Two-point functions with an invariant Planck scale and thermal effects

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Nonlinear deformations of relativistic symmetries at the Planck scale are usually addressed in terms of modified dispersion relations. We explore here an alternative route by directly deforming the two-point functions of an underlying field theory. The proposed deformations depend on a length parameter (Planck length) and preserve the basic symmetries of the corresponding theory. We also study the physical consequences implied by these modifications at the Planck scale by analyzing the response function of an accelerated detector in Minkowski space, an inertial one in de Sitter space, and also in a black hole spacetime.

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I. INTRODUCTION

One of the most profound principles of physics is the principle of relativity, which establishes the physical equivalence of all inertial observers. Discovered by Galileo in his studies on the laws of motion, the principle of relativity was employed by Einstein, assuming as fundamental physical laws the equations of electrodynamics, to propose the special theory of relativity. The theory predicts a relation between the energy and momentum of any body given by the well-known expression

$$E^2 - p^2c^2 = m^2c^4 \ . \tag{1}$$

The theory was later generalized in two directions. One was the incorporation of the quantum uncertainty principle, which after many years gave rise to the well established framework of quantum field theory. One of the basic features of the relativistic quantum theory is the existence of vacuum fluctuations, typically of the form

$$\langle \Phi(x_1)\Phi(x_2)\rangle \sim \frac{1}{4\pi^2}\frac{\hbar}{(x_1-x_2)^2} \ , \tag{2}$$

as $x_1 \rightarrow x_2$. On the other hand the principle of relativity was generalized by Einstein to assume the physical equivalence of all freely falling observers, when gravity is present, and culminated in the formulation of the general theory of relativity. One of the main consequences of it is the possibility of deforming the causal structure of Minkowsky spacetime. This happens, typically, in a gravitational collapse producing a Schwarzschild black hole

$$ds^2 = -(1-\frac{2GM}{c^2r})c^2dt^2 + \frac{dr^2}{(1-\frac{2GM}{c^2r})} + r^2d\Omega^2 \ , \tag{3}$$

or, in a cosmological context, in a de Sitter spacetime

$$ds^2 = -c^2dt^2 + e^{2Ht}d\vec{x}^2 \ , \tag{4}$$

with a cosmological constant $\Lambda = 3H^2/c^2$.

The construction of a fully consistent quantum gravity theory, incorporating both fundamental theories in some limit, is one of the basic open problems of theoretical physics. Despite this, there appears to be some robust results when quantum field theory and general relativity are combined at the semiclassical level [1, 2]. The thermal properties associated with black hole and cosmological horizons [3, 4], with temperature

$$T = \frac{\kappa}{2\pi} \frac{\hbar}{c k_B} \; , \tag{5}$$

where κ is the surface gravity of the corresponding horizon, emerge as very deep and robust results, and are likely to be fundamental to unravel the basic features of a quantum theory of gravity. The vacuum fluctuations (2) play a crucial role in the derivation of these thermal results.

A phenomenological way to explore the properties of a quantum gravity theory, which is expected to incorporate the Planck scale $l_P = (G\hbar/c^3)^{1/2}$,[1] is by modifying one of the basic equations of special relativity, namely, the dispersion relation (1), to incorporate the Planck scale. At low energies the dispersion relation is very well approximated by the standard one, but when the energy or momentum saturates the Planck scale the physics potentially departs from special relativity. There are two different attitudes in doing this. One is to allow a violation of the principle of relativity by distinguishing a particular frame to which the modified dispersion relation refers. This line of thought, motivated by condensed-matter analogies, was originally aimed at explaining gamma ray phenomenology [6] and has led to the formulation of generally covariant theories of gravity coupled to a dynamical timelike vector field [7] (see, also [8]). A different attitude is to preserve the principle of relativity, equivalence of all inertial frames, by allowing a nonlinear action of Lorentz transformations keeping then invariant the modified dispersion relation. This perspective was pushed forward in [9, 10] and has some important connections to previous mathematical studies concerning quantum deformations of the Poincaré group [11].

In any case, although the modified dispersion relations are designed to be applied to microscopic particles, nothing prevents them from being applied to macroscopic bodies, as in conventional special relativity. The Planck scale is then saturated immediately, giving rise to some sort of paradox. Although this issue can be bypassed by adding extra hypotheses, it strongly suggests that one should explore alternative routes to deform special relativity. In this sense, one could consider the deformation *ab initio* of an intrinsic quantum mechanical object, with no classical analogue and, therefore, with no possibility of being applied to conventional macroscopic bodies. On the other hand, an important lesson that follows from working with deformed dispersion relations and non-linear realizations of the Lorentz symmetry in momentum space, is the difficulty or perhaps impossibility of getting a proper realization of the kinematical symmetry in position space (independent of the energy-momentum degrees of freedom). The non-linear action in the spacetime has been implemented in the (eight-dimensional) extended phase-space. In other words, one must double the number of dimensions to find a non-linear realization of the Lorentz symmetry. We thus find the following problem. How can one naturally consider quantum objects, depending on eight variables, on which Lorentz transformations may act non-linearly? The two-point correlation functions of matter fields seem to be the simplest such quantum objects. The aim of this paper is to explore this possibility. Firstly, as a legitimate mathematical problem. But also by looking carefully at its physical consequences concerning the thermal properties of horizons.

In section II, we shall briefly review the main aspects of the standard approach to introduce non-linear actions of Lorentz symmetry via deformed dispersion relations. This will help the reader to better understand our proposal, in section III, to deform the action of the kinematical symmetry via two-point correlation functions. In this context, the natural thermal effect to be analyzed with deformed actions of the Lorentz symmetry is the acceleration radiation in Minkowski space. This will be done in section IV by analyzing the response function of an accelerated detector with a modified two-point function. In sections V and VI we shall extend our considerations to de Sitter space and black holes, respectively. Finally, in section VII, we shall briefly summarize the main conclusions.

