HIGHER ORDER ACTION FOR THE INTERACTION OF THE STRING WITH THE DILATON

M. A. Lledó* and A. Restuccia

Departamento de Física. Universidad Simón Bolívar.

Apartado 89000, Caracas 1080-A. Venezuela.

Abstract

The theory of the string in interaction with a dilaton background field is analyzed. In the action considered, the metric in the world sheet of the string is the induced metric, and the theory presents second order time derivatives. The canonical formalism is developed and it is showed that first and second class constraints appear. The degrees of freedom are the same than for the free bosonic string. The light cone gauge is used to reduce to the physical modes and to compute the physical hamiltonian.

* e-mail: mlledo@dino.conicit.ve

1. Introduction.

We consider the theory of the bosonic string in interaction with an scalar field, the dilaton. Much work has been done on the problem of constructing string theories on general background fields [1]. The theory describing the interaction of the bosonic string with the metric, the antisymmetric field and the dilaton, via the Polyakov approach, was recently studied in the excellent paper by Buchbinder, Fradkin, Lyakhovich y Pershin [2]. The theory they consider is given by the action

$$S = -\int d^2\xi\sqrt{-g}\{\frac{1}{2}g^{ab}\partial_a x^\mu \partial_b x^\nu G_{\mu\nu}(x) + \frac{1}{2}\epsilon^{ab}\partial_a x^\mu \partial_b x^\nu A_{\mu\nu}(x) +^{(2)} R\Phi(x)\}. \tag{1}$$

$G_{\mu\nu}(x)$ is the D-dimensional metric, $A_{\mu\nu}(x)$ is the antisymmetric field and $\Phi(x)$ is the dilaton. $g_{ab}(\xi)$ is the metric of the world-sheet of the string, and it is considered here as a variable, independent of the embeding $x^\mu(\xi)$. It is a Polyakov-type action. ${}^{(2)}R$ is the curvature of the two dimensional submanifold, associated to the metric g_{ab}. As it is well known, this theory can be consistently quantized provided the external fields satisfy certain restrictions. If the only background field is $G_{\mu\nu}(x)$, the theory is consistent in $D = 26$ and the metric satisfies up to linear order in the curvature the Einstein equations. Nevertheless if the dilaton is different from zero (different from constant, indeed) the critical dimension is $D = 25$ and the Einstein equations are modified. The dilaton, as expected, changes notably the classical and quantum behaviour of the system. On one hand the field equations of action (1) imply that the metric of the world sheet of the string is the induced metric only if $\Phi(x) = \text{ctt}$. (In two dimensions the last term in (1) is the Euler characteristic when $\Phi(x) = \text{ctt}$, so it becomes irrelevant to the field equations). So the presence of the dilaton changes this geometrical interpretation of the action. On the other hand, the degrees of freedoom of the theory are not the same in both situations. If $\Phi(x) \neq \text{ctt}$, the degrees of freedoom are $D-1$ (the space-time has dimension D); if $\Phi(x) = \text{ctt}$ the degrees of freedoom are $D-2$, as in the free bosonic string. This is because the term proportional to the curvature breaks the invariance of the action under the rescaling of the metric, unless it is an irrelevant, total derivative. So it appears another degree of freedoom, and the limit $\Phi(x) \rightarrow$ ctt is not smooth. The free bosonic string is not a good starting point in order to make a perturbative treatment of the background fields. In Ref. [2] the problem is solved considering a string in interaction with a non trivial dilaton as the base for the perturbative treatment.

The theory we are considering has a similar action, but now the metric $g_{ab}(\xi)$ is not an independent variable, but it corresponds to the induced metric on the two dimensional surface,

$$g_{ab}(\xi) = \partial_a x^\mu \partial_b x^\nu G_{\mu\nu}(x), \tag{2}$$

and the geometrical interpretation is guaranteed. It is a Nambu-Goto type action. Obviously both actions are not equivalent. The last one is more complicated, since the term containing the dilaton has higher derivatives. Indeed we will restrict ourselves to the case $G_{\mu\nu}(x) = \eta_{\mu\nu}$ and $A_{\mu\nu}(x) = 0$, it is, we will retain the dilaton as the only non trivial background field. This interaction is complex enough, and we expect a better understanding of the modifications that the dilaton produces compared with the free string.

