Integration of massive states as contractions of non linear σ-models.

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Abstract

We consider the contraction of some non linear σ-models which appear in effective supergravity theories. In particular we consider the contractions of maximally symmetric spaces corresponding to $N = 1$ and $N = 2$ theories, as they appear in certain low energy effective supergravity actions with mass deformations.

The contraction procedure is shown to describe the *integrating out* of massive modes in the presence of interactions, as it happens in many supergravity models after spontaneous supersymmetry breaking.

1 Introduction

Supergravity theories with mass deformations have recently received some attention because of their relation to flux compactifications (for a review see, *e.g.*, [1]) or Scherk–Schwarz generalized dimensional reduction [2].

For $N \geq 2$ local supersymmetry, the supergravity theories admit mass deformations that always correspond to gauged supergravities [14, 15]. The mass parameters may be chosen in such a way that a low energy effective Lagrangian for the massless sector can be singled out by deleting the massive modes.

This procedure is usually discussed in the framework of consistent truncations of field theories [16], but here we want to show that the same phenomenon may arise as well as a *contraction*. The basic argument is that the limiting situation of a mass scale asymptotically large is equivalent to the contraction of some group structure.

Suppose the group structure is a non-linear σ-model related to a maximally symmetric space G/H where G is non compact and H its maximally compact subgroup [3, 4]. One can make an Inönü-Wigner contraction [5] of the group G with respect to a subgroup G'. Let $H' = H \cap G'$. We can induce a contraction G/H to a manifold which will have G'/H' embedded in it. The contracted manifold has the same dimension as the original one (as it happens for contractions of algebras and groups), but with a metric that will be essentially different. An example of this are the contractions of the hyperboloid $\mathrm{SU}(1,1)/\mathrm{U}(1)$. If the contraction is made with respect to the subgroup $\mathrm{U}(1)$ one obtains the flat metric, while if the contraction is made with respect to $\mathrm{SO}(1,1)$ one obtains an hyperbolic sheet, with one translational isometry. We will see in detail how to compute the metrics in these and other cases.

There are other types of contractions that do not fit in the scheme described above, but that may have physical interest. If G/H is a symmetric space of the non compact type, it inherits a group structure through the Iwasawa decomposition of G

$$G = G_S \times H.$$

Then $G/H \approx G_S$ is the solvable Lie group associated to G [3, 4, 6]. Note that G_S depends on the real (non compact) form of G. We can then consider contractions in which G/H goes to $G'/H' \ltimes \mathbb{R}^n$ with $\dim(G/H) = \dim(G'/H') + n$, independently of the fact that G' is a subgroup of G or not.

The physical interpretation of these contractions is as a (super)-Higgs mechanism [8], where the massive modes are described by $\mathbb{R}^n$ degrees of freedom while the fields which remain massless are in G'/H' [10]. Indeed, because of the semidirect product structure, it is always consistent to set to zero (which, in this case, would correspond to integrate out) the elements of $\mathbb{R}^n$, since $\mathbb{R}^n$ is an invariant subalgebra of $G'/H' \ltimes \mathbb{R}^n$ [11]. We will consider several examples and discuss their physical applications.

The paper is organized as follows:

In Section 2 we describe the solvable algebras related to symmetric spaces

$$\frac{\mathrm{SO}(1, 1+n)}{\mathrm{SO}(1+n)}, \quad \frac{\mathrm{SU}(1, 1+n)}{\mathrm{U}(1) \times \mathrm{SU}(1+n)}, \quad \frac{\mathrm{SO}(2, 2+n)}{\mathrm{SO}(2) \times \mathrm{SO}(2+n)},$$

$$\frac{\mathrm{SU}(2, 2+n)}{\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2+n)},$$

and how these algebras are embedded one into the other. We also compute the metric of these spaces in the solvable parametrization. We show a couple of examples where these spaces are related to one another by gauging some isometries in the corresponding supergravity models followed by an integration of the massive modes.

In Section 3 we study some contractions of the solvable algebras introduced and we show how they are related among themselves. We compute the contracted metric by first giving a *deformation* of it in terms of a parameter ϵ. The limits $\epsilon \to 1$ and $\epsilon \to 0$ correspond to the original and the contracted spaces respectively. For an arbitrary $\epsilon \neq 0$, the groups are isomorphic but we will see that it is not possible, in general, to reabsorb the parameter into a redefinition of the coordinates of the coset space. This means that the spaces at $\epsilon \neq 0$ are not isometric. We will show this phenomenon in detail. We will see how it is possible to interpret the gauging and integrating procedure of the examples treated in Section 2 as a contraction followed by a quotienting by a submanifold.

In Section 4 we describe the super Higgs phenomenon associated to an effective N=2 supergravity theory with scalar manifold

$$\frac{\mathrm{SU}(1, 1+n)}{(\mathrm{U}(1) \times \mathrm{SU}(1+n))} \times \frac{\mathrm{SU}(2, 2+n)}{(\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2+n))},$$

relating it to the contraction procedure described in previous sections.

In the Appendix we explain in more detail the parametrization chosen to study these sigma models.

2 Symmetric spaces, solvable parametrizations and embeddings

We first illustrate the calculation of the solvable Lie algebra associated to a symmetric space of the non compact type with the simple example in our list. Essentially one has to diagonalize simultaneously the elements of the maximal abelian subalgebra in the space $\mathfrak{p}$ of the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p}, \quad \mathfrak{g} = \mathrm{Lie}(G), \quad \mathfrak{h} = \mathrm{Lie}(H).$$

2.1 Solvable parametrization of $\mathrm{SO}(1, 1+n)/\mathrm{SO}(1+n)$.

We consider the Lie algebra of $\mathrm{SO}(1, 1+n)$, $\mathfrak{so}(1, 1+n)$. In the fundamental representation, an element of it is given by

$$X = \begin{pmatrix} & & & b_1 \\ & A & & \vdots \\ & & & b_{n+1} \\ b_1 \cdots b_{n+1} & & 0 \end{pmatrix}, \qquad A = -A^T$$

A is an antisymmetric $(n+1) \times (n+1)$ matrix. The Cartan decomposition of $\mathfrak{g} = \mathfrak{so}(1, 1+n)$ is

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \qquad \mathfrak{k} = \mathfrak{so}(1+n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & b \\ b^T & 0 \end{pmatrix} \right\}.$$

It is easy to see that the coset has rank one. We choose the element

$$H = \begin{pmatrix} & & & 0 & 0 \\ & 0 & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 \cdots 0 & 0 & 1 \\ 0 \cdots 0 & 1 & 0 \end{pmatrix}$$

as the generator of the maximal abelian subalgebra in $\mathfrak{p}$. We must diagonalize h to obtain the reduced root pattern. This is easier by noting the following decomposition

$$\mathfrak{so}(1, n+1) \to \mathfrak{so}(1, 1) + \mathfrak{so}(n) + \mathbf{n}^+ + \mathbf{n}^-,$$

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where

$$\mathfrak{so}(1,1) = \mathrm{span}\{H\}, \quad \mathbf{n}^\pm = \left\{ \begin{pmatrix} & & & b_1 & \mp b_1 \\ & 0 & & \vdots & \vdots \\ & & & b_n & \mp b_n \\ -b_1 & \cdots & -b_n & 0 & 0 \\ \mp b_1 & \cdots & \mp b_n & 0 & 0 \end{pmatrix} \right\},$$

and a vector in $\mathbf{n}^\pm$ has charge ± 1 with respect to H. In this decomposition the algebra shows a $\mathfrak{so}(1,1)$ grading ($\mathfrak{so}(n)$ has degree 0), and $\mathbf{n}^\pm$ are nilpotent (in particular, abelian) subalgebras. The solvable Lie algebra associated to the coset $\mathrm{SO}(1,1+n)/\mathrm{SO}(1+n)$ is then

$$\mathfrak{solv}\left(\frac{\mathrm{SO}(1,1+n)}{\mathrm{SO}(1+n)}\right) = \mathrm{span}\{H\} \ltimes \mathbf{n}^+, \tag{1}$$

with commutation rules

$$[H, X_i] = X_i, \quad i = 1, \ldots n \qquad \text{(the rest zero)}.$$

Finally, the Iwasawa decomposition of the Lie algebra is

$$\mathfrak{so}(1, 1+n) = \mathfrak{so}(1+n) + \mathfrak{solv}\left(\frac{\mathrm{SO}(1,1+n)}{\mathrm{SO}(1+n)}\right).$$

We choose a coset representative of the following form

$$L = e^{u_i X_i} e^{\varphi H}, \qquad L^{-1} = e^{-\varphi H} e^{-u_i X_i}.$$

We will see that this kind of splitting of the generators is specially useful. The pull back of the Maurer-Cartan form on the group to the coset space, $L^{-1} dL$, decomposes as

$$L^{-1} dL = (L^{-1} dL)_{\mathfrak{k}} + (L^{-1} dL)_{\mathfrak{p}}.$$

The first term is the connection on the K-bundle $G \to G/K$, with $K = \mathrm{SO}(1+n)$ (spin bundle and spin connection) and the second term is the vielbein of G/K.

The metric is then computed as

$$ds^2 = \langle (L^{-1} dL)_{\mathfrak{p}}, (L^{-1} dL)_{\mathfrak{p}} \rangle$$

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where $\langle \, , \, \rangle$ is the Cartan-Killing form on $\mathfrak{g}$. Using the relation

$$e^{\alpha X} Y e^{-\alpha X} = Y e^{\alpha \beta}, \quad \text{provided} \quad [X, Y] = \beta Y,$$

it is easy to see that the metric becomes

$$ds^2 = d\varphi^2 + e^{-2\varphi} \sum_i du_i{}^2. \tag{2}$$

This metric has the translational isometries $u_i \to u_i + c_i$ which are a maximal abelian ideal of the solvable Lie algebra (see Appendix A). In this case the ideal is $\mathcal{I} = \text{span}\{X_i\}$.

It is now easy to go to the largest space in our list. It has only rank two and the rest of the solvable algebras can be seen as subalgebras of this. In fact we have a chain of embeddings of the solvable Lie algebras which implies a chain of embeddings of the corresponding symmetric spaces.

