(\nu)}(x,p-k/2)\Lambda Tr(\lambda\widehat{F}^{(e)}(y,k'))>] \tag{18}$$

In the latter equations, the symbol Tr means the trace in spin indices.

Let us now introduce the electron-neutrino $A^{(\nu e)}$and neutrino-electron $B^{(e\nu)}$correlation functions :

$$A^{(\nu e)ab}_{ijkl}(x,x',p,p')\equiv<\widehat{F}^{(\nu)ab}_{ij}(x,p)\widehat{F}^{(e)}_{kl}(x',p')>-F^{(\nu)ab}_{ij}(x,p)F^{(e)}_{kl}(x',p') \tag{19}$$
$$B^{(e\nu)ab}_{ijkl}(x,x',p,p')\equiv<\widehat{F}^{(e)}_{ij}(x,p)\widehat{F}^{(\nu)ab}_{kl}(x',p')>-F^{(e)}_{ij}(x,p)F^{(\nu)ab}_{kl}(x',p') \tag{20}$$

Then Eqs. (17) and (18) can be rewritten as

$$\left[\gamma(\partial-2ip)+2iM\right]F^{(\nu)}(x,p)=-\frac{iG_F\sqrt{2}}{(2\pi)^4}\int d^4yd^4kd^4k'e^{ik(y-x)}$$
$$[\Lambda Tr(\lambda B(y,x,k',p-k/2))+\Omega Tr(\omega B(y,x,k',p-k/2))+$$
$$\Lambda Tr(\lambda F^{(e)}(y,k'))F^{(\nu)}(x,p-k/2)+\Omega Tr(\omega F^{(e)}(y,k'))F^{(\nu)}(x,p-k/2)] \tag{21}$$

$$F^{(\nu)}(x,p)\left[\gamma(\partial+2ip)-2iM\right]=\frac{iG_F\sqrt{2}}{(2\pi)^4}\int d^4yd^4kd^4k'e^{-ik(y-x)}$$
$$[\Lambda Tr(A(x,y,p-k/2,k')\lambda)+\Omega Tr(A(x,y,p-k/2,k')\omega)+$$
$$\Lambda F^{(\nu)}(x,p-k/2)Tr(\lambda F^{(e)}(y,k'))+\Omega F^{(\nu)}(x,p-k/2)Tr(\omega F^{(e)}(y,k'))] \tag{22}$$

The two-point correlation functions defined above are not independent. In fact, one can prove that they are related by :

$$A^{(\nu e)ab*}_{ijkl}(x,x',p,p')=\gamma^0_{lp}\gamma^0_{jr}B^{(e\nu)ba}_{pqrs}(x',x,p',p)\gamma^0_{qk}\gamma^0_{si} \tag{23}$$

which can be written, in a short way, as[4]

$$A^{(\nu e)\dagger}(x,x',p,p')=B^{(e\nu)}(x',x,p',p) \tag{24}$$

This implies that Eq. (22) is actually the Hermitian conjugate of Eq. (21).

III. SYSTEM IN EQUILIBRIUM. HARTREE APPROXIMATION

In order to obtain some insight into the physical meaning of the neutrino Wigner function we will, in this section, investigate the situation when both the electrons and the neutrinos are in equilibrium, which we characterize by all statistical magnitudes as being time-space translationally invariant. This means that one-point functions can not depend on x , while two-point correlation functions can only depend on the difference of coordinates, i. e. we assume that :

$$F^{(e)}(x,p)=F^{(e)}(p) \tag{25}$$
$$F^{(\nu)}(x,p)=F^{(\nu)}(p) \tag{26}$$

and

$$A^{(\nu e)}(x,x',p,p')=A^{(\nu e)}(x-x',p,p')$$
$$B^{(e\nu)}(x,x',p,p')=B^{(e\nu)}(x-x',p,p') \tag{27}$$

By defining the Fourier transform

$$\widetilde{A}^{(\nu e)}(k,p,p')=(2\pi)^{-4}\int d^4x\, e^{-ikx}A^{(\nu e)}(x,p,p') \tag{28}$$

(analogously for $B^{(e\nu)}$), we obtain the equilibrium equations for the neutrino Wigner function

$$\left[p\gamma-M-\frac{G_F}{\sqrt{2}}\int d^4k'\left(\Lambda Tr\left(\lambda F^{(e)}(k')\right)+\Omega Tr\left(\omega F^{(e)}(k')\right)\right)\right]F^{(\nu)}(p)=$$
$$\frac{G_F}{\sqrt{2}}\int d^4kd^4k'\left[\Lambda Tr\left(\lambda\widetilde{B}^{(e\nu)}(k,k',p+k/2)\right)+\Omega Tr\left(\omega\widetilde{B}^{(e\nu)}(k,k',p+k/2)\right)\right] \tag{29}$$

$$F^{(\nu)}(p)\left[p\gamma-M-\frac{G_F}{\sqrt{2}}\int d^4k'\left(Tr\left(\lambda F^{(e)}(k')\right)\Lambda+Tr\left(\omega F^{(e)}(k')\right)\Omega)\right)\right]=$$
$$\frac{G_F}{\sqrt{2}}\int d^4kd^4k'\left[\Lambda Tr\left(\widetilde{A}^{(\nu e)}(k,p+k/2,k')\lambda\right)+\Omega Tr\left(\widetilde{A}^{(\nu e)}(k,p+k/2,k')\omega\right)\right] \tag{30}$$

As before, Eq. (30) turns out to be the Hermitian conjugate of Eq. (29). This set of equations is obviously not complete, and one should add the equations which are satisfied by the correlation functions in looking for such a

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[4]For a matrix having both spin indices and generation indices, its Hermitian conjugate is obtained by interchanging the generation indices and then taking the Hermitian conjugate in spin space.

complete set. However, in doing so there automatically appear *three-point* correlation functions. This procedure can be infinitely continued, so that one obtains, instead of a closed set, an infinite *hierarchy* similar to the BBGKY (after Bogoliubov-Born- Green- Kirkwood-Yvon) hierarchy of classical systems [21], [22]. For classical systems, one usually truncates this infinite chain by neglecting correlations of order higher than a given one, usually by showing that higher orders correspond to more rapid variations in space and time. The next step consists then in incorporating perturabatively the next-order correlations. We will use here the analogy with the classical situation, and will first examine the situation at the lowest order, i. e. when all kind of correlations are neglected. Such approximation is commonly referred to as the *Hartree approximation.* As we will show, the neutrino dispersion relations which arise from this approximation correspond to modifications of the neutrino propagator at the one-loop level [10].

