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Bäcklund transformations in 2D dilaton gravity*

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Abstract

We give a Bäcklund transformation connecting a generic 2D dilaton gravity theory to a generally covariant free field theory. This transformation provides an explicit canonical transformation relating both theories.

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Motivated by the two-dimensional model of black holes dynamics introduced by Callan, Giddings, Harvey and Strominger (CGHS) [1], a lot of works in 2D dilaton gravity has been developed from different viewpoints. A crucial property to understand the CGHS model is given by the fact that it can be mapped, via an off-shell canonical transformation, into a theory of free fields with a Minkowskian target space [2]. This, in turns, implies that theory can be quantized using different approaches to deal with the anomalies [2, 3]. It has been proved in Ref. [4] (see [5] for details) that this property of the CGHS model is also valid for a generic model of 2D dilaton gravity. Based on properties of the classical equations of motion it was shown that there exist a canonical transformation converting a generic model into a free field theory with a Minkowskian target space. However the explicit form of the transformation is unknown except for those models which can be explicitly solved [6].

The aim of this letter is to provide a more explicit form for the canonical transformations and bypass the problem of solving the classical equations of motion. To this end we shall introduce a different perspective to that used in [4, 5]. The idea is to construct a Bäcklund transformation relating the equations of motion of a 2D dilaton gravity model to free field equations. To obtain the Bäcklund transformation we shall also demand that the Hamiltonian and momentum constraints of the dilaton-gravity theory are mapped into those of a generally covariant free field theory. With this requirement the Bäcklund transformation can be promoted to a canonical transformation.

Our starting point is the action functional describing a 2D dilaton gravity model

$$S = \int d^2x\sqrt{-g}\left(R\phi + 4\lambda^2 V(\phi) - \frac{1}{2}(\nabla f)^2\right) , \tag{1}$$

where $V(\phi)$ is an arbitrary function of the dilaton field and f is a scalar matter field. The above expression represent a generic model because one can get rid of the kinetic term of the dilaton by a conformal reparametrization of the fields and bring the action into the form (1). In conformal gauge $ds^2 = -e^{2\rho}dx^+dx^-$, the equations of motion derived from the action (1) are

$$\begin{aligned} 2\partial_+\partial_-\rho + \lambda^2\frac{d}{d\phi}V(\phi)e^{2\rho} &= 0\,, && (2)\\ \partial_+\partial_-\phi + \lambda^2 V(\phi)e^{2\rho} &= 0\,, && (3)\\ \partial_+\partial_- f &= 0\,, && (4)\\ -\partial_\pm^2\phi + 2\partial_\pm\phi\partial_\pm\rho - \frac{1}{2}(\partial_\pm f)^2 &= 0\,. && (5)\end{aligned}$$

By a rather involved manipulation of these equations it was shown in [4, 5] that, irrespective of the form of the potential, the solutions define a canonical transformation mapping the theory (1) into a free field theory with constraints $C_\pm = \pm\frac{1}{2}(H \pm P)$ taking the form

$$C_\pm = \Pi_\pm X^{\pm\prime} \pm \frac{1}{4}(\pi_f \pm f')^2 , \tag{6}$$

where $(\Pi_\pm, X^\pm)$ and (π_f, f) are canonically conjugate variables and H and P are the Hamiltonian and momentum constraints. Obviously the pure gravity and matter sectors are separately equivalent to free fields. From now we shall restrict our analysis to the pure dilaton-gravity sector. A further linear canonical transformation [7]

$$2\Pi_\pm = -(\pi_0 + \pi_1) \mp (r^{0\prime} - r^{1\prime}) \,, \quad (7)$$

$$2X^{\pm\prime} = \mp(\pi_0 - \pi_1) - (r^{0\prime} + r^{1\prime}) \,, \quad (8)$$

converts finally the constraints into those of a free field theory with a Minkowskian target space

$$C_\pm = \pm\frac{1}{4}\left[(\pi_0 \pm r^{0\prime})^2 - (\pi_1 \mp r^{1\prime})^2\right] \,. \quad (9)$$

As we have already mentioned it is in general difficult to get an explicit expression for this canonical equivalence. In this letter we shall adopt an alternative approach to improve this situation. We shall consider the canonical transformation of the CGHS model introduced in [2] and reinterpret it as a Bäcklund transformation. In this new context we shall be able to generalize this Bäcklund transformation for a generic model of dilaton gravity. The Bäcklund transformation will define then an explicit canonical transformation.

