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A Note on Einstein Gravity on AdS_3 and Boundary Conformal Field Theory *

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Abstract

We find a simple relation between the first subleading terms in the asymptotic expansion of the metric field in AdS_3 , obeying the Brown-Henneaux boundary conditions, and the stress tensor of the underlying Liouville theory on the boundary. We can also provide an more explicit relation between the bulk metric and the boundary conformal field theory when it is described in terms of a free field with a background charge.

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A crucial property of 2+1 dimensional gravity with a negative cosmological constant $\Lambda = -\frac{1}{\ell^2}$, described by the action

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} (R + \frac{2}{\ell^2}), \quad (1)$$

is that the asymptotic symmetry of the theory is the conformal algebra with a central charge given by [1]

$$c = \frac{3\ell}{2G}. \quad (2)$$

Recently Strominger [2] has used this result to derive the Bekenstein-Hawking area formula for the 3d black holes [3] by counting the number of states of the conformal field theory at infinity with central charge via Cardy's formula [4]. Other approaches [5] describe excitations associated to a conformal field theory at the horizon or at any value of the black hole radius [6].

Taking into account that (1) can be reformulated [7, 8] as a Chern-Simons gauge theory with gauge group $SL(2, R) \otimes SL(2, R)$, and using WZW reduction, one can argue that the asymptotic dynamics of (1) is described by Liouville theory [9]. The aim of this letter is to further elucidate the relation between the gravity theory and the underlying conformal field theory.

Following [1, 2] we assume the following asymptotic behaviour of the metric

$$g_{+-} = -\frac{r^2}{2} + \gamma_{+-}(x^+, x^-) + \mathcal{O}(\frac{1}{r}), \quad (3)$$

$$g_{\pm\pm} = \gamma_{\pm\pm}(x^+, x^-) + \mathcal{O}(\frac{1}{r}), \quad (4)$$

$$g_{\pm r} = \frac{\gamma_{\pm r}(x^+, x^-)}{r^3} + \mathcal{O}(\frac{1}{r^4}), \quad (5)$$

$$g_{rr} = \frac{\ell^2}{r^2} + \frac{\gamma_{rr}(x^+, x^-)}{r^4} + \mathcal{O}(\frac{1}{r^5}), \quad (6)$$

where $x^\pm \equiv \frac{t}{\ell} \pm \theta$, and θ and r are the angular and radial coordinates. We have introduced explicitly the first subleading terms in the asymptotic expansion of the metric field and we want to investigate how they are related to the conformal field theory on the boundary. The infinitesimal diffeomorphisms $\zeta^a(r, t, \theta)$ preserving (3-6) are of the form [1, 2]

$$\zeta^+ = 2T^+ + \frac{\ell^2}{r^2} \partial_-^2 T^- + \mathcal{O}(\frac{1}{r^4}), \quad (7)$$

$$\zeta^- = 2T^- + \frac{\ell^2}{r^2} \partial_+^2 T^+ + \mathcal{O}\left(\frac{1}{r^4}\right), \quad (8)$$

$$\zeta^r = -r(\partial_+ T^+ + \partial_- T^-) + \mathcal{O}\left(\frac{1}{r}\right), \quad (9)$$

where the functions $T^\pm(x^\pm)$ verify the conditions $\partial_\pm T^\mp = 0$. Those diffeomorphisms with $T^\pm = 0$ should be considered as "gauge transformations". Therefore, if one consider the diffeomorphisms

$$\zeta^\pm = \frac{\alpha^\pm}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right), \quad (10)$$

$$\zeta^r = \frac{\alpha^r}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (11)$$

where α^\pm and α^r are arbitrary functions of x^+ and x^- , it is not difficult to see that the variables $\gamma_{\pm\pm}$, $\gamma_{+-} - \frac{1}{4\ell^2} \gamma_{rr}$ are the only gauge invariant quantities. Moreover, the equations of motion imply that

$$0 = R + \frac{6}{\ell^2} = -\frac{8}{r^2 \ell^2} (\gamma_{+-} - \frac{1}{4\ell^2} \gamma_{rr}) + \mathcal{O}\left(\frac{1}{r^3}\right), \quad (12)$$

and this requires that

$$\gamma_{+-} - \frac{1}{4\ell^2} \gamma_{rr} = 0 \quad (13)$$

The remaining equations of motion $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{\ell^2} g_{\mu\nu}$ lead to the equations

$$\partial_- \gamma_{++} = 0, \quad (14)$$

$$\partial_+ \gamma_{--} = 0, \quad (15)$$

In addition one can make, using the gauge transformations (10) y (11), the following consistent gauge choice

$$\gamma_{\pm r} = 0, \quad (16)$$

$$\gamma_{rr} = 0, \quad (17)$$

so the physical degrees of freedom are described by two chiral functions $\gamma_{\pm\pm}(x^\pm)$:

$$ds^2 = \frac{\ell^2}{r^2} dr^2 - r^2 dx^+ dx^- + \gamma_{++} (dx^+)^2 + \gamma_{--} (dx^-)^2 + \mathcal{O}\left(\frac{1}{r}\right), \quad (18)$$