The Wigner function of electrons in equilibrium can be calculated using standard techniques. Following [23] one has

$$F^{(e)}(p) = (2\pi)^{-3}\delta(p^2 - m_e^2)\left[\theta(p^0)f_e^+(p) + \theta(-p^0)f_e^-(p)\right](\gamma p + m_e) \tag{31}$$

with m_e the electron mass, p^0 the time-like component of the electron four-momentum p and $\theta(x)$ the step function. The functions $f_e^+(p)$, $f_e^-(p)$ are the Fermi-Dirac occupation numbers of electrons and positrons, respectively. In the frame where the matter fluid is at rest, they read as[5] :

$$f_e^{\pm}(p) = \frac{1}{e^{\beta(E\mp\mu_e)} + 1} \tag{32}$$

(in Appendix A a similar calculation is shown for neutrinos) , where $E = \sqrt{\vec{p}^2 + m_e^2}$, μ_e is the electron chemical potential and β the inverse temperature (we set the Boltzmann constant $k_B = 1$). We can now calculate the traces and integrals appearing in Eqs. (29),(30). After some algebra, it is easily obtained

$$Tr\int d^4k\lambda^\mu F^{(e)}(k) = \left\{ \begin{array}{c} Tr\int d^4k\lambda^0 F^{(e)}(k) = g_v n \\ Tr\int d^4k\lambda^i F^{(e)}(k) = 0; \quad i = 1,2,3 \end{array} \right. \tag{33}$$

Analogously

$$Tr\int d^4k\omega^\mu F^{(e)}(k) = \left\{ \begin{array}{c} Tr\int d^4k\omega^0 F^{(e)}(k) = \widetilde{g}_v n \\ Tr\int d^4k\omega^i F^{(e)}(k) = 0; \quad i = 1,2,3 \end{array} \right. \tag{34}$$

Let us now define the matrix (in flavor space) :

$$\Phi = \left(\begin{array}{cc} \widetilde{V} & 0 \\ 0 & V_n \end{array} \right). \tag{35}$$

here, $V_n = \sqrt{2}G_F g_v n$ is the effective potential for neutral currents, and $\widetilde{V} = V + V_n$, where $V = \sqrt{2}G_F n$ is the corresponding potential for charged currents, with $n = 4\int d^4k(2\pi)^{-3}\delta(k^2 - m_e^2)(\theta(p^0)f_e^+(k) + \theta(-p^0)f_e^-(k))k^0 \equiv n_e - n_{\bar{e}}$ the electron (minus positron) number density. With these notations, and neglecting correlations, Eq. (29) can be cast under the form :

$$[\gamma p - M - \gamma^0\frac{1}{2}(1-\gamma^5)\Phi]F(p) = 0 \tag{36}$$

(since electrons have been integrated out, the neutrino superscript in Wigner functions will be omitted in what follows, in order to make notations simpler). It is easily recognized in Eq. (36) the appearance of the left-handed chirality projector $P_L = \frac{1}{2}(1-\gamma^5)$, as a consequence of left-handed interactions. Let us also introduce the right-handed projector $P_R = \frac{1}{2}(1+\gamma^5)$. With the help of these two projectors, one can define the following components of the neutrino Wigner function :

$$\begin{array}{l} F_L = P_L F P_R \\ F_R = P_R F P_L \\ F_{RL} = P_R F P_R \\ F_{LR} = P_L F P_L \end{array} \tag{37}$$

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[5]Making the hypothesis that matter is at rest is equivalent to consider a particular Lorentz frame such that the fluid four-velocity is $u^\mu = (1,0,0,0)$. Results in other frames can be obtained by restoring the four-velocity u^μ, as discussed in [16].

If we apply P_L and P_R on the left and right of Eq. (36) we arrive to the set of equations

$$\begin{aligned} \gamma p F_{RL} - M F_L &= 0 \\ \gamma p F_{LR} - M F_R - \gamma^0 \Phi F_{LR} &= 0 \\ \gamma p F_R - M F_{LR} &= 0 \\ \gamma p F_L - M F_{RL} - \gamma^0 \Phi F_L &= 0 \end{aligned} \tag{38}$$

By combining the above equations, one finally arrives to :

$$(p^2 - M^2 - \gamma p \gamma^0 \Phi) F_L(p) = 0 \tag{39}$$

$$(p^2 - M^2 - \gamma^0 p \gamma M \Phi M^{-1}) F_R(p) = 0 \tag{40}$$

An important remark must be made. We are here considering a hypothetical situation where neutrinos had time enough to equilibrate with the background matter. Under this assumption, right-handed neutrinos can be produced by different mechanisms, such as spin-flip or pair production. However, production rates are suppressed by a factor m_ν/E , where E is the neutrino energy and m_ν its mass. In the astrophysical scenarios we are considering, the production rate of right-handed neutrinos is small, and therefore they can be neglected. By this reason, we will concentrate on the left-handed component $F_L(p)$. Consistently with this approximation, neutrinos will be treated under the extreme relativistic limit $m_\nu/E << 1$. This will imply that the neutrino field can be considered, approximately, as consisting on negative-helicity neutrinos and positive-helicity antineutrinos. For Wigner functions, this is shown in Appendix B, where the Wigner function will be explicitly calculated in the interacting case.

The dispersion relation obtained from Eq. (39) can be diagonalized, and one obtains the well-known expressions for masses and mixing angles in matter : this is also studied in Appendix B.

IV. NON-EQUILIBRIUM SYSTEM. TRANSPORT EQUATION.

In this section, we investigate the evolution of neutrinos when deviations from equilibrium situations arise. More precisely, we will consider that neutrinos are created and propagate through the matter background. This implies that Eqs. (25) - (27) will not be imposed, and therefore time and spatial variations have to be considered. This represents an additional difficulty in solving the system of equations for Wigner functions *and* correlation functions. For the moment, we will only consider a simple case, where correlations are neglected. This will serve us to investigate the possibilities of Wigner function techniques in deriving neutrino transport equations, and will allow in the future to study more complicated frameworks. As we will see in this section, the equations arising in this context are appropriate to deal with neutrino propagation and flavor conversion in the Sun.

We return to Eq. (21), and assume that electrons can be locally characterized by their temperature and chemical potential, in such a way that Eq. (31) is still valid.

By performing the same procedure as in Eq. (39), we can derive an equation for $F_L(x,p)$, which is now

$$[\Box - 4(p^2 - M^2) - 4ip_\mu \partial^\mu + 2i\Phi(x)\gamma^\mu\gamma^0\partial_\mu + 4\Phi(x)\gamma^\mu p_\mu \gamma^0 + 2i\gamma^\mu \left(\partial_\mu \Phi(x)\right)\gamma^0] F_L(x.p) = 0 \tag{41}$$

The next step is achieved by decomposing the complete Wigner function $F(x.p)$ into the Dirac algebra. By using the chirality projectors, as in Eq. (37) one can write $F_L(x,p)$ under the form

$$F_L(x,p) = \frac{1}{2}(1-\gamma_5) f_{L\mu}(x,p)\gamma^\mu \tag{42}$$

where $f_{L\mu}(x,p)$ is a matrix in flavor space[6], and transforms as a Lorentz vector. If the left-handed projector P_L is used again, we see that $f_{L\mu}(x,p)$ can be expressed as

$$f_L^{\mu,ab}(x,p) = \frac{1}{2} Tr[F_L^{ab}(x,p)\gamma^\mu] = \frac{1}{2}(2\pi)^{-4} \int d^4y\, e^{-ipy} < \widehat{\overline{\nu}}_L^b(x+\frac{1}{2}y)\gamma^\mu \widehat{\nu}_L^a(x-\frac{1}{2}y) > \tag{43}$$

[6]We notice from here that $f_{L\mu}(x,p)$ is an hermitian matrix.

