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Abstract

A method to perform two loop calculations in Eichten's effective field theory for heavy quarks is developed. The anomalous dimension of the axial current for static heavy quarks is calculated at two loops. For $N = 3$, we get $\gamma_A^{(2)} = -\frac{1}{36}\left[\frac{127}{2} + 94\,\xi(2) - 5\,N_f\right]$. This result is very important to understand completely the physical significance of the lattice measurement of the decay constant of the B meson. The two loop correction generated by $\gamma_A^{(2)}$ turns out to be small, less than 1.5% for four quark flavours, so that the value of the decay constant of the B meson does not need in practice renormalization group improvement. As a subproduct, the self-energy renormalization constant at two loop level for a heavy static quark and that for the coupling of a heavy quark to a gluon are obtained.

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1 Introduction

There is increasing evidence that quantum chromodynamics (QCD) describes the dynamics of hadrons. Heavy quarks, i.e. quarks with masses much larger than the QCD scale (Λ_{QCD}), play an important role in studying its non–perturbative confinement dynamics. Systems involving two heavy quarks ($Q\,\bar{Q}$) as charmonium ($c\,\bar{c}$) and bottonium ($b\,\bar{b}$), has been described in an accurate way by a nonrelativistic potential for distances under 1 Fermi and perturbative QCD at shorter distances [1]. Lattice QCD has also been used to calculate succesfully the dynamics of those heavy systems [2]. However, heavy–light quark systems ($Q\,\bar{q}$), such as the B mesons ($(b\bar{u})$, $(b\bar{d})$, $(b\bar{s})$) and, of course, the T mesons when discovered, are not so well understood as heavy–heavy quark systems are. The physical importance of the determination of the properties of heavy–light systems is twofold. On the one hand, they will help to obtain, or at least to constrain, the values of the various free parameters of QCD, as the poorly–known Kobayashi–Maskawa matrix element $\mid V_{ub} \mid$ and the effects of CP violation. On the other hand, some interactions beyond the Standard Model could be detected and, in some sense, understood from the theoretical interpretation of the large $B_d^0 - \bar{B}_d^0$ mixing measured by ARGUS (for an excelent review see [3]). The crucial difference between these systems and the purely heavy quark systems is that the former cannot be treated by nonrelativistic potencial methods owing to the relativistic light quark. Perturbative methods are unable to deal with their long distance dynamics due to the fact that the light quark typical momentum μ is of order of Λ_{QCD} and so it is subject to QCD non–perturbative effects. While presenting at present some systematic errors that may obscure the results obtained by this method [4], Lattice QCD offers among all other non–perturbative approaches, the unique possibility of computing with the same method many different matrix elements which are of interest in the fenomenology of the Standard Model and in the study of the strong interactions including heavy quark systems [5]. However, one is placed before a dilemma: on the one hand, heavy quark systems should be studied on the lattice but, on the other hand, to avoid lattice artifacts it must be satisfied $m_Q\, a \ll 1$ which is not true for the B meson (typically $a^{-1} \sim 2 \div 3$ Gev). In other words, to propagate the b quark on a lattice of spacing a, it is necessary that a be much smaller than the b quark Compton's wave $1/m_b$, which in practice is not available at the present status of computer resources. The basic idea to solve this problem is to realize that we do not need to put a dynamical heavy quark on the lattice, as was shown by Eichten [2, 6]. In fact, Eichten's proposal was to consider the heavy quark Q in a meson as a static color source which could be treated non-relativistically. Then, expanding its propagator in powers of Λ_{QCD}/m_Q, any Green's function and accordingly all relevant physical quantities, can be calculated, to some order in $1/m_Q$, only from the knowledge of the light–quark propagator and the gluon fields, which can be estimated on a lattice [6]. The usual procedure is to consider only the leading term $1/m_Q$ although some calculations, e.g. the determination of the leading spin dependent potencials for $(Q\,\bar{Q})$ systems, have been made including the next–to–leading term $1/m_Q^2$ [2]. Eichten's idea can be formulated as a low-energy effective field theory for heavy quarks which should describe its long distance physics (long distances compared to $1/m_Q$) [7, 8]. The most important difference between the effective and the complete QCD theory is that the heavy quark propagates only in time so that both its propagator and its interaction with gluons involve just the zero component of its field. This fact makes Feynman integrals to be more difficult to solve due to the lost of manifestaly Lorentz invariance.

In this paper, we will present a method, indeed a mixture of well–known Feynman diagram calculating techniques, to compute two loop integrals with static heavy quark lines which have not been studied in a systematic way so far. As an application, we will discus the $\overline{MS}$ continuum renormalization of the axial current for heavy–light quarks at two loops and study how its value influences the physical signification of the lattice measurement of the B–meson decay constant f_B. A parallel study of the renormalization of the operator which determines the B–parameter of the B meson is in process now and will be published elsewhere.

The plan of the paper is as follows. In the next section, we present briefly the effective field theory for heavy quarks. Section 3 is devoted to state the strategy of the calculation of the anomalous dimension of the axial current whose details will be discussed in the next section. Section 5 contains a summary of our final results. In Section 6, we analyse the importance of the axial current renormalization in the determination of the B–meson properties. Finally, our conclusions are present in Section 7. In addition, we have included three appendices where the techniques we have used in the calculation are deeply disscused. Many new results for one loop integrals with a heavy quark line are tabulated in Tables 1 to 3.

2 The effective field theory for heavy quarks

This section is devoted to present a résumé of the effective field theory for heavy quarks at low energies [7], its motivation and its formalism.

Consider the physical effects on hadron systems associated with mass scales much larger than the QCD scale Λ_{QCD}. In studing the matrix elements of operators containing both heavy quark, Q, and light quark, q, fields one encounters large logarithms of the type of $\ln(m_Q/\mu)$, where μ is the typical three momentum of the light quark. It, μ, must be of order of the QCD scale Λ_{QCD} because the reduced mass of heavy–light systems is, due to the heavy quark, close to the light quark mass, that in turn is of order of the dynamical scale of QCD. These logarithms come from Feynman diagrams with virtual loop momenta p in the region $\Lambda_{QCD} < p < m_Q$. In other words, m_Q plays the role of an ultraviolet cut–off so that the Feynman integrals can depend logarithmically on it. The objective is to extract the dependence of these operators on the heavy quark mass analytically, i.e. to sum the logarithms $\ln(m_Q/\mu)$, otherwise the estimates of their matrix elements using either nonrelativistic quark model or lattice QCD with a static heavy quark [6], would have an obscure physical significance. These logarithmic corrections has already been calculated in the complete theory for f_B and the B–parameter by Voloshin and Shifman [9] and also by Politzer and Wise [10]. The latter used a somewhat complicated momentum space substraction involving examining intermediate loop momentum ranges. Caswell, Lapage and Thacker [11] proposed a nonrelativistic approach to this problem with a momentum cut–off well bellow the heavy quark mass m_Q. Following Politzer and Wise [12], we think that these methods are inadequate to perform two loop and even one loop calculations involving heavy quarks. The alternative method they suggested was to expand the heavy propagator in powers of $1/m_Q$ doing calculations at zeroth order in this expansion. This idea was first proposed by Eichten [6] in order to derive heavy quark potentials for $(Q\bar{Q})$ systems on the lattice. Now, this technique will become a very useful tool to compute analitically the dependence on a large mass scale of relevant matrix elements when other scales are much smaller than it. The leading term of this expansion, corresponding to a heavy quark with $m_Q \rightarrow \infty$ so that it only propagates in time, is [6]

$$\begin{aligned} S_Q^0(x,y) &= -i\,P(x_0,y_0)\,\delta^3(\vec{x}-\vec{y})\,\Big[\theta(x^0-y^0)\,e^{-im_b(x_0-y_0)}\,\frac{1+\gamma_0}{2} \\ &+ \theta(y^0-x^0)\,e^{-im_b(y_0-x_0)}\,\frac{1-\gamma_0}{2}\Big] \end{aligned} \tag{1}$$

where the path ordered exponential $P(x_0,y_0)$ is given by

$$P(x_0,y_0) = P\,\exp\left[i\,g\int_{y^0}^{x^0} dz^0\,B_a^0(\vec{x},z^0)\,\frac{\lambda^a}{2}\right] \tag{2}$$

with $B^0(x)$ being the time component of the external gauge field and λ^a are the generators of the $SU(N)$ group in its fundamental representation normalized by $tr(\lambda^a\lambda^b) = 2\,\delta_{ab}$. The full propagator $S_Q(x,y)$ can now be calculated perturbatively solving the Dirac equation

$$\left(i\,\not\partial_\mu + ig \left(\frac{\lambda_a}{2}\right) \not A^a_\mu - m_Q \right) S_Q(x,y) = \delta^4(x-y) \tag{3}$$

in powers of $1/m_Q$ making use of Eq.(1) as its basic solution [6]. Notice that the whole calculation is done in position space.

The idea of an effective field theory for heavy quarks that would allow us to perform perturbation theory in momentum space, is due to Eichten and Hill [7] and generalizated by Georgi [8]. The goal is to obtain a lagrangian density from which the heavy quark propagator can be derived. Again, the starting point is that a heavy quark in a QCD bound state carries most of the energy and momentum of the system. As $m_Q \to \infty$, the heavy quark in the rest frame of the heavy–light system is nearly at rest and nearly on shell. Then, its equation of motion (3) simplies to

$$\left(i\gamma^0\partial_0 + ig \left(\frac{\lambda_a}{2}\right) \gamma^0 A^a_0 - m_Q \right) S_Q(x,y) = \delta^4(x-y) \tag{4}$$

Equation (4) can be derived from the lagrangian density

$$\begin{aligned}\mathcal{L}(x) \;&=\; \frac{i}{2}\overline{Q}_\alpha(x)\gamma^0\partial_0 Q_\alpha(x) - \frac{i}{2}\left[\partial_0\overline{Q}_\alpha(x)\right]\gamma^0 Q_\alpha(x)\\ &-\; m_Q\overline{Q}_\alpha(x)Q_\alpha(x) + \frac{1}{2}g\overline{Q}_\alpha(x)\lambda^a_{\alpha\beta}\gamma^0 Q_\beta(x)B^0_a(x)\end{aligned} \tag{5}$$

The now trivial dependence on the heavy quark mass, m_Q, can be eliminated redefining the heavy quark field as follows

$$Q_\alpha(x) \longrightarrow e^{im_Q x^0\gamma^0}\,Q_\alpha(x) \tag{6}$$

Then the heavy quark lagragian $\mathcal{L}$ becomes the Eichten-Hill-Georgi Lagrangian $\mathcal{L}_{EHG}$

$$\begin{aligned}\mathcal{L}_{EHG}(x) \;&=\; \frac{i}{2}\overline{Q}_\alpha(x)\gamma^0\partial_0 Q_\alpha(x) - \frac{i}{2}\left[\partial_0\overline{Q}_\alpha(x)\right]\gamma^0 Q_\alpha(x)\\ &+\; \frac{1}{2}g\overline{Q}_\alpha(x)\lambda^a_{\alpha\beta}\gamma^0 Q_\alpha(x)B^0_a(x)\end{aligned} \tag{7}$$

in the rest frame of the heavy–light quark system. The field $\frac{1}{2}(1+\gamma^0)\,Q_\alpha$ annihilates heavy quarks and $\frac{1}{2}(1-\gamma^0)\,Q_\alpha$ creates heavy anti–quarks. As can be seen, this lagrangian does not preserve the Lorentz invariance but it can be generalizated to an arbitrary reference frame integrating in velocity degrees of freedom [8]. The reader should not be misled by the Eq.(7) as it is *just an effective theory not a real one.* Indeed, as all effective theory, we must match the static theory with the complete theory at a scale of the order of the heavy quark mass, m_Q.

