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# The Minkowski and conformal superspaces

R. Fioresi [1]
*Dipartimento di Matematica, Università di Bologna*
*Piazza di Porta S. Donato, 5.*
*40126 Bologna. Italy.*
e-mail: fioresi@dm.UniBo.it

M. A. Lledó
*Departament de Física Teòrica, Universitat de València and IFIC*
*C/Dr. Moliner, 50, E-46100 Burjassot (València), Spain.*
e-mail: Maria.Lledo@ific.uv.es

and V. S. Varadarajan
*Department of Mathematics, UCLA.*
*Los Angeles, CA, 90095-1555, USA*
e-mail: vsv@math.ucla.edu

**Abstract**

We define complex Minkowski superspace in 4 dimensions as the big cell inside a complex flag supermanifold. The complex conformal supergroup acts naturally on this super flag, allowing us to interpret it as the conformal compactification of complex Minkowski superspace. We then consider real Minkowski superspace as a suitable real form of the complex version. Our methods are group theoretic, based on the real conformal supergroup and its Lie superalgebra.

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# Contents

# 1 Introduction

The superfield approach to supersymmetric physical theories is based on Minkowski superspace time. In the classical world the Poincaré group, which acts on Minkowski space, is a symmetry of any physically sensible theory. The Poincaré group is a non-semisimple group. The smallest simple group containing the Poincaré group in dimension 4 is the conformal group $\mathrm{SO}(4,2)$, or its double covering $\mathrm{SU}(2,2)$. Conformal symmetry is usually broken in nature. But fields of zero mass are conformally invariant, so there may be regimes where this symmetry can be restored. Minkowski space time is not enough to support conformal symmetry, and it needs to be *compactified* by adding points at infinity to obtain a space with an action of the conformal group. It turns out that the compactified Minkowski space is in fact a real form of the complex Grassmannian of two planes in a complex four dimensional vector space (see Ref. [1] for a careful treatment of this picture.)

The complex Grassmannian is a homogeneous space for the complex group $\mathrm{SL}(4,\mathbb{C})$ and so has a cellular decomposition derived from the Bruhat decomposition of the complex group. Complex Minkowski space time is thus the *big cell* inside the Grassmannian. One can go from conformal symmetry to Poincaré symmetry and *vice versa* by deleting or adding the points at infinity. In a way, the conformal description may be useful even when the conformal group is not a symmetry. This was one of the justifications behind the seminal paper by Penrose about what he called the *twistor* approach to Minkowski space time [2].

In the super world this picture can be generalized completely. Supertwistors were introduced in Ref. [8]. The purpose of this paper is to give a mathematically careful description of this generalization. We concentrate on Minkowski space time of dimension 4 and signature (1,3)[2] and consider its extension to a superspace of odd dimension 4, which means, in terms known to physicist, that we have $N = 1$ supersymmetry carried by a Majorana spinor of four components.

The case of even dimension four is singled out for physical reasons. Most of the treatment can be appropriately generalized to higher dimensions (and other signatures). Nevertheless the occurrence of Grassmannian and flag manifolds is peculiar to dimension four. The extension to higher supersymmetries does not present a problem, and we have chosen $N = 1$ for concrete-

[2] This is the signature of the flat metric in Minkowski space.

ness.

The discovery that superconformal spacetime is a *flag* super manifold, rather than a Grassmannian supermanifold, as one may naively be led to believe from what happens in the classical case[3], is due to Manin [1] and Kotrla and Niederle [12]. Working inside a flag super manifold is necessary to obtain the correct real form. This marks a deep difference between the classical and the super versions. We refer also to Ref. [13] for a quite complete review on superspaces associated with four dimensional Minkowski spacetime.

It is our goal to give a group theoretical view of this construction. While it is standard in the classical case, the supergroup version presents many peculiarities that have to be worked out. We construct the super flag explicitly as a quotient space of the superconformal group, keeping always the action of the supergroup explicit. This leads us to a general discussion of homogeneous spaces of supergroups which had not been rigourously treated in the literature before. We obtain the homogeneous space $G/H$, with $G \subset H$, both supergroups, as the unique supermanifold satisfying some natural properties.

The definition of Minkowski and conformal superspaces is not arbitrary, since they are determined by physical requirements. In fact, we provide in precise form the requirements that determine uniquely the Poincaré and translation superalgebras inside the conformal superalgebra, both over the complex and the real fields. The correct real form is obtained as the fixed points of a conjugation that we give explicitly. This solves the problem at the infinitesimal level, since the dual of the translation superalgebra can be identified with the infinitesimal Minkowski superspace.

The passage to the global theory is done by constructing the flag supermanifold as an homogeneous superspace, reducing infinitesimally to the Lie algebra decomposition found before. The Minkowski superspace is then the big cell inside the flag supermanifold. The real form is obtained by lifting the conjugation found in the Lie superalgebra to the supergroup and then reducing it to the homogeneous space. We also obtain the real form of the Minkowski superspace from the real form of the flag suepermanifold. This provides a significantly different and more natural description than the one appearing in the above mentioned papers [1, 12].

We want the paper to be as self contained as possible, so we include a review of the classical case in Section 2 and we explain the language of

[3]A Grassmannian supermanifold would be useful in describing *chiral fields* [11], which are not real.

supergeometry in some detail in the Appendix A. Also, we have tried to work out every result explicitly in coordinates, as to make contact with the physics notation.

The paper is organized as follows.

In Section 2 we review the construction of classical Minkowski space time in four dimensions. We start with the complex version, viewed as the big cell inside the Grassmannian of two planes in $\mathbb{C}^4$. The complex conformal group is $\mathrm{SL}(4,\mathbb{C})$, which acts naturally on the Grassmannian. One can see the complexified Poincaré group (including dilations) as the subgroup of $\mathrm{SL}(4,\mathbb{C})$ leaving the big cell invariant. Then the Grassmannian is the conformal compactification of the complex Minkowski space. We then describe the conjugation that leads to the correct real form of the complex Minkowski space time.

In Section 3 we take the classical theory of homogeneous spaces and extend it to the super setting. We make a precise definition of what a homogeneous space of a supergroup is and, using the results of Appendices A.1 and A.2, we give to it the structure of supermanifold, together with an action of the supergroup on it. We also find its functor of points.

In Section 4 we consider the problem at the infinitesimal level. The Poincaré Lie superalgebra can be defined through a set of natural conditions. Then we show that there exists a unique subalgebra of the conformal superalgebra satisfying these conditions. In fact, the conformal superalgebra splits as the direct sum (as vector spaces) of the Poincaré superalgebra plus another subalgebra, the translation superalgebra of Minkowski space time, as it arises in physics (from where it takes the name).

The infinitesimal treatment is crucial to find the adequate real form, which is not completely straightforward in the super case. When passing to the global theory in Section 5 we will see that, in order to obtain the desired real form, it is not enough to consider a super Grassmannian, but we need a super flag manifold [1, 12]. In fact, one could generalize the classical construction naively in the complex case, but the reality properties of the spinor representations prevent it from having a real structure. These peculiarities depend heavily on the dimension and signature of the Minkowski space (see for example Ref. [5] for an account of super Poincaré and conformal superalgebras in arbitrary dimension and signature).

We can lift the conjugation found at infinitesimal level to the supergroup. In particular, it preserves the super Poincaré group, and it descends to the superflag seen as a homogeneous space. It also preserves the big cell thus defining the standard Minkowski superspace.

In Appendices A.1 and A.2 we recall some facts on superspaces and supermanifolds which will be needed through the discussion. We emphasize the functorial approach: a supermanifold can be given in terms of a representable functor, its *functor of points*. In particular, we provide a representability criterion which allows us to determine if a functor is the functor of points of a supermanifold. This result is new. We illustrate the general theory with two examples that are essential for us: the Grassmannian and the flag supermanifolds.

In Appendix A.3 we give a brief account on super Lie groups. Finally, given the importance of real forms of supermanifolds in this work, we devote Appendix A.4 to defining them carefully.

# 2 The complex Minkowski space time and the Grassmannian G(2,4)

This section is a summary of the facts that we will need from Ref. [14], Section 3.3. All the details and proofs of our statements, and many others, can be found there.

The basic idea is to describe complex Minkowski space time as a big cell of the Grassmannian $G(2,4)$ by finding the action of the Poincaré group as the subgroup of $\mathrm{GL}(4,\mathbb{C})$ which leaves the big cell invariant.

We consider the Grassmannian $G(2,4)$, the set of two dimensional subspaces or planes in $\mathbb{C}^4$. A plane can be given by two linearly independent vectors and we will denote it as $\pi = (a,b) = \mathrm{span}\{a,b\}$, where $a$ and $b$ are column vectors. Obviously if $\mathrm{span}\{a,b\} = \mathrm{span}\{a',b'\}$ they define the same point of the Grassmannian. There is a transitive action of $\mathrm{GL}(4,\mathbb{C})$ on $G(2,4)$,

$$g \in \mathrm{GL}(4,\mathbb{C}), \qquad g\pi = (ga, gb).$$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of $\mathbb{C}^4$ and consider the plane $\pi_0$ spanned by the vectors $e_1$ and $e_2$. The isotropy group of $\pi_0$ is

$$P_0 = \left\{ \begin{pmatrix} L & M \\ 0 & R \end{pmatrix} \in \mathrm{GL}(4,\mathbb{C}) \right\},$$

with $L, M, R$ being $2 \times 2$ matrices, $L$ and $R$ invertible. We have then that $G(2,4) \approx \mathrm{GL}(4,\mathbb{C})/P_0$.

It is possible to use SL$(4,\mathbb{C})$ instead, since its action is also transitive.

Since the vectors $a$ and $b$ spanning a plane are linearly independent, the $4\times 2$ matrix

$$\pi=(a,b)=\begin{pmatrix} a_1 & b_1\\ a_2 & b_2\\ a_3 & b_3\\ a_4 & b_4\end{pmatrix}$$

has rank 2, and at least one of the 6 independent minors $y_{ij}=a_ib_j-b_ia_j$, $i\neq j$ is different from zero. The set

$$U_{ij}=\{(a,b)\ /\ y_{ij}\neq 0\} \tag{1}$$

is an open set of $G(2,4)$. It is easy to see that a plane in $U_{ij}$ can always be represented by

$$a=e_i+\alpha e_k+\gamma e_l, \qquad b=e_j+\beta e_k+\delta e_l,$$
$$\text{with } k\neq l \text{ and } k,l\neq i,j.$$

The six open sets $\{U_{ij}\}_{i<j}$ are a cover of $G(2,4)$, and in each open set the numbers $(\alpha,\beta,\gamma,\delta)$ are coordinates, so $U_{ij}\approx\mathbb{C}^4$. It will be convenient to organize the coordinates as a $2\times 2$ matrix

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \qquad \pi=\begin{pmatrix}\mathbb{1}\\ A\end{pmatrix}, \tag{2}$$

so $U_{ij}\approx M_2(\mathbb{C})$.

## 2.1 The Plücker embedding and the Klein quadric

We are going to see how $G(2,4)$ is embedded in the projective space $\mathbb{C}P^5$. We consider the vector space $E=\Lambda^2(\mathbb{C}^4)\approx\mathbb{C}^6$, with basis $\{e_i\wedge e_j\}_{i<j}$, $i,j=1,\ldots,4$. For a plane $\pi=(a,b)$ we have

$$a\wedge b=\sum_{i<j}y_{ij}e_i\wedge e_j.$$

A change of basis $(a',b')=(a,b)u$, where $u$ is a complex $2\times 2$ non singular matrix, produces a change

$$a'\wedge b'=\det(u)a\wedge b,$$

so the image $\pi \to [a \wedge b]$ is well defined into the projective space $P(E) \approx \mathbb{C}P^5$. It is called the Plücker map and it is not hard to see that it is an embedding. The quantities $y_{ij}$ are the *homogeneous Plücker coordinates*, and the image of the Plücker map can be identified with the solution of $p \wedge p = 0$ for $p \in E$. In coordinates, this reads as

$$y_{12}y_{34} + y_{23}y_{14} + y_{31}y_{24} = 0, \qquad y_{ij} = -y_{ji}. \tag{3}$$

Equation (3) is a quadric in $\mathbb{C}P^5$, the Klein quadric $K$, and we have just seen that $K = G(2,4)$.