II.   DEFORMED DISPERSION RELATIONS AND NONLINEAR LORENTZ ACTIONS

Given a modified dispersion relation of the general form (from now on we shall assume $c = 1$)

$$E^2 f^2(E, p) - p^2 g^2(E, p) = m^2 \; , \tag{6}$$

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[1] For a general discussion and earlier references, see [5]

the easiest way to modify the action of Lorentz transformations on the energy-momentum p^μ is by introducing a nonlinear invertible map U between the physical quantities $(-E, p_i)$ and an auxiliary energy-momentum $(-\epsilon, \Pi_i)$

$$U : \; p_\mu \equiv (-E, p_i) \; \to \Pi_\mu \equiv (-\epsilon, \Pi_i) = (-f(E,p)E, \; g(E,p)p_i) \; . \tag{7}$$

While the auxiliary vector $(-\epsilon, \Pi_i)$ transforms linearly under the Lorentz group, the physical energy-momentum transforms as

$$L(p^\mu) = [U^{-1} \; L \; U](p^\mu) \; . \tag{8}$$

The simplest choice for the function U^{-1} is

$$
\begin{aligned}
E &= \frac{\epsilon}{1 + l_P \hbar^{-1} \epsilon} \\
p_i &= \frac{\Pi_i}{1 + l_P \hbar^{-1} \epsilon} \; ,
\end{aligned}
\tag{9}
$$

as was proposed in [10] (it has a deformed dispersion relation of the form $(E^2 - p^2)/(1 + E l_P \hbar^{-1})^2 = m^2$). In the limit $l_P \to 0$ the auxiliary energy-momentum coincides with the physical one and the nonlinear action becomes the standard linear transformation. Note also that, in special relativity the energy ϵ is unbounded. Using arbitrarily large boosts one can get $\epsilon \to +\infty$, but then the energy in the deformed theory is saturated to the observer-independent Planck scale $E \to \hbar/l_P$.

The above framework for nonlinear actions on momentum space can be extended to the full (covariant) phase-space, parameterized by (p^μ, x^μ), by further extending the map U

$$U : \; (p_\mu, x^\nu) \to (\Pi_\mu, X^\nu) \; , \tag{10}$$

where

$$X^\mu = X^\mu(x^\nu, p_\rho) \; . \tag{11}$$

The above action $X^\mu(x^\nu, p_\rho)$, and the corresponding one between physical variables $x'^\mu = (x^\nu, p_\rho)$, can be determined uniquely by imposing some extra physical condition. There are essentially two different proposals in the literature. One demands that plane-waves remain solutions of free field theories [12]. The other proposal requires that the full transformation be a canonical one on phase-space [13, 14, 15].

As remarked in the introduction, the proposal for realizing the bold idea of deforming a relativistic theory in terms of deformed dispersion relations has an apparent drawback. The deformation is intended to be relevant for subatomic particles and in such a way that either the energy and/or the momentum of the particle is saturated at the Planck scale. But a formulation in phase space applies equally to microscopic and macroscopic objects (or, in other words, to quantum and classical objects). In the latter case it is very easy to reach Planck energies, but Nature does not seem to deform the established kinematical symmetries in this situation. This fact motivates the approach of next section. Instead of working initially in phase-space and deformed dispersion relations we shall put forward an alternative way to further explore the physical consequences of deforming the action of kinematic symmetries. Instead of taking, as the starting point, a modification of dispersion relations we shall consider, ab initio, a modification of the two-point function of a free field theory, which makes sense only at the quantum level.

III. NONLINEAR ACTIONS AND DEFORMED TWO-POINT FUNCTIONS

A. Conformal field theories

To illustrate our approach it is convenient to consider the largest kinematical symmetry allowed by a relativistic theory in a generic d-dimensional flat spacetime. This is the conformal group $SO(d,2)$. In a conformal field theory [16] there is a particular set of fields with well-defined transformation laws under conformal transformations $x \to x'$:

$$\Phi_j(x) \to \left| \frac{\partial x'}{\partial x} \right|^{\Delta_j/d} \Phi_j(x') \; , \tag{12}$$

where $|\frac{\partial x'}{\partial x}|$ stands for the jacobian of the transformation and Δ_j is the dimension (or conformal weight) of $\Phi_j(x)$. Restricting attention to the two-point correlation function of a single field, covariance under the transformation (12)

implies[2]

$$\langle \Phi_j(x_1)\Phi_j(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x_1}^{\Delta_j/d}\left|\frac{\partial x'}{\partial x}\right|_{x_2}^{\Delta_j/d}\langle \Phi_j(x_1')\Phi_j(x_2')\rangle \ . \tag{13}$$

As is well-known, invariance under Lorentz transformations, translations, and dilations largely restricts the form of the two-point function. One gets

$$\langle \Phi_j(x_1)\Phi_j(x_2)\rangle = \frac{C_j}{(x_1-x_2)^{2\Delta_j}} \ , \tag{14}$$

where C_j is a constant related to the normalization of the field. A typical example is a massless scalar field in $d=4$, for which $\Delta = 1$ and then

$$\langle \Phi(x_1)\Phi(x_2)\rangle = \frac{1}{4\pi^2}\frac{\hbar}{(x_1-x_2)^2} \ , \tag{15}$$

