FTUV/97-25

IFIC/97-25

Canonical Equivalence of a Generic 2D Dilaton Gravity Model and a Bosonic String Theory *

J. Cruz † and J. Navarro-Salas ‡.

Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC. Facultad de Física, Universidad de Valencia, Burjassot-46100, Valencia, Spain.

Abstract

We show that a canonical transformation converts, up to a boundary term, a generic 2D dilaton gravity model into a bosonic string theory with a Minkowskian target space.

PACS number(s): 04.60.Kz, 04.60.Ds

---

*Work partially supported by the *Comisión Interministerial de Ciencia y Tecnología* and *DGICYT*

†CRUZ@LIE.UV.ES

‡JNAVARRO@LIE.UV.ES

The interest of studying two-dimensional theories of gravity has been growing in the last years. The main motivation is to study quantum gravitational effects in a more simplified setting. In two dimensions the Einstein tensor vanishes identically and a natural analogue of the Einstein equations is given by the constant curvature equation. This equation can be obtained from a local action [1] if a scalar field ϕ is introduced in the theory

$$S = \int d^2x \sqrt{-g} \left(R + 4\lambda^2 \right) \phi \, . \tag{1}$$

The constant curvature equation can also be derived from the non-local Polyakov action [2], which can be converted into a local one by introducing a scalar field ϕ

$$S = \int d^2x \sqrt{-g} \left(2R\phi + (\nabla\phi)^2 + 4\lambda^2 \right) \, . \tag{2}$$

More recently, the model introduced by Callan, Giddings, Harvey and Strominger (CGHS) [3]

$$S = \int d^2x \sqrt{-g} \left[e^{-2\phi} \left(R + 4 (\nabla\phi)^2 + 4\lambda^2 \right) - \frac{1}{2} (\nabla f)^2 \right] \, , \tag{3}$$

where f is a massless scalar field, has been extensively studied because it describes the formation and evaporation of black holes in a simple way [4]. The gravitational part of the action (3) can be seen as a simplification of the spherically reduced Hilbert-Einstein action in four-dimensions [4]

$$S = \int d^2x \sqrt{-g} e^{-2\phi} \left(R + 2 (\nabla\phi)^2 + 2\lambda^2 e^{2\phi} \right) \, , \tag{4}$$

where the 4D metric ds_4^2 is related to the 2D metric ds^2 by

$$ds_4^2 = ds^2 + \frac{e^{-2\phi}}{\lambda^2} d\Omega^2 \, . \tag{5}$$

All the above models are particular cases of a large class of dilaton-gravity theories considered in [5]. It is well-known that, by a conformal reparametrization of the fields, one can eliminate the kinetic term for the dilaton and rewrite the action in the form [5, 6]

$$S = \int d^2x \sqrt{-g} \left(R\phi + V\left(\phi\right) \right) \ , \tag{6}$$

where $V(\phi)$ is an arbitrary function of the scalar field. For the Jackiw-Teitelboim model (1) we have $V = 4\lambda^2\phi$, while $V = 2\lambda^2 e^{-2\phi}$ for the induced gravity (2), $V = 4\lambda^2$ for the CGHS model (3) and $V = \frac{2\lambda^2}{\sqrt{\phi}}$ for the spherically reduced Einstein gravity (4).

In [7] it was shown that a non-local canonical transformation converts the constraints of the CGHS theory into those of a bosonic string theory with a Minkowskian target space. So, the hamiltonian and momentum constraint functions read as follows

$$H = \frac{1}{2}\left(\pi_0^2 + \left(r^{0\prime}\right)^2\right) - \frac{1}{2}\left(\pi_1^2 + \left(r^{1\prime}\right)^2\right) \ , \tag{7}$$

$$P = \pi_0 r^{0\prime} + \pi_1 r^{1\prime} \ , \tag{8}$$

and the corresponding quantum constraints $\hat{C}_\pm = \pm\frac{1}{2}\left(\hat{H} \pm \hat{P}\right)$ generate an anomalous algebra

$$\begin{aligned} \left[\hat{C}_\pm(x), \hat{C}_\pm(\tilde{x})\right] &= i\left(\hat{C}_\pm(x) + \hat{C}_\pm(\tilde{x})\right)\delta'(x-\tilde{x}) \\ &\mp \frac{i}{24\pi}\left(c_0 + c_1\right)\delta'''(x-\tilde{x}) \ . \end{aligned} \tag{9}$$

It was also pointed out in [7] that the value of the central charge $c = c_0 + c_1$ depends on how the vacuum is defined. The positively signed gravity variable contributes with $c_0 = 1$ but the negatively signed one gives a negative contribution $c_1 = -1$ in the Schrodinger representation [7]. Therefore, both contributions cancel and the theory can be quantized without obstruction. In fact, in terms of the geometrical variables, or in the gauge-theoretical formulation [8], explicit expressions for the wave functions have been obtained [6, 9]. However solutions to the quantum constraints of pure gravity were obtained in [6] for a generic model of 2D dilaton gravity. This suggests that the equivalence of the CGHS model and a conformal theory of two free scalar fields with opposite contributions to the hamiltonian constraint could also be valid for a general dilaton gravity theory.