Let $Q$ be the quadratic form in $E$ defining the Klein quadric,

$$Q(y) = y_{12}y_{34} + y_{23}y_{14} + y_{31}y_{24}.$$

For any $g \in \mathrm{GL}(4,\mathbb{C})$ we have that $Q(gy) = \det(g)Q(y)$, so if $g \in \mathrm{SL}(4,\mathbb{C})$ then it preserves the quadratic form $Q$. Then, the action of $\mathrm{SL}(4,\mathbb{C})$ on $\mathbb{C}^6 \approx \lambda^2(\mathbb{C}^4)$ defines a map $\mathrm{SL}(4,\mathbb{C}) \to \mathrm{SO}(6,\mathbb{C})$. Its kernel is $\pm 1$. Since both groups have the same dimension this means that $\mathrm{SL}(4,\mathbb{C})$ is the spin group of $\mathrm{SO}(6,\mathbb{C})$.

For concreteness, let us consider $U_{12}$. A plane $\pi \in U_{12}$ can always be represented as

$$\pi = \begin{pmatrix} \mathbb{1} \\ A \end{pmatrix}, \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

In particular, $\pi_0 = \mathrm{span}\{e_1, e_2\}$ has $A = 0$. The Plücker coordinates are

$$(y_{12}, y_{23}, y_{31}, y_{14}, y_{24}, y_{34}) = (1, -\alpha, -\beta, \delta, -\gamma, \alpha\delta - \beta\gamma). \tag{4}$$

Let us denote by $\pi_\infty$ the plane spanned by $e_3$ and $e_4$, so $\mathbb{C}^4 = \pi_0 \oplus \pi_\infty$. We say that a plane is finite if it has no intersection with $\pi_\infty$. The set $U_{12}$ can also be characterized as the set of finite planes, and it is a dense open set in $G(2,4) \simeq K$. It is called the *big cell* of $G(2,4)$, and we will denote it by $K^\times$.

We describe now the complement of $U_{12}$ in $G(2,4)$. A plane that has non-empty intersection with $\pi_\infty$ can always be represented by two independent vectors $a \in \pi_\infty$ and $b$,

$$a = a_3e_3 + a_4e_4, \qquad b = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4,$$

so $y_{12} = 0$. Substituting in (3) we obtain

$$y_{23}y_{14} + y_{31}y_{24} = 0. \tag{5}$$

This represents the closure of a cone in $\mathbb{C}^4$. Let us see this:

1. If $y_{34} \neq 0$ we may choose $y_{34} = 1$ and the remaining coordinates may take any value, except for the constraint (5). This is an affine cone in $\mathbb{C}^4$.

2. If $y_{34} = 0$, then one of the remaining $y$'s must be different from zero. The solutions of (5) are taken up to multiplication by a complex number, so we have a quadric in $\mathbb{C}P^3$. With these points we obtain the closure of the cone in 1.

We are going to see that the complex Poincaré group sits inside $\mathrm{GL}(4, \mathbb{C})$ as the subgroup $P$ that leaves $U_{12}$ invariant. So we will have a natural identification of $U_{12}$ with the complex Minkowski space time. Consequently, $G(2,4)$ is a compactification of Minkowski space time by the addition of the cone described above.

In fact, one can see that $P = P_\infty$, the group that leaves $\pi_\infty$ invariant. In the representation that we have used,

$$\pi_\infty = \begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix},$$

and $P_\infty$ can be written as

$$P_\infty = \left\{ \begin{pmatrix} L & 0 \\ NL & R \end{pmatrix}, \quad L, R \text{ invertible} \right\}.$$

The bottom left entry is an arbitrary $2 \times 2$ matrix. It can be written as $NL$, with $N$ arbitrary, since $L$ is invertible.

The action on $U_{12}$ is

$$A \mapsto N + RAL^{-1}, \tag{6}$$

so $P$ has the structure of semidirect product $P = M_2 \ltimes H$, where $M_2$, the set of $2 \times 2$ matrices $N$, are the translations and

$$H = \left\{ \begin{pmatrix} L & 0 \\ 0 & R \end{pmatrix}, \quad L, R \in \mathrm{GL}(2, \mathbb{C}), \quad \det L \cdot \det R = 1 \right\}.$$

The subgroup $H$ is the direct product $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^\times$. But $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ is the spin group of $\mathrm{SO}(4, \mathbb{C})$, the complexified Lorentz group, and $\mathbb{C}^\times$ acts as a dilation.

$G(2,4)$ is then a compactification of the Minkowski space time that is equivariant under the action of the conformal group.

## 2.2 The real form

The real Minkowski space time is obtained by taking the set of fixed points under a certain conjugation of the complex Minkowski space time. This can be done compatibly with our construction. We first consider the following conjugation of $E = \Lambda^2(\mathbb{C}^4)$:

$$\theta : (y_{12}, y_{23}, y_{31}, y_{14}, y_{24}, y_{34}) \;\mapsto\; (\bar{y}_{12}, \bar{y}_{23}, \bar{y}_{24}, \bar{y}_{14}, \bar{y}_{31}, \bar{y}_{34}).$$

The conjugation $\theta$ is also well defined when passing to the projective space $P(E)$. The quadratic form $Q$ defining the Klein quadric satisfies

$$\overline{Q(y)} = Q\left(\theta(y)\right).$$

The set of fixed points of $\theta$,

$$E^\theta = \{y \in E \;\mid\; \theta(y) = y\} = \{y \in \Lambda^2(\mathbb{C}^4) \;/\; y_{12}, y_{23}, y_{34}, y_{14} \text{ are real and } \; y_{31} = \bar{y}_{24}\}$$

is a real vector space. We denote by $Q_R$ the quadratic form $Q$ restricted to $E^\theta$. The form $Q_R$ is a real quadratic form with signature (4,2):

$$Q_R(y) = y_{12}y_{34} + y_{23}y_{14} + y_{31}\bar{y}_{31}.$$

Let $K = \{[y] \in P(E) \mid Q(y) = 0\}$. $\theta$ acts on $K$. Let $K_R$ be the fixed point set of $K$ under $\theta$. One can prove that $K_R$ is the image under the projection to $P(E)$ of the set of zeros of $Q_R$ on $E^\theta$.

The conjugation $\theta$ also acts on the big cell $K^\times = \{[y] \in K \mid y_{12} \neq 0\}$. According to (4), this action is just the hermitian conjugate

$$\theta : A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\mapsto\; A^\dagger = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}.$$

The fixed point set is $K_R^\times = \{A \in M_2(\mathbb{C}) \mid A^\dagger = A\}$.

To the complex group $\mathrm{SO}(6, \mathbb{C})$ acting on $K$ there corresponds the real form $\mathrm{SO}(4, 2)$ acting on $K_R$ and its spin group $\mathrm{SU}(2, 2)$. The hermitian form on $\mathbb{C}^4$ left invariant by this real form is

$$(u, v) = u^\dagger F v, \qquad F = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad u, v \in \mathbb{C}^4.$$

We can now, as in the complex case, compute the subgroup of SU(2,2) that leaves $K_R^\times$ invariant:

$$\begin{pmatrix} L & M \\ NL & R \end{pmatrix} \begin{pmatrix} I \\ A \end{pmatrix} = \begin{pmatrix} L + MA \\ NL + RA \end{pmatrix}, \qquad \begin{pmatrix} L & M \\ NL & R \end{pmatrix} \in \mathrm{SU}(2,2)$$

so $L + MA$ must be invertible for all hermitian $A$. The result is that the subgroup $P_R$ leaving $K_R^\times$ invariant is

$$P_R = \left\{ \begin{pmatrix} L & 0 \\ NL & L^{\dagger^{-1}} \end{pmatrix}, \; N \text{ hermitian and } \det(L) \in \mathbb{R} \right\},$$

and its action on $K_R^\times$ is

$$A \mapsto N + L^{\dagger^{-1}} A L^{-1}.$$

We see that $P_R$ is the Poincaré group ($\det(L) = 1$) times dilations. This completes the interpretation of $K_R^\times$ as the real Minkowski space.

# 3 Homogeneous spaces for Lie supergroups

A Lie supergroup is a supermanifold having a group structure, that is a multiplication map $\mu : G \times G \longrightarrow G$ and an inverse $i : G \longrightarrow G$ satisfying the usual diagrams. As in the ordinary theory to a Lie super group is associated a Lie superalgebra consisting of the left invariant vector fields and identified with the tangent space to the supergroup at the identity. For more details see [14] and the Appendix A.3.

We are interested in the construction of homogeneous spaces for Lie supergroups. Let $G$ be a Lie supergroup and $H$ a closed Lie subsupergroup. We want to define a supermanifold $X \equiv G/H$ with reduced manifold as $X_0 = G_0/H_0$, and to construct a morphism $\pi : G \longrightarrow X$ such that the following properties are satisfied:

1. The reduction of $\pi$ is the natural map $\pi_0 : G_0 \longrightarrow X_0$.

2. $\pi$ is a submersion.

3. There is an action $\beta$ from the left of $G$ on $X$ reducing to the action of $G_0$ on $X_0$ and compatible with the action of $G$ on itself from the left through $\pi$:

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle 1\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\beta} & X \end{array}$$

4. The pair $(X, \pi)$, subject to the properties 1, 2, and 3 is unique up to isomorphism. The isomorphism between two choices is compatible with the actions, and it is also unique.

## 3.1 The supermanifold structure on $X = G/H$

Let $G$ be a Lie supergroup of dimension $m|n$ and $H$ a closed sub supergroup of dimension $r|s$. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. For each $Z \in \mathfrak{g}$, let $D_Z$ be the left invariant vector field on $G$ defined by $Z$ (see the Appendix A for more details). For $x_0 \in G_0$ let $\ell_{x_0}$ and $r_{x_0}$ be the left and right translations of $G$ given by $x_0$. We denote by $i_{x_0} = \ell_{x_0} \circ r_{x_0}^{-1}$ the inner automorphism defined by $x_0$. It fixes the identity and induces the transformation $\mathrm{Ad}_{x_0}$ on $\mathfrak{g}$.

For any open subset $U \subset G_0$ and any sub-superalgebra $\mathfrak{k}$ of $\mathfrak{g}$ we define $\mathcal{O}_\mathfrak{k}(U)$ by

$$\mathcal{O}_\mathfrak{k}(U) = \{f \in \mathcal{O}_G(U) \mid D_Z f = 0 \text{ on } U \text{ for all } Z \in \mathfrak{k}\}.$$

Then $\mathcal{O}_\mathfrak{k}$ is a subsheaf of $\mathcal{O}_G$ (and, in particular, so is $\mathcal{O}_\mathfrak{h}$). On the other hand, for any open subset $W \subset G_0$, invariant under right translations by elements of $H_0$, we put

$$\mathcal{O}_{H_0}(W) = \{f \in \mathcal{O}_G(W) \mid f \text{ is invariant under } r_{x_0} \text{ for all } x_0 \in H_0\}.$$

If $H_0$ is connected we have

$$\mathcal{O}_{H_0}(W) = \mathcal{O}_{\mathfrak{h}_0}(W).$$

For any open set $W_0 \subset X_0 = G_0/H_0$ with $W = \pi_0^{-1}(W_0)$ we put

$$\mathcal{O}_X(W_0) = \mathcal{O}_{H_0}(W) \cap \mathcal{O}_\mathfrak{h}(W).$$

Clearly $\mathcal{O}_X(W_0) = \mathcal{O}_\mathfrak{h}(W)$ if $H_0$ is connected. The subsheaf $\mathcal{O}_X$ is a supersheaf on $X_0$. We write $X \equiv G/H$ for the ringed superspace $(X_0, \mathcal{O}_X)$. Our aim is to prove that $X$ is a supermanifold and that it has all the properties 1 to 4 listed previously.

It is clear that the left action of the group $G_0$ on $X_0$ leaves $\mathcal{O}_X$ invariant and so it is enough to prove that there is an open neighborhood $W_0$ of $\pi_0(1) \equiv \bar{1}$ with the property that $(W_0, \mathcal{O}_X|_{W_0})$ is a super domain, i. e., isomorphic to an open submanifold of $k^{p|q}$.

We will do this using the local Frobenius Theorem A.9. Also, we identify $\mathfrak{g}$ with the space of all left invariant vector fields on $G_0$, thereby identifying the tangent space of $G$ at every point canonically with $\mathfrak{g}$ itself.