where $(x_1-x_2)^2 \equiv -(T_2-T_1)^2 + (X_2-X_1)^2 + (Y_2-Y_1)^2 + (Z_2-Z_1)^2$.[3] In $d=2$ the formalism should be refined since then the global conformal group $SO(2,2)$ is enlarged to the infinite-dimensional group of local transformation $x^\pm \to x'^\pm(x^\pm)$, where $x^\pm = t \pm x$ are null coordinates. Relevant fields (usually called *primary fields*) should have a weight Δ_+ for $x^+ \to x'^+(x^+)$ and another one Δ_- with respect to $x^- \to x'^-(x^-)$. Typical examples are the derivatives $\partial_\pm\Phi$ of a two-dimensional massless scalar field Φ. In this case we also have $\Delta_+/d = 1$ and $\Delta_-/d = 0$ for $\partial_+\Phi$, and the opposite weights for $\partial_-\Phi$. This implies that

$$\langle \partial_\pm\Phi(x_1)\partial_\pm\Phi(x_2)\rangle = -\frac{1}{4\pi}\frac{\hbar}{(x_1^\pm - x_2^\pm)^2} \ . \tag{16}$$

B. Deforming the conformal two-point functions

Let us now deform the action of the kinematical symmetry on two-point functions mimicking the scheme followed above for modified dispersion relations. We can introduce an invertible map U defined as

$$U : \ \langle \phi_j(x_1)\phi_j(x_2)\rangle \ \to \langle \Phi_j(x_1)\Phi_j(x_2)\rangle \ . \tag{17}$$

The action of conformal transformations on the correlations $\langle \Phi_j(x_1)\Phi_j(x_2)\rangle$ induce, via U, an action on the (physical) two-point functions $\langle \phi_j(x_1)\phi_j(x_2)\rangle$. This can be seen with the following example. Let us choose the function U^{-1} similarly as in (9)

$$U^{-1} : \ \langle \phi_j(x_1)\phi_j(x_2)\rangle = \frac{\langle \Phi_j(x_1)\Phi_j(x_2)\rangle}{1 - l_P^2\hbar^{-1}\langle \Phi_j(x_1)\Phi_j(x_2)\rangle} \ , \tag{18}$$

where the constant $l_P^2\hbar^{-1}$, which (up to a factor c^{-3}) turns out to be equivalent to Newton's constant G, is naturally chosen for dimensional reasons. Accordingly, the deformed action of conformal transformations is

$$\langle \phi_j(x_1)\phi_j(x_2)\rangle \ \to \ \frac{|\frac{\partial x'}{\partial x}|^{\Delta_j/d}(x_1)|\frac{\partial x'}{\partial x}|^{\Delta_j/d}(x_2)\langle \Phi_j(x_1')\Phi_j(x_2')\rangle}{1 - l_P^2\hbar^{-1}|\frac{\partial x'}{\partial x}|^{\Delta_j/d}(x_1)|\frac{\partial x'}{\partial x}|^{\Delta_j/d}(x_2)\langle \Phi_j(x_1')\Phi_j(x_2')\rangle} \ . \tag{19}$$

1. Case $d=4$

For the massless scalar field in four dimensions the above formulas leads to the following deformed two-point function

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{\hbar}{4\pi^2(x_1-x_2)^2 - l_P^2} \ , \tag{20}$$

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[2] Unless explicitly stated otherwise, all expectation values are computed with respect to the vacuum state $|0\rangle$.

[3] When the two-point function is regarded as a distribution one should replace, as usual, $T_2 - T_1$ by $(T_2 - T_1 - i\epsilon)$

and a modified action of conformal transformations

$$\frac{1}{4\pi^2(x_1-x_2)^2-l_P^2} \quad \rightarrow \quad \frac{|\frac{\partial x'}{\partial x}|^{1/4}(x_1)|\frac{\partial x'}{\partial x}|^{1/4}(x_2)}{4\pi^2(x_1-x_2)^2-l_P^2|\frac{\partial x'}{\partial x}|^{1/4}(x_1)|\frac{\partial x'}{\partial x}|^{1/4}(x_2)} \quad . \tag{21}$$

Note that for the deformed two-point function an invariant (Planck) scale emerges as

$$\langle \phi(x_1)\phi(x_2)\rangle|_{x_1\to x_2} \approx -\frac{\hbar}{l_P^2} = -\frac{1}{G} \quad . \tag{22}$$

Furthermore, that invariant quantity is not necessarily tied to the expectation value in the vacuum state. If instead we have a different quantum state Ψ, the Hadamard condition for the relativistic theory, namely the universality of the short distance behavior

$$\langle \Psi|\Phi(x_1)\Phi(x_2)|\Psi\rangle|_{x_1\to x_2} \sim \langle \Phi(x_1)\Phi(x_2)\rangle|_{x_1\to x_2} \quad , \tag{23}$$

ensures that

$$\langle \Psi|\phi(x_1)\phi(x_2)|\Psi\rangle|_{x_1\to x_2} \approx -\frac{\hbar}{l_P^2} = -\frac{1}{G} \quad . \tag{24}$$

We would like to remark that the invariant (observer-independent) scale $l_P^2\hbar^{-1}$ acts as a natural regulator for the two-point functions. This admits a nice physical interpretation. When we probe quantum field theory at scales $(x_1-x_2)^2 \gg l_P^2$, gravity is negligible ($G \to 0$) and the Green functions seem to diverge like $\sim 1/(x_1-x_2)^2$. For point separations of order l_P, the role of gravity in constraining the space-time structure can no longer be neglected ($G \neq 0$), which results in $l_P^2\hbar^{-1}$ providing a natural cutoff for the Green functions.