In a recent work [10] a first step in this direction was done by constructing explicit canonical transformations which convert the Jackiw-Teitelboim model and the model with an exponential (Liouville) potential into a bosonic string theory with a Minkowskian target space. In this paper we shall extend the results of Ref [10] for a general model of 2D dilaton gravity. We shall demonstrate the existence of a canonical transformation mapping, up to a boundary term, a generic model of 2D dilaton gravity into a bosonic string theory with a flat target space of indefinite signature.

Let us consider the action (6) minimally coupled to a massless scalar field f. Parametrizing the two-dimensional metric as [6]

$$g_{\mu\nu} = e^{2\rho} \begin{pmatrix} v^2 - u^2 & v \\ v & 1 \end{pmatrix} , \tag{10}$$

the hamiltonian form of the action read as

$$S = \int d^2x \left(\pi_\rho \dot{\rho} + \pi_\phi \dot{\phi} + \pi_f \dot{f} - uH - vP \right) , \tag{11}$$

where the constraint functions H and P are given by

$$H = -\frac{1}{2}\pi_\rho \pi_\phi + 2\left(\phi'' - \phi'\rho'\right) - e^{2\rho} V\left(\phi\right) + \frac{1}{2}\left(\pi_f^2 + f'^2\right) , \tag{12}$$

$$P = \rho' \pi_\rho - \pi'_\rho + \phi' \pi_\phi + \pi_f f' \, . \tag{13}$$

Our strategy to prove the existence of a canonical transformation converting an arbitrary 2D dilaton-gravity model into a bosonic string theory will follow the procedure used in [10]. In doing so we should find the general solution to the equations of motion of the model in conformal gauge $ds^2 = -e^{2\rho} dx^+ dx^-$,

$$8 e^{-2\rho} \partial_+ \partial_- \rho = -V'(\phi) \; , \tag{14}$$

$$-4 e^{-2\rho} \partial_+ \partial_- \phi = V(\phi) \; , \tag{15}$$

$$\partial_\pm^2 \phi - 2 \partial_\pm \phi \partial_\pm \rho = T^f_{\pm\pm} = \frac{1}{2} (\partial_\pm f)^2 \; , \tag{16}$$

in terms of four arbitrary chiral functions and employ it to construct a canonical transformation mapping the theory into a parametrized scalar field theory [11]. However a general solution to this system of equations remains elusive, so we shall first consider the situation when one chiral sector of matter is trivial. In this case one can explicitly relate the fields ρ, ϕ and $\partial_\pm \phi$ with two chiral functions. This relation turns out to be sufficient to show the existence of a canonical transformation which converts the constraint functions (12-13) into those of a parametrized chiral scalar field theory. This result can be extended immediately to the general situation, without imposing any restriction to the matter energy-momentum tensor. Finally, a canonical transformation relating a parametrized scalar field theory to a bosonic string theory with a Minkowskian target space will complete the proof.

Therefore, let us start our analysis considering a generic theory with the restriction $T^f_{--} = 0$. It is not difficult to check that the solution to the equations (14-16) can be alternatively expressed as the solution to the equations

$$e^{-2\rho}\partial_-\phi = a\,, \tag{17}$$

$$\partial_+\phi = A - \frac{1}{4a}J(\phi)\ , \tag{18}$$

where $\frac{dJ(\phi)}{d\phi} = V(\phi)$. The functions A, a are related to the non-trivial component of the energy momentum tensor in the following way

$$T^f_{++} = \partial_+ A + \frac{A}{a}\partial_+ a\,. \tag{19}$$

Equation (18) defines ϕ implicitly as a functional $\phi = \phi(A, a, \beta)$ where β is a function of the x^- coordinate which appears as a constant of integration. We introduce now a definition which will be useful in the following. The symbol $\sim$ affecting any functional of the chiral functions A, a, β means that they are converted into $\tilde{A}, \tilde{a}, \tilde{\beta}$ which are now arbitrary (not chiral) functions and that the possible derivatives or integrations have been replaced according to the rule $\partial_\pm \longrightarrow \pm\partial_x$ $\left(\partial_\pm^{-1} \longrightarrow \pm\partial_x^{-1}\right)$. Taking into account that the dependence of ϕ on β must be ultralocal it is straightforward to prove that $\left(\tilde{\phi}\right)' = \widetilde{\phi'}$ and $\left(\widetilde{\partial_-\phi}\right)' = \widetilde{(\partial_-\phi)'}$. Following the lines of [10] we consider now a transformation to the new set of variables $\tilde{A}, \tilde{a}, \tilde{b}$