On $G_0$ we have a distribution spanned by the vector fields in $\mathfrak{h}$. We denote it by $\mathcal{D}_{\mathfrak{h}}$.

On each $H_0$-coset $x_0H_0$ we have a supermanifold structure which is a closed sub supermanifold of $G$. It is an integral manifold of $\mathcal{D}_{\mathfrak{h}}$, i. e. the tangent space at any point is the subspace $\mathfrak{h}$ at that point. By the local Frobenius theorem there is an open neighborhood $U$ of 1 and coordinates $x_i$, $1 \leq i \leq n$ and $\theta_\alpha$, $1 \leq \alpha \leq m$ on $U$ such that at each point of $U$, $\mathcal{D}_{\mathfrak{h}}$ is spanned by $\partial/\partial x_i, \partial/\partial\theta_\alpha (1 \leq i \leq r, 1 \leq \alpha \leq s)$. Moreover, from the theory on $G_0$ we may assume that the slices $L(\mathbf{c}) := \{(x_1, \ldots, x_n) \mid \ x_j = c_j, \ \ r+1 \leq j \leq n\}$ are open subsets of distinct $H_0$-cosets for distinct $\mathbf{c} = (c_{r+1}, \ldots, c_n)$. These slices are therefore supermanifolds with coordinates $x_i$, $\theta_\alpha$, $1 \leq i \leq r$, $1 \leq \alpha \leq s$. We have a sub-supermanifold $W'$ of $U$ defined by $x_i = 0$ with $1 \leq i \leq r$ and $\theta_\alpha = 0$ with $1 \leq \alpha \leq s$. The map $\pi_0 : G_0 \longrightarrow X_0$ may be assumed to be a diffeomorphism of $W_0'$ with its image $W_0$ in $X_0$ and so we may view $W_0$ as a superdomain, say $W$. The map $\pi_0$ is then a diffeomorphism of $W'$ with $W$. What we want to show is that $W = (W_0, \mathcal{O}_X|_{W_0})$.

**Lemma 3.1.** *The map*

$$\begin{array}{ccc} W' \times H & \xrightarrow{\gamma} & G \\ w, h & \longrightarrow & wh \end{array}$$

*is a super diffeomorphism of $W' \times H$ onto the open sub-supermanifold of $G$ with reduced manifold the open subset $W_0'H_0$ of $G_0$.*

*Proof.* The map $\gamma$ in question is the informal description of the map $\mu \circ (i_{W'} \times i_H)$ where $i_M$ refers to the canonical inclusion $M \hookrightarrow G$ of a sub-supermanifold of $G$ into $G$, and $\mu : G \times G \longrightarrow G$ is the multiplication morphism of the Lie supergroup $G$. We shall use such informal descriptions without comment from now on.

It is classical that the reduced map $\gamma_0$ is a diffeomorphism of $W_0' \times H_0$ onto the open set $U = W_0' H_0$. This uses the fact that the cosets $wH_0$ are distinct for distinct $w \in W_0'$. It is thus enough to show that $d\gamma$ is surjective at all points of $W_0' \times H_0$. For any $h \in H_0$, right translation by $h$ (on the second factor in $W' \times H$ and simply $r_h$ on $G$) is a super diffeomorphism commuting with $\gamma$ and so it is enough to prove this at $(w, 1)$. If $X \in \mathfrak{g}$ is tangent to $W'$ at $w$ and $Y \in \mathfrak{h}$, then

$$d\gamma(X, Y) = d\gamma(X, 0) + d\gamma(0, Y) = d\mu(X, 0) + d\mu(0, Y) = X + Y.$$

Hence the range of $d\gamma$ is all of $\mathfrak{g}$ since, from the coordinate chart at 1 we see that the tangent spaces to $W'$ and $wH_0$ at $w$ are transversal and span the tangent space to $G$ at $w$ which is $\mathfrak{g}$. This proves the lemma. ■

**Lemma 3.2.** *We have*

$$\gamma^* \mathcal{O}_X \big|_{W_0} = \mathcal{O}_{W'} \otimes 1,$$

*where* $\gamma^* : \mathcal{O}_G \longrightarrow \gamma^* \mathcal{O}_{W' \times H}$.

*Proof.* To ease the notation we drop the open set in writing a sheaf superalgebra, that is we will write $\mathcal{O}_X$ instead of $\mathcal{O}_X(U)$.

We want to show that for any $g$ in $\mathcal{O}_X\big|_U$, $\gamma^* g$ is of the form $f \otimes 1$ and that the map $g \longmapsto f$ is bijective with $\mathcal{O}_{W'}$. Now $\gamma^*$ intertwines $D_Z (Z \in \mathfrak{h})$ with $1 \otimes D_Z$ and so $(1 \otimes D_Z)\gamma^* g = 0$. Since the $D_Z$ span all the super vector fields on $H_0$ it follows using charts that for any $p \in H_0$ we have $\gamma^* g = f_p \otimes 1$ locally around $p$ for some $f_p \in \mathcal{O}_{W'}$. Clearly $f_p$ is locally constant in $p$. Hence $f_p$ is independent of $p$ if $H_0$ is connected. If we do assume that $H_0$ is connected, the right invariance under $H_0$ shows that $f_p$ is independent of $p$. In the other direction it is obvious that if we start with $f \otimes 1$ it is the image of an element of $\mathcal{O}_X\big|_U$. ■

**Theorem 3.3.** *The superspace* $(X_0, \mathcal{O}_X)$ *is a supermanifold.*

*Proof.* At this stage by the previous lemmas we know that $(X_0, \mathcal{O}_X)$ is a super manifold at $\bar{1}$. The left invariance of the sheaf under $G_0$ shows this to be true at all points of $X_0$. The proof that $(X_0, \mathcal{O}_X)$ is a super manifold is finished. ■

## 3.2 The action of $G$ on $X$

Clearly $G_0$ acts on $X$ but there is more: there is a natural action of the supergroup $G$ itself on $X$. We shall now describe how this action comes about.

**Proposition 3.4.** *There is a map $\beta : G \times X \longrightarrow X$ such that the following diagram*

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle 1\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\beta} & X \end{array}$$

*commutes.*

*Proof.* Let $\alpha := \pi \circ \mu : G \times G \longrightarrow X$. The action of $G_0$ on $X_0$ shows that such a map $\beta_0$ exists at the reduced level. So it is a question of constructing the pull-back map

$$\beta^* : \mathcal{O}_X \longrightarrow \mathcal{O}_{G\times X}$$

such that

$$(1 \times \pi)^* \circ \beta^* = \alpha^*.$$

Now $\pi^*$ is an *isomorphism* of $\mathcal{O}_X$ onto the sheaf $\mathcal{O}_G$ restricted to a sheaf on $X$ ($W \longmapsto \mathcal{O}_G(\pi_0^{-1}(W))$), an so to prove the *existence and uniqueness* of $\beta^*$ it is a question of proving that $\alpha^*$ and $(1 \times \pi)^*$ have the same image in $\mathcal{O}_{G\times G}$. It is easy to see that $(1\times\pi)^*$ has as its image the subsheaf of sections $f$ killed by $1 \otimes D_X (X \in \mathfrak{h})$ and invariant under $1 \times r(h)(h \in H_0)$. It is not difficult to see that this is also the image of $\alpha^*$. ∎

We tackle now the question of the uniqueness of $X$ (point 4 at the beginning of Section 3).

**Proposition 3.5.** *Let $X'$ be a super manifold with $X'_0 = X_0$ and let $\pi'$ be a morphism $G \longrightarrow X'$. Suppose that*

*(a) $\pi'$ is a submersion.*

*(b) The fibers of $\pi'$ are the super manifolds which are the cosets of $H$.*

*Then there is a natural isomorphism*

$$X \simeq X'.$$

*Proof.* Indeed, from the local description of submersions as projections it is clear that for any open $W_0 \subset X_0$, the elements of $\pi'^*(\mathcal{O}_{X'}(W_0))$ are invariant under $r(H_0)$ and killed by $D_X (X \in \mathfrak{h})$. Hence we have a natural map $X' \longrightarrow X$ commuting with $\pi$ and $\pi'$. This is a submersion, and by dimension considerations it is clear that this map is an isomorphism. ■

We now turn to examine the functor of points of a quotient.

## 3.3 The functor of points of $X = G/H$

We are interested in a characterization of the functor of points of a quotient $X = G/H$ in terms of the functor of points of the supergroups $G$ and $H$.

**Theorem 3.6.** *Let $G$ and $H$ be supergroups as above and let $Q$ be the sheafification of the functor: $T \longrightarrow G(T)/H(T)$. Then $Q$ is representable and it is the functor of points of the homogeneous space supermanifold $X = G/H$ constructed above.*

*Proof.* In order to prove this result we can use the uniqueness property which characterizes the homogeneous space $X$ described above. So we only need to prove the two facts: 1. $Q$ is representable and 2. $\pi : G \longrightarrow Q$ is a submersion, the other properties being clear.

To prove that $Q$ is representable we use the criterion in Theorem A.13. The fact that $Q$ has the sheaf property is clear by its very definition. So it is enough to prove there is a open subfunctor around the origin (then by translation we can move it everywhere). But this is given by $W \cong W' \times H$ constructed above. The fact $\pi$ is a submersion comes by looking at it in the local coordinates given by $W$. ■

**Example 3.7.** *The flag supermanifold as a quotient.* Let $G = \mathrm{SL}(m|n)$ the complex special linear supergroup as described in Appendix A.3. There is a natural action of $G$ on $V = \mathbb{C}^{m|n}$:

$$\begin{array}{ccc} G(T) \times V(T) & \longrightarrow & V(T) \\ (g_{ij}, v_k) & \longrightarrow & \sum_j g_{ij} v_j \end{array}$$

This action extends immediately to the flag supermanifold $F = F(r|s, p|q; m|n)$ of $r|s$, $p|q$ subspaces in $\mathbb{C}^{m|n}$ described in detail in Example A.15. Let us fix a flag $\mathcal{F} = \{\mathcal{O}_T^{r|s} \subset \mathcal{O}_T^{p|q}\}$ and consider the map:

$$\begin{array}{ccc} G(T) & \longrightarrow & F(T) \\ g & \longrightarrow & g \cdot \mathcal{F}. \end{array}$$

The stabilizer subgroup functor is the subfunctor of GL(4|1) given as

$$H(T) = \{g \in \mathrm{SL}(m|n)(T) \ | \ g \cdot \mathcal{F} = \mathcal{F}\} \subset \mathrm{SL}(m|n)(T).$$

Using the submersion Theorem one can see that this functor is representable by a group supermanifold (see Section 5.1 for more details).

The flag supermanifold functor $F$ is isomorphic to the functor $G/H$ defined as the sheafification of the functor $T \longrightarrow G(T)/H(T)$. In fact we have that locally $G(T)$ acts transitively on the set of direct summands of fixed rank $\mathcal{O}_T$, hence the map $g \longrightarrow g \cdot \mathcal{F}$ is surjective. Hence by Theorem 3.6 we have that as supermanifolds $F \cong G/H$.

The general flag is treated similarly. ■

# 4 The super Poincaré and the translation superalgebras

We want to give a description of the Poincaré superalgebra[4] and the translation superalgebras, as subalgebras of the conformal (Wess-Zumino) superalgebra. We will describe these superalgebras in detail, over the complex and the real fields.

Results concerning these algebras have been known in the physics literature for some time (see for example Ref. [15], the reader can compare the approaches). In our exposition, we try to extract the properties that uniquely select the Poincaré superalgebra as the suitable supersymmetric extension of the Poincaré algebra. This will be the natural setting to construct the global theory in Section 5.

## 4.1 The translation superalgebra

We first describe the translation superalgebra as it comes from physics.

The translation superalgebra is a Lie superalgebra $\mathfrak{n} = \mathfrak{n}_0 + \mathfrak{n}_1$ such that $\mathfrak{n}_0$ is an abelian Lie algebra of dimension 4, which acts trivially on $\mathfrak{n}_1$, also of dimension 4. We have different descriptions depending if the ground field is $\mathbb{R}$ or $\mathbb{C}$:

[4]For us the Poincaré algebra and Poincaré superalgebra are meant to contain also the dilation generator.