Let us now calculate the new Feynman rules. We will use them in the following sections to evaluate the Feynman diagrams needed to obtain the anomalous dimension of the heavy–light quark axial current. From Eq.(7), the bare propagator for a heavy static quark in momentum space is simply

$$i\,S_{\alpha\beta}(p) = \delta_{\alpha\beta}\,\frac{i}{p^0\gamma^0 + i\epsilon} \tag{8}$$

Also the vertex of an incoming and an outgoing quark with a gluon, the so-called fermionic vertex, must be modified in the limit of infinite quark mass. In fact, as the quark does not propagate in space, only the time component of the gauge field can interact with it. The Feynman rule in momentum space for the heavy quark vertex is

$$\int d^4x\, d^4y\, e^{-ikx}\, e^{-ipy} < 0 \mid T\left(Q_\beta(x)\,\bar{Q}_\alpha(y)\,B^\mu_a(0)\right) \mid 0 > \longrightarrow i\,g\,\gamma^0 g^{\mu 0}\left(\frac{\lambda_a}{2}\right)_{\beta\alpha} \tag{9}$$

where the Green's function must be considered with amputed legs. Other Feynman rules are identical to those of the complete theory (see for example [13]). Notice that momentum must be conserved in the effective theory even at vertices involving heavy quarks [6].

The logarithms $\ln(m_Q/\mu)$ can be displayed explicitly using this effective theory in momentum space. The starting point is that as in the static field theory the heavy quark mass m_Q is effectively taken as infinite, the Green functions will contain divergences coming from $m_Q \to \infty$. Since m_Q plays the role of a cut-off in systems with energy scale much lower than it, the effects of a very large mass will appear as poles of the type of $1/\epsilon$ with $\epsilon = (D-4)/2$. Therefore, the anomalous dimension of a heavy quark operator will be determined by the properties of Feynman diagrams as the loop momenta go to infinity. In this framework, standard dimensional regularization can be used to simplify as much as possible the evaluation of the double and simple poles of the diagrams and a mass independent renormalization scheme as $\overline{MS}$ can be used to renormalize them. Having renormalized the corresponding operator, its logarithmic dependence on the heavy quark mass, m_Q, can be summed and factorized.

3 The strategy of the calculation

In this section we describe the technique to compute the two loop contribution to the anomalous dimension of the axial current with a heavy quark. The same result will also hold for the vector current.

Consider the axial current operator

$$A_\nu(x) \equiv \bar{b}(x)\,\gamma_\nu\gamma_5\,q(x) \tag{10}$$

and its two point bare Green's function

$$G^{(0)}(x,y,z) \equiv < 0 \mid T\left\{ Q^{(0)}_{\alpha}(x)\, A^{(0)}_{\nu}(y)\, \bar{q}^{(0)}_{\beta}(z) \right\} \mid 0 > \tag{11}$$

where $Q(x)$ and $q(x)$ are heavy and light quark fields. A (0) superscript denotes bare fields. If both quarks were light, as the axial current is partially conserved so that it does not get renormalized, the Green's function $G^{(0)}$ would be renormalized as

$$G(\mu) \equiv < 0 \mid T\left\{ Q_{\alpha}(x)\, A_{\nu}(y)\, \bar{q}_{\beta}(z) \right\} \mid 0 > = (\, Z_Q\, Z_q\,)^{-1/2}\, G^{(0)} \tag{12}$$

with

$$Q^{(0)}(x) \; = \; \sqrt{Z_Q}\, Q(x) \qquad\qquad q^{(0)}(x) \; = \; \sqrt{Z_q}\, q(x) \tag{13}$$

and μ the subtraction point.

The Green's function in the effective theory, $\overline{G}(\mu)$, will be different from that of the complete theory, G. In fact, in the former the scale μ is much smaller than m_Q and so logarithmic terms depending on μ will arise. The axial current (10) in the effective theory, $\overline{A}_{\nu}(\mu)$, is thus a different operator from that in the complete theory with different renormalization properties. We can write mathematically this fact introducing the factor $C(\mu)$ defined by [10]

$$A_{\nu} \; = \; C(\mu)\, \overline{A}_{\nu}(\mu) \tag{14}$$

Hence, $C(\mu)$ is the factor that we will have to take into account so that the physical consequences of both theories agree each other. At $\mu = m_Q$, both theories must be equal, therefore

$$C(m_Q) \; = \; 1 \, + \, O\left(\frac{\alpha_s(m_Q)}{\pi}\right) \tag{15}$$

As we discussed in Section 2, the Green functions of the effective theory $\overline{G}(\mu)$ will contain factors like $1/\epsilon$ which are not compensated by wave function renormalization factors. In order to render $\overline{G}(\mu)$ finite, an additional axial current renormalization, denoted by $\overline{Z}_A$, is needed

$$\overline{G}(\mu) \; = \; \left(\overline{Z}_Q\, Z_q\right)^{-1/2}\, \overline{Z}_A\, \overline{G}^{(0)} \; \equiv \; \overline{Z}_{\overline{G}}\, \overline{G}^{(0)} \tag{16}$$

where $\overline{Z}_Q$ is the wave function renormalization constant for the heavy quark in the effective theory, defined by

$$\overline{S}_Q(x) \equiv < 0 \mid T\, (Q_{\alpha}(x)\, \bar{Q}_{\beta}(0)) \mid 0 > = \left(1/\sqrt{\overline{Z}_Q}\right)^2\, \overline{S}^{(0)}_Q \tag{17}$$

Notice that the wave function renormalization constant for the light quark in the effective theory Z_q is the same as that in the complete theory which is an old well–known result [14, 15]. In Eq.(16)

we have used that $\overline{A}_\nu$ renormalizes multiplicatively because there is no other operator of dimension three with the same quantum numbers.

Futhermore, differentiating Eq.(14) with respect to μ and bearing in mind that A_ν is independent of the renormalization point μ, we find that $C(\mu)$ must satisfy the renormalisation group equation

$$\left[\mu \frac{d}{d\mu} - \gamma_A\right] C(\mu) = 0 \tag{18}$$

where $\gamma_A = -\mu \frac{d}{d\mu} \ln \overline{Z}_A$ is the anomalous dimension of the axial current. If we write

$$\gamma_A = \left(\frac{\alpha}{\pi}\right) \gamma_A^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 \gamma_A^{(2)} + \cdots \tag{19}$$

solving (18) up to order $O(\alpha_s^2)$ we obtain

$$\begin{aligned} C(\mu) &= C(\mu') \exp\left[-\int_{g(\mu)}^{g(\mu')} d\bar{g} \frac{\gamma_A(\bar{g})}{\beta(\bar{g})}\right] \\ &= \left(\frac{\bar{\alpha}_s(\mu')}{\bar{\alpha}_s(\mu)}\right)^{-\frac{\gamma_A^{(1)}}{\beta_1}} \left\{1 + \left[\frac{\bar{\alpha}_s(\mu)}{\pi} - \frac{\bar{\alpha}_s(\mu')}{\pi}\right] \left[\frac{\gamma_A^{(2)}}{\beta_1} - \frac{\beta_2 \gamma_A^{(1)}}{\beta_1^2}\right]\right\} \end{aligned} \tag{20}$$

where $\alpha \beta = \mu \frac{d}{d\mu} \alpha$ and the coefficients β_1 and β_2 are defined as

$$\beta = \left(\frac{\alpha}{\pi}\right) \beta_1 + \left(\frac{\alpha}{\pi}\right)^2 \beta_2 + \cdots \tag{21}$$

so that $\beta_1 = -11/2 + 1/3\, N_f$, $\beta_2 = -51/4 + 19/12\, N_f$ [16] and N_f is the number of unfrozen quark flavours. Notice that the β function in the effective theory is the same as that in the complete QCD theory [10]. This fact has been assumed explicitly in deriving Eq.(20) and will be treated more deeply in Section 5.

The procedure to obtain $\overline{Z}_A$ is now clear:

1. we will calculate the Green function $\overline{G}$ at two loop level in the effective field theory, that is using the Feynman rules in Eq.(8) and Eq.(9). We are interested in its polynomial pole.
2. we will do the same with the heavy quark propagator $\overline{S}_Q$ in the effective theory.
3. we will have to renormalize the results of points 1 and 2 substracting the corresponding counterterms in the effective theory.
4. finally, we will use Eq.(17) and Eq.(16) to obtain $\overline{Z}_Q$ and $\overline{Z}_A$.

4 The calculation

The diagrams that contribute to $\overline{G}^{(4)}(k)$ and $\overline{S}_Q^{(4)}(k)$ are sketched in Fig. 1. In the first part of this section, we present the method we have used to compute Feynman diagrams with static heavy quark lines (hereafter heavy Feynman integrals). The calculation is performed in the Feynman gauge using dimensional regularization to manipulate divergences. We want to draw the reader's attention to two essential characteristics of the effective theory: the presence of heavy quark propagators (see Eq.(8)) that makes heavy Feynman integrals non–invariant under Lorentz rotations, and the fact that gluon lines connecting two fermionic vertices being at least one of them a heavy quark, carries only the zeroth component of the gauge field (see Eq.(9)). These properties will generate some new problems which do not appear in standard Feynman integrals. In some cases, however, they may simplify the calculation. For example, diagram number A.3 in Fig. 1 is exactly zero owing to the three gluon vertex for zeroth component gluons vanishes.

The second part of this section is devoted to the renormalization of the two loop diagrams. It is just conjectured, not demostrated, that the effective field theory is renormalizable [17]. As we will demostrate, the diagrams in Fig. 1 can be renormalized in the $\overline{MS}$ squeme, given rise to quatities that do not depend on logarithms of neither k^2 nor $k^{0\,2}$. Furthermore, the result for each diagram has the structure

$$\begin{aligned} I - I_C &= \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{-k^2}{4\pi\mu^2}\right)^{2\epsilon} \left[\frac{A}{\epsilon^2} + \frac{B}{\epsilon} + \ldots\right] \\ &- \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{-k^2}{4\pi\mu^2}\right)^{\epsilon} \left[\frac{A_C}{\epsilon^2} + \frac{B_C}{\epsilon} + \ldots\right] \end{aligned} \tag{22}$$

where the second term represents the counter diagram and it turns out that diagram by diagram the relation

$$A_C = 2\,A \tag{23}$$

is satisfied. Therefore, the pole part does not depend on μ as it should be

$$I - I_C = \left(\frac{\alpha}{\pi}\right)^2 \left[-\frac{A}{\epsilon^2} + \frac{B - B_C}{\epsilon} + \ldots\right] \tag{24}$$

4.1 A method to calculate heavy Feynman integrals

Self–energy diagrams A.1 to A.6 in Fig. 1 are much simpler to calculate that those for the vertex Green's function. In fact, every two loop integral that appears in diagrams A.1 to A.6 can be performed applying one loop results repeatedly. For instance, diagram A.2 gives rise to the integral

$$\int \frac{d^Dp}{(2\pi)^D} \frac{d^Dq}{(2\pi)^D} \frac{1}{p^2 q^2 (p+q+k)^0 (p+k)^0 (q+k)^0} \tag{25}$$

Using the identity

$$k^0 = (p+k)^0 + (q+k)^0 - (p+q+k)^0 \tag{26}$$

we get integrals which value can be obtained from the results for one loop integrals in Table 1.

