Neutrino-pair radiation form neutron star crusts: Collective effects

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We present a method to calculate $\nu\bar{\nu}$ energy losses from neutron star crusts, which automatically takes into account for collective effects, and allows to calculate the total emissivity without a separate consideration of particular processes. We show that the formula we obtain describe the known results for the emissivity due to plasmon decay and Bremsstrahlung from degenerate electrons, when one of this processes dominates. In the case of low temperatures, our formula gives a suppression of the electron vector weak-current contribution to $\nu\bar{\nu}$ Bremsstrahlung, due to the collective effects discussed in this paper.

I. INTRODUCTION

According to modern theoretical scenarios, within the first hundred years of existence of a neutron star the cooling of its interior layers occurs in an extremely nonuniform way. Due to the so-called "fast cooling processes" in nuclear matter, the core of a neutron star cools down very fast, while the external crust cools down much slowly, by heat diffusion towards the inside and by neutrino radiation [1]. The time necessary for thermal relaxation crucially depends on the intensity of neutrino and antineutrino radiation from the crust of the neutron star, which influences the temperature of the stellar surface, and thus the gamma and X-ray which are observed. It is well-known that radiation of neutrinos and antineutrinos from the neutron star crust is caused dominantly by $\nu\bar{\nu}$ Bremsstrahlung of degenerate electrons, and by plasmon decay into neutrino pairs. The Bremsstrahlung process, operating in a liquid or crystalline phase of the crust, has been previously studied by many authors [2] - [8]. These calculations have been performed neglecting collective interactions of electrons with the neutrino field. When the temperature of the crust is smaller than the electron plasma frequency, such an approximation is not justified because, for the scenario under consideration, the wavelength λ of radiated neutrinos and antineutrinos is larger than the electron Debye screening distance D_e. By undergoing a quantum transition, the radiating electron electromagnetically induces a motion of other electrons inside the Debye sphere around itself. The weak current of perturbed electrons inside the Debye sphere generates neutrinos coherently with the initial electron, therefore screening its vector weak coupling with the neutrino field. Since this collective effect takes place when the neutrino wave-length is of the order, or larger, than the Debye screening distance, the relevant parameter of the problem is $k^2 D_e^2$, where k is the momentum carried out by the neutrino-pair[1]. The screening effect was demonstrated in [9] for neutrino-pair emission due to electron-phonon scattering in a crystalline crust, where the condition $k^2 D_e^2 \ll 1$ is fulfilled. In this limiting case, valid when the temperature T is much smaller than the electron plasma frequency ω_{pe}, the effective vector weak-current of electrons is totally screened by the plasma polarization, which dramatically modifies the neutrino emissivity.

In the present paper, we introduce a method of calculation for $\nu\bar{\nu}$ energy losses from a neutron star crust, which incorporates collective effects in a degenerate plasma at arbitrary temperatures, limited only by the degeneration condition $T \ll \mu_e$, where μ_e is the electron chemical potential. This condition holds during the cooling epoch described above.

Collective effects appear if one takes into account the possibility of virtual photon-exchange among electrons in the plasma. To obtain the neutrino emissivity, we use the fluctuation-dissipation theorem in order to relate the weak current-current correlation function to the imaginary part of the exact retarded polarization functions of the plasma. This procedure, as we will see, has the advantage that collective effects are automatically included, and do not need a separate consideration. The obtained formula for the neutrino energy losses includes contribution of

[1]In what follows we use the system of units $\hbar = c = 1$ and the Boltzmann constant $k_B = 1$. The fine-structure constant is $\alpha = e^2 = 1/137$.

plasmon decay into neutrino pairs, as well as the process of neutrino-pair emission due to electron collisions with nuclei and phonons. We identify the latter mechanism as the Bremsstrahlung process from electrons. Thus, the total neutrino-pair emissivity of the crust, caused by $\nu\bar{\nu}$ decay of plasmons and Bremsstrahlung from electrons is represented by a unique expression.

This paper is organized as follows. In Sect. 2 we discuss the reference status of the problem, and derive some formulae describing the rate of $\nu\bar{\nu}$ Bremsstrahlung, which we use in the following considerations. In Sect. 3, we show how the same $\nu\bar{\nu}$ Bremsstrahlung rate can be obtained by the use of the Optical Theorem, and calculate the imaginary parts of the retarded polarization tensors of a liquid or crystallin crust. In order to incorporate collective effects, in Sec.4 we generalize the results obtained by the Optical Theorem by replacing the imaginary parts of the polarization tensors by the weak current-current correlation function, according to the Fluctuation-Dissipation Theorem. This correlation function is obtained by summation of all diagrams with allowance of intermediate photon-exchange among the electrons. In Sect. 5 we derive a general formula for the $\nu\bar{\nu}$ emissivity, which includes the energy losses due to electron Bremsstrahlung and plasmon decay. Some limiting cases and numerical tests are presented in Sect. 6 in order to demonstrate the validity of the obtained general formula. Discussion of the results and conclusions are shown in Sect. 7.

II. STATEMENT OF THE PROBLEM

We use the Standard Model of weak interactions, and consider low-energy electrons, which are typical for neutron star interiors. Therefore the in-vacuum weak interaction of electrons with the neutrino field can be written in a point-like current-current approach

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \int j^{\mu}(x) J_{\mu}(x) d^4x, \tag{1}$$

where G_F is the Fermi coupling constant, and

$$j^{\mu} = \bar{\nu} \gamma^{\mu} (1-\gamma_5) \nu$$

is the neutrino current. The vacuum weak current of an electron is of the standard form, which is a sum of vector and axial-vector pieces

$$J_{\mu} = \bar{\psi} (C_V \gamma_{\mu} - C_A \gamma_{\mu} \gamma_5) \psi. \tag{2}$$

Here, ψ stands for electron field; $C_V = \frac{1}{2} + 2\sin^2\theta_W$, $C_A = \frac{1}{2}$ stand for emission of electron neutrinos, whereas $C_V' = -\frac{1}{2} + 2\sin^2\theta_W$, $C_A' = -\frac{1}{2}$ are to be used for muon and tau neutrinos; θ_W is the Weinberg angle.

The differential rate of $\nu\bar{\nu}$ production takes the form

$$d\Gamma = \frac{G_F^2}{2} \frac{1}{(2\pi)^6} \int \frac{d^3p}{2E} \int \frac{d^3p'}{2E'} f(E) \left[1 - f(E')\right]$$
$$\mathrm{Tr}\left(M_{\mu}^{\dagger} M_{\nu}\right) \mathrm{Tr}\left(j^{\mu} j^{\nu *}\right) \frac{d^3k_1}{2\omega_1 (2\pi)^3} \frac{d^3k_2}{2\omega_2 (2\pi)^3}. \tag{3}$$

Here M_{μ} is the matrix element of the weak transition current for the electron in the Bremsstrahlung process. The symbol $\mathrm{Tr}\left(M_{\mu}^{\dagger} M_{\nu}\right)$ includes statistical averaging and summation over initial and final states of the background (see below). Summation over all initial $p = (E, \mathbf{p})$ and final $p' = (E', \mathbf{p}')$ states of the electron has to take into account the Pauli principle via the appropriate blocking factors, with the Fermi distribution function $f(E)$ of degenerate electrons. Finally, $k_1 = (\omega_1, \mathbf{k}_1)$, $k_2 = (\omega_2, \mathbf{k}_2)$ are the neutrino and antineutrino four-momenta, respectively. To simplify the calculations in what follows, we consider an ultrarelativistic electron gas, which is typical for the dominant volume of the neutron star crust.

The matrix element for the weak transition current of the electron has the following form [3]:

$$M^{\mu} = \bar{u}(p') \left[\gamma^{\mu} (C_V - C_A\gamma_5) G(p'+K) \gamma^{\alpha} + \gamma^{\alpha} G(p-K) \gamma^{\mu} (C_V - C_A\gamma_5)\right] u(p)$$
$$\times \int_{-\infty}^{\infty} dt e^{-iq_0 t} \langle f| A_{\alpha}(-\mathbf{q}, \mathbf{t}) b^{\dagger}(p', \sigma') b(p, \sigma) c^{\dagger}(k_1) d^{\dagger}(k_2) |i\rangle . \tag{4}$$

In this equation, $p = (E, \mathbf{p})$ and $p' = (E', \mathbf{p}')$ are the initial and final momenta of the electron. The operators b, c, d refer to electrons, neutrinos and antineutrinos, respectively. In this process, the momentum and energy transferred to the background are $\mathbf{q} = \mathbf{p} - \mathbf{p}' - \mathbf{k}$ and $q_0 = E - E' - \omega$, where $K = (\omega, \mathbf{k})$ is the total four-momentum of the neutrino pair : $K = k_1 + k_2$. . Within the Coulomb gauge, used in this section, the electromagnetic potential has only a non-vanishing (scalar) component.