2.2 Solvable parametrization of $\mathrm{U}(2, 2+n)/\mathrm{U}(2) \times \mathrm{U}(2+n)$.

An element of the Lie algebra $\mathfrak{su}(2, 2+n)$ can be written as

$$X = \begin{pmatrix} A_{2\times 2} & B_{2\times(2+n)} \\ B^\dagger_{(2+n)\times 2} & D_{(2+n)\times(2+n)} \end{pmatrix}, \qquad A^\dagger = -A, \quad C^\dagger = C,$$

and the Cartan decomposition of $\mathfrak{su}(2, 2+n) = \mathfrak{h} + \mathfrak{p}$ is

$$\mathfrak{h} = \left\{ \begin{pmatrix} A_{2\times 2} & 0 \\ 0 & D_{(2+n)\times(2+n)} \end{pmatrix} \right\}, \qquad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & B_{2\times(2+n)} \\ B^\dagger_{(2+n)\times 2} & 0 \end{pmatrix} \right\}.$$

A maximal abelian subalgebra of $\mathfrak{p}$ has dimension 2, and so the coset has rank two. We can choose for example, as maximal abelian subalgebra, the one generated by the matrices

$$H_+ = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad H_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & i & 0 & 0 & \cdots & 0 \\ 0 & -i & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The solvable algebra can be shown to be generated by

$$\mathfrak{s}_4 \;\; = \;\; \mathfrak{solv}\left(\frac{\mathrm{U}(2,2+n)}{\mathrm{U}(2)\times\mathrm{U}(2+n)}\right) = \mathrm{span}\{H_+, H_-\} +$$

$$\mathrm{span}\{Z^{ia}, Y^{ia}, T^{2,0}, T^{0,2}, S_\alpha^{(1,1)}, S_\alpha^{(1,-1)}\}, \qquad (3)$$

$$\text{where} \quad i = 1,2, \quad a = 1,\ldots n, \quad \alpha = 1,2,$$

with commutation rules

$$
\begin{aligned}
[Z^{ia}, Z^{jb}] &= \epsilon^{ij}\delta^{ab}T^{(2,0)} \\
[Y^{ia}, Y^{jb}] &= \epsilon^{ij}\delta^{ab}T^{(0,2)} \\
[Z^{ia}, Y^{jb}] &= \delta^{ab}(\delta^{ij}S_2^{(1,1)} + \epsilon^{ij}S_1^{(1,1)}) \\
[Y^{ia}, S_1^{(1,-1)}] &= Z^{ia} \\
[Y^{ia}, S_2^{(1,-1)}] &= \epsilon^{ij}Z^{ja} \\
[T^{(0,2)}, S_\alpha^{(1,-1)}] &= 2S_\alpha^{(1,1)} \\
[S_\alpha^{(1,1)}, S_\beta^{(1,-1)}] &= \delta_{\alpha\beta}T^{(2,0)} \\
[H_+, Z^{ia}] &= Z^{ia} \\
[H_-, Y^{ia}] &= Y^{ia}.
\end{aligned}
\qquad (4)
$$

The rest of the commutators with the Cartan generators H_+ and H_- are indicated by the superindices (h_+, h_-). All the other commutators are zero.

Based on this solvable algebra, we choose the following parametrization for the coset representative of $\mathrm{SU}(2,2+n)/(\mathrm{SU}(2)\times\mathrm{SU}(2+n)\times\mathrm{U}(1))$ [1]:

$$L(t, \tilde{t}, \tilde{s}_\alpha, s_\alpha, z_{ia}, y_{ia}, \psi, \phi) = A(t, \tilde{t}, \tilde{s}_\alpha, z_{1a})B(s_\alpha, z_{2a}, y_{ia})C(\psi, \phi) \qquad (5)$$

where

$$
\begin{aligned}
A &= \exp\left(tT^{(2,0)} + \tilde{t}T^{(0,2)} + \tilde{s}_\alpha S_\alpha^{(1,1)} + z_{1a}Z^{1a}\right) \\
B &= \exp\left(s_1 S_1^{(1,-1)}\right)\exp\left(s_2 S_2^{(1,-1)}\right)\exp(z_{2a}Z^{2a})\exp(y_{2a}Y^{2a})\exp(y_{1a}Y^{1a}) \\
C &= \exp\left(\psi H_+ + \phi H_-\right)
\end{aligned}
$$

The Maurer Cartan form is

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[1]In the appendix we show that the generators $T^{(2,0)}, T^{(0,2)}, S_\alpha^{(1,1)}, Z^{1a}$ correspond to true global *translational isometries*.

$$L^{-1}dL = e^{-2\psi}(s_\alpha d\tilde{s}_\alpha + (s_1^2 + s_2^2)d\tilde{t} + dt + z_{2a}dz_{1a})T^{(2,0)} + e^{-2\phi}(d\tilde{t} - y_{1a}dy_{2a})T^{(0,2)} +$$

$$+ e^{-(\phi+\psi)}(y_{1a}y_{2a}ds_1 + \frac{1}{2}(y_1^2 + y_2^2)ds_2 + d\tilde{s}_1 + 2s_1d\tilde{t} - \epsilon_{ij}y_{ia}dz_{ja})S_1^{(1,1)} +$$

$$+ e^{-(\phi+\psi)}(y_{1a}y_{2a}ds_2 - \frac{1}{2}(y_1^2 + y_2^2)ds_1 + d\tilde{s}_2 + 2s_2d\tilde{t} + \delta_{ij}y_{ia}dz_{ja})S_2^{(1,1)} +$$

$$+ e^{-\psi}(-y_{1a}ds_1 + y_{2a}ds_2 + dz_{1a})Z^{1a} + e^{-\psi}(-y_{2a}ds_1 - y_{1a}ds_2 + dz_{2a})Z^{2a} +$$

$$+ e^{\phi-\psi}ds_\alpha S_\alpha^{(1,-1)} + e^{-\phi}dy_{ia}Y^{ia} + d\psi H_+ + d\phi H_- \tag{6}$$

The metric of the coset is computed now as $\langle (L^{-1}dL)_{\mathfrak{p}}, (L^{-1}dL\rangle_{\mathfrak{p}})$. On the tangent space to the identity, this gives the following inner product:

$$\langle X, X \rangle = 1, \qquad \text{for } X = H_\pm, T^{(2,0)}, T^{(0,2)}$$

$$\langle X, X \rangle = \frac{1}{2}, \qquad \text{for } X = S_1^{(1,1)}, S_2^{(1,1)}, S_1^{(1,-1)}, S_2^{(1,-1)}, Y^{1a}, Y^{2a}, Z^{1a}, Z^{2a},$$

and the rest zero.

For $n = 0$ we obtain the reduced expression

$$ds^2 = d\phi^2 + d\psi^2 + e^{-4\psi}dtdt + 2e^{-4\psi}s_1dtd\tilde{s}_1 + 2e^{-4\psi}s_2dtd\tilde{s}_2$$

$$+2e^{-4\psi}(s_2^2 + s_1^2)dtd\tilde{t} + \frac{1}{2}(e^{-2(\psi+\phi)} + 2e^{-4\psi}s_1^2)d\tilde{s}_1d\tilde{s}_1$$

$$+2e^{-4\psi}s_2s_1d\tilde{s}_1d\tilde{s}_2 + 2e^{-4\psi}s_1(e^{2(\psi-\phi)} + s_2^2 + s_1^2)d\tilde{s}_1d\tilde{t}$$

$$+\frac{1}{2}(e^{-2(\psi+\phi)} + 2e^{-4\psi}s_2^2)d\tilde{s}_2d\tilde{s}_2$$

$$+2e^{-4\psi}s_2(e^{2(\psi-\phi)} + s_2^2 + s_1^2)d\tilde{s}_2d\tilde{t}$$

$$+\frac{1}{2}e^{2(\phi-\psi)}ds_2ds_2 + \frac{1}{2}e^{2(\phi-\psi)}ds_1ds_1$$

$$+e^{-4(\psi+\phi)}\left(e^{2\psi} + e^{2\phi}(s_1^2 + s_2^2)\right)^2 d\tilde{t}d\tilde{t}. \tag{7}$$

For arbitrary n we obtain [2] (sum over repeated indices is understood, and we have used the short-hand notation $y_1^2 = y_{1a}y_{1a}$):

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[2]For this and the rest of the calculations of different metrics we have used the program Wolfram Research, Inc., Mathematica, Version 5.1, Champaign, IL (2004).

$$ds^2 = d\phi^2 + d\psi^2 + e^{-4\psi}dtdt + 2e^{-4\psi}s_1dtd\tilde{s}_1 + 2e^{-4\psi}s_2dtd\tilde{s}_2$$

$$+2e^{-4\psi}z_{2a}dtdz_{1a} + 2e^{-4\psi}(s_2^2 + s_1^2)dtd\tilde{t} + \frac{1}{2}(e^{-2(\psi+\phi)} + 2e^{-4\psi}s_1^2)d\tilde{s}_1d\tilde{s}_1$$

$$+2e^{-4\psi}s_2s_1d\tilde{s}_1d\tilde{s}_2 + \frac{1}{2}e^{-2(\psi+\phi)}(y_1^2 + y_2^2)d\tilde{s}_1ds_2 + e^{-2(\psi+\phi)}y_{1a}y_{2a}d\tilde{s}_1ds_1$$

$$-e^{-2(\psi+\phi)}y_{1a}d\tilde{s}_1dz_{2a} + (2e^{-4\psi}s_1z_{2a} + e^{-2(\psi+\phi)}y_{2a})d\tilde{s}_1dz_{1a}$$

$$+2e^{-4\psi}s_1(e^{2(\psi-\phi)} + s_2^2 + s_1^2)d\tilde{s}_1d\tilde{t} + \frac{1}{2}(e^{-2(\psi+\phi)} + 2e^{-4\psi}s_2^2)d\tilde{s}_2d\tilde{s}_2$$

$$+e^{-2(\psi+\phi)}y_{1a}y_{2a}d\tilde{s}_2ds_2 - \frac{1}{2}e^{-2(\psi+\phi)}(y_1^2 + y_2^2)d\tilde{s}_2ds_1$$

$$+e^{-2(\psi+\phi)}y_{2a}d\tilde{s}_2dz_{2a} + (2e^{-4\psi}s_2z_{2a} + e^{-2(\psi+\phi)}y_{1a})d\tilde{s}_2dz_{1a}$$

$$+2e^{-4\psi}s_2(e^{2(\psi-\phi)} + s_2^2 + s_1^2)d\tilde{s}_2d\tilde{t}$$

$$+\frac{1}{8}e^{-2(\psi+\phi)}(4e^{4\phi} + 4e^{2\phi}(y_1^2 + y_2^2) + 4(y_{1a}y_{2a})^2 + (y_1^2 + y_2^2)(y_1^2 + y_2^2))ds_\alpha ds_\alpha$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}(2e^{2\phi}y_{1b} + (-2(y_{1a}y_{2a})y_{2b} + (y_1^2 + y_2^2)y_{1b}))ds_2dz_{2b}$$

$$+\frac{1}{2}e^{-2(\psi+\phi)}(2e^{2\phi}y_{2b} + (2(y_{1a}y_{2a})y_{1b} + (y_1^2 + y_2^2)y_{2b}))ds_2dz_{1b}$$

$$+e^{-2(\psi+\phi)}(y_2^2s_1 + 2y_{2a}s_2y_{1a} + s_1y_1^2)ds_2d\tilde{t}$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}(2e^{2\phi}y_{2b} + (2(y_{1a}y_{2a})y_{1b} + (y_1^2 + y_2^2)y_{2b}))ds_1dz_{2b}$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}(2e^{2\phi}y_{1b} + (-2(y_{1a}y_{2a})y_{2b} + (y_1^2 + y_2^2)y_{1b}))ds_1dz_{1b}$$

$$-e^{-2(\psi+\phi)}(y_1^2s_2 - 2y_{1a}s_1y_{2a} + s_2y_2^2)ds_1d\tilde{t}$$

$$-\frac{1}{2}e^{-2(\phi+\psi)}\epsilon_{ij}\epsilon_{mn}(y_{ia}y_{jb})dz_{ma}dz_{nb}$$

$$+\frac{1}{2}e^{-2(\psi+\phi)}(e^{2\phi}\delta_{ab} + (y_{1a}y_{1b} + y_{2a}y_{2b}))dz_{2a}dz_{2b}$$

$$-e^{-2(\psi+\phi)}(2y_1s_1 - 2s_2y_2)dz_2d\tilde{t}$$

$$+\frac{1}{2}e^{-4\psi}(e^{2\psi}\delta_{ab} + 2z_{2a}z_{2b} + e^{2(\psi-\phi)}(y_{1a}y_{1b} + y_{2a}y_{2b}))dz_{1a}dz_{1b}$$

$$+(2e^{-4\psi}(s_2^2 + s_1^2)z_{2a} + 2e^{-2(\psi+\phi)}(y_{1a}s_2 + s_1y_{2a}))dz_{1a}d\tilde{t}$$

$$+e^{-4(\psi+\phi)}(e^{2\psi} + e^{2\phi}(s_1^2 + s_2^2))^2d\tilde{t}d\tilde{t} - 2e^{-4\psi}y_{1a}d\tilde{t}dy_{2a}$$

$$+\frac{1}{2}e^{-4\phi}(e^{2\phi}\delta_{ab} + 2y_{1a}y_{1b})dy_{2a}dy_{2b} + \frac{1}{2}e^{-2\phi}dy_{1a}dy_{1a} \tag{8}$$

