Describing the Nucleon and its Form Factors at High q^2

B. DESPLANQUES[1]*, B. SILVESTRE-BRAC[1]†, F. CANO[2]‡, P. GONZALEZ[3,4]§ and S. NOGUERA[3]¶
[1]Institut des Sciences Nucléaires, UMR CNRS-UJF, F-38026 Grenoble Cedex, France
[2]Dipartamento di Fisica, Universitá degli Studi di Trento, I-38050 Povo (Trento), Italy
[3]Departamento de Física Teórica, Universidad de Valencia, E-46100 Burjassot (Valencia), Spain
[4]IFIC, Centro Mixto Universidad de Valencia-CSIC, E-46100 Burjassot (Valencia), Spain

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Abstract

The nucleon form factors are calculated using a non-relativistic description in terms of constituent quarks. The emphasis is put on the reliability of present numerical methods used to solve the three-body problem in order to correctly reproduce the expected asymptotic behavior of form factors. Nucleon wave functions obtained in the hyperspherical formalism or employing Faddeev equations have been considered. While a q^{-8} behavior is expected at high q for a quark-quark force behaving like $\frac{1}{r}$ at short distances, it is found that the hypercentral approximation in the hyperspherical formalism ($K=0$) leads to a q^{-7} behavior. An infinite set of waves is required to get the correct behavior. Solutions of the Faddeev equations lead to the q^{-8} behavior. The amplitude of the corresponding term however depends on the number of partial waves retained in the Faddeev amplitude. The convergence to the asymptotic behavior has also been studied. Sizeable departures are observed in some cases at squared momentum transfers as high as $50\,(\mathrm{GeV/c})^2$. It is not clear whether these departures are of the order $\frac{1}{q}$ or $\frac{1}{q^2}\log q$ relatively to the dominant

*E-mail address: desplanq@isn.in2p3.fr
†E-mail address: silvestre@isn.in2p3.fr
‡E-mail address: cano@xaloc.ific.uv.es
§E-mail address: gonzalep@evalvx.ific.uv.es
¶E-mail address: Santiago.Noguera@uv.es

contribution and whether the bad convergence results from truncations in the calculations. From a comparison with the most complete Faddeev results, a q^2 validity range is obtained for the calculation made in the hyperspherical formalism or in the Faddeev approach with the minimum number of amplitudes.

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1 Introduction

A great interest is currently devoted to the nucleon form factor and, in particular, to its behavior at high momentum transfers. Indeed, from QCD in the perturbative regime [1], one expects the nucleon form factor, $G_M(q^2)$ (or $F_1(q^2)$), to scale like q^{-4}, up to log terms. Experimentally, this behavior seems to be reached quite rapidly, around $10\,(\mathrm{GeV/c})^2$ [2].

Many theoretical works tend to make a bridge between the low and high momentum domains [3, 4]. They deal with the non-perturbative regime of QCD, where predictions are more difficult, and rely on non-relativistic calculations for the lower part of the momentum range. Among the questions that may be raised, there is the sensitivity of the predictions to the theoretical framework, including particular techniques, or the rapidity of the asymptotic behavior onset. Obviously, in view of the large momentum transfers that the asymptotic regime supposes, a definite statement would require a relativistic treatment. Some works along these lines are in progress [3, 4, 5, 6]. We nevertheless believe that a non-relativistic calculation may be of some help to provide qualitative, if not quantitative answers to the above questions. We will concentrate on them in the following.

Quite simple descriptions of nucleons in terms of constituent quarks have relied on the harmonic oscillator wave function [7]. In such models , the calculated form factors drop exponentially to zero beyond $q^2 = 3\,(\mathrm{GeV/c})^2$ [8]. Curiously, it has been sometimes deduced from this result that a constituent quark model could not give rise to a power law behavior for form factors at high q^2. Better non-relativistic descriptions of the nucleon involve solving the Schrödinger equation, using for instance the hyperspherical harmonic formalism or the Faddeev equations. These approaches are often considered as exact ones, or almost. Calculations performed with the same quark-quark force both in Valencia (hyperspherical formalism) [9] and Grenoble (Faddeev equations) [10] have evidenced discrepancies in the binding energy of the low lying baryons of the order of a few MeV to be compared to a total contribution of the order of 1 GeV for the kinetic energy and the non-constant part of the potential [11]. The small difference may be due to approximations made in either approach: the restriction to the lowest values of the grand orbital, K, in the hyperspherical formalism and the number of amplitudes in the Faddeev approach. In such conditions, it is tempting to go further and calculate the charge and magnetic form factors of both the proton and the neutron and see whether the discrepancy remains at the same level as for the binding energy. It is what we did, with the idea to check the sensitivity of the results to the approach and, within each of them, to the truncations that are currently made in the calculations. We did it also with the aim to compare the results to the asymptotic power law in q^{-8}, which is expected in non-relativistic approaches with Coulomb or Yukawa type potentials [12]. While doing so, we have been led to elaborate simple models to understand our results. In view of their possible usefulness, some of them will be presented. A few remarks as for deducing the force between quarks from the nucleon electromagnetic form factor [13], or about higher order QCD corrections will be made.

The plan of the paper will be as follows. The second section is devoted to reminding the argument for a q^{-8} power law asymptotic behavior of the nucleon form factors in a non-relativistic approach. The origin of the difference with the q^{-4} QCD behavior is briefly explained. The third section shows the importance of the description of the wave function at short distances for an accurate prediction of the form factor. This is done on an hydrogenic type two-body system. In the fourth section, we give a few details as to the calculation of the nucleon wave function in the hyperspherical formalism or using Faddeev equations. It includes general features concerning these approaches as well as a few numerical results pertinent to the nucleon. Results for the form factors calculated with the quark-quark force of Bhaduri et al. [14] and from different approaches are presented in Sect. 5. The onset of their asymptotic behavior is discussed. A detailed discussion about understanding some of the previous results is made in Sects. 6 and 7. In Sect. 8, we consider a few corrections that should be accounted for to make a realistic comparison with experiment. These include intrinsic quark form factor for the lower q^2 domain of the nucleon form factors, the consideration of interaction models with some three-body forces, an improved description of the spin-spin force and relativistic effects for the higher q^2 domain. As most of the work presented here deals with a non-relativistic picture, there is no need to introduce the extra variable Q^2, which is often introduced in relativistic approaches to remedy the inconvenience of a possibly negative squared 4-momentum transfer. The following notation $q^2 = \vec{q}^2$, where q^2 is also equal to the quantity Q^2 in the Breit frame, is therefore adopted.

2 Power law expectations for the nucleon form factor

Predictions for the form factor of two-and three-body systems at high q have been made long ago by Alabiso and Schierholz [12] in the case of spinless constituents, assuming a non-relativistic as well as a relativistic treatment. For our purpose, we remind some of their results that may be useful for the following.

Electron scattering on the nucleon is represented in Fig. 1. The kinematics is pertinent to a high momentum transfer process. In the initial nucleon at rest, quarks have a small momentum, essentially zero, while in the final state the three-quarks share equally the momentum transferred to the nucleon, $\vec{q}$, and therefore carry the momentum $\frac{\vec{q}}{3}$. Electron scattering on the nucleon involves many diagrams that differ by their time ordering and

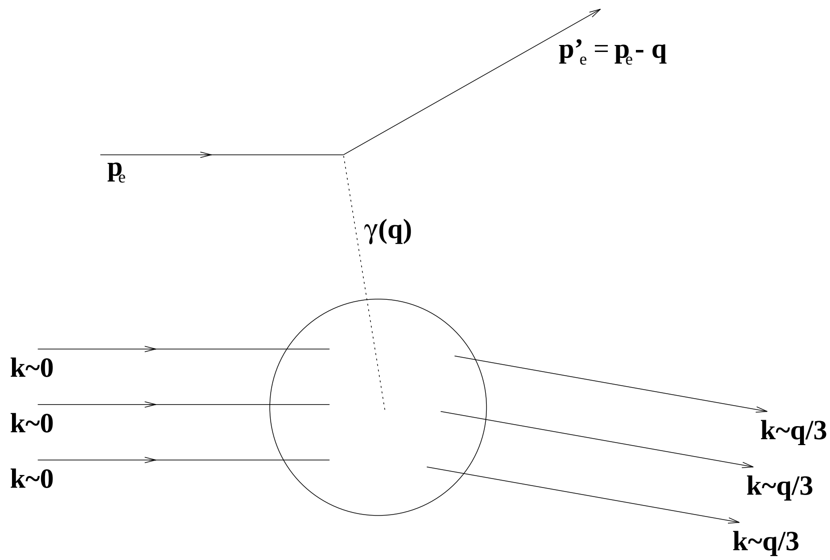

Figure 1: Representation of electron scattering on a nucleon at rest (laboratory system). The kinematics relative to quarks indicated in the figure is that of a high momentum transfer where the internal momentum of quarks within the nucleon can be neglected.

include the exchange of two gluons at least. Two of them are represented in Fig. 2. The first one, (a), is most often shown. The virtual photon transfers to a quark, essentially at rest, a momentum $\vec{q}$ that is shared with the other two quarks by the successive exchange of two gluons. This diagram can be considered as representing a final state interaction. The second diagram, (b), is completely symmetric of the first one in time. It corresponds to an interaction effect in the initial state and has the advantage to show that the form factor at high $\vec{q}$ is sensitive to the high momentum components of the nucleon wave function, a feature that is not so transparent on the first diagram. We insist on this point because interaction effects in the initial and final states have often been considered on a different footing in the past [15], with the obvious idea to simplify some calculations.

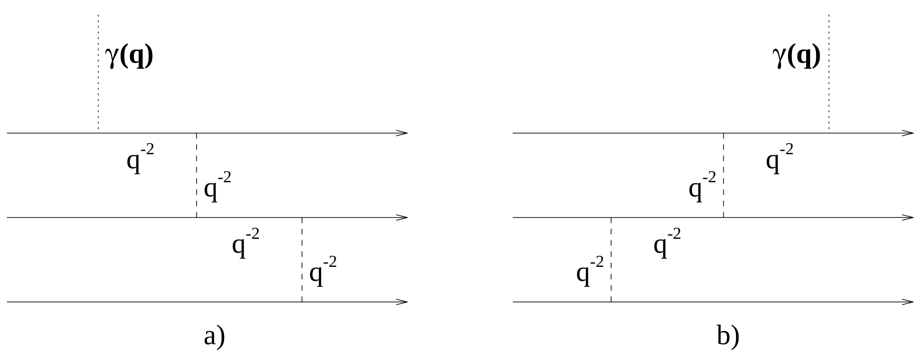

Figure 2: Representation of two processes with gluon exchange contributions relevant to the nucleon form factor at high momentum transfer. The first one (a) corresponds to an interaction in the final state and the second (b) to an interaction in the initial state. The various $\frac{1}{q^2}$ factors that contribute to the asymptotic $\frac{1}{q^8}$ form factor in the non-relativistic approach and arise from the gluon or quark propagators are indicated in the figure. Many other diagrams with different orderings of the gluon exchanges contribute. Some of them are shown in Fig. 9.

As expected from the previous observation, the form factor at high $\vec{q}$ is directly proportional to the high momentum component of the wave function, and not to its square. Let's also mention that the contribution of the two diagrams, (a) and (b), and similar ones not shown in Fig. 2, tend to cancel each other in an inelastic charge scattering process where some energy, but no momentum, is transferred to the system (this simply stems from the orthogonality of the states under consideration).

Examination of Fig. 2 provides a quick estimate of the behavior of the form factor (or the $\gamma^* N \to N$ amplitude) at high q^2. Each gluon propagator introduces a factor $\frac{1}{q^2}$. Furthermore, in the non-relativistic limit, each intermediate quark also introduces a factor $\frac{1}{q^2}$. Hence the form factor is expected to have the following behavior:

$$F_N^{n.r.}(q^2)_{q^2 \to \infty} \propto q^{-8} \tag{1}$$

The above behavior may be invalidated if the interaction contains terms which do not scale like the square inverse power of the momentum at high momenta, such as a gaussian type force, or also if higher order effects are not small enough, in which case larger q^2 should be considered. This is an important issue, which is aimed to be considered in Sects. 6, 7, 8.2 .

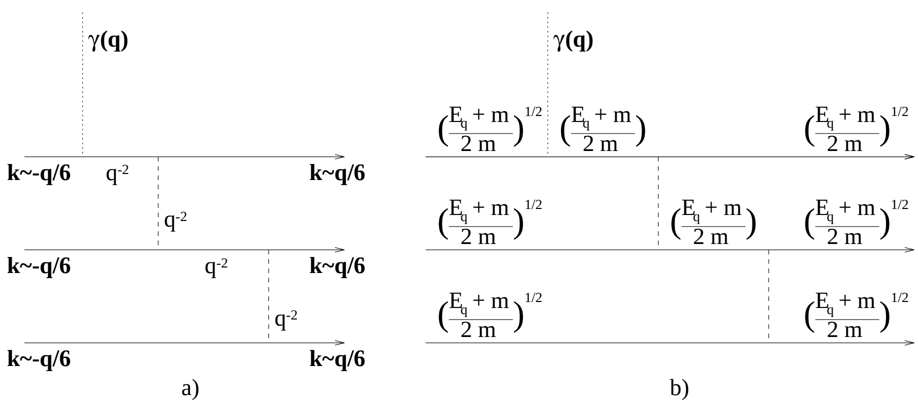

Figure 3: Representation in the Breit frame of a process contributing to the asymptotic form factor for a system of spin 1/2 particles exchanging spin 1 bosons in both a non-relativistic approach (nucleon case: $\frac{1}{q^8}$ asymptotic behavior, diagram a) and a relativistic one ($\frac{1}{q^4}$ asymptotic behavior, diagram b). The diagram b) only shows the extra factors responsible for a difference with the first case. For simplicity, the same q factor appears everywhere, but it should be understood that the appropriate fraction of q is to be used, depending on the diagram and on the position in that diagram. The representation in the lab. frame is also possible, but some caution is required in determining the asymptotic behavior as the time component of the four-momentum transfer, q_0, varies like q^2. Results for the non-relativistic case also hold for spinless particles exchanging spinless bosons.

The difference with the QCD power law expectation, q^{-4}, deserves some explanation. First of all, we notice that the form factor of a system of three spinless particles, interacting via usual scalar boson exchanges and treated relativistically, does have a behavior given by Eq. (1), as can be checked from Fig. 2 with the quark propagators substituted by scalar particle propagators and gluon exchanges substituted by scalar boson exchanges. However, if one considers spin 1 boson exchange, at each particle-boson-particle vertex, a factor q is introduced as a consequence of the vector coupling and a q^{-4} behavior is obtained. In the non-relativistic limit, Eq. (1) holds for both types of boson exchanges. For quarks, spin $\frac{1}{2}$, the exchanges of scalar or vector bosons are also equivalent in the non-relativistic limit, giving rise to a q^{-8} power law (Fig. 3a). In the relativistic treatment, the spin 1 case evidences five extra factors, q, related to the definition of the quark spinors (see Fig. 3b), of which one has to be absorbed in the definition of the initial and final nucleon spinors. These factors, which arise from boosting the nucleon at rest, have a well defined origin and, therefore, can be accounted for, at least approximately, when a more realistic estimate of the form factor or a comparison with experiment is intended to be performed. They should not be confused with factors originating from the dynamics. The case of spin zero bosons coupling to spin $\frac{1}{2}$ particles may also be considered. The above analysis needs to be refined but a q^{-4} behavior is expected too.

Summarizing this section, a non-relativistic calculation of the nucleon form factor does provide a power law q^{-8} at high q^2 (provided that the interaction between quarks is mediated by the usual exchange of bosons). The difference with the prediction, q^{-4}, is due to both relativity and the nature of the QCD interaction, which involves the coupling of spin 1 bosons to spin $\frac{1}{2}$ quarks.

3 Asymptotic form factor of a two-body system

It is well known that the form factor of a two-body system at high momentum transfer is very sensitive to its short-range description. In a few cases, a close relation can be established. Some results that may be found in the literature are reminded here for an hydrogenic type atom (see for instance [16]). We emphasize points that could be relevant for the understanding of results concerning the three-quark system.

Discarding the center of mass motion, the wave function of the first s-state of the hydrogenic atom obeys the following Schrödinger equation:

$$\left(\frac{\vec{p}^2}{2m_r} - \frac{\alpha}{r} - E\right)\ \psi(\vec{r}) = 0, \tag{2}$$

where m_r is the reduced mass, $m_r = \frac{m_1 m_2}{m_1+m_2}$, - $\frac{\alpha}{r}$ the Coulomb interaction and $r = |\vec{r}_1 - \vec{r}_2|$. The normalized solution is given by:

$$\psi(\vec{r}) = \frac{2}{\sqrt{4\pi}}(\alpha m_r)^{3/2} e^{-\alpha m_r r}, \tag{3}$$

while the corresponding binding energy is $E = -\frac{\alpha^2 m_r}{2}$. Assuming now that only the particle 1 carries a unit charge, the form factor of this state is easily calculated:

$$F(\vec{q}^2) = \int d\vec{r}\ \psi^2(r)\ e^{i\vec{q}.\vec{r}\frac{m_2}{m_1+m_2}} = \frac{1}{(1+\frac{\vec{q}^2}{4\alpha^2 m_1^2})^2}. \tag{4}$$

This expression shows that the form factor scales like q^{-4} at high momentum transfers. A direct relation of this behavior to a perturbative calculation is obtained in momentum space. There, the form factor $F(\vec{q}^2)$ gets the expression:

$$F(\vec{q}^2) = \int \frac{d\vec{k}}{(2\pi)^3}\, \varphi(\vec{k})\, \varphi(\vec{k} + \vec{q}\, \frac{m_2}{m_1+m_2}), \tag{5}$$

where $\varphi(\vec{k})$ is the Fourier transform of $\psi(r)$:

$$\varphi(\vec{k}) = \int d\vec{r}\, e^{-i\vec{k}.\vec{r}}\, \psi(\vec{r}) = \sqrt{4\pi}\frac{4(\alpha m_r)^{5/2}}{(k^2+\alpha^2 m_r^2)^2} = \varphi(-\vec{k}). \tag{6}$$

As $\varphi(\vec{k})$ is concentrated at small values of $\vec{k}$, two domains contribute equally to the integral in Eq. (5), around $\vec{k}=0$, and $\vec{k} = -\vec{q}\,\frac{m_2}{m_1+m_2}$, which allows one to write:

$$F(\vec{q}^2) \simeq 2\,\varphi(\vec{q}\,\frac{m_2}{m_1+m_2}) \int \frac{d\vec{k}}{(2\pi)^3}\, \varphi(\vec{k}). \tag{7}$$

Now, the integral in (7) is nothing but the wave function in r-space at the origin, $\psi(0)$, while $\varphi(\vec{k})$ obeys the Schrödinger equation, which in momentum space reads:

$$(\frac{\vec{k}^2 - 2m_r E}{2m_r})\, \varphi(\vec{k}) = \int \frac{d\vec{k'}}{(2\pi)^3} \frac{4\pi\alpha}{(\vec{k}-\vec{k}')^2}\, \varphi(\vec{k}'). \tag{8}$$

Again using the property that $\varphi(\vec{k})$ is concentrated at small values of k allows one to write:

$$\varphi(\vec{k})_{k\to\infty} = \frac{8\pi\alpha m_r}{k^4}\psi(0). \tag{9}$$

Gathering results given by Eqs. (7) and (9), one gets:

$$F(\vec{q}^2)_{\vec{q}^2\to\infty} = \frac{16\pi\alpha m_r}{q^4}\,(\frac{m_1+m_2}{m_2})^4\, \psi^2(0), \tag{10}$$

which is in agreement with Eq. (4) in the same limit and, at the same time, shows the sensitivity of the form factor to the radial wave function at the origin. This one contains non-perturbative effects.

A more precise statement can be made by looking back at the expression of the form factor given by Eq. (4) and making an expansion of $\psi(r)$ around the origin:

$$\psi^2(\vec{r}) = \psi^2(0) + 2\,r\,\psi(0)\,\psi'(0) + ... \tag{11}$$

Inserting this expression in Eq. (4), one obtains from the first term a $\delta(\vec{q})$ function, which obviously vanishes at high q. From terms beyond the first derivative, dots in Eq. (11), one obtains contributions that tend to zero faster than q^{-4} (dimensional argument). As a dominant contribution at high q, one is therefore left with:

$$F(\vec{q}^2)_{q^2\to\infty} = -\frac{16\pi}{q^4}(\frac{m_1+m_2}{m_2})^4\, \psi(0)\psi'(0). \tag{12}$$

In order to get this result, we employed the relation:

$$[\int dr\, r^3\, j_0(qr)]_{q\neq 0} = \lim_{\epsilon\to 0}[\int dr r^3 j_0(qr) e^{-\epsilon r}]_{q\neq 0} = -\frac{2}{q^4}. \tag{13}$$

Equation (12) shows the sensitivity of the form factor at high q to the radial wave function at the origin, but also to its derivative at the same point. The calculation of the form factor at high q therefore supposes to correctly determine the slope of the wave function at the origin, which, itself, is determined by the Coulomb potential, independently of the energy of the state under consideration. This can be checked on the first radial, $l = 0$, excitation, whose wave function and form factor are respectively given by:

$$\psi^*(\vec{r}) = \frac{1}{\sqrt{4\pi}} \frac{(\alpha m_r)^{3/2}}{\sqrt{2}} e^{-\frac{1}{2}\alpha m_r r} \, (1 - \frac{\alpha m_r r}{2}) \tag{14}$$

$$F^*(\vec{q}^2) = \frac{(1 - \frac{q^2}{(\alpha m_1)^2})(1 - 2\frac{q^2}{(\alpha m_1)^2})}{(1 + \frac{q^2}{(\alpha m_1)^2})^4}. \tag{15}$$

In the limit $q \to \infty$, the above form factor exhibits for the $1/q^4$ term a coefficient different from that obtained for the ground state form factor, $F(\vec{q}^2)$, given by Eq. (4), but this only reflects the difference in the value of the square of the wave functions at the origin for the states under consideration, $\frac{(\alpha m_r)^3}{8\pi}$ and $\frac{(\alpha m_r)^3}{\pi}$. For a general potential, the q^{-4} behavior of the form factor holds provided that the interaction inserted in the Schrödinger equation is as singular at small distances as a Coulomb or Yukawa potential ($\propto \frac{1}{r}$). It has to do with the fact that the first derivative of the wave function at the origin is determined by this piece of the interaction. It is lost when the singularity is weaker. It can be checked for instance that the form factor corresponding to the wave function, $e^{-br}(1 + br)$, cooked up in such a way to behave like $1 + 0(r^2)$ at small distances has no q^{-4} component at large q, the first non-zero component being q^{-6}. An other interesting example is the gaussian wave function, $\psi(r) = \pi^{-3/4} e^{-b^2 r^2/2}$, whose form factor:

$$F^G(q^2) = e^{-q^2/4b'^2}, \quad \text{with } b' = b\,\frac{m_1 + m_2}{m_2}, \tag{16}$$

has obviously no q^{-4} component at large q, as expected from Eq. (12).

