gr-qc/0505110

Einstein-Planck Formula, Equivalence Principle and Black Hole Radiance

Alessandro Fabbri[a] * and **José Navarro-Salas**[b] †

a) Dipartimento di Fisica dell'Università di Bologna and INFN sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy

b) Departamento de Física Teórica and IFIC(CSIC), Universidad de Valencia, Dr. Moliner 50, Burjassot-46100, Valencia, Spain

Abstract

The presence of gravity implies corrections to the Einstein-Planck formula $E = h\nu$. This gives hope that the divergent blueshift in frequency, associated to the presence of a black hole horizon, could be smoothed out for the energy. Using simple arguments based on Einstein's equivalence principle we show that this is only possible if a black hole emits, in first approximation, not just a single particle, but thermal radiation.

*e-mail: fabbria@bo.infn.it

†email: jnavarro@ific.uv.es

One hundred years ago Einstein proposed [1] an explanation for the photoelectric effect putting forward the revolutionary relation $E = h\nu$ suggested by Planck [2] five years before. The quantum nature of radiation not only emerged in its emission by matter, as suggested by the spectrum of blackbody radiation; light is also transmitted and absorbed by matter in quanta. A particularly important virtue of the Einstein-Planck formula is that it is compatible with the theory of Special Relativity [3], also proposed by Einstein in the same miraculous year 1905. The fundamental relation $E = h\nu$ turned out to be compatible, as stressed in [4], with the new emerging view of spacetime, which, in contrast, was closely tied to the Maxwell wave-view of light. A Lorentz transformation along the x axis with velocity v modifies the phase $\nu(t + x/c)$ of a plane wave, travelling in the opposite direction, to $\nu'(t' + x'/c)$, where $\nu' = \nu\sqrt{\frac{1+v/c}{1-v/c}}$. This Doppler shift exactly coincides with that relating the energy E of a massless particle in the inertial frame (t, x) (i.e., the time component of the four-vector cp^μ) with the corresponding one in the frame (t', x'): $E' = E\sqrt{\frac{1+v/c}{1-v/c}}$. Note that in order to reach large frequencies like $\nu' \sim cl_P^{-1}$ (where l_P is the Planck length), with respect to an inertial observer at rest, we would need planckian energies and therefore extremely high velocities ($v \simeq c$).

However, we can *a priori* encounter very large frequencies from the gravitational Doppler effect. Assuming a spherically symmetric compact stellar object producing, in its exterior, a Schwarzschild geometry (from now on we use geometrized units $G = 1 = c$)

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega \ , \tag{1}$$

the relation between the frequencies measured by static observers at $r = const.$ (ν') and at infinity (ν) is $\nu' = \frac{\nu}{\sqrt{1-\frac{2M}{r}}}$. If the gravitational source is compact enough (i.e., a black hole) to allow for $r \to 2M$, a pulse of radiation with frequency ν at infinity has a history such that its frequency measured by a free-falling observer that falls from rest at (t_0, r_0) can grow up without bound at that point

$$\nu' = \frac{\nu}{\sqrt{1 - \frac{2M}{r_0}}} \ , \tag{2}$$

as $r_0 \to 2M$. If the free-falling observer is not at rest and has some inwards radial velocity the above result is corrected by an additional kinematical Doppler effect and the final blueshift is greater. This happens when the observer falls from rest from infinity, in which case

$$\nu' \approx \frac{\nu}{(1 - \frac{2M}{r_0})} , \tag{3}$$

as the observer approaches the horizon.

This leads to the paradoxical result that the emitted quanta of radiation could have had an arbitrary large amount of energy in the vicinity of the black hole horizon for free-falling observers. This conclusion, however, is based on the implicit assumption that the relation between energy and frequency is maintained, without corrections, in the presence of gravity. In this essay we shall show, at a heuristic level and using simple arguments based on the equivalence principle, that finiteness of the energy outflux is not compatible with the black hole emitting just a single particle, but thermal Hawking radiation [5].

When a gravitational field is present the changes of coordinates involved in the analysis are not longer Lorentz transformations. Restricting ourselves to the relevant $(t - r)$ sector we note that the frequency simply transforms as

$$\nu' = \frac{du}{d\xi^-} \ \nu , \tag{4}$$

where ξ^- is the locally inertial (outgoing) null coordinate and $u = t - r^*$ ($r^* \equiv r + 2M \ln[(r - 2M)/2M]$) is the corresponding one at infinity. The quotient of frequencies ν'/ν measures the different inertial time rates at the emission and detection points, and this is, indeed, how one gets (2) and (3).

