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Free field realization of cylindrically symmetric Einstein gravity *

J. Cruz †, A. Miković ‡ and J. Navarro-Salas §

Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC. Facultad de Física, Universidad de Valencia, Burjassot-46100, Valencia, Spain

Abstract

Cylindrically reduced Einstein gravity can be regarded as an $SL(2,R)/SO(2)$ sigma model coupled to 2D dilaton gravity. By using the corresponding 2D diffeomorphism algebra of constraints and the asymptotic behaviour of the Ernst equation we show that the theory can be mapped by a canonical transformation into a set of free fields with a Minkowskian target space. We briefly discuss the quantization in terms of these free-field variables, which is considerably simpler than in the other approaches.

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†E-mail address: cruz@lie.uv.es
‡E-mail address: mikovic@lie.uv.es. On leave of absence from Institute of Physics, P.O.Box 57, 11001 Belgrade, Yugoslavia
§E-mail address: jnavarro@lie.uv.es

Two Killing vector reductions of 4D Einstein equations are exactly integrable 2D models [1, 2, 3, 4, 5], and therefore offer an interesting arena to investigate the quantization of the gravitational field [6, 7, 8]. These models are described by a 4D line-element of the form

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \varphi(x)\Lambda_{ab}(x)d\chi^a d\chi^b \quad , \tag{1}$$

where x^μ are the 2D coordinates, $\partial/\partial\chi^a$ are the Killing vectors and $\det\Lambda = 1$. The corresponding Einstein equations can be derived from an action for a 2D dilaton gravity coupled to an $SL(2,R)/SO(2)$ coset space σ-model [9]

$$S = \int d^2x\sqrt{-g}\varphi\left[R - \frac{1}{4}tr(\Lambda^{-1}\nabla^\mu\Lambda\Lambda^{-1}\nabla_\mu\Lambda)\right] . \tag{2}$$

Depending on the choice of the coordinates and the spatial topology one can describe cylindrical gravitational waves $(x^\mu = (t,r),\ \chi^a = (z,\phi))$, axisymmetric stationary solutions $(x^\mu = (z,r),\ \chi^a = (t,\phi))$ or Gowdy cosmologies $(x^\mu = (t,x),\ \chi^a = (y,z)$ $x,y,z \in S^1)$ where t is a time coordinate while r,ϕ,z are cylindrical coordinates and x,y,z are Cartesian coordinates.

The usual approach to study (2) is to work in the reduced phase space formalism ($\varphi = r$ gauge for cylindrical waves), so that the complete dynamics is contained in the Ernst equation for the matrix Λ

$$\nabla_\mu(\varphi\Lambda^{-1}\nabla^\mu\Lambda) = 0 . \tag{3}$$

Note that (3) has a duality symmetry, so that if

$$\Lambda = \frac{1}{\Delta}\begin{pmatrix} h^2+\Delta^2 & h \\ h & 1 \end{pmatrix} \tag{4}$$

is a solution then

$$\tilde\Lambda = \frac{\Delta}{r}\begin{pmatrix} h^2+\frac{r^2}{\Delta^2} & \tilde h \\ \tilde h & 1 \end{pmatrix} \tag{5}$$

is also a solution, provided that ${}^*d\tilde h = \frac{r}{\Delta^2}dh$. This symmetry implies that $d\tilde s^2$ can have the asymptotic behaviour of a flat metric in cylindrical coordinates, i.e. $d\tilde s^2 \sim -dt^2 + dr^2 + r^2d\phi^2 + dz^2$, which is relevant for the case of cylindrical waves, which we are going to study in detail.

The aim of this letter is to show that there is a considerable advantage if one does not fix the gauge completely, and consequently uses the special properties of the 2D diffeomorphism algebra of the constraints. The special role of 2D diffeomorphisms has been already recognized in [8], although in that work the reduced phase space formalism and an abelianized form of the constraints was used. In this paper we employ the full constraint structure, which generalizes the technique used for the case of cylindrical gravitational waves with one polarization [10], when the matrix Λ takes the diagonal form

$$\Lambda = \begin{pmatrix} e^f & 0 \\ 0 & e^{-f} \end{pmatrix} \ . \tag{6}$$

In that case it is possible to show that a canonical transformation exists which maps the constraints into a free-field form. We will show that the same can be done in the case of two polarizations, so that the theory is mapped into a theory of four free fields.

