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On the normalization of Killing vectors and energy conservation in two-dimensional gravity [*]

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Abstract

We explicitly show that, in the context of a recently proposed 2D dilaton gravity theory, energy conservation requires the "natural" Killing vector to have, asymptotically, an unusual normalization. The Hawking temperature T_H is then calculated according to this prescription.

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Two-dimensional dilaton gravity theories are considered interesting toy models for studying the issues connected with quantization of gravity. In particular, the use of exactly solvable models makes it possible to follow, analytically, the semiclassical evolution of evaporating black holes. In the context of the one discovered by RST [1], semiclassical version of the CGHS theory [2], black holes evaporate completely and the final end-point geometry is the vacuum state. It makes sense, then, to check whether the same conclusions hold in more general theories. Quite interestingly, it was shown recently [3] that two models related by a conformal rescaling of the metric may indeed describe very different physics: in one model, the so called exponential model introduced in [4], the evaporation process never ceases, whereas in the rescaled theory black holes evaporate completely leaving an everywhere regular end-point geometry that may be regarded as the semiclassical ground state (as it happens, for instance, in [5]). Several interesting questions are raised from this analysis, both in the classical and the semiclassical theory. In this note we will restrict to the classical analysis of black hole formation in the rescaled theory [3] and show that energy conservation implies a nontrivial normalization for the asymptotic timelike Killing vector associated to "natural" observers at rest at infinity.

Let us start by reminding briefly what happens in the CGHS theory [2], described by the action

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[e^{-2\phi} \left(R + 4(\nabla\phi)^2 + 4\lambda^2 \right) - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 \right] . \qquad (1)$$

In the Kruskal gauge the static black hole solutions are

$$ds^2 = \frac{-dx^+ dx^-}{\frac{M}{\lambda} - \lambda^2 x^+ x^-}, \quad e^{-2\phi} = \frac{M}{\lambda} - \lambda^2 x^+ x^- , \qquad (2)$$

where M is identified with the ADM mass. A black hole can be formed from the vacuum by sending in a shock-wave at $x^+ = x_0^+$, with $T_{++} = \frac{M}{\lambda x_0^+} \delta(x^+ - x_0^+)$.

The line element, that for $x^+ < x_0^+$ was given by $ds^2 = \frac{dx^+ dx^-}{\lambda^2 x^+ x^-}$, in the future of the shock-wave becomes

$$ds^2 = -\frac{dx^+ dx^-}{\frac{M}{\lambda} - \lambda^2 x^+ (x^- + \frac{M}{\lambda^3 x_0^+})} . \tag{3}$$

Provided we introduce the asymptotically minkowskian coordinate $\lambda x^+ = e^{\lambda \sigma^+}$ energy conservation is then simply given by the fact that

$$E = \int d\sigma^+ T_{\sigma^+ \sigma^+} = \lambda \int dx^+ x^+ T_{++} \tag{4}$$

equals the mass M of the black hole formed.

The rescaled exponential model of [3], namely

$$S = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[R\phi + \frac{\beta e^{\beta \phi}}{e^{\beta \phi} - 1} (\nabla \phi)^2 \right. $$
$$\left. + \frac{4\lambda^2}{\beta} e^{\beta \phi} \left(e^{\beta \phi} - 1 \right) - \frac{1}{2} \sum_{1=1}^{N} (\nabla f_i)^2 \right] , \tag{5}$$

has the following black hole solutions in Kruskal gauge

$$ds^2 = \frac{-dx^+ dx^-}{\frac{1}{\beta} - \frac{\lambda^2}{C} - \frac{C}{\beta} x^+ x^-} , \tag{6}$$

$$e^{-\beta \phi} = \frac{\lambda^2 \beta}{C} + C x^+ x^- . \tag{7}$$

It is useful, in this context, to consider a process a bit more general than the one described earlier. Let us study what happens when we send a shock-wave to a black hole characterized by $C = C_{in}$ (the case $C_{in} = \lambda^2 \beta$, corresponding to the Minkowski ground state, was considered in [3]) . The "out" solution will be given by

$$ds^2 = \frac{-dx^+ dx^-}{\frac{1}{\beta} - \frac{\lambda^2}{C_{out}} - \frac{C_{out}}{\beta} (x^+ + \Delta^+)(x^- + \Delta^-)} , \tag{8}$$

and by imposing continuity to the metric across the shock-wave line $x^+ = x_0^+$, we can determine the value of $\Delta^\pm$

$$\Delta^+ = x_0^+ \left(\frac{C_{in}}{C_{out}} - 1 \right) , \tag{9}$$

$$\Delta^- = \frac{\lambda^2 \beta}{x_0^+ C_{in}} \left(\frac{1}{C_{in}} - \frac{1}{C_{out}} \right) . \tag{10}$$