As in every two loop calculation, we will need many different types of one loop heavy integrals that it would be well to know in a very early stage of the work. Appendix C is devoted to this crucial task. For the most part, one loop heavy integrals can be derived from a basic general one involving a hypergeometric function with a complicated argument. However, we have also used the Gegenbauer polynomial method (see Appendix A) as well as the integration by parts technique (see Appendix B) to evaluate some specially intricate one loop heavy integrals. In Table 1, Table 2 and Table 3 we give the values of many one loop integrals up to order ϵ^0. Higher orders of ϵ can also be calculated but they are not needed here. As an example of the use of these results, consider the calculation of the anomalous dimension at one loop level. The heavy quark propagator with amputed legs, i.e. the heavy quark self-energy, at one loop level is

$$\begin{aligned} \overline{S}_Q^{(2)}(k) &= -g^2 C_2(R)\, \delta_{\alpha\beta}\, \gamma^0 \int \frac{d^Dp}{(2\pi)^D} \frac{\left[1-(1-a)\frac{p^{0\,2}}{p^2}\right]}{p^2 (p+k)^0} \\ &= \left(\frac{\alpha}{\pi}\right) \frac{C_2(R)}{4} \left(\frac{4k^{0\,2}}{4\pi\mu^2}\right)^\epsilon (3-a)\frac{1}{\epsilon}\left[1+\epsilon(\gamma - 4/(3-a))\right] ik^0\gamma^0 \end{aligned} \tag{27}$$

where $C_R = N^2 - 1/2N$, N being the number of colors. The values of the intermediate integrals have been taken from Table 1 and a is the gauge parameter. In turn, in Table 2, we can found immediately the values of the one loop integrals that arise in the vertex Green's function $\overline{G}^{(2)}(k)$ again with amputed legs

$$\begin{aligned} \overline{G}^{(2)}(k) &= -i g^2 C_2(R)\, \delta_{\alpha\beta} \left\{ \int \frac{d^Dp}{(2\pi)^D} \frac{\gamma^0\, \not p}{p^2 (p-k)^2 p^0} - (1-a) \int \frac{d^Dp}{(2\pi)^D} \frac{(p-k)^0 (\not p - \not k)\, \not p}{p^2 (p-k)^4 p^0} \right\} \\ &= -\left(\frac{\alpha}{\pi}\right) \frac{C_2(R)}{4} \left(\frac{-k^2}{4\pi\mu^2}\right)^\epsilon a\frac{1}{\epsilon}\left[1+\epsilon(\gamma-(1+1/a))\right] \end{aligned} \tag{28}$$

Finally, the light quark self-energy is well-known. It can be found, for example, in ref. [13]

$$S_q^{(2)}(k) = -\left(\frac{\alpha}{\pi}\right) \frac{C_2(R)}{4} \left(\frac{-k^2}{4\pi\mu^2}\right)^\epsilon a\,\frac{1}{\epsilon}\left[1+\epsilon(\gamma-1)\right] i\,\not k \tag{29}$$

From Eq.(27) to Eq.(29) along with Eq.(16)

$$\begin{aligned}
\overline{Z}_Q &= 1 - \left(\frac{\alpha}{\pi}\right) \frac{C_2(R)}{4} (3-a) \frac{1}{\epsilon} \\
\overline{Z}_{\overline{G}} &= 1 + \left(\frac{\alpha}{\pi}\right) \frac{C_2(R)}{4} (3-a) \frac{1}{\epsilon} \\
\overline{Z}_A &= 1 + \left(\frac{\alpha}{\pi}\right) \frac{3\, C_2(R)}{8} \frac{1}{\epsilon}
\end{aligned} \tag{30}$$

which agree with the renormalization constants obtained in refs. [7, 9, 10]. Notice that $\overline{Z}_A$ is gauge invariant at one loop level, as it should be.

Consider now the vertex diagrams in Fig. 1. For instance, the two loop heavy integrals corresponding to diagrams B.1 and B.5 are in the Feynman gauge

$$\int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{\gamma^0 \, \not{p} \gamma^0 \, \not{q}}{p^2 \, q^2 \, (p-k)^2 \, (p-q)^2 \, p^0 \, q^0} \tag{31}$$

and

$$\int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{[-(2p^0 - q^0 - k^0)\gamma^0 + (2\,\not{p} - \not{q} - \not{k})] \, \not{q}}{q^2 \, (q-k)^2 \, (p-k)^2 \, (p-q)^2 \, p^0 \, q^0} \tag{32}$$

respectively. Diagram B.2 leads to an integral whith the same numerator as B.1 but the heavy propagator p^0 being replace by the three momentum one $(q-p+k)^0$. There arises also integrals with only one heavy propagator as in diagrams B.4 and B.6. Even a three heavy propagator denominator appears in diagram B.3, although it can be easily reduced to a sum of integrals with two heavy propagators. Diagrams B.7 to B.9 can be performed with no aditional difficulties making use o one loop results in Tables 1 to 3. On the calculation we have used a mixture of well-known Feynman diagram calculating techniques: integration by parts and Gegenbauer polynomial series. The former has been used to reduce the power degree of the propagators in the denominator of the integrals so that they could be written as a sum of, one hopes, much more simple integrals. In turn, these derived integrals can be evaluated using repeatedly the one loop results in Table 1 to Table 3, parametrizating à la Feynman some propagators or even applying again integration by parts. Notice that we are just interested in the pole polynomials of the diagrams not in their finite parts. On the other hand, the Lorentz structure of the vertex Green function $\overline{G}(k)$ allows many forms factors

$$\overline{G}(k) = A(k) + B(k) \frac{k^0 \gamma^0}{\sqrt{-k^2}} + C(k) \frac{k^i \gamma^i}{\sqrt{-k^2}} + \cdots \tag{33}$$

among which we just have to calculate $A(k)$ because the pole polinomial of the dimensionless Green function $\overline{G}(k)$ cannot depend on k after renormalization. This fact enables us to discard integrals which give rise to form factors different from A. However, often there arises integrals that cannot be completely reduced by integration by parts, for example (31). Then, we will evaluate them expanding the propagators, including the heavy ones, as Gegenbauer polinomial series which allow us to treate separately their radial and angular integrations. Using the ortogonality properties of Gegenbauer polynomials, we usualy can obtain the $1/\epsilon$ poles of the integrals. The details can be found in Appendix A and Appendix B.

4.2 Renormalization

The contributions to the two loop anomalous dimension of the axial current are obtained evaluating the diagrams of Fig. 1 and subtracting the corresponding two loop counterterm diagrams. In the minimal subtruction scheme the latter are obtained by retaining only the $1/\epsilon$ parts in the subdiagrams. Therefore, we have to calculate some one loop superficially divergent diagrams in the effective theory, namely quark self–energy, gluon self–energy, ...etc, retaining only its $1/\epsilon$ part. The pieces in the Lagrangian added to cancel these one loop divergences will give rise to diagrams that subtructed to the two loop diagrams of Fig. 1 will eliminate their logarithmic momentum dependence. In practice, these counterterm diagrams are obtained inserting the counterterm vertices in all possible ways in the one loop diagrams for the corresponding Green's function. Now, a remark should be made about the counterterms in the effective theory. As in it the quark propagator and gluon–quark vextex are different from those in the complete theory, we will have to recalculate some of them. In fact, we need to know the residue of the single pole of the heavy quark self–energy ($\overline{Z}_Q$), of the vertex of a heavy and a light quark with an axial current insertion ($\overline{Z}_{\overline{G}}$) and of the heavy quark gluon vertex ($\overline{Z}_{\overline{\Gamma}}$). The first and the second ones have been already calculated in Eq.(27) and Eq.(28), respectively. For $\overline{Z}_{\overline{\Gamma}}$ the Green's function

$$\overline{\Gamma}(x,y) \equiv < 0 \mid T\left\{ Q_\beta(x)\, B^0_a(0)\, \bar{Q}_\alpha(y) \right\} \mid 0 > = \overline{Z}_{\overline{\Gamma}} \overline{\Gamma}^{(0)} \tag{34}$$

with amputed legs must be calculated at one loop level. It is not difficult to get

$$\begin{aligned} \overline{\Gamma}^{(2)}(k) &= i g^3 \left(\frac{\lambda^a}{2}\right)_{\beta\alpha} \gamma^0 \left\{ \frac{1}{2N} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2\,(p-k)^{0\,2}} \left[1 - (1-a)\frac{p^{0\,2}}{p^2} \right] \right. \\ &+ \left. (1-a)\, N \int \frac{d^D p}{(2\pi)^D} \frac{p^0\,[\,p^{0\,2} - p^2\,]}{p^6\,(p-k)^0} \right\} \\ &= i g \gamma^0 \left(\frac{\lambda^a}{2}\right)_{\beta\alpha} \left\{ -\left(\frac{\alpha}{\pi}\right) \frac{1}{16} \left[\,(3+a)\,C_2(G) - 4\,(3-a)\,C_2(R)\,\right] \frac{1}{\epsilon} \right\} \end{aligned} \tag{35}$$

where $C_G(R) = N$ and again we have made use of Table 1. As we mentioned above, the proper heavy fermionic vertex at one loop level (35) is different from that for the light quark [13]

$$\Gamma^{(2)}(k) = i\,g\,\gamma^{\mu}\left(\frac{\lambda^{a}}{2}\right)_{\beta\alpha}\left\{-\left(\frac{\alpha}{\pi}\right)\frac{1}{16}\left[(3+a)\,C_2(G) + 4a\,C_2(R)\right]\frac{1}{\epsilon}\right\} \tag{36}$$

These are the one loop divergent diagrams we will need to renormalize the two loop diagrams in Fig. 1 whose final contributions are tabulated in Table 5 and Table 6.