In the latter equation, $\widehat{\nu}_L$ is the left-handed component of the neutrino field. We next analyze the equation of motion obeyed by $f_{L\mu}(x,p)$. In order to do this, we substitute Eq. (42) into Eq. (41). After some algebra, we obtain the equations

$$\begin{aligned} -\frac{1}{4}\Box f_L^0 + (p^2 - M^2) f_L^0 - \Phi(x)\left(p^0 f_L^0 + \vec{p}\cdot\vec{f}_L\right) \\ + ip_\mu \partial^\mu f_L^0 - \frac{i}{2}\frac{\partial}{\partial t}\left(\Phi(x) f_L^0\right) + \frac{i}{2}\vec{\nabla}\left(\Phi(x)\cdot\vec{f}_L\right) = 0 \end{aligned} \tag{44}$$

$$\begin{aligned} -\frac{1}{4}\Box \vec{f}_L + (p^2 - M^2)\vec{f}_L - \Phi(x)\left(p^0 \vec{f}_L + \vec{p} f_L^0\right) + \frac{1}{2}\vec{\nabla}\times\left(\Phi(x)\vec{f}_L\right) \\ + ip_\mu \partial^\mu \vec{f}_L - \frac{i}{2}\frac{\partial}{\partial t}\left(\Phi(x)\vec{f}_L\right) + \frac{i}{2}\vec{\nabla}\left(\Phi(x) f_L^0\right) + i\Phi(x)\vec{p}\times\vec{f}_L = 0 \end{aligned} \tag{45}$$

Eqs. (44) and (45) are the basic transport equations to be solved on a general situation, with the help of appropriate boundary conditions. We will investigate the consequences of this set of equations in a future work. For the moment, as a test, we will show that, under the circumstances usually considered when the MSW is studied, we reproduce the known equations for this effect.

V. MSW EFFECT

We consider neutrinos moving along a straight line (for example, the radial direction of the star). According to this, we assume that $\vec{f}_L$ is parallel to $\vec{p}$. This allows us to write

$$\vec{f}_L(x,p) \equiv \vec{p} f(x,p) \tag{46}$$

where $f(x,p)$ is a new function. We also introduce, for convenience,

$$f_L^0(x,p) \equiv |\vec{p}| g(x,p) \tag{47}$$

Next, we assume that neutrinos are ultrarelativistic, and that the effective potentials $\widetilde{V}$ and V_n in Eq.(35) satisfy

$$\widetilde{V}, V_n \ll p^0 \sim |\vec{p}| \sim 1MeV \tag{48}$$

The last condition will concern the characteristic scale of spatial and time variations of the neutrino distribution function. This scale has a macroscopic size R, at least of the order of 1 Km, or even more (in the case of the resonant zone in the Sun, for example). In this case, we can make the following estimate :

$$\frac{\partial f_L^\mu}{\partial t} \sim |\vec{\nabla} f_L^\mu| \sim \frac{f_L^\mu}{R} \tag{49}$$

When $R = 1$ Km, then $1/R \sim 10^{-16} MeV$. If we are interested on variations of the distribution function on the scale R, then, together with hypothesis Eq. (48) we can simplify Eqs. (44) and (45) to give

$$\begin{aligned} \left[p^2 - M^2 - (p^0 + |\vec{p}|)\Phi(x)\right](f+g) + ip_\mu\partial^\mu(f+g) = 0 \\ \left[p^2 - M^2 - (p^0 - |\vec{p}|)\Phi(x)\right](f-g) + ip_\mu\partial^\mu(f-g) = 0 \end{aligned} \tag{50}$$

In the latter equations, $f(x,p)$ and $g(x,p)$ are *Hermitian* matrices (in flavor space). We can perform a *local* transformation for each of them in such a way that both become diagonal (and therefore real). In this way, one can easily check that two possibilities are open for the system Eq. (50) to have non-trivial solutions. The two possibilities are

$$\det\left[p^2 - M^2 - (p^0 \pm |\vec{p}|)\Phi(x)\right] = 0 \tag{51}$$

We recognize in Eq. (51) the dispersion relations for neutrinos and antineutrinos, as described in Appendix B within the Hartree equilibrium hypothesis. However, quantities in the latter equation depend on the coordinate x . This means that the neutrino mass eigenstates can be obtained locally from the Hartree approximation (which gives the same as the MSW effect). We then have, for neutrinos, $\det\left[p^2 - M^2 - (p^0 + |\vec{p}|)\Phi(x)\right] = 0$. This implies the condition $f(x,p) = g(x,p)$. Therefore, $f_L^0(x,p) = |\vec{p}| f(x,p) \simeq p^0 f(x,p)$, and we conclude that

$$f_L^\mu(x,p) \simeq p^\mu f(x,p) \tag{52}$$

within the same approximation. Finally, we have for Eq. (42)

$$F_L(x,p) = \frac{1}{2}(1-\gamma_5) p_\mu \gamma^\mu f(x,p) \tag{53}$$

where $f(x,p)$, for ultra-relativistic neutrinos, obeys the following equation of motion

$$\left[p^2 - M^2 - 2\Phi(x) p^0\right] f + i p_\mu \partial^\mu f = 0 \tag{54}$$

We have discussed above the possibility of making a local transformation which diagonalizes $f(x,p)$. We have exploited the fact that, under these circumstances, it becomes a real matrix. However, from the physical point of view, it is more convenient to introduce a different local transformation, in such a way that the factor inside the brackets in the latter equation becomes diagonal, i.e. we consider an unitary transformation given by the matrix $U_M(x)$

$$\widetilde{f} = U_M^\dagger f U_M \tag{55}$$

such that

$$U_M^\dagger(x)[(p^2 - M^2) - 2\Phi(x) p^0] U_M(x) = p^2 - \widetilde{M}^2(x) \tag{56}$$

where $\widetilde{M}^2(x) \equiv U_M^\dagger(x)\left[M^2 + 2\Phi(x)p^0\right] U_M(x)$ is a diagonal matrix, which contains the local mass eigenvalues [7]:

$$\widetilde{M}^2(x) = \begin{pmatrix} \widetilde{M}_1^2(x) & 0 \\ 0 & \widetilde{M}_2^2(x) \end{pmatrix} \tag{57}$$

On the other hand, one finds

$$U_M^\dagger(x)\left(i\partial^\mu p_\mu f(x,p)\right) U_M(x) = i\partial^\mu p_\mu \tilde{f}(x,p) - [\tilde{f}(x,p), U_M^\dagger(x) i\partial^\mu p_\mu U_M(x)] \tag{58}$$

Thus the equation of motion reads

$$i\partial^\mu p_\mu \tilde{f}(x,p) + (p^2 - \widetilde{M}^2(x))\tilde{f}(x,p) - i[\tilde{f}(x,p), U_M^\dagger(x) p_\mu \partial^\mu U_M(x)] = 0 \tag{59}$$