We note, by comparison to the CGHS theory, that an additional shift has appeared, namely Δ^+. The energy momentum tensor has the expression

$$T_{++}^f = \frac{\Delta^- C_{in} C_{out}}{\lambda^2 \beta^2} \delta(x^+ - x_0^+) = \frac{C_{out}}{\beta x_0^+} \left(\frac{1}{C_{in}} - \frac{1}{C_{out}} \right) \delta(x^+ - x_0^+) . \tag{11}$$

In order to calculate the energy of this shock-wave we must write down T_{++}^f in asymptotically minkowskian coordinates. However, due the presence of a non-vanishing Δ^+ an ambiguity arises. The asymptotically minkowskian coordinate σ^+ is different for the in and out solutions, namely

$$\sqrt{\frac{C_{in}}{\beta}} x^+ = e^{\sqrt{\frac{C_{in}}{\beta}} \sigma^+} , \qquad \sqrt{\frac{C_{out}}{\beta}} (x^+ + \Delta^+) = e^{\sqrt{\frac{C_{out}}{\beta}} \tilde{\sigma}^+} , \tag{12}$$

and the energy momentum tensor in terms of these coordinates becomes

$$T_{\sigma^+ \sigma^+}^f = \frac{1}{\beta} \sqrt{\frac{C_{in}}{\beta}} \left(\frac{C_{out}}{C_{in}} - 1 \right) \delta(\sigma^+ - \sigma_0^+) , \tag{13}$$

$$T_{\tilde{\sigma}^+ \tilde{\sigma}^+}^f = \frac{1}{\beta} \sqrt{\frac{C_{out}}{\beta}} \left(1 - \frac{C_{in}}{C_{out}} \right) \delta(\tilde{\sigma}^+ - \tilde{\sigma}_0^+) , \tag{14}$$

giving, potentially, two different results for the energy of the shock wave. Furthermore, neither of these results can be compatible with energy conservation because they do not satisfy the basic requirement

$$E = f(C_{out}) - f(C_{in}) , \tag{15}$$

for some function f, which expresses that the energy of the shock wave must be equal to the difference between the ADM masses of the in and out solutions. We can avoid this problem if we just state that the energy of the shock-wave is measured by the observer in which the metric asymptotically behaves as

$$ds^2 = -\frac{C}{\lambda^2 \beta} d\sigma^+ d\sigma^- . \tag{16}$$

These new coordinates are defined by

$$\frac{C_{in}}{\lambda\beta}x^+ = e^{\frac{C_{in}}{\lambda\beta}\sigma^+}, \qquad \frac{C_{out}}{\lambda\beta}(x^+ + \Delta^+) = e^{\frac{C_{out}}{\lambda\beta}\tilde{\sigma}^+}, \tag{17}$$

and now the energy momentum tensor becomes

$$T^f_{\sigma^+\sigma^+} = \frac{1}{\lambda\beta^2}(C_{out} - C_{in})\delta(\sigma^+ - \sigma_0^+), \tag{18}$$

$$T^f_{\tilde{\sigma}^+\tilde{\sigma}^+} = \frac{1}{\lambda\beta^2}(C_{out} - C_{in})\delta(\tilde{\sigma}^+ - \tilde{\sigma}_0^+). \tag{19}$$

This new result has the following two desired properties:

- The energy of the shock-wave does not depend on the choice of in or out observers (this can be easily understood by noting that we now have $\frac{d\sigma^+}{d\tilde{\sigma}^+} = 1$ at the shock-wave) ;

- It satisfies the consistency condition (15).