While the first derivative of the wave function at the origin is determined by the $1/r$ term of the potential, the second one furthermore depends on the "binding energy", i.e. the total mass of the system minus the masses of the constituents and other constant terms (see Eq. (96) for the corresponding situation in the three-body case). It is therefore state dependent. With this respect, we should mention that hadronic systems to be discussed here are sensitive to a confining potential. This one does not directly influence the calculation of the second derivative at $r = 0$, but it does indirectly through its contribution to the total energy of the system. As will be seen in Sects. 4.3 and 6.1, this one is making the second derivative smaller than for a pure Coulomb problem, with possibly an opposite sign in some cases. This feature may have consequences for the rapidity of the onset of the asymptotic power law behavior of the form factor of hadronic systems and for the manner how it occurs.

4 Description of the nucleon wave function in terms of quarks

Determining a nucleon wave function in terms of quarks has been done in many papers, see refs. [17, 18] for general presentations and refs. [7, 10, 19, 20, 21, 22, 23, 24, 25, 26, 27] for more specific works. We nevertheless remember a few details relative to our own calculations [11] and, especially, to some of the approximations that have been made. The quark-quark force employed in our calculations is that of Bhaduri et al. [14]:

$$\begin{aligned} V_{BHA} &= \frac{1}{2}\sum_{i<j}\left(-\frac{\kappa}{r_{ij}} + \frac{r_{ij}}{a^2} - D + \frac{\kappa_\sigma}{m_i m_j}\frac{\exp(-r_{ij}/r_0)}{r_0^2 r_{ij}}\vec{\sigma}_i.\vec{\sigma}_j\right) \\ &\equiv \sum_{i<j} V_{ij} \equiv \sum_{i<j} V(\vec{r_i} - \vec{r_j}), \end{aligned} \tag{17}$$

where $\kappa = \kappa_\sigma = 102.67\,\mathrm{MeV\,fm}$, $a = 0.0326\,\mathrm{MeV}^{-1/2}\,\mathrm{fm}^{1/2}$, $r_0 = 0.4545\,\mathrm{fm}$, $m_i = m_j = m_q = 337\,\mathrm{MeV}$, $D = 913.5\,\mathrm{MeV}$. With these definitions, distances are expressed in units of fermi and the potential in MeV. For characterizing the strength of the spin-spin force, we will most often refer to the following quantity:

$$\kappa'_\sigma = \frac{\kappa_\sigma}{(r_0 m_q)^2} = 1.66\,\kappa_\sigma. \tag{18}$$

Fitted on the meson spectrum, the above interaction provides a reasonably good account of the baryon spectrum [10], although it misses the Roper resonance. It is reminded however that the choice of the model is not essential here and that references to other quark interaction models will be made in any case, see Sects. 4.3 and 8.2.

Having neither spin-orbit nor tensor component, the force given by Eq. (17) has the particular feature to conserve the spin. The baryon wave function can thus be factorized into a spin and an orbital part. The simplification may be a drawback with some respects, for a discussion of the helicity conservation in QCD for instance. Here, on the contrary, it represents an advantage as it avoids unnecessary admixture of the effects we are looking at with other ones that are irrelevant for our purpose.

Four spin-isospin quark wave functions with the spin and isospin of the nucleon, $S = 1/2, T = 1/2$, are available. In notations of ref. [10], they are:

$$\begin{aligned} |S> &= \frac{1}{\sqrt{2}}|\chi^0\eta^0 + \chi^1\eta^1>, \\ |A> &= \frac{1}{\sqrt{2}}|\chi^0\eta^1 - \chi^1\eta^0>, \\ |MS> &= \frac{1}{\sqrt{2}}|\chi^0\eta^0 - \chi^1\eta^1>, \\ |MA> &= \frac{1}{\sqrt{2}}|\chi^0\eta^1 + \chi^1\eta^0>, \end{aligned} \tag{19}$$

where $\chi^0(\eta^0)$ and $\chi^1(\eta^1)$ correspond to quarks 1 and 2 coupled with an intermediate spin (isospin) equal respectively to 0 and 1. All of them have a definite character under the exchange of quarks 1 and 2 (symmetric : $|S>$ and $|MS>$, antisymmetric: $|A>$ and $|MA>$). Under the exchange of quarks 1, 2 and 3, the spin-isospin wave functions $|S>$ and $|A>$ are respectively symmetric and antisymmetric. The two other ones, $|MS>$ and $|MA>$, have a mixed character and transform into a combination of each other under the exchange of quarks 1 and 3, or 2 and 3, see Eqs. (127, 128). These wave functions have to be combined with spatial wave functions that have appropriate transformation properties under the exchange of quarks 1, 2 and 3 : complete symmetry for $|S>$, complete antisymmetry for $|A>$ and mixed symmetry for $|MS>$ and $|MA>$. The antisymmetry is ensured, as well known, by the color wave function, which we omit to write down as its effect factorizes out. The total wave function of momentum $\vec{P}$ may thus be written:

$$\Psi_{\vec{P}}(\vec{r}_1,\vec{r}_2,\vec{r}_3) = \psi_S(\vec{r}_1,\vec{r}_2,\vec{r}_3)\,|S> + \psi_A(\vec{r}_1,\vec{r}_2,\vec{r}_3)\,|A>$$
$$+\frac{1}{\sqrt{2}}\,\left(\psi_{MS}(\vec{r}_1,\vec{r}_2,\vec{r}_3)\,|MS> + \psi_{MA}(\vec{r}_1,\vec{r}_2,\vec{r}_3)\,|MA>\right). \tag{20}$$

It obeys the Schrödinger equation:

$$\left(\frac{\vec{p}_1^{\,2}}{2m_q}+\frac{\vec{p}_2^{\,2}}{2m_q}+\frac{\vec{p}_3^{\,2}}{2m_q}+V(\vec{r}_1-\vec{r}_2)+V(\vec{r}_2-\vec{r}_3)+V(\vec{r}_3-\vec{r}_1)-E\right)$$
$$\Psi_{\vec{P}}(\vec{r}_1,\vec{r}_2,\vec{r}_3)=0. \tag{21}$$

For the considered potential, the center of mass motion factorizes out. Being irrelevant for the description of the nucleon, we omit from now on the corresponding phase factor as well as any reference to the total momentum, implicitly assuming that the system is at rest, with $\vec{P}=0$. As to the internal wave functions $\psi_{S,A,MS,MA}(\vec{r}_1,\vec{r}_2,\vec{r}_3)$, they may be expressed in terms of Jacobi variables, $\vec{r}_1-\vec{r}_2$ and $\vec{r}_3-\frac{\vec{r}_1+\vec{r}_2}{2}$, or any combination of them, see end of App. C. The normalization of the nucleon wave function is chosen to be:

$$\int d(\vec{r}_1-\vec{r}_2)\;d(\vec{r}_3-\frac{\vec{r}_1+\vec{r}_2}{2})\,\Big(\psi_S^2(\vec{r}_1,\vec{r}_2,\vec{r}_3)+\psi_A^2(\vec{r}_1,\vec{r}_2,\vec{r}_3)$$
$$+\frac{1}{2}(\psi_{MS}^2(\vec{r}_1,\vec{r}_2,\vec{r}_3)+\psi_{MA}^2(\vec{r}_1,\vec{r}_2,\vec{r}_3))\Big)=1. \tag{22}$$

The symmetry properties of $\psi_{MS}(\vec{r}_1,\vec{r}_2,\vec{r}_3)$ and $\psi_{MA}(\vec{r}_1,\vec{r}_2,\vec{r}_3)$ under the exchange of particles 1 and 3, or 2 and 3:

$$\psi_{MS}(\vec{r}_3,\vec{r}_2,\vec{r}_1) = -\frac{1}{2}\psi_{MS}(\vec{r}_1,\vec{r}_2,\vec{r}_3)+\frac{\sqrt{3}}{2}\,\psi_{MA}(\vec{r}_1,\vec{r}_2,\vec{r}_3),$$
$$\psi_{MA}(\vec{r}_3,\vec{r}_2,\vec{r}_1) = +\frac{1}{2}\psi_{MA}(\vec{r}_1,\vec{r}_2,\vec{r}_3)+\frac{\sqrt{3}}{2}\,\psi_{MS}(\vec{r}_1,\vec{r}_2,\vec{r}_3), \tag{23}$$

also imply the relation:

$$\int d(\vec{r}_1-\vec{r}_2)d(\vec{r}_3-\frac{\vec{r}_1+\vec{r}_2}{2})\,\Big(\psi_{MS}^2(\vec{r}_1,\vec{r}_2,\vec{r}_3)-\psi_{MA}^2(\vec{r}_1,\vec{r}_2,\vec{r}_3)\Big)=0. \tag{24}$$

4.1 Faddeev approach

In the Faddeev approach, the wave function $\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ is written as a sum of 3 terms:

$$\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \Psi_{12,3} + \Psi_{13,2} + \Psi_{23,1}, \tag{25}$$

where $\Psi_{12,3}$ is symmetric in the exchange of particles 1 and 2 while $\Psi_{13,2}$ and $\Psi_{23,1}$ are obtained from $\Psi_{12,3}$ by performing the corresponding permutations, so that $\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ is symmetrical in the exchange of particles 1, 2, and 3. The amplitude $\Psi_{12,3}$ obeys the equation:

$$(\frac{\vec{p}_1^2}{2m_q} + \frac{\vec{p}_2^2}{2m_q} + \frac{\vec{p}_3^2}{2m_q} + V(\vec{r}_1 - \vec{r}_2) - E)\Psi_{12,3} = -V(\vec{r}_1 - \vec{r}_2)(\Psi_{13,2} + \Psi_{32,1}). \tag{26}$$

For calculations, $\Psi_{12,3}$ is developped on the spin-isospin basis (19), or an equivalent one, and the spatial part, which in the present case has a total orbital angular momentum $L = 0$, is decomposed in terms of the spherical harmonics relative to the Jacobi variables $\vec{x} = \vec{r}_1 - \vec{r}_2$ and $\vec{y} = \frac{2}{\sqrt{3}}(\vec{r}_3 - \frac{\vec{r}_1+\vec{r}_2}{2})$:

$$\psi_{12,3}(\vec{r}_1 - \vec{r}_2, \vec{r}_3 - \frac{\vec{r}_1 + \vec{r}_2}{2}) = \sum_{\lambda=0}^{\infty} \sum_{m=-\lambda}^{\lambda} Y_\lambda^{m*}(\hat{x})\, Y_\lambda^m(\hat{y})\, \varphi_\lambda(x, y). \tag{27}$$

The sum includes even and odd values of λ. Through determining combinations with well defined symmetry character (see equation below), these values will be associated with spin-isospin wave functions that have the same parity under the exchange of particles 1 and 2, respectively ($|S>, |MS>$) and ($|A>, |MA>$). In practice, only a finite number is retained. The minimum is to retain the lowest value, $\lambda = 0$, which gives two amplitudes in $\Psi_{12,3}$ corresponding to the two spin-isospin components, $|\chi^0\eta^0>$ and $|\chi^1\eta^1>$, or equivalently $|S>$ and $|MS>$. The restriction to these two components, $\varphi_0^0(x, y)$ and $\varphi_0^1(x, y)$, respectively associated with the spin-isospin wave functions $|\chi^0\eta^0>$ and $|\chi^1\eta^1>$, is motivated by the expected dominance of the interaction in these channels, which should be somewhat corrected however for the long range of the gluon exchange. As it may be useful, we give here the full expression of the wave function:

$$\begin{aligned}\Psi_{\mathrm{F}}(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \quad & \left[\frac{\psi_{12,3}^0 + \psi_{12,3}^1}{\sqrt{2}} + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right] |S> \\ & + \left[\frac{\psi_{12,3}^0 - \psi_{12,3}^1}{\sqrt{2}} - \frac{1}{2}(1 \leftrightarrow 3) - \frac{1}{2}(2 \leftrightarrow 3)\right] |MS> \\ & + \frac{\sqrt{3}}{2} \left[\frac{\psi_{32,1}^0 - \psi_{32,1}^1}{\sqrt{2}} - (1 \leftrightarrow 2)\right] |MA>,\end{aligned} \tag{28}$$

where, in the two component case:

$$\begin{aligned}\psi_{12,3}^{0,1} &= \varphi_0^{0,1}(x, y), \\ \psi_{32,1}^{0,1} &= \varphi_0^{0,1}(\,|\frac{1}{2}\vec{x} + \frac{\sqrt{3}}{2}\vec{y}|\,, |-\frac{1}{2}\vec{y} + \frac{\sqrt{3}}{2}\vec{x}|\,), \\ \psi_{13,2}^{0,1} &= \varphi_0^{0,1}(\,|\frac{1}{2}\vec{x} - \frac{\sqrt{3}}{2}\vec{y}|\,, |-\frac{1}{2}\vec{y} - \frac{\sqrt{3}}{2}\vec{x}|\,).\end{aligned} \tag{29}$$

While values of λ higher than 0 are neglected in the above minimal calculation of the Faddeev amplitude $\Psi_{12,3}$, their contribution to the total wave function are not totally neglected. They appear through the exchange terms $\Psi_{32,1}$ and $\Psi_{13,2}$, whose expansion in terms of spherical harmonics $\sum Y_{\lambda'}^{m'*}(\hat{x})\ Y_{\lambda'}^{m'}(\hat{y})$ and spin-isospin wave functions, in principle infinite, has been limited to 10 terms in calculating the observables (form factors) presented in this paper. A calculation consisting in retaining 8 amplitudes in the expansion of the Faddeev amplitude $\Psi_{12,3}$ (not to be confused with the wave function) has also been performed. The comparison with the 2 amplitude calculation will provide an interesting test. Indeed an important issue is to know whether some truncation in the Faddeev amplitude affects the power law behavior of the form factor at high momentum transfers.

4.2 Hyperspherical formalism approach

In the hyperspherical formalism, a different choice of the variables is made (see App. A). The modulus of the Jacobi variables are expressed in terms of an hyperradius, ρ, and an extra angle, ϕ:

$$\begin{aligned} \frac{x}{\sqrt{2}} &= \rho_{12} = \rho\ \sin\phi, \\ \frac{y}{\sqrt{2}} &= \ \rho_3 \ = \rho\ \cos\phi. \end{aligned} \tag{30}$$

In terms of these variables, the total wave function reads:

$$\Psi_{\rm HH} \equiv |\psi(\rho,\Omega)> = \sum_{K,L,sym} \psi_{K,L}(\rho)\ \left[Y_{[K,sym]}^{(L,M_L)}(\Omega)\ |sym>\right]_{J^P}, \tag{31}$$

where $|sym>$ represents the spin-isospin wave functions given by (19) and Ω the set of the various angles relative to the unit vectors, $\hat{\rho}_{12}$ and $\hat{\rho}_3$ together with the hyperspherical angle ϕ. $Y_{[K,sym]}^{(L,M_L)}(\Omega)$ stands for the hyperspherical harmonic (HH) of definite symmetry, orbital angular momentum and quantum numbers specified by K. The hyperradial wave functions, $\psi_{K,L}(\rho)$, satisfy a (infinite) set of coupled equations written in detail in App. A.

In practice, as for solving the Faddeev equation, a truncation is required. Two amplitudes have been retained. They involve the totally symmetric spin-isospin wave function $|S>$ and a combination of the mixed ones, $|MS>$ and $|MA>$. They respectively have the lowest allowed K value, 0 and 2. By identifying

$$\psi_{0,L=0}(\rho) \equiv \psi_1(\rho), \tag{32}$$

$$\psi_{2,L=0}(\rho) \equiv \psi_3(\rho), \tag{33}$$

an explicit expression of the wave function is:

$$\begin{aligned} &\Psi_{\rm HH}(\vec{r}_1,\vec{r}_2,\vec{r}_3) \simeq \frac{1}{\pi\sqrt{\pi}}\left[\psi_1(\rho)|S>\right. \\ &\left.+\sqrt{2}\ \psi_3(\rho)\left((\cos^2\phi-\sin^2\phi)|MS> -2\ \sin\phi\ \cos\phi\ \hat{\rho}_{12}.\hat{\rho}_3|MA>\right)\right]. \end{aligned} \tag{34}$$

The front factor ensures the normalization over the various angles,

$$\int \frac{d\Omega}{\pi^3} = 1, \tag{35}$$

and the radial wave functions are normalized as:

$$\int d\rho\, \rho^5\, (\psi_1^2(\rho) + \psi_3^2(\rho)) = 1. \tag{36}$$

As for the Faddeev approach, one may wonder whether the truncation of the wave function has some consequence for the prediction of the form factor at high momentum transfers. From now on, it is remarked that the description of the system with the totally symmetric radial wave function $\psi_1(\rho)$ is more economical but poorer than the corresponding one in the Faddeev approach, first bracket on the r.h.s. of Eq. (28). The latter contains extra terms that, in the hyperspherical formalism, suppose to introduce components with $K = 4,\, 6.....$ This can be seen by realizing that the completely symmetric hyperspherical harmonic for $K = 0$ only contains $Y_{\lambda=0}(\hat{x})\, Y_{\lambda=0}(\hat{y})$, whereas the Faddeev amplitudes $\Psi_{32,1}$ and $\Psi_{13,2}$ in the simplest case (2 amplitude calculation) contain terms with $\lambda \neq 0$. Let's also notice that the radially symmetric component, $\psi_1(\rho)$, is the dominant one, while the mixed symmetry component, $\psi_3(\rho)$, which is smaller, is directly determined by the spin-spin interaction.

4.3 Static properties and wave functions

We give in Table 1 a few results relative to the nucleon wave function calculated with the Bhaduri et al.'s force in the different approaches : Faddeev with 2 and 8 amplitudes and hyperspherical formalism. They concern the mass of the nucleon and its square matter radius, the proton and neutron charge squared radii and the mixed symmetry probability. Results where the spin-spin force in the hyperspherical formalism is neglected are also given. The comparison of the results incorporating the spin-spin interaction does not evidence much difference for the mass or matter radius. The sensitivity to the approximation is more important for the neutron charge squared radius and the mixed symmetry probability, which both involve the spin-spin interaction at the first and second order respectively.

As we are interested in the form factor at high q, which depends on the description of the system at short distances, we also show some wave functions. This is done only for the hyperspherical formalism where the graphical presentation of the results is much simpler than for the Faddeev approach since it reduces to the wave functions $\psi_1(\rho)$ and $\psi_3(\rho)$. They are given in Fig. 4. Examination of $\psi_1(\rho)$ at short distances evidences at first sight an exponential behavior. This one is approximately given by:

$$\psi_1(\rho) = \alpha_1 e^{-\beta_1 \rho}, \tag{37}$$

with $\alpha_1 = 19.95\,\mathrm{fm}^{-3}$ and $\beta_1 = 1.715\,\mathrm{fm}^{-1}$.

The slope, β_1, which is entirely determined by the Coulomb like part of the potential (including the spin-spin part which also behaves as $\frac{1}{r}$ at small r), is close to what is

Table 1: Static properties pertinent to the description of the nucleon, as calculated from different approaches with the same force: Faddeev equations with 2 and 8 amplitudes (first and second columns, F) and hyperspherical harmonic formalism (HH) with a restriction to the K values, 0 and 2 (third column). Results calculated in the hyperspherical harmonic formalism with $K=0$ are given separately in the fourth column for a different force. This one only includes the Coulomb and confining parts of the Bhaduri et al.'s potential (no spin-spin component). The listed quantities are successively the mass, the square matter radius, the mixed symmetry probability, the proton and neutron square "charge" radius, and the density of the nucleon at the origin.

	(F) $2A$	(F) $8A$	(HH) $\kappa_{\sigma'} = 1.66\,\kappa$	(HH) $\kappa_{\sigma'} = 0$
M_N(MeV)	1031	1020	1039	1201
$< r_m^2 >$ (fm^2)	0.219	0.219	0.218	0.255
$P_{S'}$	1.0%	2.1%	1.5%	0
$< r_p^2 >$ (fm^2)	0.234	0.243	0.238	0.255
$< r_n^2 >$ (fm^2)	-0.015	-0.024	-0.020	0
$\psi^2(0)$(fm^{-6})	386	386	398	136

theoretically expected (see App. A):

$$\frac{8\sqrt{2}}{5\pi}(\kappa + \kappa'_\sigma)\, m_q = 1.702\,\mathrm{fm}^{-1}. \tag{38}$$

Some departure from the exponential form expected in a pure Coulombian problem is however observed, which is seen in particular on the wave function, $\psi_1(\rho)$, calculated when the spin-spin part of the interaction is turned off (see Fig. 4). For this case, a sizeable change occurs in the slope of the wave function at short distances when going from $\rho = 0$ to $\rho = 0.25\,$fm. It can lead to specific features in the form factor at high q, which, as reminded in Sect. 3 for the two-body case, is sensitive to that part of the wave function. Numerically, this one up to ρ^2 terms is given by:

$$\psi_1(\rho) = \alpha'_1(1 - \beta'_1\rho + \gamma'_1\rho^2), \tag{39}$$

with $\alpha'_1 = 11.65\,\mathrm{fm}^{-3}, \beta'_1 = 0.645\,\mathrm{fm}^{-1}$ and $\gamma'_1 = -2.0\,\mathrm{fm}^{-2}$.
The value of β'_1 is close to the theoretical expectation, $\frac{8\sqrt{2}}{5\pi}\kappa m_q = 0.640\,\mathrm{fm}^{-1}$. The value of α'_1 is smaller than the value of α_1 in (37). This is in direct relation with the absence of the short range attraction that the spin-spin interaction provides on the average. As to the γ'_1 coefficient, it evidences a striking departure from its value in the pure Coulombian problem, $\gamma = \frac{\beta^2}{2} = 0.20\,\mathrm{fm}^{-2}$ (according to the notation employed in App. A), since it

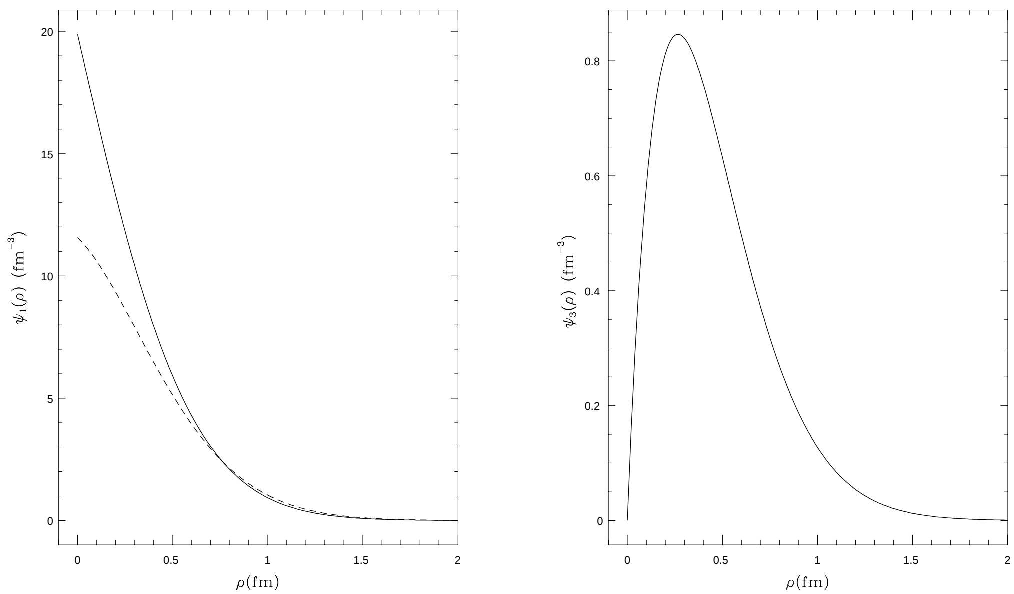

Figure 4: Radial wave functions $\psi_1(\rho)$ and $\psi_3(\rho)$ obtained in the hyperspherical formalism with the Bhaduri et al.'s force, Eq. (17) (continuous line). The wave function $\psi_1(\rho)$ in the case where the spin-spin force is turned off is also shown (dashed line).

has an opposite sign and is one order of magnitude larger. It depends on the constant terms appearing in Eq. (88) and should be compared to the theoretical expectation given by Eq. (96):

$$-\frac{m_q}{6}(E - 3m_q + \frac{3}{2}D) + \frac{5}{12}(\frac{8\sqrt{2}}{5\pi}\kappa m_q)^2 = -2.1\,\mathrm{fm}^{-2}. \tag{40}$$

Through E, it mainly involves the non-perturbative part of the potential (17), arising from the confinement. In comparison, the role played by the perturbative part due to one-gluon exchange, represented by the last term in the l.h.s. of (40), is negligible.