1. In the real case, let $\mathfrak{l}_0$ be the Lie algebra of the Lorentz group, $\mathfrak{l}_0 = \mathfrak{so}(3,1)$. The subspace $\mathfrak{n}_1$ is a real spin module for $\mathfrak{l}_0$. It is the sum of the two inequivalent complex spin modules $\mathfrak{n}_1 = S^+ \oplus S^-$, and satisfies a reality condition[5].

2. In the complex case, $\mathfrak{l}_0$ is the complexification of the Lorentz algebra, $\mathfrak{l}_0 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. The subspace$\mathfrak{n}_1$ is, in physicists notation, the $\mathfrak{l}_0$-module $\mathfrak{n}_1 \simeq D(1/2,0) \oplus D(0,1/2)$.

In both cases the odd commutator is a non zero symmetric $\mathfrak{l}_0$-map from $\mathfrak{n}_1 \times \mathfrak{n}_1$ into $\mathfrak{n}_0$ which is $\mathfrak{l}_0$-equivariant.

These properties identify uniquely the translation superalgebra in both cases, real and complex.

The translation superalgebra will be identified in Section 5.1 with the dual of the infinitesimal Minkowski superspace time. In Subsection 4.2 we will see that it is the complement in the conformal superalgebra of the Poincaré superalgebra.

## 4.2 The complex Poincaré superalgebra.

Let us restrict our attention to the complex field. The classical complex Minkowski space time, as described in Section 2, has a natural action of the Poincaré group naturally embedded into the conformal group. Its compactification, $G(2,4)$, carries a natural action of the complex conformal group $\mathrm{SL}(4,\mathbb{C})$.

In the super geometric infinitesimal setting the complex conformal (Wess-Zumino) superalgebra is $\mathfrak{g} = \mathfrak{sl}(4|1)$. We want to define the Poincaré Lie superalgebra $\mathfrak{p}$ as a generalization of the Lie algebra of the Poincaré group introduced in Section 2. We want $\mathfrak{p}$ to be a Lie subsuperalgebra in $\mathfrak{g}$ subject to the following natural conditions:

1. $\mathfrak{p}$ is a parabolic superalgebra, i. e. it contains a Borel sub-superalgebra of $\mathfrak{g}$ (the Borel subsuperalgebra is normalized so that its intersection with $\mathfrak{g}_0$ is the standard Borel of $\mathfrak{g}_0$).

[5]On $S^+ \oplus S^-$ there is a conjugation $\sigma$ commuting with the action of the spin group. This conjugation exchanges the spaces $S^+$ and $S^-$, and in a certain basis is just the complex conjugation. The spinors $s$ satisfying the reality condition $\sigma(s) = s$ provide an irreducible real representation of the spin group. They are called Majorana spinors in the physics literature.

2. $\mathfrak{p} \cap \mathfrak{g}_0$ is the parabolic subalgebra of $\mathfrak{g}_0$ that consists of matrices of the form

$$\begin{pmatrix} L & 0 & 0 \\ M & R & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $L, M, R \in \mathcal{M}_2(\mathbb{C})$ are $2 \times 2$ matrices.

3. There is a sub superalgebra $\mathfrak{n} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}$, and $\mathfrak{n}$ is a translation superalgebra in the sense described above. The Lorentz algebra$\mathfrak{l}_0$ is the subalgebra of $\mathfrak{p} \cap \mathfrak{g}_0$ consisting of matrices of the form

$$\mathfrak{l}_0 = \left\{ \begin{pmatrix} L & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L, R \in \mathcal{M}_2(\mathbb{C}) \right\},$$

and the group $L_0$ is the simply connected group associated to $\mathfrak{l}_0$, namely

$$L_0 = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y \in \mathrm{SL}(2, \mathbb{C}) \right\}.$$

In particular this amounts to ask that $(\mathfrak{g}/\mathfrak{p})_1$ as an $L_0$-module is isomorphic to $D(1/2, 0) \oplus D(0, 1/2)$.

It turns out that these conditions determine uniquely the Poincaré superalgebra $\mathfrak{p}$ inside the conformal superalgebra $\mathfrak{g}$.

**Lemma 4.1.** *There exists a unique subalgebra $\mathfrak{p}$ in the conformal superalgebra $\mathfrak{g}$ satisfying the conditions 1 to 3 above, namely $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}$ where*

$$\mathfrak{p} = \left\{ \begin{pmatrix} L & 0 & 0 \\ M & R & \alpha \\ \beta & 0 & c \end{pmatrix} \right\}, \qquad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & A & \gamma \\ 0 & 0 & 0 \\ 0 & \delta & 0 \end{pmatrix} \right\}. \tag{7}$$

*Here $L, M, R$ and $A$ are $2 \times 2$ matrices, $\gamma$ and $\alpha$ are $1 \times 2$, $\delta$ and $\beta$ are $2 \times 1$, and $c$ is a scalar and the supertrace condition for $\mathrm{SL}(4|1)$ is $c = \mathrm{tr}(L) + \mathrm{tr}(R)$.*

From now on we refer to $\mathfrak{p}$ as the *Poincaré Lie superalgebra* or *Poincaré superalgebra* for shortness.

*Proof.* It is easy to check that $\mathfrak{p}$ and $\mathfrak{n}$ are Lie superalgebras. This can be done by matrix calculation, but it is easier (and better suited for the higher dimensional case) to look at these as composed of root spaces with respect to the Cartan subalgebra $\mathfrak{h}$ where $\mathfrak{h}$ consists of diagonal matrices

$$H = \left\{ \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{pmatrix} \right\} \qquad \text{with} \;\; a_5 = a_1 + a_2 + a_3 + a_4.$$

We treat the $a_i$ as the linear functions $H \longmapsto a_i$, for $1 \leq i \leq 5$. Then $\mathfrak{p}$ is the sum of $\mathfrak{h}$ and the root spaces for the roots

$$\begin{array}{cl} \pm(a_1 - a_2),\ \pm(a_3 - a_4),\ a_3 - a_i,\ a_4 - a_i,\ a_5 - a_i & \text{for } i = 1, 2; \\ a_j - a_5 & \text{for } j = 3, 4; \end{array}$$

while $\mathfrak{n}_0, \mathfrak{n}_1$ are the respective sums of root spaces for the roots

$$a_j - a_i, \;\; \text{and} \;\; a_5 - a_i,\ a_j - a_5 \quad \text{for } i = 3, 4,\ j = 1, 2.$$

The root description above implies easily that $0 \neq [\mathfrak{n}_1, \mathfrak{n}_1] \subset \mathfrak{n}_0$, and also that $\mathfrak{n}_0$ acts trivially on $\mathfrak{n}_1$, i.e., $[\mathfrak{n}_0, \mathfrak{n}_1] = 0$.

To verify the module structure of $\mathfrak{n}_1$ under $L_0$ it is more convenient to use the matrix description. The formula

$$\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 0 & \delta & 0 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x\gamma \\ 0 & 0 & 0 \\ 0 & \delta y^{-1} & 0 \end{pmatrix}$$

shows that the action is

$$(\gamma, \delta) \longmapsto (x\gamma, \delta y^{-1}) \tag{8}$$

which gives

$$\mathfrak{n}_1 \simeq D(1/2, 0) \oplus D(0, 1/2).$$

There are 4 other parabolic sub-superalgebras as one can see examining the complete list in the general case in [10] p. 51. They are defined as the sets of matrices of the following four different forms:

$$\mathfrak{p}_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & a_{55} \end{pmatrix} \right\}, \quad \mathfrak{p}_2 = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & 0 & a_{55} \end{pmatrix} \right\},$$

$$\mathfrak{p}_3 = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & \alpha_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & \alpha_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & \alpha_4 \\ \beta_1 & 0 & 0 & 0 & a_{55} \end{pmatrix} \right\}, \quad \mathfrak{p}_4 = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \alpha_1 \\ a_{21} & a_{22} & 0 & 0 & \alpha_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & \alpha_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & \alpha_4 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix} \right\}.$$

However each they fail condition 3. The verifications are the same as above and are omitted.

**Observation 4.2.** We can define a form on $\mathfrak{n}_0$ by taking the determinant:

$$q : \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longmapsto \det(A).$$

The odd commutator map is (using the notation of Lemma 4.1)

$$((\gamma, \delta), (\gamma', \delta')) \longmapsto \begin{pmatrix} 0 & \gamma\delta' + \gamma'\delta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and under the transformation (8) becomes

$$A = \gamma\delta' + \gamma'\delta \longmapsto x(\delta\gamma' + \delta'\gamma)y^{-1},$$

showing that the form $q(\gamma, \delta) = \det(\gamma\delta' + \gamma'\delta)$ is preserved. ■

## 4.3 The real conformal and Poincaré superalgebras

We shall now show that this whole picture carries over to $\mathbb{R}$. We shall construct a *conjugation* $\sigma$ of the super Lie algebra $\mathfrak{g} = \mathfrak{sl}(4|1)$ with the following properties:

1. $\mathfrak{g}^\sigma$, the set of fixed points of $\sigma$, is the Wess-Zumino super conformal Lie algebra, $\mathfrak{su}(2,2|1)$.
2. $\sigma$ leaves $\mathfrak{p}$ and $\mathfrak{n}$ invariant as well as the even and odd parts of $\mathfrak{g}$.
3. $\sigma$ preserves the big cell $\mathfrak{c} := \mathfrak{n} \cap \mathfrak{g}_0$.
4. $\mathfrak{c}^\sigma$ consists of all $2 \times 2$ skew hermitian matrices.
5. The group $L_0^\sigma$ consists of all matrices of the form

$$\begin{pmatrix} x & 0 \\ 0 & x^{\dagger -1} \end{pmatrix},$$

and the action on $\mathfrak{c}^\sigma$ is

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & xAx^\dagger \\ 0 & 0 \end{pmatrix}.$$

The function

$$q : \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \longmapsto \det(A)$$

is the Minkowski metric; traditionally one takes the hermitian matrices, but skew hermitian serve equally well: one just has to multiply by $i$.

To construct $\sigma$ we proceed as in Ref. [14], pp. 112-113, but using a slight variant of the construction. There, $F$ was taken in block diagonal form as

$$\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

We now take

$$F = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

where the blocks are all $2 \times 2$ matrices.

**Proposition 4.3.** *There is a conjugation $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the properties (1)-(5) listed above. It is given by*

$$\sigma : \begin{pmatrix} X & \mu \\ \nu & x \end{pmatrix} \longmapsto \begin{pmatrix} -FX^\dagger F & iF\nu^\dagger \\ i\mu^\dagger F & -\bar{x} \end{pmatrix} \tag{9}$$

*where $X$ is a $4 \times 4$ matrix, $\mu$ is $4 \times 1$, $\nu$ is $1 \times 4$, and $x$ is a scalar.*

*Proof.* It is a simple calculation to check that $\sigma$ is an antilinear map satisfying $\sigma^2 = \mathbb{1}$ (so it is a conjugation) and that it preserves the super bracket on $\mathfrak{g}$. We shall now verify Properties (1–5). In fact the verification of Properties (2–5) is just routine and is omitted. The reason why we take $F$ in the off diagonal form is to satisfy the requirement that $\sigma$ leaves $\mathfrak{p}$ and $\mathfrak{n}$ invariant. That Property (1) is also true is because the matrix $F$ defines a hermitian form of signature $(2,2)$, as does the original choice. The fact that the corresponding real form is transformable to the one where $F$ is chosen in the diagonal form is proved on page 112 of Ref. [14], with $F$ replaced by any hermitian matrix of signature $(2,2)$. ■

On the Poincaré superalgebra $\mathfrak{p}$ and on the super translation algebra $\mathfrak{n}$ the conjugation $\sigma$ is given explicitly by:

$$\sigma : \begin{pmatrix} L & 0 & 0 \\ M & R & \alpha \\ \beta & 0 & c \end{pmatrix} \longmapsto \begin{pmatrix} -R^\dagger & 0 & 0 \\ -M^\dagger & -L^\dagger & i\beta^\dagger \\ i\alpha^\dagger & 0 & -\bar{c} \end{pmatrix} \tag{10}$$

$$\sigma : \begin{pmatrix} 0 & A & \gamma \\ 0 & 0 & 0 \\ 0 & \delta & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & A^\dagger & i\delta^\dagger \\ 0 & 0 & 0 \\ i\alpha^\dagger & 0 & -\bar{c} \end{pmatrix}. \tag{11}$$

Hence the reality conditions read:

$$R^\dagger = -L, \quad M = -M^\dagger, \quad \alpha = i\beta^\dagger, \quad c = -\bar{c}, \quad A = -A^\dagger, \quad \delta = \gamma^\dagger.$$

# 5 The global theory

We want to extend the infinitesimal results of the previous section to obtain the Minkowski superspace time as the big cell inside a certain super flag manifold, realized as homogeneous space for the super conformal group.