The canonical transformation for the CGHS theory proposed in [2] makes use of the following auxiliary canonical variables η^0, η^1, p_0, p_1 defined by

$$r^a = \frac{1}{\sqrt{2}}\eta^a \,, \quad (10)$$

$$\pi_a = \sqrt{2}(p_a - \frac{1}{2}\epsilon_{ab}\eta^{b\prime}) \,, \quad (11)$$

where ϵ_{ab} is the antisymmetric tensor with $\epsilon_{01} = -1$. In terms of the canonical variables (η^a, p_a) the constraints have the form

$$H = \eta^{0\prime}p_1 + \eta^{1\prime}p_0 - (p_1^2 - p_0^2) \,, \quad (12)$$

$$P = \eta^{0\prime}p_0 + \eta^{1\prime}p_1 \,. \quad (13)$$

The canonical transformation is then defined by the following relations

$$\eta^0 = \frac{1}{2\lambda}e^{-\rho}\left(\pi_\rho \sinh\theta - 2\phi' \cosh\theta\right) \,, \quad (14)$$

$$\eta^1 = \frac{1}{2\lambda}e^{-\rho}\left(\pi_\rho \cosh\theta - 2\phi' \sinh\theta\right) \,, \quad (15)$$

$$p_0 = 2\lambda e^\rho \sinh\theta \,, \quad (16)$$

$$p_1 = -2\lambda e^\rho \cosh\theta \,, \quad (17)$$

where $\theta = \frac{1}{2}\int_{-\infty}^x d\tilde{x}\pi_\phi$. This transformation is canonical because it can be obtained from a generating functional. It is interesting to point out now that the above field redefinition can be regarded as a Bäcklund transformation connecting the dilaton-gravity equations (2), (3) for the CGHS model with free field equations

$$\partial_+\partial_-\eta^0 = 0 = \partial_+\partial_-\eta^1 \,. \tag{18}$$

We want now to generalize the above transformation for a generic model. It is easy to see that a transformation of the form

$$\begin{aligned}
\eta^{0\prime} &= \tilde{F}^{-1}\left[(H+\tilde{F}^2)\cosh\theta + P\sinh\theta\right] \,, && (19)\\
\eta^{1\prime} &= \tilde{F}^{-1}\left[(H+\tilde{F}^2)\sinh\theta + P\cosh\theta\right] \,, && (20)\\
p_0 &= -\tilde{F}\sinh\theta \,, && (21)\\
p_1 &= \tilde{F}\cosh\theta \,, && (22)
\end{aligned}$$

where $\tilde{F}$ and θ are arbitrary functions, bring the constraints of a generic theory to the form (12), (13). Because the canonicity of the transformation requires that the fields η^0 and η^1 verify free field equations, a natural generalization of the Bäcklund transformation defined by (14)-(17) is given by the following ansatz

$$\begin{aligned}
\eta^{0\prime} &= -\frac{1}{2\lambda}F^{-1}e^{-\rho}\left[(H+4\lambda^2F^2e^{2\rho})\cosh\theta + P\sinh\theta\right] \,, && (23)\\
\dot{\eta}^1 &= -2\lambda F e^{\rho}\cosh\theta \,, && (24)
\end{aligned}$$

where

$$\begin{aligned}
H(\rho,\phi) &= -2\dot{\phi}\dot{\rho} + 2(\phi'' - \phi'\rho') - 4\lambda^2 V(\phi)e^{2\rho} \,, && (25)\\
P(\rho,\phi) &= -2(\dot{\phi}\rho' - \dot{\phi}' + \phi'\dot{\rho}) \,, && (26)\\
F(\rho,\phi) &= \exp\left\{\frac{\lambda^2}{2}\partial_+^{-1}\partial_-^{-1}\left(\frac{d}{d\phi}V(\phi)e^{2\rho}\right)\right\} \,, && (27)\\
\theta(\rho,\phi) &= -\int_{-\infty}^{x} d\tilde{x}(\dot{\rho} + \frac{\dot{F}}{F}) \,. && (28)
\end{aligned}$$

The main goal of this letter is the following result:

The transformation defined by (23), (24) is a Bäcklund transformation that relates the solutions ρ, ϕ of a generic dilaton gravity theory and the solutions η^0, η^1 of a free field theory.