The energy transfer to the background, q_0, is small with respect to the momentum transfer $|\mathbf{q}|$, and is therefore neglected in the electron matrix element. On the other hand, the momentum transfer to the background, of the order $|\mathbf{q}| \sim p_F$, where p_F is the electron Fermi momentum, is large with respect to the neutrino-pair momentum, therefore we assume that $q = p - p' - k \simeq p - p'$. Keeping this in mind, we can use the well-known method of soft photons [13]. With the aid of commutation rules and the Dirac equation, one can represent the matrix element as follows

$$M^{\mu} = \left(\frac{p'^{\mu}}{p'K} - \frac{p^{\mu}}{pK}\right) \bar{u}\left(p'\right) \gamma^{0} \left(C_V - C_A \gamma_5\right) u\left(p\right) \times \int_{-\infty}^{\infty} dt e^{-iq_0 t} \left\langle f\right| A_0\left(-\mathbf{q}, \mathbf{t}\right) b^{\dagger}\left(p', \sigma'\right) b\left(p, \sigma\right) c^{\dagger}\left(k_1\right) d^{\dagger}\left(k_2\right) \left|i\right\rangle . \tag{5}$$

After squaring and taking the trace, and performing summation over final states and thermal averaging over initial states of the background we obtain

$$\mathrm{Tr}\left(M^{\dagger\mu} M^{\nu}\right) = 4\left(C_V^2 + C_A^2\right) \int dq_0 d^3q \delta\left(\mathbf{p} - \mathbf{p}' - \mathbf{q} - \mathbf{k}\right) \delta\left(E - E' - q_0 - \omega\right) \left(2EE' - pp'\right) \left(\frac{p'^{\mu}}{p'K} - \frac{p^{\mu}}{pK}\right) \left(\frac{p'^{\nu}}{p'K} - \frac{p^{\nu}}{pK}\right) \left\langle A_0 A_0 \right\rangle_{q_0, \mathbf{q}} , \tag{6}$$

The Fourier transform of the AA correlation function can be written with the aid of the dynamic form factor $S\left(q_0, \mathbf{q}\right)$:

$$\left\langle A_0 A_0 \right\rangle_{q_0, \mathbf{q}} = \frac{\left(4\pi e^2 Z\right)^2}{\left(q^2 \varepsilon\left(\mathbf{q}\right)\right)^2} S\left(q_0, \mathbf{q}\right) . \tag{7}$$

The dielectric function of a degenerate ultrarelativistic electron gas [14] can be taken in the static limit

$$\varepsilon\left(\mathbf{q}\right) = 1 + \frac{1}{\mathbf{q}^2 D_e^2} \left(\frac{2}{3} + \frac{1 - 3x^2}{6x} \ln\left|\frac{1+x}{1-x}\right| + \frac{x^2}{3} \ln\left|\frac{x^2}{1-x^2}\right|\right) , \tag{8}$$

where

$$x = \frac{|\mathbf{q}|}{2p_F} . \tag{9}$$

Using the approximations discussed above, we can rewrite Eq. (6) as follows

$$\mathrm{Tr}\left(M^{\dagger\mu} M^{\nu}\right) = 4\left(C_V^2 + C_A^2\right) \int dq_0 d^3q \left(\frac{4\pi e^2 Z}{q^2 \varepsilon\left(q\right)}\right)^2 S\left(q_0, \mathbf{q}\right) \left(2EE' - pp'\right) \times \left(\frac{p'^{\mu}}{p'K} - \frac{p^{\mu}}{pK}\right) \left(\frac{p'^{\nu}}{p'K} - \frac{p^{\nu}}{pK}\right) \delta\left(\mathbf{p} - \mathbf{p}' - \mathbf{q}\right) \delta\left(E - E' - q_0 - \omega\right) . \tag{10}$$

In a liquid crust, the energy transfer to the background is negligible due to the large nucleus mass M_i. In this case we can neglect q_0 everywhere except in the dynamic form factor. This allows the integration over dq_0 to be performed. We obtain

$$\mathrm{Tr}\left(M^{\dagger\mu} M^{\nu}\right)_{\mathsf{liquid}} = 8\pi N_i \left(C_V^2 + C_A^2\right) \int d^3q \left(\frac{4\pi e^2 Z}{q^2 \varepsilon\left(q\right)}\right)^2 S_{st}\left(\mathbf{q}\right) \left(2EE' - pp'\right) \times \left(\frac{p'^{\mu}}{p'K} - \frac{p^{\mu}}{pK}\right) \left(\frac{p'^{\nu}}{p'K} - \frac{p^{\nu}}{pK}\right) \delta\left(\mathbf{p} - \mathbf{p}' - \mathbf{q}\right) \delta\left(E - E' - \omega\right) . \tag{11}$$

The static structure factor of ions is defined as

$$S_{st}\left(\mathbf{q}\right) = \frac{1}{N_i} \int \frac{dq_0}{2\pi} S\left(q_0, \mathbf{q}\right) , \tag{12}$$

where N_i is the number density of ions. We use the structure factor calculated in [15] for a one-component classical plasma, which is a tabulated function of the non-ideality parameter Γ of the ionic component of the plasma.

In the case of a crystallin crust, the dynamic form factor can be written, in the one-phonon approximation [3] :

$$S(q_0,\mathbf{q}) = 2\pi \frac{N_i}{M_i} e^{-2W(q)} \sum_{\mathbf{s}\lambda\mathbf{K}} \frac{(\mathbf{q}\cdot\hat{\mathbf{e}}_{\lambda\mathbf{s}})}{2\omega_{\lambda\mathbf{s}}} \left[\frac{\delta(q_0-\omega_{\lambda\mathbf{s}})\,\delta_{\mathbf{q},\mathbf{s}-\mathbf{K}}}{1-e^{-\omega_{\lambda\mathbf{s}}/T}} + \frac{\delta(q_0+\omega_{\lambda\mathbf{s}})\,\delta_{\mathbf{q},-\mathbf{s}-\mathbf{K}}}{e^{-\omega_{\lambda\mathbf{s}}/T}-1} \right]. \tag{13}$$

Here $\omega_{\lambda\mathbf{s}}$ is the phonon frequency, which depends on the wave vector $\mathbf{s}$. Each phonon mode λ is defined by its polarization vector $\hat{\mathbf{e}}_{\lambda\mathbf{s}}$. The $\mathbf{K}$'s are reciprocal lattice vectors. The Debye-Waller factor is of the form

$$2W(q) = \sum_{\mathbf{s}\lambda\mathbf{K}} \frac{(\mathbf{q}\cdot\hat{\mathbf{e}}_{\lambda\mathbf{s}})^2}{2N_iM_i\omega_{\lambda\mathbf{s}}} \coth\left(\frac{\omega_{\lambda\mathbf{s}}}{2T}\right) \tag{14}$$

III. TREATMENT BY THE OPTICAL THEOREM

With the help of the Optical Theorem, the rate Eq. (3) of $\nu\bar{\nu}$ production can be written as

$$d\Gamma = \frac{G_F^2}{2} \frac{2}{\exp\left(\frac{\omega}{T}\right)-1} \operatorname{Im}\left[\Pi_{\mu\nu}(K)\operatorname{Tr}\left(j^\mu j^{\nu *}\right)\right] \frac{d^3k_1}{2\omega_1(2\pi)^3} \frac{d^3k_2}{2\omega_2(2\pi)^3}, \tag{15}$$

The exact, irreducible retarded polarization tensor $\Pi^{\mu\nu}$ of the medium, represents the sum of compact diagrams which include inside the electromagnetic interactions of the electron with nuclei, and have ends at the weak vertex $(C_V\gamma_\mu - C_A\gamma_\mu\gamma_5)$. This tensor is the following sum of vector and axial pieces

$$\Pi^{\mu\nu}(K) = \frac{C_V^2}{4\pi e^2}\Pi_V^{\mu\nu} + \frac{C_A^2}{4\pi e^2}\Pi_A^{\mu\nu}. \tag{16}$$

We traditionally include an extra-factor $4\pi e^2$ in the definition of all polarization tensors. By this reason, the factor $1/4\pi e^2$ has been included in Eq. (16). Following these notations, the vector-vector tensor $\Pi_V^{\mu\nu}$ is the retarded tensor for the electromagnetic polarization of the plasma. In the absence of external magnetic fields, the parity-violating axial-vector polarization does not contribute to the rate of neutrino-pair production. In fact, by inserting $\int d^4K\delta^{(4)}(K-k_1-k_2)=1$ in this equation, and making use of the Lenard's integral

$$\begin{aligned} &\int \frac{d^3k_1}{2\omega_1}\frac{d^3k_2}{2\omega_2}\delta^{(4)}(K-k_1-k_2)\operatorname{Tr}\left(j^\mu j^{\nu *}\right) \\ &= \frac{4\pi}{3}\left(K_\mu K_\nu - K^2 g_{\mu\nu}\right)\theta\left(K^2\right)\theta\left(K^0\right), \end{aligned} \tag{17}$$

where $\theta(x)$ is the Heaviside step function, Eq. (15) can be written as

$$d\Gamma = \frac{4\pi}{3}\frac{G_F^2}{2}\frac{2}{\exp\left(\frac{\omega}{T}\right)-1}\operatorname{Im}\Pi^{\mu\nu}(K)\left(K_\mu K_\nu - K^2 g_{\mu\nu}\right)\theta\left(K^2\right)\theta\left(K^0\right)\frac{d^4K}{(2\pi)^6}. \tag{18}$$

Since the axial-vector polarization has to be an antisymmetric tensor, its contraction in (18) with the symmetric tensor $K_\mu K_\nu - K^2 g_{\mu\nu}$ vanishes.