## 5.1 The complex super flag $F = F(2|0, 2|1, 4|1)$

Let $G = \mathrm{SL}(4|1)$ be the complex super special linear group as described in Appendix A.3. The natural action of $G$ on $V = \mathbb{C}^{4|1}$ extends immediately to the flag supermanifold (see Example A.16) $F = F(2|0, 2|1; 4|1)$ of $2|0$, $2|1$ subspaces in $\mathbb{C}^{4|1}$. Let us fix a flag $\mathcal{F} = \{\mathcal{O}_T^{2|0} \subset \mathcal{O}_T^{2|1}\}$ and consider the map:

$$\begin{aligned} G(T) &\longrightarrow F(T) \\ g &\longrightarrow g \cdot \mathcal{F}. \end{aligned}$$

The stabilizer subgroup functor is the subfunctor of $G$ given as

$$H(T) = \{g \in G(T) \ | \ g \cdot \mathcal{F} = \mathcal{F}\} \subset G(T).$$

One can readily check it consists of all matrices in $G(T)$ of the form:

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} & \gamma_{15} \\ g_{21} & g_{22} & g_{23} & g_{24} & \gamma_{25} \\ 0 & 0 & g_{33} & g_{34} & 0 \\ 0 & 0 & g_{43} & g_{44} & 0 \\ 0 & 0 & \gamma_{53} & \gamma_{54} & g_{55} \end{pmatrix}.$$

This functor is representable by a group supermanifold by the Submersion Theorem A.8. In fact, as we shall see presently, the map $g \mapsto g \cdot \mathcal{F}$ is a submersion.

We wish to describe explicitly $F$ as the functor of points of the supermanifold quotient of the supergroups $G$ and $H$ as it was constructed in Section 3. The homogeneous space $X = G/H$ constructed in Section 3 is proven to be unique once three conditions are verified, namely:

1. The existence of a morphism $\pi : G \longrightarrow X$, such that its reduction is the natural map $\pi_0 : G_0 \longrightarrow X_0$.

2. $\pi$ is a submersion and the fiber of $\pi$ over $\pi_0(1)$ is $H$. Since $\pi$ is a submersion the fiber is well defined as a super manifold.

3. There is an action from the left of $G$ on $X$ reducing to the action of $G_0$ on $X_0$ and compatible with the action of $G$ on itself from the left through $\pi$.

Conditions (1) and (3) are immediate in our case. The only thing that we have to check is that the map

$$\begin{array}{ccc} G(T) & \xrightarrow{\pi} & F(T) \\ g & \longrightarrow & g \cdot \mathcal{F} \end{array}$$

is a submersion. We have to verify that at all topological points $\pi$ has surjective differential. It is enough to do the calculation of the differential at the identity element $e \in G(T)$.

The calculation, being local, takes place inside the big cell of $U \subset F$, which lies inside the product of big cells $U_1 \times U_2$ of the Grassmannians $G_1 = G(2|0;4|1)$, $G_2 = G(2|1;4|1)$.

We want to give local coordinates in $U_1 \times U_2$ and in $U$. We know by Example A.15 that they are affine spaces. So let us write, in the spirit of (2), local coordinates for $U_1$ and $U_2$ as

$$\left( \begin{pmatrix} I \\ A \\ \alpha \end{pmatrix}, \begin{pmatrix} I & 0 \\ B & \beta \\ 0 & 1 \end{pmatrix} \right) \in U_1(T) \times U_2(T), \qquad T \in (\text{smfld}),$$

where $I$ is the identity, $A$ and $B$ are $2 \times 2$ matrices with even entries and $\alpha = (\alpha_1, \alpha_2)$, $\beta^t = (\beta_1, \beta_2)$ are rows with odd entries.

An element of $U_1$ is inside $U_2$ if

$$A = B + \beta\alpha, \tag{12}$$

so we can take as coordinates for a flag in the big cell $U$ the triplet $(A, \alpha, \beta)$. We see then that $U$ is an affine $4|4$ superspace. Equation (12) is also known as *twistor relation*, see Ref. [1].

In these coordinates, $\mathcal{F}$ becomes

$$\left( \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \approx (0, 0, 0).$$

We want to write the map $\pi$ in these coordinates. In a suitable open subset near the identity of the group we can take an element $g \in G(T)$ as

$$g = \begin{pmatrix} g_{ij} & \gamma_{i5} \\ \gamma_{5j} & g_{55} \end{pmatrix}, \qquad i, j = 1, \dots 4.$$

Then, we can write an element $g \cdot \mathcal{F} \in G_1 \times G_2$ as:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \\ g_{31} & g_{32} \\ g_{41} & g_{42} \\ \gamma_{51} & \gamma_{52} \end{pmatrix}, \begin{pmatrix} g_{11} & g_{12} & \gamma_{15} \\ g_{21} & g_{22} & \gamma_{25} \\ g_{31} & g_{32} & \gamma_{35} \\ g_{41} & g_{42} & \gamma_{45} \\ \gamma_{51} & \gamma_{52} & g_{55} \end{pmatrix} \approx \begin{pmatrix} I \\ WZ^{-1} \\ \rho_1 Z^{-1} \end{pmatrix}, \begin{pmatrix} I & 0 \\ VY^{-1} & (\tau_2 - WZ^{-1}\tau_1)a \\ 0 & 1 \end{pmatrix}, \tag{13}$$

where

$$\rho_1 = \begin{pmatrix} \gamma_{51} & \gamma_{52} \end{pmatrix}, \quad W = \begin{pmatrix} g_{31} & g_{32} \\ g_{41} & g_{42} \end{pmatrix}, \quad Z = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

$$\tau_1 = \begin{pmatrix} \gamma_{15} \\ \gamma_{25} \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} \gamma_{35} \\ \gamma_{45} \end{pmatrix}, \quad d = (g_{55} - \nu Z^{-1}\mu_1)^{-1}$$

$$V = W - g_{55}^{-1}\tau_2\rho_1, \quad Y = Z - g_{55}^{-1}\tau_1\rho_1.$$

Finally the map $\pi$ in these coordinates is given by:

$$g \mapsto \left(WZ^{-1}, \rho_1 Z^{-1}, (\tau_2 - WZ^{-1}\tau_1)d\right).$$

At this point one can compute the super Jacobian and verify that at the identity it is surjective. Let $\bar{e}$ be the image of the identity in $F$, $\bar{e} \in U$. It is easy to see that the matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ \tilde{W} & 0 & \tilde{\tau}_2 \\ \tilde{\rho}_1 & 0 & 0 \end{pmatrix}$$

map isomorphically onto the image of $d\pi$. These are just the transpose of the matrices inside $\mathfrak{n}$ in (7). We then obtain that the translation superalgebra can be interpreted as the dual of the infinitesimal Minkowski superspace.

It is not difficult to show that the subgroup of $G = \mathrm{SL}(4|1)$ that leaves the big cell invariant is the set of matrices in $G$ of the form

$$\begin{pmatrix} L & 0 & 0 \\ NL & R & R\chi \\ d\varphi & 0 & d \end{pmatrix}, \tag{14}$$

with $L, N, R$ being $2 \times 2$ even matrices, $\chi$ and odd $1 \times 2$ matrix, $\varphi$ a $2 \times 1$ odd matrix and $d$ a scalar. This is then the complex Poincaré supergroup, whose Lie algebra is $\mathfrak{p}$ in (7). The action of the supergroup on the big cell can be written as

$$\begin{aligned} A &\longrightarrow R(A + \chi\alpha)L^{-1} + N, \\ \alpha &\longrightarrow d(\alpha + \varphi)L^{-1}, \\ \beta &\longrightarrow d^{-1}R(\beta + \chi). \end{aligned}$$

If the odd part is zero, then the action reduces to the one of the Poincaré group on the Minkowski space in (6).

We then see that the big cell of the flag supermanifold $F(2|0, 4|0, 4|1)$ can be interpreted as the complex super, Minkowski space time, being the flag its *superconformal compactification*.

## 5.2 The real Minkowski superspace

We want to construct a real form of the Minkowski superspace that has been described in Section 5.1.

We start by explicitly computing the real form of SL(4|1) that corresponds to the $\sigma$ defined in Section 7. We shall compute it on $G \equiv \mathrm{GL}(4|1)$, but it is easy to check that the conjugation so defined leaves $G_1 \equiv \mathrm{SL}(4|1)$ invariant.

In Appendix A.4 it is explained that in order to obtain a real form of $G$ we need a natural transformation $\rho$ from $G$ to its complex conjugate $\bar{G}$. Let $R$ be a complex ringed super space. We start by defining

$$\begin{array}{ccc} G(R) & \longrightarrow & \bar{G}(R) \\ \\ g = \begin{pmatrix} D & \tau \\ \rho & d \end{pmatrix} & \longrightarrow & g^\theta = \begin{pmatrix} D^\dagger & j\rho^\dagger \\ j\tau^\dagger & \bar{d} \end{pmatrix} \end{array}.$$

Here we use $j$ for either $i$ or $-i$. To get the Lie algebra conjugation $\sigma$ we shall eventually choose $j = i$, but at this stage we need not specify which sign we take.

**Lemma 5.1.** *We have*

$$(hg)^\theta = g^\theta h^\theta.$$

*Proof.* It is important to note that we have taken the following convention: if $\theta$ and $\xi$ are odd variables, then

$$\overline{\theta\xi} = \bar{\theta}\bar{\xi}. \tag{15}$$

This convention is opposed to the one used in physics, namely

$$\overline{\theta\xi} = \bar{\xi}\bar{\theta},$$

but as it is explained in Ref. [6], (15) is the one that makes sense functorially. According to this convention, then for matrices $X, Y$ with *odd* entries

$$(\overline{XY})^T = -(\bar{Y})^T(\bar{X})^T.$$

Then the lemma results from direct calculation. ∎

We are ready to define the involution which gives the real form of $G$. We will denote it by $\xi$:

$$\begin{array}{ccc} G(R) & \xrightarrow{\xi} & \bar{G}(R) \\ g & \longrightarrow & g^\xi := L(x^\theta)^{-1}L \end{array} \qquad L = \begin{pmatrix} F & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have that $(hg)^\xi = h^\xi g^\xi$ and that $(g^\xi)^\xi = g$, so it is a conjugation.

**Lemma 5.2.** *The map $x \mapsto x^\xi$ is a natural transformation. It defines a ringed space involutive isomorphism $\rho : G \longrightarrow \bar{G}$ which is $\mathbb{C}$-antilinear.*

*Proof.* This is a direct check. ■

**Proposition 5.3.** *The topological space $G^\xi$ consisting of the points fixed by $\rho$ has a real supermanifold structure and the supersheaf is composed of those functions $f \in \mathcal{O}_G$ such that $\overline{\xi^*(f)} = f$.*

*Proof.* Immediate from the definitions in Appendix A.4. ■

We now wish to prove that the involution $\xi$ that gives the real form for $G$ corresponds to the involution $\sigma$ constructed at the Lie algebra level in Section 4.3.