The energy of the quanta is constructed in a different way. To simplify things let us assume spherical symmetry. The energy of a (spherical) outgoing narrow pulse of radiation can be defined as $E = \int \langle\Psi|T_{uu}|\Psi\rangle du$ at infinity, whereas $E' = \int \langle\Psi|T_{\xi^-\xi^-}|\Psi\rangle d\xi^-$ at the horizon. $|\Psi\rangle$ represents the quantum state of the radiation. Therefore the crucial point is to unravel how the quantities $\langle\Psi|T_{uu}|\Psi\rangle$ and $\langle\Psi|T_{\xi^-\xi^-}|\Psi\rangle$, which represent the luminosity of the radiated particle (i.e., energy per unit proper time) at different spacetime points, are related. Note that these quantities are related to the expectation values of the four-dimensional stress-energy tensor $\bar{T}_{ab}$ by the expression $\bar{T}_{ab} = T_{ab}/4\pi r^2$.

We can estimate the relation between $\langle\Psi|T_{uu}|\Psi\rangle$ and $\langle\Psi|T_{\xi^-\xi^-}|\Psi\rangle$ neglecting the backscattering of the radiation between the emission and detection points. Moreover, putting aside the angular coordinates, the stress-energy tensor can be identified directly with the corresponding normal-ordered expression in the locally inertial frame[‡] [6]

$$\langle\Psi|T_{\xi^-\xi^-}|\Psi\rangle \approx \langle\Psi| : T_{\xi^-\xi^-} : |\Psi\rangle \ . \tag{5}$$

Therefore, we can apply the rules of planar conformal invariance [7], and then a simple answer emerges

$$\langle\Psi| : T_{uu} : |\Psi\rangle = \left(\frac{d\xi^-}{du}\right)^2 \langle\Psi| : T_{\xi^-\xi^-} : |\Psi\rangle - \frac{\hbar}{24\pi}\{\xi^-, u\} \ , \tag{6}$$

where $\{\xi^-, u\} = \frac{d^3\xi^-}{du^3}/\frac{d\xi^-}{du} - \frac{3}{2}\left(\frac{d^2\xi^-}{du^2}/\frac{d\xi^-}{du}\right)^2$ is the Schwarzian derivative. Note that, generically, the relation should be of the form

$$\langle\Psi|T_{uu}|\Psi\rangle = \left(\frac{d\xi^-}{du}\right)^2 \langle\Psi|T_{\xi^-\xi^-}|\Psi\rangle + \hbar C(u, \xi^-; g_{\mu\nu}, \Psi) \tag{7}$$

where C, which depends on the initial and final points, the background metric and the quantum state, represents a correction to the flat-space formula. So we have

$$E' \equiv \int \langle\Psi|T_{\xi^-\xi^-}|\Psi\rangle d\xi^- = \int \frac{du}{d\xi^-}\langle\Psi|T_{uu}|\Psi\rangle du + \hbar \int \left(\frac{du}{d\xi^-}\right)^2 C d\xi^- \ , \tag{8}$$

For instance, for a one-particle (wave-packet) state of frequency around ν and peaked about the time $u \approx u_0$ we have $E \approx h\nu$ and therefore

$$E' = \frac{du}{d\xi^-}|_{u_0} h\nu + \dots \ , \tag{9}$$

where the leading term fits expression (3). The corrections can be worked out entirely in the approximation we are considering. Neglecting the backscattering of the radiation we get, for the one-particle state considered before,

[‡]We neglect this way subleading contributions coming from the spatial curvature of the metric.

$$E' = \frac{du}{d\xi^-}|_{u_0} h\nu + \frac{\hbar}{24\pi}\int \left(\frac{du}{d\xi^-}\right)^2 \{\xi^-, u\} d\xi^- \ . \tag{10}$$

Note that the term involving the Schwarzian derivative in (10) turns out to be independent of the particular quantum state. This is due to the fact that we are considering the "emission" point very close to the horizon. Were it not located close to the horizon, the correction would be, in general, state-dependent. Therefore, and just at the vicinity of the horizon, this term can be interpreted as a vacuum energy contribution.

We also remark that the correction codifies the fact that the locally inertial coordinates at the two points are different, and not related by Lorentz transformations (for them the Schwarzian derivative vanishes and we recover the usual relation $E' = h\nu'$). However, we must note that to evaluate (10) we need to know, in addition to the first derivative of the relation $\xi^- = \xi^-(u)$ (as required to relate the frequencies), the second and also the third derivatives. To get rid of the acceleration of a point-particle at a given point, as invoked by Einstein's equivalence principle, it is enough to know the first and second derivatives. A simple calculation leads to

$$\xi^- = \frac{d\xi^-}{du}|_{u_0}[(u-u_0) - \frac{M}{2r_0^2}(u-u_0)^2 + O((u-u_0)^3)] \ , \tag{11}$$