2. Case $d=2$

The generic nonlinear realization (19) can be displayed more explicitly for the deformed correlations of the $d=2$ model mentioned above. The deformed correlations derived from (16) and (18) are then

$$\langle \partial_\pm\phi(x_1)\partial_\pm\phi(x_2)\rangle = \frac{-\hbar}{4\pi(x_1^\pm-x_2^\pm)^2+l_P^2} \quad . \tag{25}$$

The deformed action of conformal transformations reads as

$$\langle \partial_\pm\phi(x_1)\partial_\pm\phi(x_2)\rangle \quad \rightarrow \quad \frac{\frac{dx'^\pm}{dx^\pm}(x_1)\frac{dx'^\pm}{dx^\pm}(x_2)\langle \Phi_\pm(x_1')\Phi_\pm(x_2')\rangle}{1-l_P^2\hbar^{-1}\frac{dx'^\pm}{dx^\pm}(x_1)\frac{dx'^\pm}{dx^\pm}(x_2)\langle \Phi_\pm(x_1')\Phi_\pm(x_2')\rangle} \quad . \tag{26}$$

Therefore

$$\frac{-\hbar}{4\pi(x_1^\pm-x_2^\pm)^2+l_P^2} \quad \rightarrow \quad \frac{-\hbar\frac{dx'^\pm}{dx^\pm}(x_1)\frac{dx'^\pm}{dx^\pm}(x_2)}{4\pi(x_1'^\pm-x_2'^\pm)^2+l_P^2\frac{dx'^\pm}{dx^\pm}(x_1)\frac{dx'^\pm}{dx^\pm}(x_2)} \quad , \tag{27}$$

which clearly shows the emergence of an invariant Planck scale. For instance, under a boost of rapidity ξ: $x^\pm \to x'^\pm = e^{\pm\xi}x^\pm$, the deformed two-point functions are

$$\frac{-\hbar}{4\pi(x_1^\pm-x_2^\pm)^2+l_P^2} \quad \rightarrow \quad \frac{-\hbar e^{\pm2\xi}}{4\pi(x_1'^\pm-x_2'^\pm)^2+l_P^2e^{\pm2\xi}} \quad , \tag{28}$$

so all inertial observers agree for the coincident-point limit $-\frac{\hbar}{l_P^2}$ of these correlation functions.

C. Massive scalar field

The approach presented above for conformal field theories can be extended easily to other theories. To illustrate this let us consider a massive scalar field in four dimensions. One can deform the corresponding two-point function in

a way parallel to that used for conformal theories. Taking into account that the two-point function for a relativistic massive scalar field in four-dimensions is

$$\langle\Phi(x_1)\Phi(x_2)\rangle = \frac{m\hbar}{4\pi^2\sqrt{(x_1-x_2)^2}}K_1(m\sqrt{(x_1-x_2)^2}) \ , \tag{29}$$

where K_1 is a modified Bessel function, the deformed two-point function reads

$$\langle\phi(x_1)\phi(x_2)\rangle \equiv \frac{\langle\Phi_m(x_1)\Phi_m(x_2)\rangle}{1-l_P^2\hbar^{-1}\langle\Phi_m(x_1)\Phi_m(x_2)\rangle} = \frac{m\hbar K_1(m\sqrt{(x_1-x_2)^2})}{4\pi^2\sqrt{(x_1-x_2)^2}-ml_P^2K_1(m\sqrt{(x_1-x_2)^2})} \ . \tag{30}$$

Note that, since the short-distance behavior of (29) coincides with that of a massless field, the invariant (Planck) scale emerges in the same way

$$\langle\phi(x_1)\phi(x_2)\rangle|_{x_1\to x_2} \approx -\frac{\hbar}{l_P^2} \ . \tag{31}$$

D. Relation to other approaches

It is interesting to stress that a method to modify the two-point function in a Lorentz-invariant way has also been suggested in [17], by invoking some sort of path-integral duality. In terms of the Schwinger's proper time formalism, this approach is equivalent to deforming the symmetric two-point function

$$\langle\Phi(x_1)\Phi(x_2)\rangle = \int_{-\infty}^{+\infty} ds\, e^{-im^2s}K(x_1,x_2;s) = \frac{\hbar}{4\pi^2(x_1-x_2)^2} \ , \tag{32}$$

where $K(x_1,x_2;s)$ is the heat kernel of the matter field Φ, as follows

$$\langle\phi(x_1)\phi(x_2)\rangle = \int_{-\infty}^{+\infty} ds\, e^{-im^2s}e^{il_P^2/(4\pi^2s)}K(x_1,x_2;s) \ , \tag{33}$$

where $K(x_1,x_2;s)$ is the same relativistic heat kernel. For a massless field the above proposal for deforming the Green functions and that of (18) leads to the same deformed two-point function. However, one does not get the same result for generic massive fields.

IV. RESPONSE FUNCTION OF AN ACCELERATED DETECTOR: THE ROLE OF PLANCK SCALE

A natural way to show that the notion of particle is, in general, observer-dependent was proposed in [23]. The particle content of the vacuum perceived by an observer with trajectory $x = x(\tau)$ and equipped with a detector, can be analyzed by considering the interaction of the matter field $\Phi(x)$ with the detector, modeled by the interaction lagrangian (see, for instance, [1])

$$g\int d\tau\, m(\tau)\Phi(x(\tau)) \ , \tag{34}$$

where $m(\tau)$ represents the detector's monopole moment and g is the strength of the coupling. It is assumed that the detector has some internal energy eigenstates $|E\rangle$, providing the internal matrix elements $\langle E|m(0)|E_0\rangle$ with the detector ground state $|E_0\rangle$. For a general trajectory, the detector will not remain in its ground state $|E_0\rangle$, but will undergo a transition to an excited state $|E\rangle$. To first order in perturbation theory, the transition amplitude from $|E_0\rangle|0_M\rangle$ to $|E\rangle|\psi\rangle$, where $|0_M\rangle$ is the Minkowski vacuum state of the scalar field and $|\psi\rangle$ an arbitrary field state, is given by

$$g\int d\tau\langle E\psi|m(\tau)\Phi(x(\tau))|E_0 0_M\rangle \ . \tag{35}$$

The probability for the detector to make the transition from $|E_0\rangle$ to $|E\rangle$ (summing over all final states of the scalar field) is then given by

$$g^2|\langle E|m(0)|E_0\rangle|^2 F(E-E_0) \ , \tag{36}$$

where $F(E - E_0)$ is the so-called response function of the detector

$$F(E - E_0) = \int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\tau_2 e^{-i(E-E_0)(\tau_1 - \tau_2)/\hbar} \langle 0_M | \Phi(x(\tau_1)) \Phi(x(\tau_2)) | 0_M \rangle \ . \tag{37}$$