A. Polarization tensors

To specify the components of the polarization tensors, we select a basis constructed from the following orthogonal four-vectors

$$h^\mu \equiv \frac{(\omega,\mathbf{k})}{\sqrt{K^2}}, \qquad l^\mu \equiv \frac{(k,\omega\mathbf{n})}{\sqrt{K^2}}, \tag{19}$$

where the space-like unit vector $\mathbf{n}=\mathbf{k}/k$ is directed along the electromagnetic wave-vector $\mathbf{k}$. Thus, the longitudinal basis tensor can be chosen as $L^{\rho\mu} \equiv -l^\rho l^\mu$, with normalization $L_\rho^\rho = 1$. The transverse (with respect to $\mathbf{k}$) components of $\Pi^{\rho\mu}$ have a tensor structure proportional to the tensor $T^{\rho\mu} \equiv (g^{\rho\mu} - h^\rho h^\mu + l^\rho l^\mu)$, where $g^{\rho\mu} = \mathrm{diag}(1,-1,-1,-1)$ is the signature tensor. This choice of $T^{\rho\mu}$ allows us to describe the two remaining directions orthogonal to h and l. Therefore, the transverse basis tensor has normalization $T_\rho^\rho = 2$. One can also check the following orthogonality relations: $l_\rho T^{\rho\mu} = 0$, as well as $k_i T^{i\mu} = 0$, and $K_\rho L^{\rho\mu} = K_\rho T^{\rho\mu} = 0$. In this basis, the vector-vector polarization tensor has the following form

$$\Pi_V^{\rho\mu}(K) = \pi_l(K) L^{\rho\mu} + \pi_t(K) T^{\rho\mu}, \tag{20}$$

where the longitudinal polarization function is defined as $\pi_l(K) = \left(1 - \omega^2/k^2\right) \Pi_V^{00}$ and the transverse polarization function is found to be $\pi_t(K) = (g_{\rho\mu}\Pi_V^{\rho\mu} - \pi_l)/2$. The axial-vector polarizations have to be antisymmetric tensors[2]. They can be written as

$$\Pi_{AV}^{\rho\mu}(K) = \pi_{AV}(K)\, i h_\lambda \epsilon^{\rho\mu\lambda 0}, \qquad \Pi_{VA}^{\rho\mu}(K) = \pi_{VA}(K)\, i h_\lambda \epsilon^{\rho\mu\lambda 0}, \tag{21}$$

where $\epsilon^{\rho\mu\lambda 0}$ is the completely antisymmetric tensor $\left(\epsilon^{0123} = +1\right)$; $\pi_{AV}(K)$ and $\pi_{VA}(K)$ are the axial-vector polarization functions of the medium. As for the axial term, it must be a symmetrical tensor. The most general expression for this tensor is, therefore

$$\Pi_A^{\mu\nu}(K) = \pi_l(K) L^{\mu\nu} + \pi_t(K) T^{\mu\nu} + \pi_A(K) g^{\mu\nu}. \tag{22}$$

Real parts of the retarded polarization tensors are the same as the real parts of time-ordered (causal) polarizations, which can be written in the one-loop approximation as

$$\Pi^{\mu\rho} = 4\pi i e^2 \,\mathrm{Tr}\left[\int \frac{d^4p}{(2\pi)^4}\, \gamma^\mu \hat{G}(p)\, \gamma^\rho \hat{G}(p+K)\right], \tag{23}$$

$$\Pi_{VA}^{\mu\rho} = 4\pi i e^2 \,\mathrm{Tr}\left[\int \frac{d^4p}{(2\pi)^4}\, \gamma^\mu \hat{G}(p)\, \gamma^\rho \gamma_5 \hat{G}(p+K)\right], \tag{24}$$

$$\Pi_A^{\mu\rho} = 4\pi i e^2 \,\mathrm{Tr}\left[\int \frac{d^4p}{(2\pi)^4}\, \gamma^\mu \gamma_5 \hat{G}(p)\, \gamma^\rho \gamma_5 \hat{G}(p+K)\right]. \tag{25}$$

Here, $\hat{G}(p)$ is the in-medium electron propagator, which includes the Pauli principle restrictions. This approximation has been studied by different authors. It corresponds to a collisionless plasma, and thus the one-loop polarization functions are real-valued in the case $K^2 > 0$, when Landau damping is not possible. In an ultrarelativistic, strongly-degenerate electron plasma, the longitudinal and transverse polarization functions take the form[3] [10]:

$$\mathrm{Re}\,\pi_l = \frac{1}{D_e^2}\left(1 - \frac{\omega^2}{k^2}\right)\left(1 - \frac{\omega}{2k}\ln\frac{\omega+k}{\omega-k}\right), \tag{26}$$

$$\mathrm{Re}\,\pi_t = \frac{3}{2}\omega_{pe}^2\left[1 + \left(\frac{\omega^2}{k^2} - 1\right)\left(1 - \frac{\omega}{2kv_F}\ln\frac{\omega+k}{\omega-k}\right)\right]. \tag{27}$$

The electron plasma frequency and the Debye screening distance are defined as

$$\omega_{pe}^2 = \frac{4\pi n_e e^2}{\mu_e}, \qquad \frac{1}{D_e^2} = 3\omega_{pe}^2, \tag{28}$$

with $\mu_e \simeq p_F$ and n_e being the chemical potential (the Fermi energy) and the number density, respectively, of degenerate electrons. The axial-vector one-loop polarization tensor is given by

[2]We consider also the axial-vector polarization because it will be used in the next Section.

[3]Our Eq. (26) differs from Eq. (A39) of the Ref. [10] by an extra factor $\left(\omega^2/k^2 - 1\right)$ because our basis l^μ, h^μ is different from that used by Braaten and Segel. All components of the complete tensor Eq. (20) identically coincide with that obtained in [10] for the degenerate case. By the same reason, an extra factor $\sqrt{K^2}$ appears in the π_{VA} expression (29).

$$\mathrm{Re}\,\pi_{VA}\left(K\right)=\frac{2e^{2}}{\pi}p_{F}\sqrt{K^{2}}\left(1-\frac{\omega^{2}}{k^{2}}\right)\left(1-\frac{\omega}{2k}\ln\frac{\omega+k}{\omega-k}\right). \tag{29}$$

Imaginary parts of polarization tensors appear at next order of corrections, when electron collisions with ambient particles are included. The partial contribution of the electron scattering off nuclei to the imaginary part of the retarded polarizations can be obtained by comparing the differential rate of the $\nu\bar{\nu}$ production Eq. (18) with Eq. (3). We arrive to

$$\frac{2}{\exp\left(\frac{\omega}{T}\right)-1}\,\mathrm{Im}\left[\frac{C_{V}^{2}}{4\pi e^{2}}\Pi_{V}^{\mu\nu}+\frac{C_{A}^{2}}{4\pi e^{2}}\Pi_{A}^{\mu\nu}\right]=\sum_{i,f}\mathrm{Tr}\left(M^{\dagger\mu}M^{\nu}\right), \tag{30}$$

where summation over initial and final states of the electron should be understood as described above. This formula can be identified with the optical theorem. By using the basis expansion Eqs. (20-22) and the following relations

$$L_{\mu\nu}L^{\mu\nu}=1,\quad T_{\mu\nu}T^{\mu\nu}=2,\quad L_{\mu\nu}T^{\mu\nu}=0,\quad L_{\mu\nu}g^{\mu\nu}=1,\quad T_{\mu\nu}g^{\mu\nu}=2 \tag{31}$$

we obtain

$$\mathrm{Im}\,\pi_{l}=2\pi e^{2}\frac{\exp\left(\frac{\omega}{T}\right)-1}{\left(C_{V}^{2}+C_{A}^{2}\right)}\sum_{i,f}\left(L_{\mu\nu}-h_{\mu}h_{\nu}\right)\mathrm{Tr}\left(M^{\dagger\mu}M^{\nu}\right), \tag{32}$$

$$\mathrm{Im}\,\pi_{t}=\pi e^{2}\frac{\exp\left(\frac{\omega}{T}\right)-1}{\left(C_{V}^{2}+C_{A}^{2}\right)}\sum_{i,f}\left(T_{\mu\nu}-2h_{\mu}h_{\nu}\right)\mathrm{Tr}\left(M^{\dagger\mu}M^{\nu}\right), \tag{33}$$

$$\mathrm{Im}\,\pi_{A}=2\pi e^{2}\frac{\exp\left(\frac{\omega}{T}\right)-1}{C_{A}^{2}}\sum_{i,f}h_{\mu}h_{\nu}\,\mathrm{Tr}\left(M^{\dagger\mu}M^{\nu}\right). \tag{34}$$