Let us write explicitly the matrices $\tilde{f}(x,p)$ and $U_M(x)$ by defining

$$U_M(x) \equiv \begin{pmatrix} \cos\theta_M(x) & -\sin\theta_M(x) \\ \sin\theta_M(x) & \cos\theta_M(x) \end{pmatrix} \tag{60}$$

$$\tilde{f}(x,p) \equiv \begin{pmatrix} \tilde{f}^{11}(x,p) & \tilde{f}^{12}(x,p) \\ \tilde{f}^{21}(x,p) & \tilde{f}^{22}(x,p) \end{pmatrix} \tag{61}$$

where the functions $\tilde{f}^{11}(x,p)$ and $\tilde{f}^{22}(x,p)$ are real, while $\tilde{f}^{12}(x,p)$ and $\tilde{f}^{21}(x,p)$ are the complex conjugate of one another. After substituting Eqs. (57),(60) and (61) into Eq. (59), one obtains

$$ip^\mu \partial_\mu \widetilde{f}^{11}(x,p) + \left(p^2 - \widetilde{M}_1^2(x)\right)\widetilde{f}^{11}(x,p) - ip^\mu \partial_\mu \theta_M(x)\left(\widetilde{f}^{12}(x,p) + \widetilde{f}^{21}(x,p)\right) = 0 \tag{62}$$

[7] We use a tilde to represent magnitudes in the interacting eigenstates basis, as mentioned in Section 2.

$$ip^\mu\partial_\mu\widetilde{f}^{12}(x,p)+\left(p^2-\widetilde{M}_1^2(x)\right)\widetilde{f}^{12}(x,p)+ip^\mu\partial_\mu\theta_M(x)\left(\widetilde{f}^{11}(x,p)-\widetilde{f}^{22}(x,p)\right)=0 \tag{63}$$

$$ip^\mu\partial_\mu\widetilde{f}^{21}(x,p)+\left(p^2-\widetilde{M}_2^2(x)\right)\widetilde{f}^{21}(x,p)+ip^\mu\partial_\mu\theta_M(x)\left(\widetilde{f}^{11}(x,p)-\widetilde{f}^{22}(x,p)\right)=0 \tag{64}$$

$$ip^\mu\partial_\mu\widetilde{f}^{22}(x,p)+\left(p^2-\widetilde{M}_2^2(x)\right)\widetilde{f}^{22}(x,p)+ip^\mu\partial_\mu\theta_M(x)\left(\widetilde{f}^{12}(x,p)+\widetilde{f}^{21}(x,p)\right)=0 \tag{65}$$

A remark is in order. As can be seen from the above equations, it is not possible, in a general situation out of equilibrium, to perform a transformation that makes both the neutrino Wigner function *and* the mass matrix diagonal. However, local mass eigenstates can be used as a useful physical basis to simplify the equations, as done in this section.

We can write the latter system of equations in a more familiar way. By taking the real part on the first and last one, we readily arrive to the conditions

$$\begin{aligned}\left[p^2-\widetilde{M}_1^2(x)\right]\widetilde{f}^{11}(x,p)&=0\\ \left[p^2-\widetilde{M}_2^2(x)\right]\widetilde{f}^{22}(x,p)&=0\end{aligned} \tag{66}$$

which imply that $\widetilde{f}^{11}(x,p)$ ($\widetilde{f}^{22}(x,p)$) is non-vanishing only when $p^2=\widetilde{M}_1^2(x)$ ($p^2=\widetilde{M}_2^2(x)$). Similarly, from the two remaining equations it can be easily deduced that the functions $\widetilde{f}^{12}(x,p)$ and $\widetilde{f}^{21}(x,p)$ have to vanish unless the condition $p^2=\frac{1}{2}\left[\widetilde{M}_1^2(x)+\widetilde{M}_2^2(x)\right]$ is fulfilled. Therefore, in the equations of motion for these functions we can substitute

$$\begin{aligned}p^2-\widetilde{M}_1^2(x)&\to\frac{1}{2}\Delta(x)\\ p^2-\widetilde{M}_2^2(x)&\to-\frac{1}{2}\Delta(x)\end{aligned} \tag{67}$$

where $\Delta(x)\equiv\widetilde{M}_2^2(x)-\widetilde{M}_1^2(x)$ is the in-medium neutrino mass difference. Furthermore, in the ultra-relativistic case, we can approximate the operator $p^\mu\partial_\mu$ in *all* the Eqs. (62)-(65) by

$$p^\mu\partial_\mu\simeq|\vec{p}|\frac{\partial}{\partial t}+|\vec{p}|\frac{\partial}{\partial x}=|\vec{p}|D \tag{68}$$

We have defined $D=\frac{\partial}{\partial t}+\frac{\partial}{\partial x}$. After some manipulations, Eqs. (62)-(65) can be cast under the simple matrix form

$$iD\widetilde{f}(x,t)=[\widetilde{f}(x,t),\widetilde{H}(x)] \tag{69}$$

Here,

$$\widetilde{H}(x)=\begin{pmatrix}\frac{\Delta(x)}{4|\vec{p}|} & -i\theta'_M(x)\\ i\theta'_M(x) & -\frac{\Delta(x)}{4|\vec{p}|}\end{pmatrix} \tag{70}$$

is the *effective Hamiltonian* in the mass eigenstates basis, and $\theta'_M(x)=\frac{d}{dx}\theta_M(x)$. Eq. (69) can also be written in the flavor basis, if we undo the transformation introduced in Eq. (55), with the definitions

$$f(x,t)\equiv\begin{pmatrix}f^{ee}(x,t) & f^{e\mu}(x,t)\\ f^{\mu e}(x,t) & f^{\mu\mu}(x,t)\end{pmatrix}=U_M(x)\widetilde{f}(x,t)U_M^\dagger(x) \tag{71}$$

After a straightforward calculation, we arrive to the equation

$$iDf(x,t)=[f(x,t),H(x)] \tag{72}$$

with

$$H(x)=\begin{pmatrix}\alpha(x) & \beta\\ \beta & -\alpha(x)\end{pmatrix} \tag{73}$$

is the Hamiltonian in the flavor basis, and we introduced the notations

$$\begin{aligned} \alpha(x) &= (\Delta_0 \cos 2\theta - A(x))/4|\vec{p}| \\ \beta &= \Delta_0 \sin 2\theta / 4|\vec{p}| \end{aligned} \tag{74}$$

where $A(x) = 2|\vec{p}|\sqrt{2}G_F\, n(x)$ is the induced mass due to charged currents, $\Delta_0 = m_2^2 - m_1^2$ is the vacuum neutrino mass difference and θ the vacuum mixing angle.