2.3 Chain of embeddings

We have the following chain of solvable Lie algebras

$$
\begin{aligned}
\mathfrak{s}_4 &= \mathfrak{solv}\left(\frac{\mathrm{SU}(2,2+n)}{\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(2+n)}\right) \qquad (\text{see (3)}), \\
\mathfrak{s}_3 &= \mathfrak{solv}\left(\frac{\mathrm{SO}(2,2+n)}{\mathrm{SO}(2)\times\mathrm{SO}(2+n)}\right) = \mathrm{span}\{H_+, H_-\} + \mathrm{span}\{Z^{1a}, Y^{1a}, S_2^{(1,1)}, S_2^{(1,-1)}\}, \\
\mathfrak{s}_2 &= \mathfrak{solv}\left(\frac{\mathrm{SU}(1,1+n)}{\mathrm{U}(1)\times\mathrm{SU}(1+n)}\right) = \mathrm{span}\{H_+ + H_-\} + \mathrm{span}\{Z^{1a}, Y^{1a}, S_2^{(1,1)}\}, \\
\mathfrak{s}_1 &= \mathfrak{solv}\left(\frac{\mathrm{SO}(1,1+n)}{\mathrm{SO}(1+n)}\right) = \mathrm{span}\{H_+ + H_-\} + \mathrm{span}\{Y^{1a}\}, \tag{9}
\end{aligned}
$$

with $\mathfrak{s}_i \subset_{\mathrm{sub}} \mathfrak{s}_{i+1}$. Following the same procedure than in the previous examples, one can show that these solvable Lie algebras correspond to the following chain of symmetric spaces:

$$
\frac{\mathrm{SO}(1,1+n)}{\mathrm{SO}(1+n)} \subset \frac{\mathrm{SU}(1,1+n)}{\mathrm{U}(1+n)} \subset \frac{\mathrm{SO}(2,2+n)}{\mathrm{SO}(2)\times\mathrm{SO}(2+n)} \subset \frac{\mathrm{SU}(2,2+n)}{\mathrm{U}(2)\times\mathrm{SU}(2+n)} \tag{10}
$$

Notice that in this chain we have

$$
\frac{G_i}{H_i} \subset \frac{G_{i+1}}{H_{i+1}}
$$

with

$$
\begin{aligned}
G_1 \subset G_2, && H_1 \subset H_2, \\
G_3 \subset G_4, && H_3 \subset H_4, \\
G_1 \subset G_3, && H_1 \subset H_3, \\
G_2 \subset G_4, && H_2 \subset H_4,
\end{aligned}
$$

but G_2 is not in G_3 nor H_2 in H_3 for generic n.

The solvable parametrization (5) allows us to compute the metric of the spaces in (10) by imposing different restrictions on (8).

For the coset $\mathrm{SO}(2, 2+n)/(\mathrm{SO}(2) \times \mathrm{SO}(2+n))$, we have

$$z_{2a} = y_{2a} = t = \tilde{t} = \tilde{s}_1 = s_2 = 0,$$

so the metric is

$$
\begin{aligned}
ds^2 &= d\phi^2 + d\psi^2 + \frac{1}{2}e^{-2(\psi+\phi)}d\tilde{s}_2 d\tilde{s}_2 - \frac{1}{2}e^{-2(\psi+\phi)}y_1^2 d\tilde{s}_2 ds_1 \\
&\quad + e^{-2(\psi+\phi)}y_{1a}d\tilde{s}_2 dz_{1a} + \frac{1}{8}e^{-2(\psi+\phi)}(4e^{4\phi} + (y_1^2)^2 + 4e^{2\phi}y_1^2)ds_1 ds_1 \\
&\quad - \frac{1}{2}e^{-2(\psi+\phi)}y_{1a}(2e^{2\phi} + y_1^2)ds_1 dz_{1a} + \frac{1}{2}e^{-4\psi}(e^{2\psi}\delta_{ab} + e^{2(\psi-\phi)}y_{1a}y_{1b})dz_{1a}dz_{1b} \\
&\quad + \frac{1}{2}e^{-2\phi}dy_{1a}dy_{1a}. \quad\quad (11)
\end{aligned}
$$

Imposing the constraints $s_1 = \phi - \psi = 0$, we obtain the metric on $\mathrm{SU}(1, 1+n)/\mathrm{U}(1+n)$,

$$
\begin{aligned}
ds^2 &= 2d\phi^2 + \frac{1}{2}e^{-4\phi}d\tilde{s}_2 d\tilde{s}_2 + e^{-4\phi}y_{1a}d\tilde{s}_2 dz_{1a} \\
&\quad + \frac{1}{2}e^{-4\phi}(e^{2\phi}\delta_{ab} + y_{1a}y_{1b})dz_{1a}dz_{1b} + \frac{1}{2}e^{-2\phi}dy_{1a}dy_{1a}, \quad\quad (12)
\end{aligned}
$$

and imposing $z_{1a} = \tilde{s}_2 = 0$, we obtain the metric for $\mathrm{SO}(1, 1+n)/\mathrm{SO}(1+n)$:

$$ds^2 = 2d\phi^2 + \frac{1}{2}e^{-2\phi}dy_{1a}dy_{1a} \quad\quad (13)$$

which, up to a rescaling of the coordinates, corresponds to (2).

We can further impose $y_{1n} = 0$ to obtain the same form than (13) but with $a = 1, \ldots n-1$. It is the metric of $\mathrm{SO}(1, n)/\mathrm{SO}(n)$.

2.4   Truncations and integration of massive modes

Let us consider a sigma model described by the metric (13). As we have seen, this model has n translational isometries corresponding to the coordinates y_{1a}. We may consider gauging one of these isometries, say y_{1n}. We introduce a gauge field $A = A_\mu dx^\mu$ and substitute dy_{1n} by the covariant differential

$$Dy_{1n} = dy_{1n} + gA.$$

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We redefine the connection by a gauge transformation

$$\hat{A} = A + \frac{1}{g}dy_{1n},$$

which will not change the kinetic term for A. Substituting this definition in the metric we obtain

$$ds^2 = 2d\phi^2 + \frac{1}{2}e^{-2\phi}\sum_{a=1}^{n-1}dy_{1a}dy_{1a} + \frac{1}{2}e^{-2\phi}g^2\hat{A}^2.$$

We see that the effect of the gauging is absorbing the field y_{1n} to give mass to the gauge vector. Moreover, in this model $\hat{A}$ is decoupled from the rest of fields (except for the warping factor $e^{-2\phi}$), so setting $\hat{A} = 0$ is consistent with the equations of motion. After the truncation the sigma model becomes $SO(1,n)/SO(n)$. This is explained by the mathematical identity

$$\mathfrak{solv}\left(\frac{SO(1,1+n)}{SO(1+n)}\right) = \mathfrak{solv}\left(\frac{SO(1,1+n-k)}{SO(1+n-k)}\right) \ltimes \mathbb{R}^k$$

which is a consequence of (1).

We want to consider now the model $SU(1,2)/U(2)$, with metric (12) for $n = 1$. Note that $\tilde{s}_2$ and z_1 are translational isometries. As before, we can gauge them by introducing abelian connections A^1, A^2 with covariant differentials

$$\begin{aligned} d\tilde{s}_2 &\rightarrow D\tilde{s}_2 = d\tilde{s}_2 + k_1 A^1 \\ dz_1 &\rightarrow Dz_1 = dz_1 + k_2 A^2. \end{aligned}$$

We define the gauge-transformed connections

$$\begin{aligned} \hat{A}^1 &= A^1 + \frac{1}{k_1}d\tilde{s}_2 \\ \hat{A}^2 &= A^2 + \frac{1}{k_2}dz_1. \end{aligned}$$

By substituting this definition, we can see that in the metric there will appear the terms

$$ds^2 = \cdots + \frac{1}{2}e^{-4\phi}(k_1)^2(\hat{A}^1)^2 + \cdots + \frac{1}{2}e^{-2\phi}(k_2)^2(\hat{A}^2)^2 + \cdots$$

So, as before, the effect of the gauging has been to give mass to the vectors by absorbing the modes associated to the translational isometries.

Nevertheless, in this case other interactions are present. By assuming that the mass of the vectors is big enough we can take their kinetic terms to zero, and then we obtain algebraic equations for $\hat{A}^1$, $\hat{A}^2$. A straightforward calculation shows that, after the elimination of these fields the metric that remains is $SO(1,2)/SO(2)$, that is eq. (13) with $n = 1$.

The difference between the two models here described is that in the first case the integration of the massive modes is exact (that is, it is a consistent truncation of the theory), while in the second case a limiting process is involved (masses $\rightarrow \infty$).

In the next section we will see that these integrations can be modeled by a *contraction* of the metric of the initial manifold, followed by a *quotienting* of the manifold by a submanifold.