A direct evaluation of formulae Eq. (32-34) yield, for the liquid crust,

$$\mathrm{Im}\,\pi_{l}\left(\omega,k\right)=-\frac{2}{\pi}\omega_{pe}^{2}Z\alpha^{2}\frac{p_{F}}{k}F_{l}\left(u,\Gamma\right), \tag{35}$$

$$\mathrm{Im}\,\pi_{t}\left(\omega,k\right)=-\frac{1}{\pi}\omega_{pe}^{2}Z\alpha^{2}\frac{p_{F}}{k}F_{0}\left(u,\Gamma\right)-\frac{1}{2}\,\mathrm{Im}\,\pi_{l}, \tag{36}$$

$$\mathrm{Im}\,\pi_{A}=0. \tag{37}$$

The functions $F_{l}\left(u,\Gamma\right)$ and $F_{0}\left(u,\Gamma\right)$ are defined by the following integrals

$$F_{l}^{\mathsf{liquid}}\left(u,\Gamma\right)=\frac{1}{u}\int_{0}^{2p_{F}}\frac{dq\,S_{st}\left(q\right)}{\left|\varepsilon\left(q\right)\right|^{2}q}\left(1-x^{2}\right)\frac{1}{x^{2}}\left[1-\frac{1}{2}\frac{\left(1-u^{2}\right)}{ux\zeta}\ln\frac{\zeta+xu}{\zeta-xu}\right] \tag{38}$$

$$F_{0}^{\mathsf{liquid}}\left(u,\Gamma\right)=\int_{0}^{2p_{F}}\frac{dq\,S_{st}\left(q\right)}{\left|\varepsilon\left(q\right)\right|^{2}q}\left(1-x^{2}\right)\left(\frac{1}{x\zeta}\ln\frac{\zeta+xu}{\zeta-xu}\right), \tag{39}$$

with $x=q/2p_{F}$, and $\zeta=\sqrt{1-u^{2}(1-x^{2})}$. Since we consider point-like nuclei, the functions $F_{l}\left(u,\Gamma\right)$ and $F_{0}\left(u,\Gamma\right)$ depend only on the variable $u=k/\omega$ and the non-ideality parameter of the ionic component of the plasma [11]:

$$\Gamma=\frac{Z^{2}e^{2}}{aT}=0.02254\frac{Z^{5/3}}{T_{9}}\frac{p_{F}}{m_{e}}, \tag{40}$$

where a is the ion-sphere radius. Expressions Eq. (38) and Eq. (39) are valid for $\Gamma<172$. Above this value, the crust crystallizes. For the crystallin crust we obtain the more complicate expressions :

$$F_l^{\mathsf{cryst}}(u,\omega,\Gamma) = \frac{1}{N_i u\omega}\int_0^{2p_F}\frac{dq}{|\varepsilon(q)|^2 q}\frac{1-x^2}{x^2}\left[1-\frac{1}{2}\frac{(1-u^2)}{ux\zeta}\ln\frac{\zeta+xu}{\zeta-xu}\right]$$
$$\times\left[\exp\left(\frac{\omega}{T}\right)-1\right]\int_{-\infty}^{\infty}dq_0\frac{(\omega+q_0)}{\exp\left(\frac{\omega+q_0}{T}\right)-1}S(q_0,q) \tag{41}$$

$$F_0^{\mathsf{cryst}}(u,\omega,\Gamma) = \frac{1}{N_i\omega}\int_0^{2p_F}\frac{dq}{|\varepsilon(q)|^2 q}\frac{1-x^2}{x\zeta}\ln\frac{\zeta+xu}{\zeta-xu}$$
$$\times\left[\exp\left(\frac{\omega}{T}\right)-1\right]\int_{-\infty}^{\infty}dq_0\frac{(\omega+q_0)}{\exp\left(\frac{\omega+q_0}{T}\right)-1}S(q_0,q), \tag{42}$$

Inserting the explicit form Eq. (13) of the one-phonon dynamic form factor, and performing the integration over q_0 we obtain

$$F_l^{\mathsf{cryst}}(u,\omega,\Gamma) = \frac{\pi}{M_i u\omega}\left[\exp\left(\frac{\omega}{T}\right)-1\right]\sum_{\mathbf{s}\lambda\mathbf{K}}\frac{1}{\omega_{\lambda\mathbf{s}}}$$
$$\times\int_0^{2p_F}\frac{dq\, e^{-2W(q)}\left(\mathbf{q}\cdot\hat{\mathbf{e}}_{\lambda\mathbf{s}}\right)}{|\varepsilon(q)|^2 q}\frac{1-x^2}{x^2}\left(1-\frac{1-u^2}{2ux\zeta}\ln\frac{\zeta+xu}{\zeta-xu}\right)$$
$$\times\left[\frac{\delta(\mathbf{q}-\mathbf{s}+\mathbf{K})}{1-e^{-\omega_{\lambda\mathbf{s}}/T}}\frac{(\omega+\omega_{\lambda\mathbf{s}})}{\exp\left(\frac{\omega+\omega_{\lambda\mathbf{s}}}{T}\right)-1}+\frac{\delta(\mathbf{q}+\mathbf{s}+\mathbf{K})}{e^{-\omega_{\lambda\mathbf{s}}/T}-1}\frac{(\omega-\omega_{\lambda\mathbf{s}})}{\exp\left(\frac{\omega-\omega_{\lambda\mathbf{s}}}{T}\right)-1}\right] \tag{43}$$

$$F_0^{\mathsf{cryst}}(u,\omega,\Gamma) = \frac{\pi}{M_i\omega}\left[\exp\left(\frac{\omega}{T}\right)-1\right]\sum_{\mathbf{s}\lambda\mathbf{K}}\frac{1}{\omega_{\lambda\mathbf{s}}}$$
$$\times\int_0^{2p_F}\frac{dq\, e^{-2W(q)}\left(\mathbf{q}\cdot\hat{\mathbf{e}}_{\lambda\mathbf{s}}\right)}{|\varepsilon(q)|^2 q}\frac{1-x^2}{x\zeta}\ln\frac{\zeta+xu}{\zeta-xu}$$
$$\times\left[\frac{\delta(\mathbf{q}-\mathbf{s}+\mathbf{K})}{1-e^{-\omega_{\lambda\mathbf{s}}/T}}\frac{(\omega+\omega_{\lambda\mathbf{s}})}{\exp\left(\frac{\omega+\omega_{\lambda\mathbf{s}}}{T}\right)-1}+\frac{\delta(\mathbf{q}+\mathbf{s}+\mathbf{K})}{e^{-\omega_{\lambda\mathbf{s}}/T}-1}\frac{(\omega-\omega_{\lambda\mathbf{s}})}{\exp\left(\frac{\omega-\omega_{\lambda\mathbf{s}}}{T}\right)-1}\right], \tag{44}$$

IV. INCLUDING COLLECTIVE EFFECTS

By the Fluctuation-Dissipation Theorem, the function

$$\Phi^{\mu\nu}(K) = \frac{2}{\exp\left(\frac{\omega}{T}\right)-1}\,\mathrm{Im}\,\Pi^{\mu\nu}(K) \tag{45}$$

should be identified with the Fourier transform of the correlation function of two distant weak-currents in the plasma. However, the above approximation, with $\Pi^{\mu\nu}$ being the irreducible polarization tensor, does not take into account for collective effects in the correlation function. To generalize the correlation function to this case, we should also include the exchange of an intermediate virtual photon between electrons. This can be done by summation of the two diagrams shown in Fig. 1, where the thick dashed line is the in-medium photon propagator $D^{\rho\lambda}(K)$, defined as the infinite sum of graphs shown in Fig. 2. In this figure, the thin dashed-line represents $\mathrm{D}^0_{\rho\lambda}(K)$ - the photon propagator in vacuum. According to this series, the exact photon propagator in the medium satisfies the Dyson's equation

$$D_{\lambda\rho}(K) = \mathrm{D}^0_{\lambda\rho}(K) + \frac{1}{4\pi}\mathrm{D}^0_{\lambda\nu}(K)\,\Pi_V^{\nu\mu}(K)\,D_{\mu\rho}(K) \tag{46}$$

Within the Lorentz gauge, $\mathrm{D}^0_{\lambda\nu}(K)$ has the following form

$$\mathrm{D}^0_{\lambda\rho}(K) = \frac{4\pi}{K^2}\left(g_{\lambda\rho}-h_\lambda h_\rho\right), \tag{47}$$

The compact block $\Pi_V^{\mu\nu}(K)$ is the retarded polarization tensor of the plasma, given by Eq. (20). The retarded propagator of the in-medium photon has the same tensor structure as the vector-vector polarization tensor. The solution to the Dyson's equation is of the form:

$$D_{\lambda\rho}(K) = D_l(K) L_{\lambda\rho} + D_t(K) T_{\lambda\rho} \tag{48}$$

with

$$D_l(K) = \frac{4\pi}{K^2 - \pi_l(K)}, \qquad D_t(K) = \frac{4\pi}{K^2 - \pi_t(K)}. \tag{49}$$

Thus, the sum of graphs of the Fig. 1 gives the new tensors :

$$\begin{aligned} \tilde{\Pi}_V^{\mu\nu}(\mathbf{k},\omega) &= \Pi_V^{\mu\nu}(\mathbf{k},\omega) + \frac{1}{4\pi}\Pi_V^{\mu\lambda}(\mathbf{k},\omega) D_{\lambda\rho}(\mathbf{k},\omega)\Pi_V^{\rho\nu}(\mathbf{k},\omega), \\ \tilde{\Pi}_A^{\mu\nu}(\mathbf{k},\omega) &= \Pi_A^{\mu\nu}(\mathbf{k},\omega) + \frac{1}{4\pi}\Pi_{AV}^{\mu\lambda}(\mathbf{k},\omega) D_{\lambda\rho}(\mathbf{k},\omega)\Pi_{VA}^{\rho\nu}(\mathbf{k},\omega). \end{aligned} \tag{50}$$

