Unified Graphical Summary of Neutrino Mixing Parameters

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The neutrino mixing parameters are presented in a number of different ways by the various experiments, e.g. SuperKamiokande, K2K, SNO, KamLAND and Chooz and also by the Particle Data Group. In this paper, we argue that presenting the data in terms of $\sin^2\theta$, where θ is the mixing angle appropriate for a given experiment has a direct physical interpretation. For current atmospheric, solar and reactor neutrino experiments, the $\sin^2\theta$'s are effectively the probability of finding a given flavor in a particular neutrino mass eigenstate. The given flavor and particular mass eigenstate varies from experiment to experiment, however, the use of $\sin^2\theta$ provides a unified picture of all the data. Using this unified picture we present a graphical way to represent these neutrino mixing parameters which includes the uncertainties. All of this is performed in the context of the present experimental status of three neutrino oscillations.

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Neutrino flavor transitions have been observed in atmospheric, solar, reactor and accelerator experiments. Transitions for at least two different E/L's (neutrino energy divided by baseline) are seen. To explain these transitions, extensions to the Standard Model of particle physics are required. The simplest and most widely accepted extension is to allow the neutrinos to have masses and mixings, similar to the quark sector, then these flavor transitions can be explained by neutrino oscillations.

This picture of neutrino masses and mixings has recently come into sharper focus with the salt data presented by the SNO collaboration[1]. When combined with the KamLAND experiment[2] and other solar neutrino experiments[3, 4] the range of allowed values for the solar mass squared difference, δm^2_{sol}, and the mixing angle, θ_{sol}, are reported as

$$\begin{aligned} 6.6\times10^{-5}\mathrm{eV}^2 < \quad &\delta m^2_{sol} \quad < 8.7\times10^{-5}\mathrm{eV}^2 \\ 0.33 < \ &\tan^2\theta_{sol} \ < 0.50 \end{aligned} \tag{1}$$

at the 90 % confidence level. Also maximal mixing, $\tan^2\theta_{sol} = 1$, has been ruled out at greater than 5 σ. The solar data is consistent with $\nu_e \to \nu_\mu$ and/or ν_τ.

The atmospheric data from SuperKamiokande has changed only slight in the past year with a preliminary new analysis presented at EPS conference[5] and is consistent with the K2K long baseline experiment[6]. The ranged of allowed values for the atmospheric mass squared difference, δm^2_{atm} and the mixing angle, θ_{atm}, are reported as

$$\begin{aligned} 1.3\times10^{-3}\mathrm{eV}^2 < \quad &\delta m^2_{atm} \quad < 3.0\times10^{-3}\mathrm{eV}^2 \\ 0.91 < \ &\sin^2 2\theta_{atm} \ \leq 1 \end{aligned} \tag{2}$$

at the 90 % confidence level. The atmospheric data is consistent with $\nu_\mu \to \nu_\tau$.

The best constraint on the involvement of the ν_e at the atmospheric δm^2 comes from the Chooz reactor experiment and this puts a limit on the mixing angle associated with these oscillations, θ_{chz}, reported as

$$0 \leq \ \sin^2 2\theta_{chz} \ < 0.1 \tag{3}$$

at the 90 % confidence level at $\delta m^2_{atm} = 2.5\times10^{-3}\mathrm{eV}^2$. This constraint depends on the precise value of δm^2_{atm} with a stronger (weaker) constraint at higher (lower) allowed values of δm^2_{atm}.

The only part of this picture which is still blurry is the $\bar\nu_\mu \to \bar\nu_e$ flavor transitions reported by the LSND experiment[8] at large mass squared differences, 0.2 eV^2 < δm^2_{lsnd} < 1 eV^2. Within the next couple of years the MiniBooNE experiment[9] will bring this part of the picture into focus. If the flavor transitions claimed by the LSND experiment are confirmed to be neutrino oscillations then a major revision of this picture maybe necessary[10]. Given that the LSND observation is unconfirmed and doesn't appear to fit easily into our current neutrino oscillation picture we have decided to wait for confirmation by MiniBooNE before trying to incorporate this result into the neutrino mixing parameter summary reported here.