3   Contractions of groups and coset spaces

Contraction of a Lie algebra with respect to a subalgebra.
We describe the Inönü-Wigner contraction of an algebra with respect to a subalgebra. Let $\mathfrak{g}$ be an arbitrary, finite dimensional Lie algebra with commutator $[\,,\,]$ and let $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, with $\mathfrak{g}_1$ a subalgebra. We define the following family of linear maps

$$
\begin{aligned}
\phi_\epsilon : \mathfrak{g} &\longrightarrow \mathfrak{g} \\
x = x_1 \oplus x_2 &\longrightarrow x = x_1 \oplus \epsilon x_2,
\end{aligned}
$$

labelled by a real parameter ϵ, In matrix form, the map and its inverse ($\epsilon \neq 0$) are block-diagonal

$$
\phi_\epsilon = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \epsilon \mathbb{1} \end{pmatrix}, \qquad \phi_\epsilon^{-1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \frac{1}{\epsilon}\mathbb{1} \end{pmatrix}.
$$

We can define a new commutator

$$
[X, Y]_\epsilon = \phi_\epsilon^{-1}([\phi_\epsilon(X), \phi_\epsilon(Y)]), \qquad X, Y \in \mathfrak{g}.
$$

$[\,,\,]_\epsilon$ is a *deformed bracket*, but of a simple form, since for $\epsilon \neq 0$ is, by construction, isomorphic to the bracket with $\epsilon = 1$. We define the *contraction* of

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$\mathfrak{g}$ with respect to the subalgebra $\mathfrak{g}_1$ as a Lie algebra with the same supporting vector space $\mathfrak{g}_c \approx \mathfrak{g}$ and with commutator

$$[X, Y]_c = \lim_{\epsilon \to 0} \phi_\epsilon^{-1}([\phi_\epsilon(X), \phi_\epsilon(Y)]), \qquad X, Y \in \mathfrak{g}. \tag{14}$$

This bracket is well defined but, since ϕ_ϵ is not invertible, $[\,,\,]_c$ will not be, in general, isomorphic to the original bracket.

Representations of the contracted algebra.

We consider now a representation of $\mathfrak{g}$ on a finite dimensional vector space W

$$R(X) : W \to W, \qquad X \in \mathfrak{g}$$

and assume that $W = W_1 \oplus W_2$ with W_1 an invariant subspace under the action of the subalgebra $\mathfrak{g}_1$. As before, we define a one parameter family of linear maps

$$\psi_\epsilon : W \qquad \longrightarrow \qquad W$$

$$w = w_1 \oplus w_2 \quad \longrightarrow \quad w = w_1 \oplus \epsilon w_2,$$

so

$$\psi_\epsilon = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \epsilon \mathbb{1} \end{pmatrix}, \qquad \psi_\epsilon^{-1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \frac{1}{\epsilon} \mathbb{1} \end{pmatrix}.$$

Let us denote

$$R_\epsilon(X) = \psi_\epsilon^{-1} \circ R(\phi_\epsilon(X)) \circ \psi_\epsilon, \qquad X \in \mathfrak{g}.$$

R_ϵ is a representation of the deformed algebra. It is easy to check that the map R_c

$$R_c(X) = \lim_{\epsilon \to 0} R_\epsilon(X)$$

is a representation of $\mathfrak{g}_c$ on W.

Notice that $\psi_\epsilon = \phi_\epsilon$ for the adjoint representation.

Generalized contractions

The map ϕ_ϵ can in fact be more general than the one considered before, the only constraint being that the bracket in (14) is well defined. The conditions for this to happen were studied in Ref. [7] and are called *generalized Inönü-Wigner contractions*. They are also a particular example of *algebra expansions* [9].

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We will use particular examples of generalized contractions where the brackets can be seen explicitly to have a well defined limit. We will not describe the general theory of these contractions, for which we refer to the original paper, Ref. [7].

3.1 Deformations and contractions of the metric: some examples.

As we have seen, we can always contract an algebra $\mathfrak{g}$ with respect to a subalgebra $\mathfrak{g}'$. The contracted algebra, $\mathfrak{g}_c$ will have always the structure of a semidirect product

$$\mathfrak{g}_c = \mathfrak{g}' \ltimes \mathbb{R}^n.$$

In the chain (9) we have described subalgebras of $\mathfrak{s}_4$, so we can contract each algebra $\mathfrak{s}_i$ with respect to $\mathfrak{s}_j$ with $j < i$

Since the solvable Lie algebras are related to the corresponding symmetric spaces, we are going to define a procedure to contract the symmetric spaces. We will start with a representation R_ϵ of the deformed Lie algebra, and compute the coset representative as in (5) with this new representation. From this, one can compute a deformed vielbein and a deformed metric. This procedure will introduce the parameter ϵ in the metric, so we will have a uniparametric family of metrics. Then, we can take the limit $\epsilon \to 0$.

We are interested in the simple examples presented in 2.4. The first one is trivial, since the contraction of $\mathfrak{s}_1$ for arbitrary n by the subalgebra $\mathfrak{s}_1$ for $n - 1$ has no effect, giving again $\mathfrak{s}_1$ for n.

Let us see how this works with the next example. We start with the algebra $\mathfrak{solv}(\mathrm{SU}(1,2)/\mathrm{U}(2))$ (which is $\mathfrak{s}_2$ for $n = 1$) and we will work out the contraction with respect to $\mathfrak{solv}(\mathrm{SO}(1,2)/\mathrm{SO}(2))$ which is ($\mathfrak{s}_1$ for $n = 1$).

$$\begin{align}
\mathfrak{s}_2(n = 1) &= \operatorname{span}\{H_1 = H_+ + H_-\} + \operatorname{span}\{Z^1, Y^1, S_1^{(1,1)}\}, \tag{15}\\
\mathfrak{s}_1(n = 1) &= \operatorname{span}\{H_1\} + \operatorname{span}\{Y^1\}. \tag{16}
\end{align}$$

It is convenient to write explicitly the commutation rules

$$[H_1, Z^1] = Z^1, \quad [H_1, Y^1] = Y^1, \quad [H_1, S_1^{(1,1)}] = 2S_1^{(1,1)}, \quad [Z^1, Y^1] = S_1^{(1,1)}.$$

The deformed algebra is

$$[H_1, Z^1]_\epsilon = Z^1, \quad [H, Y^1]_\epsilon = Y^1, \quad [H_1, S_1^{(1,1)}]_\epsilon = 2S_1^{(1,1)}, \quad [Z^1, Y^1]_\epsilon = \epsilon S_1^{(1,1)} \to 0.$$

The contracted algebra has the property that the only elements in $\mathfrak{g}'$ that act on the abelian factor $\mathbb{R}^n$ are the elements of the commuting subalgebra of $\mathfrak{p}$. In our case this subalgebra is $H_1 = H_+ + H_-$. This property will translate in a particularly simple form of the metric.

We consider the three dimensional representation (induced from the fundamental of $\mathfrak{su}(1,2)$). We decompose the representation space as

$$\mathbb{C}^3 = V_1 \oplus V_2, \qquad V_1 = \{ \begin{pmatrix} v_1 \\ 0 \\ v_3 \end{pmatrix} \}, \quad V_2 = \{ \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \},$$

being V_1 an invariant subspace under the subalgebra (16)and consider the linear map $\psi_\epsilon(e_1 \oplus e_2) = e_1 \oplus \epsilon e_2$. Then we have a three dimensional representation of the deformed algebra,

$$R_\epsilon(H_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$R_\epsilon(Z^1) = \begin{pmatrix} 0 & i & 0 \\ i\epsilon^2 & 0 & -i\epsilon^2 \\ 0 & i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix},$$

$$R_\epsilon(Y^1) = \begin{pmatrix} 0 & 1 & 0 \\ -\epsilon^2 & 0 & \epsilon^2 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$R_\epsilon(S_1^{(1,1)}) = -\epsilon \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix} \rightarrow 0.$$

We compute now the vielbein and the metric in the way that we indicated in section 2. Notice that the Euclidean metric that we put on the solvable Lie algebra with the deformed bracket is *the same* than the one for $\epsilon = 1$. In this way the normal metric Lie algebras (in the terminology of Ref. [6]) are not isomorphic, nor are isometric the corresponding Riemannian spaces. We obtain then a true deformation of the metric.

The result is

$$\begin{aligned} ds^2 &= 2d\phi^2 + \frac{1}{2}e^{-4\phi}d\tilde{s}_2 d\tilde{s}_2 + \epsilon^2 e^{-4\phi}y_1 d\tilde{s}_2 dz_1 \\ &\quad + \frac{1}{2}e^{-4\phi}(e^{2\phi} + \epsilon^4 y_1^2)dz_1 dz_1 + \frac{1}{2}e^{-2\phi}dy_1 dy_1, \end{aligned} \qquad (17)$$

which can be compared with (12) for $\epsilon = 1$.

For $\epsilon \to 0$ we get

$$ds^2 = \left(2d\phi^2 + \frac{1}{2}e^{-2\phi}dy_1 dy_1 \right) + \frac{1}{2}e^{-2\phi}dz_1 dz_1 + \frac{1}{2}e^{-4\phi}d\tilde{s}_2 d\tilde{s}_2. \quad (18)$$

The first two factors correspond to (13). The remaining modes appear decoupled except for warping factors of type $e^{a\phi}/2$. Then, imposing the constraints $z_1 = 0 = \tilde{s}_2$ is always a consistent truncation of the contracted sigma model (18). We see with this simple example that integrating out massive modes can be geometrically modeled by a contraction of the sigma model, followed by a quotienting by the decoupled modes.

It is instructive to compute Ricci tensor of the deformed metric (17). We obtain (in the ordered basis $\phi, y_1, z_1, \tilde{s}_2$)

$$R^a_b(\epsilon) = \begin{pmatrix} -6 & 0 & 0 & 0 \\ 0 & -2(2+\epsilon^4) & 0 & 0 \\ 0 & 0 & -2(2+\epsilon^4) & 0 \\ 0 & 0 & 8y_1\epsilon^2(\epsilon^4-1) & 2(-4+\epsilon^4) \end{pmatrix} \quad (19)$$

We see that for arbitrary ϵ it is not an Einstein space. In the relevant limits

$$R^a_b(1) = \begin{pmatrix} -6 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}, \qquad R^a_b(0) = \begin{pmatrix} -6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -8 \end{pmatrix}$$

For $\epsilon = 1$ we have an Einstein space, but not for arbitrary ϵ. It becomes clear from this simple example that the deformation cannot be reabsorbed by a change of coordinates.