By substituting Eqs. (20-22) , (48), and (49) into Eqs. (50) we obtain

$$\tilde{\Pi}_V^{\mu\nu} = \frac{K^2\pi_l}{K^2-\pi_l}L^{\mu\nu} + \frac{K^2\pi_t}{K^2-\pi_t}T^{\mu\nu}, \tag{51}$$

$$\tilde{\Pi}_{AV}^{\mu\nu} = \frac{K^2}{K^2-\pi_t}\Pi_{AV}^{\mu\nu}, \qquad \tilde{\Pi}_{VA}^{\mu\nu} = \frac{K^2}{K^2-\pi_t}\Pi_{VA}^{\mu\nu}, \tag{52}$$

$$\tilde{\Pi}_A^{\mu\nu} = \pi_l L^{\mu\nu} + \pi_t T^{\mu\nu} + \pi_A g^{\mu\nu} - \frac{\pi_{AV}\pi_{VA}}{K^2-\pi_t}\epsilon^{\mu\lambda\sigma 0}\epsilon^{\rho\nu\tau 0}g_{\lambda\rho}h_\sigma h_\tau. \tag{53}$$

Replacement of the irreducible tensors in Eq. (16) by the generalized expressions (51-53) results in the following weak current-current correlation tensor

$$\begin{aligned} \tilde{\Phi}^{\mu\nu}(K) &= \frac{2}{\exp\left(\frac{\omega}{T}\right)-1}\,\mathrm{Im}\left[\frac{C_V^2}{4\pi e^2}\left(\frac{K^2\pi_l}{K^2-\pi_l}L^{\mu\nu} + \frac{K^2\pi_t}{K^2-\pi_t}T^{\mu\nu}\right)\right. \\ &\left. + \frac{C_A^2}{4\pi e^2}\left(\pi_l L^{\mu\nu} + \pi_t T^{\mu\nu} - \frac{\pi_{AV}\pi_{VA}}{K^2-\pi_t}\epsilon^{\mu\lambda\sigma 0}\epsilon^{\rho\nu\tau 0}g_{\lambda\rho}h_\sigma h_\tau\right)\right]. \end{aligned} \tag{54}$$

V. ENERGY LOSS DUE TO NEUTRINO-PAIR RADIATION

We consider the emissivity of neutrino pairs from the plasma , i.e. the total energy which is emitted into neutrino pairs per unit volume and time. Taking into account collective effects, to the lowest order in G_F, the emissivity is given by the following formula:

$$Q = \frac{G_F^2}{2}\frac{4\pi}{3}\frac{1}{(2\pi)^6}\sum_\nu\int_0^\infty d\omega\,\omega\int_{k<\omega} d^3k\,\left(K_\mu K_\nu - K^2 g_{\mu\nu}\right)\tilde{\Phi}^{\mu\nu}(K). \tag{55}$$

The symbol $\sum_\nu$ indicates that summation over the three neutrino types has to be performed, with the corresponding values of C_V and C_A, as explained in Section 2. In what follows, this summation will be understood in any formula of the emissivity. Contraction of the correlation function (54) with the neutrino-pair tensor $K_\mu K_\nu - K^2 g_{\mu\nu}$, as indicated in Eq. (55), yields:

$$\begin{aligned} &\left(K_\mu K_\nu - K^2 g_{\mu\nu}\right)\tilde{\Phi}^{\mu\nu}(K) = -K^2\frac{1}{4\pi e^2}\frac{2}{\exp\left(\frac{\omega}{T}\right)-1}\times \\ &\mathrm{Im}\left[C_V^2 K^2\left(\frac{\pi_l}{K^2-\pi_l} + 2\frac{\pi_t}{K^2-\pi_t}\right) + 2C_A^2\frac{k^2}{K^2}\frac{\pi_{AV}\pi_{VA}}{K^2-\pi_t} + C_A^2\left(\pi_l + 2\pi_t\right)\right]. \end{aligned}$$

Taking the imaginary part of the right-hand side of this equation, after a lengthy (although straightforward) calculation, we obtain the following formula :

$$\left(K_\mu K_\nu - K^2 g_{\mu\nu}\right)\tilde{\Phi}^{\mu\nu}(K) = K^2 \frac{1}{4\pi e^2}\frac{2}{\exp\left(\frac{\omega}{T}\right)-1}\times$$
$$\left[C_V^2 K^4\left(\frac{|\mathrm{Im}\,\pi_l|}{\left(K^2-\mathrm{Re}\,\pi_l\right)^2+\left(\mathrm{Im}\,\pi_l\right)^2}+2\frac{|\mathrm{Im}\,\pi_t|}{\left(K^2-\mathrm{Re}\,\pi_t\right)^2+\left(\mathrm{Im}\,\pi_t\right)^2}\right)\right.$$
$$+C_A^2\left(|\mathrm{Im}\,\pi_l|+2\,|\mathrm{Im}\,\pi_t|\right)$$
$$\left.+2C_A^2\frac{k^2}{K^2}\frac{|\mathrm{Im}\,\pi_t|}{\left(K^2-\mathrm{Re}\,\pi_t\right)^2+\left(\mathrm{Im}\,\pi_t\right)^2}\left(\mathrm{Re}\,\pi_{AV}\right)^2\right]. \quad (56)$$

Here it is assumed that $\mathrm{Im}\,\pi_A = \mathrm{Im}\,\pi_{AV} = 0$, and we incorporated the fact that the imaginary parts of retarded polarization functions are negative for $\omega > 0$, so that $-\mathrm{Im}\,\pi_{l,t} = |\mathrm{Im}\,\pi_{l,t}|$.

As the polarization functions Eqs. (26, 27) and Eqs. (35, 36) are independent of the direction of the vector **k**, integration over angles can be done in Eq. (55), and we obtain the $\nu\bar{\nu}$ emissivity from the plasma as follows

$$Q = \frac{G_F^2}{48\pi^5 e^2}\int_0^\infty d\omega\,\omega\int_0^\omega dk\,\frac{k^2K^2}{\exp\left(\frac{\omega}{T}\right)-1}\times$$
$$\left[C_V^2 K^4\left(\frac{|\mathrm{Im}\,\pi_l|}{\left(K^2-\mathrm{Re}\,\pi_l\right)^2+\left(\mathrm{Im}\,\pi_l\right)^2}+2\frac{|\mathrm{Im}\,\pi_t|}{\left(K^2-\mathrm{Re}\,\pi_t\right)^2+\left(\mathrm{Im}\,\pi_t\right)^2}\right)\right.$$
$$+C_A^2\left(|\mathrm{Im}\,\pi_l|+2\,|\mathrm{Im}\,\pi_t|\right)$$
$$\left.+2C_A^2\frac{k^2}{K^2}\left(\mathrm{Re}\,\pi_{AV}\right)^2\frac{|\mathrm{Im}\,\pi_t|}{\left(K^2-\mathrm{Re}\,\pi_t\right)^2+\left(\mathrm{Im}\,\pi_t\right)^2}\right]. \quad (57)$$

This formula describes the $\nu\bar{\nu}$ emissivity from the plasma with inclusion of collective effects due to the plasma polarization. As we included the in-medium photon contribution to the correlation function, Eq. (57) describes the energy losses due to plasmon decay and $\nu\bar{\nu}$ Bremsstrahlung from electrons. Due to collective effects, the contribution to these particular processes through the vector weak-current can not be separated from one another. This can be understood as follows. The vector transition current of the electron is responsible for production of neutrino pairs and on-shell photons in the medium as well. When a virtual photon participating in the $\nu\bar{\nu}$ Bremsstrahlung goes on shell, it has such a large life-time, that decay of the photon into neutrino pairs can be dually interpreted as free plasmon decay, or as $\nu\bar{\nu}$ Bremsstrahlung from electrons. This interference does not permit to separate the partial contributions. However, such an interference can be neglected in some limiting cases, when one of the processes strongly dominates.

VI. LIMITING CASES AND NUMERICAL TESTS

A. Moderate temperatures

At moderate temperatures $T \sim \omega_{pe}$, plasmon decay into neutrino-pairs strongly dominates the neutrino energy losses from the degenerate plasma of a liquid crust [16]. In this case, the imaginary parts of the polarization functions result only in a minor widening of the plasmon spectra due to plasmon capture and creation in electron collisions with nuclei. By taking in Eq. (57) the imaginary part of polarization functions equal to zero, we obtain:

$$Q = Q_{l\to\nu\bar{\nu}} + Q_{t\to\nu\bar{\nu}}, \quad (58)$$

where

$$Q_{l\to\nu\bar{\nu}} \equiv \frac{G_F^2 C_V^2}{48\pi^4 e^2}\int_0^\infty \frac{d\omega\,\omega}{\exp\left(\frac{\omega}{T}\right)-1}\int_0^\omega dk\,k^2K^6\delta\left(K^2-\pi_l\right), \quad (59)$$

$$Q_{t\to\nu\bar{\nu}} \equiv \frac{G_F^2}{24\pi^4 e^2}\int_0^\infty \frac{d\omega\,\omega}{\exp\left(\frac{\omega}{T}\right)-1}\int_0^\omega dk\,k^2\left(C_V^2K^6+C_A^2k^2\pi_{AV}^2\right)\delta\left(K^2-\pi_t\right). \quad (60)$$

As it follows from the poles of Eq. (49), longitudinal and transverse eigen-photon modes satisfy the dispersion relations

$$\omega^2 - k^2 - \mathrm{Re}\,\pi_{l,t}(\omega, k) = 0, \tag{61}$$

therefore Eqs. (59) and (60) are easily interpreted as the emissivity due to decay of on-shell longitudinal (l) and transverse (t) plasmons, respectively, into neutrino pairs. We will now show that, in fact, Eqs. (59) and (60) give the known results for the neutrino pair emissivity from longitudinal and transverse plasmons, respectively.