When dealing with particular examples, one finds useful to introduce the response rate function

$$\dot{F}(E - E_0) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-i(E-E_0)\Delta\tau/\hbar} \langle 0_M | \Phi(x(\tau_1)) \Phi(x(\tau_2)) | 0_M \rangle \ . \tag{38}$$

To work out the above expressions one should be careful when dealing with the typical short-distance singularity of the two-point function. Usually one considers the "$i\epsilon$-prescription" for the two-point function.

For a uniformly accelerated trajectory in Minkowski spacetime

$$t = \frac{1}{a} \sinh a\tau \ , \quad x = \frac{1}{a} \cosh a\tau \ , \tag{39}$$

where a is the acceleration, the two-point function, which for simplicity is chosen for a massless scalar field, becomes

$$\langle 0_M | \Phi(x(\tau_1)) \Phi(x(\tau_2)) | 0_M \rangle = \frac{-\hbar(\frac{a}{2})^2}{4\pi^2 \sinh^2 \frac{a}{2}(\Delta\tau - i\epsilon)} \ . \tag{40}$$

Unlike for inertial trajectories, the response rate function $\dot{F}(E - E_0)$ of a uniformly accelerated observer does not vanish, and turns out to be

$$\dot{F}(E - E_0) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-i(E-E_0)\Delta\tau/\hbar} \frac{-\hbar(\frac{a}{2})^2}{4\pi^2 \sinh^2 \frac{a}{2}(\Delta\tau - i\epsilon)} = \frac{1}{2\pi} \frac{E - E_0}{e^{2\pi(E-E_0)/\hbar a} - 1} \ . \tag{41}$$

This result tells us that a uniformly accelerated observer in Minkowski space feels himself immersed in a thermal bath at the temperature $k_B T = \frac{a\hbar}{2\pi}$. Note that even if one considers the massive scalar, this result still holds. The mass dependence of the two-point function does not affect the value of the integral, which is only sensitive to the short-distance behavior, already captured by the massless field.

We would like to remark that the use of the "$i\epsilon-$prescription" in treating the two-point function in the distributional sense guarantees the vanishing of the response function for an inertial observer. However, there is an alternative, and more natural, way to enforce the vanishing of the response function for inertial detectors. Essentially, we can calibrate the detector with the accelerated (Rindler) vacuum $|0_R\rangle$

$$\dot{F}(E - E_0) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-i(E-E_0)\Delta\tau} [\langle 0_M | \Phi(x(\tau_1)) \Phi(x(\tau_2)) | 0_M \rangle - \langle 0_R | \Phi(x(\tau_1)) \Phi(x(\tau_2)) | 0_R \rangle] \ . \tag{42}$$

The integrand is now a smooth function, thanks to the universal short-distance behavior of the two-point function, and the result is equivalent to (41)

$$\dot{F}(E - E_0) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-i(E-E_0)\Delta\tau/\hbar} \left[\frac{-\hbar(\frac{a}{2})^2}{4\pi^2 \sinh^2 \frac{a}{2}\Delta\tau} + \frac{\hbar}{4\pi^2(\Delta\tau)^2} \right] = \frac{1}{2\pi} \frac{E - E_0}{e^{2\pi(E-E_0)/\hbar a} - 1} \ . \tag{43}$$

Although both expressions are mathematically equivalent, they lead to different results when the correlation functions are deformed, as we will see later.

A. Deforming the two-point function

Let us now explore what happens when one deforms the standard two-point function according to (30). For simplicity we shall considerer the case of a massless scalar field, which leads to

$$\langle 0_M | \phi(x_1) \phi(x_2) | 0_M \rangle = \frac{\hbar}{4\pi^2(x_1 - x_2)^2 - l_P^2} \ . \tag{44}$$

The rate $\dot{F}(w)$, where we define $w = E/\hbar$ and set $E_0 = 0$ for simplicity, can be worked out in a parallel way to the standard relativistic theory. The novelty is that one has now modified two-point functions. Therefore (38) should be replaced by (42) and, accordingly, (43) by

$$\dot{F}_{l_P}(w) = -\frac{\hbar}{4\pi^2} \int_{-\infty}^{+\infty} d\Delta\tau \, e^{-iw\Delta\tau} \left[\frac{1}{(2/a)^2 \sinh^2(\frac{a}{2}\Delta\tau) + l_P^2/4\pi^2} - \frac{1}{(\Delta\tau)^2 + l_P^2/4\pi^2} \right] . \tag{45}$$

The final result is then

$$\dot{F}_{l_P}(w) = \frac{\hbar}{2\pi} \left[\frac{we^{\pi w/a}}{(e^{2\pi w/a} - 1)} \frac{\sinh[\frac{w}{a}(\theta - \pi)]}{\frac{w}{a}\sin\theta} + \frac{\pi e^{-wl_P}}{l_P} \right] , \tag{46}$$

where $\theta \equiv 2\arcsin\left(\frac{l_P a}{4\pi}\right)$. The thermal Planckian spectrum is smoothly recovered in the limit $l_P \to 0$. In fact, the rate $\dot{F}_{l_P}(w)$ can be expanded as

$$\dot{F}_{l_P}(w) \approx \frac{\hbar}{2\pi} \left[\frac{w}{e^{2\pi w/a} - 1} - \frac{l_P a^2}{32\pi} + O(l_P^3) \right] . \tag{47}$$