This looks as a surprising result, but there are general arguments which suggest that this is possible. The Poisson bracket algebra of the constraints for (2) is the 2D diffeomorphism algebra. This algebra admits representations quadratic in canonical variables

$$G_0 = \frac{1}{2}\left(\eta^{ij}P_iP_j + \eta_{ij}Q^{i\prime}Q^{j\prime}\right) \quad , \quad G_1 = P_iQ^{i\prime} \quad , \tag{7}$$

where G_0 is the hamiltonian constraint and G_1 is the spatial diffeomorphism constraint. The prime denotes differentiation with respect to the spatial coordinate (r in our case), $i,j = 1,...,n$ and η_{ij} is a flat Minkowskian metric. The quadratic representations are not possible in higher dimensions, since then the constraint algebra has structure functions which are not constant. The representation (7) implies that Q^i are free fields and since (2) is an integrable 2D theory one then expects to find a canonical transformation from the initial canonical variables to (P_i, Q^i) variables. This argument also explains why it was possible to find free field canonical variables for many examples of integrable 2D dilaton gravity theories [11, 10]. Also note that by writing $g_{\mu\nu} = e^{2\rho}\hat{g}_{\mu\nu}$, where $\hat{g}_{\mu\nu}$ is a fixed background metric, the action (2) becomes a nonlinear σ-model with a four-dimensional target space

$$S = \int d^2x \sqrt{-\hat{g}}\left[G_{ij}(X)\hat{\nabla}_\mu X^i \hat{\nabla}^\mu X^j + \Phi(X)\hat{R}\right] \ . \tag{8}$$

The general covariance of (2), or equivalently the background independence of (8), implies that the theory is a conformally invariant field theory. In fact, the couplings $G_{ij}(X)$ and $\Phi(X)$ satisfy the lowest order β-functions equations in the standard loop expansion for dilaton-gravity. Since in the abelian case this conformal field theory is a free-field theory, one also expects that this will happen in the non-abelian case. In the abelian case (6) the Ernst equation reduces to the cylindrical Laplace equation and the theory can be explicitly solved. Furthermore, the asymptotic behaviour of the solutions allows one to show that the theory is equivalent to a theory of three free fields with a Minkowskian target-space. Although the Ernst equation cannot be explicitly solved in the general case, the asymptotic behaviour of the solutions in the weak coupling regime $\varphi \to \infty$ is enough to show that the underlying conformal field theory is still described by a set of free fields.

The SL(2,R)/SO(2) coset space can be parametrized in the following way

$$\Lambda = \begin{pmatrix} e^f + h^2 e^{-f} & e^{-f}h \\ e^{-f}h & e^{-f} \end{pmatrix} \ . \tag{9}$$

This is the parametrization (4) with $\Delta = e^f$. The action (2) then becomes

$$S = \int d^2x\sqrt{-g}\varphi \left[R - \frac{1}{2}\left(\nabla f\right)^2 - \frac{1}{2}e^{-2f}\left(\nabla h\right)^2\right] . \tag{10}$$

The corresponding equations of motion in the conformal gauge are given by

$$\partial_+\partial_-\varphi = 0 , \tag{11}$$

$$4\partial_+\partial_-\rho + \partial_+ f\partial_- f + e^{-2f}\partial_+ h\partial_- h = 0 , \tag{12}$$

$$\partial_+(\varphi\partial_- f) + \partial_-(\varphi\partial_+ f) + 2\varphi e^{-2f}\partial_+ h\partial_- h = 0 , \tag{13}$$

$$\partial_+(\varphi e^{-2f}\partial_- h) + \partial_-(\varphi e^{-2f}\partial_+ h) = 0 , \tag{14}$$

$$C_\pm = 2\partial_\pm^2\varphi - 4\partial_\pm\varphi\partial_\pm\rho + \varphi\left[(\partial_\pm f)^2 + e^{-2f}(\partial_\pm h)^2\right] = 0 , \tag{15}$$