By expanding $K^2 - \mathrm{Re}\,\pi_t$ around ω_t one can approximate

$$\omega^2 - k^2 - \mathrm{Re}\,\pi_{l,t}(\omega, k) \simeq \frac{1}{Z_{l,t}}\left(\omega^2 - \omega_{l,t}^2(k)\right) \tag{62}$$

with

$$Z_{l,t}(k) \equiv \left(1 - \left.\frac{\partial \pi_{l,t}(\omega, k)}{\partial \omega^2}\right|_{\omega^2 = \omega_{l,t}^2}\right)^{-1}. \tag{63}$$

In this way

$$\delta\left(\omega^2 - k^2 - \mathrm{Re}\,\pi_{l,t}\right) = \frac{Z_{l,t}(k)}{2\omega_{l,t}(k)}\delta\left(\omega - \omega_{l,t}(k)\right), \tag{64}$$

where $\omega_{l,t}(k)$ is the solution (for $\omega > 0$) of the dispersion equation (61) for longitudinal or transverse photons. Substitution into Eqs. (59) and (60) results in

$$\begin{aligned}
Q_{l\to\nu\bar{\nu}} &= C_V^2 \frac{G_F^2}{96\pi^4 e^2}\int_0^\infty d\omega \frac{1}{\exp\left(\frac{\omega}{T}\right) - 1}\int_0^\omega dk\, k^2\, Z_l(k) K^6 \delta\left(\omega - \omega_l(k)\right) \\
Q_{t\to\nu\bar{\nu}} &= \frac{G_F^2}{48\pi^4 e^2}\int_0^\infty d\omega \int_0^\omega dk\, k^2 \frac{Z_t(k)}{\exp\left(\frac{\omega}{T}\right) - 1}\left[C_V^2 K^6 + C_A^2 k^2\, \overset{2}{\mathrm{Re}}\,\pi_{AV}(\omega, k)\right]\delta\left(\omega - \omega_t(k)\right).
\end{aligned} \tag{65}$$

The integral over ω is trivial. As for the integral over k, one has to remember that the condition $K^2 > 0$ implies that, for longitudinal modes, k can not be larger than the limiting value $k_{\max}$ [10]. Finally, we obtain

$$\begin{aligned}
Q_{l\to\nu\bar{\nu}} &= \frac{G_F^2 C_V^2}{96\pi^4 e^2}\int_0^{k_{\max}} dk\, \frac{k^2 Z_l(k)}{\exp\left(\frac{\omega_l(k)}{T}\right) - 1}\left(\omega_l^2(k) - k^2\right)^3, \\
Q_{t\to\nu\bar{\nu}} &= \frac{G_F^2}{48\pi^4 e^2}\int_0^\infty dk \frac{k^2 Z_t(k)}{\exp\left(\frac{\omega_t(k)}{T}\right) - 1}\left[C_V^2\left(\omega_t^2(k) - k^2\right)^3 + C_A^2 k^2\, \mathrm{Re}\,\pi_{AV}^2\left(\omega_t(k), k\right)\right].
\end{aligned} \tag{66}$$

These formulae coincide with the known expressions [10] for neutrino pair emission from plasmons.

B. High-temperature limit

We now consider a degenerate electron gas in the case of high temperatures $T \gg \omega_{pe}$. Then the emissivity of Bremsstrahlung, which increases as T^6, strongly overcomes the emissivity due to plasmon decay, which only goes as T^3 (see , e.g. [10]). At such temperatures, the dominant contribution comes from $K^2 \gg \mathrm{Re}\,\pi_{l,t} \sim \omega_{pe}^2$. Thus, neglecting $\mathrm{Re}\,\pi_{l,t}$ and $\mathrm{Im}\,\pi_{l,t}$ in the denominators of Eq. (57), and considering also $\left(\mathrm{Re}\,\pi_{AV}\right)^2 \ll K^4$, we obtain

$$Q_{Br} = \frac{G_F^2\left(C_V^2 + C_A^2\right)}{48\pi^5 e^2}\int_0^\infty \frac{d\omega\, \omega}{\exp\left(\frac{\omega}{T}\right) - 1}\int_0^\omega dk\, k^2 K^2\left(\left|\mathrm{Im}\,\pi_l\right| + 2\left|\mathrm{Im}\,\pi_t\right|\right). \tag{67}$$

Performing this integral, with $\mathrm{Im}\,\pi_{l,t}$ defined by Eqs. (35, 36), we arrive to the known result [11] for the $\nu\bar{\nu}$ Bremsstrahlung emissivity:

$$\begin{aligned}
Q_{Br} = &\frac{4\pi}{189} G_F^2\left(C_V^2 + C_A^2\right) e^4 N_i Z^2 T^6 \\
&\times \int_0^{2p_F} \frac{dq\, S(q)}{\left|\varepsilon(q)\right|^2 q}\left(1 - x^2\right)\int_0^1 du \frac{u\left(1 - u^2\right)}{x\zeta}\ln\frac{\zeta + xu}{\zeta - xu}.
\end{aligned} \tag{68}$$

The latter integral can be done analytically (see also [7]) :

$$\int_0^1 du \frac{u\left(1-u^2\right)}{x\zeta} \ln \frac{\zeta + xu}{\zeta - xu} = \frac{2}{3}\frac{1}{1-x^2}\left(1 + \frac{2x^2}{1-x^2}\ln x\right), \tag{69}$$

and we finally obtain

$$Q_{Br} = \frac{8\pi}{567} G_F^2 \left(C_V^2 + C_A^2\right) e^4 N_i Z^2 T^6 \int_0^{2p_F} \frac{dq\, S(q)}{\left|\varepsilon(q)\right|^2 q} \left(1 + \frac{2x^2}{1-x^2}\ln x\right). \tag{70}$$

C. Low-temperature limit: collective effects

In the case of low temperatures $T \ll \omega_{pe}$, the occupation numbers of eigen-photon modes in the medium are exponentially suppressed, and the typical energy of a neutrino-antineutrino pair is of the order of the medium temperature. Under this situation we can neglect $K^2 \sim T^2$ with respect to $\mathrm{Re}\,\pi_{l,t} \sim \omega_{pe}^2$ in the denominators of Eq. (57) .

Now, Eq. (57) takes the form

$$Q = Q^V + Q^A. \tag{71}$$

These terms describe the vector (Q^V) and an axial (Q^A) weak-current contributions

$$Q^V \equiv \frac{G_F^2 C_V^2}{48\pi^5 e^2} \int_0^\infty \frac{d\omega\, \omega}{\exp\left(\frac{\omega}{T}\right) - 1} \int_0^\omega dk\, k^2 K^2 \left(\frac{K^4}{\mathrm{Re}^2\, \pi_l} \left|\mathrm{Im}\,\pi_l\right| + 2\frac{K^4}{\mathrm{Re}^2\, \pi_t} \left|\mathrm{Im}\,\pi_t\right|\right), \tag{72}$$

$$Q^A \equiv \frac{G_F^2 C_A^2}{48\pi^5 e^2} \int_0^\infty \frac{d\omega\, \omega}{\exp\left(\frac{\omega}{T}\right) - 1} \int_0^\omega dk\, k^2 K^2 \left(\left|\mathrm{Im}\,\pi_l\right| + 2\left|\mathrm{Im}\,\pi_t\right|\right). \tag{73}$$

In the latter equation we neglected terms which are proportional to $\mathrm{Re}\,\pi_{AV} / \mathrm{Re}\,\pi_t \sim T/\mu_e \ll 1$.

As mentioned above, the imaginary parts of polarization tensors are due to electron collisions with nuclei. Thus, Eqs. (72, 73) should be understood as the neutrino-pair Bremsstrahlung from electrons. Substituting in Eq. (72) the explicit form of the polarizations, and performing integrations over dk and $d\omega$ we obtain, for a liquid crust, the following expression for the axial weak-current contribution

$$Q_{Br}^A = \frac{8\pi}{567} G_F^2 C_A^2 e^4 N_i Z^2 T^6 \int_0^{2p_F} \frac{dq\, S(q)}{\left|\varepsilon(q)\right|^2 q} \left(1 + \frac{2x^2}{1-x^2}\ln x\right) \tag{74}$$

which coincides with the known result Eq. (70) for the axial term in the ultrarelativistic case, while the contribution of the vector weak current contains in the integrand the additional, small factors, $K^4 / \mathrm{Re}^2\, \pi_{l,t} \sim T^4/\omega_{pe}^4$ and therefore decreases along with the temperature as T^{10}. Thus, at low temperatures, the contribution of the vector weak current to emissivity is suppressed due to collective effects, as discussed in [9].