The behavior of the component, $\psi_3(\rho)$, at short distances also deserves some comments. An approximate expression is:

$$\psi_3(\rho) = \alpha_3\,\rho\, e^{-\beta_3 \rho} \tag{41}$$

with $\alpha_3 = 8.4\,\mathrm{fm}^{-4}$, $\beta_3 = 3.76\,\mathrm{fm}^{-1}$. The value of α_3 may be compared to the theoretical expectation given by Eq. (101):

$$\frac{32}{35\pi}\kappa'_\sigma m_q \alpha_1 = 8.23\,\mathrm{fm}^{-4}. \tag{42}$$

As to the β_3 parameter, it has no well determined interpretation. While the exponential factor in Eq. (41) turns out to be a (very!) good approximation to the numerical solution at small ρ, a more complete theoretical analysis indicates that the second term of the expansion of the solution should contain a well defined term, $\rho^2 \log \rho$, besides a ρ^2 term. While the former is determined by the knowledge of α_3, α_1 and β_1, the latter is related to the solution of the homogenous Eq. (100) and cannot be similarly determined. The insertion of the ρ^2 term in the l.h.s. of this equation indeed leads to an indertermination for its coefficient.

The comparison of the phenomenological parameters β_1, β'_1, γ'_1 and α_3 with the theoretical expectations evidences some slight discrepancy. We believe that this one has its origin in the numerical methods used to solve Eqs. (99, 100). Notwithstanding it, we think that the results show the adequacy of the Numerov algorithm that we employed to solve the equations of the HH approach and determine the short range part of the hyperradial wave function.

5 Expression and calculation of the nucleon form factors

Having determined the nucleon wave functions, we now consider the form factors that can be calculated from them. The charge and magnetic form factors of the proton and the neutron, which represent four quantities, are the object of a current interest. They can be obtained from the charge and magnetization densities. In the approximation used for describing the wave function, where the total orbital momentum of the nucleon is $L = 0$, in agreement with the quark-quark force we used, the starting point is given by the following formulas:

$$< N|\frac{1+\tau^z}{2}G_E^p(\vec{q}^2) + \frac{1-\tau^z}{2}G_E^n(\vec{q}^2)|N > (2\pi)^3\delta(\vec{P}^f - \vec{P}^i - \vec{q}) = \\ < N(\vec{P}^f)|\sum_i(\frac{1}{2}(\frac{1}{3}+\tau_i^z)e^{i\vec{q}.\vec{r}_i})|N(\vec{P}^i) >, \quad (43)$$

$$< N|\vec{\sigma}(\frac{1+\tau^z}{2}\frac{G_M^p(\vec{q}^2)}{2m_N} + \frac{1-\tau^z}{2}\frac{G_M^n(\vec{q}^2)}{2m_N})|N > (2\pi)^3\delta(\vec{P}^f - \vec{P}^i - \vec{q}) = \\ < N(\vec{P}^f)|\sum_i \frac{1}{2m_q}\frac{\vec{\sigma}_i}{2}(\frac{1}{3}+\tau_i^z)e^{i\vec{q}.\vec{r}_i}|N(\vec{P}^i) > . \quad (44)$$

Using the symmetry of the nucleon wave function, the sum over the three-quarks appearing in (43, 44) can be restricted to one of them, the matrix element being multiplied by 3. The third quark is here chosen as the expression of our wave functions for mixed symmetry states have a definite symmetry property under the exchange of particles 1 and 2. After performing the algebra relative to spin and isospin operators, one is left with the following expressions:

$$G_E^p(\vec{q}^2) = (< S|O|S > +\frac{1}{2} < MS|O|MS >) + \sqrt{2} < S|O|MS > \\ +(< A|O|A > +\frac{1}{2} < MA|O|MA >) - \sqrt{2} < A|O|MA >, \quad (45)$$

$$G_E^n(\vec{q}^2) = -\sqrt{2} < S|O|MS > +\sqrt{2} < A|O|MA >, \quad (46)$$

$$G_M^p(\vec{q}^2)\,\frac{m_q}{m_N} = (< S|O|S + \frac{1}{2} < MS|O|MS >) + \sqrt{2} < S|O|MS > \\ -\frac{1}{3}(< A|O|A > +\frac{1}{2} < MA|O|MA >) + \frac{\sqrt{2}}{3} < A|O|MA >, \quad (47)$$

$$G_M^n(\vec{q}^2)\,\frac{m_q}{m_N} = -\frac{2}{3}(< S|O|S > +\frac{1}{2} < MS|O|MS >) - \frac{\sqrt{2}}{3} < S|O|MS >$$

$$+\frac{2}{3}(< A|O|A > +\frac{1}{2} < MA|O|MA >) + \frac{\sqrt{2}}{3} < A|O|MA > . \qquad (48)$$

The matrix elements are defined as:

$$< X|O|Y >= \int d(\frac{\vec{r}_1 - \vec{r}_2}{2})\, d(\vec{r}_3 - \frac{\vec{r}_1 + \vec{r}_2}{2})$$
$$\psi_X(\vec{r}_1, \vec{r}_2, \vec{r}_3)\, e^{i\vec{q}.(\frac{2}{3}\vec{r}_3 - \frac{\vec{r}_1+\vec{r}_2}{3})}\, \psi_Y(\vec{r}_1, \vec{r}_2, \vec{r}_3), \qquad (49)$$

where X, Y stand for the different symmetry character of the components of the wave function, S, A, MS and MA. The expressions (45-48) have been written in such a way to emphasize their dependence on four independent quantities. As one of them involves the completely antisymmetric component $\psi_A(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ which is expected to be small (when it is not simply put to zero as a result of approximating the wave function as in the hyperspherical formalism or the two amplitude Faddeev calculations), one should approximately obtain the following relation:

$$\frac{1}{3}(\, G_E^p(\vec{q}^2) - G_E^n(\vec{q}^2)\,) - G_M^p(\vec{q}^2)\, \frac{m_q}{m_N} - G_M^n(\vec{q}^2)\, \frac{m_q}{m_N} = 0. \qquad (50)$$

This is trivially satisfied for:

$$G_E^p(\vec{q}^2) = f(\vec{q}^2), \qquad G_E^n(\vec{q}^2) = 0,$$
$$G_M^p(\vec{q}^2)\frac{m_q}{m_N} = f(\vec{q}^2), \quad G_M^n(\vec{q}^2)\frac{m_q}{m_N} = -\frac{2}{3} f(\vec{q}^2), \qquad (51)$$

a choice which is sometimes made in the litterature, $f(\vec{q}^2)$ being given most often a dipole expression. The validity of Eq. (50) has been tested in ref. [28] for both the Faddeev wave functions used in the present work and the measurements. At $\vec{q}^2 = 0$, expressions different from Eqs. (45-48) may be found in the literature [9]. They correspond to another type of component $|A>$ with a non-zero orbital angular momentum (1 instead of 0). In the harmonic oscillator model, the two kinds of states represent at least a $2\hbar\omega$ ($L = 1$) and a $6\hbar\omega$ ($L = 0$) excitations respectively.

The results for the form factors, $G_E^p(\vec{q}^2)$, $G_E^n(\vec{q}^2)$, $G_M^p(\vec{q}^2)$ and $G_M^n(\vec{q}^2)$ are given in Fig. 5, for transferred momenta up to $\vec{q}^2 = 5\,(\mathrm{GeV/c})^2$. In each case, the results are shown for the Faddeev calculations with two and eight amplitudes and for the hyperspherical formalism (with the two waves $K = 0$ and $K = 2$). For $G_E^p(\vec{q}^2)$, we also give results for a gaussian wave function corresponding to a matter radius close to that obtained in the other approaches. It is seen that the Faddeev calculations with two and eight amplitudes significantly differ from each other, especially for the charge neutron form factor. The eight amplitude results compare reasonably well with the results obtained in the hyperspherical formalism up to $\vec{q}^2 = 3\,(\mathrm{GeV/c})^2$. Some discrepancy appears beyond this value, as seen in this figure. One also sees that the result using a gaussian wave function quickly differs from the above ones (a factor of about 2 at $\vec{q}^2 = 5\,(\mathrm{GeV/c})^2$) and tends to 0 more rapidly.

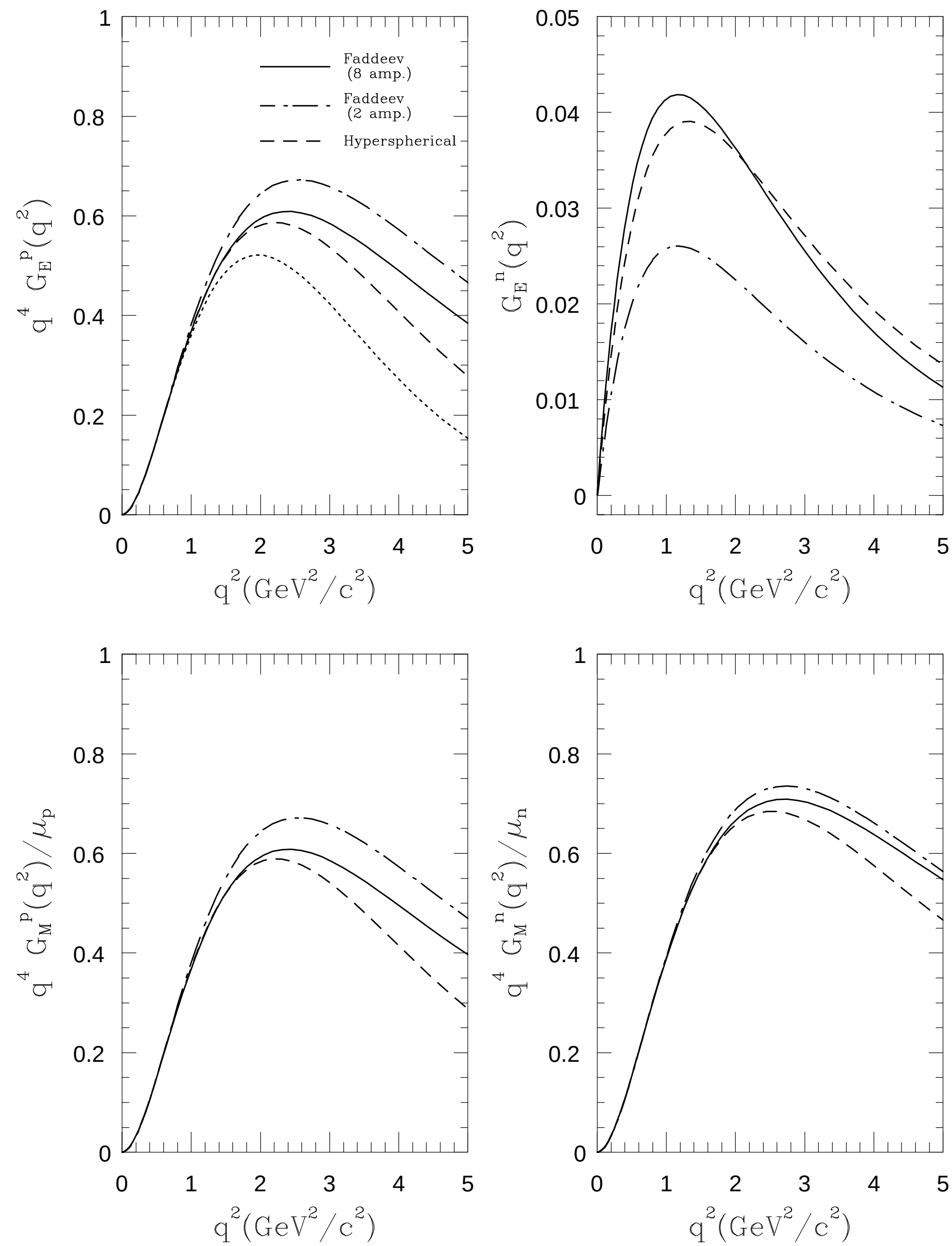

Figure 5: Electric and magnetic form factors of the proton and the neutron up to $\vec{q}^2 = 5\,(\mathrm{GeV/c})^2$. Except for $G_E^n(\vec{q}^2)$, where it is not necessary, they have been multiplied by a factor q^4 to emphasize the differences in the higher q domain. Results are presented for wave functions calculated with the Faddeev equations (2 and 8 amplitudes) and the hyperspherical formalism (K=0 and 2). A comparison with results provided by a gaussian wave function can be made by looking at the upper-left figure (dotted line).

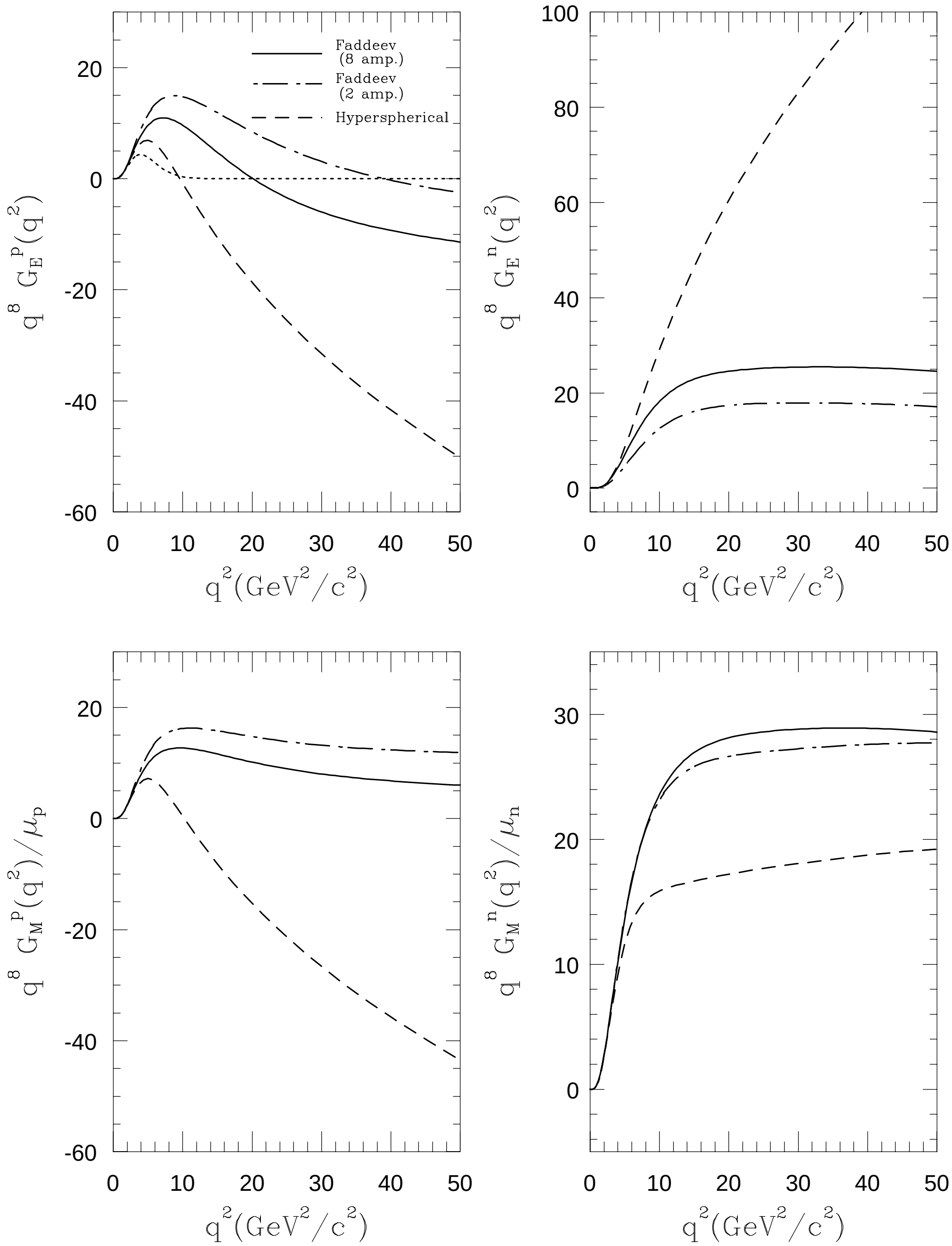

Figure 6: Same as in Fig. 5, but up to $\vec{q}^2 = 50\,(\mathrm{GeV/c})^2$. Results have been multiplied by q^8 in all cases to emphasize the asymptotic behavior.

In Fig. 6, we present results that concern the asymptotic behavior of the form factors. They have been divided by q^{-8}, which is the expected behavior at high q (Sect. 2). These results, which should tend to a constant number, are shown up to $\vec{q}^2 = 50\,(\mathrm{GeV/c})^2$. Examination of the figures indeed indicates that, roughly, some asymptotic value seems to be reached. This demonstrates that constituent quark models, contrary to what is sometimes claimed, do lead to a power law behavior of form factors. This is achieved provided that they incorporate at least a minimal description of the short range correlations produced by gluon exchanges. Looking in more details, one may however notice that the results obtained with the hyperspherical formalism are somewhat higher and that their convergence to some asymptotic value is less clear than for the Faddeev approach. In this case, a slight variation with $\vec{q}^2$ is still visible, suggesting that the asymptotic behavior may not be reached yet. These different features have led us to make a more refined analysis.

As a side remark, let us mention that the smallness of the form factors at the highest values of q^2 under consideration (of the order of $10^{-5} - 10^{-6}$) may cast doubt about their numerical accuracy. Tests for which an analytical result was available (see next section) has revealed that this accuracy was surprisingly good and could not be responsible for the above departures. In fact from a dimensional argument, the form factor given by Eq. (49) is expected to at least contain a factor, $\frac{1}{q^6}$ (see App. B), which already explains for a large part the smallness of the form factor. Some of the observed departures may however be due to the lack of accuracy in solving equations to obtain wave functions.

6 Discussion of the results

Due to the difficulty to perform extensive calculations with the Faddeev approach, the discussion made below will essentially concerns the results obtained with the hyperspherical harmonic formalism. We will nevertheless attempt to establish some relationship between the two approaches. On the other hand, instead of working with the proton and neutron, charge and magnetic form factors, we will deal with matrix elements involving components of the wave function with a given symmetry, namely:

$$< S|O|S >, \;\; < S|O|MS >, \;\; < MS|O|MS >, \;\; < MA|O|MA >, \tag{52}$$

whose expressions are given by Eq. (49). The reason to make this alternative presentation of the results is the possibility that they may not all have the same asymptotic behavior. This is expected from other three-body calculations where it was shown that, in momentum space, the component ψ_{MS} was going to zero more rapidly than ψ_S [29]. This result was however obtained with separable forces and, actually, for the Faddeev amplitude.

6.1 Calculations with the hyperspherical harmonics

The advantage of the hyperspherical harmonic formalism is to provide a simple, but approximate expression of the wave function at short distances, which furthermore can be used for analytic calculations of the form factors. Thus, from the expressions $\psi_1(\rho)$

and $\psi_3(\rho)$ given by Eqs. (37) and (41), one gets for the various matrix elements listed above:

$$< S|O|S > \;=\; 405\sqrt{6}\,\frac{\alpha_1^2\beta_1}{(q^2+6\beta_1^2)^{7/2}} \qquad \rightarrow \qquad \frac{7.88}{q^7}(\mathrm{GeV/c})^7, \tag{53}$$

$$< S|O|MS > \;=\; -2835\sqrt{6}\frac{\alpha_1\alpha_3 q^2}{(q^2+6(\frac{\beta_1+\beta_3}{2})^2)^{9/2}} \qquad \rightarrow -\frac{13.5}{q^7}(\mathrm{GeV/c})^7, \tag{54}$$

$$< MS|O|MS > = 7290\frac{\alpha_3^2}{q^8}\left(64-\frac{175\sqrt{6}\beta_3\, q^{10}+..}{(q^2+6\beta_3^2)^{11/2}}\right) \rightarrow \frac{75.6}{q^8}(\mathrm{GeV/c})^8, \tag{55}$$

$$< MA|O|MA > = -2430\frac{\alpha_3^2}{q^8}\left(64-\frac{168\sqrt{6}\beta_3\, q^{10}+..}{(q^2+6\beta_3^2)^{11/2}}\right) \rightarrow -\frac{25.2}{q^8}(\mathrm{GeV/c})^8. \tag{56}$$

For the two last matrix elements, only the terms dominant at high q and the first corrections to it have been retained here. A complete expression is given in App. B. An expression identical to Eq. (53) was obtained in [23, 30], but on a pure phenomenological basis, without relation to some microscopic dynamics.