In Section 2. we describe the canonical formalism for higher derivatives. In Section 3. we apply it to the string in interaction with the dilaton, obtaining the primary constraints. In Section 4. we compute the secondary constraints. In Section 5. the first class constraints are covariantly separated from the second class constraints and the degrees of freedoom of the theory are computed. In Section 6. we proceed to fix the light cone gauge and to compute the Hamiltonian, comparing with other approaches. By using the light cone gauge we do not need to impose any restriction on the background fields. It is known that starting from the lagrangean approach of a string in a background, the light cone gauge is an admissible gauge provided the background is restricted. In particular $G_{\mu\nu}$ must be a pp-wave. However, in the phase space approach we follow in this paper, no conditions on Killing vectors of the background is required. The problem is that the only dependence on the transverse momentum becomes non quadratic because the dependence of the background with the coordinate x^-. This feature does not allow the functional integration of the transverse momentum to recover the lagrangean approach for arbitrary background. However, this condition on the background is not a requirement of the canonical formulation. Finally, in Section 7. we establish our conlusions.

2. Higher derivatives.

We use the generalization of the canonical formalism to higher derivatives proposed in Ref. [3][4]. For clearness, we briefly resum here the resut for a Lagrangean depending on second order time derivatives.

We consider a physical system on an n-dimmensional configuration space. Let $L(q,\dot{q},\ddot{q})$ be the Lagrangean of the system depending on the coordinates $(q^1,\dots,q^n)$ and their time derivatives of order one and two. The Euler Lagrange equations are obtained applying the Hamilton principle to the functional action,

$$S(q) = \int_{t_i}^{t_f} L(q,\dot{q},\ddot{q})dt. \tag{3}$$

This is equivalent to extremizing the constrained functional,

$$R(q,u,v) = \int_{t_i}^{t_f} L(q,u,v)dt, \tag{4}$$

subject to

$$\dot{q} = u, \quad \dot{u} = v. \tag{5}$$

The constraints are regular, so we can apply the Lagrange theorem and consider the unconstrained functional

$$\int_{t_i}^{t_f} [L(q,u,v) - p(u-\dot{q}) - \pi(v-\dot{u})]dt, \tag{6}$$

where we have introduced Lagrange multipliers p and π. The canonical moments of the coordinates (q,u) are the corresponding Lagrange multipliers (p,π). The

Hamiltonian can be read off from (6) as a function of two sets of canonically conjugated variables (q, p) and (u, π) and a set of non canonical ones v,

$$H(q, u, p, \pi, v) = pu + \pi v - L(q, u, v). \tag{7}$$

Doing independent variations of all the variables, one obtains canonical equations of motion for the canonical coordinates, and in addition one obtains the set of equations

$$\frac{\partial H}{\partial v} = 0. \tag{8}$$

If the Lagrangean is singular, the Hessian

$$\frac{\partial^2 H}{\partial v^i \partial v^j} = -\frac{\partial^2 L}{\partial v^i \partial v^j} \tag{9}$$

has rank $r < n$, and some of the equations (8) are primary constraints. The Dirac procedure to compute the complete set of constraints follows as usual, if constraints are regular. The components of v which cannot be calculated play the same role as the Lagrange multipliers associated to first class constraints.

In the next section we apply the formalism to the string in interaction with de dilaton.

3. Canonical action.

We denote $\xi^0 = \tau$ and $\xi^1 = \sigma$. A dot means a derivative with respect to τ and a prime a derivative with respect to σ. D is the dimension of space-time, whose metric is flat, with signature $(1, -1, ..., -1)$. We assume the conditions $g_{00} = \dot{x}^2 > 0$ y $g_{11} = x'^2 < 0$ hold. We denote ${}^{(2)}R$ simply by R, since there is not possibility of confussion.

The lagrangean action is

$$S = -\int d^2\xi \sqrt{-g}(1 + \alpha \Phi R). \tag{10}$$

If the space-time metric is flat, the curvature can be expressed in terms of the second fundamental form of the surface as

$$R = s_a^{ia} s_b^{ib} - s_b^{ia} s_a^{ib}. \tag{11}$$

where $s^i_{ab} \quad i = 1, \dots D-2$ are the components of the second fundamental form in an orthonormal base of the $D-2$ normal vectors to the surface, n^i_μ.