In Fig. 3, we show the result of a numerical calculation of the neutrino-pair emissivity from a liquid crust with a density ρ such that $\rho Y_e = 7.3 \times 10^9 gr\ cm^{-3}$ (Y_e is the electron fraction per baryon), consisting on nuclei with an atomic number $Z = 30$ embedded into a degenerate ultrarelativistic electron gas, which is in good agrement with the total neutrino emissivity, as given by the sum of the plasmon decay contribution, calculated in [16], and the "standard" Bremsstrahlung contribution calculated in [11]. One readily observes from this figure, that our calculation coincides with the standard Bremsstrahlung emissivity, when this is the dominant process (at high temperatures), and with plasmon decay at lower temperatures.

Collective effects are dramatic at low temperatures $T \ll \omega_{pe}$, when plasmon decay is negligible, which are typical for the crystallin crust. However, the above criterion $\Gamma > 172$ for crystallization of the crust corresponds to the case where an infinite time scale is allowed to attain thermal equilibrium. In actual stellar evolution, only a finite time scale is allowed for temperature variation. Therefore, the ionic system is likely in an amorphous-crystal state for $172 < \Gamma \lesssim 210$ [17]. Thus, in practice, the liquid state of the crust persists up to $\Gamma \lesssim 210$. At such temperatures, the plasmon decay is exponentially suppressed, and Bremsstrahlung becomes again dominating. In this case, collective effects dramatically suppress the vector weak-current contribution to neutrino-pair Bremsstrahlung from electrons in the liquid crust.

Unfortunately, the structure factor is not tabulated for the liquid ionic system in this domain of Γ. For this reason, in order to make these effects more apparent, we considered a liquid crust with the same value of ρY_e, consisting on nuclei with $Z = 10$. Such a crust remains liquid for temperatures much smaller than the plasma frequency. The result of our calculation is plotted in Fig. 4, where we show separately the Q^V and Q^A pieces, in comparison with plasmon decay. The Q^A contribution to the emissivity shows the standard behavior $\sim T^6$, while the Q^V piece drops much faster with temperature, as T^{10}. As we discussed above, such a behavior is a consequence of collective effects, which result on the screening of the vector weak-current of electrons in the plasma.

In Fig. 5, we have plotted the ratio R of the total emissivity, as calculated from our formulae, to the sum of the standard Bremsstrahlung plus plasmon decay (taken from [11] and [16]). As it readily seen from this figure, this ratio abruptly decreases at low temperatures, and tends to the expected value [9].

$$\frac{Q}{Q_0} = \frac{C_A^2 + 2C_A'^2}{C_V^2 + C_A^2 + 2\left(C_V'^2 + C_A'^2\right)} = 0.448. \tag{75}$$

VII. CONCLUSIONS

We have shown a method of calculation for the neutrino-pair energy losses from the crust of a neutron star, which has the advantage that collective effects are automatically included, and do not need a separate consideration. The formula we obtained for the neutrino-pair emissivity, Eq. (57), includes the contribution of plasmon decay into neutrino pairs, as well as neutrino-pair emission due to electron collisions with nuclei and phonons. Some limiting cases, considered in Sect. 6, demonstrate that this formula describes the known results for energy losses due to plasmon decay and neutrino-pair Bremsstrahlung of electrons, when one of these processes dominates, at moderate or high temperatures. Our formula takes into account collective effects, which in the case of low temperatures manifest in a suppression of the vector weak-current contribution to the emissivity, in accordance with the results in [9]. Some numerical tests for moderate and high temperatures show a good agreement of our calculated total emissivity, with that obtained by summation of the separate contributions of plasmon decay and electron Bremsstrahlung obtained by different authors. In general, Eq. (57) can also be used for a non-degenerate plasma, if the corresponding polarization functions are inserted. Thus, Eq. (57) is able to incorporate neutrino-pair energy losses from the electron plasma caused by processes other than plasmon decay and neutrino-pair Bremsstrahlung. This can be done by improvement of the imaginary parts of the retarded polarization functions, with inclusion of different physical processes leading to damping of the plasmon in the medium under consideration.

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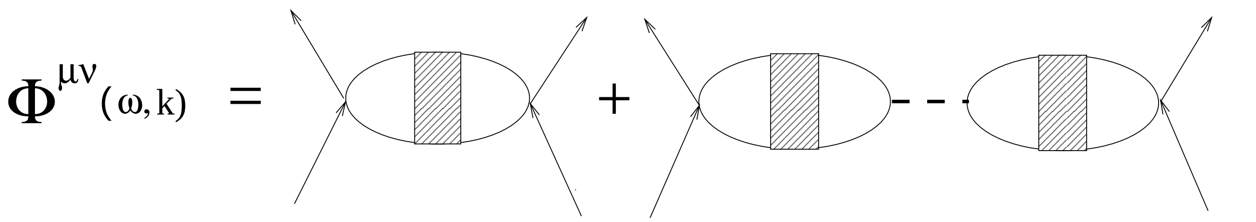

FIG. 1. Diagrams contributing to the weak correlation function, when collective effects are included. Shadow rectangles on electron loops represent intermediate electron interactions with nuclei (in the liquid crust) or with phonons (in the crystall). The thick dashed line on the second diagram stands for an in-medium photon (see next figure).

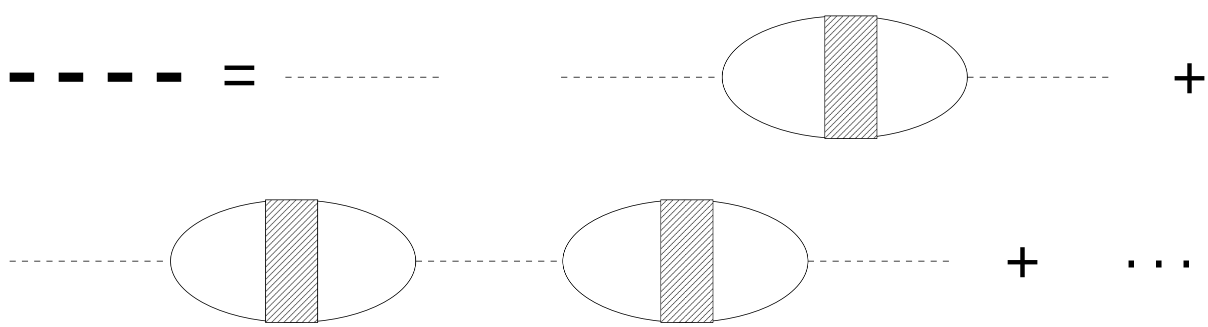

FIG. 2. Series of diagrams defining the in-medium photon propagator (shown as a thick dashed line). Thin dashed lines correspond to the in-vacuum photon propagator.

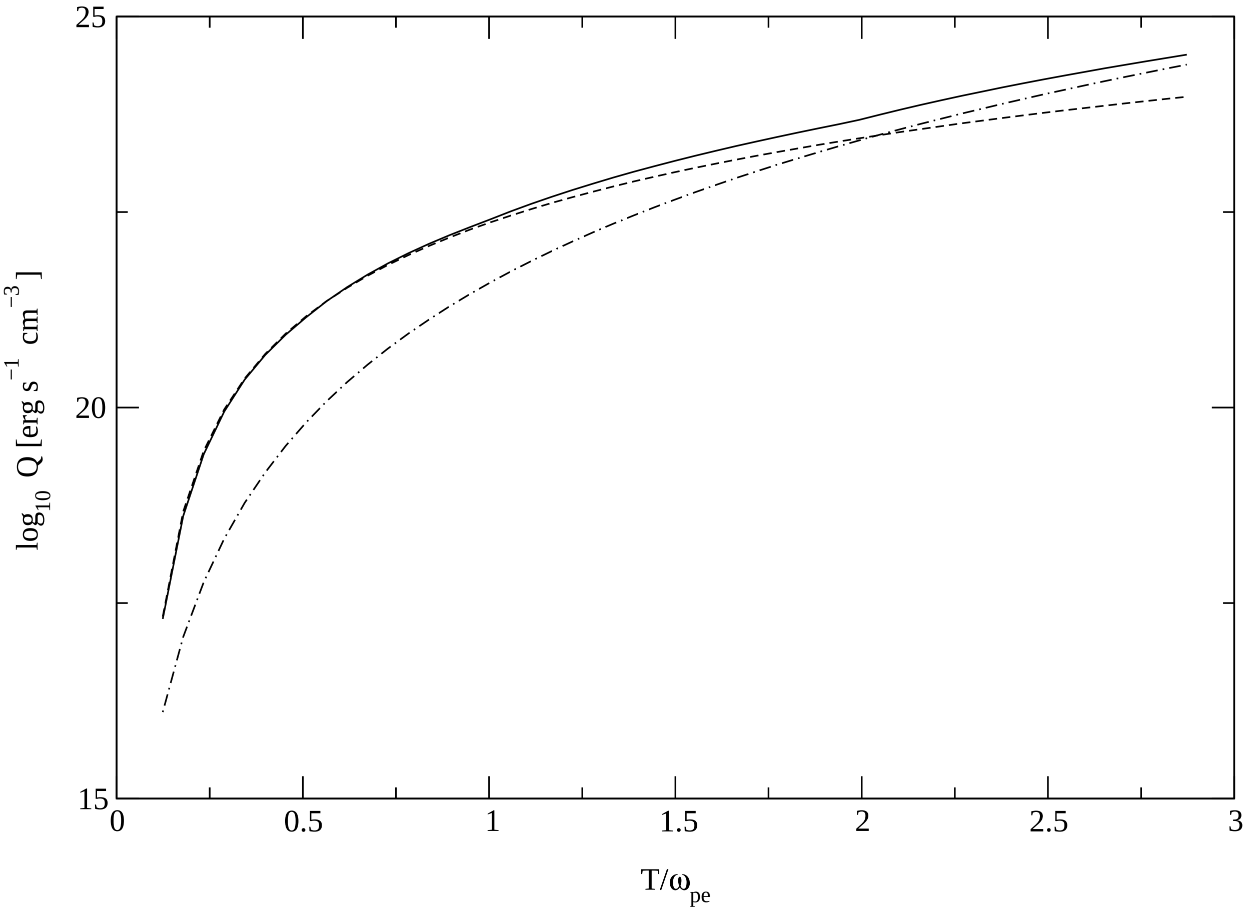

FIG. 3. Total neutrino-pair emissivity, calculated from our Eq. 57 (solid line), in comparison with "standard" (i.e., without collective effects) plasmon decay (dashed line) and Bremsstrahlung (dashed-dotted line). All curves are for a density corresponding to $\rho Y_e = 7.3 \times 10^9 grcm^{-3}$ and an atomic number $Z = 30$ for ions.