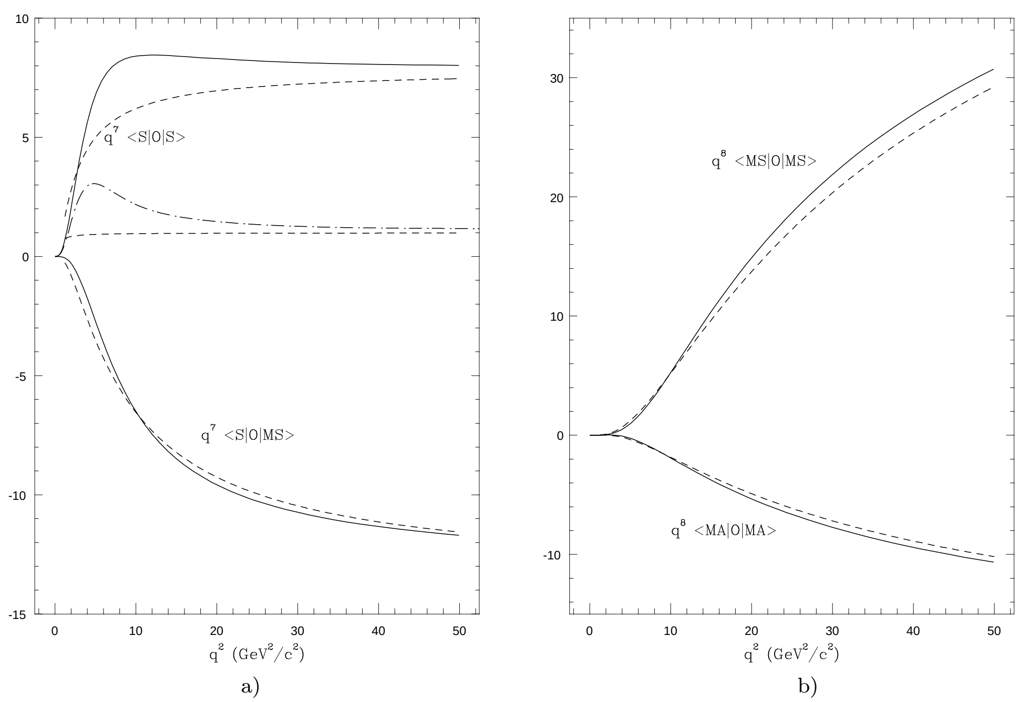

Figure 7: Form factors calculated in the hyperspherical formalism up to $q^2 = 50\,(\mathrm{GeV/c})^2$. The form factors, $< S|O|S >$ and $< S|O|MS >$, have been multiplied by q^7 (Fig. a) and the form factors, $< MS|O|MS >$ and $< MA|O|MA >$, by a factor q^8 (Fig. b). In each figure, we present the form factors obtained from the numerical wave functions, $\psi_1(\rho)$ and $\psi_3(\rho)$ (continuous line), as well as the analytic ones, Eqs. (53-56) (dashed line). The $< S|O|S >$ form factor in absence of spin-spin interaction is also shown in Fig. a (dash-dotted line for the numerical calculation and dashed line for the analytical one).

Being a good representation of the short range behavior of the calculated wave functions, the parametrizations of the components, $\psi_1(\rho)$ and $\psi_3(\rho)$, given by Eqs. (37) and (41) should provide accurate expectations for the form factor at high q in the hyperspherical harmonic approach considered here. By comparing the form factors given by Eqs. (53-56) to those calculated with the actual wave functions, it is possible to determine the range of q^2 where the description of the wave functions by the parametrizated ones, Eqs. (37, 41), becomes relevant, as well as the role of non-perturbative effects they do not account for. The results are presented in Figs. 7a, b. They have been multiplied by a factor q^7 for $< S|O|S >$ and $< S|O|MS >$ and q^8 for $< MS|O|MS >$ and $< MA|O|MA >$. These factors are chosen in accordance with the asymptotic behavior expected from Eqs. (53-56). The results so obtained can be directly compared to the asymptotic numerical values given in Eqs. (53-56). Figure 7a also contains the result for the form factor, $< S|O|S >$, when the spin-spin part of the Bhaduri et al.'s force is removed.

Examination of the figures shows a good agreement between the form factors given by Eqs. (53-56) and those obtained with the calculated wave functions, $\psi_1(\rho)$ and $\psi_3(\rho)$, in the range, $q^2 > 30\,(\mathrm{GeV/c})^2$. With the help of the analytic expression, it is possible to have a discussion on the onset of the asymptotic behavior.

These results evidence two striking features. First, the form factors $< S|O|S >$ and $< S|O|MS >$ at very high q behave like q^{-7}, instead of q^{-8} as it is expected in a complete calculation [12]. The q^{-7} behavior can be understood using counting rules, similarly to the derivation of the q^{-8} in the general case: the propagator of the set of three-quarks in the hyperspherical harmonic formalism introduces a factor q^{-2}, while the interaction, given by the Fourier transform of the $\frac{1}{\rho}$ hyperradial potential provides a factor q^{-5}. Second, the form factors $< MS|O|MS >$ and $< MA|O|MA >$ behave like q^{-8}, but the asymptotic behavior is far to be reached in the range around $50\,(\mathrm{GeV/c})^2$ (a factor 2-3 is missing).

The explanation for the wrong behavior of the form factors $< S|O|S >$ and $< S|O|MS >$ is to be looked for in the approximation consisting in retaining the lowest K values in the expansion of the wave function. Let us examine for example the spatial wave function for $K = 0$, $\psi_1(\rho)$. Depending on the variable $\rho = \sqrt{\rho_{12}^2 + \rho_3^2}$, at small ρ_{12} ($\rho_{12} << \rho_3$) it behaves like :

$$\psi_1(\rho)_{\rho_{12}\to 0} = \psi_1(\rho_3) + \frac{1}{2}\psi_1'(\rho_3)\frac{\rho_{12}^2}{\rho_3} + ... \qquad (57)$$

The appearance of ρ_{12}^2 as the first non zero term in the expansion cannot account properly for the short range correlations of particles 1 and 2, the third particle being a spectator. As one obtains in the two-body case (see Sect. 3), a term linear in ρ_{12} is expected. To get the right behavior, one should sum up an infinite set of contributions with K ranging from 0 to ∞. Furthermore, a rapid convergence is not ensured at all.

In the case without spin-spin force ($\psi_3(\rho) = 0$), we looked at some contributions of the next terms in the hyperspherical harmonic expansion corresponding to $K = 4$ and 6, Eq. (102)). The first one contributes to $< S|O|S >$ three times as much as the wave $K = 0$ in the asymptotic domain, with the same sign. The contribution of the $K = 6$ wave, which is antisymmetric in the ρ_{12} and ρ_3 variables and, thus, introduces a richer structure in the wave function, could correct the result in the right direction. This is in agreement with the fact that reproducing the correct asymptotic behavior requires a somewhat fine description of the wave function at short distances, especially with respect to the ϕ hyperspherical angle.

Another related aspect is worthwhile to be mentioned. Up to a factor, the coefficients of the q^{-7} term in Eq. (53) and Eq. (54), $\alpha_1.\alpha_1\beta_1$ and $\alpha_1.\alpha_3$ respectively, have a form similar to Eq. (12), namely the product of the wave function at the origin, α_1, which by itself contains non-perturbative effects, and its derivative at the same point, $\alpha_1\beta_1$ or α_3. Factorizing out the wave function at the origin, α_1, the last quantities, β_1 and $\frac{\alpha_3}{\alpha_1}$, for which an analytic expression can be obtained from Eqs. (38) and (42), are seen to be linearly dependent on the strength of the dominant part of the interaction at short distances, determined by κ and κ'_σ. This is not in accordance with the argument reminded in Sect. 2, which implies two gluon exchange and, therefore, a dependence on the square of these κ factors.

On the other hand, from the smallness of the mixed symmetry state (1.5%), it may be inferred that its contribution to the form factor is negligible. This is true at small momentum transfers, but not at high ones, as shown in Fig. 7a or on the analytic expressions given by Eqs. (53, 54). Taking into account the expressions of α_3/α_1, Eq. (42) and β_1, Eq. (38), it is found that the matrix elements $< S|O|MS >$ and $< S|O|S >$ at high q are quite comparable. Indeed the ratio tends to a constant of the order of unity:

$$(\frac{< S|O|MS >}{< S|O|S >})_{q\to\infty} = -2\sqrt{2}\,\frac{\kappa'_\sigma}{\kappa+\kappa'_\sigma} \simeq -1.8 \tag{58}$$

The last aspect we want to discuss about the form factors, $< S|O|S >$ and $< S|O|MS >$, is the role of non perturbative effects. In a pure Coulombian problem and in the approximation where only the $K = 0$ wave is retained, the form factor approaches its asymptotic value from below. Examination of Fig. 7a shows the opposite feature, as a consequence of having incorporated into the ρ^2 term of the description of the wave function at short distances a contribution depending on the energy of the system, see Eq. (40), which indirectly involves the confining interaction. The above feature, which is strongly enhanced when the spin-spin force is neglected (see Fig. 7a), will be also seen in other calculations presented below. The observed structure at $q^2 = 4\,(\mathrm{GeV/c})^2$ in the case without spin-spin force has to do with the change in the second derivative of the wave function around $\rho = 0.3\,\mathrm{fm}$ that can be seen in Fig. 4. An immediate consequence is to make the onset of the asymptotic behavior to be reached quicker.

It is also interesting to look at the two other form factors, $< MS|O|MS >$ and $< MA|O|MA >$, although it is more difficult to rely on theoretical expectations, due to the lack of interpretation for the parameter β_3 entering the short-range description of the component $\psi_3(\rho)$, Eq. (41), on which they depend. We already noticed that these form factors behave like q^{-8} at high q (see analytical calculations, Eqs. (55, 56)). On this basis, one may think that their role should be negligible at high q, in comparison to $< S|O|S >$ or $< S|O|MS >$. This is hardly verified ($< MS|O|MS >= 0.55 < S|O|S >$ at $q^2 = 50\,(\mathrm{GeV/c})^2$).

Moreover, the values of these form factors multiplied by q^8, which should tend to a constant, are still increasing in the range $q^2 \simeq 50\,(\mathrm{GeV/c})^2$ and far to have reached the asymptotic value (larger by a factor 2).

In fact, assuming that the asymptotic behavior was reached in this range led us to the preliminary conclusion that this form factor was behaving like q^{-7}, like $< S|O|S >$ and $< S|O|MS >$. It is only from the examination of the analytical calculation, Eqs. (53-56), that we could determine that the asymptotic behavior of the form factors, $< MS|O|MS >$ and $< MA|O|MA >$, was q^{-8} and understand, as well, the other features they evidence. In particular, as is seen from Eqs. (55, 56), the next to leading order contribution in q^{-8} contains an extra factor q^{-1}, instead of q^{-2} for the form factors, $< S|O|S >$ or $< S|O|MS >$. The contribution is about 60% of the dominant term at $q = 7\,(\mathrm{GeV/c})$ and, since it is destructive, it provides an overall reduction of the form factor by a factor 2-3. This feature strongly delays the onset of the asymptotic behavior for the form factors, $< MS|O|MS >$ and $< MA|O|MA >$. Equations (55, 56) suggest that the asymptotic behavior is obtained within a few % at values of q^2 as large as 10000 $(\mathrm{GeV/c})^2$ (!).

So, we can conclude that calculating reliably the form factors at values of q^2 as high as those considered here supposes not only an accurate determination of the wave function at short distances, but also on the whole range. Calculations we did up to $q^2 = 200\,(\mathrm{GeV/c})^2$ have revealed moderate but regular oscillations around the asymptotic behavior of the form factor $< S|O|S >$, explaining a slight plateau that can hardly be seen in Fig. 7a around $q^2 = 40\,(\mathrm{GeV/c})^2$, but shows up in the numerical values. From the checks we did, it turns out that these oscillations are in relation with the value of the hyperradius where the inner and outer Numerov solutions of the Schrödinger equations were matched. At this point, the third derivative of the numerical solution evidences a change in sign (giving rise to the oscillations) that does not seem to have any physical meaning.

6.2 Calculations in the Faddeev approach

One may wonder whether observations made about the calculations performed in the hyperspherical harmonic approach apply to those in the Faddeev approach, for which we have a priori no precise benchmark as far as the asymptotic behavior of form factors is concerned. From examining the corresponding results presented in Fig. 8, we may notice that the convergence of the form factors $< S|O|S >$ and $< S|O|MS >$ to some asymptotic behavior assumed to be q^{-8} is the same as for the hyperspherical harmonic calculations with respect to q^{-7}: same rate and from above for the first one. As to the form factors $< MS|O|MS >$ and $< MA|O|MA >$ $(+2 < A|O|A >)$, the absolute value of their product by q^8, especially for the second one, tends to steadily increase around $q^2 = 50\,(\mathrm{GeV/c})^2$, quite similarly to the results obtained with the hyperspherical harmonic approach. Due to the way calculations were performed, the contribution of the fully antisymmetric spin-isospin state to the last matrix element, $< A|O|A >$, could not be easily separated in the case of the 8 amplitude Faddeev approach where it is not zero. Its contribution is in any case quite small and should not affect the discussion relative to the matrix element, $< MA|O|MA >$, to which it is admixed (what we reminded between parentheses).

Two more observations are in order. There is no evidence that, for $q \to \infty$, the form factors, $< MS|O|MS >$ or $< MA|O|MA >$, go to zero more rapidly than the form factors, $< S|O|S >$ or $< S|O|MS >$, as is expected from some calculations (see Sect. 7.1

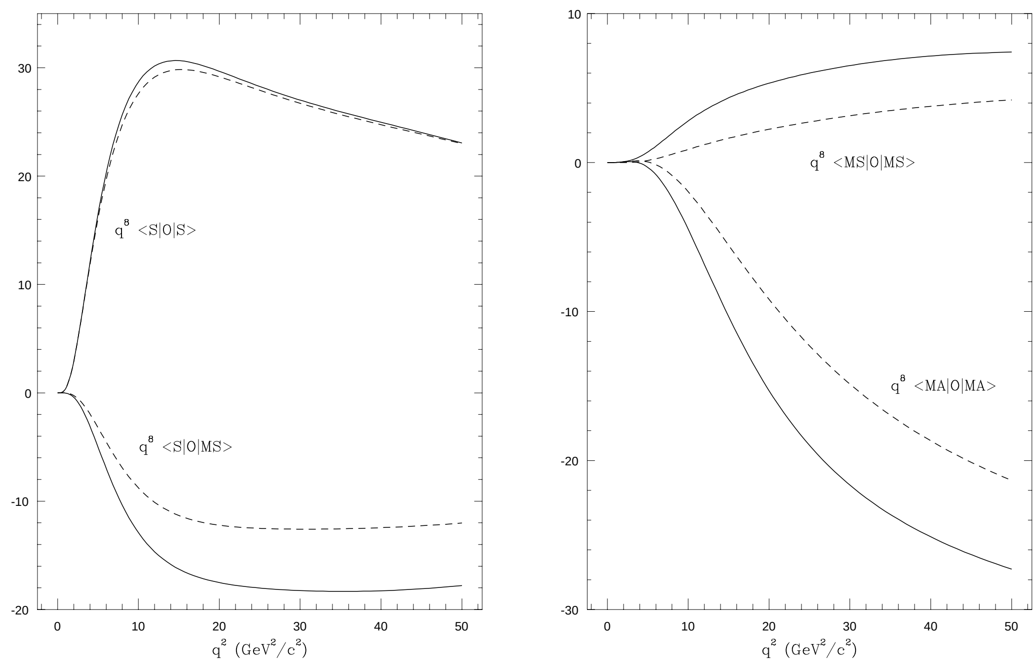

Figure 8: Form factors calculated in the Faddeev approach up to $q^2 = 50\,(\mathrm{GeV/c})^2$. The dashed and continuous lines respectively represent the results for the 2 and 8 amplitude calculations. The form factors, $< S|O|S >$, $< S|O|MS >$, $< MS|O|MS >$ and $< MA|O|MA >$, have been multiplied by a factor q^8. The values, which these quantities are expected to converge to, are given in Eqs. (74).

below). On the other hand, on the basis of the same calculations, the form factors in the Faddeev approach are expected to go to zero more rapidly and to be smaller in absolute value at high q than in the hyperspherical harmonic approach with the truncation space chosen. The comparison of the results indicates that this conclusion is supported in some part for the form factors, $< S|O|S >$ and $< S|O|MS >$, (a factor 0.4 at $q^2 = 50\,(\mathrm{GeV/c})^2$ for $< S|O|S >$ for instance), but not for $< MA|O|MA >$ $(+2 < A|O|A >)$ (a factor 2.5). At this point, the slow convergence of this matrix element to an asymptotic value prevents one to make definite statements.

Another question of interest concerns the convergence of the form factors to their asymptotic behavior depending on the Faddeev calculation, two or eight amplitudes in the present case. Apart for a change in scale for the form factors involving the mixed symmetry state, the examination of our results does not evidence any significant change in the asymptotic behavior. The increase with q of the quantity, $q^8.(< MA|O|MA > (+2 < A|O|A >)\,)$, is slightly less with the eight amplitude than with the two amplitude Faddeev calculation around $q^2 = 50\,(\mathrm{GeV/c})^2$. In any case, and contrary to the hyperspherical harmonic approach, there is no evidence at this point that the power law asymptotic behavior of form factors is affected by the truncation of the Faddeev type calculation which, in the simplest case (two amplitudes), already involves a more structured wave function. Only the overall size may depend on the truncation. In fact, an examination of the mixed symmetry components of the wave function at small distances for the 2 and 8 amplitude cases, around $\rho = 0.05\,\mathrm{fm}$, shows a discrepancy of a few percents only, without any relation to what is observed in the calculations performed for momentum transfers up to

$q^2 = 50\,(\mathrm{GeV/c})^2$. This is another indication that the asymptotic behavior is far to be reached for the matrix elements $< MS|O|MS >$ and $< MA|O|MA >$.

7 Wave function approach to the asymptotic form factors

Partly to reduce the gap between the two approaches we discussed, partly to answer some of the questions raised by the results obtained in the Faddeev approach, we here present further developments. The aim is to get an estimate of the asymptotic form factor as accurate as possible, starting from the observation that the wave function at the origin is rather well determined and independent on the approach (see Table 1). Such a program can be in principle performed because, similarly to the two-body case discussed in Sect. 3, the form factor at high q is expected to only depend on the gluon exchange force, once the wave function at the origin is known. The expression for the form factors reads in momentum space:

$$< X|O|Y >= \int \frac{d\vec{\kappa}_{12}}{(2\pi)^3} \frac{d\vec{\kappa}_3}{(2\pi)^3}\; \psi_X(\vec{\kappa}_{12}, \vec{\kappa}_3)\; \psi_Y(\vec{\kappa}_{12}, \vec{\kappa}_3 + \sqrt{\frac{2}{3}}\; \vec{q}). \tag{59}$$

Being essentially interested in the asymptotic behavior, some approximation can be done.

We follow the qualitative lines developped by Alabiso and Schierholz [12], whose approach essentially extends to the three-body system that one reminded for the two-body case in Sect. 3. Noting that the wave function is peaked at small values of the argument, two regions of the integration range over the variable $\vec{\kappa}_3$ are important: $\vec{\kappa}_3 = 0$ and $\vec{\kappa}_3 = -\sqrt{\frac{2}{3}}\vec{q}$. One thus gets :

$$\begin{aligned} < X|O|Y >_{q\to\infty} =& \left(\int \frac{d\vec{\kappa}_{12}}{(2\pi)^3} \frac{d\vec{\kappa}_3}{(2\pi)^3}\; \psi_X(\vec{\kappa}_{12}, \vec{\kappa}_3)\right)\; \psi_Y(0, \sqrt{\frac{2}{3}}\; \vec{q}) \\ &+ \left(\int \frac{d\vec{\kappa}_{12}}{(2\pi)^3} \frac{d\vec{\kappa}_3}{(2\pi)^3}\; \psi_Y(\vec{\kappa}_{12}, \vec{\kappa}_3)\right)\; \psi_X(0, -\sqrt{\frac{2}{3}}\; \vec{q}). \end{aligned} \tag{60}$$

Remembering that the integral is nothing but the value of the wave function in configuration space at the origin, and that only $\psi_S(\rho = 0) \neq 0$, we come to the conclusion that two among the four matrix elements $< S|O|S >$, $< S|O|MS >$, $< MS|O|MS >$ and $< MA|O|MA >$ have a non zero contribution at the dominant order, which is given by $\psi_Y(\vec{\kappa}_{12} = 0, \vec{\kappa}_3 = \sqrt{\frac{2}{3}}\,\vec{q})$ or $\psi_X(\vec{\kappa}_{12} = 0, \vec{\kappa}_3 = -\sqrt{\frac{2}{3}}\,\vec{q})$. They are:

$$< S|O|S >_{q\to\infty} = 2\psi_S^{(r)}(\rho = 0)\; \psi_S^{(k)}(\vec{\kappa}_{12} = 0, \vec{\kappa}_3 = \sqrt{\frac{2}{3}}\vec{q}), \tag{61}$$

$$< S|O|MS >_{q\to\infty} = \psi_S^{(r)}(\rho = 0)\; \psi_{MS}^{(k)}(\vec{\kappa}_{12} = 0, \vec{\kappa}_3 = \sqrt{\frac{2}{3}}\vec{q}). \tag{62}$$

The superscripts, r and k, remind that the corresponding wave functions, which have been defined previously, refer to different spaces.

7.1 Improving upon the hyperspherical harmonics results

One can check that when Eqs. (61) and (62) are applied to the numerical solutions of the Bhaduri potential at short distances obtained in the hypersperical harmonic formalism, Eqs. (37) and (41), the asymptotic behavior of the form factors, Eqs. (53) and (54), is reproduced. Concerning $< MS|O|MS >$ and $< MA|O|MA >$, the next to dominant order approach confirms the asymptotic behavior obtained in Eqs. (55) and (56).

In order to improve the calculation of the form factors in the hyperspherical approach, we can try to get a better determination of the wave function for high values of the momenta, what can be done iteratively from a zeroth order wave function. This one is chosen as the Fourier transform of a wave function of the form $\psi_1(\rho)/\pi^{3/2} = \bar{\alpha} e^{-\bar{\beta}\rho}/\pi^{3/2}$ (it may correspond for instance to a Coulomb like potential or to the solution of the Bhaduri et al. potential at short distances Eq. (37), in which case, $\bar{\alpha} = \alpha_1$, $\bar{\beta} = \beta_1$). It reads:

$$\begin{aligned} \psi^{(0)}(\vec{\kappa}_{12}, \vec{\kappa}_3) &= \int d\vec{\rho}_{12}\, d\vec{\rho}_3\, e^{-i(\vec{\kappa}_{12}.\vec{\rho}_{12} + \vec{\kappa}_3.\vec{\rho}_3)}\, \frac{\bar{\alpha}}{\pi^{3/2}}\, e^{-\bar{\beta}\sqrt{\rho_{12}^2+\rho_3^2}} \\ &= \frac{N^{1/2}}{(\kappa_{12}^2 + \kappa_3^2 + \bar{\beta}^2)^{7/2}}, \end{aligned} \tag{63}$$

where

$$\vec{\kappa}_{12} = \frac{\vec{p}_1 - \vec{p}_2}{\sqrt{2}}, \quad \vec{\kappa}_3 = \sqrt{\frac{2}{3}}\, (\vec{p}_3 - \frac{\vec{p}_1 + \vec{p}_2}{2}), \quad N^{1/2} = 120\, \pi^{3/2}\, \bar{\alpha}\, \bar{\beta}.$$

The quantities, $\vec{p}_1, \vec{p}_2$ and $\vec{p}_3$, represent the momenta of the quarks 1, 2 and 3. Representing only the high momentum behavior of the wave function, $\psi^{(0)}(\vec{\kappa}_{12}, \vec{\kappa}_3)$ and subsequent wave functions don't fulfill the usual normalization condition, which would imply a relation between $\bar{\alpha}$ and $\bar{\beta}$, see Eq. (98).