$$\xi^+ = \frac{d\xi^+}{dv}|_{v_0}[(v-v_0) + \frac{M}{2r_0^2}(v-v_0)^2 + O((v-v_0)^3)] \ , \tag{12}$$

where $v = t + r^*$ and r_0 is given by the relation $(v_0 - u_0)/2 = r_0 + 2M\ln[(r_0 - 2M)/2M]$. Since now we are working with extended objects (quantum wave packets) it should not be surprising that higher-order conditions emerge. We have to make use of the local Lorentz frame of a free-falling observer [8]. This in practice requires one to select the normal Riemann coordinates. Every time-like or null geodesic passing through the preferred point (u_0, v_0) is a straight line in that frame. So the expression (11) should be improved by adding the corresponding third order. The calculation leads to

$$\xi^- = \frac{d\xi^-}{du}|_{u_0}[(u-u_0) - \frac{M}{2r_0^2}(u-u_0)^2 - \frac{M(r_0-3M)}{6r_0^4}(u-u_0)^3 + O((u-u_0)^4)] \ . \tag{13}$$

Now we can estimate the flux of energy in the vicinity of the horizon. For points close to the horizon $r_0 \to 2M$ $(u_0 \to +\infty)$, we find that

$$\begin{aligned} \langle\Psi| : T_{\xi^-\xi^-} : |\Psi\rangle|_{\xi^-\approx 0} &= \left(\frac{du}{d\xi^-}\right)^2 [\langle\Psi| : T_{uu} : |\Psi\rangle + \frac{\hbar}{24\pi}\{\xi^-, u\}]|_{u_0} \\ &\approx \left(\frac{du}{d\xi^-}\right)^2 [\langle\Psi| : T_{uu} : |\Psi\rangle|_{u_0} - L + A(1-\frac{2M}{r_0})^2] , \quad (14) \end{aligned}$$

where $L = \frac{\hbar}{768\pi M^2}$ and $A = \frac{\hbar}{128\pi M^2}$. Note that, remarkably, the value obtained for L is sensitive to all the three terms explicitly written in the expansion (13)[§].

Since the energy flux of a single particle at infinity $\langle\Psi| : T_{uu} : |\Psi\rangle|_{u_0}$ is peaked around the point u_0 it can never balance the constant term L in (14). This is in fact a realization of Heisenberg's uncertainty principle. Therefore, although the relation between energy and frequency is modified (see eq. (10)) the final result is qualitatively unchanged because the value of $\langle\Psi| : T_{\xi^-\xi^-} : |\Psi\rangle$ at the horizon is necessarily divergent. We can interpret this outcome by saying that if we require finiteness of the energy flux then this implies that a black hole can never radiate out a single particle. This conclusion still holds for a generic many particle state in the Fock space. Only when the emitted radiation gives a (constant) *thermal* luminosity $\langle\Psi| : T_{uu} : |\Psi\rangle = \frac{\pi}{12\hbar}T^2$, with Hawking temperature $T = \hbar/8\pi M$, both constant terms can be summed up to produce a finite energy flux at the black hole horizon. Similar arguments apply for the finiteness of the correlation $\langle\Psi| : T_{\xi^-\xi^-}(1) :: T_{\xi^-\xi^-}(2) : |\Psi\rangle$ at the horizon, which also requires the thermal correlation for $\langle\Psi| : T_{uu}(1) :: T_{uu}(2) : |\Psi\rangle$.

We finally stress that if one neglects the third order term in (13) and keeps only the usual first and second terms as requested for the definition of locally inertial coordinates, the divergent blueshift cannot be removed. The thermal nature of the radiation is closely related to the explicit form of the "normal coordinates" as one approaches the classical horizon. Finally, let us mention that backreaction effects will certainly modify the location and structure of the horizon and therefore the associated radiation. It may be interesting to explore their physical implications within the picture offered in this essay.

[§]When backscattering is included the constant L and also A are modified by grey-body factors.

References

[1] A. Einstein, *Annalen der Physick* 17 (1905) 132.

[2] M. Planck, *Verhandlungen der Deutschen Physikalischen Gesellschaft* 2 (1900) 237.

[3] A. Einstein, *Annalen der Physick* 17 (1905) 891.

[4] L. de Broglie *Rev. Mod. Phys.* 21 (1949) 345. A. Sommerfeld, *Optics*, Academic Press, New-York (1964).

[5] S.W. Hawking, *Commun. Math. Phys.* 43 (1975) 199

[6] A. Fabbri and J. Navarro-Salas, *Modeling black hole evaporation*, Imperial College Press-World Scientific, London (2005)

[7] A.A. Belavin, A.M. Polyakov and A.B, Zamolodchikov, *Nucl. Phys.* B241 (1984) 333

[8] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, Freeman, San Francisco (1973)