5 The results

In this section we present our final results in a unified way. From Table 5 and Table 6, it is simple to derive the expresions for $\overline{S}_{Q}^{(4)}(k)$ and $\overline{G}^{(4)}(k)$. The corresponding renormalization constants in the Feynman gauge are

$$\begin{aligned} Z_{\overline{G}} &= 1 + \left(\frac{\alpha}{\pi}\right)\frac{C_2(R)}{2}\frac{1}{\epsilon} + \left(\frac{\alpha}{\pi}\right)^{2}\frac{C_2(R)}{64}\left\{\left[18\,C_2(G) - 8\,C_2(R) - 8\,T(R)\,N_f\right]\frac{1}{\epsilon^{2}}\right. \\ &+ \left.\left[\left(\frac{37}{3}C_2(G) - C_2(R) - \frac{20}{3}T(R)N_f\right) + \xi(2)\left(14\,C_2(G) - 8\,C_2(R)\right)\right]\frac{1}{\epsilon}\right\} \end{aligned} \tag{37}$$

$$\begin{aligned} Z_{Q} &= 1 - \left(\frac{\alpha}{\pi}\right)\frac{C_2(R)}{2}\frac{1}{\epsilon} - \left(\frac{\alpha}{\pi}\right)^{2}\frac{C_2(R)}{64}\left\{\left[18\,C_2(G) - 8\,C_2(R) - 8\,T(R)\,N_f\right]\frac{1}{\epsilon^{2}}\right. \\ &- \left.\left[\frac{76}{3}C_2(G) - \frac{32}{3}T(R)N_f\right]\frac{1}{\epsilon}\right\} \end{aligned} \tag{38}$$

where $T(R) = 1/2$ for the $SU(N)$ gauge group. These equations are the main results of this work. As it is well–known, the residue of the double pole $1/\epsilon^{2}$ of a renormalization constant is completely determinated by the residues of its single pole at one loop level. In fact, writting a renormalization constant Z as

$$Z = 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{\alpha}{\pi}\right)^{m}\frac{Z_n^{(m)}}{\epsilon^{n}} \tag{39}$$

the renormalization group allows us to write [13]

$$\begin{aligned} Z_2^{(2)} &= -\frac{1}{4}\left[\gamma_Z^{(1)} - \frac{11}{6}C_2(G) + \frac{2}{3}T(R)\,N_f\right. \\ &+ \left.\left(\left(\frac{13}{3} - a\right)\frac{C_2(G)}{4} - \frac{2}{3}T(R)\,N_f\right)a\,\frac{\partial}{\partial a}\right]Z_1^{(1)} \end{aligned} \tag{40}$$

where $\gamma_Z = -\mu\frac{d}{d\mu}\ln Z$, is the anomalous dimension of Z. If

$$\gamma_Z = \left(\frac{\alpha}{\pi}\right)\gamma_Z^{(1)} + \left(\frac{\alpha}{\pi}\right)^{2}\gamma_Z^{(2)} + \cdots \tag{41}$$

we have

$$\gamma_Z^{(1)} = -2\,Z_1^{(1)} \qquad \gamma_Z^{(2)} = -4\,Z_1^{(2)} \tag{42}$$

Using the renomalization constants at one loop level for an arbitrary gauge given in Eq.(30), it is very easy to checked that both $\overline{Z}_G$ and $\overline{Z}_Q$ pass the test of the double pole.

In order to obtain the renormalization constant for the axial current, $\overline{Z}_A$, we need to know that for the light quark self-energy at two loops. It has been calculated by Egoryan and Tarasov [14] and confirmed by Tarrach [15]

$$
\begin{aligned}
Z_q &= 1+\left(\frac{\alpha}{\pi}\right)\frac{C_2(R)}{4}\frac{1}{\epsilon}+\left(\frac{\alpha}{\pi}\right)^2\frac{C_2(R)}{64}\left\{\left[4\,C_2(G)+2\,C_2(R)\right]\frac{1}{\epsilon^2}\right.\\
&+ \left.\left[17\,C_2(G)-3\,C_2(R)-4\,T(R)N_f)\right]\frac{1}{\epsilon}\right\}
\end{aligned}
\tag{43}
$$

From Eq.(16), Eq.(37), Eq.(38) and Eq(43), we get the renormalization constant for the axial current

$$
\begin{aligned}
\overline{Z}_A &= 1+\left(\frac{\alpha}{\pi}\right)\frac{3\,C_2(R)}{8}\frac{1}{\epsilon}+\left(\frac{\alpha}{\pi}\right)^2\frac{C_2(R)}{64}\left\{\left[11\,C_2(G)+\frac{9}{2}C_2(R)-4\,T(R)\,N_f\right]\frac{1}{\epsilon^2}\right.\\
&+ \left.\left[\left(\frac{49}{6}C_2(G)-\frac{5}{2}C_2(R)-\frac{10}{3}T(R)N_f\right)+\xi(2)\left(14\,C_2(G)-8\,C_2(R)\right)\right]\frac{1}{\epsilon}\right\}
\end{aligned}
\tag{44}
$$

and its anomalous dimension

$$
\gamma_A^{(1)} = \frac{3\,C_2(R)}{4} = -1 \tag{45}
$$

$$
\begin{aligned}
\gamma_A^{(2)} &= -\frac{C_2(R)}{32}\left[\left(\frac{49}{3}C_2(G)-5\,C_2(R)-\frac{20}{3}T(R)\,N_f\right)\right.\\
&+ \left.8\,\xi(2)\left(\frac{7}{2}C_2(G)-2\,C_2(R)\right)\right] = -\frac{1}{36}\left[\frac{127}{2}+94\,\xi(2)-5\,N_f\right]
\end{aligned}
\tag{46}
$$

where the values in the righthand side are for $N=3$.

As we mentioned in Section 3, during our whole calculation we have assumed that the β function for heavy quarks is the same up to two loops as that for light quarks, i.e. the strong coupling of a gluon to the heavy quark runs the same way as the strong coupling for the light fields. This is a consequence of gauge invariance. Politzer and Wise [10] have verified this explicitly at one loop level using the method of the background gauge field. Now, we want to repeat the calculation but in a much easier way. The coupling α of a heavy quark to a gluon renormalizes

$$
\alpha_0 = \overline{Z}_\Gamma^2\, Z_{3YM}^{-1}\, \overline{Z}_Q^{-2}\,\alpha \equiv \overline{Z}_\alpha\,\alpha \tag{47}
$$

where Z_{3YM} is the renormalization constant for the gluon self-energy [14]

$$
Z_{3YM} = 1-\left(\frac{\alpha}{\pi}\right)\left\{\frac{C_2(R)}{8}\left(\frac{13}{3}-a\right)-\frac{T(R)}{3}N_f\right\}\frac{1}{\epsilon} \tag{48}
$$

and $\overline{Z}_{\Gamma}$ has been calculated explicitly at one loop level in Eq.(35). Hence

$$\overline{Z}_{\alpha} = 1 + \left(\frac{\alpha}{\pi}\right) \left[\frac{11}{12} C_2(G) - \frac{T(R)}{3} N_f\right] \frac{1}{\epsilon} \tag{49}$$

which generates the usual β function. Finally, for completeness, $\overline{Z}_{\Gamma}$ can be obtained at two loop level using the Slavnov–Taylor identity

$$\frac{\overline{Z}_{\Gamma}}{\overline{Z}_Q} = \frac{\tilde{Z}_3}{\tilde{Z}_1} \tag{50}$$

where $\tilde{Z}_1$ and $\tilde{Z}_3$ are the renormalization constants for the ghost self–energy and ghost–gluon vertex respectively which can be found in [13, 14]. In the Feynman gauge we found

$$\begin{aligned} \overline{Z}_{\Gamma} &= 1 + \left(\frac{\alpha}{\pi}\right) \frac{1}{16} [(3+a) C_2(G) - 4(3-a) C_2(R)] \frac{1}{\epsilon} \\ &+ \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{64} \left\{ \left[\frac{17}{2} C_2^2(G) + 8 C_2^2(R) - 26 C_2(R) C_2(G) + 8 C_2(R) T(R) N_f \right.\right. \\ &- \left. 2 C_2(G) T(R) N_f \right] \frac{1}{\epsilon^2} + \frac{1}{3} \left[\frac{67}{4} C_2^2(G) - 76 C_2(R) C_2(G) \right. \\ &+ \left.\left. 32 C_2(R) T(R) N_f - 5 C_2(G) T(R) N_f \right] \frac{1}{\epsilon} \right\} \end{aligned} \tag{51}$$

Notice that we have used the ghost renormalization constants derived in the complete theory. This is correct because as the ghost only interacts with the gluon, the presence of a heavy quark is irrelevant at two loop level. Notice also that the one loop term of $\overline{Z}_{\Gamma}$ in Eq.(51) agrees with that we have obtained directly in Eq.(35), as it should be.

6 The decay constant of the B meson

The decay constant of the B meson f_B is one of the most important quantities in studing heavy–light quark systems. There has been some attempts to measure f_B on the lattice [18, 19] although the values obtained have rather big systematic errors. In order to get the physical value of f_B in the continuum, it is required to compute the radiative corrections for the axial current (10), which is the interpolating operator for the B meson, on the lattice à la Eichten, i.e. in the limit $m_Q \to \infty$, and in the $\overline{MS}$ continuum renormalization scheme in the complete QCD theory, i.e. for finite m_Q. Both problems have been studied in detail in refs. [7] and [20]. The basic point is to realize that the use of the static quark propagator of Eq.(1) corresponds to take as the largest mass–scale in the calculations not the ultraviolet cut–off a^{-1}, but the heavy quark mass, m_Q. The calculation involves two steps. In a first step the conection between the effective operator on the lattice and that in the continuum at the renormalisation point $\mu = a^{-1}$ is determined. Despite some theoretical differences, the relationship is [20]

$$\overline{A}_{\rho}^{\overline{MS}}(\mu = a^{-1}) = \left(1 - 8.76\,\frac{\tilde{\alpha}_s(a^{-1})}{\pi}\right) \overline{A}_{\rho}^{LATT}(a^{-1}) \tag{52}$$

In a second step, how to pass from the continuum operator in the effective theory to that in the complete QCD is needed. As we have discussed in Section 2, this requires to sum the logarithms $\ln(m_Q/\mu)$, i.e. to know the factor $C(\mu)$. This problem has been solved in refs. [9, 10] at one loop level. Using Eq.(20) and our result for the two loop anomalous dimension of the axial current (45)

$$\begin{aligned} Z_f &= \left\{\frac{\tilde{\alpha}_s(m_b)}{\tilde{\alpha}_s(a^{-1})}\right\}^{1/\beta_1} \left(1 - \frac{\tilde{\alpha}_s(a^{-1})}{\pi} 8.76\right) C(m_b, g(m_b)) \\ &\times \left\{1 + \frac{\tilde{\alpha}_s(a^{-1}) - \tilde{\alpha}_s(m_b)}{\pi} \left(\frac{\gamma_A^{(2)}}{\beta_1} - \frac{\beta_2\,\gamma_A^{(1)}}{\beta_1^2}\right)\right\} \end{aligned} \tag{53}$$

where $f_B^{CONT} \equiv Z_f\, f_B^{LATT}$. If $\alpha_s(a^{-1}) - \alpha_s(m_Q)$ is of order $\alpha_s^2(m_Q)$, it will be sufficient to know the anomalous dimension up to one loop level. This has been the assumption implicitly made in every lattice calculation to date. It leads to $Z_f \simeq 0.8$. This supposition appears to be logical for the values of the lattice spacing which are usually used, $a^{-1} \sim 2 - 3$ GeV, but it has not been established as a fact up to now. Therefore, in order to understand completely the results extracted from the lattice and make larger extrapolations, the two loop anomalous dimension is needed [19]. In Table 6 we have tabulated the value of the two loop correction to Z_f

$$\triangle Z_f \equiv \frac{\tilde{\alpha}_s(a^{-1}) - \tilde{\alpha}_s(m_b)}{\pi} \left(\frac{\gamma_A^{(2)}}{\beta_1} - \frac{\beta_2\,\gamma_A^{(1)}}{\beta_1^2}\right) \tag{54}$$

for some values of N_f, number of unfrozen quark flavours, Λ_{QCD} and for $a^{-1} = 2, 4$ and 6 Gev. As can be seen, in all cases it is small. For $N_f = 4$, $\triangle Z_f$ is less than 2%. Therefore, the impact of $\gamma_A^{(2)}$ in the constant Z_f is weak, less than 1.5%, and in practice no renormalization group improvement up to order $O(\alpha/\pi)$ is needed to relate the matrix elements of A_ν and $\overline{A}_\nu(a^{-1})$.