The orthonormal vectors satisfy

$$\begin{aligned} &(n^i n^j) - \delta^{ij} = 0 \\ &(n^i x') = 0 \\ &(n^i \dot{x}) = 0. \end{aligned} \tag{12}$$

and the second fundamental form is

$$s^i_{ab} = D_a D_b x^\mu n^i_\mu = x^\mu_{,a,b} n^i_\mu. \tag{13}$$

We have different expressions for the curvature. First, we can express it in terms of the covariant derivatives D_a, independently of the normal vectors,

$$R = g^{ab} g^{cd} D_a D_b x^\mu D_c D_d x_\mu - g^{ab} g^{cd} D_a D_c x^\mu D_b D_d x_\mu, \tag{14}$$

or in terms of them,

$$R = \frac{2}{g}(s^i_{00} s^i_{11} - (s^i_{01})^2) = \frac{2}{g}\big((x^\mu_{,0,0} n^i_\mu)(x^\nu_{,1,1} n^i_\nu) - (x^\nu_{,0,1} n^i_\nu)^2\big). \tag{15}$$

Both expressions are equivalent. The normal vectors $n^{i\mu}$ can be considered as independent variables only if constraints (12) are introduced in the action with Lagrange multipliers

$$\begin{aligned} &-\int d^2\xi \{\sqrt{-g} - \frac{2\alpha\Phi}{\sqrt{-g}}[(\ddot{x} n^i)(x' n^i) - (\dot{x}' n^i)^2] + \\ &+ \lambda_{ij}\big((n^i n^j) - \delta^{ij}\big) - \mu_i (n^i x') - \nu_i (n^i \dot{x})\}, \end{aligned} \tag{16}$$

where $\lambda_{ij}, \ \mu_i, \eta_i$ are the Lagrange multipliers associated with the constraints defining the new variables as an orthonormal system of normal vectors to the surface. This considerably simplifies the problem.

Let us compute the Euler-Lagrange equations. When varying the n^i one obtains relations which allow to compute the Lagrange multipliers. If these relations are introduced in the equation which results from varying x^μ, one obtains

$$
\begin{aligned}
&[\frac{(\dot{x}x')\dot{x}^\mu}{\sqrt{-g}} - \frac{\dot{x}^2 x'^\mu}{\sqrt{-g}}]' + [\frac{(\dot{x}x')x'^\mu}{\sqrt{-g}} - \frac{x'^2\dot{x}^\mu}{\sqrt{-g}}]^{\cdot} + \frac{2\alpha R\partial^\mu\Phi}{\sqrt{-g}} + \\
&+ [\frac{2\alpha\Phi'}{\sqrt{-g}}D_0\dot{x}^\mu - \frac{2\alpha\dot{\Phi}}{\sqrt{-g}}D_1\dot{x}^\mu]' + [\frac{2\alpha\dot{\Phi}}{\sqrt{-g}}D_1 x'^\mu - \frac{2\alpha\Phi'}{\sqrt{-g}}D_0 x'^\mu]^{\cdot} = 0.
\end{aligned}
\tag{17}
$$

The auxiliary variables $n^{i\mu}$ are eliminated. The two first terms in (17) are the field equations of the free bosonic string. The remaining terms depend on $\partial_\mu\Phi$, so, if the dilaton is a constant, the theory is equivalent to the free bosonic string.

The canonical analysis follows as in the previous section. We introduce new variables u^μ, v^μ, $p_\mu, \pi_\mu, m_{i\mu}, \gamma^{i\mu}$. The canonical action is

$$
S = \int d^2\xi\left[p\dot{x} + \pi\dot{u} + m_{i\mu}\dot{n}^{i\mu} - \mathcal{H}(x,u,n,\gamma,p,\pi,m)\right], \tag{18}
$$

where

$$
\begin{aligned}
&\mathcal{H}(x,u,n,\gamma,p,\pi,m) = \mathcal{H}_0 + v^\mu(\pi_\mu - \frac{2\alpha\Phi}{\sqrt{-g}}(x"n^i)n^i_\mu) + \gamma^{i\mu}m_{i\mu} + \\
&+ \lambda_{ij}\big((n^i n^j) - \delta^{ij}\big) - \mu_i(n^i x') - \nu_i(n^i\dot{x}),
\end{aligned}
\tag{19}
$$

and

$$
\mathcal{H}_0 = pu + \sqrt{-g} + \frac{2\alpha\Phi}{\sqrt{-g}}(u'n^i)^2. \tag{20}
$$

The variables $(v^\mu, \gamma^{i\mu}, \lambda_{ij}, \mu_i, \nu_i)$ act as multipliers. From here, we can read the primary constraints of the theory. For the following analysis, it is convenient to consider the decomposition of the Lagrange multiplier v^μ, in terms of its normal and tangential components,

$$
v^\mu = \omega^i n^{i\mu} + \Lambda_1 u^\mu + \Lambda_2 x'^\mu. \tag{21}
$$

In such way,

$$
v^\mu(\pi_\mu - \frac{2\alpha\Phi}{\sqrt{-g}}(x"n^i)n^i_\mu) = \omega^i(\pi_i n^i - \frac{2\alpha\Phi}{\sqrt{-g}}(x"n^i)) + \Lambda_1\pi u + \Lambda_2\pi x'. \tag{22}
$$