where $C_\pm = \frac{1}{2}(G_0 \pm G_1)$ are the constraint equations. The free field equation (11) has the obvious solution

$$\varphi = A_+(x^+) + A_-(x^-) , \tag{16}$$

with $A_\pm$ two arbitrary chiral functions, while the conformal factor ρ can be expressed as

$$\rho = a_+(x^+) + a_-(x^-) + \frac{1}{4}\int_{x^+}^{\infty} dy^+ \int_{-\infty}^{x^-} dy^- \left[\partial_+ f\partial_- f + e^{-2f}\partial_+ h\partial_- h\right] , \tag{17}$$

where $a_\pm$ are other two arbitrary chiral functions. By inserting the last two expressions into the constraints, we obtain

$$\begin{aligned} C_\pm &= 2\partial_\pm^2 A_\pm - 4\partial_\pm A_\pm\partial_\pm a_\pm \\ &\quad + \partial_\pm A_\pm \int_{\mp\infty}^{x^\mp} dy^\mp \left[\partial_+ f\partial_- f + e^{-2f}\partial_+ h\partial_- h\right] \\ &\quad + \varphi\left[(\partial_\pm f)^2 + e^{-2f}(\partial_\pm h)^2\right] = 0 . \end{aligned} \tag{18}$$

The Bianchi identities $\partial_\mp C_\pm = 0$ imply that $\partial_\mp P_\pm = 0$, where

$$\begin{aligned} P_\pm &= \frac{1}{2}\partial_\pm A_\pm \int_{\mp\infty}^{x^\mp} dy^\mp \left[\partial_+ f\partial_- f + e^{-2f}\partial_+ h\partial_- h\right] \\ &\quad + \frac{1}{2}\varphi\left[(\partial_\pm f)^2 + e^{-2f}(\partial_\pm h)^2\right] . \end{aligned} \tag{19}$$

$P_\pm(x^\pm)$ can be evaluated by taking the limits $x^\mp \to \mp\infty$, since then the integral terms vanish, and hence

$$P_\pm = \lim_{x^\mp \to \mp\infty} \frac{1}{2}\varphi\left[(\partial_\pm f)^2 + e^{-2f}(\partial_\pm h)^2\right] . \tag{20}$$

The main difference with respect to the abelian case ($h = 0$) is that the equations (13) and (14) for the fields f and h can not be solved explicitely. However, for our purposes it is sufficient to know the asymptotic behaviour of the solutions and this can be found without solving the equations.

If we perform a change of variables $f\sqrt{\varphi} = \tilde{F},\ h\sqrt{\varphi} = \tilde{H}$ where $\tilde{F}, \tilde{H}$ and their derivatives are bounded in the limit $\varphi \to \infty$ and $A\left(x^+\right)$ and $B\left(x^-\right)$ are monotonic increasing (decreasing) functions which go as $x^+(-x^-)$ when $x^\pm \to \infty$ $(x^- \to -\infty)$, then the equations (13) and (14) can be written as

$$\partial_+\partial_-\tilde{F} + O\left(\frac{1}{\sqrt{\varphi}}\right) = 0\,, \tag{21}$$

$$\partial_+\partial_-\tilde{H} + O\left(\frac{1}{\sqrt{\varphi}}\right) = 0\,. \tag{22}$$

Therefore in the limit $\varphi \to \infty$, one has $f \sim \frac{F}{\sqrt{A_+ + A_-}}$ and $h \sim \frac{H}{\sqrt{A_+ + A_-}}$, where F and H are bounded free fields with bounded derivatives. In the abelian case this is explicitly realized because the exact solution for the field f,

$$\begin{aligned} f &= \frac{1}{2}\int_{-\infty}^{\infty} d\lambda\, J_0\left(\frac{\lambda}{2}(A_+ + A_-)\right)\Big[B_+(\lambda)e^{i\frac{\lambda}{2}(A_+ - A_-)} \\ &\quad + B_-(\lambda)e^{-i\frac{\lambda}{2}(A_+ - A_-)}\Big]\,, \end{aligned} \tag{23}$$