The almost-standard way to incorporate the atmospheric, solar, reactor and accelerator data into the three neutrino picture is to assign the small mass-squared splitting associated with the solar δm^2 to the splitting between the mass eigenstates labeled 1 and 2. Because of the effects of matter on solar neutrinos, we already know that the mass eigenstate with the larger electron neutrino component has the smaller mass. We label this state 1 and the heavier state with the smaller electron neutrino component state 2. Therefore the solar δm^2 can be iden-

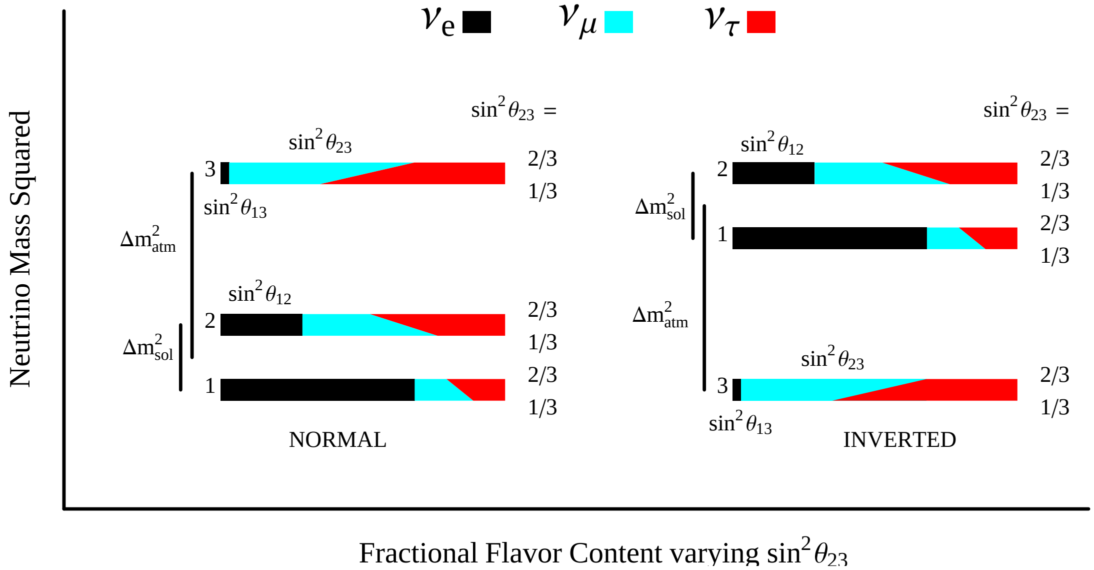

FIG. 1: The range of probability of finding the α-flavor in the i-th mass eigenstate as indicated for the two different mass hierarchies as $\sin^2\theta_{23}$ varies over its allowed range at the 90% C.L. The bottom of the bars is for the minimum allowed value of $\sin^2\theta_{23} \approx 1/3$ and the top of the bars is for the maximum value of $\sin^2\theta_{23} \approx 2/3$. The other mixing parameters are held fixed: $\sin^2\theta_{12} = 0.30$, $\sin^2\theta_{13} = 0.03$ and $\delta = \frac{\pi}{2}$.

tified[1] as

$$\delta m^2_{21} \equiv m^2_2 - m^2_1 = \delta m^2_{sol} > 0. \tag{4}$$

The large mass splitting associated with the atmospheric δm^2 is therefore the splitting between the mass eigenstate labeled 3 and the more closely spaced 1 and 2 mass eigenstates. From the Chooz experiment we know that the 3-mass eigenstate has a very small electron neutrino component. The sign of the splitting of this state from the solar doublet, 1 and 2, is unknown. Thus,

$$|\delta m^2_{32}| = \delta m^2_{atm} > 0. \tag{5}$$

Therefore both mass eigenstates in the solar doublet, 1 and 2, could have smaller mass than the 3 mass eigenstate (this is usually called the normal hierarchy) or a larger mass than the 3-mass eigenstate (the inverted hierarchy). The determination of the sign of this splitting, i.e the sign of δm^2_{32}, is one of the important unknowns within the neutrino sector.