$$\phi = \tilde{\phi}\,, \tag{20}$$

$$\pi_\phi = -2\widetilde{\dot{\rho}} = \frac{1}{4\tilde{a}}V\left(\tilde{\phi}\right) + \frac{\tilde{a}'}{\tilde{a}} - \widetilde{\left(\frac{\partial_-^2\phi}{\partial_-\phi}\right)}\,, \tag{21}$$

$$\rho = \frac{1}{2}\log\frac{\widetilde{\partial_-\phi}}{\tilde{a}}\,, \tag{22}$$

$$\pi_\rho = -2\widetilde{\dot{\phi}} = -2\tilde{A} + \frac{1}{2\tilde{a}}J\left(\tilde{\phi}\right) - 2\widetilde{\partial_-\phi}\,, \tag{23}$$

After a long computation the symplectic 2-form on the unconstrained phase space

$$\omega = \int dx\left(\delta\phi\wedge\delta\pi_\phi + \delta\rho\wedge\delta\pi_\rho + \delta f\wedge\delta\pi_f\right)\ , \tag{24}$$

becomes (from now on the exterior product will be omited)

$$\omega = \int dx \left(2\frac{\delta\tilde{a}}{\tilde{a}}\delta\tilde{A} + \delta f \delta\pi_f \right) + \omega_b \,, \tag{25}$$

where ω_b is a boundary term

$$\omega_b = \int d \left(\delta\tilde{\phi}\frac{\delta\tilde{a}}{\tilde{a}} + \delta\tilde{\phi}\frac{\delta\widetilde{\partial_-\phi}}{\widetilde{\partial_-\phi}} \right) \,, \tag{26}$$

and the light-cone combinations $C^{\pm} = \pm\frac{1}{2}(H \pm P)$ of the constraints (12-13) turn out to be

$$C^+ = \tilde{A}' + \tilde{A}\frac{\tilde{a}'}{\tilde{a}} + \frac{1}{4}\left(\pi_f + f'\right)^2 \tag{27}$$

$$C^- = 0 \,. \tag{28}$$

Equation (28) is consistent with the assumption of chirality $\left(T^f_{--} = 0\right)$. At this point, it is clear that defining

$$X^+ = \log\tilde{a}\tilde{A}, \qquad \Pi_+ = 2\tilde{A} \,, \tag{29}$$

the 2-form (25) becomes

$$\omega = \int dx \left(\delta X^+ \delta\Pi_+ + \delta f \delta\pi_f\right) + \omega_b \,, \tag{30}$$

and, therefore, (X^+, Π_+) become canonical coordinates for the gravity-sector up to a boundary term ω_b. Moreover the constraints (27-28) are now the constraints of a parametrized chiral scalar field theory

$$C^+ = \Pi_+ X^{+\prime} + \frac{1}{4}\left(\pi_f + f'\right)^2 \,, \tag{31}$$

$$C^- = 0 \,. \tag{32}$$

We shall now show that the canonical equivalence of a chiral 2D dilaton gravity model and a parametrized chiral scalar field theory can be extended to the non-chiral situation. In the general case, the classical solutions are parametrized by four arbitrary chiral functions $A(x^+), a(x^+), B(x^-)$, $b(x^-)$. We can choose A, a, B, b in such a way that when $T^f_{--} = 0$ the equations of motion are equivalent to (17-18) and when $T^f_{++} = 0$ they are equivalent to

$$\partial_- \phi = B - \frac{1}{4b} J(\phi) \ , \tag{33}$$

$$e^{-2\rho} \partial_+ \phi = b \ . \tag{34}$$

The classical solution $\phi = \phi(A, a; B, b)$, $\rho = \rho(A, a; B, b)$ defines a transformation: $\phi = \tilde{\phi}$, $\pi_\phi = -2\widetilde{\dot{\rho}}$, $\rho = \tilde{\rho}$, $\pi_\rho = -2\widetilde{\dot{\phi}}$, which reduces to (20-23) for $B = 0$ and to the analogous chiral transformation for $A = 0$. It is then clear that the only possible form for the constraint functions consistent with the previous result is