Thermality is maintained until some frequency scale Ω, which can be estimated by requiring that the correction does not exceed the leading term in the above approximated expression. A simple calculation leads to the condition $32\pi\Omega e^{-2\pi\Omega/a} \approx l_P a^2$. Planck-scale effects could potentially emerge at the scale Ω, which is roughly some orders above $T = a/2\pi$, if the acceleration is not very high (in comparison with the Planck scale). This shows that the thermal spectrum is essentially robust against trans-Planckian physics. We would like to remark that the robustness of the effect can also be explained within the standard relativistic field theory framework [18] [4]

Note that, if one ignores the calibration term, and naively works out the response rate as

$$-\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} d\Delta\tau \, e^{-iw\Delta\tau} \frac{\hbar}{(2/a)^2 \sinh^2(\frac{a}{2}\Delta\tau) + l_P^2/4\pi^2} , \tag{48}$$

the resulting expression

$$\frac{w\hbar}{2\pi} \frac{e^{\pi w/a}}{(e^{2\pi w/a} - 1)} \frac{\sinh[\frac{w}{a}(\theta - \pi)]}{\frac{w}{a}\sin\theta} , \tag{49}$$

largely departs from the thermal spectrum. Even worse, for an inertial observer, $a = 0$, the response rate does not vanish, as one should expect according to the principle of relativity. To produce a physically sound result one must necessarily subtract the naive "inertial" contribution, as in (43), replacing (48) by (45).

V. THERMAL PROPERTIES OF DE SITTER SPACE

It is well known that a geodesic observer in de Sitter space (4) feels a thermal bath of particles at a temperature $k_B T = \frac{H\hbar}{2\pi}$, where $H = \sqrt{\frac{\Lambda}{3}}$. One can derive this result [4] in a parallel way to acceleration radiation effect of the previous section. We shall now consider a scalar field in de Sitter space

$$(\Box - \frac{m^2}{\hbar^2} - \xi R)\Phi = 0 , \tag{50}$$

where m is the mass and ξ stands for the coupling to the curvature. The response rate function is also given by the general expression

$$\dot{F}(w) = \int_{-\infty}^{+\infty} d\Delta\tau \, e^{-iw\Delta\tau} \langle \Phi(x(\tau_1))\Phi(x(\tau_2)) \rangle , \tag{51}$$

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[4] In terms of modified dispersion relations the robustness of the effect has been addressed in [19]

where now the expectation value is understood with respect to the (global) de Sitter vacuum $|0_{dS}\rangle$, invariant under the $SO(4,1)$ isometries of the de Sitter spacetime. For simplicity, without loss of generality, we shall restrict our discussion to conformal coupling $\xi = 1/6$ and $m = 0$. In this situation the two-point function takes the simple form

$$\langle \Phi(x_1)\Phi(x_2)\rangle = \frac{H^2 \eta_1 \eta_2 \hbar}{4\pi^2[-(\eta_1 - \eta_2 - i\epsilon)^2 + (\vec{x}_1 - \vec{x}_2)^2]} , \tag{52}$$

where $\eta = -H^{-1}e^{-Ht}$ is the conformal time, for which the metric takes the form

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + d\vec{x}^2) . \tag{53}$$

Freely falling detectors with trajectories

$$t = \tau \qquad \vec{x} = \vec{x}_0 , \tag{54}$$

will have a response function with rate

$$\dot{F}(w) = \int_{-\infty}^{+\infty} d\Delta\tau e^{-iw\Delta\tau} \frac{-\hbar H^2}{16\pi^2 \sinh^2 \frac{H}{2}(\Delta\tau - i\epsilon)} . \tag{55}$$

This is exactly the same integral as (41), producing now thermal radiation at the temperature $k_B T = \frac{H\hbar}{2\pi}$.

A. Deforming the two-point function

Following the same strategy as for the uniformly accelerated observer in Minkowski, we shall now deform the two-point function preserving the de Sitter invariance $SO(4,1)$ instead of the Poincaré invariance of the (global) Minkowskian vacuum. We can chose the function U as in previous sections, which leads to

$$U^{-1}: \ \langle \phi(x_1)\phi(x_2)\rangle = \frac{\langle \Phi(x_1)\Phi(x_2)\rangle}{1 - l_P^2 \hbar^{-1}\langle \Phi(x_1)\Phi(x_2)\rangle} = \frac{H^2 \eta_1 \eta_2 \hbar}{4\pi^2[-(\eta_1 - \eta_2 - i\epsilon)^2 + (\vec{x}_1 - \vec{x}_2)^2] - l_P^2 H^2 \eta_1 \eta_2} . \tag{56}$$

Armed with this deformed two-point function we can evaluate the new response rate function. As in (42), the integral for $\dot{F}_{l_P}(w)$ involves the difference between the above correlation function, evaluated in the global de Sitter vacuum, and the two-point function evaluated in the vacuum of a comoving observer. The two-point function of the comoving (inertial) observer $\langle 0_I|\Phi(x(\tau_1))\Phi(x(\tau_2))|0_I\rangle$ can be easily obtained by summing over modes evaluated along the observer's trajectory. To find the modes, one must solve the field equation

$$(\Box - \frac{1}{6}R)\Phi(\tilde{t}, \tilde{x}) = 0 , \tag{57}$$

where $R = 4\Lambda = 12H^2$, using the static line element

$$ds^2 = -(1 - H^2\tilde{r}^2)d\tilde{t}^2 + \frac{d\tilde{r}^2}{1 - H^2\tilde{r}^2} + \tilde{r}^2 d\Omega^2 , \tag{58}$$

where

$$\tilde{t} = -\frac{1}{2H}\ln[H^2(\eta^2 - r^2)] , \quad \tilde{r} = -\frac{r}{H\eta} . \tag{59}$$