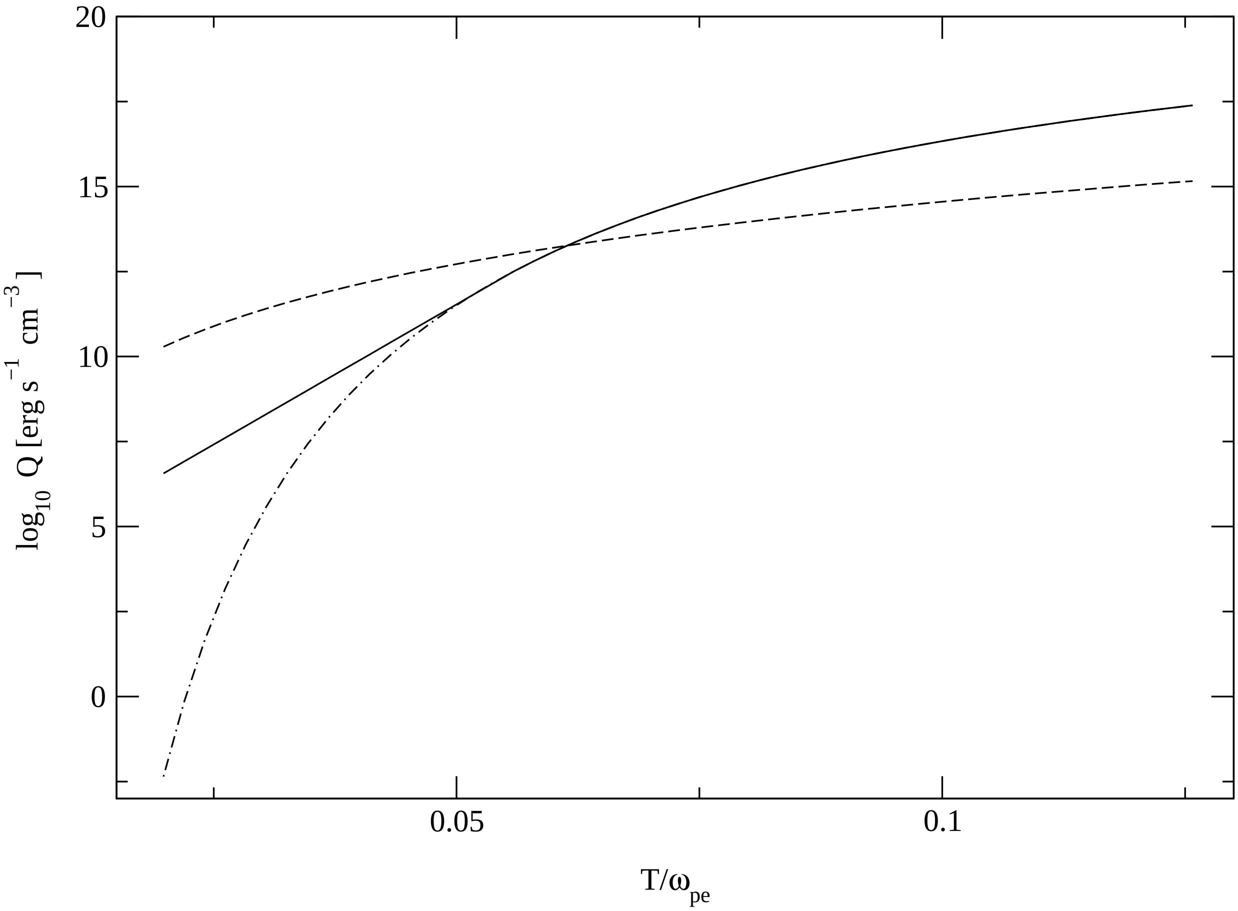

FIG. 4. Separate vector (solid line) and axial (dashed line) contributions to the neutrino-pair emissivity. At low temperatures, the vector weak-current contribution is suppressed with respect to the axial term due to collective effects. At moderate temperatures, when plasmon decay dominates, the vector weak-current contribution coincides with the plasmon decay curve (dashed-dotted line). All curves correspond the same value of ρY_e as in the previous figure, and an atomic number $Z = 10$ for ions.

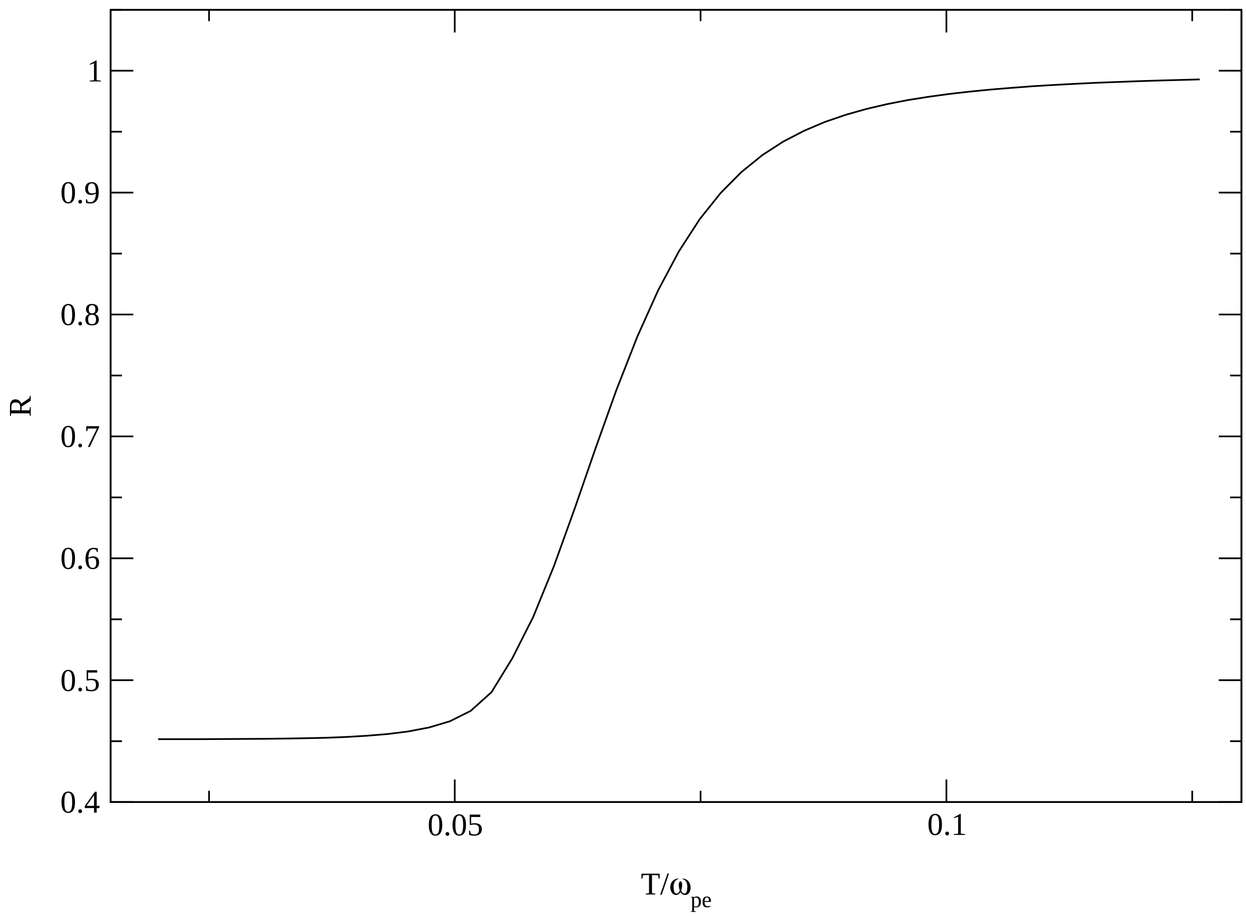

FIG. 5. The ratio of the total neutrino-pair emissivity, as obtained from our formula, to the sum of standard plasmon decay and Bremsstrahlung. Plasma conditions are the same as in the previous figure.