And the primary constraints are,

$$
\begin{aligned}
A^{ij} &:= (n^i n^j) - \delta^{ij} = 0 \\
B^i &:= (n^i x') = 0 \\
C^i &:= (n^i u) = 0 \\
D_{i\mu} &:= m_{i\mu} = 0 \\
\varphi^i &:= \pi_i n^i - \frac{2\alpha\Phi}{\sqrt{-g}}(x" n^i) = 0 \\
\psi_1 &:= \pi u = 0 \\
\psi_2 &:= \pi x' = 0.
\end{aligned}
\tag{23}
$$

Apart from the constraints determining the auxiliary variables, the moment π is completely constrained. The Hamiltonian is not zero on the primary constraints.

In the next section, we compute the secondary constraints.

4. Secondary constraints.

We compute the Poisson bracket of the Hamiltonian with all primary constraints. From the conservation of $A^{ik}, B^i, C^i, D_{j\nu}$ one obtains the Lagrange multipliers,

$$
\lambda_{kj} = -\frac{2\alpha\Phi}{\sqrt{-g}}(n^j u')(n^k u') \tag{24}
$$

$$
\begin{aligned}
\eta_j = &-\frac{2\alpha\Phi}{(-g)^{-3/2}} 2(n^j u')[((u'x')(ux') - x'^2(uu')]- \\
&- \omega^j \frac{2\alpha\Phi}{(-g)^{-3/2}}[((ux'')x'^2 - (ux')(x'x'')]
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\mu_j = &-\frac{2\alpha\Phi}{(-g)^{-3/2}} 2(n^j u')[((u'u')(ux') - u^2(x'u')]+ \\
&+ \omega^j \frac{2\alpha\Phi}{(-g)^{-3/2}}[((ux'')(ux') - u^2(x'x'')].
\end{aligned}
\tag{26}
$$

If $\gamma^{i\nu}$ is decomposed as

$$
\gamma^{i\nu} = \alpha^i u^\nu + \beta^i x'^\nu + \epsilon^{ik} n^{k\nu}, \tag{27}
$$

then one obtains

$$\alpha^i = \frac{\omega^i x'^2 - (n^i u')(ux')}{-g} \tag{28}$$

$$\beta^i = \frac{-\omega^i (ux') + (n^i u') u^2}{-g}. \tag{29}$$

The antisymmetric part of ϵ^{ik} remains undetermined, while the symmetric part is zero.

$$\epsilon^{\bar{ik}} = \frac{1}{2}(\epsilon^{ik} + \epsilon^{ki}) = 0 \tag{30}$$

The conservation of Ψ_1, Ψ_2 gives two secondary constraints,

$$\Psi_3 := \mathcal{H}_0 = pu + \sqrt{-g} + \frac{2\alpha\Phi}{\sqrt{-g}}(u'n^i)^2 = 0 \tag{31}$$

$$\Psi_4 := \pi u' + px' = 0 \tag{32}$$

It shows that the Hamiltonian is zero.The conservation of φ^i gives another secondary constraint,

$$\begin{aligned} \zeta^i := & -pn^i + \frac{2\alpha\Phi}{\sqrt{-g}}(n^i u'') + + \frac{2\alpha}{\sqrt{-g}}\big[2\Phi'(n^i u') - \dot{\Phi}(n^i x'')\big] + \\ & + \frac{2\alpha\Phi}{(-g)^{-3/2}}\big[(x'' n^i)\big(u^2(x'u') - (ux')(uu')\big) - (u'n^i)\big((ux')(ux'') - \\ & - u^2(x'x'') - 2(ux')(u'x') - 2x'^2(uu')\big)\big] = 0. \end{aligned} \tag{33}$$