Along the observer's trajectory ($\tilde{r} = 0$) only the s-wave modes[5] contribute, and the two-point function becomes

$$\langle 0_I|\Phi(x(\tau_1))\Phi(x(\tau_2))|0_I\rangle = \hbar \int_0^\infty dw \frac{1}{4\pi^2} w e^{-iw\Delta\tau} = -\frac{\hbar}{4\pi^2(\Delta\tau - i\epsilon)^2} . \tag{60}$$

The corresponding deformed two-point function is

$$\langle 0_I|\phi(x(\tau_1))\phi(x(\tau_2))|0_I\rangle = -\frac{\hbar}{4\pi^2(\Delta\tau)^2 + l_P^2} . \tag{61}$$

It is now immediate to see that the final result is equivalent to that of the accelerated detector in Minkowski, equation (46), with the replacement $a \to H$.

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[5] For completeness, these modes are $\Phi_{w,l=0}(\tilde{t}, \tilde{r}, \theta, \varphi) = \frac{1}{2\pi\sqrt{w}} e^{-iw\tilde{t}} \frac{1}{\tilde{r}} \sin\left[\frac{w}{H}\tanh^{-1}(\tilde{r}H)\right]$

VI. BLACK HOLE RADIANCE AND CONFORMAL SYMMETRY

In this section we shall stress that the Hawking effect can be analyzed in the general framework presented in section II. Although there are not global isometries in the spacetime of a collapsing star, a powerful symmetry emerges in the near horizon region. This region is effectively two-dimensional and conformally invariant, as can be seen easily by writing the wave equation of a scalar field in a Schwarzschild background. Expanding the field in spherical harmonics

$$\Phi(x^\mu) = \sum_{l,m} \frac{\Phi_l(t,r)}{r} Y_{lm}(\theta, \varphi) \; , \tag{62}$$

the four-dimensional Klein–Gordon equation for Φ is then converted into a two-dimensional wave equation, for each angular momentum component

$$\Big(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}} - V_l(r) \Big) \Phi_l(t,r) = 0 \; , \tag{63}$$

with the potential $V_l(r) = \big(1 - \frac{2GM}{r}\big)\big[l(l+1)/r^2 + 2GM/r^3\big]$, and $r^* \equiv r + 2GM \ln|r/2GM - 1|$ is the radial tortoise coordinate. In the near-horizon limit $r \to 2GM$ ($r^* \to -\infty$) the potential vanishes and (63) becomes the two-dimensional free wave equation, which in null coordinates ($u \equiv t - r^*$, $v \equiv t + r^*$) reads $\partial_u \partial_v \Phi_l = 0$. This equation is very important since it exhibits the emergence of the two-dimensional conformal invariance $u \to u' = u'(u)$, $v \to v' = v'(v)$, which is at the heart of the thermodynamic properties of black holes. This symmetry plays a pivotal role in unraveling the universal behavior of black hole entropy (see, for instance [20, 21, 22]) and in the derivation of Hawking radiation. To make explicit this latter fact we shall rewrite the number of particles in terms of the two-point functions of a two-dimensional conformal field theory. To be more precise, in the conventional analysis in terms of Bogolubov coefficients, one first performs the integration in distances (i.e., evaluation of the scalar product between "in" and "out" modes) and leaves to the end the sum of "in" modes. In contrast, we can invert the order and perform first the sum of "in" modes, which naturally leads to introduce the two-point function of the matter field, and leave the integration in distances to the end

$$\langle in|N_i^{out}|in\rangle = \sum_k \beta_{ik}\beta_{ik}^* = -\sum_k (u_i^{out}, u_k^{in*})(u_i^{out*}, u_k^{in}) =$$

$$= \sum_k \left(\int_\Sigma d\Sigma_1^\mu \, u_i^{out}(x_1) \overleftrightarrow{\partial}_\mu u_k^{in}(x_1) \right) \left(\int_\Sigma d\Sigma_2^\nu \, u_i^{out*}(x_2) \overleftrightarrow{\partial}_\nu u_k^{in*}(x_2) \right) \; , \tag{64}$$

where Σ is an initial Cauchy hypersurface. Taking into account that

$$\langle in|\Phi(x_1)\Phi(x_2)|in\rangle = \hbar \sum_k u_k^{in}(x_1) u_k^{in*}(x_2) \; , \tag{65}$$

we obtain a simple expression for the particle production number in terms of the two-point function

$$\langle in|N_i^{out}|in\rangle = \hbar^{-1} \int_\Sigma d\Sigma_1^\mu d\Sigma_2^\nu [u_i^{out}(x_1) \overleftrightarrow{\partial}_\mu][u_i^{out*}(x_2) \overleftrightarrow{\partial}_\nu] \langle in|\Phi(x_1)\Phi(x_2)|in\rangle \; , \tag{66}$$

where the "$i\epsilon$−prescription" is assumed to regulate the integrand. Alternatively, one can subtract $\langle out|N_i^{out}|out\rangle \equiv 0$ from (64) to obtain

$$\langle in|N_i^{out}|in\rangle = \hbar^{-1} \int_\Sigma d\Sigma_1^\mu d\Sigma_2^\nu [u_i^{out}(x_1) \overleftrightarrow{\partial}_\mu][u_i^{out*}(x_2) \overleftrightarrow{\partial}_\nu] \left[\langle in|\Phi(x_1)\Phi(x_2)|in\rangle - \langle out|\Phi(x_1)\Phi(x_2)|out\rangle\right] \; , \tag{67}$$

which is now regular provided $|in\rangle$ and $|out\rangle$ satisfy the Hadamard condition[6].