We consider now the Inönu-Wigner contraction of $\mathfrak{s}_4$ with respect to $\mathfrak{s}_3$. For simplicity, we take $n = 1$, so we have $\mathfrak{s}_4 = \mathfrak{s}_3 + \mathfrak{g}$ where

$$\mathfrak{s}_3 = \mathrm{span}\{H_+, H_-, S_2^{(1,1)}, S_1^{(1,-1)}, Y^1, Z^1\}$$
$$\mathfrak{g} = \mathrm{span}\{T^{(2,0)}, T^{(0,2)}, S_1^{(1,1)}, S_2^{(1,-1)}, Y^2, Z^2\}$$

Differently from the first example, we use the adjoint representation to introduce the parameter ϵ. The result for the metric is

$$ds^2 = d\phi^2 + d\psi^2 + e^{-4\psi}dtdt + 2e^{-4\psi}s_1 dt d\tilde{s}_1 + 2e^{-4\psi}s_2 dt d\tilde{s}_2$$

$$+2e^{-4\psi}z_2 dt dz_1 + 2e^{-4\psi}(s_2^2\epsilon^2 + s_1^2)dt d\tilde{t} + \frac{1}{2}(e^{-2(\psi+\phi)} + 2e^{-4\psi}s_1^2)d\tilde{s}_1 d\tilde{s}_1$$

$$+2e^{-4\psi}s_2 s_1 d\tilde{s}_1 d\tilde{s}_2 + \frac{1}{2}e^{-2(\psi+\phi)}(y_1^2 + y_2^2\epsilon^2)d\tilde{s}_1 ds_2 + e^{-2(\psi+\phi)}y_1 y_2 d\tilde{s}_1 ds_1$$

$$-e^{-2(\psi+\phi)}y_1 d\tilde{s}_1 dz_2 + (2e^{-4\psi}s_1 z_2 + e^{-2(\psi+\phi)}y_2)d\tilde{s}_1 dz_1$$

$$+2e^{-4\psi}s_1(e^{2(\psi-\phi)} + s_2^2\epsilon^2 + s_1^2)d\tilde{s}_1 d\tilde{t} + \frac{1}{2}(e^{-2(\psi+\phi)} + 2e^{-4\psi}s_2^2)d\tilde{s}_2 d\tilde{s}_2$$

$$+e^{-2(\psi+\phi)}y_1 y_2\epsilon^2 d\tilde{s}_2 ds_2 - \frac{1}{2}e^{-2(\psi+\phi)}(y_1^2 + y_2^2\epsilon^2)d\tilde{s}_2 ds_1$$

$$+e^{-2(\psi+\phi)}y_2\epsilon^2 d\tilde{s}_2 dz_2 + (2e^{-4\psi}s_2 z_2 + e^{-2(\psi+\phi)}y_1)d\tilde{s}_2 dz_1$$

$$+2e^{-4\psi}s_2(e^{2(\psi-\phi)}\epsilon^2 + s_2^2\epsilon^2 + s_1^2)d\tilde{s}_2 d\tilde{t}$$

$$+\frac{1}{8}e^{-2(\psi+\phi)}(4e^{4\phi} + y_1^4 + (2\epsilon^2 + 4\epsilon^4)y_1^2 y_2^2 + y_2^4\epsilon^4 + 4e^{2\phi}(y_1^2 + y_2^2\epsilon^4))ds_2 ds_2$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}y_1(2e^{2\phi} + y_1^2 - (2\epsilon^4 - \epsilon^2)y_2^2)ds_2 dz_2 +$$

$$\frac{1}{2}e^{-2(\psi+\phi)}y_2(2e^{2\phi}\epsilon^2 + (2\epsilon^2 + 1)y_1^2 + y_2^2\epsilon^2)ds_2 dz_1$$

$$+e^{-2(\psi+\phi)}(y_2^2 s_1\epsilon^2 + 2y_2 s_2 y_1\epsilon^4 + s_1 y_1^2)ds_2 d\tilde{t}$$

$$+\frac{1}{8}e^{-2(\psi+\phi)}(4e^{4\phi} + y_1^4 + (2\epsilon^2 + 4)y_1^2 y_2^2 + y_2^4\epsilon^4 + 4e^{2\phi}(y_1^2 + y_2^2))ds_1 ds_1$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}y_2(2e^{2\phi} + (2 + \epsilon^2)y_1^2 + y_2^2\epsilon^4)ds_1 dz_2$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}y_1(2e^{2\phi} + y_1^2 - (2 - \epsilon^2)y_2^2)ds_1 dz_1$$

$$-e^{-2(\psi+\phi)}(y_1^2 s_2\epsilon^2 - 2y_1 s_1 y_2 + s_2 y_2^2\epsilon^4)ds_1 d\tilde{t} + \frac{1}{2}e^{-2(\psi+\phi)}(e^{2\phi} + y_1^2 + y_2^2\epsilon^4)dz_2 dz_2$$

$$-e^{-2(\psi+\phi)}(2y_1 s_1 - 2s_2 y_2\epsilon^4)dz_2 d\tilde{t} + \frac{1}{2}e^{-4\psi}(e^{2\psi} + 2z_2^2 + e^{2(\psi-\phi)}(y_1^2 + y_2^2))dz_1 dz_1$$

$$+(2e^{-4\psi}(s_2^2\epsilon^2 + s_1^2)z_2 + 2e^{-2(\psi+\phi)}(y_1 s_2\epsilon^2 + s_1 y_2))dz_1 d\tilde{t}$$

$$(e^{-4\phi} + 2e^{-2(\phi+\psi)}(s_1^2 + s_2^2\epsilon^4) + e^{-4\psi}(s_1^4 + 2s_1^2 s_2^2\epsilon^2 + s_2^4\epsilon^4))d\tilde{t}d\tilde{t} - 2e^{-4\phi}y_1 d\tilde{t}dy_2$$

$$+\frac{1}{2}e^{-4\phi}(e^{2\phi} + 2y_1^2)dy_2 dy_2 + \frac{1}{2}e^{-2\phi}dy_1 dy_1 + e^{-2(\phi+\psi)}y_1 y_2(\epsilon^2 - 1)dz_1 dz_2)$$

$$+\frac{1}{2}e^{-2\psi}y_1 y_2(1 - \epsilon^2)(2 + e^{-2\phi}(y_1^2 + y_2^2\epsilon^2))ds_1 ds_2, \tag{20}$$

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which can be compared with (8) for $\epsilon = 1$. For $\epsilon = 0$ it becomes

$$
\begin{aligned}
ds^2 \;=\;& (d\phi^2 + d\psi^2 + \frac{1}{2}e^{-4\psi}(e^{-2(\phi-\psi)} + 2s_2^2)d\tilde{s}_2^2 \\
& -\frac{1}{2}e^{-2(\phi+\psi)}y_1^2 d\tilde{s}_2 ds_1 + (e^{-2(\phi+\psi)}y_1 + 2e^{-4\psi}s_2 z_2)d\tilde{s}_2 dz_1 \\
& +\frac{1}{8}e^{-2(\phi+\psi)}(4e^{4\phi} + y_1^4 + 4y_1^2 y_2^2 + 4e^{2\phi}(y_1^2 + y_2^2))ds_1^2 \\
& -\frac{1}{2}e^{-2(\phi+\psi)}y_1(2e^{2\phi} + y_1^2 - 2y_2^2)ds_1 dz_1 \\
& +\frac{1}{2}e^{-4\psi}(e^{2\psi} + e^{-2(\phi-\psi)}(y_1^2 + y_2^2) + 2z_2^2)dz_1^2 + \frac{1}{2}e^{-2\phi}dy_1^2) \\[6pt]
& +e^{-4\psi}dt^2 + e^{-4(\phi+\psi)}(e^{2\psi} + e^{2\phi}s_1^2)^2 d\tilde{t}^2 + \frac{1}{2}e^{-4\psi}(e^{-2(\phi-\psi)} + 2s_1^2)d\tilde{s}_1^2 \\
& +\frac{1}{8}e^{-2(\phi+\psi)}(2e^{2\phi} + y_1^2)^2 ds_2^2 + \frac{1}{2}e^{-4\phi}(e^{2\phi} + 2y_1^2)dy_2^2 \\
& +\frac{1}{2}e^{-2(\phi+\psi)}(e^{2\phi} + y_1^2)dz_2^2 + 2e^{-4\psi}s_1^2 dt d\tilde{t} + 2e^{-4\psi}s_1 dt d\tilde{s}_1 \\
& +2e^{-4\psi}s_2 dt d\tilde{s}_2 + 2e^{-4\psi}z_2 dt dz_1 + 2e^{-2(\phi+2\psi)}s_1(e^{2\psi} + e^{2\phi}s_1^2)d\tilde{t}d\tilde{s}_1 \\
& +2e^{-4\psi}s_1^2 s_2 d\tilde{t}d\tilde{s}_2 + 2e^{-2(\phi+\psi)}s_1 y_1 y_2 d\tilde{t}ds_1 + e^{-2(\phi+\psi)}s_1 y_1^2 d\tilde{t}ds_2 \\
& -2e^{-4\phi}y_1 d\tilde{t}dy_2 + 2e^{-2(\phi+2\psi)}s_1(e^{2\psi}y_2 + e^{2\phi}s_1 z_2)d\tilde{t}dz_1 \\
& -2e^{-2(\phi+\psi)}s_1 y_1 d\tilde{t}dz_2 + 2e^{-4\psi}s_1 s_2 d\tilde{s}_1 d\tilde{s}_2 + e^{-2(\phi+\psi)}y_1 y_2 d\tilde{s}_1 ds_1 \\
& +\frac{1}{2}e^{-2(\phi+\psi)}y_1^2 d\tilde{s}_1 ds_2 + e^{-4\psi}(e^{-2(\phi-\psi)}y_2 + 2s_1 z_2)d\tilde{s}_1 dz_1 - e^{-2(\phi+\psi)}y_1 d\tilde{s}_1 dz_2 \\
& +\frac{1}{2}e^{-2(\phi+\psi)}y_1(2e^{2\phi} + y_1^2)y_2 ds_1 ds_2 \\
& -e^{-2(\phi+\psi)}y_2(e^{2\phi} + y_1^2)ds_1 dz_2 - \frac{1}{2}e^{-2(\phi+\psi)}y_1(2e^{2\phi} + y_1^2)ds_2 dz_2 \\
& +\frac{1}{2}e^{-2(\phi+\psi)}y_1^2 y_2 ds_2 dz_1 - e^{-2(\phi+\psi)}y_1 y_2 dz_1 dz_2
\end{aligned}
\tag{21}
$$

We can compare the first 5 lines of (21) with (11). They are different, but the extra terms are zero when imposing the constraints

$$
z_2 = y_2 = t = \tilde{t} = \tilde{s}_1 = s_2 = 0.
$$

This means that there is an isometric embedding of $SO(2,3)/(SO(2) \times SO(3))$ in the manifold with the metric (21). We can improve this result by making use of a generalized contraction, that gives a simpler contracted metric. We will do that in the next section.

3.2 Generalized contractions: some examples.

Generalized contraction of $U(2,3)/(U(2) \times U(3))$

We consider the following decomposition of s_4,

$$\mathfrak{s}_4 = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3, \quad \text{where}$$
$$\mathfrak{g}_0 = \text{span}\{H_+, H_-, S_2^{(1,1)}, S_1^{(1,-1)}, Y^1, Z^1, T^{(2,0)}\}$$
$$\mathfrak{g}_1 = \text{span}\{S_1^{(1,1)}\} \qquad \mathfrak{g}_2 = \text{span}\{T^{(0,2)}, Z^2\} \qquad \mathfrak{g}_3 = \text{span}\{S_2^{(1,-1)}, Y^2\}$$

and the linear map

$$
\begin{array}{ccc}
\mathfrak{s}_4 & \xrightarrow{\phi_\epsilon} & \mathfrak{s}_4 \\
e_0 + e_1 + e_2 + e_3 & \longrightarrow & e_0 + \epsilon e_1 + \epsilon^2 e_2 + \epsilon^3 e_3
\end{array}
\qquad \text{with } e_i \in \mathfrak{g}_i.
$$

Equation (14) gives a deformed bracket that has a well defined limit when $\epsilon \to 0$. We write here the contracted bracket. The only surviving commutators from (4) when $\epsilon \to 0$ are

$$[Z^1, Y^1] = S_2^{(1,1)}$$
$$[Y^1, S_1^{(1,-1)}] = Z^1$$
$$[H_+, Z^i] = Z^i$$
$$[H_-, Y^i] = Y^i,$$

so the contracted algebra has as a subalgebra

$$\mathfrak{solv}\left(\frac{SO(2,3)}{SO(2) \times SO(3)}\right) = \text{span}\{H_+, H_-\} + \text{span}\{Z^1, Y^1, S_2^{(1,1)}, S_1^{(1,-1)}\}$$

in semidirect product with $\mathbb{R}^6 = \text{span}\{Z^2, Y^2, T^{(0,2)}, T^{2,0}, , S_1^{(1,1)}, S_2^{(1,-1)}\}$, where the only generators that act on $\mathbb{R}^6$ are H_+ and H_-.