In the three neutrino scenario the mixing matrix which relates the flavor states, $\alpha = (e,\ \mu,\ \tau)$, to the mass eigenstates, i=(1, 2, 3), is called the Pontecorvo-Maki-Nakagawa-Sakata matrix (PMNS)[13], that is

$$|\nu_\alpha\rangle = U_{\alpha,i}\ |\nu_i\rangle \tag{6}$$

[1] Our convention is that $\delta m^2_{ji} \equiv m^2_j - m^2_i$.

With the above identification of the mass eigenstates the standard PMNS matrix parametrization is given by[11]

$$U_{\alpha i} = \tag{7}$$

$$\begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{13} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{13}c_{23} \end{pmatrix}$$

where s_{ij} and c_{ij} are shorthand for $\sin\theta_{ij}$ and $\cos\theta_{ij}$ respectively. The factorized form of this PMNS mixing matrix, eqn(7), is useful because it allows an identification between the mixing angles of the three component picture and those of the two component analysis reported by the experiments. Thus,

$$\theta_{23} \cong \theta_{atm}, \quad \theta_{12} \cong \theta_{sol}, \quad \text{and} \quad \theta_{13} \cong \theta_{chz}. \tag{8}$$

This is an excellent approximation because of the smallest of the ratio of the solar to atmospheric δm^2, $\delta m^2_{sol}/\delta m^2_{atm}$, *and* $\sin^2\theta_{chz}$. A recent detailed 3 neutrino analysis can be found in [12]. The phase factor δ, $(0 \leq \delta < 2\pi)$, doesn't occur in any two component analysis however this phase allows for the possibility of CP violation in three neutrino oscillations.