Let us apply this scheme in the formation process of a spherically symmetric black hole (for details see [25]) and restrict the "out" region to I^+. The "in" region is, as usual, defined by I^-. At I^+ the radial plane-wave modes are

$$u_{wlm}^{out}(t,r,\theta,\phi)|_{I^+} \sim u_{wl}^{out}(u) \frac{Y_l^m(\theta,\phi)}{r} \; , \tag{68}$$

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[6] The two-point distribution should have (for all physical states) a short-distance structure similar to that of the ordinary vacuum state in Minkowski space: $(2\pi)^{-2}(\sigma + 2i\epsilon t + \epsilon^2)^{-1}$, where $\sigma(x_1, x_2)$ is the squared geodesic distance [24].

where $u_{wl}^{out}(u) = \frac{e^{-iwu}}{\sqrt{4\pi w}}$. Integrating the angular degrees of freedom the emission rate, evaluated at I^+, takes the form

$$\langle in|\dot{N}_{wlm}^{out}|in\rangle = \frac{|t_l(w)|^2}{\pi\hbar w}\int_{-\infty}^{\infty} d(u_1 - u_2)e^{-iw(u_1-u_2)}\langle in|\partial_u\Phi_l(x_1)\partial_u\Phi_l(x_2)|in\rangle \ , \tag{69}$$

where $t_l(w)$ represent the transmission coefficients associated with the potential barrier and

$$\langle\partial_u\Phi_l(x_1)\partial_u\Phi_l(x_2)\rangle = -\frac{\hbar}{4\pi}\frac{\frac{dv}{du}(u_1)\frac{dv}{du}(u_2)}{[v(u_1) - v(u_2) - i\epsilon]^2} \tag{70}$$

is the two-point function of a two-dimensional conformal field theory of the primary fields $\partial_u\Phi$ (see subsection III A) transformed under the conformal rescaling[7] $u(v) \approx v_H - \kappa^{-1}\ln\kappa(v_H - v)$, where $\kappa = 1/4GM$. Performing the integration in $\Delta u \equiv u_1 - u_2$ we recover the Planckian spectrum and the particle production rate

$$\langle in|\dot{N}_{wlm}^{out}|in\rangle = \frac{|t_l(w)|^2}{e^{2\pi\kappa^{-1}w} - 1} \ . \tag{71}$$

Now that Hawking radiation has been derived using two-point functions, it is worth comparing equations (38), (51), and (69). In the three cases, the two-point function is evaluated in the coordinates of the detector. Let us see now how deformations of the two-point function would affect Hawking radiation provided we use again the same deformation function U. In this case, as explained in subsection III B 2, the effect of the Planck scale is encapsulated in the deformed correlation function

$$\langle\partial_u\phi_l(x_1)\partial_u\phi_l(x_2)\rangle = -\frac{\hbar}{4\pi}\frac{\frac{dv}{du}(u_1)\frac{dv}{du}(u_2)}{[v(u_1) - v(u_2)]^2 + l_P^2\frac{dv}{du}(u_1)\frac{dv}{du}(u_2)} \ , \tag{72}$$

which preserves the conformal invariance by construction. Following the analogy with the case of the accelerated detector in Minkowski space and the inertial one in de Sitter space, one should use expression (67) to evaluate the emission rate in the modified theory. As the final result we find again the function (46), up to the overall factor $w\hbar/2\pi$ and the grey-body coefficients, with the replacement $a \to \kappa$

$$\langle in|\dot{N}_{wlm}^{out}|in\rangle = |t_l(w)|^2\left[\frac{e^{\pi w/\kappa}}{(e^{2\pi w/\kappa} - 1)}\frac{\sinh[\frac{w}{\kappa}(\theta - \pi)]}{\frac{w}{\kappa}\sin\theta} + \frac{\pi e^{-wl_P}}{wl_P}\right] \ . \tag{73}$$

where $\theta \equiv 2\arcsin\left(\frac{l_P\kappa}{4\pi}\right)$.

VII. CONCLUSIONS

In this paper we have offered a unified view of the typical thermal effects of quantum field theory in curved space aiming at showing their robustness against trans-Planckian physics. We have shown that all these effects can be easily derived in terms of two-point correlation functions, which allowed us to explore the effects of Planck scale physics by suitable deformations of such functions. The deformations proposed here are somehow parallel the approach presented in [10], where dispersion relations were modified while keeping the principle of relativity. In our case, the two-point functions were deformed respecting the symmetries of the original theory: Lorentz symmetry for the acceleration radiation effect, de Sitter $SO(4,1)$ symmetry for the Gibbons-Hawking effect, and the two-dimensional conformal symmetry for the Hawking effect.

One of the advantages of our approach is that it is relatively straightforward to modify the theory maintaining the relevant symmetries of the problem and, since our basic objects are two-point functions, the physical consequences can be easily evaluated. On the other hand, it is worth noting that using the result (46), which corresponds to the acceleration radiation problem, we have been able to show that the three effects described are robust under Planck-scale deformations. This fact together with the elegant kinematic method of connecting the acceleration, de Sitter and black hole radiation given in [26] may support our view that none of the semiclassical thermal effects

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[7] Note that, due to the particular spacetime geometry of a gravitational collapse, there is always a reflection at $r = 0$ which transforms $v \to u$. This explains why this transformation is of the form $v \to v(u)$.

depends on ultra-high-energy physics.

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