**Proposition 5.4.** *The conjugation $\xi$ of $G$ defined above induces on* $\mathrm{Lie}(G)$ *the conjugation*

$$\begin{pmatrix} X & \mu \\ \nu & x \end{pmatrix} \longrightarrow \begin{pmatrix} -FX^\dagger F & -jF\nu^\dagger \\ -j\mu^\dagger F & -\bar{x} \end{pmatrix},$$

*which for $j = -i$ coincides with $\sigma$ in (9).*

*Proof.* We have to compute the tangent map at the identity, so we shall write

$$\begin{pmatrix} D & \tau \\ \rho & d \end{pmatrix} \approx \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} X & \mu \\ \nu & x \end{pmatrix}, \qquad \begin{pmatrix} X & \mu \\ \nu & x \end{pmatrix} \in \mathfrak{g},$$

up to first order in $\varepsilon$. We now compute $x^\xi$ up to first order in $\varepsilon$,

$$x^\xi \approx \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon L \begin{pmatrix} -X^\dagger & -j\nu^\dagger \\ -j\mu^\dagger & -\bar{x} \end{pmatrix} L,$$

from which the result follows. ■

We have defined a real form of $G = \mathrm{GL}(4|1)$ which is also well defined on $G_1 = \mathrm{SL}(4|1)$ and agrees with the real form of the Lie superalgebra discussed in Section 4.3. Also, it is easy to check that it reduces to a conjugation on the

Poincaré supergroup (14). We can compute the conjugation on an element of the Poincaré supergroup:

$$g = \begin{pmatrix} L & 0 & 0 \\ M & R & R\chi \\ d\varphi & 0 & d \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} L^{-1} & 0 & 0 \\ -R^{-1}ML^{-1} + \chi\varphi L^{-1} & R^{-1} & -\chi d^{-1} \\ -\varphi L^{-1} & 0 & d^{-1} \end{pmatrix}$$

$$g^{\xi} = \begin{pmatrix} {R^{\dagger}}^{-1} & 0 & 0 \\ -{L^{\dagger}}^{-1}M^{\dagger}{R^{\dagger}}^{-1} - {L^{\dagger}}^{-1}\varphi^{\dagger}\chi^{\dagger} & {L^{\dagger}}^{-1} & -j{L^{\dagger}}^{-1}\varphi^{\dagger} \\ -j\bar{d}^{-1}\chi^{\dagger} & 0 & \bar{d}^{-1} \end{pmatrix}.$$

It follows that the fixed points are those that satisfy the conditions:

$$L = {R^{\dagger}}^{-1}, \qquad \chi = -j\varphi^{\dagger}, \qquad ML^{-1} = -(ML^{-1})^{\dagger} - j{L^{\dagger}}^{-1}\varphi^{\dagger}\varphi L^{-1}, \quad (16)$$

which reduce to $\sigma$ (restricted to the super Poincaré subalgebra) in (9) at the Lie algebra level.

To get a more familiar form for the reality conditions, we observe that the last equation in (16) can be cast as

$$M'L^{-1} \equiv ML^{-1} + \frac{1}{2}j{L^{\dagger}}^{-1}\varphi^{\dagger}\varphi L^{-1}, \qquad M' = -M'^{\dagger}.$$

This is just an odd translation, and amounts to multiply $g$ on the right by the group element

$$g' = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ -\frac{1}{2}jR^{-1}{L^{\dagger}}^{-1}\varphi^{\dagger}\varphi L^{-1} & \mathbb{1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We want now to compute the real form of the big cell. The first thing to observe is that the real form is well defined on the quotient space $G/H$ (the superflag), where $H$ is the group described in Section 5.1 and consists of elements in $G$ stabilizing a certain flag. In fact, one can check as we did for the Poincaré supergroup that $H$ is stable under $\sigma$.

Notice that a point of the big cell $(A, \alpha, \beta)$ can be represented by an element of the group

$$g = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ A & \mathbb{1} & \beta \\ \alpha & 0 & 1 \end{pmatrix},$$

since

$$g \begin{pmatrix} \mathbb{1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ A \\ \alpha \end{pmatrix}, \qquad g \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ A & \beta \\ \alpha & 1 \end{pmatrix} \approx \begin{pmatrix} \mathbb{1} & 0 \\ A - \beta\alpha & \beta \\ 0 & 1 \end{pmatrix}.$$

We first compute the inverse,

$$g^{-1} = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ -A + \beta\alpha & \mathbb{1} & -\beta \\ -\alpha & 0 & 1 \end{pmatrix},$$

and then $g^\xi$

$$g^\xi = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ -A^\dagger - \alpha^\dagger\beta^\dagger & \mathbb{1} & -j\alpha^\dagger \\ -j\beta^\dagger & 0 & 1 \end{pmatrix}.$$

The element $g^\xi$ is already in the desired form, so the real points are given by

$$A = -A^\dagger - j\alpha^\dagger\alpha, \qquad \beta = -j\alpha^\dagger.$$

We can make a convenient change of coordinates,

$$A' \equiv A + \frac{1}{2} j\alpha^\dagger\alpha,$$

so the reality condition is

$$A' = -A'^\dagger,$$

and we recover the same form than in Section (4.3) for the (purely even) Minkowski space time.

# A Supergeometry

In this section we want to recall the basic definitions and results in supermanifold theory. For more details see Refs. [1, 6, 14], which use a language similar to ours.

## A.1 Basic definitions

For definiteness, we take the ground field to be $k = \mathbb{R}, \mathbb{C}$. A *superalgebra* $A$ is a $\mathbb{Z}_2$-graded algebra, $A = A_0 \oplus A_1$. The subspace $A_0$ is an algebra, while $A_1$ is an $A_0$-module. Let $p(x)$ denote the parity of an homogeneous element $x$,

$$p(x) = 0 \text{ if } x \in A_0, \qquad p(x) = 1 \text{ if } x \in A_1.$$

The superalgebra $A$ is said to be *commutative* if for any two homogeneous elements $x, y$

$$xy = (-1)^{p(x)p(y)} yx$$

The category of commutative superalgebras will be denoted by (salg). From now on all superalgebras are assumed to be commutative unless otherwise specified.

**Definition A.1.** A *superspace* $S = (S_0, \mathcal{O}_S)$ is a topological space $S_0$ endowed with a sheaf of superalgebras $\mathcal{O}_S$ such that the stalk at a point $x \in S_0$ denoted by $\mathcal{O}_{S,x}$ is a local superalgebra[6] for all $x \in S_0$. More generally we speak also of *ringed superspace* whenever we have a topological space and a sheaf of superalgebras. ■

**Definition A.2.** A *morphism* $\phi : S \longrightarrow T$ of superspaces is given by $\phi = (\phi_0, \phi^*)$, where $\phi_0 : S_0 \longrightarrow T_0$ is a map of topological spaces and $\phi^* : \mathcal{O}_T \longrightarrow \phi_0^* \mathcal{O}_S$ is such that $\phi_x^*(\mathbf{m}_{\phi_0(x)}) = \mathbf{m}_x$ where $\mathbf{m}_{\phi_0(x)}$ and $\mathbf{m}_x$ are the maximal ideals in the stalks $\mathcal{O}_{T,\phi_0(x)}$ and $\mathcal{O}_{S,x}$ respectively. ■

**Example A.3.** The superspace $k^{p|q}$ is the topological space $k^p$ endowed with the following sheaf of superalgebras. For any open subset $U \subset k^p$

$$\mathcal{O}_{k^{p|q}}(U) = \mathcal{O}_{k^p}(U) \otimes k[\xi_1 \dots \xi_q],$$

where $k[\xi_1 \dots \xi_q]$ is the exterior algebra (or *Grassmann algebra*) generated by the $q$ variables $\xi_1 \dots \xi_q$ and $\mathcal{O}_{k^p}$ denotes the $C^\infty$ sheaf on $k^p$ when $k = \mathbb{R}$ and the complex analytic sheaf on $k^p$ when $k = \mathbb{C}$ . ■

**Definition A.4.** A *supermanifold* of dimension $p|q$ is a superspace $M = (M_0, \mathcal{O}_M)$ which is locally isomorphic to $k^{p|q}$, i. e. for all $x \in M_0$ there exist an open set $V_x \subset M_0$ and $U \subset k^p$ such that:

$$\mathcal{O}_M|_{V_x} \cong \mathcal{O}_{k^{p|q}}|_U.$$

[6]A local superalgebra is a superalgebra with a maximal ideal.

A *morphism* of supermanifolds is simply a morphism of superspaces.

The classical manifold $M_0$ underlying the supermanifold $M$, is the *reduced space* of $M$. Its sheaf is given at any open set $U$ as $\mathcal{O}_M(U)$ modulo the nilpotent elements. ■

The theory of supermanifolds resembles very closely the classical theory. One can, for example, define tangent bundles, vector fields and the differential of a morphism similar to the classical case.

**Definition A.5.** Let $M = (M_0, \mathcal{O}_M)$ be a supermanifold. A *tangent vector* $X_m$ at $m \in M_0$ is a (super) derivation $X_m : \mathcal{O}_{M,m} \longrightarrow k$. This allows usto define the tangent space $T_mM$ of $M$ at $m$ and the tangent bundle $TM$. *Super vector fields* are sections of the tangent bundle. If $f : M \longrightarrow N$ is a morphism, we define its differential $(df)_m : T_mM \longrightarrow T_{f(m)}N$ as $(df)_m(X_m)\alpha = X_m(f^*(\alpha))$.

■

We summarize here some of the results that we will need later, sending the reader to [14], Sections 4.2, 4.3, 4.4 and to [4], Chapter 4. These papers treat only the $C^\infty$ case; the complex analytic case is done similarly.

**Definition A.6.** Let $f : M \longrightarrow N$ be a supermanifold morphism with $f_0 : M_0 \longrightarrow N_0$ the underlying classical morphism on the reduced spaces. We say that $f$ is a *submersion* at $m \in M_0$ if $f_0$ is an ordinary submersion and $(df)_m$ is surjective. Likewise, $f$ is an *immersion* at $m$ if $f_0$ is an immersion and $(df)_m$ is injective. Finally $f$ is a *diffeomorphism* at $m$ if it is a submersion and an immersion.

When we say $f$ is a submersion (resp. immersion or diffeomorphism) we mean $f$ a submersion at all points of $M_0$. ■

As in the classical setting, submersions and immersions have the usual local models. For more details see [14] p. 148.

**Definition A.7.** We say that $N$ is an *open* sub-supermanifold of $M$ if $N_0$ is an open submanifold of $M_0$ and $\mathcal{O}_N$ is the restriction of $\mathcal{O}_M$ to $N$.

We say that $N$ is a *closed* sub-supermanifold of $M$ if $N_0$ is a closed submanifold of $M_0$ and there exists a map $f : N \hookrightarrow M$ which is an immersion and such that $f_0$ is the classical embedding $N_0 \subset M_0$ . ■

Closed sub supermanifolds can be determined by using the super version of the submersion theorem.

**Theorem A.8.** *Submersion Theorem. Let $f : M \longrightarrow N$ be a submersion and let $P_0 = f_0^{-1}(n)$ for $n \in N_0$. Then $P_0$ admits a supermanifold structure. Locally we have that for $p \in P_0$, $\mathcal{O}_{P,p} = \mathcal{O}_{M,p}/f^*(I_n)$, where $I_n$ is the ideal in $\mathcal{O}_{N,n}$ of elements vanishing at $n$. Moreover,*

$$\dim P = \dim M - \dim N.$$

*Proof.* (Sketch). For an open subset $V$ of $P_0$, sections are defined as assignments $q \longmapsto s(q)$ where $s(q) \in \mathcal{O}_{P,p}(q)$ for all $q$ and locally on $V$ these arise from sections of $\mathcal{O}_M$. By the local description of submersions as projections it is seen easily that this defines the structure of a super manifold on $P$. Note that $\dim(P) = \dim(M) - \dim(N)$. ∎

Frobenius theorem plays a fundamental role in constructing sub supermanifolds of a given manifold.

**Theorem A.9.** *Let $M$ be a supermanifold and let $\mathcal{D}$ be an integrable super distribution on $M$ of dimension $r|s$ (i.e. a subbundle, locally a direct factor of the tangent bundle of $M$).*

*Local Frobenius Theorem. Then at each point there exists a coordinate system $(x, \xi)$ such that the distribution at that point is spanned by $\partial_{x_i}$, $\partial_{\xi_\alpha}$, with $1 \leq i \leq r$ and $1 \leq \alpha \leq s$.*

*Global Frobenius Theorem. Then at each point there exists a unique maximal supermanifold $N$ such that $TN = \mathcal{D}$.*

*Proof.* See Ref. [14] 4.7, p. 157 and Ref. [4] Chapter 4. ∎

We now turn to a different and alternative way to introduce supermanifolds, that is the functor of points approach.

## A.2 The functor of points

In supergeometry the functor of points approach is particularly useful since it brings back the geometric intuition to the problems, leaving the cumbersome sheaf notation in the background.