For instance, the standard BTZ black hole solutions can be brought, via gauge transformations, to the form (18) with

$$\gamma_{++} = 2G\ell(M\ell + J), \quad (19)$$

$$\gamma_{--} = 2G\ell(M\ell - J), \quad (20)$$

It is interesting to note that, when either $\gamma_{++} = 0$ or $\gamma_{--} = 0$, the omitted terms in the expansion (18) vanish (in the appropriate gauge) and the corresponding BTZ solutions describe extremal black hole geometries. Hence

$$ds^2 = \frac{\ell^2}{r^2} dr^2 - r^2 dx^+ dx^- + \gamma_{++} (dx^+)^2 \quad (21)$$

is an exact solution.

We want now to identify the fields $\gamma_{\pm\pm}$ in terms of appropriate conformal fields on the boundary. To this end, let us determine the conformal transformation properties of these fields. The action of the diffeomorphisms (7-9) on the metric (18) induces the following transformation law

$$\delta_{T^\pm} \gamma_{\pm\pm} = 2(T^\pm \partial_\pm \gamma_{\pm\pm} + 2\gamma_{\pm\pm} \partial_\pm T^\pm) - \ell^2 \partial_\pm^3 T^\pm. \quad (22)$$

It is then clear that the variables $\gamma_{\pm\pm}$ are proportional, up to a constant $\frac{c}{24}$, to the stress tensor components $\Theta_{\pm\pm}$ of the underlying conformal field theory with central charge $c = \frac{3\ell}{2G}$

$$\Theta_{\pm\pm} = \frac{1}{4\ell G} \gamma_{\pm\pm} + \frac{\ell}{16G}, \quad (23)$$

The above relation can also be confirmed by working out the conserved charges $J[\xi]$ given in [1]

$$J[\xi] \propto \lim_{r \rightarrow \infty} \int d\phi \left\{ \frac{\ell}{r} \xi^\pm + \frac{r^3}{\ell^3} \xi^\pm (g_{rr} - \frac{\ell^2}{r^2}) + \frac{1}{\ell} \left(\frac{\xi^\pm}{r} + \xi_{,r}^\pm \right) (g_{\phi\phi} - r^2) + 2\xi^\phi \pi_\phi^r \right\}, \quad (24)$$

in terms of the metric (18). One obtains

$$J[\xi] \propto \int d\phi \{ T^+ (4\gamma_{++} + \ell^2) + T^- (4\gamma_{--} + \ell^2) \}, \quad (25)$$

which corresponds to the conserved charges associated to the stress tensor (23) of a conformal field theory on a cylinder with central charge $c = \frac{3\ell}{2G}$. Note that the shift term $\frac{c}{24}$ in (23) arises from changing variables from the sphere to the cylinder $z = e^{\frac{t}{\ell} + i\theta}$. The Fourier components $L_n(\bar{L}_n)$ of $\Theta_{++}(\Theta_{--})$ obey the Virasoro algebra

$$i \{ L_n, L_m \} = (n-m)L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n,-m}, \quad (26)$$

$$i \{ \bar{L}_n, \bar{L}_m \} = (n-m)\bar{L}_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n,-m}, \quad (27)$$

$$\{ L_n, \bar{L}_m \} = 0, \quad (28)$$

with central charge $c = \frac{3\ell}{2G}$.

Our aim now is to provide a more explicit description of the expansion (3-6). A simple and natural way to do this is to realize that the conformal symmetry also arises from a conformal analysis of infinity. The conformally related metric $d\bar{s}^2 = \frac{e^{2\phi}}{r^2} ds^2$ where ds^2 is the elementary black hole solution with $M = J = 0$

$$ds^2 = -r^2 dx^+ dx^- + \frac{\ell^2}{r^2} dr^2, \quad (29)$$

and $e^{2\phi}$ is a positive function depending on the coordinates x^\pm , induces a metric on the boundary $r \rightarrow \infty$