From the system of equations (72), we first obtain that

$$iD\left[f^{ee}(x,t) + f^{\mu\mu}(x,t)\right] = 0 \tag{75}$$

which implies that the total number of electron plus muon neutrinos is conserved during the propagation, as expected. We can then write $f^{ee}(x,t) + f^{\mu\mu}(x,t) = K$, where K is constant during the propagation : $DK = 0$. By manipulating the equations, one can also derive a third-order differential equation for $f^{ee}(x,t)$, which reads as

$$\alpha D^3 f^{ee}(x,t) - \alpha' D^2 f^{ee}(x,t) + 4\alpha(\alpha^2+\beta^2) D f^{ee}(x,t) - 4\alpha'\beta^2\left(f^{ee}(x,t) - \frac{K}{2}\right) = 0 \tag{76}$$

where $\alpha' = \frac{d}{dx}\alpha$. The latter equation coincides with the one derived by Mikheyev and Smirnov [7] to describe the evolution of the survival probability of an electron neutrino in a non-constant density medium. This agrees with our interpretation of $f^{ee}(x,t)$ as giving the distribution function (proportional to the number density for a given momentum) of electron neutrinos, and analogously for $f^{\mu\mu}(x,t)$ as the distribution function of muon neutrinos. All known results for the MSW effect within the situation considered here can be reproduced from the above equations.

Eq. (72) defines the evolution of flavor distribution functions, and can be compared to the corresponding results derived from other treatments. As we mentioned in the introduction, there are (to our knowledge) two different approaches to neutrino propagation in dense media which make use of some kind of distribution functions. Both methods are based on perturbation techniques, assuming that the Hamiltonian can be separated into two terms : $H = H_0 + H_{int}$, with H_{int} considered as a small perturbation. They both arrive to an equation with a term which is second order in G_F and corresponds to a non-trivial (i.e., non-diagonal in flavor) Boltzmann collision integral of neutrinos interacting with other particles in the background. This second-order term is absent in our approach, at least when correlations are neglected. Within this approximation, and neglecting small derivative terms[8], the evolution equations of Ref. [11] coincides with Eq. (72). A similar comparison can be made using the results derived in [13], [14].

VI. CONCLUSIONS

In this paper we have studied propagation of two neutrino species in dense media with flavor oscillations. We used an scheme which is based on the introduction of relativistic Wigner functions and correlation functions. Our aim is to develop relativistic kinetic equations, and to analyze the possibility that such approach will correctly describe both the relativistic dispersion relations of in-medium neutrinos, as well as the appropriate evolution equations for neutrino distribution functions. This allows us to treat in-medium neutrino masses and kinetic equations on an equal foot. We considered the medium as consisting on electrons, although other constituents can be incorporated in a straightforward way.

By writing the equations of motion for Wigner operators and taking statistical averages, one arrives to an infinite chain of equations of the BBGKY hierarchy-type. This infinite chain has to be broken at some level in order to obtain a solution, and we have considered here the lowest-level of approximation, which consists in neglecting correlations (the so-called Hartree approximation). We have first examined the situation in equilibrium, which reproduces the well-known results for relativistic dispersion relations [10]. Next, we assumed that the electron background is in equilibrium, although its density needs not to be constant, and neutrinos propagate out of equilibrium. This gives rise to a set of kinetic equations which have to be solved, in a general situation, by taking the appropriate boundary conditions.

[8] As discussed in [11] , these terms should only kept in the equations when strong inhomogeneities of size L are present, such that $|\vec{p}|L \sim 1$. See also [24] and [25].

In order to obtain some insight into the above equations, we have considered with some detail the usual MSW scenario, in which neutrinos are relativistic and the scale of space-time variations in distribution functions have macroscopic values (such as flavor conversion in the Sun or newly-born neutron stars). In this case, our system of equations reduce to the results obtained by other authors with the use of perturbation techniques [11], [13] when only the first-order correction is considered. However, none of these methods has been showed to incorporate the correct neutrino dispersion relations.

Now the question which arises is wether the inclusion of correlations into our scheme will lead to the same results as in the previous references, also for the second-order terms. Indeed, this seems to be the case, since correlations turn out to be proportional to the coupling constant G_F, and substitution into the BBGKY hierarchy will give corrections of the order G_F^2, as one can see from Eqs. (21) and (22). A more detailed study of the kinetic equations developed here will elucidate this question and, perhaps, give rise to new phenomena in the physics of neutrinos in dense media. This will be the subject of a future work.

APPENDIX A: FREE SYSTEM WITH ONE AND TWO GENERATIONS

In this appendix we give explicit formulae for the non-interacting neutrino Wigner functions. We first consider the case with only one generation, and we will generalize these results to the case of two neutrino generations with a non-diagonal mass matrix. Neutrinos are assumed to be in equilibrium. Of course, equilibrium can not be reached in the non-interacting case, but one can imagine a situation where a very weak interaction is added in order for the system to reach equilibrium. This hypothetical interaction can then be turned off without changing the equilibrium properties of the system.

We start with one generation of free neutrinos in equilibrium. The Wigner function then verifies the equations :

$$\begin{aligned}(\gamma p - m)F^{(\nu)}(p) = 0 \\ F^{(\nu)}(p)(\gamma p - m) = 0\end{aligned} \tag{A1}$$

where m is the neutrino mass and $F^{(\nu)}(p)$ the equilibrium Wigner function. As in previous cases, the above equations are the Hermitian conjugate of one another. We next multiply the first one by $(\gamma p + m)$, which gives

$$(p^2 - m^2)F^{(\nu)}(p) = 0 \tag{A2}$$

and this implies that the Wigner function vanishes whenever the condition $p^2 - m^2 = 0$ is not fulfilled. We introduce the Grand Canonical density matrix operator

$$\widehat{\rho} = Z^{-1} e^{-\beta\,(\widehat{H} - \mu\,\widehat{L})}; \qquad Z = Tr\, e^{-\beta\,(\widehat{H} - \mu\,\widehat{L})} \tag{A3}$$

where $\widehat{H}$ is the Hamiltonian, $\widehat{L}$ the lepton number operator and μ the neutrino chemical potential.

We obtain, after standard quantization of the neutrino field in the helicity-states basis:

$$\begin{aligned} F_{ij}^{(\nu)}(p) &= (2\pi)^{-4} \int d^4y\, e^{-ipy} < \widehat{\bar{\nu}}_j(x + y/2)\widehat{\nu}_i(x - y/2) >= \\ &= (2\pi)^{-4} \int d^4y\, e^{-ipy} (2\pi)^{-3} \int d^3\vec{k} \sum_{\lambda=\pm} [< \widehat{N}_\lambda(\vec{k}) > \bar{u}_j^\lambda(\vec{k}) u_i^\lambda(\vec{k}) e^{iky} - < \widehat{\bar{N}}_\lambda(\vec{k}) > \bar{v}_j^\lambda(\vec{k}) v_i^\lambda(\vec{k}) e^{-iky}] \\ &= (2\pi)^{-3} \int d^3\vec{k} \sum_{\lambda=\pm} [< \widehat{N}_\lambda(\vec{k}) > \bar{u}_j^\lambda(\vec{k}) u_i^\lambda(\vec{k}) \delta(p - k) - < \widehat{\bar{N}}_\lambda(\vec{k}) > \bar{v}_j^\lambda(\vec{k}) v_i^\lambda(\vec{k}) \delta(p + k)] \end{aligned} \tag{A4}$$

where $\widehat{N}_\lambda(\vec{k})$ ($\widehat{\bar{N}}_\lambda(\vec{k})$) is the number operator for neutrinos (antineutrinos) with momentum $\vec{k}$ and helicity λ[9]. The notations $u^\lambda(\vec{k}), v^\lambda(\vec{k}), ...$ for spinors and $\widehat{a}^\dagger(\vec{k}), \widehat{a}_\lambda(\vec{k}), ...$ for creation and destruction operators have their usual meaning. The above averages can be calculated using standard techniques, giving :

$$< \hat{N}_\lambda(\vec{k}) >= Tr(\widehat{\rho}\widehat{a}_\lambda^\dagger(\vec{k})\widehat{a}_\lambda(\vec{k})) = \frac{1}{e^{\beta\,(E_k - \mu)} + 1} \equiv f(k) \tag{A5}$$