Proof: First of all, we observe the relation that there exists between the functions F and θ. Taking derivatives in expression (27) we can rewrite (2) as

$$\partial_+\partial_-(\rho + \ln F) = 0 \,. \tag{29}$$

Using (28) we then obtain the following identities

$$\begin{aligned}
\partial_+(\rho + \ln F) &= -\partial_+\theta \,, && (30)\\
\partial_-(\rho + \ln F) &= \partial_-\theta \,. && (31)
\end{aligned}$$

Now we take derivatives in (23), (24)

$$\begin{aligned}
\partial_+\partial_-\eta^{0\prime} = \tfrac{-1}{2\lambda}F^{-1}e^{-\rho}\,\{&\partial_+\partial_-P\sinh\theta + \partial_+\partial_-H\cosh\theta + \\
&[F^2e^{2\rho}\left(\partial_+(\rho+\ln F)\partial_-\theta + \partial_-(\rho+\ln F)\partial_+\theta + \partial_+\partial_-\theta\right) + \\
&\partial_+H\partial_-\theta + \partial_-H\partial_+\theta - \partial_+P\partial_-(\rho+\ln F) - \partial_-P\partial_+(\rho+\ln F) - \\
&P\left(\partial_+\partial_-(\rho+\ln F) - \partial_+(\rho+\ln F)\partial_-(\rho+\ln F) - \partial_+\theta\partial_-\theta\right) - \\
&H\left(\partial_+(\rho+\ln F)\partial_-\theta + \partial_-(\rho+\ln F)\partial_+\theta - \partial_+\partial_-\theta\right)]\sinh\theta + \\
&[F^2e^{2\rho}\left(\partial_+\partial_-(\rho+\ln F) + \partial_+(\rho+\ln F)\partial_-(\rho+\ln F) + \partial_+\theta\partial_-\theta\right) + \\
&\partial_+P\partial_-\theta + \partial_-P\partial_+\theta - \partial_+H\partial_-(\rho+\ln F) - \partial_-H\partial_+(\rho+\ln F) - \\
&H\left(\partial_+\partial_-(\rho+\ln F) - \partial_+(\rho+\ln F)\partial_-(\rho+\ln F) - \partial_+\theta\partial_-\theta\right) - \\
&P\left(\partial_+(\rho+\ln F)\partial_-\theta + \partial_-(\rho+\ln F)\partial_+\theta - \partial_+\partial_-\theta\right)]\cosh\theta\}\ , \qquad (32)
\end{aligned}$$

$$\begin{aligned}
\partial_+\partial_-\dot\eta^1 = -2\lambda Fe^{\rho}\,\{&[\partial_+(\rho+\ln F)\partial_-\theta + \partial_-(\rho+\ln F)\partial_+\theta + \partial_+\partial_-\theta]\cosh\theta + \\
&[\partial_+\partial_-(\rho+\ln F) + \partial_+(\rho+\ln F)\partial_-(\rho+\ln F) + \partial_+\theta\partial_-\theta]\sinh\theta\}\ , \qquad (33)
\end{aligned}$$

and taking into account (29), (30), (31) the above expressions become

$$\begin{aligned}
\partial_+\partial_-\eta^{0\prime} &= \frac{-1}{2\lambda}F^{-1}e^{-\rho}\,\{\partial_+\partial_-P\sinh\theta + \partial_+\partial_-H\cosh\theta + \\
&\quad [\partial_+(H-P)\partial_-\theta + \partial_-(H+P)\partial_+\theta]\sinh\theta + \\
&\quad [\partial_+(P-H)\partial_-\theta + \partial_-(P+H)\partial_+\theta]\cosh\theta\}\ , \qquad (34)\\
\partial_+\partial_-\dot\eta^1 &= 0\,. \qquad (35)
\end{aligned}$$

Finally, using the Bianchi identities $\partial_\pm(H \mp P) = 0$, we see that the r.h.s. of (34) vanishes and then

$$\begin{aligned}
\partial_+\partial_-\eta^0 &= 0\,, \qquad (36)\\
\partial_+\partial_-\eta^1 &= 0\,. \qquad (37)
\end{aligned}$$

Moreover, the above derivation also work on the other way around, so if η^0, η^1 satisfy the free field equations (36), (37) then ρ, ϕ satisfy the equations of motion (2), (3).