$$d\bar{s}_b = -e^{2\phi} dx^+ dx^-, \quad (30)$$

The 2d conformal symmetry (7-8) acts naturally on the metric (30) preserving the conformal structure. This is in fact a particular case of the more general AdS_{d+1}/CFT_d correspondence suggested in [10] and elaborated in [11, 12]. In Ref. [13] the Weyl anomaly for conformal field theories described via the supergravity action in $d=2,4,6$ has been calculated regularising the gravitational action in a generally covariant way. In the case $d=2$ one recovers the well-known trace anomaly $\langle T^\alpha_\alpha \rangle = \frac{c}{24\pi} R$. However, it was pointed out in [14] that in quantising a $2d$ conformal field one can alternatively preserve the Weyl symmetry and partially break diffeomorphism invariance. Therefore, the trace anomaly disappears but one encounters that the stress tensor does not obey the covariant conservation law:

$$\langle \nabla_\mu T^{\mu\nu} \rangle = -\frac{c}{48\pi} \partial_\nu R(\sqrt{-g}g^{\alpha\beta}). \quad (31)$$

In conformal coordinates this equation implies that $\langle T^{\mu\nu} \rangle$ transforms according to the Virasoro anomaly [15]. Our approach is related to this second viewpoint because, as we have already mentioned, the diffeomorphism generated by $L_n, n \neq 0, \pm 1$ should not be considered as "pure gauge transformations" due to the Virasoro anomaly. Therefore it is natural in our scheme to pick up a trivial metric on the boundary to represent the unique conformal equivalence class. However, due to the breakdown of general covariance in the boundary, we cannot choose a particular form of the flat metric, but instead we have to consider the general form of a flat metric. All this means that ϕ should be a

free field. Going back to the bulk part of AdS₃ we see immediately that the metric

$$ds^2 = -e^{2\phi} r^2 dx^+ dx^- + \frac{\ell^2}{r^2} dr^2, \quad (32)$$

induces the boundary metric (30) and satisfy Einstein 's equations. However (32) does not satisfy the asymptotic conditions (3-6). To recover these conditions we have to transform (32) in an appropriated way. A redefinition of the radial coordinate $r \rightarrow r e^{-\phi}$ brings the metric into the form

$$\begin{aligned} ds^2 = & - r^2 dx^+ dx^- + \frac{\ell^2}{r^2} dr^2 - \frac{2\ell^2}{r} (\partial_+ \phi dx^+ dr + \partial_- \phi dx^- dr) \\ & + \ell^2 (\partial_+ \phi)^2 (dx^+)^2 + \ell^2 (\partial_- \phi)^2 (dx^-)^2 \\ & + 2\ell^2 (\partial_+ \phi) (\partial_- \phi) dx^+ dx^-, \end{aligned} \quad (33)$$

The above metric fulfils the conditions (3,4,6), but to satisfy the requirement (5) it is necessary to do a second transformation $x^\pm \rightarrow x^\pm + \frac{\ell^2}{r^2} \partial_\mp \phi$. We then get

$$\begin{aligned} ds^2 = & - r^2 dx^+ dx^- + \frac{\ell^2}{r^2} dr^2 + (4 \frac{\ell^4}{r^4} \partial_+ \phi \partial_- \phi + \mathcal{O}(\frac{1}{r^6})) dr^2 \\ & + (2\ell^2 (\partial_+ \phi \partial_- \phi - \partial_+ \partial_- \phi) + \mathcal{O}(\frac{1}{r^2})) dx^+ dx^- \\ & + (\ell^2 ((\partial_+ \phi)^2 - \partial_+^2 \phi) + \mathcal{O}(\frac{1}{r^2})) (dx^+)^2 \\ & + (\ell^2 ((\partial_- \phi)^2 - \partial_-^2 \phi) + \mathcal{O}(\frac{1}{r^2})) (dx^-)^2 \\ & + \mathcal{O}(\frac{1}{r^3}) dx^+ dr + \mathcal{O}(\frac{1}{r^3}) dx^- dr, \end{aligned} \quad (34)$$

where the omitted terms can be computed in a recursive way. Therefore we have

$$\gamma_{\pm\pm} = \ell^2 ((\partial_\pm \phi)^2 - \partial_\pm^2 \phi), \quad (35)$$

$$\gamma_{+-} = \ell^2 (\partial_+ \phi \partial_- \phi - \partial_+ \partial_- \phi), \quad (36)$$

$$\gamma_{rr} = 4\ell^4 (\partial_+ \phi \partial_- \phi), \quad (37)$$

$$(38)$$

but, as we have already mentioned, only the terms $\gamma_{\pm\pm}$, $\gamma_{+-} - \frac{1}{4\ell^2} \gamma_{rr}$ are gauge invariant.