For an interaction dominated by a Coulomb force (or a Yukawa one) and for the wave function given by Eq. (63), it is possible to calculate analytically the wave function by an iteration procedure up to first order. Thus from the Schrödinger equation,

$$|\psi >= \frac{1}{E-T}\, V|\psi >= \frac{1}{E-T}\, (V_{12} + V_{13} + V_{23})|\psi >, \tag{64}$$

we can write (as it is done in the Faddeev formalism):

$$\psi = \psi_{12} + \psi_{13} + \psi_{23}, \tag{65}$$

so that

$$\psi_{ij}^{(1)} = \frac{1}{E^{(0)} - T} V_{ij} \psi^{(0)}. \tag{66}$$

The contribution due to the force between particles 1 and 2, which can be identified to the Faddeev amplitude $\psi_{12,3}$, Eq. (27), but in momentum space, reads:

$$\begin{aligned} \psi_{12}^{(1)}(\vec{\kappa}_{12}, \vec{\kappa}_3) &= \frac{2\sqrt{2}}{15\pi}(\kappa + \kappa'_\sigma)\, m_q \\ &\times \Bigg(\frac{3\, N^{1/2}}{4\kappa_{12}(\kappa_{12}^2 + \kappa_3^2 + \bar{\beta}^2)^{7/2}}\, \log(\frac{\sqrt{\kappa_{12}^2 + \kappa_3^2 + \bar{\beta}^2} + \kappa_{12}}{\sqrt{\kappa_{12}^2 + \kappa_3^2 + \bar{\beta}^2} - \kappa_{12}}) \end{aligned}$$

$$+\frac{3}{2}\frac{N^{1/2}}{(\kappa_{12}^2+\kappa_3^2+\bar{\beta}^2)^3(\kappa_3^2+\bar{\beta}^2)}+\frac{N^{1/2}}{(\kappa_{12}^2+\kappa_3^2+\bar{\beta}^2)^2(\kappa_3^2+\bar{\beta}^2)^2}\Bigg). \tag{67}$$

The total wave function is obtained by adding to (67) the contributions where the role of particles 1 and 3 or 2 and 3 are exchanged, corresponding to the interaction of the pairs 23 or 13. As a consistency check, it can be verified that averaging the total wave function over the hyperspherical angle ϕ with the appropriate weight factor in the differential element of volume, $\sin^2\phi\cos^2\phi$, allows one to recover the zeroth order wave function (63). This is achieved provided that the following condition is fulfilled for a Coulomb potential with strength, $\kappa+\kappa'_\sigma$:

$$\bar{\beta}=\frac{8\sqrt{2}}{5\pi}\,(\kappa+\kappa'_\sigma)\,m_q. \tag{68}$$

Expression (67) can also be used to calculate the mixed symmetry component. In such a case, the separate amplitudes with the pair of particles 12 having spin 0 and 1 should be associated with the appropriate strength of the force, $\kappa+3\kappa'_\sigma$ and $\kappa-\kappa'_\sigma$. The Yukawa nature of the spin-spin force may also be accounted for. In our opinion, the wave function so obtained may be usefully compared to the two amplitude Faddeev calculation presented in Sect. 4.1. It certainly has not all the physics contained in this one, especially that related to the confinement but it probably has a large part of the physics relative to the description of the form factor at high momentum transfers. In this respect, it is noticed that the amplitude (67) behaves like κ_3^{-8} when κ_3 goes to ∞, while the zeroth order amplitude (63) behaves like κ_3^{-7}. As it will be shown below, this property determines the behavior of the form factor at high q for the dominant contributions.

The expression of the total wave function (28) together with that of the individual contributions given by (67) for one of them can be used to improve the form factors calculated in the hyperspherical formalism. In comparison with the most complete 8 amplitude Faddeev calculation, the form factors, $<S|O|S>$ and $<S|O|MS>$, obtained on the basis of Eqs. (61, 62), are found to be smaller by a factor 2-3 in the asymptotic domain around 50 (GeV/c)^2. As to the ratio, $\frac{<S|O|MS>}{<S|O|S>}$, an expectation can be obtained in the limit where one retains the dominant contribution (due to that part of the total wave function involving the terms obtained from Eq. (67) by exchanging the role of particles 1 and 3 or 2 and 3):

$$\left(\frac{<S|O|MS>}{<S|O|S>}\right)_{q\to\infty}=-\frac{1}{\sqrt{2}}\,\frac{\kappa'_\sigma}{\kappa+\kappa'_\sigma}\simeq-0.45. \tag{69}$$

It is found too small by a factor 1.5. Notice that the above factor in Eq. (69) is smaller than in Eq. (58) by a factor 4, showing once more the failure of the hyperspherical harmonic formalism to make accurate predictions in the present domain when it is restricted to the lower K values. As to the other form factors, $<MS|O|MS>$ and $<MA|O|MA>$, a prediction made in the spirit of Eqs. (61, 62), but accounting for the minimal dependence of wave functions on $\vec{\kappa}_{12}$ or $\vec{\kappa}_3$, leads to a $q^{-10}\log q$ dependence of these quantities at high q. This implies that they should tend to zero with q more rapidly than the matrix elements, $<S|O|S>$ and $<S|O|MS>$, while the full calculations rather indicate an opposite situation around $q^2=50\,(\mathrm{GeV/c})^2$. The slow convergence to the asymptotic behavior we do find in some cases, especially for the form factors, $< MS|O|MS >$ and $< MA|O|MA >$, calculated in the hyperspherical harmonic formalism, invites to some caution in concluding from this discrepancy.

In order to understand these results and their difference with the most complete ones, let us note that to get a q^{-8} asymptotic dependence of the form factors, one has to pick up terms proportional to ρ^2 in the expansion of the product of the initial and final wave functions at small distances. This follows from dimensional arguments (the elementary volume involves six powers of ρ). To get such terms, one can combine ρ^2 terms coming from the wave function in the initial (final) state with the zeroth order wave function (the wave function at the origin) in the final (initial) state. The contribution of these ones to the form factors are likely to be accounted for by Eqs. (61, 62). One can also combine terms of the first order in ρ in each state. These ones have obviously no contribution to the asymptotic expressions of the form factors given by Eqs. (61, 62).

The failure of Eqs. (61, 62) to account for the asymptotic behavior can also be analyzed by considering the structure of the wave functions in momentum space. With a wave function as simple as (63), the dominance of the low momenta components in the matrix element (59), which is essential to get Eqs. (61, 62), necessarily applies to either wave function of the initial or the final state. With the wave function (67), which results from a first iteration over the previous one, but contains different factors, other possibilities are offered. The dominance of the contribution of low momentum components may involve one factor in the initial state wave function with respect to some integration variable and a factor in the final state wave function with respect to another integration variable. This is illustrated by the following contribution of the dominant part of a matrix element (Eq. (59)) involving the Faddeev amplitudes $\psi_{13,2}\ \psi_{23,1}$:

$$\left[\int d\vec{\kappa}_{12}\ d\vec{\kappa}_3\ \frac{1}{(\kappa_{12}^2+\kappa_3^2+\bar{\beta}^2)^2\ \ ((-\frac{1}{2}\vec{\kappa}_3-\frac{\sqrt{3}}{2}\vec{\kappa}_{12})^2+\bar{\beta}^2)^2}\right.$$

$$\left.\frac{1}{((-\frac{1}{2}(\vec{\kappa}_3+\sqrt{\frac{2}{3}}\vec{q})+\frac{\sqrt{3}}{2}\vec{\kappa}_{12})^2+\bar{\beta}^2)^2\ \ (\kappa_{12}^2+(\vec{\kappa}_3+\sqrt{\frac{2}{3}}\vec{q})^2+\bar{\beta}^2)^2}\right]\simeq$$

$$\left[\int \frac{1}{(\kappa_{12}'^2+\kappa_3'^2+\bar{\beta}^2)^2\ (\kappa_3'^2+\bar{\beta}^2)^2(\ (-\frac{1}{2}(\vec{\kappa}_3'+\sqrt{\frac{2}{3}}\vec{q})-\frac{\sqrt{3}}{2}\vec{\kappa}_{12}')^2+\bar{\beta}^2)^2}\right.$$

$$\left.\frac{1}{(\ (\frac{1}{2}\vec{\kappa}_{12}'+\frac{\sqrt{3}}{2}\vec{\kappa}_3')^2+(-\frac{1}{2}\vec{\kappa}_3'+\frac{\sqrt{3}}{2}\vec{\kappa}_{12}'+\sqrt{\frac{2}{3}}\vec{q})^2+\bar{\beta}^2)^2} d\vec{\kappa}_{12}'\ d\vec{\kappa}_3'\right]_{\kappa_{12}',\kappa_3'\sim\bar{\beta}}$$

$$+\left[\int \frac{1}{(\kappa_{12}''^2+\kappa_3''^2+\bar{\beta}^2)^2\ (\kappa_3''^2+\bar{\beta}^2)^2\ (\ (-\frac{1}{2}(\vec{\kappa}_3''-\sqrt{\frac{2}{3}}\vec{q})-\frac{\sqrt{3}}{2}\vec{\kappa}_{12}'')^2+\bar{\beta}^2)^2}\right.$$

$$\left.\frac{1}{(\ (\frac{1}{2}\vec{\kappa}_{12}''+\frac{\sqrt{3}}{2}\vec{\kappa}_3'')^2+(-\frac{1}{2}\vec{\kappa}_3''+\frac{\sqrt{3}}{2}\vec{\kappa}_{12}''-\sqrt{\frac{2}{3}}\vec{q})^2+\bar{\beta}^2)^2} d\vec{\kappa}_{12}''\ d\vec{\kappa}_3''\right]_{\kappa_{12}'',\kappa_3''\sim\bar{\beta}}$$

$$+(\frac{2}{\sqrt{3}})^3\left[\int \frac{1}{(\ (\frac{1}{3}(\vec{\kappa}_3'+2\vec{\kappa}_3''+\sqrt{\frac{2}{3}}\vec{q})^2+\kappa_3'^2+\bar{\beta}^2)^2\ (\kappa_3'^2+\bar{\beta}^2)^2\ (\kappa_3''^2+\bar{\beta}^2)^2}\right.$$

$$\left.\frac{1}{(\,(\frac{1}{3}(\vec{\kappa}_3'' + 2\vec{\kappa}_3' - \sqrt{\frac{2}{3}}\vec{q})^2 + \kappa_3''^2 + \bar{\beta}^2)^2} d\vec{\kappa}_3' \, d\vec{\kappa}_3''\right]_{\kappa_3', \kappa_3'' \sim \bar{\beta}} . \qquad (70)$$

The two first terms on the r.h.s. of (70) are nothing but a contribution similar to that given in Eq. (60). They correspond to the processes represented in Fig. 9a. The third term involves the short range radial dependence of wave functions of both the initial and final states. It corresponds to the contribution of a diagram like that drawn in Fig. 9d.

The integration in the above equation can be performed in the limit of large q, where the q^{-8} dependence factorizes out. The result is:

$$\left(\frac{128}{3\pi} + 81(\frac{2}{\sqrt{3}})^3\right)\frac{\pi^4}{q^8\bar{\beta}^2}, \qquad (71)$$

where the two terms respectively correspond to the two first terms and the third one on the r.h.s. of Eq. (70). It is worthwhile to make a comparison with the contribution of a "diagonal term", $\psi_{13,2}\ \psi_{13,2}$, or $\psi_{23,1}\ \psi_{23,1}$, given by:

$$\frac{16}{3\pi}\frac{\pi^4}{q^8\bar{\beta}^2}. \qquad (72)$$

It is seen that the contribution from a non-diagonal term, Eq. (71), is more efficient to contribute at high q, roughly a factor 50 larger. This is in relation with the expectation that the form factor at high q requires a momentum transfer from the quark which is stroken by the photon to the two other quarks. It is likely that a more complete calculation will produce for the "diagonal term" an extra q^{-1} factor at least, which the above suppression factor partly accounts for (see Sect. 7.3 below).

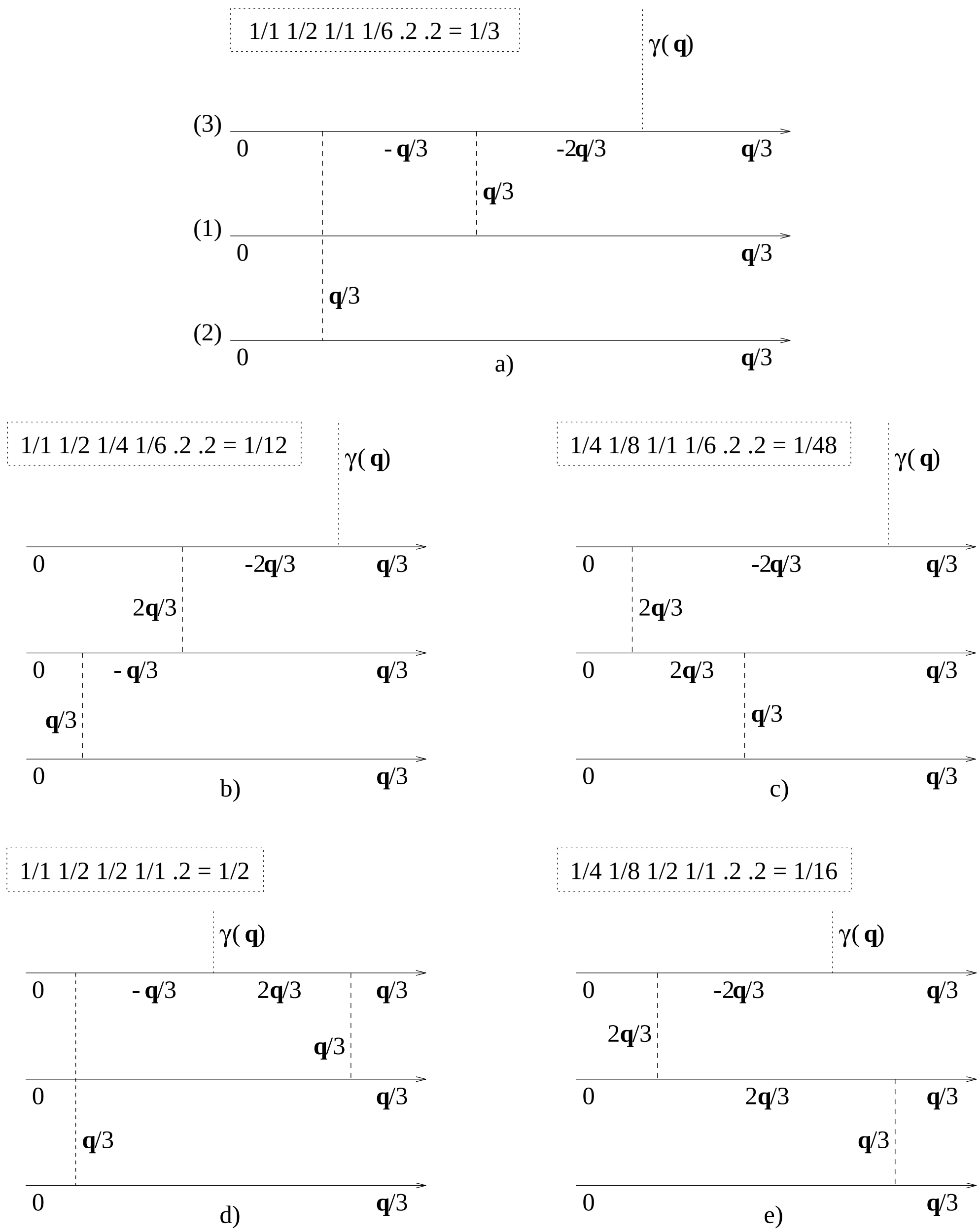

Figure 9: Representation of diagrams contributing to the asymptotic form factor with the indication of various factors entering the gluon or quark propagators. These ones, which can be written as $1/(n\,(\frac{\vec{q}}{3})^2)$, are represented in the figure by $1/n$, the factor $(\frac{\vec{q}}{3})^2$ being factored out. Reading diagram a) from left to right for instance, one explicitly gets: $n = 1$ for the first gluon propagator, $n = 1 + 1$ for the first quark propagator, $n = 1$ for the second gluon propagator and $n = 4 + 1 + 1$ for the second quark propagator. The weight accounting for undisplayed diagrams is also indicated: a factor 2 for similar ones where the role of the final and initial states is exchanged and, in most cases, a factor 2 for similar diagrams where the role of particles 1 and 2 are exchanged. Factors accounting for the different spin-isospin components of the wave function are not incorporated here (rather see Table 2).

7.2 Further development

From Eq. (59), and the wave functions provided by Eq. (67), one can now make a more complete calculation of the dominant term at high q, including some contribution due to the spin-spin force. The different form factors then read:

$$\begin{aligned}
< S|O|S > &= \frac{729\sqrt{3}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8} \\
&\qquad [\frac{512}{15\pi}(\frac{1}{\sqrt{3}}+\frac{1}{48}\log\frac{2+\sqrt{3}}{2-\sqrt{3}})\ +\ 9\,](\kappa+\kappa'_\sigma)^2, \\
< S|O|MS > &= -\frac{729\sqrt{6}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8} \\
&\qquad [\frac{32}{5\pi}(\frac{1}{\sqrt{3}}+\frac{1}{36}\log\frac{2+\sqrt{3}}{2-\sqrt{3}})\ +\ \frac{15}{4}](\kappa+\kappa'_\sigma)\, 2\kappa'_\sigma, \\
< MS|O|MS > &= \frac{729\sqrt{3}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8}\ [\,0\ +\ 3\,](2\kappa'_\sigma)^2, \\
< MA|O|MA > &= \frac{729\sqrt{3}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8}\ [\,0\ -\ 12\,](2\kappa'_\sigma)^2,
\end{aligned} \tag{73}$$

The first term in the squared brackets corresponds to the improvement essentialy based on Eqs. (61, 62) discussed in Sect. 7.1. The second one corresponds to a further improvement, which goes beyond these equations and was discussed at the end of the same section. The two terms have the same origin as the two first terms in the r.h.s. of Eq. (70) on the one hand and the third one on the other. They can be shown to have a close relation with the diagrams of Fig. 9 (see below).

A slightly different but more complete expression, as far as the asymptotic behavior and the effect of the spin-spin force are concerned, can be obtained. We start from a zeroth order, Eq. (115), strongly peaked at zero momenta, and proceed to two iterations obtaining a wave function, Eq. (120), which is more appropriate than Eq. (67) as far as the short range correlation is concerned (no average over the ϕ angle), but is not normalizable. Then, the asymptotic behavior is given by :

$$\begin{aligned}
< S|O|S > &= \frac{729\sqrt{3}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8}\ [7((\kappa+\kappa'_\sigma)^2-2\kappa'^2_\sigma)\, C+9(\kappa+\kappa'_\sigma)^2] \\
&= (3.1C+17.9)\,(\mathrm{GeV/c})^8\, q^{-8} \\
&= (19.8,\, 21.0)\,(\mathrm{GeV/c})^8\, q^{-8}, \\
< S|O|MS > &= -\frac{729\sqrt{6}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8}\ [\frac{9}{4}\,C+\frac{15}{4}](\kappa+\kappa'_\sigma)\, 2\kappa'_\sigma \\
&= -(7.9C+13.2)\,(\mathrm{GeV/c})^8\, q^{-8} \\
&= -(17.9,\, 21.1)\,(\mathrm{GeV/c})^8\, q^{-8}, \\
< MS|O|MS > &= \frac{729\sqrt{3}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8}\ [\,0\ +\ 3\,](2\kappa'_\sigma)^2 \\
&= 9.3\,(\mathrm{GeV/c})^8\, q^{-8},
\end{aligned}$$

$$
\begin{aligned}
< MA|O|MA > &= \frac{729\sqrt{3}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8}\,[\,0\ -\ 12\,](2\kappa'_\sigma)^2, \\
&= -37.2\,(\mathrm{GeV/c})^8\, q^{-8}. \qquad (74)
\end{aligned}
$$

The factor C accounts for extra non-perturbative effects, not included in the value of the wave function at the origin, $\bar{\alpha}$ (see appendix C). Although there is no close relationship, differences between (73) and (74) can give some insight on the effect of the truncation of the Faddeev amplitude (remember that the wave function from (67) can be assimilated to the two amplitude Faddeev calculation). It is noticed that they only affect the first term of each bracket. The main difference is represented by the contribution of the term $-2\kappa'^2_\sigma$ to $< S|O|S >$. This one could not occur in results presented in Eq. (73), which were obtained using the fully spatially symmetric component of the wave function, Eq. (63), as a zeroth order. The absence of difference for the second term in the bracket is due to the fact that, once the wave function at the origin, $\bar{\alpha}$, is determined, the calculation involves the first order terms of an expansion of the wave function around the origin. These terms, see Eqs. (113, 114), are completely determined by the $\frac{1}{r}$ behavior of the potential at short distances. The numerical values have been calculated using $\bar{\alpha} = 19.64\,\mathrm{fm}^{-3}$, $m_q = 337\,\mathrm{MeV}$, $\kappa = 102.67\,\mathrm{MeV\,fm}$ and $\kappa'_\sigma = 1.66\,\kappa$, the range given in the brackets being determined by the values $C = 0.6$ and $C = 1.0$. This range can be directly compared to the values that form factors shown in Fig. 8 are taking in the asymptotic domain.

Table 2: Detail of the coefficients entering Eq. (74) with indication of the two interacting quark pairs and the diagram of Fig. 9 they correspond to.

	$< S\|O\|S >$	$\frac{< S\|O\|MS >}{\sqrt{2}}$	$< MS\|O\|MS >$	$< MA\|O\|MA >$
$V_{23}V_{13}O$ a	$\frac{16}{3}$	$-\frac{8}{3}$	0	0
$V_{12}(V_{13}+V_{23})O$ b	$\frac{4}{3}$	$\frac{1}{3}$	0	0
$(V_{13}+V_{23})V_{12}O$ c	$\frac{1}{3}$	$\frac{1}{12}$	0	0
sum	7	-$\frac{9}{4}$	0	0
$V_{23}OV_{13}$ d	8	-4	4	-12
$(V_{13}+V_{23})OV_{12}$ e	1	$\frac{1}{4}$	-1	0
sum	9	-$\frac{15}{4}$	3	-12

The different factors appearing in Eq. (74) can also be directly obtained, up to an overall factor, from considering the five types of diagrams shown in Fig. 9. The detail is given in Table 2. Three of them (a, b, c) involve two gluon exchanges in the initial or the final state wave functions. The two other ones (d, e) involve one-gluon exchange in both the initial and the final state wave functions. These two type contributions correspond to the first and second terms in the squared bracket appearing in Eqs. (74). They involve the wave function at the origin multiplied by its second derivative, $\psi^{(r)}(0)\,\psi^{(r)''}(0)$, for the first one and the product of the derivatives at the origin, $\psi^{(r)'}(0)\,\psi^{(r)'}(0)$, for the second one. These expressions are those that are relevant for determining the asymptotic behavior of the form factor for the three-body case, while the expression, $\psi^{(r)}(0)\,\psi^{(r)'}(0)$, is appropriate for the two-body case (incidentally, this expression works for the hyperspherical harmonic calculation limited to the wave $K = 0$ (see Eq. (53) together with Eqs. (37, 38)). We finally remark that, at high q, the combination of the various form factors, which corresponds to a scalar-isoscalar quark density, has the peculiarity to factorize into a simple form:

$$< S|O|S > + \frac{1}{2}(< MS|O|MS > + < MA|O|MA >) =$$
$$\frac{729\sqrt{3}}{\pi}\,\frac{(\bar{\alpha}\,m_q)^2}{q^8}\,[\,7\,C \;+\; 9\,]\,(\,(\kappa+\kappa'_\sigma)^2 - 2\kappa'^2_\sigma\,). \tag{75}$$

7.3 Remaining discrepancies

The comparison of the Faddeev results in the asymptotic domain with the expectation given by the matrix element $< S|O|S >$ of Eq. (74) improves over that made previously in this section (7.1). On the other hand, the results of the eight amplitude Faddeev calculations are definitively closer to what is expected from (74) than the two amplitude Faddeev calculations. The remaining discrepancies may have their origin either in the truncation of the total Faddeev amplitude or in the fact that the asymptotic behavior has not been reached yet, unless both are related. At present, it is not clear whether the discrepancies should correspond to corrections of the order, $\frac{1}{q}$, as in the case of hyperspherical harmonic results obtained previously, see Eqs. (55) and (56), or of the order, $\frac{1}{q^2}$ (up to $\log q$ terms). Numerically, either one is possible in the higher range of momentum transfers we explored.