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[9]Operators will be considered in normal order.

$$< \widehat{\bar{N}}_\lambda(\vec{k}) >= Tr(\widehat{\rho}\widehat{b}^\dagger_\lambda(\vec{k})\widehat{b}_\lambda(\vec{k})) = \frac{1}{e^{\beta\,(E_k+\mu)}+1} \equiv \bar{f}(k) \tag{A6}$$

with $E_k \; = \sqrt{m^2+\vec{k}^2}$. Let us define the matrices

$$\begin{aligned} \Sigma^\lambda_{ij}(\vec{k}) &= \bar{u}^\lambda_j(\vec{k})u^\lambda_i(\vec{k}) \\ \bar{\Sigma}^\lambda(\vec{k}) &= -\bar{v}^\lambda_j(-\vec{k})v^\lambda_i(-\vec{k}) \end{aligned} \tag{A7}$$

More explicitly, if we introduce a coordinate system in such a way that $\vec{k} = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ one has

$$\Sigma^+(\vec{k}) = \frac{1}{2E_k}\begin{pmatrix} m & -E_k-k \\ -E_k+k & m \end{pmatrix}\mu(\vec{k}) \tag{A8}$$

$$\Sigma^-(\vec{k}) = \frac{1}{2E_k}\begin{pmatrix} m & -E_k+k \\ -E_k-k & m \end{pmatrix}\rho(\vec{k}) \tag{A9}$$

$$\bar{\Sigma}^+(\vec{k}) = \frac{1}{2E_k}\begin{pmatrix} m & E_k\;-k \\ E_k\;+k & m \end{pmatrix}\mu(\vec{k}) \tag{A10}$$

$$\bar{\Sigma}^-(\vec{k}) = \frac{1}{2E_k}\begin{pmatrix} m & E_k\;+k \\ E_k\;-k & m \end{pmatrix}\rho(\vec{k}) \tag{A11}$$

where

$$\mu(\vec{k}) = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}\sin\theta\, e^{-i\phi} \\ \frac{1}{2}\sin\theta\, e^{i\phi} & \sin^2\frac{\theta}{2} \end{pmatrix} \tag{A12}$$

and $\rho(\vec{k}) = \mu(-\vec{k})$. By substituting into Eq. (A4) one obtains, after some algebra

$$F^{(\nu)}(p) = (2\pi)^{-3}2E_p\delta(p^2-M^2)\sum_{\lambda=\pm}[\theta(p^0)f(p)\Sigma^\lambda(\vec{p}) + \theta(-p^0)\bar{f}(p)\bar{\Sigma}^\lambda(\vec{p})] \tag{A13}$$

By performing the sum over helicities, we finally arrive to the expression

$$F^{(\nu)}(p) = (\gamma p + m)f_W(p) \tag{A14}$$

where the scalar function $f_W(p)$ is given by

$$f_W(p) = \frac{1}{4m}Tr\left[F^{(\nu)}(p)\right] = (2\pi)^{-3}\delta(p^2-m^2)\left[\theta(p^0)f(p) + \theta(-p^0)\bar{f}(p)\right] \tag{A15}$$

The generalization of the above formulae to more than one neutrino flavor is done by using mass eigenstates as an intermediate step. As defined in Section 2, we use a prime to represent such states. In the case of non-interacting neutrinos, they arise from diagonalization of the free mass matrix M. The equation of motion for the neutrino Wigner function is now, in the flavor basis

$$(\gamma p - M)F^{(\nu)}(p) = 0 \tag{A16}$$

The free mass eigenstates are related to the flavor states through a matrix

$$U \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{A17}$$

in the way

$$\widehat{\nu}(x) = U\widehat{\nu}'(x) \tag{A18}$$

and one has

$$M = UM'U^\dagger \tag{A19}$$

where M' is the eigenvalues matrix

$$M' = diag(m_1, m_2) \tag{A20}$$

The Wigner function in the mass eigenstates basis, $F'^{(\nu)}(p)$ is related to $F^{(\nu)}(p)$ trough

$$F^{(\nu)}(p) = UF'^{(\nu)}(p)U^\dagger \tag{A21}$$

and will obey the diagonal equation

$$(p^2 - M'^2)F'^{(\nu)}(p) = 0 \tag{A22}$$

By quantizing the massive fields $\widehat{\nu}'^a(x)$ ($a = 1, 2$ for two generations) we obtain the result

$$\begin{aligned} F'^{(\nu)ab}_{ij}(p) &= (2\pi)^{-4} \int d^4y\, e^{-ipy} < \widehat{\bar{\nu}}'^b_j(x + y/2)\widehat{\nu}'^a_i(x - y/2) >= \\ &= (2\pi)^{-3} \int d^3\vec{k} \sum_{\lambda=\pm} [< \widehat{N}^a_\lambda(\vec{k}) > \bar{u}^{a,\lambda}_j(\vec{k})u^{a,\lambda}_i(\vec{k})\delta(p - k) - \\ &< \widehat{\bar{N}}^a_\lambda(\vec{k}) > \bar{v}^{a,\lambda}_j(\vec{k})v^{a,\lambda}_i(\vec{k})\delta(p + k)]\delta_{a,b} \qquad a, b = 1, 2 \end{aligned} \tag{A23}$$

with similar notations as before. The superscript a has been added to label different mass eigenstates. In this basis, then, the Wigner function is diagonal, as expected from Eq. (A22). The Wigner functions appearing in the latter equation are given by

$$\begin{aligned} F'^{(\nu)11}(p) &= (2\pi)^{-3}\delta(p^2 - m_1^2)\left[\theta(p^0)f_1(p) + \theta(-p^0)\bar{f}_1(p)\right](\gamma p + m_1) \\ F'^{(\nu)22}(p) &= (2\pi)^{-3}\delta(p^2 - m_2^2)\left[\theta(p^0)f_2(p) + \theta(-p^0)\bar{f}_2(p)\right](\gamma p + m_2) \end{aligned} \tag{A24}$$

and we introduced the notations

$$f_a(p) = \frac{1}{e^{\beta(E^a_p - \mu)} + 1} \tag{A25}$$

$$\bar{f}_a(p) = \frac{1}{e^{\beta(E^a_p + \mu)} + 1} \tag{A26}$$

($a = 1, 2$) with $E^a_p \equiv \sqrt{m_a^2 + \vec{p}^2}$. We notice that the chemical potential is the same for both generations of neutrinos in Eq. (A25). This is so because, in our model, there is only a conserved current : the electron plus muon lepton number. A similar comment can be made regarding antineutrinos, Eq. (A26). The Wigner function in the flavor basis is then obtained from Eq. (A21).