We use the adjoint representation of the deformed algebra to compute

the deformed metric. The result is

$$ds^2 = d\phi^2 + d\psi^2 + e^{-4\psi}dtdt + 2e^{-4\psi}s_1\epsilon dtd\tilde{s}_1 + 2e^{-4\psi}s_2\epsilon^3 dtd\tilde{s}_2$$

$$+2e^{-4\psi}z_2\epsilon^2 dtdz_1 + 2e^{-4\psi}(s_2^2\epsilon^2 + s_1^2\epsilon^8)dtd\tilde{t} + \frac{1}{2}(e^{-2(\psi+\phi)} + 2e^{-4\psi}s_1^2\epsilon^2)d\tilde{s}_1 d\tilde{s}_1$$

$$+2e^{-4\psi}s_2 s_1\epsilon^4 d\tilde{s}_1 d\tilde{s}_2 + \frac{1}{2}e^{-2(\psi+\phi)}(y_1^2\epsilon^2 + y_2^2\epsilon^8)d\tilde{s}_1 ds_2 + e^{-2(\psi+\phi)}y_1 y_2\epsilon^2 d\tilde{s}_1 ds_1$$

$$-e^{-2(\psi+\phi)}y_1\epsilon d\tilde{s}_1 dz_2 + (2e^{-4\psi}s_1 z_2\epsilon^3 + e^{-2(\psi+\phi)}y_2\epsilon^2)d\tilde{s}_1 dz_1$$

$$+2e^{-4\psi}s_1(e^{2(\psi-\phi)}\epsilon + s_2^2\epsilon^9 + s_1^2\epsilon^3)d\tilde{s}_1 d\tilde{t} + \frac{1}{2}(e^{-2(\psi+\phi)} + 2e^{-4\psi}s_2^2\epsilon^6)d\tilde{s}_2 d\tilde{s}_2$$

$$+e^{-2(\psi+\phi)}y_1 y_2\epsilon^6 d\tilde{s}_2 ds_2 - \frac{1}{2}e^{-2(\psi+\phi)}(y_1^2 + y_2^2\epsilon^6)d\tilde{s}_2 ds_1$$

$$+e^{-2(\psi+\phi)}y_2\epsilon^5 d\tilde{s}_2 dz_2 + (2e^{-4\psi}s_2 z_2\epsilon^5 + e^{-2(\psi+\phi)}y_1)d\tilde{s}_2 dz_1$$

$$+2e^{-4\psi}s_2(e^{2(\psi-\phi)}\epsilon^5 + s_2^2\epsilon^{11} + s_1^2\epsilon^5)d\tilde{s}_2 d\tilde{t}$$

$$+\frac{1}{8}e^{-2(\psi+\phi)}(4e^{4\phi} + y_1^4\epsilon^4 + (4\epsilon^{12} + 2\epsilon^{10})y_1^2 y_2^2 + y_2^4\epsilon^{16} + 4e^{2\phi}(y_1^2\epsilon^2 + y_2^2\epsilon^{12}))ds_2 ds_2$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}y_1(2e^{2\phi}\epsilon + y_1^2\epsilon^3 - (2\epsilon^{11} - \epsilon^9)y_2^2)ds_2 dz_2 +$$

$$\frac{1}{2}e^{-2(\psi+\phi)}y_2(2e^{2\phi}\epsilon^6 + (\epsilon^2 + 2\epsilon^6)y_1^2 + y_2^2\epsilon^{10})ds_2 dz_1$$

$$+e^{-2(\psi+\phi)}(y_2^2 s_1\epsilon^9 + 2y_2 s_2 y_1\epsilon^{11} + s_1 y_1^2\epsilon^3)ds_2 d\tilde{t}$$

$$+\frac{1}{8}e^{-2(\psi+\phi)}(4e^{4\phi} + y_1^4 + (4\epsilon^4 + 2\epsilon^6)y_1^2 y_2^2 + y_2^4\epsilon^{12} + 4e^{2\phi}(y_1^2 + y_2^2\epsilon^2))ds_1 ds_1$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}y_2(2e^{2\phi}\epsilon + (2\epsilon^3 + \epsilon^5)y_1^2 + y_2^2\epsilon^{11})ds_1 dz_2$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}y_1(2e^{2\phi} + y_1^2 - (2\epsilon^4 - \epsilon^6)y_2^2)ds_1 dz_1$$

$$-e^{-2(\psi+\phi)}(y_1^2 s_2\epsilon^5 - 2y_1 s_1 y_2\epsilon^3 + s_2 y_2^2\epsilon^{11})ds_1 d\tilde{t} + \frac{1}{2}e^{-2(\psi+\phi)}(e^{2\phi} + y_1^2\epsilon^2 + y_2^2\epsilon^{10})dz_2 dz_2$$

$$-e^{-2(\psi+\phi)}(2y_1 s_1\epsilon^2 - 2s_2 y_2\epsilon^{10})dz_2 d\tilde{t} + \frac{1}{2}e^{-4\psi}(e^{2\psi} + 2z_2^2\epsilon^4 + e^{2(\psi-\phi)}(y_1^2 + y_2^2\epsilon^2))dz_1 dz_1$$

$$+(2e^{-4\psi}(s_2^2\epsilon^{10} + s_1^2\epsilon^4)z_2 + 2e^{-2(\psi+\phi)}(y_1 s_2\epsilon^3 + s_1 y_2\epsilon^5))dz_1 d\tilde{t}$$

$$+(e^{4\psi} + e^{4\phi}(s_1^4\epsilon^4 + 2s_1^2 s_2^2\epsilon^{10} + s_2^4\epsilon^{16}) + 2e^{2(\phi+\psi)}(s_1^2\epsilon^2 + s_2^2\epsilon^{10}))d\tilde{t}d\tilde{t} - 2e^{-4\phi}y_1\epsilon d\tilde{t}dy_2$$

$$+\frac{1}{2}e^{-4\phi}(e^{2\phi} + 2y_1^2\epsilon^2)dy_2 dy_2 + \frac{1}{2}e^{-2\phi}dy_1 dy_1 + e^{-2(\phi+\psi)}y_1 y_2(\epsilon^5 - \epsilon^3)dz_1 dz_2$$

$$+\frac{1}{2}e^{-2\psi}y_1 y_2(2(\epsilon^2 - \epsilon^6) + (\epsilon^4 - \epsilon^6)(y_1^2 + y_2^2\epsilon^6))ds_1 ds_2. \tag{22}$$

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And for $\epsilon = 0$ it becomes

$$
\begin{aligned}
ds^2 =\ & (d\phi^2 + d\psi^2 + \frac{1}{2}e^{-2(\psi+\phi)}d\tilde{s}_2^2 - \frac{1}{2}e^{-2(\psi+\phi)}y_1^2 d\tilde{s}_2 ds_1 \\
& + e^{-2(\psi+\phi)}y_1 d\tilde{s}_2 dz_1 + \frac{1}{8}e^{-2(\psi+\phi)}(4e^{4\phi} + y_1^4 + 4e^{2\phi}y_1^2)ds_1^2 \\
& - \frac{1}{2}e^{-2(\psi+\phi)}y_1(2e^{2\phi} + y_1^2)ds_1 dz_1 + \frac{1}{2}e^{-4\psi}(e^{2\psi} + e^{2(\psi-\phi)}y_1^2)dz_1^2 + \frac{1}{2}e^{-2\phi}dy_1^2) \\
& + e^{-4\psi}dt^2 + e^{-4\phi}d\tilde{t}^2 + \frac{1}{2}e^{-2(\phi+\psi)}d\tilde{s}_1^2 + \frac{1}{2}e^{2(\phi-\psi)}ds_2^2 \\
& + \frac{1}{2}e^{-2\phi}dy_2^2 + \frac{1}{2}e^{-2\psi}dz_2^2
\end{aligned}
$$

The first three lines reproduce (11) for $n = 1$, and the rest of the terms are flat up to factors $e^{(a\phi+b\psi)}$. The physical meaning of this limit remains unclear at this moment, but it relates two different sigma models in what can be a generalized procedure of integrating out some modes.

Generalized contraction of $U(2, 1 + n)/(U(2) \times U(1 + n))$

We show here another example of generalized contraction that has an application in a physically interesting theory.

Let us denote $H_1 = H_+ + H_-$. Then the commutation rules of $\mathfrak{s}_2$ are

$$
[H_1, Z^{1a}] = Z^{1a}, \quad [H_1, Y^{1a}] = Y^{1a}, \quad [H_1, S_2^{(1,1)}] = 2S_2^{(1,1)}, \quad [Z^{1a}, Y^{1b}] = S_2^{(1,1)}.
$$

Consider the subalgebras of $\mathfrak{s}_4$

$$
\mathfrak{s}_2' = \mathrm{span}\{H_+, Z^{ia}, T^{(2,0)}\} \qquad \mathfrak{s}_2'' = \mathrm{span}\{H_-, Y^{ia}, T^{(0,2)}\},
$$

with commutation rules

$$
\begin{aligned}
&[H_+, Z^{ia}] = Z^{ia}, \quad [H_+, T^{(2,0)}] = 2T^{(2,0)}, \quad [Z^{ia}, Z^{jb}] = \delta^{ab}\epsilon^{ij}T^{(2,0)}, \quad \text{for } \mathfrak{s}_2', \\
&[H_-, Y^{ia}] = Y^{ia}, \quad [H_-, T^{(0,2)}] = 2T^{(0,2)}, \quad [Y^{ia}, Y^{jb}] = \delta^{ab}\epsilon^{ij}T^{(0,2)}, \quad \text{for } \mathfrak{s}_2''.
\end{aligned}
$$