$$C^\pm = \Pi_\pm X^{\pm\prime} \pm \frac{1}{4} \left(\pi_f \pm f'\right)^2 \ , \tag{35}$$

where X^+, Π_+ are given by (29) and $X^- = \log \tilde{b}\tilde{B}$, $\Pi_- = 2\tilde{B}$. All we need now is to see that the transformation $(\phi, \pi_\phi, \rho, \pi_\rho) \longrightarrow (X^\pm, \Pi_\pm)$ is canonical up to a boundary term. This follows immediately because the unique expression for ω in terms of $X^\pm, \Pi_\pm$ which leads to the hamiltonian equations of motion in conformal gauge, $\partial_\mp X^\pm = \partial_\mp \Pi_\pm = 0$, and is the integral of a scalar density is (omiting the matter contribution and boundary terms)

$$\omega = \int dx \left(\delta X^+ \delta \Pi_+ + \delta X^- \delta \Pi_-\right) \ . \tag{36}$$

A further linear canonical transformation [12]

$$2\Pi_\pm = -\left(\pi_0 + \pi_1\right) \mp \left(r^{0\prime} - r^{1\prime}\right) \ , \tag{37}$$

$$2X^{\pm\prime} = \mp \left(\pi_0 - \pi_1\right) - \left(r^{0\prime} + r^{1\prime}\right) \ , \tag{38}$$

converts the constraints of a parametrized scalar field theory into those a bosonic string theory with a Minkowskian target space. The boundary term ω_b of the symplectic form can be treated in two different ways. Either impossing appropriated boundary conditions to the $X^{\pm}$ fields to cancel it when the spatial section is closed [10] or introducing new variables to account for the asymptotic behaviour of the fields when the spatial section is open [13, 14]. In the absence of matter fields the equivalence of a dilaton gravity model and a set of two free fields of opposite signature explains why there is no obstruction in the quantization of a generic model in the functional Schrodinger approach [6]. When matter fields are present the quantum constraints require a modification to cancel the anomaly. Remarkably, the addition of a $X^{\pm}$-dependent term to the quantum constraints (35) (motivated by a covariant factor ordering [12]) produces a cancellation of the anomaly allowing to solve all the Dirac quantum conditions [14, 15] maintaining the unitarity of the theory. Therefore it is possible to consistently quantize a matter-coupled dilaton gravity theory in the functional Schrodinger approach.

We thank J. M. Izquierdo, A. Mikovic, D. J. Navarro, M. Navarro and C. F. Talavera for useful comments. J.C. acknowledges the Generalitat Valenciana for a FPI fellowship.

References

[1] R. Jackiw, C. Teitelboim in *"Quantum theory of gravity"*, ed. S. Christensen (Adam Hilger, Bristol, 1984)

[2] A. M. Polyakov, *Phys. Lett.* B 163, 207 (1981).

[3] C. G. Callan, S. B. Giddings, J. A. Harvey and A. Strominger, *Phys. Rev.* D 45, 1005 (1992).

[4] A. Strominger, *"Les Houches Lectures on Black Holes"* hep-th/9501071.

[5] T. Banks and M. O′Loughlin, *Nucl. Phys.* B 362, 649 (1991).

[6] D. Louis-Martinez, J. Gegenberg and G. Kunstatter, *Phys. Lett.* B 321, 193 (1994).

[7] D. Cangemi, R. Jackiw and B. Zwiebach, *Ann. Phys* (N.Y) 245, 408 (1996).

[8] D. Cangemi and R. Jackiw, *Phys. Rev. Lett.* 69, 233 (1992).

[9] D. Cangemi and R. Jackiw, *Phys. Lett.* B 337, 271 (1994).

[10] J. Cruz, J. M. Izquierdo, D. J. Navarro and J. Navarro-Salas, *" Free fields via Canonical Transformations of Matter-coupled 2D Dilaton Gravity Models "* hep-th/9704168

[11] K. V. Kuchar, *J. Math. Phys.* 17, 801 (1976).

[12] K. V. Kuchar, *Phys. Rev.* D 39 (1989) 1579 ; *Phys. Rev.* D 36, 2263 (1989). K. V. Kuchar and C. G. Torre, *J. Math. Phys.* 30,1769 (1989).

[13] A. Barvinsky and G. Kunstatter, *Phys. Lett.* B389, 231 (1996).

[14] K. V. Kuchar, J. D. Romano and M. Varadarajan, *Phys. Rev.* D 55, 795 (1997).

[15] E. Benedict, R. Jackiw and H. J. Lee, *Phys. Rev.* D 54, 6213 (1996).