We must point out that this choice of coordinates implies that the normalization of the time-like Killing vector $\xi = \partial_t$ at spatial infinity is $-\frac{C}{\lambda^2\beta}$ instead of the more standard one $\xi^2 = -1$. Now we must determine whether this result agrees with the energy conservation. In doing so we need an expression for the ADM mass of the static solutions (6), (7). Let us make use of the formula [6] (see also [7])

$$M_{ADM} = \frac{F_0}{2}\left\{\int^\phi dD(s)V(s)e^{-\int^s dt\frac{H(t)}{D'(t)}} - (\nabla D(\phi))^2 e^{-\int^\phi dt\frac{H(t)}{D'(t)}}\right\} \tag{20}$$

for the ADM mass of a static solution arising from the action

$$S = \int d^2x\sqrt{-g}\left[D(\phi)R + H(\phi)(\nabla\phi)^2 + V(\phi)\right], \tag{21}$$

where in our case

$$D(\phi) = \phi, \tag{22}$$

$$H(\phi) = \frac{\beta e^{\beta\phi}}{e^{\beta\phi} - 1}, \tag{23}$$

4

$$V(\phi) = \frac{4\lambda^2}{\beta} e^{\beta\phi}(e^{\beta\phi} - 1) \,. \tag{24}$$

Applying formula (20) to the static solutions (6), (7) we get

$$M_{ADM} = F_0 \left(\frac{2C}{\beta^3} + constant \right) \,. \tag{25}$$

The constant F_0 is related to the normalization at infinity of the time-like Killing vector. Actually we have that

$$\lim_{x \to \infty} \xi^2 = - \left(\frac{2F_0}{\beta} \sqrt{\frac{C}{\beta}} \right)^2 \,. \tag{26}$$

So if we take the same normalization we used when we calculated the energy of the shock-wave, we have $F_0 = \frac{\beta}{2\lambda}$ and therefore

$$M_{ADM} = \frac{C}{\lambda\beta^2} + constant \,. \tag{27}$$

This way the energy of the shock-wave, given by the formulas (18), (19) is just

$$E = M_{ADM}(out) - M_{ADM}(in) \,, \tag{28}$$

and the energy is exactly conserved. Thus we are led to conclude that the only possible way to achieve energy conservation in this model is to pick up a "mass dependent" normalization at infinity of the time-like Killing vector instead of the standard one.

As a straightforward application of the above considerations, we now determine the Hawking temperature T_H of the black holes (6). The usual argument of determining the period of Euclidean time $\tau = it$ in order for the metric to be free of conical singularities at the event horizon gives

$$T_H = \frac{C}{2\pi\lambda\beta} = \frac{1}{2\pi}(\lambda + \beta M) \,. \tag{29}$$

In the last equality, according to [3], the constant appearing in (27) has been chosen in such a way that $M = 0$ in the ground state $C = \lambda^2\beta$. The key

observation in obtaining this formula is that we used $t = \frac{\sigma^+ + \sigma^-}{2}$, where, again, $\sigma^\pm$ are defined requiring the metric to be asymptotically of the form (16). We note, finally, that the same result can be inferred by calculating the near-horizon evaporation flux in the "out" region (8), i.e.

$$< T^f_{\tilde{\sigma}^- \tilde{\sigma}^-} > |_h = -\frac{N}{24} \{\sigma^-, \tilde{\sigma}^-\} |_h = \frac{N}{48} \frac{C^2}{\lambda^2 \beta^2} = \frac{N}{12} \pi^2 T_H^2 , \qquad (30)$$

where $\{\sigma^-, \tilde{\sigma}^-\}$ is the Schwartzian derivative and σ^-, $\tilde{\sigma}^-$ are defined by

$$-\frac{C_{in}}{\lambda \beta} x^- = e^{\frac{-C_{in}}{\lambda \beta} \sigma^-} , \qquad -\frac{C_{out}}{\lambda \beta} (x^- + \Delta^-) = e^{-\frac{C_{out}}{\lambda \beta} \tilde{\sigma}^-} . \qquad (31)$$

The aim of this note has been to point out the unusual normalization required for the Killing vector to produce a definition of the energy compatible with its conservation in a physical process.

Acknowledgements

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