We have that $\mathfrak{s}_2 \simeq \mathfrak{s}_2' \simeq \mathfrak{s}_2''$ but $[\mathfrak{s}_2', \mathfrak{s}_2''] \neq 0$, so $\mathfrak{s}_2' \oplus \mathfrak{s}_2''$ is not a subalgebra of $\mathfrak{s}_4$. Nevertheless, one can find a generalized contraction of $\mathfrak{s}_4$ which has $\mathfrak{s}_2' \oplus \mathfrak{s}_2''$ as a subalgebra. We consider the decomposition,

$$
\mathfrak{s}_4 = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2
$$

$$
\mathfrak{g}_0 = \mathrm{span}\{H_+, H_-\}, \quad \mathfrak{g}_1 = \mathrm{span}\{Y^{ia}, Z^{ia}, S_\alpha^{(1,1)}\}, \quad \mathfrak{g}_2 = \{T^{(0,2)}, T^{(2,0)}, S_\alpha^{(1,-1)}\},
$$

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and the linear map

$$
\begin{array}{ccc}
\mathfrak{s}_4 & \xrightarrow{\phi_\epsilon} & \mathfrak{s}_4 \\
e_0 + e_1 + e_2 & \longrightarrow & e_0 + \epsilon e_1 + \epsilon^2 e_2
\end{array}
\qquad e_i \in \mathfrak{g}_i.
\tag{23}
$$

The contracted Lie algebra, with commutator given by 14) is well defined. It is worthy to see the commutators of the contracted algebra:

$$
\begin{aligned}
&[Z^{ia}, Z^{jb}]_\epsilon = \epsilon^{ij} \delta^{ab} T^{(2,0)} \\
&[Y^{ia}, Y^{jb}]_\epsilon = \epsilon^{ij} \delta^{ab} T^{(0,2)} \\
&[Z^{ia}, Y^{jb}]_\epsilon = \epsilon \, \delta^{ab} (\delta^{ij} S_2^{(1,1)} + \epsilon^{ij} S_1^{(1,1)}) \\
&[Y^{ia}, S_1^{(1,-1)}]_\epsilon = \epsilon^2 \, Z^{ia} \\
&[Y^{ia}, S_2^{(1,-1)}]_\epsilon = \epsilon^2 \, \epsilon^{ij} Z^{ja} \\
&[T^{(0,2)}, S_\alpha^{(1,-1)}]_\epsilon = \epsilon^3 \, 2 S_\alpha^{(1,1)} \\
&[S_\alpha^{(1,1)}, S_\beta^{(1,-1)}]_\epsilon = \epsilon \, \delta_{\alpha\beta} T^{(2,0)} \\
&[H_+, Z^{ia}]_\epsilon = Z^{ia} \\
&[H_-, Y^{ia}]_\epsilon = Y^{ia}
\end{aligned}
\tag{24}
$$

showing explicitly $\mathfrak{s}_2' \oplus \mathfrak{s}_2''$ as a subalgebra when $\epsilon \to 0$. We use the adjoint representation to compute the metric, as in the previous examples. The result is

$$ds^2 = d\phi^2 + d\psi^2 + e^{-4\psi}dtdt + 2e^{-4\psi}\epsilon s_1 dtd\tilde{s}_1 + 2e^{-4\psi}\epsilon s_2 dtd\tilde{s}_2$$

$$+2e^{-4\psi}z_{2a}dtdz_{1a} + 2e^{-4\psi}\epsilon^4(s_2^2 + s_1^2)dtd\tilde{t} + \frac{1}{2}(e^{-2(\psi+\phi)} + 2e^{-4\psi}\epsilon^2 s_1^2)d\tilde{s}_1 d\tilde{s}_1$$

$$+2e^{-4\psi}\epsilon^2 s_2 s_1 d\tilde{s}_1 d\tilde{s}_2 + \frac{1}{2}e^{-2(\psi+\phi)}\epsilon^3(y_1^2 + y_2^2)d\tilde{s}_1 ds_2 + e^{-2(\psi+\phi)}\epsilon^3 y_{1a}y_{2a}d\tilde{s}_1 ds_1$$

$$-e^{-2(\psi+\phi)}\epsilon y_{1a}d\tilde{s}_1 dz_{2a} + \epsilon(2e^{-4\psi}s_1 z_{2a} + e^{-2(\psi+\phi)}y_{2a})d\tilde{s}_1 dz_{1a}$$

$$+2e^{-4\psi}\epsilon^3 s_1(e^{2(\psi-\phi)} + \epsilon^2(s_2^2 + s_1^2))d\tilde{s}_1 d\tilde{t} + \frac{1}{2}(e^{-2(\psi+\phi)} + 2e^{-4\psi}\epsilon^2 s_2^2)d\tilde{s}_2 d\tilde{s}_2$$

$$+e^{-2(\psi+\phi)}\epsilon^3 y_{1a}y_{2a}d\tilde{s}_2 ds_2 - \frac{1}{2}e^{-2(\psi+\phi)}\epsilon^3(y_1^2 + y_2^2)d\tilde{s}_2 ds_1$$

$$+e^{-2(\psi+\phi)}\epsilon y_{2a}d\tilde{s}_2 dz_{2a} + \epsilon(2e^{-4\psi}s_2 z_{2a} + e^{-2(\psi+\phi)}y_{1a})d\tilde{s}_2 dz_{1a}$$
$$+2e^{-4\psi}\epsilon^3 s_2(e^{2(\psi-\phi)} + \epsilon^2(s_2^2 + s_1^2))d\tilde{s}_2 d\tilde{t}$$
$$+\frac{1}{8}e^{-2(\psi+\phi)}(4e^{4\phi} + 4e^{2\phi}\epsilon^4(y_1^2 + y_2^2) + 4\epsilon^6(y_{1a}y_{2a})^2 + \epsilon^6(y_1^2 + y_2^2)(y_{1b}^2 + y_{2b}^2))ds_\alpha ds_\alpha$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}\epsilon^2(2e^{2\phi}y_{1b} + \epsilon^2(-2(y_{1a}y_{2a})y_{2b} + (y_1^2 + y_2^2)y_{1b}))ds_2 dz_{2b}$$

$$+\frac{1}{2}e^{-2(\psi+\phi)}\epsilon^2(2e^{2\phi}y_{2b} + \epsilon^2(2(y_{1a}y_{2a})y_{1b} + (y_1^2 + y_2^2)y_{2b}))ds_2 dz_{1b}$$

$$+e^{-2(\psi+\phi)}\epsilon^6(y_2^2 s_1 + 2y_{2a}s_2 y_{1a} + s_1 y_1^2)ds_2 d\tilde{t}$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}\epsilon^2(2e^{2\phi}y_{2b} + \epsilon^2(2(y_{1a}y_{2a})y_{1b} + (y_1^2 + y_2^2)y_{2b}))ds_1 dz_{2b}$$

$$-\frac{1}{2}e^{-2(\psi+\phi)}\epsilon^2(2e^{2\phi}y_{1b} + \epsilon^2(-2(y_{1a}y_{2a})y_{2b} + (y_{1a}^2 + y_{2a}^2)y_{1b}))ds_1 dz_{1b}$$

$$-e^{-2(\psi+\phi)}\epsilon^6(y_{1a}^2 s_2 - 2y_{1a}s_1 y_{2a} + s_2 y_{2a}^2)ds_1 d\tilde{t}$$

$$-\frac{1}{2}e^{-2(\phi+\psi)}\epsilon_{ij}\epsilon_{mn}\epsilon^2(y_{ia}y_{jb})dz_{ma}dz_{nb}$$

$$+\frac{1}{2}e^{-2(\psi+\phi)}(e^{2\phi}\delta_{ab} + \epsilon^2(y_{1a}y_{1b} + y_{2a}y_{2b}))dz_{2a}dz_{2b}$$

$$-e^{-2(\psi+\phi)}\epsilon^4(2y_1 s_1 - 2s_2 y_2)dz_2 d\tilde{t}$$

$$+\frac{1}{2}e^{-4\psi}(e^{2\psi}\delta_{ab} + 2z_{2a}z_{2b} + e^{2(\psi-\phi)}\epsilon^2(y_{1a}y_{1b} + y_{2a}y_{2b}))dz_{1a}dz_{1b}$$

$$+\epsilon^4(2e^{-4\psi}(s_2^2 + s_1^2)z_{2a} + 2e^{-2(\psi+\phi)}(y_{1a}s_2 + s_1 y_{2a}))dz_{1a}d\tilde{t}$$
$$+e^{-4(\psi+\phi)}(e^{4\psi} + e^{2(\phi+\psi)}\epsilon^6(s_1^2 + s_2^2) + e^{4\phi}\epsilon^8(s_1^2 + s_2^2)^2)d\tilde{t}d\tilde{t} - 2e^{-4\phi}y_{1a}d\tilde{t}dy_{2a}$$
$$+\frac{1}{2}e^{-4\phi}(e^{2\phi}\delta_{ab} + 2y_{1a}y_{1b})dy_{2a}dy_{2b} + \frac{1}{2}e^{-2\phi}dy_{1a}dy_{1a} \tag{25}$$

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In the contraction limit $\epsilon \to 0$ the metric reduces to

$$
\begin{aligned}
ds^2 = \ & \big(d\phi^2 + e^{-4\phi}d\tilde{t}d\tilde{t} - 2e^{-4\phi}y_{1a}d\tilde{t}dy_{2a} + \\
& \frac{1}{2}e^{-4\phi}(e^{2\phi}\delta_{ab} + 2y_{1a}y_{1b})dy_{2a}dy_{2b} + \frac{1}{2}e^{-2\phi}dy_{1a}dy_{1a}\big) + \\
& \big(d\psi^2 + e^{-4\psi}dtdt + 2e^{-4\psi}z_{2a}dtdz_{1a} + \\
& \frac{1}{2}e^{-4\psi}(e^{2\psi}\delta_{ab} + 2z_{2a}z_{2b})dz_{1a}dz_{1b} + \frac{1}{2}e^{-2\psi}dz_{2a}dz_{2a}\big) + \\
& +\frac{1}{2}e^{-2(\psi+\phi)}d\tilde{s}_\alpha d\tilde{s}_\alpha + \frac{1}{2}e^{-2(\psi+\phi)}ds_\alpha ds_\alpha.
\end{aligned}
$$

By comparison with (12) we can see, after a suitable renaming of the coordinates and a rescaling by a global constant factor, that this is the metric on

$$
\left(\frac{\mathrm{SU}(1,1+n)}{\mathrm{U}(1+n)} \times \frac{\mathrm{SU}(1,1+n)}{\mathrm{U}(1+n)} \right) \ltimes \mathbb{R}^4.
$$

4 Super Higgs mechanism in Supergravity: geometric interpretation

We consider an $N = 2$ supergravity model coupled to $n + 2$ hypermultiplets and $n + 1$ vector multiplets. This model comes from a compactification of type IIB supergravity on the $N = 4$ orientifold $T^6/\mathbb{Z}_2$ [17, 18]. Such model, when certain fluxes are turned on, has an $N = 3$ phase obtained after the integration of the massive gravitino multiplet. The theory describing the $N = 3$ massles modes can be further Higgsed to an $N = 2$ phase by turning on other suitable fluxes and further integration [19]. The scalar manifold for the $N = 2$ theory is [10]

$$
\mathcal{M}_Q \times \mathcal{M}_{SK} = \frac{\mathrm{U}(2,2+n)}{\mathrm{U}(2) \times \mathrm{U}(2+n)} \times \frac{\mathrm{U}(1,1+n)}{\mathrm{U}(1) \times \mathrm{U}(1+n)}.
$$

Here n refers to the brane degrees of freedom. The special geometry and symplectic basis that describe this model have been discussed in Ref. [12].