To construct the fully generalized canonical transformation we introduce the canonically conjugated momenta $\pi_\rho = -2\dot\phi$, $\pi_\phi = -2\dot\rho$, $p_0 = \dot\eta^0$, $p_1 = \dot\eta^1$ as independent variables. Then we get

$$\begin{aligned}
\eta^{0\prime} &= \frac{-1}{2\lambda}F^{-1}e^{-\rho}\left[(H + 4\lambda^2F^2e^{2\rho})\cosh\theta + P\sinh\theta\right]\ , \qquad (38)\\
\eta^{1\prime} &= \frac{-1}{2\lambda}F^{-1}e^{-\rho}\left[(H + 4\lambda^2F^2e^{2\rho})\sinh\theta + P\cosh\theta\right]\ , \qquad (39)\\
p_0 &= 2\lambda Fe^{\rho}\sinh\theta\,, \qquad (40)\\
p_1 &= -2\lambda Fe^{\rho}\cosh\theta\,, \qquad (41)
\end{aligned}$$

where H, P are given by

$$H = -\frac{1}{2}\pi_\rho \pi_\phi + 2(\phi'' - \rho'\phi') - 4\lambda^2 V(\phi) e^{2\rho}\,, \tag{42}$$

$$P = \pi_\rho \rho' - \pi_\rho' + \pi_\phi \phi'\,, \tag{43}$$

F is given by (27) and $\theta = \frac{1}{2}\int_{-\infty}^{x} d\tilde{x}(\pi_\phi - 2\frac{\dot{F}}{F})$. We have also seen that this transformation maps the constraints (42), (43) into the previous free form (12), (13). It is well known that a Bäcklund transformation can be viewed as a canonical transformation. This is so because there are no other expressions for the Poisson brackets that reproduce the Hamiltonian equations of motion for the free fields η^a, p_a.

The CJZ transformation for the CGHS model is recovered when $F(\rho,\phi) = 1$. Then $\theta = \frac{1}{2}\int_{-\infty}^{x} d\tilde{x}\pi_\phi$ and (38)- (41) read as

$$\eta^{0\prime} = \frac{1}{2\lambda} e^{-\rho}\left[(\frac{1}{2}\pi_\rho\pi_\phi - 2(\phi'' + \rho'\phi'))\cosh\theta - (\pi_\rho\rho' - \pi_\rho' + \pi_\phi\phi')\sinh\theta\right] \tag{44}$$

$$\eta^{1\prime} = \frac{1}{2\lambda} e^{-\rho}\left[(\frac{1}{2}\pi_\rho\pi_\phi - 2(\phi'' + \rho'\phi'))\sinh\theta - (\pi_\rho\rho' - \pi_\rho' + \pi_\phi\phi')\cosh\theta\right] \tag{45}$$

$$p_0 = 2\lambda e^{\rho}\sinh\theta\,, \tag{46}$$

$$p_1 = -2\lambda e^{\rho}\cosh\theta\,. \tag{47}$$

It is easy to see that (44),(45) leads to (14), (15).

For the model with an exponential (Liouville) potential $V = e^{\beta\phi}$ the function F is also local. The transformations (38)-(41) with $F = e^{-\beta\phi}$ provides an alternative canonical transformation for the induced 2D Polyakov gravity, which differs from the one obtained by using the classical solutions [8]. Another interesting example is the Jackiw-Teitelboim model ($V(\phi) = \phi$). In this case one cannot find a local expression for the function F. However, taking into account that ρ verifies a Liouville equation the field $\rho + \ln F$ coincide with the well known free field associated, via a canonical transformation, to a Liouville field [9].

In this letter we have constructed a Bäcklund transformation relating a generic 2D dilaton-gravity model to a generally covariant free field theory. This way we have provided an explicit canonical transformation connecting both theories.

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