APPENDIX B: NEUTRINO MASSES AND WIGNER FUNCTIONS IN THE HARTREE APPROXIMATION

We now discuss with some detail the dispersion relations and neutrino masses which arise from the equations of motion of the Wigner function within the Hartree approximation. In order to simplify the notations as much as possible, and to obtain insight into the problem as well, we start with one neutrino flavor (let us consider electron neutrinos). Then, Eq. (36) takes the simpler form

$$[\gamma p - m - \gamma^0 \frac{1}{2}(1 - \gamma^5)\widetilde{V}]F^{(\nu)}(p) = 0 \tag{B1}$$

Dispersion relations are the necessary conditions for Eq. (B1) to have solutions other than the trivial one $F^{(\nu)}(p) = 0$. One can compute the determinant for this equation by setting $p = (p_0, 0, 0, p_z)$. The result then is

$$\det[\gamma p - m - \gamma^0 \frac{1}{2}(1 - \gamma^5)\widetilde{V}] = \left[p_0^2 - p_z^2 - m^2 - (p_z + p_0)\,\widetilde{V}\right]\left[p_0^2 - p_z^2 - m^2 + (p_z - p_0)\,\widetilde{V}\right] \tag{B2}$$

By equating to zero this determinant, and solving for p_0, one obtains four solutions : $p_0 = E_+$, $p_0 = E_-$, $p_0 = -\bar{E}_+$ and $p_0 = -\bar{E}_-$, where

$$E_+ \equiv \frac{\widetilde{V}}{2} + \sqrt{(\frac{\widetilde{V}}{2} - p_z)^2 + m^2} \tag{B3}$$

$$E_- \equiv \frac{\widetilde{V}}{2} + \sqrt{(\frac{\widetilde{V}}{2} + p_z)^2 + m^2} \tag{B4}$$

$$\bar{E}_+ \equiv -\frac{\widetilde{V}}{2} + \sqrt{(\frac{\widetilde{V}}{2} - p_z)^2 + m^2} \tag{B5}$$

$$\bar{E}_- \equiv -\frac{\widetilde{V}}{2} + \sqrt{(\frac{\widetilde{V}}{2} + p_z)^2 + m^2} \tag{B6}$$

In order to identify the above results, we start from Eq. (B1) and use the projection operators to obtain an equation for $F_L^{(\nu)}(p)$, as in section 3. One then arrives to

$$(p^2 - m^2 - \gamma p \gamma^0 \widetilde{V}) F_L^{(\nu)}(p) = 0 \tag{B7}$$

We take into account the following identity

$$\gamma p \gamma^0 = \begin{pmatrix} p^0 + \vec{p} \cdot \vec{\sigma} & 0 \\ 0 & p^0 - \vec{p} \cdot \vec{\sigma} \end{pmatrix} \tag{B8}$$

This means that the positive-helicity component $F_L^+(p)$ has to obey the relationship

$$[p^2 - m^2 - \widetilde{V}(p_0 - |\vec{p}|)] F_L^+(p) = 0 \tag{B9}$$

The solutions for the dispersion relation $p^2 - m^2 - \widetilde{V}(p_0 - |\vec{p}|) = 0$ are given by $p_0 = E_+$ and $p_0 = -\bar{E}_+$. One can also make an analogous study for $F_L^-(p)$, which gives $p_0 = E_-$ and $p_0 = -\bar{E}_-$ as possible solutions. This allows us to interpret Eqs. (B3)-(B6) as corresponding to neutrinos (E) and antineutrinos ($\bar{E}$) of positive and negative helicity (subindexes $+$ and $-$, respectively). One can check that these solutions, obtained from the Hartree approximation, coincide with the ones obtained from the poles of the neutrino propagator at the one-loop level (see, for example, [10]).

We now concentrate on ultra-relativistic neutrinos, and consider densities as occur in normal or compact stars. Under these conditions, we can neglect terms such as $\widetilde{V}|\overrightarrow{p}|$ and m^2 by comparison to $|\overrightarrow{p}|^2$ and, in this way, the dispersion relations give us the neutrinos effective masses for each degree of freedom, which are

$$M_+^2 = m^2 \tag{B10}$$

$$M_-^2 = m^2 + 2\widetilde{V}|\overrightarrow{p}| \tag{B11}$$

$$\overline{M}_+^2 = m^2 - 2\widetilde{V}|\overrightarrow{p}| \tag{B12}$$

$$\overline{M}_-^2 = m^2 \tag{B13}$$

We used the same notations as for the energies. In these equations we see that positive polarization neutrinos and negative polarization antineutrinos behave, approximately, as free particles.

The above results can be generalized to the case of two neutrino flavors in a straightforward way. Let us start from Eq. (39) and transform it to the mass eigenstates basis in vacuum, as discussed in Appendix A for the non-interacting case. We then arrive to the following equation

$$(p^2 - M'^2 - \gamma p \gamma^0 \Phi') F_L'^{(\nu)}(p) = 0 \tag{B14}$$

Following the discussion we made above for one generation, we readily obtain the equation for the negative-helicity component (we replace p_0 to E_- for neutrinos)

$$(p^2 - M'^2 - (E_- + |\vec{p}|)\Phi') F_L^{-\prime(\nu)}(p) = 0 \tag{B15}$$

The latter equation suggests us to define an *effective mass matrix* $\mathcal{M}_-$ as follows

$$\mathcal{M}_-^2 \equiv M'^2 + (E_- + |\vec{p}|)\Phi' \tag{B16}$$

If flavor mixing exists, this matrix is non-diagonal. More explicitly, we find

$$\mathcal{M}_-^2 = \begin{pmatrix} (V_n + V\cos^2\theta)(E_- + |\vec{p}|) + m_1^2 & -V\sin\theta\cos\theta(E_- + |\vec{p}|) \\ -V\sin\theta\cos\theta(E_- + |\vec{p}|) & (V_n + V\sin^2\theta)(E_- + |\vec{p}|) + m_2^2 \end{pmatrix} \tag{B17}$$