The $N = 2$ model can also be Higgsed to N=1,0 phases by still turning on fluxes. This corresponds in the supergravity language to gauge two translational isometries of the quaternionic manifold. In the parametrization that

we have used before (25) it is manifest that the coordinates $\tilde{s}_\alpha$ correspond to two translational isometries, generated by $S_\alpha^{(1,1)}$, and we used the two bulk vector fields to gauge them.

When the gauge interactions are switched on, the σ-model lagrangian gets modified by the minimal coupling prescription, with

$$d\tilde{s}_\alpha \rightarrow D\tilde{s}_\alpha = d\tilde{s}_\alpha + k_{\alpha,\Lambda}A^\Lambda, \qquad \Lambda = 0, 1, \ldots n+1$$

and we may choose the constants $k_{1,0} \neq 0$, $k_{2,1} \neq 0$ and the rest zero. Then the Higgs mechanism takes place, as we described in Section 2.4, with the $\tilde{s}_\alpha$ contributing to the longitudinal components of the massive vectors

$$\hat{A}_\mu^0 = A_\mu^0 + \frac{1}{k_{1,0}}\partial_\mu\tilde{s}_1, \qquad \hat{A}_\mu^1 = A_\mu^1 + \frac{1}{k_{2,1}}\partial_\mu\tilde{s}_2.$$

From Eq. (6) we can see that the kinetic term of these modes is

$$ds^2 = \quad \cdots + e^{-4\psi}\left(s_\alpha D\tilde{s}_\alpha + (s_1^2 + s_2^2)d\tilde{t} + dt + z_{2a}dz_{1a}\right)^2 +$$
$$\frac{1}{2}e^{-2(\psi+\phi)}\left(D\tilde{s}_1 + y_{1a}y_{2a}ds_1 + \frac{1}{2}(y_1^2 + y_2^2)ds_2 + 2s_1 d\tilde{t} - \epsilon_{ij}y_{ia}dz_{ja}\right)^2 +$$
$$\frac{1}{2}e^{-2(\psi+\phi)}\left(D\tilde{s}_2 + y_{1a}y_{2a}ds_2 - \frac{1}{2}(y_1^2 + y_2^2)ds_1 + 2s_2 d\tilde{t} + \delta_{ij}y_{ia}dz_{ja}\right)^2 + \cdots$$

After the substitution
$$D\tilde{s}\alpha \rightarrow k_{\alpha,\Lambda}\hat{A}^\Lambda = B^\Lambda$$

the kinetic term for the vectors remains unchanged, while mass terms appear for the vectors $\hat{A}^\Lambda$, with masses $k_{0,1}$ and $k_{1,2}$. The modes $\tilde{s}_\alpha$ disappear from the Lagrangian.

In the large mass limit, the massive fields $B_\mu^\Lambda = m\hat{A}_\mu^\Lambda$ appear in the Lagrangian through expressions of the type

$$(B_\mu + f_\mu)^2,$$

where f_μ is some interaction of the massless modes [10]. The B's are Lagrange multipliers and their equations of motions make these terms to vanish.

The $N = 2$ gauged theory has a scalar potential stabilizing two additional scalars which in our parametrization, correspond to the coordinates s_α [13]. These fields acquire also a mass through the potential. In the large mass

limit, these fields become Lagrange multipliers and the potential is such that their field equations set them to zero.

After performing these integrations, the metric becomes the one of the symmetric space

$$\frac{U(1, 1+n)}{U(1) \times U(1+n)} \times \frac{U(1, 1+n)}{U(1) \times U(1+n)}.$$

Now we see that this example fits with the contraction performed in Section 3.2. We can see that the terms set to zero in the metric by taking the limit $\epsilon \to 0$ are precisely the terms eliminated by the integration procedure. The modes that have become massive are the modes in $\mathbb{R}^4$ in the decomposition

$$\mathfrak{solv}\left(\frac{U(2, 2+n)}{U(2) \times U(2+n)}\right) \to \mathfrak{solv}\left(\frac{U(1, 1+n)}{U(1) \times U(1+n)} \times \frac{U(1, 1+n)}{U(1) \times U(1+n)}\right) \ltimes \mathbb{R}^4.$$

Since $\mathbb{R}^4$ is an invariant subgroup of the contracted group, the quotient

$$\mathfrak{solv}\left(\frac{U(1, 1+n)}{U(1) \times U(1+n)} \times \frac{U(1, 1+n)}{U(1) \times U(1+n)}\right) \ltimes \mathbb{R}^4 / \mathbb{R}^4$$

$$\approx \mathfrak{solv}\left(\frac{U(1, 1+n)}{U(1) \times U(1+n)} \times \frac{U(1, 1+n)}{U(1) \times U(1+n)}\right),$$

is a (solvable) group, associated to the symmetric space. So in the geometrical picture the integration of the massive modes is again modeled by a contraction and a quotient by an invariant subgroup.

A About solvable Lie algebras and translational isometries

A Lie algebra $\mathfrak{s}$ is solvable if the chain of ideals

$$\mathfrak{s}^{(0)} = \mathfrak{s}, \quad \mathfrak{s}^{(1)} = [\mathfrak{s}, \mathfrak{s}], \dots, \mathfrak{s}^{(p)} = [\mathfrak{s}^{(p-1)}, \mathfrak{s}^{(p-1)}], \dots$$

has $\mathfrak{s}^{(p+1)} = 0$ for some integer p. It is possible to prove [4] that a Lie algebra $\mathfrak{s}$ is solvable if and only if there is a chain of ideals $\mathfrak{i}_{i+1} \subset \mathfrak{i}_i$ with $\mathfrak{i}_i / \mathfrak{i}_{i+1}$ an abelian algebra, $\mathfrak{i}_0 = \mathfrak{s}$ and $\mathfrak{i}_{p+1} = 0$ for some p. It is clear that $\mathfrak{i}_p = 0$ is an abelian ideal.

Example A.1

As an example, let us consider $\mathfrak{s} = \mathfrak{s}_4$ so

$$
\begin{aligned}
\mathfrak{i}_0 &= \mathfrak{s}_4 \\
\mathfrak{i}_1 = [\mathfrak{i}_0, \mathfrak{i}_0] &= \operatorname{span}\{Z^{ia}, Y^{ia}, T^{2,0}, T^{0,2}, S_\alpha^{(1,1)}, S_\alpha^{(1,-1)}\} \\
\mathfrak{i}_2 = [\mathfrak{i}_1, \mathfrak{i}_1] &= \operatorname{span}\{Z^{ia}, T^{2,0}, T^{0,2}, S_\alpha^{(1,1)}\} \\
\mathfrak{i}_3 = [\mathfrak{i}_2, \mathfrak{i}_2] &= \operatorname{span}\{T^{2,0}\}.
\end{aligned}
$$

Notice that it is possible to substitute $\mathfrak{i}_3$ in the chain by the maximal abelian ideal

$$
\mathfrak{i}'_3 = \operatorname{span}\{T^{2,0}, T^{0,2}, S_\alpha^{(1,1)}, Z^{1a}\},
$$

or by this other one (with the same dimension)

$$
\mathfrak{i}''_3 = \operatorname{span}\{T^{2,0}, T^{0,2}, S_\alpha^{(1,1)}, Z^{2a}\},
$$

so the chain is not unique.

∎

One can also show that $\mathfrak{s}^{(1)}$ is a nilpotent Lie algebra. The unique simply connected group associated to a nilpotent Lie algebra is exponential (the exp map is a diffeomorphism of the Lie algebra into the Lie group) [4].

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{t}$ an abelian subalgebra. Let $\{X_i\}$ be a basis of $\mathfrak{t}$ and $\{Y_\alpha\}$ a basis of a complementary space to $\mathfrak{t}$. In a neighborhood of the identity, we have the exponential map

$$
L(u^i, v^\alpha) = e^{u^i X_i} e^{v^\alpha Y_\alpha}. \tag{26}
$$

The Maurer-Cartan form is

$$
L^{-1} dL = e^{-v^\alpha Y_\alpha} X_i e^{v^\alpha Y_\alpha} du^i + e^{-v^\alpha Y_\alpha} d(e^{v^\alpha Y_\alpha}).
$$

From this expression, one can see that the local expression of the Maurer-Cartan form does not depend on the coordinates u^i. Whenever the group G with Lie algebra $\mathfrak{g}$ is diffeomorphic to $\mathbb{R}^n \times M$, with $\mathbb{R}^n$ parametrized by u^i (in other words, the coordinates u^i are global), we will say that the generators X_i are *translational isometries*.

We consider now the solvable algebras associated to the non compact symmetric spaces by the Iwasawa decomposition and explore the translational isometries in the corresponding symmetric spaces. The solvable Lie algebras are always a semidirect product

$$\mathfrak{s} = \mathfrak{a} \ltimes \mathfrak{n},$$

where $\mathfrak{a}$ is abelian (it contains the non compact Cartan elements) and $\mathfrak{s}^{(1)} = \mathfrak{n}$ is the nilpotent part. The non compact symmetric spaces are simply connected, so they are, in each case, the unique simply connected group associated to the corresponding solvable algebra. We denote it by

$$S = A \ltimes N, \quad \text{with} \quad \mathrm{Lie}(A) = \mathfrak{a} \text{ and } \mathrm{Lie}(N) = \mathfrak{n},$$

A and N being simply connected as well[3] (and hence, exponential). As a manifold,

$$S = A \times N = \exp(\mathfrak{a}) \times \exp(\mathfrak{n}). \tag{27}$$

Let us now consider the factor N in (27). We want to prove that the generators in the abelian ideal are translational isometries. Let $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$, with $\mathfrak{n}_1$ an abelian ideal and $\mathfrak{n}_2$ any complementary subspace. We have that the map

$$\mathfrak{n} \quad \longrightarrow \quad N$$
$$(X_1, X_2) \quad \longrightarrow \quad \exp(X_1 + X_2)$$

is a diffeomorphism. We want to show that, equally, the map

$$\mathfrak{n} \quad \longrightarrow \quad N$$
$$(X_1, X_2) \quad \longrightarrow \quad \exp X_1 \exp X_2$$

is a diffeomorphism. It is enough to prove that obvious that any element $\exp(Y_1 + Y_2)$ can be written as $\exp X_1 \exp X_2$ for some $X_i \in \mathfrak{n}_i$. We notice that

$$\exp X_1 \exp X_2 = \exp(X_1 + X_2 + \frac{1}{2}[X_1, X_2] + \cdots) = \exp(X_2 + X_1'), \text{ with } X_1' \in \mathfrak{n}_1.$$

We take $Y_2 = X_2$, and the equation $Y_1 = X_1'$ can be solved for some $X_1 \in \mathfrak{n}_1$.

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