The transformation law (22) can be reproduced if ϕ transforms as

$$\delta\phi = \ell^2 \{2\partial_+\phi T^+ + 2\partial_-\phi T^- + \partial_+T^+ + \partial_-T^-\}, \quad (39)$$

Redefining now the scalar field $\bar{\phi} = \sqrt{\frac{\ell}{2G}}\phi$ the stress tensor $\Theta_{\pm\pm}$ takes the form

$$\Theta_{\pm\pm} = \frac{1}{2} [(\partial_{\pm}\bar{\phi})^2 - \sqrt{\frac{\ell}{2G}}\partial_{\pm}^2\bar{\phi}] + \frac{\ell}{16G}, \quad (40)$$

which is similar to that arising in Liouville theory. However, it is well-known [16, 17] that a Bäcklund transformation convert the Liouville theory into an improved free field theory. It is just this free field theory which emerges in this approach. The quantum stress tensor (40) gives rise to a central charge

$$c = 1 + 3Q^2, \quad (41)$$

where the background charge is given by $Q = \sqrt{\frac{\ell}{2G}} + 2\sqrt{\frac{2G}{\ell}}$. In a semiclassical description $\ell \gg G$ the central charge (41) reproduces the classical value (2). It is also worthwhile to remark that when ϕ is a chiral field there is not subleading terms in (34) and one recovers solutions of the form (21).

If we choose a boundary metric of constant curvature λ to represent the given conformal equivalence class

$$d\bar{s}_b = -e^{2\phi_L} dx^+ dx^-, \quad (42)$$

,where ϕ_L obeys a Liouville equation $\partial_-\partial_+\phi_L = \frac{\lambda}{8}e^{2\phi_L}$, the extension of the metric to the bulk part is more involved than (32). Moreover, the first gauge invariant subleading terms in the asymptotic expansion are

$$\gamma_{\pm\pm} = \ell^2((\partial_{\pm}\phi_L)^2 - \partial_{\pm}^2\phi_L), \quad (43)$$

$$\gamma_{+-} - \frac{1}{4\ell^2}\gamma_{rr} = -\ell^2(\partial_+\partial_-\phi_L - \frac{\lambda}{8}e^{2\phi_L}), \quad (44)$$

So we also recover the usual expression for the stress tensor of Liouville theory. Therefore, and according to this, the freedom in picking a metric on the conformal structure seems to be related to canonical transformations of the boundary theory. The boundary metric (32) is the most natural one to relate the bulk gravity theory to the boundary conformal field theory.

In this paper we have provided an explicit relation between the "would-be gauge" degrees of freedom of the gravity theory and the underlying boundary conformal field theory. To do this we have made use of the holographic correspondence AdS/CFT of [12] for the theory (1). However, instead of breaking Weyl invariance on the boundary as in [12, 13], we sacrifice partially diffeomorphism invariance to make contact with the asymptotic boundary conditions of [1, 2]. This way we have provided an explicit relation between the bulk metric on AdS₃, verifying the boundary conditions of [1, 2] and an improved free field theory on the boundary. This result raises a question also pointed out by Carlip [18] concerning the derivation of the BTZ black hole entropy of Ref [2]. The Cardy's formula for the asymptotic density of states of a conformal field theory with central charge c

$$\log \rho(\Delta, \bar{\Delta}) \sim 2\pi \sqrt{\frac{c\Delta}{6}} + 2\pi \sqrt{\frac{c\bar{\Delta}}{6}} \quad (45)$$

where Δ and $\bar{\Delta}$ are the eigenvalues of the two Virasoro generators L_0 and \bar{L}_0 , assumes that the lowest eigenvalues $(\Delta_0, \bar{\Delta}_0)$ of L_0 and \bar{L}_0 vanish. In general, the above formula is still essentially valid if one replace the central charge "c" by the so-called effective central charge $c_{eff} = c - 24\Delta_0$ [19]. However, the minimum value of L_0 (\bar{L}_0) is not zero for the improved free field $\bar{\phi}$, in fact $c_{eff} = 1$ as one should expect by a direct counting of states. Moreover, for normalizable (macroscopic) states [20] of Liouville theory we also have $c_{eff} = 1$. To properly account for the black hole entropy one should include states of imaginary momentum for the improved free field, or microscopic states in the terminology of Liouville theory.

In a recent paper [21] it is claimed that stringy degrees of freedom account for the full density of states giving rise the Bekenstein-Hawking entropy of BTZ black holes.

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