The conservation of Ψ_3 and Ψ_4 is satisfied trivially. The conservation of ζ^i gives an equation for ω^i,

$$F^{ik}\omega^k + G^i = 0, \tag{34}$$

where

$$\begin{aligned} F^{ik} = & \delta^{ik}\frac{x'^2}{\sqrt{-g}} + \frac{2\alpha\Phi'}{(-g)^{3/2}}\big[(uu')x'^2 - (ux')(u'x')\big] - \delta^{ik}\frac{2\alpha\dot{\Phi}}{(-g)^{3/2}}\big[(ux'')x'^2 - \\ & - (ux')(x''x')\big] - \delta^{ik}\Big(\frac{2\alpha\Phi'}{\sqrt{-g}}\Big)' - \frac{2\alpha}{\sqrt{-g}}\big[(\partial_\mu\Phi n^{k\mu})(x''n^i) + (\partial_\mu\Phi n^{i\mu})(x''n^k)\big], \end{aligned} \tag{35}$$

and

$$G^i = \frac{u^2(x"n^i) - 2(ux')(u'n^i)}{\sqrt{-g}} + \frac{2\alpha}{\sqrt{-g}}(\partial_\mu \Phi n^{i\mu})(n^k u')^2 + \\ + 2\frac{2\alpha\Phi'}{(-g)^{3/2}}(n^i u')\big[u^2(x'u') - (ux')(uu')\big] + \\ + 2\frac{2\alpha\dot{\Phi}}{(-g)^{3/2}}(n^i u')\big[x'^2(uu') - (ux')(x'u')\big] + 2\frac{2\alpha}{\sqrt{-g}}(\partial_\mu \Phi u'^\mu)(n^i u') + \\ + \frac{2\alpha}{\sqrt{-g}}\big[2(\partial_\mu\partial_\nu\Phi x'^\mu u^\nu)(n^i u') - (\partial_\mu\partial_\nu\Phi u^\mu u^\nu) + (n^i x'')\big]. \tag{36}$$

The term independent of Φ in (35) is always different from zero, provided x' is a spatial vector. (34) is an algebraic equation which allows for the computation of ω^i (for example, one can suppose analiticity in α). It is complicated, but we are not going to use it explicitly. The important thing is that this Lagrange multiplier can be computed, and that the conservation of ζ^i gives no other secondary constraint. This is the complete set of constraints.

The constraints concerning the true variables of the theory can be resumed in two covariant expressions that do not involve the auxiliary variables. These expressions will be useful when fixing the gauge.

$$\varphi_\mu := \pi_\mu - \frac{2\alpha\Phi}{\sqrt{-g}} D_1 x'_\mu = 0 \tag{37}$$

is equivalent to $\Psi_1 = 0$, $\Psi_2 = 0$,$\varphi^i = 0$. Also,

$$\zeta_\mu = p_\mu + \frac{1}{\sqrt{-g}}\big[(ux')x'_\mu - x'^2 u_\mu\big] + \big[\frac{2\alpha(\partial_\nu\Phi u^\nu)}{\sqrt{-g}} + 2\frac{2\alpha\Phi}{(-g)^{-3/2}}\tilde{\Gamma}^1_{01}\big] D_1 x'_\mu - \\ - \big[2\frac{2\alpha\Phi}{(-g)^{-3/2}}\tilde{\Gamma}^1_{11} + 2\frac{2\alpha\Phi'}{\sqrt{-g}}\big] D_0 x'_\mu - 2\frac{2\alpha\Phi}{\sqrt{-g}}(D_0 x'_\mu)' - \frac{2\alpha\Phi}{(-g)^{-3/2}}\tilde{\Gamma}^1_{01} x''_\mu - \\ - \frac{2\alpha\Phi}{(-g)^{-3/2}}\big[(ux')x'_\mu - x'^2 u_\mu\big](u'^\nu D_0 x'_\nu) + \frac{2\alpha\Phi}{\sqrt{-g}} u"^{\perp}_{\mu} + \\ + \frac{2\alpha\Phi}{(-g)^{-3/2}}\big[u_\mu\big((ux'')(x'u') - (ux')(u'x'')\big) + x'_\mu\big(u^2(x''u') - (uu')(ux'')\big)\big] \tag{38}$$

is equivalent to $\Psi_3 = 0, \Psi_4 = 0, \zeta^i = 0$. We have used the Christoffel symbols of the metric g_{ab}, $\tilde{\Gamma}^c_{ab} = g\Gamma^c_{ab}$.

$$
\begin{aligned}
\tilde{\Gamma}^0_{00} &= x'^2(uv) - (ux')(vx') \\
\tilde{\Gamma}^1_{00} &= u^2(vx') - (ux')(uv) \\
\tilde{\Gamma}^0_{01} &= x'^2(uu') - (ux')(u'x') \\
\tilde{\Gamma}^1_{01} &= u^2(x'u') - (ux')(uu') \\
\tilde{\Gamma}^0_{11} &= x'^2(ux'') - (ux')(x'x'') \\
\tilde{\Gamma}^1_{11} &= u^2(x'x'') - (ux')(ux'').
\end{aligned} \tag{39}
$$