From examining the general expression for the form factor, Eq. (59), or more particularly the expression for the wave function obtained from Eq. (67), it is reasonable to consider that the theoretical corrections to the leading term should be of the order $\frac{1}{q^2}$ (up to $\log q$ terms) in a complete calculation. A rigorous demonstration is difficult however. Any attempt to show it from an expansion of the form factor in powers of $\frac{1}{q}$ indicates the existence of terms that linearly diverge in the limit $q \to \infty$, as it can be checked from the expression of a particular contribution, see third term in the r.h.s. of Eq. (70) (discarding the limitation $\kappa'_3, \kappa''_3 \sim \bar{\beta}$). In practice, they are cut off by a factor q, hence corrections of relative order $\frac{1}{q}$. It is possible that the total sum of these terms cancels out, leaving a correction of relative order $\frac{1}{q^2}$, while in an incomplete calculation they would partly survive. An example, though referred to the two-body case, is detailed in the appendix D.

It also provides some hint for making rigorous statements as to corrections of the relative order $\frac{1}{q^2}$ to the leading order contributions.

On the other hand, from QCD based calculations, the opinion is that corrections should be of the order $\frac{1}{q^2}$ [31], but this estimate incorporates the full quark propagator and, especially, accounts for Z-type diagrams neglected here. One cannot infer from this calculation that the present one should evidence similar features. Notice that QCD based arguments hold up to $\log q$ terms. These ones partly originate from a relativistic treatment and should not be confused with those mentioned above, which are due to the three-body nature of the problem under consideration. The absence of convergence of some form factors, $< MA|O|MA >$ in both the hyperspherical and Faddeev aproaches and $< MS|O|MS >$ in the hyperspherical approach, makes the situation more confusing. As they are sensitive to the spin-spin part, one may expect them to depend on its short range behavior, which deviates from a Coulomb type potential we considered in our theoretical analysis. Assuming that the effect of the exponential factor in the force can be approximated by exp($-\rho/r_0$) with $\rho = 3/q$, part of the large dependence of the above form factors could be explained. No similar dependence is seen in the other form factors however. Furthermore, one should not forget that we are dealing with a coupled channel problem, which can change the behavior of wave functions at short distances with respect to expectations based on perturbative arguments. In fact, this is a likely explanation for the slow convergence of the form factors, $< MA|O|MA >$ and $< MS|O|MS >$, to their asymptotic behavior.

7.4 Physical form factors

Throughout this section and the previous one, we discussed particular contributions to the physical form factors with the idea to facilitate the analysis of their asymptotic behavior. Using Eqs. (74), the expressions of the physical form factors in the asymptotic regime are:

$$
\begin{aligned}
G^p_E(\vec{q}^{\,2}) &= \frac{729\sqrt{3}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8} \\
&\qquad \Big((7C+9)(\kappa+\kappa'_\sigma)^2 - (9C+15)(\kappa+\kappa'_\sigma)\,\kappa'_\sigma - (14C+18)\kappa'^2_\sigma\Big) \\
&= (-8.1C - 14.7)\,(\mathrm{GeV/c})^8\, q^{-8} = -(19.6,\, 22.8)\,(\mathrm{GeV/c})^8\, q^{-8}, \\
G^n_E(\vec{q}^{\,2}) &= \frac{729\sqrt{3}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8}(9C+15)(\kappa+\kappa'_\sigma)\,\kappa'_\sigma \\
&= (11.2C + 18.6)\,(\mathrm{GeV/c})^8\, q^{-8} = (25.3,\, 29.8)\,(\mathrm{GeV/c})^8\, q^{-8}, \\
G^p_M(\vec{q}^{\,2})/\mu_p &= \frac{729\sqrt{3}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8}\,\frac{m_N}{m_q\,\mu_p} \\
&\qquad \Big((7C+9)(\kappa+\kappa'_\sigma)^2 - (9C+15)(\kappa+\kappa'_\sigma)\,\kappa'_\sigma - (14C-14)\kappa'^2_\sigma\Big) \\
&= (-8.2C + 10.2)\,(\mathrm{GeV/c})^8\, q^{-8} = (5.3,\, 2.0)\,(\mathrm{GeV/c})^8\, q^{-8}, \\
G^n_M(\vec{q}^{\,2})/\mu_n &= \frac{729\sqrt{3}}{\pi}\frac{(\bar{\alpha}\, m_q)^2}{q^8}\,\frac{m_N}{m_q\,\mu_n}
\end{aligned}
$$

$$\Big(-(\frac{14}{3}C+6)(\kappa+\kappa'_\sigma)^2+(3C+5)(\kappa+\kappa'_\sigma)\,\kappa'_\sigma+(\frac{28}{3}C-20)\kappa'^2_\sigma\Big)$$
$$=(-2.6C+32.3)\,(\mathrm{GeV/c})^8\,q^{-8}=(30.7,\,29.7)\,(\mathrm{GeV/c})^8\,q^{-8}. \qquad (76)$$

The negative sign for $G_E^p(\vec{q}^2)$, which implies a change in sign for this form factor between 0 and ∞, is in agreement with numerical calculations presented in Fig. 6. This feature, which contradicts what is generally expected from QCD, has its origin in the unrealistic value of κ'_σ in the Bhaduri et al's quark-quark force (see next section). As expected from Eq. (76) above, the proton magnetic form factor is much smaller than the neutron one in the asymptotic regime. Again, this feature is strongly sensitive to the value of κ'_σ.

Comparison of the different calculations of the form factors considered along the text with Eqs. (76) shows that the 8 amplitude Faddeev calculation is the most convergent to the asymptotic behavior. It cannot be said however that it has definitively been reached in the range $30-50\,(\mathrm{GeV/c})^2$, confirming what can be inferred from examination of Fig. 8.

8 Making the calculations of the form factors more realistic

In the previous sections, we mainly discussed the reliability of predictions with respect to technical aspects of the calculations regarding the description of the nucleon in terms of quarks. Any comparison with measurements was eluded. The reason for that is the existence of other effects to be accounted for. They are successively discussed in the next subsections.

8.1 Quark form factors

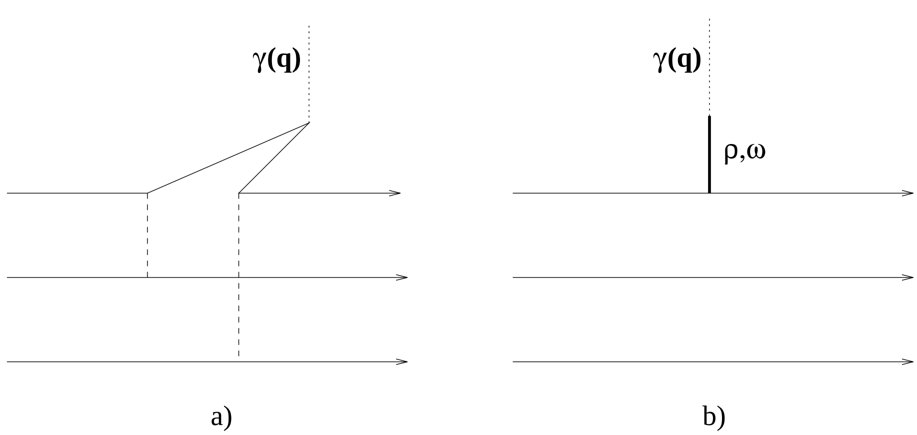

Figure 10: Extra contributions to form factors that have to be included, due to a coupling to a quark-antiquark pair (a) or a vector meson (b).

In a constituent quark model like the one we used, it is quite legitimate to account for some electromagnetic form factor of the quarks. Contributions due to the coupling of the photon to a quark-antiquark pair, shown in Fig. 10a, or to a vector meson, ω or ρ, shown in Fig. 10b, cannot be generated by the non-relativistic constituent quark model and, consequently, their effect has to be added by hand [9, 32]. Processes represented in Fig. 10b lead to a quite successful phenomenology at the nucleon (and meson) level, which is known as vector meson dominance. This can be transposed at the quark level without any change. The same statement partly holds for the contribution of the pion cloud to the proton and neutron form factors, which may be accounted for in some part in the ρ contribution. Some contribution may also come from the Darwin-Foldy term (for the charge density), but this one may be partly incorporated in the ω or ρ contribution.

For the proton charge squared radius, the latter can contribute an amount of $0.4\,\mathrm{fm}^2$ to be added to the matter squared radius, $0.25\,\mathrm{fm}^2$. The pion cloud, for that part not included in the ρ meson can contribute an extra $0.10 - 0.15\,\mathrm{fm}^2$, making the proton and neutron charge squared radius close to the measured ones. At non-zero momentum transfers, the same mechanism tends to make the prediction for the form factors closer to measurements in the range, $q^2 = 0 - 1\ (\mathrm{GeV/c})^2$ [32]. At higher momentum transfers, the hypothesis of the vector meson dominance doesn't lead to any improvement. On the contrary, assuming the full validity of this hypothesis, it adds an extra factor, q^{-2}, to the discrepancy by a factor q^{-4} between the result of non-relativistic calculations and the QCD expectations.

8.2 Improved description of the nucleon spectroscopy

An other aspect concerns the force between quarks. The Bhaduri et al.'s force is known to miss the position of the Roper resonance by a large amount. To remedy this situation, one proposal based essentially on a phenomenological basis, has been to add a three-quark force [11]. This one has not led to any significant discrepancy in the low energy phenomenology of pionic baryon decays. On the contrary, it improves the situation in many cases [9, 33]. From the expression of this force:

$$V_{II}^{(3)} = \frac{1}{2} \sum_{i \neq j \neq k} \frac{V_0}{m_i m_j m_k} \frac{e^{-m_0 r_{ij}}}{m_0 r_{ij}} \frac{e^{-m_0 r_{ik}}}{m_0 r_{ik}}, \tag{77}$$

which involves two Yukawa factors, it is expected that the form factor should have a q^{-6} asymptotic behavior, instead of q^{-8}. This might contribute to make the form factors closer to the q^{-4} behavior expected from QCD. While the above three-body force may account in a phenomenological way for omitted relativistic effects, we nevertheless believe that the effect it provides has no relationship to the boost effects that could provide the appropriate behavior and should naturally appear in a full relativistic calculation (see Sect. 8.4 below).

Calculations using the nucleon wave function obtained from solving the corresponding Schrödinger equation and the above three-body force have been performed in the hyperspherical harmonic formalism with $K = 0$ and $K = 2$. Due to a strong short range attraction (the force leads to a $\frac{1}{\rho^2}$ term, similar to the centrifugal barrier), the solution behaves approximately like $\frac{1}{\rho^{\frac{3}{2}}}$ at short distances, instead of 1 for the Bhaduri et al.'s potential [14]. It results that the dominant contribution to the form factor, $< S|O|S >$, scales like q^{-3} at large momentum transfers, differing from the expectation by three powers of q. The discrepancy has its source in the inability of the hyperspherical harmonic formalism with a limited number of terms to correctly reproduce the short range correlations between quarks produced by the three-body force given by Eq. (77). The problem is similar to that encountered for the Coulomb part of the force in the same approach, with the difference that it is more severe here. The calculated form factor is shown in Fig. 11 (dash-dotted line).

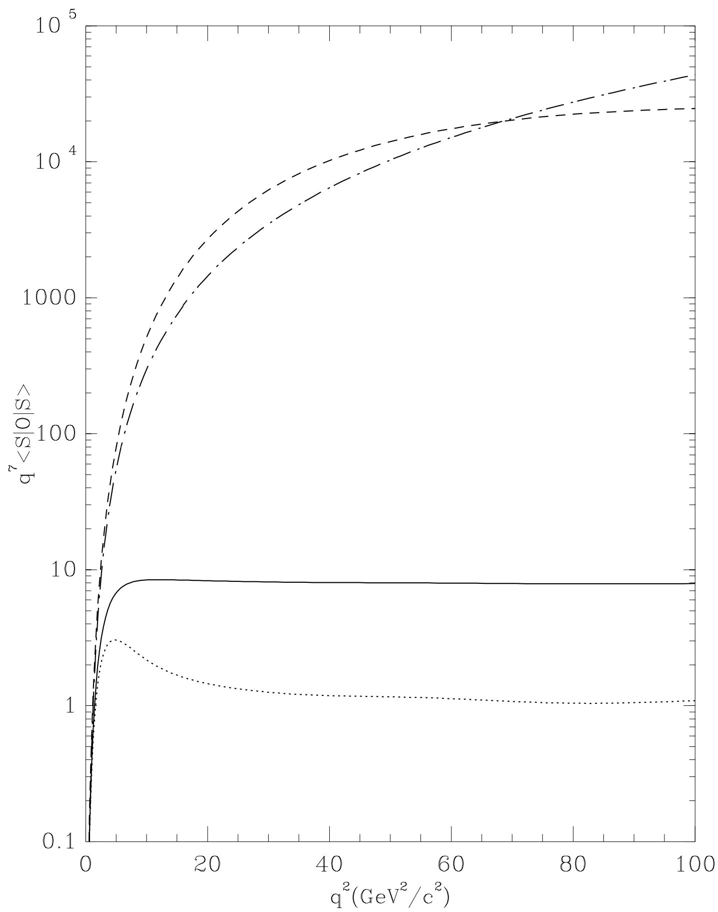

Figure 11: Form factors calculated with three-body forces and without. Those presented here correspond to the $< S|O|S >$ transition matrix element. They have been multiplied by a factor q^7 to evidence the convergence to the asymptotic value expected for two of them (potential of Bhaduri et al. and potential including a gaussian three-body force, Eq. (78), continuous and dashed lines respectively). The other curves represent results for the three-body force given by Eq. (77) (dash-dotted line) and for the Bhaduri's model in absence of spin-spin force (dotted line).

Due to the proximity of some collapse in calculating the nucleon wave function with the above mentioned three-quark force, calculations were also performed with a Gaussian type force:

$$V_{III}^{(3)} = V_0 \exp(-\sum_{i \leq j} \frac{r_{ij}^2}{\lambda^2}). \tag{78}$$

This force is much less singular at short distances than $V_{II}^{(3)}$ and tends to zero more quickly for high momentum transfers, like $\exp(-\frac{\lambda^2}{4}(\kappa_{12}^2+\kappa_3^2))$. The resulting nucleon spectroscopy is equally good [33]. The form factors so obtained do not differ much from the previous ones and, like them, overshoot those obtained with the Bhaduri et al.'s force by a large factor in the range around 50 (GeV/c)^2 (see Fig. 11, dashed line). Contrary to the other calculation, the present one roughly evidences a behavior q^{-7}. This result is expected in the hyperspherical harmonic formalism with a restriction to $K = 0$ and $K = 2$ when the Coulomb part of the potential takes over the gaussian part. Surprisingly, it does not seem to be strongly affected by the large strength of the gaussian three-body force. While the asymptotic power law is identical to that obtained with the Bhaduri et al.'s force (continuous line in Fig. 11), the overall size is changed by three orders of magnitude.

The comparison of the various results obtained from forces without and with three-body forces tends therefore to show that extra contributions to the description of the force between quarks, especially at short distances, can considerably modify the onset of the asymptotic behavior of the form factors, making it to occur at higher momentum transfers or with a different pattern. Thus, the form factors in the model involving the three-body force given by Eq. (77) reach their asymptotic behavior from below. As to the form factor calculated with the other three-body force, Eq. (78), the apparent onset of the asymptotic behavior that is seen in Fig. 11 is misleading. The product of the form factor multiplied by q^7 represented in the figure passes through a maximum and then should decrease to reach a value which is of the order of $0.7\,10^4$ (GeV/c)^7. Except for a difference in scale, the pattern exhibited by this result is quite similar to that one obtained for the Bhaduri et al.'s force without spin-spin force (dotted line in the same figure).

We would also like to notice that the difference by three orders of magnitude for the form factors at high q produced when three-body forces are included is not a distinctive feature. A similar effect [34] has been obtained with a two-body quark force resulting from meson exchange [27]. The common feature has perhaps some relationship to the small mean squared radius of the nucleon that these models predict, of the order of 0.10 fm^2.

8.3 Description of the spin-spin force

It is known that the spin-spin part of the quark-quark force we used, essentially fitted on the difference of the π- and ρ-meson masses, is too large with respect to a genuine gluon exchange force by a factor 6 [33]. Correcting for this factor would decrease the contribution of the mixed symmetry states to the form factors and, indirectly, would accelerate the convergence to their asymptotic value since these particular contributions have a slower convergence. It would also probably remove some of the sign changes that are observed for some calculated form factors, especially $G_E^p(\vec{q}^2)$ and $G_M^p(\vec{q}^2)$. These changes are not expected to occur and, furthermore, are not seen in the measurements.

The correction by a factor 6 is not the main problem however. Apart for the fact that it should be compensated by something else, the radial part of the force, which is given by a δ function in the non-relativistic limit, has been replaced by a less singular function, namely a Yukawa type potential. The advantage of this one is that it can be incorporated into a Schrödinger equation while avoiding a collapse of the solution in some cases [35]. It is noticed that the spin independent part of the force also contains a δ type force, but this one is generally neglected.

In a perturbative calculation of the asymptotic form factors, there would not be any difficulty to use a δ type force. The propagator of the gluons in Figs. 2 would be replaced by a constant, removing for each of them a q^{-2} factor in predicting the behavior of the asymptotic form factor. The result so obtained should be closer to the asymptotic QCD prediction. This feature however supposes approximations that are not supported by a more rigorous derivation of the force. The δ type force is expected to be made less singular by the presence of relativistic normalization factors, $\frac{m_q}{E^f}\,\frac{m_q}{E^i}$. The resulting non-locality is difficult to account for, and in absence of a better solution, employing a Yukawa potential instead of a δ type avoids to make unrealistic predictions.

8.4 Relativistic corrections

A complete relativistic calculation of the form factor is rather tedious. Some recipe to account for part of these effects has been used in the past. One of them consists in changing the argument of the form factor as follows [36, 37, 38]:

$$F^{r.}(q^2) = F^{n.r.}\left(\frac{q^2}{1+\frac{q^2}{4m_N^2}}\right). \tag{79}$$

This expression, which has its origin in a naive estimate of the effect due to the Lorentz contraction, leads to a constant at large q, in contradiction with the QCD expectation. It turns out to be valid only at small q [39, 40]. A modified expression, motivated by the QCD result, has also been used [41, 42].

The discussion presented in Sect. 2 indicates that the non-relativistic calculation misses an important factor due to a boost effect which mainly affects particles interacting by exchanging spin 1 bosons. This factor may be roughly of the order of:

$$\left(\frac{m_q + E_{\bar{q}}}{2m_q}\right)^4 \tag{80}$$

where $E_{\bar{q}}$ is an energy corresponding to an average momentum, $\bar{q}$, carried by the quarks in the Breit frame, see Fig. 3. At large q, this energy varies like a fraction of q and the above factor, Eq. (80), provides the q^4 factor discrepancy between the non-relativistic and relativistic calculations, while allowing one to recover the non-relativistic prediction in the limit of small momentum transfers. There is no point to insist on the approximate character of this factor. It results from an analysis of what a non-relativistic calculation partly misses and the determination of the coefficient of the q^4 factor in Eq. (80) is likely to be uncertain. Furthermore, corrections of the same order are expected from Z-type diagrams, which involve the negative energy part of the quark propagator. In absence of a more refined analysis, we will simply mention an estimate made by incorporating in our calculation the factor given by Eq. (80) together with a value of $\bar{q}$ between $\frac{q}{6}$ (momentum of initial and final quarks in the Breit frame) and $\frac{q}{2}$ (momentum of quarks in the intermediate state of Fig. 9.d). This amounts to a factor $\frac{q^4}{2304\, m_q^4}$ at high q. Disregarding the effect of the mixed symmetry component due to its uncertain character (see above), we considered the contribution of the matrix element $< S|O|S >$ to the quantities $q^4\, G_M^p(\vec{q}^2)/\mu_p$ and $q^4\, G_M^n(\vec{q}^2)/\mu_n$. Results so obtained for the Bhaduri et al.'s quark interaction model have the size that experiment suggests, $0.4\,(\mathrm{GeV/c})^4$ [43], but are off by three orders of magnitude for the other models incorporating three-quark forces.

9 Conclusion

We have studied the nucleon form factors using a non-relativistic constituent quark model. The emphasis has been put more on general features pertinent to their calculation than on a fine description of observables. A special attention has been given to the asymptotic behavior of the form factors at high momentum transfer.

Contrary to what is sometimes mentioned, it is found that the non-relativistic quark model does lead to power law form factors at high q. This power law is related to the force between quarks in momentum space and thus can be predicted if it is well defined. It is our belief that this approach, possibly corrected for exchange currents and relativity, is completely equivalent to the one used in the QCD asymptotic regime where uncorrelated wave functions are used together with an effective operator accounting for three-quarks absorbing a virtual photon while exchanging two gluons. In the present approach however, the calculation of the form factors is performed in a unique model of the nucleon and tends to cover the full range of momentum transfers from 0 to ∞.

The exact power law behavior, which is expected to be q^{-8} in the non-relativistic framework used in the present study, has been found to depend on the approximations made in treating the three-body system. While the hyperspherical harmonic formalism with a restriction to the lowest values of the grand orbital, K, is generally a good approximation for determining the binding energy, it has been found to systematically lead here to the wrong power law, q^{-7} instead of q^{-8}. To remedy this situation, the full set of K values is required. This is expected from the behavior of the wave function of two quarks at a short distance, on which it depends linearly. As far as we can see, the Faddeev approach is unsensitive to this difficulty, but we did not get the certitude that the magnitude of the asymptotic form factor is independent on the truncation of the Faddeev amplitude although it could be. In any case, this dependence seems to be small. At low momentum transfers, the form factors calculated with the 2 amplitude Faddeev approach or with the hyperspherical formalim are in good agreement with the more complete 8 amplitude Faddeev calculation up to $q^2 = 2\,(\mathrm{GeV/c})^2$ for the former and $q^2 = 3\,(\mathrm{GeV/c})^2$ for the latter.