**Definition A.10.** Given a supermanifold $X$ we define its *functor of points* as the following representable functor from the category of supermanifolds to the category of sets:

$$h_X : (\text{smfld}) \longrightarrow (\text{set}), \qquad h_X(T) = \text{Hom}(T, X).$$

■

Given two supermanifolds $X$ and $Y$, Yoneda's lemma establishes a one to one correspondence between the morphisms $X \longrightarrow Y$ and the natural transformations $h_X \longrightarrow h_Y$. This allows one to view a morphism of supermanifolds as a family of morphisms $h_X(T) \longrightarrow h_Y(T)$ depending functorially on the supermanifold $T$.

We want to give a representability criterion, which allows one to single out among all the functors from the category of supermanifolds to the category of sets those that are representable, i. e.those that are the functor of points of a supermanifold. In order to do this, we need to generalize the notion of *open submanifold* and of *open cover* to fit this more general functorial setting.

**Definition A.11.** . Let $U$ and $F$ be two functors (smfld) $\longrightarrow$ (set). The functor $U$ is a *subfunctor* of $F$ if $U(R) \subset F(R)$ for all $R \in$ (smfld). We denote it as $U \subset F$.

We say that $U$ is an *open subfunctor* of $F$ if for all natural transformations $f : h_T \longrightarrow F$ with $T \in$ (smfld) then $f^{-1}(U) = h_V$, where $V$ is open in $T$. If $U$ is also representable we say that $U$ is an *open supermanifold subfunctor*.

Let $\mathcal{U}_\alpha$ be open subfunctors of $\mathbb{R}^{m|n}$ (or $\mathbb{C}^{m|n}$). We say that $\{\mathcal{U}_\alpha\}$ is an open cover of a functor $F$ : (smfld) $\longrightarrow$ (set) if for all supermanifolds $T$ and natural transformations $f : h_T \longrightarrow F$, $f^{-1}(\mathcal{U}_\alpha) = h_{V_\alpha}$ and $V_\alpha$ cover $T$. ■

**Definition A.12.** A functor $F$ : (smfld) $\longrightarrow$ (set) is said to be *local* if it has the sheaf property, that is, if when restricted to the open sets of a given supermanifold $T$ it is a sheaf.

Notice that any functor $F$ : (smfld) $\longrightarrow$ (set) when restricted to the category of open sub supermanifolds of a given supermanifold $T$ defines a presheaf.

We are ready to state a representability criterion which gives necessary and sufficient conditions for a functor from (smfld) to (set) to be representable.

**Theorem A.13.** *Representability Criterion. Let $F$ be a functor $F$ :* (smfld) $\longrightarrow$ (set)*, such that when restricted to the category of manifolds is representable. Then the functor $F$ is representable if and only if:*
*1. $F$ is local, i. e.it has the sheaf property.*
*2. $F$ is covered by open supermanifold functors.*

*Proof*. The proof resembles closely the proof given in Ref. [7] Chapter 1, for the ordinary algebraic category. For completeness we include a sketch of it.

If $F$ is representable, $F = h_X$ and one can check directly that it has the two properties listed above. This is done in the super algebraic category for example in Ref. [4] Chapter 5.

Let $\{h_{X_\alpha}\}_{\alpha\in A}$ be the open supermanifold subfunctors that cover $F$. Define $h_{X_{\alpha\beta}} = h_{X_\alpha} \times_F h_{X_\beta}$ (This will correspond to the intersection of the two open $X_\alpha$ and $X_\beta$).

We have the commutative diagram:

$$\begin{array}{ccc} h_{X_{\alpha\beta}} = h_{X_\alpha} \times_F h_{X_\beta} & \xrightarrow{j_{\beta,\alpha}} & h_{X_\beta} \\ \Big\downarrow j_{\alpha,\beta} & & \Big\downarrow i_\beta \\ h_{X_\alpha} & \xrightarrow{i_\alpha} & F \end{array}$$

As a set we define:

$$|X| =_{\text{def}} \coprod_\alpha |X_\alpha| / \sim,$$

where $\sim$ is the following equivalence relation:

$$\begin{aligned} &\forall\, x_\alpha \in |X_\alpha|,\ x_\beta \in |X_\beta|,\ x_\alpha \sim x_\beta \Longleftrightarrow \\ &\exists\, x_{\alpha\beta} \in |X_{\alpha\beta}|,\ \text{ with } j_{\alpha,\beta}(x_{\alpha\beta}) = x_\alpha,\ j_{\beta,\alpha}(x_{\alpha\beta}) = x_\beta \end{aligned}$$

and $|Y|$ denotes the underlying topological space of a generic supermanifold $Y$.

This is an equivalence relation. The map $\pi_\alpha : |X_\alpha| \hookrightarrow |X|$ is an injective map into the topological space $|X|$.

We now need to define a sheaf of superalgebras $\mathcal{O}_X$ by using the sheaves in the open $X_\alpha$ and "gluing". Let $U$ be open in $|X|$ and let $U_\alpha = \pi_\alpha^{-1}(U)$. Define:

$$\mathcal{O}_X(U) =_{\text{def}} \{(f_\alpha) \in \coprod_{\alpha\in I} \mathcal{O}_{X_\alpha}(U_\alpha) | \quad j^*_{\beta,\gamma}(f_\beta) = j^*_{\gamma,\beta}(f_\gamma), \forall \beta, \gamma \in I\}.$$

The condition $j^*_{\beta,\gamma}(f_\beta) = j^*_{\gamma,\beta}(f_\gamma)$ simply states that to be an element of $\mathcal{O}_X(U)$, the collection $\{f_\alpha\}$ must be such that $f_\beta$ and $f_\gamma$ agree on the intersection of $X_\beta$ and $X_\gamma$ for any $\beta$ and $\gamma$.

One can check directly that $\mathcal{O}_X$ is a sheaf of superalgebras and that $h_X = F$. For more details see [4] Chapter 5. ■

We want to discuss two important examples of supermanifolds together with their functors of points, namely the Grassmannian and the flag supermanifolds. They will appear later in the definition of the Minkowski super space time. In each case, we will define a functor and then we will use Theorem A.13 to show that the functor is representable, so it is the functor of points of some supermanifold whose reduced manifold is the Grassmannian and the flag manifold respectively. We will see that our definitions of super Grassmannian and super flag coincide with Manin's ones in Ref. [1], Chapter 1.

**Remark A.14.** For the rest of this paper we use the same letter, say $X$, to denote both a supermanifold and its functor of points, as it is customary to do in the literature. So instead of writing $h_X(T)$ we will simply write $X(T)$, for $T$ a generic supermanifold.

**Example A.15.** *The Grassmannian supermanifold.*

We define the Grassmannian of $r|s$-subspaces of a $m|n$-dimensional complex vector space as the functor $\mathrm{Gr} : (\mathrm{smfld}) \longrightarrow (\mathrm{set})$ such that for any supermanifold $T$, with reduced manifold $T_0$, $\mathrm{Gr}(T)$ is the set of locally free sheaves over $T_0$ of rank $r|s$, direct summands of $\mathcal{O}_T^{m|n} =_{\mathrm{def}} \mathcal{O}_T \otimes \mathbb{C}^{m|n}$.

Equivalently $\mathrm{Gr}(T)$ can also be defined as the set of pairs $(L, \alpha)$ where $L$ is a locally free sheaf of rank $r|s$ and $\alpha$ a surjective morphism

$$\alpha : \mathcal{O}_T^{m|n} \longrightarrow L,$$

modulo the equivalence relation

$$(L, \alpha) \sim (L', \alpha') \quad \Leftrightarrow \quad L \approx L', \quad \alpha' = a \circ \alpha,$$

where $a : L \to L$ is an automorphism of $L$.

We need also to specify Gr on morphisms $\psi : R \longrightarrow T$. Given the element $(L, \alpha)$ of $\mathrm{Gr}(T)$, $\alpha : \mathcal{O}_T^{m|n} \to L$, we have the element of $\mathrm{Gr}(R)$,

$$\mathrm{Gr}(\psi)(\alpha) : \mathcal{O}_R^{m|n} = \mathcal{O}_T^{m|n} \otimes_{\mathcal{O}_T} \mathcal{O}_R \to L \otimes_{\mathcal{O}_T} \mathcal{O}_R.$$

We want to show that Gr is the functor of points of a supermanifold. By its very definition Gr is a local functor, so by Theorem A.13 we just have to show that Gr admits a cover by open supermanifold functors.

Consider the multiindex $I = (i_1, \dots, i_r|\mu_1, \dots, \mu_s)$ and the map $\phi_I : \mathcal{O}_T^{r|s} \longrightarrow \mathcal{O}_T^{m|n}$ where

$$\phi_I(x_1, \dots, x_r|\xi_1, \dots \xi_s) = \quad \begin{array}{l} m|n - \text{tuple with} \\ x_1, \dots, x_r \text{ occupying the position } i_1, \dots, i_r, \\ \xi_1, \dots, \xi_s \text{ occupying the position } \mu_1, \dots, \mu_s \\ \text{and the other positions are occupied by zero.} \end{array}$$

For example, let $m = n = 2$ and $r = s = 1$. Then $\phi_{1|2}(x, \xi) = (x, 0|0, \xi)$.

We define the subfunctors $v_I$ of Gr as follows. The set $v_I(T)$ is the set of pairs $(L, \alpha)$, $\alpha : \mathcal{O}_T^{m|n} \longrightarrow L$ (modulo the equivalence relation), such that $\alpha \circ \phi_I$ is invertible. Since we can, up to an automorphism, choose $\alpha(t) = t$ for any $t \in T_0$, this means that $(\alpha \cdot \phi)_t : \mathcal{O}_{T,t}^{r|s} \longrightarrow L_t$ is an isomorphism of free $\mathcal{O}_{T,t}$-modules.

It is not difficult to check that they are open supermanifold functors and that they cover Gr. Actually the open supermanifold functors $v_I$ are the functors of points of superspaces isomorphic to matrix superspaces of suitable dimension, as it happens in the classical case.

This is very similar to the algebraic super geometry case which is explained in detail in Ref. [4] Chapter 5. Hence, by Theorem A.13, Gr is the functor of points of a supermanifold, that we will call the *super Grassmannian of $r|s$ subspaces into a $m|n$ dimensional space.* ■

**Example A.16.** *The flag super manifold.* Let $F$ : (smfld) $\longrightarrow$ (set) be the functor such that for any supermanifold $T$, $F(T)$ is the set of all flags $S_1 \subset \cdots \subset S_i \subset \cdots \subset S_k \subset \mathcal{O}_T^{m|n}$, where the $S_i$ are locally free sheaves of rank $d_i|e_i$, direct summands of $\mathcal{O}_T^{m|n}$. We want to show that $F$ is representable, i. e.it is the functor of points of what we call the super flag of $d_1|e_1, \dots d_k|e_k$ spaces into a $m|n$-dimensional complex vector super space.

The functor $F$ is clearly local, so by Theorem A.13 we just have to show that it admits a cover by open submanifold functors. For concreteness, we will consider the super flag of $d_1|e_1, d_2|e_2$ spaces into and $m|n$ space, the general case being a simple extension of this. Consider the natural transformation $\phi : F \longrightarrow \mathrm{Gr}_1 \times \mathrm{Gr}_2$ given by

$$\begin{aligned} \phi_T : F(T) &\longrightarrow \mathrm{Gr}_1(T) \times \mathrm{Gr}_2(T) \\ S_1 \subset S_2 &\longrightarrow (S_1, S_2), \end{aligned}$$

where the $\mathrm{Gr}_i$ are functors of points of super Grassmannians.

Let $v_I^i$ be the open subfunctors of $\mathrm{Gr}_i$ described in Example A.15. Consider the subfunctors $u_{IJ} = \phi^{-1}(v_I^1 \times v_J^2)$. These are open. In fact if we have a natural transformation $\psi : h_X \longrightarrow F$, then

$$\psi^{-1}(u_{IJ}) = (\phi \circ \psi)^{-1}(v_I^1 \times v_J^2) = h_W$$

for $W$ open in $X$ (the product of open sets is open in our topology). The fact that they cover $F$ is also clear. As for the representability, since the $v_I$ are the functors of points of super spaces $\mathbb{C}^{p|q}$ for suitable $p$ and $q$, it is not hard to see that the $u_{IJ}$ will be given by algebraic relations in the $v_I$'s coordinates, hence they are also representable. By Theorem A.13 we have that $F$ is also representable.