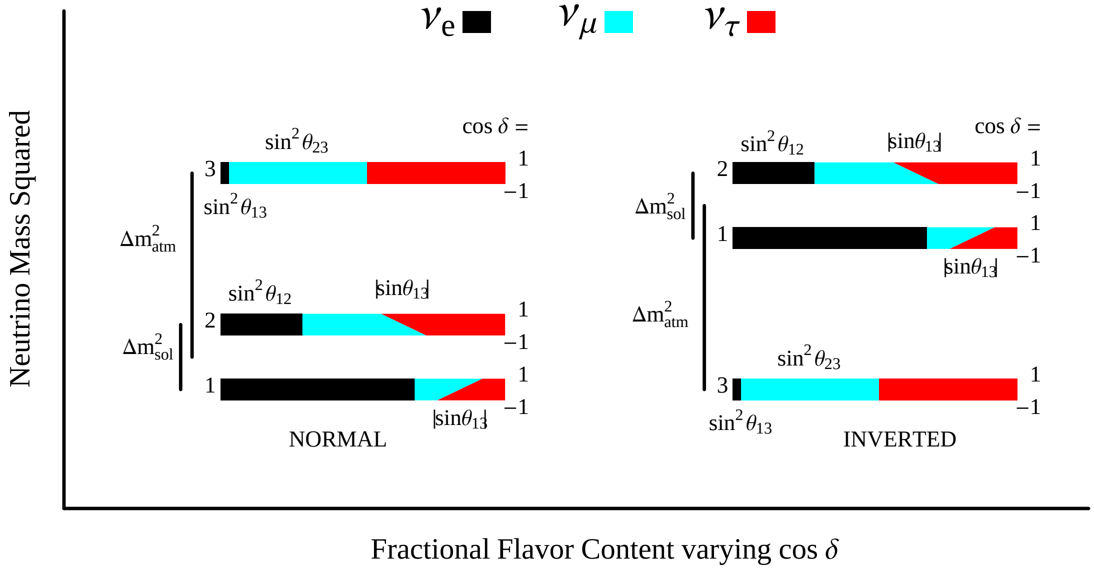

FIG. 2: The range of probability of finding the α-flavor in the i-th mass eigenstate as indicated as the CP-violating phase, δ, is varied. The bottom of the bars is for the minimum allowed value of $\cos\delta = -1$ and the top of the bars is for the maximum value of $\cos\delta = 1$. The other mixing parameters are held fixed: $\sin^2\theta_{12} = 0.30$, $\sin^2\theta_{13} = 0.03$ and $\sin^2\theta_{23} = 0.50$. The maximum to minimum variation of the fractional flavor content of μ or τ in mass eigenstates 1 or 2 is very close to $\sin\theta_{13}$. The only parameter in the PMNS mixing matrix this figure is not sensitive to is the sign of $\sin\delta$.

An extremely useful way to understand the meaning of the various mixing angles is to relate them to the probability of finding the α-flavor in the i-th mass eigenstate. This probability is given by the absolute square of the PMNS matrix elements, $|U_{\alpha i}|^2$. Thus the probability of finding ν_e in the 3-th neutrino mass eigenstate is just $\sin^2\theta_{13}$ which is known from the Chooz data to be no larger than a few per cent ($< 3\%$). Similarly the probability of finding ν_μ (ν_τ) in the 3-th mass eigenstate is just $\cos^2\theta_{13}\sin^2\theta_{23} \approx \sin^2\theta_{23}$ ($\cos^2\theta_{13}\cos^2\theta_{23} \approx \cos^2\theta_{23}$) since $\cos^2\theta_{13}$ is very close to unity. Also the probability of finding the ν_e in the 2-th mass eigenstate is just $\cos^2\theta_{13}\sin^2\theta_{12} \approx \sin^2\theta_{12}$. Since the ^8B solar neutrinos exit the sun as nearly a pure ν_2 neutrino mass eigenstate, due to matter effects[14], the measurement of the Charge Current to Neutral Current (CC/NC) ratio by SNO is a direct measurement of $\sin^2\theta_{12}$ up to small corrections.

In general the probability of finding the α-flavor in the i-th mass eigenstate, $P_\nu(\alpha,\, i)$ is given by

$$P_\nu(\alpha, i) = |U_{\alpha i}|^2 \approx \tag{9}$$

$$\begin{pmatrix} c_{12}^2 & s_{12}^2 & s_{13}^2 \\ c_{23}^2 s_{12}^2 + K s_{13}\cos\delta & c_{23}^2 c_{12}^2 - K s_{13}\cos\delta & s_{23}^2 \\ s_{23}^2 s_{12}^2 - K s_{13}\cos\delta & s_{23}^2 c_{12}^2 + K s_{13}\cos\delta & c_{23}^2 \end{pmatrix}$$

where $K = \frac{1}{2}\sin 2\theta_{12}\sin 2\theta_{23}$ ($\approx \frac{1}{2}$) and terms of order $\sin^2\theta_{13}$ have been dropped except in the (e,3) component which otherwise would be zero. Note, that up to this order the sum of each row and each column of this probability matrix adds up to one as required by unitarity. The probabilities (μ,1), (μ,2), (τ,1) and (τ,2) all depend linearly on $\sin\theta_{13}\cos\delta$ whose sign is determined by $\cos\delta$ and the magnitude can be quite significant compared to the terms independent of $\sin\theta_{13}$ in these probabilities.

Translating the mixing angle information reported by the experiments into ranges of probability of finding the α-flavor in the i-th mass eigenstate we obtain

$$\begin{aligned} 0.25 < \quad & \sin^2\theta_{12} \cong P_\nu(e,2) & < 0.33 \\ 0.35 < \quad & \sin^2\theta_{23} \cong P_\nu(\mu,3) & < 0.65 \\ & \sin^2\theta_{13} \equiv P_\nu(e,3) & < 0.03 \end{aligned} \tag{10}$$

at the 90% confidence level. Clearly, using the probability metric, i.e. $\sin^2\theta$, our current information of the solar mixing is significantly better than that of the atmospheric mixing. This occurs because $\sin^2 2\theta$ is a poor measure of $\sin^2\theta$ near $\sin^2\theta = \frac{1}{2}$. Eqn(9) and (10) can be used to calculate the ranges for all the other the probabilities with the unknown $\cos\delta$ varying from -1 to +1.