The convergence of the form factors to their asymptotic value is another feature we looked at carefully. By only considering form factors in the range around $q^2 = 50\,(\mathrm{GeV/c})^2$, it has often been difficult to guess their power law behavior. This has led us to examine the wave functions at short distances and, from them, to determine the asymptotic behavior. In a few cases, we had to conclude that form factors at $q^2 = 50\,(\mathrm{GeV/c})^2$ were off their asymptotic value by a factor 2-3. This occured especially for form factors involving the transition between mixed symmetry components, $MS \leftrightarrow MS$ and $MA \leftrightarrow MA$ and is due in these cases to sizeable next to leading order contributions, which seem to be of the relative order q^{-1} but may also be of relative order $q^{-2}\log q$ as generally expected theoretically.

Most often, the behavior of the form factor at high q is illustrated by a process where a quark, which has been stroken by a virtual photon, shares the momentum it received with the other two quarks by the successive exchange of two gluons. In practice, calculations involve several processes with different time orderings of the photon absorption and gluons exchanges. In the present work, the dominant contribution rather comes from a process where the same quark that is stroken by the photon has previously and subsequently exchanged a gluon with each of the other quarks. It is likely that this should also hold in QCD, but this has to be checked. In any case, we believe it is worthwhile to emphasize the point as some arguments and calculations have been developped without regard to this type of process. Its contribution is essentially determined by the square of the derivative of the wave function at the origin. This differs from the two-body case or the simplest calculation with the harmonics hyperspherical formalism, which only involve the first derivative.

While we have not been able to fully answer some of the questions raised in the introduction, we showed that some caution is required in dealing with the asymptotic behavior of form factors. Their onset may not occur as quickly as usually expected. Most important, it supposes an accurate determination of the three-quark wave function at short distances. Present calculations provide enough evidence that the high momentum transfer behavior in a non-relativistic approach is given by q^{-8}. We believe that the difference with the power law, q^{-4}, expected from QCD is most likely due to relativistic (boost) effects that have not been considered in the present work. Consequently, it is inappropriate to use the q^{-4} behavior to get constraints on the non-relativistic three-quark wave function. Contrary to a belief that probably originates from a non-relativistic approach or from studies with spinless particles, a q^{-4} behavior of the form factor does not imply a similar behavior of the wave function at high momenta. Along the same lines, it is inapproriate to use a truncated hyperspherical harmonic approach to analyze form factors at high momentum transfers. In either case [23, 3, 13, 30], a bias is systematically introduced, which prevents one to make relevant conclusions.

For the future, it would be worthwhile to study more carefully the next to leading order corrections to the form factors, both mathematically and numerically. The questions to be answered are the relative order of these corrections, $\frac{1}{q}$, or $\frac{1}{q^2}$ (up to $\log q$ terms), and the sensitivity to the truncation generally performed in solving the Faddeev equation. Obviously, a full relativistic calculation would be desirable. It is likely that the questions raised here will appear there too. We hope that the answers we got will be of some relevance for this more difficult problem.

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A Wave functions in the hyperspherical harmonic formalism

A.1 Jabobi coordinates

For a wide range of physical potentials (mainly two-body potentials depending on the relative coordinates), the three-body problem is conveniently treated by using Jacobi coordinates. For the equal mass (m) case, the Jacobi coordinates can be defined as corresponding to the vectors:

$$\begin{aligned} \vec{\rho}_{12} &= \frac{\vec{r}_1 - \vec{r}_2}{\sqrt{2}}, \\ \vec{\rho}_3 &= \sqrt{2/3}\,(\vec{r}_3 - \frac{\vec{r}_1 + \vec{r}_2}{2}). \end{aligned} \tag{81}$$

Altogether with the center of mass ($\vec{R}$) coordinate, they determine completely the position of the system.

The kinetic energy operator can be separated as

$$T = \sum_i \frac{\vec{p_i}^{\,2}}{2m} = \frac{\vec{P}^{\,2}}{6m} + \frac{\vec{p}^{\,2}_{\rho_{12}}}{2m} + \frac{\vec{p}^{\,2}_{\rho_3}}{2m} \equiv T_{c.m.} + T_{int}, \tag{82}$$

containing the center of mass kinetic energy, $\frac{\vec{P}^{\,2}}{6m} \equiv T_{c.m.}$, plus the internal kinetic energy, T_{int}, in terms of the Jacobi momenta. Then for potentials depending only on the Jacobi coordinates, the center of mass motion can be factorized out.

A.2 Hyperspherical coordinates and harmonics

From $\vec{\rho}_{12}$ and $\vec{\rho}_3$, one can define hyperspherical coordinates $\rho \in [0, \infty), \phi \in [0, \pi/2]$ in the form:

$$\begin{aligned} \rho_{12} &= \rho \sin\phi, \\ \rho_3 &= \rho \cos\phi, \end{aligned} \tag{83}$$

the differential volume element $d^3\vec{\rho}_{12}\, d^3\vec{\rho}_3$ being then written as:

$$\rho^5\, d\rho\, d\Omega \equiv \rho^5\, d\rho\, d\phi\, \sin^2\phi\, \cos^2\phi\, d\hat{\rho}_{12}\, d\hat{\rho}_3. \tag{84}$$

The internal kinetic energy is then expressed in terms of ρ and $\Omega \equiv \{\theta_{12}, \varphi_{12}, \theta_3, \varphi_3, \phi\}$, (where (θ, φ) are the spherical angles of $\vec{\rho}_{12}$ and $\vec{\rho}_3$) in the form:

$$T_{int} = -\frac{1}{2m}\left(\frac{\partial^2}{\partial \rho^2} + \frac{5}{\rho}\frac{\partial}{\partial \rho} + \frac{K^2(\Omega)}{\rho^2}\right), \tag{85}$$

where $K^2(\Omega)$ is an angular operator whose eigenfunctions are called the hyperspherical harmonics (HH), $Y_{[K]}(\Omega)$:

$$K^2(\Omega)\, Y_{[K]}(\Omega) = -K(K+4)\, Y_{[K]}(\Omega), \tag{86}$$

K, the so called grand orbital number, defines the parity of the HH as $(-)^K$. One can choose appropriate combinations of hyperspherical harmonics with definite values of the total orbital angular momentum $L(\vec{L} = \vec{l}_{12} + \vec{l}_3)$ and its third projection M_L and with definite spatial symmetry:

$$Y^{(L,M_L)}_{[(K,sym)]}(\Omega).$$

In terms of these, the wave function of a system of total momentum $\vec{P} = 0$ and spin and parity, J^P, reads

$$|\Psi(\rho,\Omega)>=<\rho\Omega|\Psi>= \sum_{\substack{K,L\\ sym}} \psi_{K,L}(\rho)\left[Y^{(L,M_L)}_{[K,sym]}(\Omega)|sym>\right]_{J^P} \equiv \sum_j \psi_j(\rho) Y_j(\Omega), \tag{87}$$

where $|sym>$ stands for a spin-isospin wave function of definite symmetry and the square bracket implies the coupling to total angular momentum J with the required overall symmetry. The subindex j of $\psi_j(\rho)$ and $Y_j(\Omega)$ corresponds to a simplified notation, j comprising K, angular momentum and spin-isospin variables.

The state $|\Psi(\rho,\Omega)>$ satisfies the Schrödinger equation:

$$\left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial \rho^2} + \frac{5}{\rho}\frac{\partial}{\partial \rho} + \frac{K^2(\Omega)}{\rho^2}\right) + V + 3m - E\right]|\Psi(\rho,\Omega)>= 0. \tag{88}$$

By substituting $|\Psi(\rho,\Omega)>$ by its expansion, Eq. (87), and projecting it, one gets a system of coupled equations (its number depending on the number of $Y_j(\Omega)$ terms kept in the expansion):

$$\begin{aligned}\left[-\frac{1}{2m}\left(\frac{1}{\rho^{5/2}}\frac{d^2}{d\rho^2}\rho^{5/2} - \frac{(K+2)^2 - 1/4}{\rho^2}\right) + 3m - E\right]\psi_j(\rho)&\\ + \sum_{j'} V_{jj'}(\rho)\, \psi_{j'}(\rho) = 0,&\end{aligned} \tag{89}$$

where the matrix elements of the potential are given by

$$V_{j,j'}(\rho) = \int d\Omega\, Y^*_{j'}(\Omega)\, V(\rho,\Omega,s,\tau)\, Y_j(\Omega), \tag{90}$$

In this equation, s, τ indicate the possible dependences of the potential on spin and isospin. The normalization condition is:

$$\sum_j \int d\rho\, \rho^5\, |\psi_j(\rho)|^2 = 1\,. \tag{91}$$

A.3 Details about wave functions

For the moment, we are going to truncate the expansion (87) by considering only $K = 0, 2$ terms. Furthermore, if we consider a restricted potential as the one given by (17) but without spin-spin interaction, the ground state does not contain $K = 2$ component. Then we can write:

$$|\Psi(\rho,\Omega)>_{\frac{1}{2}^+} = \frac{1}{\pi\sqrt{\pi}}\psi_1(\rho)|S>, \tag{92}$$

satisfying

$$\left[\frac{1}{2m_q}(-\frac{1}{\rho^{5/2}}\frac{d^2}{d\rho^2}\rho^{5/2} + \frac{15}{4\rho^2}) - \frac{4\sqrt{2}}{\pi}\frac{\kappa}{\rho} + \frac{16\sqrt{2}}{5\pi}\frac{\rho}{a^2} - \frac{3D}{2} + 3m_q - E\right]\psi_1(\rho) = 0\,. \tag{93}$$

The terms $-\frac{4\sqrt{2}}{\pi}\frac{\kappa}{\rho}$ and $\frac{16\sqrt{2}}{5\pi}\frac{\rho}{a^2}$ represent respectively the monopole components of the Coulomb and linear parts of the quark-quark interaction:

$$-\frac{4\sqrt{2}}{\pi}\frac{\kappa}{\rho} = -3\int d\,\Omega\left(\frac{1}{\pi\sqrt{\pi}}\right)^2 \frac{\kappa}{2\sqrt{2}\rho\sin\phi}, \tag{94}$$

$$\frac{16\sqrt{2}}{5\pi}\frac{\rho}{a^2} = 3\int d\,\Omega\left(\frac{1}{\pi\sqrt{\pi}}\right)^2 \frac{\rho\sqrt{2}\sin\phi}{2a^2}. \tag{95}$$

The relative distance r_{12} appearing in the potential (17) has been replaced by its expression in terms of the hyperspherical coordinates, $\sqrt{2}\rho\,\sin\phi$, and the factor 3 accounts for the interaction between the three pairs 12, 23 and 13 which contribute equally to (94, 95). The solution of Eq. (93) at short distance can be expanded in terms of the powers of ρ. It reads:

$$\begin{aligned}\psi_1(\rho)_{\rho\to 0} = \alpha'_1 \Bigg[1 &- \frac{8\sqrt{2}}{5\pi}\kappa\, m_q\,\rho \\ &-\frac{1}{12}\left(2m_q(E - 3m_q + \frac{3D}{2}) - 5(\frac{8\sqrt{2}}{5\pi}\kappa m_q)^2\right)\rho^2 \\ +\frac{1}{21}&\left(\frac{32\sqrt{2}}{5\pi}\frac{m_q}{a^2} + \frac{68\sqrt{2}}{15\pi}\kappa(E - 3m_q + \frac{3D}{2})m_q^2 - \frac{25}{12}(\frac{8\sqrt{2}}{5\pi}\kappa m_q)^3\right)\rho^3 + ..\Bigg],\end{aligned} \tag{96}$$

where α'_1 is a constant (the value of the wave function at the origin).

The coefficient of the term linear in ρ is determined by the Coulomb part of the force, while the energy of the system appears in the second term in ρ^2 of the expansion. The intensity of the confining potential affects the third term in ρ^3 of the expansion. It however implicitly appears in the energy E.

A particular case of Eq. (96) is that one where the confining potential is neglected. Equation (93) then reduces to that of a Coulombian type problem. The solution is given by:

$$\psi(\rho) = \alpha e^{-\beta\rho}, \tag{97}$$

with $\beta = \frac{8\sqrt{2}}{5\pi}\kappa m_q$ and $E - 3m_q + \frac{3D}{2} = -\frac{\beta^2}{2m_q} = -\frac{64}{25\pi^2}\kappa^2 m_q$.
From the normalization condition of the radial wave function given by:

$$\int d\rho\, \rho^5\, \psi^2(\rho) = 1, \tag{98}$$

one gets $\alpha = \frac{(2\beta)^3}{\sqrt{120}}$.

In the case where the spin-spin interaction is turned on, one has to solve a set of coupled equations. For our purpose here, it is enough to keep the most singular part of the spin-spin force ($\frac{\kappa'_\sigma}{r_{ij}}\,\vec{\sigma}_i.\vec{\sigma}_j$) (the complete expression has been used for the numerical calculations detailed in the main text):

$$\begin{aligned}\Bigg[\frac{1}{2m_q}\left(-\frac{1}{\rho^{5/2}}\frac{d^2}{d\rho^2}\rho^{5/2} + \frac{15}{4}\frac{1}{\rho^2}\right) - \frac{4\sqrt{2}}{\pi}(\kappa+\kappa'_\sigma)\frac{1}{\rho} + \frac{16\sqrt{2}}{5\pi a^2}\,\rho \\ -\frac{3D}{2} + 3m_q - E\Bigg]\,\psi_1(\rho) = \frac{8}{5\pi}\kappa'_\sigma\frac{1}{\rho}\,\psi_3(\rho),\end{aligned} \tag{99}$$

$$\begin{aligned}\Bigg[\frac{1}{2m_q}\left(-\frac{1}{\rho^{5/2}}\frac{d^2}{d\rho^2}\rho^{5/2} + \frac{63}{4}\frac{1}{\rho^2}\right) - \frac{4\sqrt{2}}{\pi}(\frac{38}{35}\kappa + \frac{10}{7}\kappa'_\sigma)\frac{1}{\rho} + \frac{16\sqrt{2}}{5\pi a^2}\,\frac{62}{63}\,\rho \\ -\frac{3D}{2} + 3m_q - E\Bigg]\,\psi_3(\rho) = \frac{16}{5\pi}\kappa'_\sigma\frac{1}{\rho}\,\psi_1(\rho),\end{aligned} \tag{100}$$

where $\psi_3(\rho) \equiv \psi_{K=2,L=0}(\rho)$. In the limit $\rho \to 0$, it is easy to show that to remove the $\frac{1}{\rho^2}$ singularity on the left hand side of the second equation, $\psi_3(\rho)$ has to go to zero as :

$$\psi_3(\rho)_{\rho\to 0} = \frac{32}{35\pi}\kappa'_\sigma m_q\,\rho\,\psi_1(0). \tag{101}$$

Similarly, other components in the limit $\rho \to 0$ may be determined. An expression of the wave function beyond $K = 0, 2$, i.e. including $K = 4$ and $K = 6$ terms, and limited to those terms linears in the variable ρ is the following:

$$\begin{aligned}(\Psi_{HH})_{linear} \simeq \frac{1}{\pi\sqrt{\pi}}\Bigg[1 - \frac{8\sqrt{2}}{5\pi}(\kappa+\kappa'_\sigma)\, m_q\, \rho \\ .\left(1 - \frac{2}{21}\left(\,(\cos^2\phi - \sin^2\phi)^2 + 4\cos^2\phi\,\sin^2\phi\;(\hat{\rho}_{12}.\hat{\rho}_3)^2 - \frac{1}{2}\right)\right. \\ \left.-\frac{8}{231}(\cos^2\phi - \sin^2\phi)\left((\cos^2\phi - \sin^2\phi)^2 - 12\cos^2\phi\,\sin^2\phi\;(\hat{\rho}_{12}.\hat{\rho}_3)^2\right) + ..\right)|S> \\ +\frac{32\sqrt{2}}{5\pi}\;\kappa'_\sigma m_q\,\rho\left((\cos^2\phi - \sin^2\phi)|MS> -2\,\sin\phi\,\cos\phi\;\hat{\rho}_{12}.\hat{\rho}_3|MA>\right) + ...\Bigg].\end{aligned} \tag{102}$$

B Details about calculations of the form factors

The complete expressions of the form factors calculated from the analytic expressions of the wave functions given by Eqs. (37) and (41) are as follows:

$$< S|O|S >= 405\sqrt{6}\,\frac{\alpha_1^2\beta_1}{(q^2 + 6\beta_1^2)^{7/2}}, \tag{103}$$

$$< S|O|MS >= -2835\sqrt{6}\,\frac{\alpha_1\alpha_3 q^2}{(q^2+6(\frac{\beta_1+\beta_3}{2})^2)^{9/2}}, \tag{104}$$

$$< MS|O|MS >= 7290\frac{\alpha_3^2}{q^8}\left(64-\frac{\beta_3\sqrt{6}}{(q^2+6\beta_3^2)^{11/2}}\right. \tag{105}$$
$$\left.\left[175q^{10}+574q^8\tilde{\beta}_3^2+924q^6\tilde{\beta}_3^4+792q^4\tilde{\beta}_3^6+352q^2\tilde{\beta}_3^8+64\tilde{\beta}_3^{10}\right]\right),$$

$$< MA|O|MA >= -2430\frac{\alpha_3^2}{q^8}\left(64-\frac{\beta_3\sqrt{6}}{(q^2+6\beta_3^2)^{11/2}}\right. \tag{106}$$
$$\left.\left[168q^{10}+588q^8\tilde{\beta}_3^2+924q^6\tilde{\beta}_3^4+792q^4\tilde{\beta}_3^6+352q^2\tilde{\beta}_3^8+64\tilde{\beta}_3^{10}\right]\right),$$

where $\tilde{\beta}_3=\sqrt{6}\,\beta_3$. While close expressions of form factors can be obtained in a few cases, as above, quite generally and especially for more realistic wave functions incorporating terms depending linearly on the variables, r_{12}, r_{13} and r_{23}, this is not possible.

One may however be interested to get some prediction for the asymptotic form factor. Starting from the expression of the matrix element, $< X|O|Y >$, given by Eq. (49), and after integration over the orientations of the ρ_{12} and ρ_3 variables, which can be performed in most cases, one is left with a quantity of the following form:

$$I(q)=\frac{16}{\pi}\sqrt{\frac{3}{2}}\,\frac{1}{q}\int d\phi\sin^2\!\phi\cos^2\!\phi\;\;d\rho\,\rho^4\,\sin(\sqrt{\frac{2}{3}}\,q\rho\cos\phi)\;H(\rho,\phi), \tag{107}$$

where $H(\rho,\phi)$ involves the nucleon radial wave functions. The overall factor, $\frac{16}{\pi}$, arises from a factor $(4\pi)^2$ due to the integration over the angles relative to the vectors, $\vec{\rho}_{12}$ and $\vec{\rho}_3$, and another one, $(\frac{1}{\pi\sqrt{\pi}})^2$, appearing in the normalization of the wave function, Eq. (102). Assuming that $H(\rho,\phi)$ can be expanded at small ρ as:

$$H(\rho,\phi)=F_0+\rho\,F_1(\phi)+...+\rho^n\,F_n(\phi)+..,\tag{108}$$

it is clear, from dimensional arguments, that the ρ^n term will produce a contribution to the form factor proportional to $q^{-(6+n)}$, besides $\delta(\vec{q})$ functions or its derivatives which are irrelevant at high q. To get the corresponding coefficient, a_n, one has to perform the integration over the ρ and ϕ variables:

$$a_n=\frac{16}{\pi}\sqrt[6+n]{\frac{3}{2}}\,\lim_{\epsilon\to 0}\int d\phi\sin^2\!\phi\cos^2\!\phi\;dx\,x^{4+n}\,\sin(x\cos\phi)\,F_n(\phi)\,e^{-\epsilon x}. \tag{109}$$

As in Eq. (13), the factor $e^{-\epsilon x}$ allows one to get rid of the part involving the $\delta(\vec{q})$ function or its derivatives. After integration over x, Eq. (109) becomes:

$$a_n=\frac{16}{\pi}\sqrt[6+n]{\frac{3}{2}}\,\lim_{\epsilon\to 0}(-\frac{d}{d\epsilon})^{4+n}\int d\phi\sin^2\!\phi\cos^2\!\phi\,\frac{F_n(\phi)}{\cos^2\!\phi+\epsilon^2}. \tag{110}$$

As a check, one can show from expression (37) that

$$F_n(\phi)=\alpha_1^2\,\frac{(-2\,\beta_1)^n}{n!}. \tag{111}$$

Putting it in (110) and using the relation:

$$\int d\phi \frac{\sin^2\!\phi \cos^2\!\phi}{\cos^2\!\phi + \epsilon^2} = \frac{\pi}{2}\left(\frac{1}{2} + \epsilon^2 - \epsilon\sqrt{1+\epsilon^2}\right), \tag{112}$$

the different terms of the expansion of the form factor (103) in terms of the powers of $\frac{1}{q}$, $\frac{1}{q^{6+n}}$ $(n = 0, 1, ..)$, are recovered.

C Short distance behavior of the wave function in the Faddeev formalism and asymptotic form factors

C.1 Short distance wave functions

In the Faddeev approach, and for the considered interaction given by (17), the short range behavior of the wave function (28) can be obtained from Eq. (26). So, for the amplitudes $\psi^{0,1}_{12,3}(\vec{r}_1, \vec{r}_2, \vec{r}_3)$:

$$\begin{aligned}
\psi^0_{12,3}(\vec{r}_1, \vec{r}_2, \vec{r}_3) &\propto 1 - \frac{3}{4}\,(\kappa + 3\kappa'_\sigma)\; m_q\; r_{12} + ...,\\
\psi^1_{12,3}(\vec{r}_1, \vec{r}_2, \vec{r}_3) &\propto 1 - \frac{3}{4}\,(\kappa - \kappa'_\sigma)\; m_q\; r_{12} + ...\;\;,
\end{aligned} \tag{113}$$

and for the full wave function and up to an undetermined factor:

$$\begin{aligned}
\Psi_{F,0}(\vec{r}_1, \vec{r}_2, \vec{r}_3) &\propto \left(1 - \frac{1}{4}(\kappa + \kappa'_\sigma)\; m_q\,(r_{12} + r_{13} + r_{23}) + ...\right)|S>\\
&-\frac{1}{4}\kappa'_\sigma m_q \left(((2r_{12} - r_{13} - r_{23}) + ..)\,|MS> + \sqrt{3}\,((r_{23} - r_{13}) + ..)\,|MA>\right).
\end{aligned} \tag{114}$$

It is worth to notice that from this wave function one can recover the first terms of the expansion, Eq. (102), by making the projection on appropriate hyperspherical harmonics. It is also noticed that this wave function, after averaging over all directions of $\vec{r}_{12}$, has a term linear in its modulus, r_{12}, or equivalently in $\sin\phi$, while the truncated wave function given by Eq. (102) depends on the square of $\sin\phi$. As this linear term in r_{12} corresponds to a first order in the interaction, the wave function in momentum space given by Eq. (67) should have some track of it. In fact by taking the Fourier transform of Eq. (67) it is possible to recover (113). The proportionality factor is then found to be $\frac{1}{3}\frac{\bar{\alpha}}{\pi\sqrt{\pi}}$.