■

## A.3 The linear supergroups

For completeness, we include here a brief summary of the definition of a Lie supergroup, some of its properties, and the construction of the supergroups $\mathrm{GL}(m|n)$ and $\mathrm{SL}(m|n)$. For more details see [14] and [4].

The ground field is always $k = \mathbb{R}$ or $\mathbb{C}$.

**Definition A.17.** A *Lie supergroup* is a supermanifold whose functor of points

$$G : (\text{smfld}) \longrightarrow (\text{set})$$

is group valued. ■

**Remark A.18.** Saying that $G$ is group valued is equivalent to have the following natural transformations:

1. Multiplication $\mu : G \times G \longrightarrow G$, such that $\mu \circ (\mu \times 1\!\!1) = (\mu \times 1\!\!1) \circ \mu$, i. e.

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times 1\!\!1} & G \times G \\ {\scriptstyle 1\!\!1 \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

2. Unit $e : e_k \longrightarrow G$, where $e_k$ : (smfld) $\longrightarrow$ (set), such that $\mu \circ (1\!\!1 \otimes e) = \mu \circ (e \times 1\!\!1)$, i. e.

$$\begin{array}{ccccc} G \times e_k & \xrightarrow{1\!\!1 \times e} & G \times G & \xrightarrow{e \times 1\!\!1} & e_k \times G \\ \downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow \\ G & = & G & = & G \end{array}$$

3. Inverse $i : G \longrightarrow G$, such that $\mu \circ (1\!\!1 \times i) = e \circ 1\!\!1$, i. e.

$$\begin{array}{ccc} G & \xrightarrow{(1\!\!1, i)} & G \times G \\ \downarrow & & \downarrow{\scriptstyle \mu} \\ e_k & \xrightarrow{e} & G \end{array}$$

**Example A.19.** Let $k^{m|n} = (k^m, \mathcal{O}_{k^{m|n}})$ denote the supermanifold whose reduced space is the affine space $k^m$ and with supersheaf:

$$\mathcal{O}_{k^{m|n}}(U) = \mathcal{O}_{k^m}(U) \otimes \wedge(\xi_1, \ldots, \xi_n), \qquad U \subset k^m$$

introduced in Appendix A. Its functor of points is given by:

$$\text{(smfld)} \longrightarrow \text{(set)}, \qquad k^{m|n}(T) = \text{Hom}(T, k^{m|n}).$$

The set $k^{m|n}(T)$ can be identified with the set of $m|n$-tuples $(t_1, \ldots, t_m | \theta_1, \ldots, \theta_n)$, where the $t_i$'s and $\theta_j$'s are, respectively, even and odd global sections of $\mathcal{O}_T$.

The set $k^{m|n}(T)$ is an additive group for all $T$, so $k^{m|n}$ is a Lie supergroup.
■

Given a Lie supergroup one defines its Lie superalgebra Lie($G$) as the set of left invariant vector fields, together with a natural super bracket on them.

We recall that a left invariant vector field is defined as a vector field $V$ satisfying the condition:

$$(V \otimes 1\!\!1)i^* = i^* V$$

where $\mu$ denotes the multiplication and $i(x, y) = \mu(y, x)$.

As in the ordinary case Lie($G$) can be identified with the tangent space of $G$ at the identity (which is a topological point). For more details see [4] Chapter 4, and [14] p. 276.

**Example A.20.** Let $\mathrm{GL}(m|n) : (\text{smfld}) \longrightarrow (\text{set})$ be the functor such that $\mathrm{GL}(m|n)(T)$ are the invertible $m|n \times m|n$ matrices with entries in $\mathcal{O}_T(T)$:

$$\begin{pmatrix} p_{m\times m} & q_{m\times n} \\ r_{n\times m} & s_{n\times n} \end{pmatrix}, \tag{17}$$

where the submatrices $p$ and $s$ have even entries and $q$ and $r$ have odd entries. The invertibility condition implies that $p$ and $q$ are ordinary invertible matrices.

$\mathrm{GL}(m|n)$ is the functor of points of a supermanifold (i. e. it is representable) whose reduced space is an open set $U$ in the ordinary space $k^{m^2+n^2}$, namely the matrices with invertible diagonal blocks. In fact one can readily check that the supersheaf $\mathcal{O}_{\mathrm{GL}(m|n)}$ is

$$\mathcal{O}_{\mathrm{GL}(m|n)} = \mathcal{O}_{k^{m^2+n^2|2mn}}|_U$$

Notice: the supermanifold $k^{m^2+n^2|2mn}$ can be identified with the supermatrices $m|n \times m|n$.

The subfunctor $\mathrm{SL}(m|n)$ of $\mathrm{GL}(m|n)$ consists on all matrices with Berezinian [3] equal to 1, where

$$\mathrm{Ber}\begin{pmatrix} p_{m\times m} & q_{m\times n} \\ r_{n\times m} & s_{n\times n} \end{pmatrix} = \det(s^{-1})\det(p - qsr). \tag{18}$$

The proof that $\mathrm{SL}(m|n)$ is representable uses Theorem A.8 (the submersion theorem).

It is not hard to show that $\mathrm{Lie}(\mathrm{GL}(m|n))(T)$ consists of all matrices $m|n \times m|n$ (with entries in $\mathcal{O}_T(T)$) while $\mathrm{Lie}(\mathrm{SL}(m|n))(T)$ is the subalgebra of $\mathrm{Lie}(\mathrm{GL}(m|n))$ consisting of matrices with zero supertrace.

## A.4 Real structures and real forms

We want to understand how it is possible to define real structures and real forms in supergeometry. For more details see Ref. [6] p. 92.

A major character in this game is the complex conjugate of a super manifold.

Let $M = (|M|, \mathcal{O}_M)$ be a complex manifold. The *complex conjugate* of $M$ is the manifold $\bar{M} = (|M|, \mathcal{O}_{\bar{M}})$ where $\mathcal{O}_{\bar{M}}$ is the sheaf of the antiholomorphic functions on $M$ (which are immediately defined once we have $\mathcal{O}_M$ and the complex structure on $M$). We have a $\mathbb{C}$-antilinear sheaf morphism

$$\begin{array}{ccc} \mathcal{O}_M & \longrightarrow & \mathcal{O}_{\bar{M}} \\ f & \longrightarrow & \bar{f}. \end{array}$$

In the super context it is not possible to speak directly of antiholomorphic functions and for this reason we need the following generalization of complex conjugate super manifold.

**Definition A.21.** Let $M = (|M|, \mathcal{O}_M)$ be a complex super manifold. We define a *complex conjugate* of $M$ as a complex super manifold $\bar{M} = (|\bar{M}|, \mathcal{O}_{\bar{M}})$, where now $\mathcal{O}_{\bar{M}}$ is just a supersheaf, together with a ringed space $\mathbb{C}$-antilinear isomorphism. This means that we have an isomorphism of topological spaces $|M| \cong |\bar{M}|$ and a $\mathbb{C}$-antilinear sheaf isomorphism

$$\begin{array}{ccc} \mathcal{O}_M & \longrightarrow & \mathcal{O}_{\bar{M}} \\ f & \longrightarrow & \bar{f}. \end{array}$$

■

**Example A.22.** Let $M = \mathbb{C}^{1|1} = (\mathbb{C}, \mathcal{O}_{\mathbb{C}}[\theta])$, $\bar{M} = (\mathbb{C}, \mathcal{O}_{\bar{\mathbb{C}}}[\bar{\theta}])$ where $\mathcal{O}_{\mathbb{C}}$ and $\mathcal{O}_{\bar{\mathbb{C}}}$ denote respectively the sheaf of holomorphic and antiholomorphic functions on $\mathbb{C}$. The isomorphism is

$$\begin{array}{ccc} \mathcal{O}_M & \longrightarrow & \mathcal{O}_{\bar{M}} \\ z & \mapsto & \bar{z} \\ \theta & \mapsto & \bar{\theta}. \end{array}$$

Notice that while $\bar{z}$ has a the meaning of being the complex conjugate of $z$, $\bar{\theta}$ is simply another odd variable that we introduce to define the complex conjugate.

Practically one can think of the complex conjugate super manifold as a way of giving a meaning to $\bar{f}$ the complex conjugate of a super holomorphic function. ■

We are ready to define a real structure on a complex supermanifold.

**Definition A.23.** Let $M = (|M|, \mathcal{O}_M)$ be a complex super manifold. We define a *real structure* on $M$ as an involutive isomorphism of ringed spaces

$\rho : M \longrightarrow \bar{M}$, which is $\mathbb{C}$-antilinear on the sheaves $\rho^* : \mathcal{O}_{\bar{M}} \longrightarrow \rho^*\mathcal{O}_M$, $\rho^*(\lambda f) = \bar{\lambda}\rho(f)$. We define the *real form* $M_r$ of $M$ defined by $\rho$ as the supermanifold $(M^\rho, \mathcal{O}_{M_r})$ where $M^\rho$ are the fixed points of $\rho : |M| \longrightarrow |\bar{M}| = |M|$ and $\mathcal{O}_{M_r}$ are all the functions $f \in \mathcal{O}_M|_{M^\rho}$ such that $\overline{\rho^*(f)} = f$.■

If $M$ is a complex supermanifold, one can always construct the complex conjugate $\bar{M}$ in the following way. Take $|\bar{M}| = |M|$ and as $\mathcal{O}_{\bar{M}}$ the sheaf with the complex conjugate $\mathbb{C}$-algebra structure (that is $\lambda \cdot f = \bar{\lambda} f$). In order to obtain a real structure on $M$, we need a ringed spaces morphism $M \longrightarrow \bar{M}$ with certain properties. By Yoneda's Lemma this is equivalent to give an invertible natural transformation between the functors of points:

$$\rho : M(R) \longrightarrow \bar{M}(R)$$

for all super ringed spaces $R$ satisfying the $\mathbb{C}$-antilinear condition.

We take this point of view in Section 5.2 when we discuss the real Minkowski space.

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# References


[1] Y. Manin. *Gauge field theory and complex geometry.* Springer Verlag, (1988). (Original Russian edition in 1984).

[2] R. Penrose. *Twistor algebra.* J. Math. Phys. **8**, 345-366, (1967).

[3] F. A. Berezin. *Introduction to superanalysis.* Edited by A. A. Kirillov. D. Reidel Publishing Company, Dordrecht (Holland) (1987). With an Appendix by V. I. Ogievetsky. Translated from the Russian by J. Niederle and R. Kotecký. Translation edited by Dimitri Leĭtes.

[4] L. Caston and R. Fioresi. *Mathematical Foundation of Supersymmetry.* To appear.

[5] R. D'Auria, S. Ferrara, M. A. Lledó and V. S. Varadarajan. *Spinor algebras.* J.Geom.Phys. **40**, 101-128, (2001).

[6] P. Deligne and J. Morgan. *Notes on supersymmetry (following J. Bernstein)*, in "Quantum fields and strings. A course for mathematicians", Vol 1, AMS, (1999).

[7] M. Demazure and P. Gabriel. *Groupes Algébriques, Tome 1.* Mason & Cie, éditeur. North-Holland Publishing Company, The Netherlands (1970).

[8] A. Ferber. *Supertwistors and conformal supersymmetry.* Nucl. Phys. B, **132**, 55-64, (1978).

[9] R. Hartshorne *Algebraic geometry.* (Third edition). Springer Verlag. New York, (1983).

[10] V. G. Kac. *Lie superalgebras.* Adv. in Math. **26**, 8-26, (1977).

[11] S. Ferrara, J. Wess and B. Zumino. *Supergauge multiplets and superfields.* Phys. Lett. B **51**, 239, (1974).

[12] M. Kotrla and J. Niederle. *Supertwistors And Superspace.* Czech. J.Phys. B **35**, 602, (1985).

[13] P. S. Howe and G. G. Hartwell. *A superspace survey.* Class. Quantum Grav. **12**, 1823-1880, (1995).

[14] V. S. Varadarajan. *Supersymmetry for mathematicians: an introduction.* Courant Lecture Notes, **1**. AMS (2004).

[15] V.K. Dobrev and V.B. Petkova. *On the group-theoretical approach to extended conformal supersymmetry : function space realizations and invariant differential operators.* Fortschr. d. Phys. **35**, 537-572, (1987).