where $B_\pm(\lambda)$ are arbitrary coefficients, behaves as

$$\begin{aligned} f &\sim \frac{1}{\sqrt{(A_+ + A_-)}}\int_{-\infty}^{\infty} d\lambda(\pi\lambda)^{-\frac{1}{2}}\Big[B_+(\lambda)e^{i\lambda A_+}e^{-i\frac{\pi}{4}} + B_-(\lambda)e^{-i\lambda A_+}e^{i\frac{\pi}{4}} \\ &\quad + B_+(\lambda)e^{-i\lambda A_-}e^{-i\frac{\pi}{4}} + B_-(\lambda)e^{i\lambda A_-}e^{i\frac{\pi}{4}}\Big]\,, \end{aligned} \tag{24}$$

when $A_+ + A_- \to \infty$ and therefore F is a bounded free field and $\partial_+ F, \partial_- F, ...$ are also bounded. By taking into account the asymptotic behaviour of f and h, we can obtain from (20)

$$P_\pm = \frac{1}{2}(\partial_\pm F)^2 + \frac{1}{2}(\partial_\pm H)^2\,. \tag{25}$$

If one defines

$$X^\pm = A_\pm, \qquad \Pi_\pm = -4\partial_\pm a_\pm + 2\frac{\partial_\pm^2 A_\pm}{\partial_\pm A_\pm}\,, \tag{26}$$

the constraints take a free-field form

$$C_\pm = \Pi_\pm \partial_\pm X^\pm + (\partial_\pm F)^2 + (\partial_\pm H)^2\,. \tag{27}$$

In terms of the canonical variables, (27) can be written as

$$C_\pm = \pm\Pi_\pm X'^\pm + \frac{1}{4}\left(\Pi_F \pm F'\right)^2 + \frac{1}{4}\left(\Pi_H \pm H'\right)^2\,. \tag{28}$$

By performing a canonical transformation

$$2X^{\pm\prime} = \mp(\Pi_1 - \Pi_0) - X^{0\prime} - X^{1\prime} \quad , \quad 2\Pi^{\pm} = -\Pi_0 - \Pi_1 \mp (X^{1\prime} - X^{0\prime}) \tag{29}$$

the constraints take the form (7).

We can also find the exact expressions for the free fields F and H in terms of the initial variables. In order to do this, we split the expressions (19) as $P_{\pm} = P^1_{\pm} + P^2_{\pm}$ where

$$P^1_{\pm} = \frac{1}{2}\varphi\,(\partial_{\pm} f)^2 + \frac{1}{2}\partial_{\pm}\varphi \int_{\mp\infty}^{x^{\mp}} dy^{\mp}\partial_+ f\partial_- f + \int_{\mp\infty}^{x^{\mp}} dy^{\mp}\varphi e^{-2f}\partial_{\pm} f\partial_+ h\partial_- h\,, \tag{30}$$

$$P^2_{\pm} = \frac{1}{2}\varphi e^{-2f}(\partial_{\pm} h)^2 + \frac{1}{2}\partial_{\pm}\varphi \int_{\mp\infty}^{x^{\mp}} dy^{\mp} e^{-2f}\partial_+ h\partial_- h - \int_{\mp\infty}^{x^{\mp}} dy^{\mp}\varphi e^{-2f}\partial_{\pm} f\partial_+ h\partial_- h\,. \tag{31}$$

It is easy to check that the equations of motion imply that

$$\partial_{\mp} P^1_{\pm} = \partial_{\mp} P^2_{\pm} = 0\,. \tag{32}$$

Thus we have divided $P_{\pm}$ into two free field contributions, and it is clear that we can write

$$P^1_{\pm} = \frac{1}{2}(\partial_{\pm} F)^2 \quad , \quad P^2_{\pm} = \frac{1}{2}(\partial_{\pm} H)^2\,. \tag{33}$$

These two equations can serve as the defining relations for the free fields F and H in terms of the initial variables.