The corresponding dispersion relation is then

$$\det\left(p^2 - \mathcal{M}_-^2\right) = 0 \tag{B18}$$

Eqs. (B16) and (B18)can be used to obtain the *exact* energy levels of neutrinos (and antineutrinos) as a function of the neutrino momentum. In the ultra-relativistic limit, one can approximate

$$\mathcal{M}_-^2 \simeq \begin{pmatrix} A_n & 0 \\ 0 & A_n \end{pmatrix} + \begin{pmatrix} A\cos^2\theta + m_1^2 & -A\sin\theta\cos\theta \\ -A\sin\theta\cos\theta & A\sin^2\theta + m_2^2 \end{pmatrix} \tag{B19}$$

here, $A \equiv 2|\vec{p}|V$ is the induced squared mass due to charged currents and $A_n \equiv 2|\vec{p}|V_n$ the analogous magnitude for neutral currents. We first consider the first term on the second hand of Eq. (B19) in order to find the eigenvalues of $\mathcal{M}_-^2$ (following the notations introduced in Section 2, we represent the diagonal form of $\mathcal{M}_-$ by $\tilde{M}$). Since neutral currents are diagonal in flavor, they can be added at the end. After some trivial algebra, the mass eigenvalues $\tilde{M}_1$ and $\tilde{M}_2$ are given by

$$\tilde{M}_{1,2}^2 = \frac{1}{2}(A + \Sigma) \mp \frac{1}{2}[(\Delta_0 \cos 2\theta - A)^2 + \Delta_0^2 \sin^2 2\theta]^{\frac{1}{2}} + A_n \tag{B20}$$

where $\Sigma \equiv m_2^2 + m_1^2$ and $\Delta_0 \equiv m_2^2 - m_1^2$. One can also obtain, by diagonalizing $\mathcal{M}_-^2$ the in-medium mixing angle θ_M. The well-known result

$$\sin^2 2\theta_M = \frac{\Delta_0^2 \sin^2 2\theta}{(\Delta_0 \cos 2\theta - A)^2 + \Delta_0^2 \sin^2 2\theta} \tag{B21}$$

is then reproduced.

The procedure discussed in Appendix A can be extended in order to construct the neutrino Wigner functions in the equilibrium state described by the Hartree approximation. We will only give the final result. As before, it is illustrative to consider first the case of one generation (for example, electron neutrinos). One obtains

$$F^{(\nu)}(p) = F^-(p) + F^+(p) + \bar{F}^-(p) + \bar{F}^+(p) \tag{B22}$$

where

$$\begin{aligned} F^-(p) &= (2\pi)^{-3} 2E_p \delta(p^2 - M_-^2) f(p) \Sigma^-(\vec{p}) \\ F^+(p) &= (2\pi)^{-3} 2E_p \delta(p^2 - M_+^2) f(p) \Sigma^+(\vec{p}) \\ \bar{F}^-(p) &= (2\pi)^{-3} 2E_p \delta(p^2 - \bar{M}_-^2) \bar{f}(p) \bar{\Sigma}^-(\vec{p}) \\ \bar{F}^+(p) &= (2\pi)^{-3} 2E_p \delta(p^2 - \bar{M}_+^2) \bar{f}(p) \bar{\Sigma}^+(\vec{p}) \end{aligned} \tag{B23}$$

and the effective masses are defined in Eqs. (B10)-(B13). The chiral left-handed component $F_L^{(\nu)}(p)$, as defined in Eq. (37) can be approximated by :

$$F_L^{(\nu)}(p) \simeq F_L^-(p) + \bar{F}_L^+(p) = (2\pi)^{-3} 2E_p \left[\delta(p^2 - M_-^2) f(p) \Sigma_L^-(\vec{p}) + \delta(p^2 - \bar{M}_+^2) \bar{f}(p) \bar{\Sigma}_L^+(\vec{p})\right] \tag{B24}$$

We have introduced the following definitions :

$$\Sigma_L^-(\vec{p}) \equiv P_L \Sigma^-(\vec{p}) P_R \simeq -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rho(\vec{p}) \tag{B25}$$
$$\bar{\Sigma}_L^+(\vec{p}) \equiv P_L \bar{\Sigma}^+(\vec{p}) P_R \simeq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mu(\vec{p})$$

$$\Sigma_L^+(\vec{p}) \equiv P_L \Sigma^+(\vec{p}) P_R \simeq 0$$
$$\bar{\Sigma}_L^-(\vec{p}) \equiv P_L \bar{\Sigma}^-(\vec{p}) P_R \simeq 0$$

The latter approximations hold for ultra-relativistic neutrinos. Thus, under this approximation the field contains only two degrees of freedom : neutrinos with negative helicity and antineutrinos with positive helicity.

We now go to the case of two neutrino generations. As in the free case, the Wigner function will be diagonal if considered in the interaction eigenstates basis :

$$\widetilde{F}^{(\nu)ab}(p) = \begin{pmatrix} \tilde{F}^{(\nu)11}(p) & 0 \\ 0 & \tilde{F}^{(\nu)22}(p) \end{pmatrix} \tag{B26}$$

Moreover, each one of the diagonal components can be easily obtained by taking into account the corresponding dispersion relations. One then arrives to a set of equations similar to Eqs. (B22)-(B23) for each one of the two diagonal components of the Wigner function. As in the case of one generation, we concentrate on the chiral left-handed component, which is finally approximated by

$$\widetilde{F_L}^{(\nu)ab}(p) \simeq \widetilde{F_L}^{-ab}(p) + \widetilde{\bar{F}_L}^{+ab}(p) = (2\pi)^{-3} 2E_p \delta_{ab} \left[\delta(p^2 - \tilde{M}_a^2) f_a(p) \Sigma_L^-(\vec{p}) + \delta(p^2 - \widetilde{\bar{M}}_a^2) \bar{f}_a(p) \bar{\Sigma}_L^+(\vec{p})\right] \tag{B27}$$

with $a, b = 1, 2$. The functions $f_a(p)$ and $\bar{f}_a(p)$ are the same functions defined in Eqs. (A25) and (A26), but with the replacement $m_a \to \tilde{M}_a$. The two terms in the latter equation obey the relations

$$(\gamma p - \widetilde{M}^{ab}) \widetilde{F}_L^{-ab}(p) = 0$$
$$(\gamma p - \widetilde{\bar{M}}^{ab}) \widetilde{\bar{F}}_L^{+ab}(p) = 0 \tag{B28}$$

where $\widetilde{M}^{ab}$ and $\widetilde{\bar{M}}^{ab}$ are the (diagonal) effective masses matrices for neutrinos and antineutrinos. In order to construct the Wigner Function in the flavor space, we perform an unitary transformation U_M that is defined by the mixing angle in mater θ_M . In this way we have, for example :

$$F_L^{-ee}(p) = \cos^2(\theta_M) \widetilde{F}_L^{-11}(p) + \sin^2(\theta_M) \widetilde{F}_L^{-22}(p)$$
$$F_L^{-e\mu}(p) = F_L^{-\mu e}(p) = \sin(\theta_M)\cos(\theta_M) \left[\widetilde{F}_L^{-11}(p) - \widetilde{F}_L^{-22}(p)\right]$$
$$F_L^{-\mu\mu}(p) = \sin^2(\theta_M) \widetilde{F_L}^{-11}(p) + \cos^2(\theta_M) \widetilde{F_L}^{-22}(p) \tag{B29}$$

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