7 Conclusions

We have calculated at two loop level the renormalization constant for the axial current in the effective field theory of Eichten. Its anomalous dimension turns out to be

$$\gamma_A = -\left(\frac{\alpha}{\pi}\right) - \frac{1}{36}\left[\frac{127}{2} + 94\,\xi(2) - 5\,N_f\right]\left(\frac{\alpha}{\pi}\right)^2 + \cdots \tag{55}$$

We also have determined the self–energy renormalisation constant for a heavy quark and for the coupling of a heavy quark to a gluon in this theory. The two loop anomalous dimension of the axial current turns out to be small so that the dependence on it of the value of f_B in the continuum obtained from measurement on the lattice, is in practice irrelevant. We have developed a technique to perform higher loop calculations in the effective theory for heavy quarks evaluating many new Feynman integrals involving one o more heavy quark proagators. Finally, we have demostrated that, at two loop level, the effective theory can be renormalized with only a few modifications in the counterterms involving heavy quarks. We hope that this calculation may help to understand more deeply the theory of heavy quarks that can be a source of very interesting and important results both in its continuum and discreted form.

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A The Gegenbauer's polynomial method

The Gegenbauer's method is a generalization to $D = 4 + 2\epsilon$ dimensions of the old Chebyshev polynomial expansion of Feynman propagators [21]. The goal is to avoid Feynman parametrization that for loops with more than two lines leads to very difficult parametric integrals involving hypergeometric functions and polylogarithms. It prescribes to expand p–space euclidean propagators in the denominator of massless two loop integrals in terms of the so–called Gegenbauer polinomials [22, 23, 24]. Doing that, one splits the D–dimensional integration into an often trivial radial integration and an angular one that usualy can be performed using the ortogonality properties of these polynomials. However, sometimes this method leads to so complicated angular integrals which make this technique useless. Chetyrkin *et al.* have used a x–space version of the same method to deal with non–planar diagrams, i.e. diagrams with at least a line carrying three momenta [25]. Unfortunately, this technique cannot be applied to integrals with heavy propagators because it is necessary to turn to x-space using a Fourier transformation. As it is demostrated below, it is easier to expand also the heavy quark propagator in p–space as a Gegenbauer's serie. As far as we know, this is the first work where this method is applied to non–covariant Feynman integrals. In the remainder of this section, we always consider integrals defined over euclidean space.

A.1 Summary of Gegenbauer polynomial mathematical properties

The Gegenbauer polynomials $C_n^\lambda(x)$ have as generating function [26, 27]

$$\left(1 - 2xt + t^2\right)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x)\, t^n \tag{56}$$

Their general expresion is

$$C_n^\lambda(x) = (-2)^{-n} \left(1 - x^2\right)^{-\lambda+1/2} \frac{\Gamma(2\lambda+n)\,\Gamma(\lambda+1/2)}{\Gamma(2\lambda)\,\Gamma(\lambda+1/2+n)\,n!} \frac{d^n}{dx^n} \left(1 - x^2\right)^{n+\lambda-1/2} \tag{57}$$

and they satisfy the recurrence formula

$$2\,(n+\lambda)\,x\,C_n^\lambda(x) = (n+1)\,C_{n+1}^\lambda(x) + (n-1+2\lambda)\,C_{n-1}^\lambda(x) \tag{58}$$

Their parity behaviour is done by

$$C_n^\lambda(-x) = (-)^n\, C_n^\lambda(x) \tag{59}$$

Some special and very useful cases are

$$C_0^\lambda(x) \;=\; 1$$

$$
\begin{aligned}
C_1^\lambda(x) &= 2\lambda x \\
C_2^\lambda(x) &= -\lambda + 2\lambda(\lambda+1)x^2 \\
C_n^\lambda(1) &= \frac{\Gamma(n+2\lambda)}{n!\,\Gamma(2\lambda)} \\
C_n^\lambda(0) &= \begin{cases} 0 & n = 2m+1 \\ \dfrac{(-)^m\,\Gamma(\lambda+m)}{m!\,\Gamma(\lambda)} & n = 2m \end{cases}
\end{aligned}
$$

The most important expansions we will use are [24, 27]

$$
\begin{aligned}
\frac{1}{[(p-q)^2]^\alpha} &= \frac{1}{(p\,q)^\alpha}\frac{\Gamma(\lambda)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha)}{\Gamma(k+\lambda)}\,(T(p,q))^{k+\alpha} \\
&\times\ {}_2F_1[\alpha-\lambda\,,\,\alpha+k\,;\,\lambda+k+1\,;\,T(p,q)^2]\,C_k^\lambda(\hat{p}\cdot\hat{q}) \\
\frac{1}{[(p-q)^0]^\alpha} &= \frac{(-)^{\frac{\alpha}{2}}}{p^\alpha}\frac{\Gamma(\lambda)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{(k+\lambda)\,\Gamma(k+\alpha)}{\Gamma(1+k+\lambda)\,2^k}\,\frac{(-)^{-\frac{k}{2}}}{(1-(q^0/p)^2)^{\frac{\alpha+k}{2}}} \\
&\times\ {}_2F_1\left[\frac{\alpha+k}{2}\,,\,\lambda+\frac{k-\alpha+1}{2}\,;\,\lambda+k+1\,;\,\frac{1}{(1-(q^0/p)^2)}\right]C_k^\lambda(\hat{p}\cdot\hat{e}_0) \\
\frac{1}{[p^0]^\alpha} &= \frac{(-)^{\frac{\alpha}{2}}}{p^\alpha}\frac{\Gamma(\lambda)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{(-)^{-\frac{k}{2}}\,(k+\lambda)\,\Gamma(k+\alpha)\,\sqrt{\pi}}{\Gamma(\frac{1+k+\alpha}{2})\,\Gamma(\lambda+1+\frac{k-\alpha}{2})\,2^k}\,C_k^\lambda(\hat{p}\cdot\hat{e}_0) \qquad (60)
\end{aligned}
$$

where ${}_2F_1$ is the confluent hypergeometric function, $p = |\,\vec{p}\,| = (p^2)^{\frac{1}{2}}$, $\hat{p} = \vec{p}/p$, $\hat{e}_\mu$ is an unitary vector on the direction μ and $\lambda = D/2-1$.

The ortogonality relation that they satisfy is

$$
\int d\Omega_b\, C_n^\lambda(\hat{a}\cdot\hat{b})\, C_m^\lambda(\hat{b}\cdot\hat{c}) = \delta_{mn}\,\frac{2\,\pi^{1+\lambda}}{\Gamma(1+\lambda)}\,\frac{\lambda}{\lambda+n}\,C_n^\lambda(\hat{a}\cdot\hat{c}) \qquad (61)
$$

Very often one has to perform angular integrals involving three Gegenbauer polinomials

$$
W(n,m,l) \equiv \frac{1}{\Omega_D}\int d\Omega_p\, C_n^\lambda(\hat{q}\cdot\hat{x}_1)\, C_m^\lambda(\hat{q}\cdot\hat{x}_2)\, C_l^\lambda(\hat{q}\cdot\hat{x}_3) \qquad (62)
$$

Unfortunately, W is only known for $l=1$ [25]

$$
\begin{aligned}
W(n,m,1) &= \sum_{\sigma\geq 0}\frac{\lambda\,(n-1-2\sigma+\lambda)}{(n+\lambda)\,(m+\lambda)}\,(\,\delta_{m,n-1}-\delta_{m,n+1}\,) \\
&\times\ C_1^\lambda(\hat{x}_1\cdot\hat{x}_3)\,C_{n-1-2\sigma}^\lambda(\hat{x}_1\cdot\hat{x}_2) + (m \Longleftrightarrow n\,,\,x_1 \Longleftrightarrow x_2) \qquad (63)
\end{aligned}
$$

A.2 Calculating two loop heavy integrals using the Gegenbauer method

A very important example of the use of the Gegenbauer polynomials is the two loop integral

$$
\mathcal{A} = \int \frac{d^Dp}{(2\pi)^D}\,\frac{d^Dq}{(2\pi)^D}\,\frac{1}{q^2\,(p-k)^2\,(p-q)^2\,p^0\,q^0} \qquad (64)
$$

$\mathcal{A}$ appears in three irreducible diagrams for the heavy–light quark vertex, namely B.1, B.2 and B.5. If we try to evaluate $\mathcal{A}$ integrating first over q using the formulae in Appendix C, we will have to face with a second integral now over p which involves a hypergeometric function with a complicated momentum dependent argument. However, making the expansion of the propagators in terms of Gegenbauer polinomials we render $\mathcal{A}$ almost trivial to be calculated. In fact, using the series in Eq.(60), we get

$$\begin{aligned}
\mathcal{A} &= \frac{i^2\pi}{k\,(2\pi)^{2D}}\Gamma^4(\lambda)\sum_{j_1,j_2,k_1=0}^{\infty}\sum_{j_3,j_4,k_2=0}^{\infty}(-)^{-j_1/2}(-)^{-j_3/2}\frac{1}{2^{j_1+j_3}}\\
&\times \frac{(j_1+\lambda)\Gamma(j_1+1)}{\Gamma(1+j_1/2)\Gamma(1/2+j_1/2+\lambda)}\frac{(j_3+\lambda)\Gamma(j_3+1)}{\Gamma(1+j_3/2)\Gamma(1/2+j_3/2+\lambda)}\\
&\times \frac{\Gamma(1+j_2)\Gamma(1+j_2+\lambda)\Gamma(1+k_1-\lambda)\Gamma(1+k_1+j_2)}{\Gamma(j_2+\lambda)\Gamma(1-\lambda)\Gamma(1+j_2)\,k_1!\,\Gamma(1+k_1+j_2+\lambda)}\\
&\times \frac{\Gamma(1+j_4)\Gamma(1+j_4+\lambda)\Gamma(1+k_2-\lambda)\Gamma(1+k_2+j_4)}{\Gamma(j_4+\lambda)\Gamma(1-\lambda)\Gamma(1+j_4)\,k_2!\,\Gamma(1+k_2+j_4+\lambda)}\\
&\times \int_0^\infty dp\,p^{D-4}\,[T(p,k)]^{1+j_4+2k_2}\int_0^\infty dq\,q^{D-5}\,[T(q,p)]^{1+j_2+2k_1}\\
&\times \int d\Omega_p\,C^\lambda_{j_3}(\hat{p}\cdot\hat{e_0})\,C^\lambda_{j_4}\left(\hat{p}\cdot\hat{k}\right)\int d\Omega_q\,C^\lambda_{j_1}(\hat{q}\cdot\hat{e_0})\,C^\lambda_{j_2}(\hat{q}\cdot\hat{p})
\end{aligned}\tag{65}$$

where we have made used of the definition of the hypergeometric function as a power serie. Equation (64) looks formidable but soon it will undergo a great simplification. Firstly, the radial integration can be easily performed

$$\begin{aligned}
&\int_0^\infty dp\,p^{D-4}\,[T(p,k)]^{1+j_4+2k_2}\int_0^\infty dq\,q^{D-5}\,[T(q,p)]^{1+j_2+2k_1} =\\
&\quad k^{2D-8}\frac{(-2)\,[2k_1+j_2+1]}{(D-3+2k_1+j_2)\,(D-5-2k_1-j_2)}\,\frac{(-2)\,[2k_2+j_4+1]}{(2D-6+j_4+2k_2)\,(2D-8-j_4-2k_2)}
\end{aligned}\tag{66}$$