The covariant derivatives are,

$$
\begin{aligned}
D_0 x'^\mu &= u'^\mu - \Gamma^0_{01} u^\mu - \Gamma^1_{01} x'^\mu \\
D_1 x'^\mu &= x''^\mu - \Gamma^0_{11} u^\mu - \Gamma^1_{11} x'^\mu.
\end{aligned} \tag{40}
$$

and $u''^\perp$ is the normal part to u'' given by

$$
u''^{\perp\mu} = u''^\mu - \frac{1}{-g}[(u''x')(ux') - (u''u)x'^2]u^\mu - \frac{1}{-g}[(u''u)(ux') - (u''x)u^2]x'^\mu. \tag{41}
$$

These are the constraints one would have obtained if the original action, where the auxiliary variables are substituted, had been used. We can see that all the moments, p and π can be computed in terms of the coordinates x and u, so the degrees of freedom are notably reduced. We expect some constraints to be first class in order to contemplate the reparametrization invariance of the theory.

In the next section we study the character of these constraints and compute the degrees of freedoom of the theory.

5. First and second class constraints.

Between the constraints associated to the auxiliary variables, there is a first class constraint, corresponding to the undetermined Lagrange multiplier. This constraint is

$$
D_{\hat{i}\hat{k}} = \frac{1}{2}(D_{i\mu} n^\mu_k - D_{k\mu} n^\mu_i). \tag{42}
$$

The Poisson bracket of this constraint with the remaining ones is zero. The gauge transformation it generates only afects the variables n^i, and it is given by

$$\delta_\varepsilon n^{i\mu} = \varepsilon^{\hat{ik}} n^{k\mu}. \qquad (43)$$

The meaning of this transformation is an infinitesimal rotation in the space of normal vectors. The other constraints A^{ij}, B^i, C^i,$D_{\bar{ik}}$, $D_{i\mu}u^\mu$, and $D_{i\mu}x'^\mu$ are second class. All these constraints and the gauge invariance (43) determine $n^{i\mu}$ and $m_{i\mu}$ without ambiguity, so there is no dynamical variables. The remaining constraints restrict the true variables of the theory. We are going to elucidate which of them are first class.

The Hamiltonian we have computed,

$$\mathcal{H} = \Lambda_1\psi_1 + \Lambda_2\psi_2 + \tilde{\psi}_3 \qquad (44)$$

with

$$\begin{aligned}
\tilde{\Psi}_3 = \Psi_3 &- \frac{2\alpha\Phi}{\sqrt{-g}}(n^i u')(n^j u')A_{ij} - (\frac{2\alpha\Phi}{(-g)^{-3/2}}2(n^j u')[(u'x')(ux') - x'^2(uu')]+ \\
&+\omega^j \frac{2\alpha\Phi}{(-g)^{-3/2}}[((ux'')x'^2 - (ux')(x'x"))])B_j- \\
&-(\frac{2\alpha\Phi}{(-g)^{-3/2}}2(n^j u')[((u'u')(ux') - u^2(x'u')]- \\
&-\omega^j \frac{2\alpha\Phi}{(-g)^{-3/2}}[((ux'')(ux') - u^2(x'x"))])C_j+ \\
&+\frac{1}{-g}[\omega^i x'^2 - (n^i u')(ux')](D_{i\mu}u^\mu) + \frac{1}{-g}[-\omega^i(x'u) - (n^i u')u^2](D_{i\mu}x'^\mu)+ \\
&+\omega^i\varphi^i,
\end{aligned} \qquad (45)$$

and ω^i given by equation (34), is proportional to the first class constraints. Ψ_1 and Ψ_2 are first class, as follows from direct computation. The rest is a first class constraint we will call $\tilde{\Psi}_3$. In fact, one can show that Ψ_3 and Ψ_4 are not by themselves first class constraints. In order to convert them into first class constraints, they must be corrected with terms proportional to other second class constraints. We will substitute Ψ_3 by $\tilde{\Psi}_3$ in the set of constraints we are using to describe the submanifold.