An expression for the momentum space wave function reproducing the short range behavior of the r-space wave function (114), may by useful. Starting from the following zeroth order completely symmetrical wave function (see permutation relations at the end of this appendix),

$$\begin{aligned}
\Psi^{(0)}_{12,3} + \Psi^{(0)}_{32,1} + \Psi^{(0)}_{13,2} = \frac{1}{3}\frac{\bar{\alpha}}{\pi\sqrt{\pi}}(2\pi)^6\,&(\delta(\vec{\kappa}_3)\;\delta(\vec{\kappa}_{12})\\
&+\delta(-\frac{1}{2}\vec{\kappa}_3 + \frac{\sqrt{3}}{2}\vec{\kappa}_{12})\;\delta(\frac{1}{2}\vec{\kappa}_{12} + \frac{\sqrt{3}}{2}\vec{\kappa}_3)
\end{aligned}$$

$$+\delta(-\frac{1}{2}\vec{\kappa}_3-\frac{\sqrt{3}}{2}\vec{\kappa}_{12})\,\delta(\frac{1}{2}\vec{\kappa}_{12}-\frac{\sqrt{3}}{2}\vec{\kappa}_3)\Big)\;|S>, \tag{115}$$

one can iteratively obtain a first order wave function from the momentum space Faddeev equation. For one of the amplitudes, $\Psi^{(1)}_{12,3}(\kappa_{12},\kappa_3)$ for instance:

$$\Psi^{(1)}_{12,3}(\kappa_{12},\kappa_3)=-\frac{1}{\sqrt{2}}\frac{m_q}{\kappa_{12}^2+\kappa_3^2+\beta^{(1)2}}\int\frac{d\vec{\kappa}'_{12}\,d\vec{\kappa}'_3}{(2\pi)^3}\,V(\kappa_{12},\kappa'_{12})\delta(\vec{\kappa}_3-\vec{\kappa}'_3)$$
$$\Big(\Psi^{(0)}_{12,3}(\kappa'_{12},\kappa'_3)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\Big), \tag{116}$$

where the potential, here given for definiteness for its Coulomb part and in accordance with Eq. (17), is defined as

$$V(\kappa_{12},\kappa'_{12})=-\frac{4\pi\,\kappa}{(\vec{\kappa}_{12}-\vec{\kappa}'_{12})^2}\,.$$

Then

$$\Psi^{(1)}_{12,3}+\Psi^{(1)}_{32,1}+\Psi^{(1)}_{13,2}=\frac{\bar{\alpha}}{\pi\sqrt{\pi}}\frac{4\pi}{\sqrt{2}}m_q\,C^{(1)}\,\Big(I_3\;[(\kappa+\kappa'_\sigma)|S>+2\kappa'_\sigma|MS>]$$
$$+\,I_1\;[(\kappa+\kappa'_\sigma)|S>+2\kappa'_\sigma(-\frac{1}{2}|MS>+\frac{\sqrt{3}}{2}|MA>)]$$
$$+\,I_2\;[(\kappa+\kappa'_\sigma)|S>+2\kappa'_\sigma(-\frac{1}{2}|MS>-\frac{\sqrt{3}}{2}|MA>)]\Big), \tag{117}$$

where:

$$I_3=\frac{(2\pi)^3\;\delta(\vec{\kappa_3})}{(\kappa_{12}^2+\beta^{(1)2})\kappa_{12}^2},$$
$$I_1=\frac{(2\pi)^3\;\delta(-\frac{1}{2}\vec{\kappa}_3+\frac{\sqrt{3}}{2}\vec{\kappa}_{12})}{((\frac{1}{2}\vec{\kappa}_{12}+\frac{\sqrt{3}}{2}\vec{\kappa}_3)^2+\beta^{(1)2})(\frac{1}{2}\vec{\kappa}_{12}+\frac{\sqrt{3}}{2}\vec{\kappa}_3)^2},$$
$$I_2=\frac{(2\pi)^3\;\delta(-\frac{1}{2}\vec{\kappa}_3-\frac{\sqrt{3}}{2}\vec{\kappa}_{12})}{((\frac{1}{2}\vec{\kappa}_{12}-\frac{\sqrt{3}}{2}\vec{\kappa}_3)^2+\beta^{(1)2})(\frac{1}{2}\vec{\kappa}_{12}-\frac{\sqrt{3}}{2}\vec{\kappa}_3)^2}, \tag{118}$$

and $C^{(1)}$ is a normalization constant. The quantities I_1 and I_2 are obtained from I_3 by performing a circular permutation and re-expressing the variables in terms of $\vec{\kappa}_{12}$ and $\vec{\kappa}_3$. The $\frac{1}{\kappa_{12}^4}$ behavior at high values of κ_{12} is essential to reproduce the linear dependence in $\vec{r}_{12}$ in Eq. (114).

From Eq. (116), multiplying both members by the factor κ_{12}^4, taking the limit $\kappa_{12}\to\infty$ and integrating over the variable $\vec{\kappa}_3$, one gets for the Coulomb type potential:

$$\lim_{\kappa_{12}\to\infty}\kappa_{12}^4\int d\vec{\kappa}'_3\Psi_{12,3}(\kappa_{12},\kappa'_3)$$
$$=\frac{4\pi}{\sqrt{2}}m_q(\kappa+\kappa'_\sigma)\int\frac{d\vec{\kappa}'_{12}\,d\vec{\kappa}'_3}{(2\pi)^3}\,(\Psi_{12,3}(\kappa'_{12},\kappa'_3)+(1\leftrightarrow 3)+(2\leftrightarrow 3))\,. \tag{119}$$

Up to a common factor, the l.h.s. represents the coefficient of the r_{12} term (see the first line on the r.h.s. of (114)) while the r.h.s. represents the configuration space wave function at the origin, $\Psi(0)$. Now, by using the wave function given by (117) and under the requirement of recovering the coefficient of the linear term in r in(114) one gets $\beta^{(1)} = \frac{3}{\sqrt{2}} m_q(\kappa + \kappa'_\sigma)$. In this way, the corresponding wave function can be used in calculations where this property may be required as in the calculation of the form factor in the asymptotic regime. This value also ensures that the wave function at the origin is unchanged at this order, with the result $C^{(1)} = 1$.

It is also possible to iterate over the solution given by Eq. (117) ((114) in configuration space). The amplitude, $\Psi^{(2)}_{12,3}$, so obtained, reads:

$$\Psi^{(2)}_{12,3} = \frac{\bar{\alpha}}{\pi\sqrt{\pi}} (\frac{4\pi}{\sqrt{2}} m_q)^2 \; C^{(2)}$$
$$\Big([\, (\kappa + \kappa'_\sigma)^2 (J_3 + J_1 + J_2) + (2\kappa'_\sigma)^2 (J_3 - \frac{1}{2} J_1 - \frac{1}{2} J_2)] \; |S>$$
$$+(\kappa + \kappa'_\sigma)(2\kappa'_\sigma)[(J_3 + J_1 + J_2) + (J_3 - \frac{1}{2} J_1 - \frac{1}{2} J_2)] \; |MS>$$
$$+\frac{\sqrt{3}}{2} (\kappa + \kappa'_\sigma)(2\kappa'_\sigma)(J_1 - J_2) \; |MA>$$
$$+\frac{\sqrt{3}}{2} (2\kappa'_\sigma)^2 (J_1 - J_2) \; |A> \Big) , \tag{120}$$

where J_i is related to I_i as follows:

$$J_i = \frac{1}{\kappa_{12}^2 + \kappa_3^2 + \beta^{(2)^2}} \int \frac{d\vec{\kappa}'_{12} \, d\vec{\kappa}'_3}{(2\pi)^3} \, \frac{\delta(\vec{\kappa}_3 - \vec{\kappa}'_3)}{(\vec{\kappa}_{12} - \vec{\kappa}'_{12})^2} \, I'_i \, , \tag{121}$$

with the result:

$$J_3 = \frac{1}{\kappa_{12}^2 + \beta^{(2)^2}} \, \frac{1}{\kappa_{12} \beta^{(1)}} \mathrm{arctg}(\frac{\beta^{(1)}}{\kappa_{12}}) \, \frac{(2\pi)^3 \delta(\vec{\kappa}_3)}{4\pi \beta^{(1)}},$$
$$J_1 = \frac{1}{\kappa_{12}^2 + \kappa_3^2 + \beta^{(2)^2}} \, \frac{1}{(\vec{\kappa}_{12} + \frac{1}{\sqrt{3}} \vec{\kappa}_3)^2} \, (\frac{2}{\sqrt{3}})^3 \, \frac{1}{(\frac{4}{3}\kappa_3^2 + \beta^{(1)^2}) \frac{4}{3} \kappa_3^2},$$
$$J_2 = \frac{1}{\kappa_{12}^2 + \kappa_3^2 + \beta^{(2)^2}} \, \frac{1}{(\vec{\kappa}_{12} - \frac{1}{\sqrt{3}} \vec{\kappa}_3)^2} \, (\frac{2}{\sqrt{3}})^3 \, \frac{1}{(\frac{4}{3}\kappa_3^2 + \beta^{(1)^2}) \frac{4}{3} \kappa_3^2}. \tag{122}$$

For simplicity, we omitted the dependence of the functions, I_i on the arguments, $\vec{\kappa}'_{12}$ and $\vec{\kappa}'_3$, what is reminded by a " $'$ " at I.

In the limit where $\beta^{(1)}$ can be considered as an infinitesimally small quantity, the last factors in J_1 and J_2 tend to the function $\delta(\vec{\kappa}_3)$, making the quantities J_1, J_2 and J_3 equal to each other. Equation (120) simplifies and identifies to that part of Eq. (117) corresponding to the same Faddeev amplitude, $\Psi_{12,3}$, proportional to I_3. On the other hand, the presence of the $\delta(\vec{\kappa}_3)$ in Eq. (122) has no physical foundation. It results from the iteractive character of the calculation and should disappear in a complete one, as partly realized when going from the first iteration, Eq. (117), to the second one, Eq. (120).

The parameter, $\beta^{(2)}$, may be determined by requiring that Eq. (119) is satisfied when the expression of the wave function, Eq. (120), is used. This can be done once $\beta^{(1)}$ has been determined from the previous iteration ($\beta^{(1)} = \frac{3}{\sqrt{2}} m_q(\kappa + \kappa'_\sigma)$). Disregarding this value, one may also assume $\beta^{(1)} = \beta^{(2)}$ and determine the corresponding common value of these parameters by fulfilling Eq. (119). The value of $C^{(2)}$ is determined so that the configuration space wave function at the origin, $\Psi(0)$, is equal to that one in the previous iteration or its value from a full calculation:

$$\Psi(0) = \int \frac{d\vec{\kappa}'_{12}\, d\vec{\kappa}'_3}{(2\pi)^6} \left(\Psi_{12,3}(\kappa'_{12}, \kappa'_3) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right). \tag{123}$$

For a pure Coulombian problem and depending on the approach, values $C^{(2)} = 1.0$ and $C^{(2)} = 0.66$, corresponding respectively to $\beta^{(2)} = 1.105\, \kappa m_q$ and $\beta^{(2)} = 1.403\, \kappa m_q$, are obtained, providing some uncertainty range. Ultimately, after an infinite series of iterations, the value of the parameter corresponding to $\beta^{(2)}$ should converge to that related to the binding energy. This one is expected to be equal to $\beta = \frac{8\sqrt{2}}{5\pi} \kappa m_q$ in the hypercentral approximation ($K = 0$), Eq. (97), and $\beta = 1.016\, \frac{8\sqrt{2}}{5\pi} \kappa m_q = 0.732\, \kappa m_q$ in a more complete calculation [44, 45].

C.2 Asymptotic form factors

Concerning the asymptotic behavior of the form factors, it is interesting to apply Eqs. (107-109) to the first order term of the wave function given by (114) which involves terms r_{12} and $r_{13} + r_{23}$. After integration over the various angles and up to some factor, one is left with the following expression of $F_1(\phi)$:

$$\begin{aligned} F_1(\phi) &= \sin\phi \qquad (\text{for } r_{12}), \\ &= \frac{|\sin\phi + \sqrt{3}\cos\phi|^3 - |\sin\phi - \sqrt{3}\cos\phi|^3}{6\sqrt{3}\sin\phi\,\cos\phi} \quad (\text{for } r_{13} + r_{23}). \end{aligned} \tag{124}$$

The first term in Eq. (124) can be dealt with easily. Using the relation:

$$\int d\phi\, \frac{\sin^2\!\phi\, \cos^2\!\phi}{\cos^2\!\phi + \epsilon^2} \sin\phi = \frac{2}{3} - \epsilon\,(1 + \epsilon^2)\,(\frac{\pi}{2} - \text{arctg}\,\epsilon), \tag{125}$$

it is seen that, in absence of ϵ^5 terms in the expansion, there is no contribution to the coefficient of the q^{-7} term, a_1. The second term in Eq. (124) can be written as a term similar to the previous one:

$$\int d\phi\, \frac{\sin^2\!\phi\, \cos^2\!\phi}{\cos^2\!\phi + \epsilon^2}\, \frac{1}{\sin\phi} = 1 - \epsilon\,(\frac{\pi}{2} - \text{arctg}\,\epsilon), \tag{126}$$

plus another one integrated over the range, $\frac{1}{2} < \cos\phi < 1$. As this one safely converges when $\epsilon \to 0$, it is immediate that its expansion in terms of powers of ϵ will exhibit only even powers and, therefore, will not contribute to a_1. Thus, the linear term in ρ appearing in the expansion of the correct wave function around the origin does not contribute to the form factor at the order q^{-7}, as expected. This is not a trivial result however. A linear dependence of the wave function on the variable ρ_3, for instance, which has no physical ground, but turns out to be undistinguishable from a contribution of a ρ_{12} dependence in the approximated calculation retaining the $K = 0$ wave, leads to a q^{-4} asymptotic behavior of the form factor. This indicates that the ρ_3 dependence of the wave function has to be carefully determined, otherwise it would lead to a bias in calculating form factors in the asymptotic domain.

Expression (120), which involves the effect of two gluon exchanges without any restriction, in contrast to Eq. (67), can be used to determine the asymptotic behavior of form factors. When it is averaged over the various angles, including ϕ, it allows one to recover the high momentum behavior of the wave function (63) obtained in the hyperspherical formalism, with the same front factor, independently of the value taken by $C^{(2)}$.

Throughout this paper, and especially to get the expressions given in this appendix, we employed the following permutation relations:
exchange of particles 1 and 3:

$$\begin{aligned}
\vec{\rho}_3 &\rightarrow -\frac{1}{2}\vec{\rho}_3 + \frac{\sqrt{3}}{2}\vec{\rho}_{12}, & \vec{\rho}_{12} &\rightarrow \frac{1}{2}\vec{\rho}_{12} + \frac{\sqrt{3}}{2}\vec{\rho}_3, \\
\vec{\kappa}_3 &\rightarrow -\frac{1}{2}\vec{\kappa}_3 + \frac{\sqrt{3}}{2}\vec{\kappa}_{12}, & \vec{\kappa}_{12} &\rightarrow \frac{1}{2}\vec{\kappa}_{12} + \frac{\sqrt{3}}{2}\vec{\kappa}_3, \\
|MS> &\rightarrow -\frac{1}{2}|MS> + \frac{\sqrt{3}}{2}|MA>, & |MA> &\rightarrow \frac{1}{2}|MA> + \frac{\sqrt{3}}{2}|MS>,
\end{aligned} \tag{127}$$

exchange of particles 2 and 3:

$$\begin{aligned}
\vec{\rho}_3 &\rightarrow -\frac{1}{2}\vec{\rho}_3 - \frac{\sqrt{3}}{2}\vec{\rho}_{12}, & \vec{\rho}_{12} &\rightarrow \frac{1}{2}\vec{\rho}_{12} - \frac{\sqrt{3}}{2}\vec{\rho}_3, \\
\vec{\kappa}_3 &\rightarrow -\frac{1}{2}\vec{\kappa}_3 - \frac{\sqrt{3}}{2}\vec{\kappa}_{12}, & \vec{\kappa}_{12} &\rightarrow \frac{1}{2}\vec{\kappa}_{12} - \frac{\sqrt{3}}{2}\vec{\kappa}_3, \\
|MS> &\rightarrow -\frac{1}{2}|MS> - \frac{\sqrt{3}}{2}|MA>, & |MA> &\rightarrow \frac{1}{2}|MA> - \frac{\sqrt{3}}{2}|MS>\,.
\end{aligned} \tag{128}$$

D Details about the q^{-8} asymptotic behavior of the form factors and terms beyond

We first consider in this appendix the effect of a truncation of the total wave function calculated in the Faddeev approach, which has been limited to the ten partial waves with the lowest values of the angular momentum of the pair of particles 1 and 2: $l = 0$, 1, 2, 3 and 4 for each spin state of the same pair, 0 and 1. We especially look at the contribution of the term which arises from the wave function given by Eq. (114) and involves the quantity $r_{13}\,r_{23}$. Its contribution to form factors, which is the dominant one, involves l values ranging from 0 to ∞, while the contribution of the other term $r_{12}(r_{13}+r_{23})$, which is less important, only implies $l = 0$. The first contribution may thus be sensitive to some truncation, while the second one cannot. Let's remind that the truncation under consideration is different from that on the Faddeev amplitude, for which two cases with 2 and 8 partial waves have been considered in this work.

Using an approach similar to the one presented in the appendix B, we calculated the contributions to the coefficient of the q^{-8} factor due to different waves. We separated those for even and odd values of l, corresponding respectively to the matrix elements $< S|O|S >, < S|O|MS >$ and $< MS|O|MS >$ on the one hand, and $< MA|O|MA >$ on the other. Assuming that the total sum is normalized to one, we found that the contribution for a given l is given by:

$$\frac{900(-1)^{(l+1)}}{(2l-5)(2l-3)(2l-1)(2l+3)(2l+5)(2l+7)}. \tag{129}$$

The sum thus reads for even values of l:

$$1 = 0.5714(=\frac{4}{7}) + 0.4329(=\frac{100}{231}) - 0.0040(=\frac{4}{1001}) - ... \tag{130}$$

and for odd values:

$$1 = 0.952(=\frac{20}{21}) + 0.046(=\frac{20}{429}) + ... \tag{131}$$

The above result shows that no serious discrepancy is introduced by the truncation of the total wave function we made and that the origin of the discrepancy of the calculated form factors and their expectations in the asymptotic regime has to be looked for elsewhere. On the contrary, calculations using phenomenological wave functions limited to $l = 0$ would in any case miss a sizeable contribution.

We mentioned in the text the possibility that the correction to the dominant q^{-8} term be of the relative order q^{-1}, with the result of delaying the convergence of the form factors to their asymptotic value. We here give details about an example dealing with a two-body system. This is not directly relevant to the three-body case of interest in this work, but we nevertheless believe it could cast some light on this one.

The expression of the form factor for a system of two equal mass particles is given by:

$$F(\vec{q}^2) = \int \frac{d\vec{k}}{(2\pi)^3}\, \varphi(\vec{k})\, \varphi(\vec{k} + \frac{1}{2}\vec{q}), \tag{132}$$

where $\varphi(\vec{k})$ represents the wave function of the system under consideration. For the lowest state of the hydrogenic system discussed in Sect. 3, $\varphi(\vec{k}) = \sqrt{4\pi}\frac{4(\kappa)^{5/2}}{(k^2+\kappa^2)^2}$. Hence:

$$F(\vec{q}^2) = \int \frac{d\vec{k}}{(2\pi)^3}\, \frac{64\pi\, \kappa^5}{(k^2+\kappa^2)^2((\vec{k}+\frac{1}{2}\vec{q})^2+\kappa^2)^2} = \frac{\kappa^4}{(\frac{1}{16}q^2+\kappa^2)^2}. \tag{133}$$

In the limit of large q^2, the form factor reads:

$$F(\vec{q}^2) = \frac{256\, \kappa^4}{q^4}(1 - 32\frac{\kappa^2}{q^2} + ...), \tag{134}$$

evidencing a q^{-2} correction to the leading order term. It is interesting to compare this result with that of a slightly different expression:

$$\tilde{F}(\vec{q}^2) = \int \frac{d\vec{k}}{(2\pi)^3}\, \frac{64\pi\, \kappa^5}{(k^2+\kappa^2)^2(k^2+\frac{1}{4}q^2+\kappa^2)^2}, \tag{135}$$

corresponding to the first term of an expansion of the integrand in Eq. (133) in terms of the quantity $\frac{\vec{k}.\vec{q}}{k^2+\frac{q^2}{4}+\kappa^2}$. Its expression is given by:

$$\tilde{F}(\vec{q}^2) = \frac{128\,\kappa^4}{q^4}\left(1 + \frac{\kappa}{\sqrt{\kappa^2+\frac{1}{4}q^2}} - \frac{4\kappa}{\kappa+\sqrt{\kappa^2+\frac{1}{4}q^2}}\right). \tag{136}$$

In the limit of large q, it becomes:

$$\tilde{F}(\vec{q}^2) = \frac{128\,\kappa^4}{q^4}(1 - 6\frac{\kappa}{q} + ...). \tag{137}$$

The dominant term at high q is a half of that one in the former calculation, Eq. (134), which is in relation with the fact that only the low value of k in Eq. (135) contributes to it, whereas in Eq. (132), the values of $\vec{k}$ around $-\frac{1}{2}\vec{q}$ also contribute for an equal amount. More important here, the correction to the dominant term is of relative order q^{-1} instead of q^{-2}. This is in agreement with an expansion in terms of powers of q^{-2} of the integrand in Eq. (135). Beyond the q^{-4} term which converges, the next one in q^{-6} diverges linearly and should be effectively cut-off by a factor q, hence the correction of relative order q^{-1}. The same argument should apply to the full expression of the form factor, Eq. (132), but it turns out that the different q^{-1} corrections cancel out in this case, indicating that the way the variable $\vec{q}$ appears in this expression is of special relevance for the existence of q^{-2} corrections instead of q^{-1}, as expected at first sight. In any case, the above discussion shows the difficulty to make statements in the more complicated three-body case.

The difference as to the q^{-1} corrections has probably to do with the difference in the mathematical properties of $F(\vec{q}^2)$ and $\tilde{F}(\vec{q}^2)$ when the variable q is made complex, $q \to |q|e^{i\theta}$ with θ varying from 0 to π. An absence of change is the sign that the function under consideration is a function of q^2. When the above transformation is made on $\tilde{F}(\vec{q}^2)$, a singularity occurs for $\theta = \frac{1}{2}\pi$ and the integration variable taking the value $k = \sqrt{\frac{|q|^2}{4} - \kappa^2}$. This does not occur for $F(\vec{q}^2)$, Eq. (133), due to the presence at the denominator of the term $\vec{k}.\vec{q}$ besides the quantity $k^2+\frac{q^2}{4}+\kappa^2$. The weight of the singularity is reduced by the condition that $\vec{k}$ should be orthogonal to $\vec{q}$.

One can imagine that the above results be extended for some part to the three-body case. The absence of an exact analytic expression for the three-body wave function prevents one to make definite statements however. We nevertheless expect that corrections of relative order q^{-2} should be associated to $\log q$ factors, indicating that the results for the two-body system cannot be transposed as such to the three-body system.

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