After doing that, we can analyse the ϵ–pole structure of $\mathcal{A}$. The radial integral generates a $1/\epsilon$ term for $j_4 = k_2 = 0$, otherwise it is of order of ϵ^0. On the other hand, in the denominator of Eq.(64) there are two Γ functions which argument is $1-\lambda = -\epsilon$. However, if $k_1 = k_2 = 0$ these singular Gamma functions will be canceled by the same functions in the numerator. Therefore, the ϵ–structure of $\mathcal{A}$ is

Order of $\mathcal{A}$	$k_1 = 0$, $k_2 = 0$	$k_1 = 0$, $k_2 \neq 0$	$k_1 \neq 0$, $k_2 = 0$	$k_1 \neq 0$, $k_2 \neq 0$
$j_4 = 0$	$O(1/\epsilon)$	$O(\epsilon)$	$O(\epsilon^0)$	$O(\epsilon^2)$
$j_4 \neq 0$	$O(\epsilon^0)$	$O(\epsilon)$	$O(\epsilon)$	$O(\epsilon^2)$

As we are interested only in the ϵ–pole polynomial of $\mathcal{A}$, the case we must to study is $j_4 = 0$, $k_1 = 0$ and $k_2 = 0$. Secondly, the angular integration over $\hat{q}$ is immediate using Eq.(60). The next angular integration, now over $\hat{p}$, is trivial because of in the case we are dealing with the Gegenbauer's polynomial with index j_4 desappears (see Eq.(60)) and we can use again Eq.(60). The result of these manipulations is

$$\begin{aligned}\mathcal{A}_{1/\epsilon} &= \frac{\pi\,\Omega_D^2}{(2\pi)^{2D}}\,\frac{\lambda^2\,\Gamma^4(\lambda)}{\Gamma(2\lambda)}\,k^{2D-8} \\ &\times \sum_{j_1=0}^{\infty}\frac{1}{4^{2j_1+1}}\,\frac{\Gamma^2(2j_1+2)\,\Gamma(2j_1+2\lambda+1)}{\Gamma^2(3/2+j_1)\,\Gamma^2(1+j_1+\lambda)\,\Gamma(2j_1+\lambda+1)\,\Gamma(\lambda)} \\ &\times \frac{(-2)^2\,2\,(j_1+1)}{(D-2+2j_1)\,(D-6-2j_1)\,(2D-6)\,(2D-8)}\end{aligned} \tag{67}$$

where Ω_D is the volume of an unit sphere in D-dimensions. Taking only its $1/\epsilon$ residue, we obtain

$$\mathcal{A} = -\frac{2}{(4\pi)^4}\,\frac{1}{\epsilon}\sum_{m=1}^{\infty}\frac{1}{m^2} = -\frac{2}{(4\pi)^4}\,\frac{1}{\epsilon}\,\xi(2) \tag{68}$$

which is the pole–polynomial of $\mathcal{A}$.

Two loop integrals with tensor structure of momenta can, sometimes, be calculated with this method. Consider the Minkowskian integral which arises from diagram B.1 in Fig. 1

$$\mathcal{B} = \int \frac{d^Dp}{(2\pi)^D}\,\frac{d^Dq}{(2\pi)^D}\,\frac{\not p\,\not q}{p^2\,q^2\,(p-k)^2\,(p-q)^2\,p^0\,q^0} \tag{69}$$

Notice that in the product $\not p\,\not q$ only $p^0\,q^0$ and $p^i\,q^j\,\gamma_i\,\gamma_j$, with i,j runing from 1 to $D-1$, contribute to the vertex form factor we are interested in. In fact, it is easy to check that the crossing terms p^iq^0 and p^0q^i give rise to form factors like $k^i\,O(1/\sqrt{k^2},1/k^0)$. Therefore, using that the integral of the term $p^i\,q^j$ must be symmetric under the exchange of the indices i and j, we can make the sustitution

$$\not p\,\not q \longrightarrow (p\cdot q) = \frac{1}{2}\left[p^2 + q^2 - (p-q)^2\right] \tag{70}$$

in the numerator of $\mathcal{B}$, reducing its calculation to evaluate already known integrals, including $\mathcal{A}$. The diagram B.2 can be computed the same way. Now, however, as a proof of the consistence of our two loop calculation, we would like to evaluate $\mathcal{B}$ using the Gegenbauer series given in Eq.(60). The p^0q^0 piece gives rise to an usual Feynman integral easily doable

$$\mathcal{B}_{p^0q^0} = \frac{1}{(4\pi)^4}\left(\frac{k^2}{4\pi}\right)^{2\epsilon}\frac{1}{2\epsilon^2}\left[1+2\epsilon(\gamma-5/2)\right] \tag{71}$$

The non–trivial calculation is, of course, that generated by p^iq^j. Going to Euclidean space and expanding first only the propagators involving the loop momentum q, we get

$$\mathcal{B}_{p^iq^j} = \frac{i}{(2\pi)^{2D}}\,\frac{\Gamma^2(\lambda)}{2\lambda}\sum_{j_1,j_2,k_1=0}^{\infty}(-)^{-j_1/2}\,\frac{\sqrt{\pi}}{2^{j_1}}\,\frac{(j_1+\lambda)\,\Gamma(j_1+1)}{\Gamma(1+j_1/2)\,\Gamma(1/2+j_1/2+\lambda)}$$

$$\times \frac{\Gamma(1+j_2)\Gamma(1+j_2+\lambda)\Gamma(1+k_1-\lambda)\Gamma(1+k_1+j_2)}{\Gamma(j_2+\lambda)\Gamma(1-\lambda)\Gamma(1+j_2)\,k_1!\,\Gamma(1+k_1+j_2+\lambda)}$$
$$\times \frac{(-2)\,[2k_1+j_2+1]}{(D-2+2k_1+j_2)(D-4-2k_1-j_2)} \int dp\, d\Omega_p\, p^{2D-6} \frac{\gamma_i\gamma_j \hat{p}^i \hat{p}^j}{(p-k)^2 p^0} \widetilde{W}(\hat{p}\cdot\hat{e}_0) \quad (72)$$

with

$$\widetilde{W}(\hat{p}\cdot\hat{e}_0) \equiv \frac{\lambda\,\Omega_D}{(j_1+\lambda)(j_2+\lambda)} \sum_{\sigma\geq 0} \Big[(j_1-1-2\sigma+\lambda)(\delta_{j_2,j_1-1}-\delta_{j_2,j_1+1})C^{\lambda}_{j_1-1-2\sigma}(\hat{p}\cdot\hat{e}_0)$$
$$+ \quad (j_2-1-2\sigma+\lambda)(\delta_{j_1,j_2-1}-\delta_{j_1,j_2+1})C^{\lambda}_{j_2-1-2\sigma}(\hat{p}\cdot\hat{e}_0)\Big] \quad (73)$$

where we have expressed q^i as a Gegenbauer polynomial of index 1 and then we have perfomed the resulting angular integration using Eq.(62). The numerator of the remainder integral over p is

$$\hat{p}^i\hat{p}^j\gamma_i\gamma_j = -1 + \hat{p}^{0\,2} \quad (74)$$

Hence, $B_{p^iq^j}$ can be descomposed into two pieces. The first one gives rise to the integral

$$\int dp\, d\Omega_p\, p^{2D-6} \frac{1}{(p-k)^2 p^0} \widetilde{W}(\hat{p}\cdot\hat{e}_0) \quad (75)$$

and the second one to

$$\int dp\, d\Omega_p\, p^{2D-7} \frac{\hat{p}^0}{(p-k)^2} \widetilde{W}(\hat{p}\cdot\hat{e}_0) \quad (76)$$

In order to avoid having to evaluate difficult angular integrals involving three Gegenbauer polynomials, we compute before the radial integrals because then we could discard terms which do not contribute to the pole–polynomial of B [23]. This often implies to set at least one of the indices j_1, j_2, k_1 equal to zero, which simplies strongly the angular integration which now can be done using the ortogonality relation (60). Sometimes, however, this is imposible and integrals such as (61) must be evaluated making the method useless. Fortunately, it is not our case. In fact, expanding the remainder propagators and performing only radial integrals we find

$$B_{p^iq^j} \propto \sum_{j_1,j_2,k_1=0} \sum_{j_3,j_4,k_2=0} \frac{\Gamma(k_1+1-\lambda)\Gamma(k_2+1-\lambda)}{\Gamma^2(1-\lambda)} \frac{k^{2D-8}}{(D-4-j_2-2k_1)(2D-8-j_4-2k_2)}$$
$$\times \int d\Omega_p\, C^{\lambda}_{j_4}\left(\hat{p}\cdot\hat{k}\right) \widetilde{W}(\hat{p}\cdot\hat{e}_0) \left\{A\, C^{\lambda}_{j_3}(\hat{p}\cdot\hat{e}_0) + B\, C^{\lambda}_{1}(\hat{p}\cdot\hat{e}_0)\right\} \quad (77)$$

where we have write only the terms that are important to analyse the pole structure of $B_{p^iq^j}$. All other contributions are in constants A and B. From Eq.(76) we find that $B_{p^iq^j}$ can contain a pole $1/\epsilon^2$ only for $k_1 = k_2 = 0$

$B_{p^iq^j}$	$j_2 = 0$	$j_2 \neq 0$
$j_4 = 0$	$O(1/\epsilon^2)$	$O(1/\epsilon)$
$j_4 \neq 0$	$O(1/\epsilon)$	$O(\epsilon^0)$

otherwise it is order $O(\epsilon^0)$ at least. It is easy to check that $\widetilde{W}$ vanishes if $j_2 = 0$ as it is the scalar product of two ortogonal Gegenbauer polinomials. Therefore, we just have to calculate the case $j_4 = 0$ and $j_2 \neq 0$ where angular integrations are trivial. After some elemental algebra, we get

$$B_{p^i q^j} = \frac{1}{(4\pi)^4} \frac{1}{\epsilon} \left\{ \frac{3}{2}\xi(2) + \frac{1}{2}(1 - \xi(2)) \right\} \tag{78}$$

where we have displayed explicitly the contributions coming from the two pieces of $B_{p^i q^j}$. With Eq.(70) and Eq.(77) we can obtain the final result which is exactly the same as that found using the sustitution (69).