In order to obtain the remaining first class constraints, we make an arbitrary linear combination of the second class constraints

$$< F >=< a_{kl}A^{kl} + b_k B^k + c_k C^k + d^{k\mu} D_{k\mu} + f_k \varphi^k + h_k \zeta^k + e\Psi_4 >, \qquad (46)$$

and compute the Poisson bracket of this constraint with the others. This procedure will determine the arbitrary coeficients until we have an arbitrary linear combination of first class constraints. The result is that they are all zero except

$$\begin{aligned} (d^i n^j) &= 0 \\ (d^i x') &= -(n^i x'')e \\ (d^i u) &= -(n^i u')e, \end{aligned} \qquad (47)$$

and e remains undetermined. The only first class constraint is

$$\begin{aligned} \tilde{\Psi}_4 = \Psi_4 + &+ \frac{1}{-g}[-(n^i u')x'^2 + (n^i x'')(ux')](D_{i\nu}u^\nu) + \\ &+ \frac{1}{-g}[-(n^i u')(ux') + (n^i x'')u^2](D_{i\nu}x'^\nu). \end{aligned} \qquad (48)$$

We have four first class constraints, so the degrees of freedoom of the string in interaction with the dilaton are $D-2$, the same as the free bosonic string. In the theory of Ref. [2] the interaction term breaks the Weyl invariance, so it appears an additional degree of freedoom. This problem is solved in this approach.

It is interesting to compare this result with the one for the rigid string [4][5]. The rigid string is a theory which also presents two dimensional reparametrization invariance, and presents second order derivatives in the Lagrangean. The canonical formulation leads to four first class constraints too, and the Poisson algebra of them does not contain the Virasoro algebra as a subalgebra. This result is achieved only when restricting to certain submanifold. For the rigid string no second class constraints appear, so the degrees of freedoom are twice the ones of the free string.

6. Light cone gauge.

We can impose four gauge fixing conditions. We take the light cone gauge, defined by

$$\begin{aligned} \chi_1 &= ux' \\ \chi_2 &= u^2 + x'^2 \\ \chi_3 &= x^+ - u_0^+ \tau \\ \chi_4 &= u^+ - u_0^+ . \end{aligned} \tag{49}$$

When $\Phi = 0$, the constraints reduce to

$$p = u, \quad \pi = 0 \tag{50}$$

χ_1 and χ_2 are the constraints of the free bosonic string while χ_3 and χ_4 correspond to the usual light cone gauge conditions. In this formulation one has more degrees of freedoom and we could select for χ_1 and χ_2 another conditions, different from the usual constraints px' and $p^2 + x'^2$. If the conditions were admissible, one would obtain an equivalent theory.

This gauge fixing allows the reduction to the physical modes, which are the transversal modes $x^\top, u^\top$, computing the longitudinal ones,

$$\begin{aligned} (ux') &= u^- x'^+ + u^+ x'^- - (ux')^\top \\ x'^- &= \frac{-(ux')^\top}{u^{+0}} \\ u^- &= \frac{-x'^{\top 2} + u^{\top 2}}{2u^{+0}}. \end{aligned} \tag{51}$$

We want to show now that (49) are admissible gauge fixing conditions, by computing the Lagrange multipliers. We take the total Hamiltonian, this is, proportional to all first class constraints,

$$\mathcal{H}_T = \Lambda_1 \Psi_1 + \Lambda_2 \Psi_2 + \Lambda_3 \tilde{\Psi}_3 + \Lambda_4 \tilde{\Psi}_4 \tag{52}$$

and compute the conservation of all gauge fixing conditions. The result is

$$\begin{aligned} &\Lambda_3 - 1 \\ &\Lambda_2 = -\frac{u^{+0} u'^- - (uu')^\top}{x'^{\top 2}} \\ &\Lambda_1 = -\frac{\Lambda_4' x'^{\top 2} + (u'x')^\top}{u^{+0} u^- + u^{\top 2}} \end{aligned} \tag{53}$$

and Λ_4 satisfies the equation

$$u^{+0}\Lambda_4' = -\omega^i n^{i+} - u^{+0}\frac{(u'x')^\top}{x'^{\top 2}}. \tag{54}$$

Regretfully, the light cone gauge does not leave the action in canonical form

$$S_{phys} = \int d^2\xi(-(p\dot{x})^\top - (\pi\dot{u})^\top + p^+\dot{x}^- + \pi^+\dot{u}^-), \tag{55}$$

so the computation of the Hamiltonian is not direct from here. Nevertheless, the equations of motion for the transversal modes $x^\top$, $u^\top$ are first order (in time) equations. We can compute the energy of the system as the conserved quantity associated to traslational invariance of S_{phys}. It only holds (in this gauge) if $\partial_+\Phi = 0$. This means that the light cone gauge is apropriate to compute the energy only in this case.