Therefore we have constructed a transformation leading to constraints quadratic in chiral variables and this implies that the transformation is canonical [10], since there is no other expression for the symplectic form that together with (28) reproduces the Hamiltonian equations of motion for the free fields $\Pi_{\pm}, X^{\pm}, F$ and H. Alternatively, one can examine the symplectic form on the space of solutions. It can be written as

$$\omega = \omega_+ + \omega_- = \frac{1}{2}\int_{x^-=\infty} dx^+\delta j^- + \frac{1}{2}\int_{x^+=-\infty} dx^-\delta j^+\,, \tag{34}$$

where δ stands for the exterior derivative on the space of solutions and j^{μ} is the symplectic current potential [12]. The coefficients of one half in (34) come from the reflecting boundary conditions at $r = 0$. The light-cone components of the one-form current j^{μ} can be easely calculated from the action (10)

$$j^+ = -4\varphi\partial_-\delta\rho - \varphi\partial_- f\delta f - \varphi e^{-2f}\partial_- h\delta h\,, \tag{35}$$

$$j^- = 4\partial_+\varphi\delta\rho - \varphi\partial_+ f\delta f - \varphi e^{-2f}\partial_+ h\delta h\,. \tag{36}$$

By taking into account the asymptotic behaviour of f and h it is easy to see that

$$\begin{aligned}
\omega &= \frac{1}{2}\int_{x^-=-\infty} dx^+ \left[\delta X^+ \delta\Pi_+ + \delta F_+ \delta\partial_+ F + \delta H_+ \delta\partial_+ H\right] \\
&+ \frac{1}{2}\int_{x^+=\infty} dx^- \left[\delta X^- \delta\Pi_- + \delta F_- \delta\partial_- F + \delta H_- \delta\partial_- H\right] \\
&= \int_{t=const.} dr \left[-\delta X^0 \delta\dot{X}^0 + \delta X^1 \delta\dot{X}^1 + \delta F \delta\dot{F} + \delta H \delta\dot{H}\right] , \qquad (37)
\end{aligned}$$

where $F_\pm(x^\pm), H_\pm(x^\pm)$ are the chiral parts of the free fields F and H, $(F = F_+ + F_-, H = H_+ + H_-)$ and dots represent the t derivatives.

Note that the defining relations for the free fields (26) and (33) are also valid in the case when r is compact, and therefore one can have free fields in the case of Gowdy cosmologies. In that case the corresponding free-field theory is a string theory in 4D Minkowski space.

Since the cylindrically symmetric gravity can be mapped into a set of four free fields with a Minkowskian target space, the quantization in terms of the free-field variables is considerably simpler than if one uses the observables obtained from the Ernst equation [3], since the later lead to a non-linear Yangian algebra. A less straightforward task will be finding the expectation values of the original variables, since they become complicated functionals of the free fields. The problem of expressing the original variables in terms of the observables is in general a complicated problem. However, in our case the existence of the free fields F and H implies that one can write an asymptotic series expansions

$$f = \frac{F}{\sqrt{\varphi}} + \frac{F_1}{\varphi} + \frac{F_2}{\varphi\sqrt{\varphi}} + \cdots \quad , \qquad (38)$$

and

$$h = \frac{H}{\sqrt{\varphi}} + \frac{H_1}{\varphi} + \frac{H_2}{\varphi\sqrt{\varphi}} + \cdots \quad , \qquad (39)$$

where F_i and H_i are functions of F and H, which can be determined from (33). In this way one obtains recurrence relations for higher-order F_i and H_i in terms of the lower order ones, which can be solved order by order. For example, $F_1 = -\frac{1}{2}H^2$ and $H_1 = FH$, and so on. When $H = 0$, one recovers in this way the asymptotic expansion of the Bessel function, which is the exact solution in the Abelian case. Hence the relations (38) and (39) can serve as explicit expressions for f and h in terms of the free fields F and H. Note that the asymptotic flatness of the dual metric given by (5) requires that $f \to 0$ and $\tilde{h} \to 0$ for $r \to \infty$. Since asymptotically $\partial_\pm \tilde{h} \sim \mp r dh$ and $h \sim \frac{H}{\sqrt{r}}$, we then obtain that $H = O(r^\epsilon)$, where $\epsilon < -1/2$. Note that this asymptotic behaviour for H corresponds to square integrable functions on the r line. This is relevant for the quantum case, since this asymptotics gives the Fock space representations.