A.3 Calculating one loop heavy integrals using the Gegenbauer method

Not only two loop integrals can be performed with this method but also important one loop ones do. For example, consider

$$C \equiv \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 (p-k)^2 (p^0)^{-2\epsilon}} \tag{79}$$

Expanding the propagators and performing the trivial radial and angular integrals, we get

$$\begin{aligned} C &= \frac{\Omega_D}{(2\pi)^D} \frac{\lambda \Gamma^2(\lambda)}{\Gamma(1-\lambda)\Gamma(-2\epsilon)} (-)^\epsilon k^{D-4+4\epsilon} \sum_{j_1, k_1 = 0}^{\infty} (-)^{-j_1/2} \frac{\sqrt{\pi}}{2^{j_1}} \\ &\times \frac{\Gamma(\lambda + j_1 + 1)\Gamma(1 - \lambda + k_1)\Gamma(1 + j_1 + k_1)\Gamma(j_1 - 2\epsilon)}{\Gamma(1 + j_1 + k_1 + \lambda) k_1! \Gamma(j_1 + \lambda)\Gamma(1/2 + j_1/2 - \epsilon)\Gamma(1 + j_1/2 + \lambda + \epsilon)} \\ &\times \frac{(-2)(2k_1 + j_1 + 1)}{(D - 2 + 2\epsilon + j_1 + 2k_1)(D - 4 - j_1 - 2k_1)} C_{j_1}^{\lambda}\left(\hat{k} \cdot \hat{e_0}\right) \end{aligned} \tag{80}$$

Its pole structure is very simple: only for $j_1 = k_1 = 0$, C has a $1/\epsilon$ pole, otherwise it is order ϵ^1 at least. As we are insterested in the value of C up to order ϵ^0, we will take $j_1 = k_1 = 0$. In this case, C simplies to

$$C = -\frac{1}{(4\pi)^{2+\epsilon}} k^{4\epsilon} (-)^\epsilon \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2} - \epsilon)\Gamma(2 + 2\epsilon)(1 + 2\epsilon)} \frac{1}{2\epsilon} \tag{81}$$

which leads to the same result as that calculated directly from the basic one loop formula (113) in Appendix C, as it should be. The value of C is quoted in Eq.(126).

To finish this section we will present briefly an interesting and powerful way to evaluate integrals with propagators powered to indices depending on ϵ. For instance, consider

$$\mathcal{D} \equiv \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^{-\epsilon} (p-k)^2 p^0} \tag{82}$$

The basic point is to realize that from Eq.(113) in Appendix C we can know to all orders in ϵ the integral

$$\mathcal{D}' \equiv \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p-k)^2 p^0} \tag{83}$$

Expanding the $(p-k)^2$ propagator in $\mathcal{D}'$, it is a simple exercise to found

$$\begin{aligned}\mathcal{D}' &= -\frac{2}{(4\pi)^2}k^0\frac{1}{\epsilon}\left[1+(\gamma-3/2-\ln 4\pi+\ln(k^2/\mu^2))\right] \\ &+ \frac{2}{(4\pi)^2}k\sum_{j=1}^{\infty}(-)^j\frac{1}{j(j+2)}C^1_{1+2j}(\hat{k}\cdot\hat{e}_0)\end{aligned} \tag{84}$$

Comparing Eq.(83) with the value of $\mathcal{D}'$ up to order ϵ^0 (see Table 1), we can sum the Gegenbauer serie

$$\sum_{j=1}^{\infty}(-)^j\frac{1}{j(j+2)}C^1_{1+2j}(x) = x\left[\frac{1}{2}-\ln 4x^2\right] \tag{85}$$

Equation (84) can be explicitly checked in the case $x=1$. Now, notice that $\mathcal{D}$ and $\mathcal{D}'$ differ just from their corresponding radials integrations. In fact, the value of the radial integral for $\mathcal{D}'$ can be obtained from that of $\mathcal{D}$ just making the sustitution $D \to D+2\epsilon$. This does not modify its pole structure and only the residue of $1/\epsilon$, easily computable, changes. Therefore, the ϵ^0 part of $\mathcal{D}$ is the same as that for $\mathcal{D}'$ which we already know and the value of $\mathcal{D}$ up to order of ϵ^0 can be found. It is tabulated in Table 2. The same method can be used when some index is $n-\epsilon$. In this case, the unknown integral can be obtained from that with index n following the same steps as above.

B Integration by parts Method

This method was introduced first by Chetyrkin and Tkachov [28] (see also [29]) in order to calculate β–functions in 4 loops. However, as we will demostrate, its basic idea is so simple and powerful that it can be applied to many different types of integrals including those with heavy propagators. The starting identity of this technique is

$$0 = \int \frac{d^D p}{(2\pi)^D} \frac{\partial}{\partial p^\rho} f(p,k) \tag{86}$$

where $f(p,k)$ is whatever function of the loop momentum p and of the external momentum k. Equation (85) holds in dimensional regularization [30]. In fact, it can be considered as a consequence of translational invariance of dimensionally regularised integrals in p–space [28]. Choosing appropiate functions f, Eq.(85) can generate many very important algebraic identities that would enable us to express a non–trivial n–loop integral through diagrams of much simpler structure, usually known to all orders in ϵ. The basic trick is to manage to get in the numerator of the identity (85) combinations of momenta that also appear in the denominator so that they can cancel each other. Furthermore, through this procedure, a set of recursive relations can be constructed which will make possible to evaluate diagrams with propagators powered to high indices from a few basic integrals with propagators powered to low indices. For instance, consider the integral

$$[1,p^\mu]\,[\alpha\,\beta\,\gamma] \equiv \int \frac{d^D p}{(2\pi)^D} \frac{[1,p^\mu]}{p^{2\,\alpha}\,(p-k)^{2\,\beta}\,p^{0\,\gamma}} \tag{87}$$

where here "α" stands here by the factor $p^{2\,\alpha}$ in the denominator, "β" by $(p-k)^{2\,\beta}$ and so on. This is a very convenient notation when using integration by parts. It is easy to check that

$$\frac{d}{dp^\rho}\,\{p^\rho\,[\,\alpha\,\beta\,\gamma\,]\} = (D-2\alpha-\beta-\gamma)\,[\,\alpha\,\beta\,\gamma\,] - \beta\,[\,\alpha-1\,\beta+1\,\gamma\,] + \beta\,k^2\,[\,\alpha\,\beta+1\,\gamma\,] \tag{88}$$

where here the notation of square brackets refers to the integrant rather than to the integral and repeated index ρ are understood to be summed. Using Eq.(85), we find an algebraic relation among integrals

$$(D-2\alpha-\beta-\gamma)\,[\,\alpha\,\beta\,\gamma\,] = \beta\,[\,\alpha-1\,\beta+1\,\gamma\,] - k^2\,\beta\,[\,\alpha\,\beta+1\,\gamma\,] \tag{89}$$

where we have used that $2(a\cdot b) = a^2 + b^2 - (a-b)^2$. Other useful relations can be generated using $(p-k)^\rho$, $p^\mu\,p^\rho$, $p^\mu\,(p-k)^\rho$, $g^{\mu\rho}$, $g^{0\rho}$, $p^\mu\,g^{0\rho}$ and $p^0\,g^{0\rho}$ instead of p^ρ as functions f in Eq.(85)

$$(D-\alpha-2\beta-\gamma)\,[\,\alpha\,\beta\,\gamma\,] = \alpha\,[\,\alpha+1\,\beta-1\,\gamma\,] - \gamma\,k^0\,[\,\alpha\,\beta\,\gamma+1\,] - k^2\,\alpha\,[\,\alpha+1\,\beta\,\gamma\,] \tag{90}$$

$$(D-2\alpha-\beta-\gamma+1)p^{\mu}\,[\alpha\beta\gamma] = \beta p^{\mu}\,[\alpha-1\beta+1\gamma] - k^2\beta p^{\mu}\,[\alpha\beta+1\gamma] \tag{91}$$

$$(D-\alpha-2\beta-\gamma+1)p^{\mu}\,[\alpha\beta\gamma] = \alpha p^{\mu}\,[\alpha+1\beta-1\gamma] - \gamma k^0 p^{\mu}\,[\alpha\beta\gamma+1]$$
$$-k^2\alpha p^{\mu}\,[\alpha+1\beta\gamma] + k^{\mu}\,[\alpha\beta\gamma] \tag{92}$$

$$2\alpha p^{\mu}\,[\alpha+1\beta\gamma] + 2\beta p^{\mu}\,[\alpha\beta+1\gamma] = 2\beta k^{\mu}\,[\alpha\beta+1\gamma] - \gamma g^{\mu 0}\,[\alpha\beta\gamma+1] \tag{93}$$

$$2\alpha\,[\alpha+1\beta\gamma-1] + 2\beta\,[\alpha\beta+1\gamma-1] = 2\beta k^0\,[\alpha\beta+1\gamma] - \gamma\,[\alpha\beta\gamma+1] \tag{94}$$

$$2\alpha p^{\mu}\,[\alpha+1\beta\gamma-1] + 2\beta p^{\mu}\,[\alpha\beta+1\gamma-1] = 2\beta k^0 p^{\mu}\,[\alpha\beta+1\gamma]$$
$$-\gamma p^{\mu}\,[\alpha\beta\gamma+1] + g^{\mu 0}\,[\alpha\beta\gamma] \tag{95}$$

$$2\alpha\,[\alpha+1\beta\gamma-2] + 2\beta\,[\alpha\beta+1\gamma-2] = 2\beta k^0\,[\alpha\beta+1\gamma-1]$$
$$-\gamma\,[\alpha\beta\gamma] + [\alpha\beta\gamma] \tag{96}$$

Notice that the formal manipulations of the divergent expressions we are dealing with are correct within dimensional regularization [28].

In performing some $[\alpha\beta\gamma]$ integrals, a second type of independent relations are needed because integration–by–parts equations (88) to (95) are not sufficient to reduce them to a sum of already known integrals. We can add to Eq.(88) to Eq.(95) other algebraic relations obtained by parametric differentiation with respect to the external momentum k. By parametric derivation we mean

$$\frac{\partial}{\partial k^{\mu}}\int\frac{d^Dp}{(2\pi)^D}\,f(p,k) = \int\frac{d^Dp}{(2\pi)^D}\,\frac{\partial}{\partial k^{\mu}}\,f(p,k) \tag{97}$$

which again is correct within dimensional regularization [30]. For $[\alpha\beta\gamma]$ we can get the additional relations

$$\frac{d}{dk^{\mu}}\,[\alpha\beta\gamma] = 2\beta p^{\mu}\,[\alpha\beta+1\gamma] - 2\beta k^{\mu}\,[\alpha\beta+1\gamma] \tag{98}$$

$$\frac{d}{dk^{\mu}}\,[\alpha\beta\gamma] = -2\alpha p^{\mu}\,[\alpha+1\beta\gamma] - 2\gamma g^{\mu 0}\,[\alpha\beta\gamma+1] \tag{99}$$

$$\frac{d}{dk^0}\,[\alpha\beta\gamma] = 2\beta\,[\alpha\beta+1\gamma-1] - 2\beta k^0\,[\alpha\beta+1\gamma] \tag{100}$$

$$\frac{d}{dk^0}\,[\alpha\beta\gamma] = -2\alpha\,[\alpha+1\beta\gamma-1] - 2\gamma\,[\alpha\beta\gamma+1] \tag{101}$$

In Appendix C we make use of these relations among integrals $[\alpha\beta\gamma]$ to obtain their expression for various values of the indices α, β and γ up to order of ϵ^0.