$$E = \int d\sigma\big[\frac{\delta\mathcal{L}}{\delta\dot{x}^\top}\dot{x}^\top + \frac{\delta\mathcal{L}}{\delta\dot{u}^\top}\dot{u}^\top - \mathcal{L}\big], \tag{56}$$

where

$$S_{phys} = \int d\sigma d\tau\mathcal{L}. \tag{57}$$

The result is,

$$E = \int d\sigma u^{+0}p^- \tag{58}$$

whith p^- expressed in terms of the physical modes

$$\begin{aligned} p^- =& \frac{1}{2u^{+0}}(x'^{\top 2} + u^{\top 2}) + 2\alpha\Phi x'^{\top 2}\cdot \\ &\big[(\partial_\mu\Phi x'^\mu)\big(-\frac{(x'x'')^\top}{2u^{+0}} - \frac{(uu')^\top}{u^{+0}} + \frac{x'^{\top 2}(x'u')^\top(x'u)^\top}{u^{+0}} + \frac{x'^{\top 2}(x'x'')^\top u^{\top 2}}{2u^{+0}}\big)- \\ &- (\partial_\mu\Phi u^\mu)\big(-\frac{(x'u')^\top}{2u^{+0}} - \frac{(ux'')^\top}{u^{+0}} + \frac{x'^{\top 2}(x'x'')^\top(x'u)^\top}{u^{+0}} + \frac{x'^{\top 2}(x'u')^\top u^{\top 2}}{2u^{+0}}\big)\big] \end{aligned} \tag{59}$$

The Hamiltonian only depends on Φ through his derivatives, so when $\Phi =$ctt one recovers the energy of the free bosonic string. If $\Phi \neq$ctt, in the energy appears a term proportional to α. This term produces the energy not being definite in sign.

We compare with the system treated in [2]. This system has a degree of freedoom more. The canonical coordinates one uses to describe it are $q^s = (x^\mu, \gamma)$, $p_s = (p_\mu, \pi)$, and the action is

$$S = \int d\sigma d\tau \left[p\dot{x} + \pi\dot{\gamma} - \lambda_0 T_0 - \lambda_1 T_1\right] \tag{60}$$

where T_0 and T_1 are the first class constraints which satisfy the Virasoro algebra. They are given by the following expressions

$$T_0 = \frac{1}{2} G^{rs} p_r p_s + \frac{1}{2} G_{rs} q'^r q'^s - 2(N_s q'^s)' \tag{61}$$

and

$$T_1 = p_s q^s - 2(N^s p_s)', \tag{62}$$

where we have used the notation $N_s = (-\partial_\mu \phi, 0)$, $Ns = (0, 1)$ and

$$G_{rs} = \begin{pmatrix} \eta_{\mu\nu} & -\partial_\mu \phi \\ -\partial_\nu \phi & 0 \end{pmatrix}. \tag{63}$$

If we fix the light cone gauge,

$$\begin{aligned} x^+ &= p^{+0}\tau, \\ p^+ &= p^{+0}, \end{aligned} \tag{64}$$

the physical Hamiltonian is p^-, which in terms of the physical modes is

$$\begin{gathered} p^- \left[p^{+0} - \frac{\partial^+ \phi^+ \pi}{(\partial\phi)^2}\right] = \\ \frac{1}{2}(p^{\top 2} + x'^{\top 2}) + \frac{\pi^2}{2(\partial\phi)^2} + \frac{p^{+0}\partial^- \phi\pi}{(\partial\phi)^2} + \frac{(\partial\phi p)^\top \pi}{(\partial\phi)^2} + \frac{\phi'\gamma'}{(\partial\phi)^2} - 2\phi'', \end{gathered} \tag{65}$$

and it is not positive definite.

Our conclussion is that the interaction term of the string and the dilaton must be corrected in order to obtain a consistent quantum theory.

7. Conclusions.

In this paper we obtained the canonical formulation of the string in interaction with a background dilaton field, with an action of Nambu-Goto type which has second order time derivatives. The complete set of constraints is computed, and it is found that four first class constraints appear, reflecting the reparametrization invariance of the lagrangean action. In addition, the theory is restricted by second class constraints. We decouple covariantly the first and second class constraints. However, because of the second class constraints, the covariant quantization of the system becomes intrincated. The degrees of freedoom of the theory are the same as for the free bosonic string, in distinction to the Polyakov type theory, which has only first order time derivatives. We had used the light cone gauge to reduce to the physical modes, and to compute the physical hamiltonian, which becomes indefinite in sign if the dilaton field is different from constant. It is well known that higher order terms in the curvature should be included in order to obtain the low energy approximation of a complete string theory. It is the Hamiltonian of the complete theory the one which is required to be positive definite. Our result, which clearly extend to the case when the other background fields are not trivial, shows that any conclusion based on an analysis of a truncated theory could be modified by higher order contributions. The theory is compared with other approaches.

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