Note that in the free-field approach the quantum constraints generate a 2D conformal algebra with a central charge $c = 4$, if the standard quantization of a conformal field theory is used. Consistent quantization can be then achieved via the introduction of ghost fields and background charges in order to have vanishing of the total central charge. Alternatively, if the theory is quantized in the Schrodinger representation, the value of the central charge is $c = 2$, because the scalar field with negative kinetic energy contributes with $c = -1$ to the Virasoro anomaly[11]. In order to have a consistent Dirac quantization one has to modify the quantum constraints in such a way that the anomaly cancels. The addition to the constraints of a term depending on the pure 2D dilaton gravity variables $X^{\pm}$ ensures that

$$\tilde{C}_{\pm} = C_{\pm} + \frac{2}{48\pi}\left[\frac{X^{\pm\prime\prime\prime}}{X^{\pm\prime}} - \left(\frac{X^{\pm\prime\prime}}{X^{\pm\prime}}\right)^{2}\right] , \tag{40}$$

form the constraint algebra without the anomaly [11]. The modified constraints can be solved in terms of the ”gravitationally dressed” oscillators [13], defined by

$$\hat{F}\left(X\right) = \frac{1}{2\sqrt{\pi}}\int\frac{dk}{|k|}\left[e^{ikX}\hat{a}_{F}\left(k\right) + \ h.c.\right] , \tag{41}$$

$$\hat{H}\left(X\right) = \frac{1}{2\sqrt{\pi}}\int\frac{dk}{|k|}\left[e^{ikX}\hat{a}_{H}\left(k\right) + \ h.c.\right] . \tag{42}$$

The Fourier coefficients $\hat{a}_F(k)$ and $\hat{a}_H(k)$ constitute a complete set of observables, so it would be interesting to see how they are related to the observables obtained from the Ernst equation. Note that one can construct an $SL(2,R)$ affine algebra from the Ernst equation observables, and this algebra generates the Geroch group [14, 5]. On the other hand, one can easily construct an $SL(2,R)$ affine algebra from $\hat{a}_F(k)$ and $\hat{a}_H(k)$ via the Wakimoto construction [15]. This algebra will be also a dynamical symmetry algebra. How these two algebras are related would be an interesting problem for further study.

We expect that our results can be extended to the case of an arbitrary coset space sigma model coupled to 2D dilaton gravity.

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References

[1] V. Belinskii and V. Zakharov, *Sov. Phys.* JETP 48 (1978) 985.

[2] D. Maison, *Phys. Rev. Lett.* 41 (1978) 521.

[3] D. Korotkin and H. Samtleben, *Phys. Rev. Lett.* 80 (1998) 14.

[4] A. Ashtekar and V. Hussain, gr-qc/9712053.

[5] D. Bernard and B. Julia, hep-th/9712254.

[6] K. V. Kuchar, *Phys. Rev.* D 4 (1971) 955.

[7] A. Ashtekar, *Phys. Rev. Lett.* 77 (1996) 4864.
A. Asthekar and M. Pierri, *J. Math. Phys.* 37 (1996) 6250.

[8] D. Korotkin and H. Nicolai, *Nucl. Phys.* B 475 (1996) 397.

[9] P. Breitenlohner, D. Maison and G. Gibbons, *Commun. Math. Phys.* 120 (1988) 295.

[10] J. Cruz, D. J. Navarro and J. Navarro-Salas, hep-th/9712194.

[11] D. Cangemi, R. Jackiw and B. Zwiebach, *Ann. Phys*. (N.Y.) 245 (1996) 408.

[12] Č. Crnković and E. Witten, in *"Three Hundred Years of Gravitation"*, ed. S. W. Hawking and W. Israel, C.U.P. Cambridge (1987) 676.

[13] K. V. Kuchar, J. D. Romano and M. Varadarajan, *Phys. Rev.* D 55 (1997) 795.
E. Benedict, R. Jackiw and H.-J. Lee *Phys. Rev.* D54 (1996) 6213.

[14] D. Korotkin and H. Samtleben, gr-qc/9611061.

[15] M. Wakimoto, *Commun. Math. Phys.* 104 (1986) 605.