In the past, the central value of all of these probabilities has been presented in a bar graph with a separate horizontal bar for each neutrino mass eigenstate with color and/or shading coding for each of the neutrino flavors. This is a very useful pictorial way of presenting all of the neutrino mixing data with a physical interpretation. In this letter we extend this diagram to include the range of possible probabilities allowed by the data. To do this we make use of the thickness of the bars so that the bottom of the bar represents the minimum allowed value for a particular parameter and the top of the bar represents the maximum allowed value of this parameter. With these figures one can easily see how the range of parameters effects the flavor probabilities for any given mass eigenstate. Remember that these flavor probabilities are the absolute squares of the elements of the PMNS mixing matrix. Below we discuss a few important examples of such figures.

In Fig. 1 we have held all mixing parameters fixed (the values are given in the caption), except $\sin^2\theta_{23}$ which varies through out it's allowed range. Clearly the probability of finding the ν_μ or ν_τ flavors in all of the neutrino mass eigenstates varies significantly because of the uncertainty in $\sin^2\theta_{23}$. This uncertainty in $\sin^2\theta_{23}$ is of particular importance for the proposed Long Baseline neutrino experiments searching for $\nu_\mu \to \nu_e$, JParc to SK[16] and NuMI-Off-Axis[15], as the leading term in the oscillation probability is proportional to $\sin^2\theta_{23}\sin^2\theta_{13}$.

Similarly, in Fig. 2 we have varied the CP violating phase so that $\cos\delta$ varies from -1 to +1 keep the other angles fixed. Notice that for values of $\sin^2\theta_{13}$ close to the maximum allowed by the Chooz experiment the probabilities (μ,1), (μ,2), (τ,1) and (τ,2) have significant dependence on the CP violating phase δ through the variable $\cos\delta$. The range of variation of any of these probabilities is linear in $\sin\theta_{13}$ so that $\sin^2\theta_{13}$ must be very small before this range is insignificant. Determination of the phase δ will most likely come from future Long Baseline experiments as the asymmetry between $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$, which can approach one, is directly proportional to $\sin\delta$ if the energy and baseline are tuned to a peak in the oscillations.

Other variations of this figure are possible[17] including the possibility of varying more than one variable at a time e.g. one could choose the bottom of the bars so that say $P_\nu(\mu,1)$ is minimum and the top such that $P_\nu(\mu,1)$ is maximum for the allowed ranges of all the mixing angles. Another possibility is to use the maximum range of each probability without including the correlations between these probabilities.

In summary we have argued that using $\sin^2\theta$ instead of the numerous other ways that exist in the literature allows for a unified treatment of the neutrino mixing parameters which has direct physical interpretation.

- $\sin^2\theta_{sol}$ is the probability of finding the electron neutrino flavor in the heavier of the solar doublet of neutrino mass eigenstates (labeled 2) up to small corrections and is directly measured by the SNO Charge Current to Neutral Current ratio.

- $\sin^2\theta_{atm}$ is the probability of finding the muon neutrino flavor in the isolated neutrino mass eigenstate (labeled 3) up to small corrections.

- $\sin^2\theta_{chz}$ is the probability of finding the electron neutrino flavor in the isolated neutrino mass eigenstate (labeled 3).

We have extended the usual pictorial presentation of the absolute square of the PMNS matrix elements, i.e. the probabilities of finding a given flavor in a particular mass eigenstate, so as to incorporate the uncertainties in our knowledge of the mixing parameters. Together, these two features provide a coherent, unified, graphical summary of our present knowledge of the neutrino sector.

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