Consider now the Feynman's integral corresponding to diagram B.5 in Fig. 1

$$[2p^i-q^i-k^i]\gamma_i\,\not{q}[1\,2\,3\,4\,5\,6] \equiv$$
$$\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\,\frac{[-(2p^0-q^0-k^0)\gamma^0 + (2\not{p}-\not{q}-\not{k})]\,\not{q}}{q^2(q-k)^2(p-k)^2(p-q)^2p^0q^0} \tag{102}$$

where "1" stands by q^2, "2" by $(q-k)^2$, and so on. It is worth descomposing the factor $\not q$ in the numerator into its zeroth and non-zeroth vector components: $q^0\gamma^0 + q^j\gamma_j$. In fact, it is easy to check that the piece of the numerator coming from q^0 will give rise to a contribution proportional to $k^i\gamma_i \times O(1/\sqrt{k}, 1/k^0)$. Obviously, these form factors do not contribute to the anomalous dimension and can be negleted. Therefore, the three pieces to be calculated are $p^i q^j \gamma_i \gamma_j$, $q^i q^j \gamma_i \gamma_j$ and $k^i q^j \gamma_i \gamma_j$. Regarding the first one, integration by parts with $p^\mu q^\mu (p-q)^\rho$ as function f in Eq.(85), enables us to write

$$\begin{aligned}(D-3)p^\mu q^\mu[1\,2\,3\,4\,5\,6] &= q^\mu q^\mu[1\,2\,3\,4\,5\,6] + p^\mu q^\mu[1\,2\,3^2\,4^0\,5\,6] \\ &- p^\mu q^\mu[1\,2^0\,3^2\,4\,5\,6] - p^\mu q^\mu[1\,2\,3\,4\,5^2\,6^0]\end{aligned} \tag{103}$$

where integrals in the righthand side of (102) are either already known, as the third, or much simpler than that in the lefthand side. The more difficult integral to calculate is the first one, that is the same as the second piece in the numerator of [123456]. We can write it as

$$q^i q^j \gamma_i \gamma_j = q^2 - q^{0\,2} \tag{104}$$

On one hand, the term $q^{0\,2}$ generates an integral involving a heavy propagator in the denominator whose pole part can be extracted using a Feynman's parametrization. On the other hand, the q^2 piece gives rise to an integral that looks very difficult owing to the presence of two heavy propagators, p^0 and q^0. In order to evaluate it, we can use again integration by parts that allows us to write

$$(D-4)[1^0\,2\,3\,4\,5\,6] = [1^0\,2\,3^2\,4^0\,5\,6] - [1^0\,2^0\,3^2\,4\,5\,6] - [1^0\,2\,3\,4\,5^2\,6^0] \tag{105}$$

where we have choosed in this case $(p-q)^\rho$ as function f in Eq.(85). Each integral in the righthand side of Eq.(104) can be solved using repeatedly the results for one loop integrals in Table 1. Notice that we have to expand the Gamma functions up to order $O(\epsilon^2)$ due to the factor 2ϵ in the lefthand side of Eq.(104). After some algebra, the final result is

$$[1^0\,2\,3\,4\,5\,6] = -\frac{1}{(4\pi)^4}\frac{4}{\epsilon}\,\xi(2) \tag{106}$$

The Gegenbauer method could also have been used to evaluate this integral. The final results are exactly the same as they should be. Finally, the third piece in Eq.(101) does not contribute to the pole polynomial of diagram B. 5 which turns out to be

$$[2p^i - q^i - k^i]\gamma_i \not q[1\,2\,3\,4\,5\,6] = \frac{1}{(4\pi)^4}\frac{3}{2\,\epsilon}(1 - 4\xi(2)) \tag{107}$$

C One loop integrals with heavy propagators

To perform the two loop integrals needed to obtain the anomalous dimension of the axial current, it is necessary first to evaluate many one loop integrals involving one or more static heavy quark propagators. In this Appendix, we present briefly how to calculate the most important ones.

C.1 Basic one loop heavy integrals

Consider

$$IG_2(\alpha,\beta;b^2;a) \equiv \int \frac{d^Dp}{(2\pi)^D} \frac{1}{\left[p^2 - b^2 + i\epsilon\right]^\beta \left[(p-a+i\epsilon)^0\right]^\alpha} \tag{108}$$

with $D = 4 + 2\epsilon$ the space–time dimension and α and β arbitrary real numbers. As it will be seen below, many useful one loop heavy integrals in dimensional regularization can be obtained from IG_2 as particular cases.

In order to evaluate IG_2 we first perform a Wick rotation going into a D–dimensional Euclidean space. The integrant of IG_2 is invariant under rotations in the $D-1$–dimensional subspace p_i with $i = 1 \ldots D-1$. Hence, we can integrate trivialy over the direction of the $(D-1)$–dimensional momentum $\vec{p}$

$$\begin{aligned} IG_2(\alpha,\beta;b^2;a) &= i^{\alpha-1}(-)^{-\beta} \frac{2\,\pi^{(D-1)/2}}{\Gamma((D-1)/2)\,(2\pi)^D} \int_{-\infty}^{+\infty} dp^D \frac{1}{\left[p^D - a^D\right]^\alpha} \\ &\times \int_0^\infty d\,|\,\vec{p}\,|\, \frac{|\,\vec{p}\,|^{D-2}}{\left[|\,\vec{p}\,|^2 + p^{D\,2} + b^2\right]^\beta} \end{aligned} \tag{109}$$

where $i\,p^D = p^0$ and $i\,a^D = a^0$. The radial integral can be performed by means of the well–known formula

$$\int_0^\infty dp \frac{p^{n-1}}{\left[p^2 + \rho^2\right]^m} = \frac{\Gamma(n/2)\,\Gamma(m-n/2)}{2\,\Gamma(m)} \left(\rho^2\right)^{n/2-m} \tag{110}$$

which is valid for $m - n/2 > 0$. Note that for values of m and n where the integral is ill–defined, its divergences will occur as poles of the Gamma functions. As $\Gamma(m)$ has an unique analitic continuation, the result of the integral in the left side of Eq.(109) can be defined to be, for all n and m, the right side of Eq.(109). Then, we get

$$\begin{aligned} IG_2(\alpha,\beta;b^2;a) &= (-)^{-\beta}\,i^{-\alpha+1} \frac{\Gamma(\beta-(D-1)/2)}{\Gamma(\beta)\,2^D\,\pi^{(D+1)/2}} \\ &\times \int_{-\infty}^{+\infty} dp^0 \frac{1}{\left[p^D - a^D\right]^\alpha \left[\left(p^D\right)^2 + b^2\right]^{\beta-(D-1)/2}} \end{aligned} \tag{111}$$

The integral over p^D can be performed using the residue theorem with some remarks. In fact, choosing as contour the upper semicircle, we find that the integral in Eq.(110) is equal to $2\pi i\, Res(p^D = i\,b)$, where Res is the residue of the integrant at the singularity $p^D = i\,b$. This residue is easy to calculate for positive integer powered factors in the denominator of the integrant. In fact, then its singularities are just poles whose contribution can be obtained as it is well-known, differentiating the non-singular integrant with respect to p^D at the pole. Doing that, it turns out that the residue can be written as a hypergeometric confluent function

$$
\begin{aligned}
& Res(p^D = i\,b\,;\, 1/(p^{D\,2} + b^2)^n\,(p^D + i\,a^0)^m\,) = \\
& \qquad (-i)\, i^{-m}\, \frac{\Gamma(2n+m-1)}{\Gamma(n)\,\Gamma(n+m)}\, \frac{1}{[a+b]^{2n+m-1}}\; {}_2F_1(2n+m-1\,,\, n\,;\, n+m;\, \frac{a^0 - b}{a^0 + b}) \qquad (112)
\end{aligned}
$$

where we have made use of the integral representation of the hypergeometric function which is valid for $n + m > n \geq 0$. In the case when the integral (110) is divergent, we can use Eq.(111) to define its value for all α and β. As above, divergences will show up as poles in the Gamma functions. The final expresion for IG_2 is

$$
\begin{aligned}
IG_2(\alpha,\beta;b^2;a) &= \frac{i\,(-)^{-\beta-\alpha}}{2^{D-1}\,\pi^{(D-1)/2}}\, \frac{\Gamma(\alpha + 2\beta - D)}{\Gamma(\beta)\,\Gamma(\alpha + \beta - (D-1)/2)}\, [2\,b]^{D-\alpha-2\beta} \\
&\times\; {}_2F_1\left[\alpha + 2\beta - D\,,\, \alpha\,;\, \alpha + \beta - (D-1)/2\,;\, \frac{1}{2}\left(1 - \frac{a^0}{b}\right)\right] \qquad (113)
\end{aligned}
$$

where we have made a transformation of the argument of the hypergeometric function. Notice that Eq.(112) has the correct limit as $\alpha \to 0$, namely the usual basic formula of dimensional regularization [13].

A specially useful case of Eq.(112) arises taken $b = 0$. Then, the hypergeometric function disappears and Eq.(112) simplifies to

$$
\begin{aligned}
I_2(\alpha,\beta;a) &\equiv \int \frac{d^Dp}{(2\pi)^D}\, \frac{1}{(p^2 + i\epsilon)^\beta\ [(p - a + i\epsilon)^0]^\alpha} \\
&= \frac{i\,(-)^{-\beta-\alpha}}{2^{D-1}\,\pi^{(D-1)/2}}\, \frac{\Gamma(\alpha + 2\beta - D)\,\Gamma(D - 2\beta)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma((D+1)/2 - \beta)}\, \left[a^0\right]^{D-\alpha-2\beta} \qquad (114)
\end{aligned}
$$

where we have used that

$$
{}_2F_1[a,\, b;\, c;\, 1] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad (115)
$$

In Table 1 we have listed the results for the integral I_2 for some values of α and β up to order ϵ^0. Note that their logarithmic dependence is on $4k^{0\,2}/\mu^2$ instead of the usual k^2/μ^2 factor. This is due to the non-rotacional invariance of the I_2 integrals.

Sometimes, we need to know also the value of

$$I2_3(\alpha,\beta,\gamma;k) \equiv \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^{2\alpha}(p-k)^{0\beta}p^{0\gamma}} \tag{116}$$

Using a Feynman's parametrization for the heavy propagators in Eq.(115), $I2_3$ becomes an I_2 integral which can be evaluated by means of Eq.(113)

$$I2_3(\alpha,\beta,\gamma;k) = \frac{i(-)^{-\gamma-\beta-\alpha}}{2^{D-1}\pi^{(D-1)/2}} \frac{\Gamma(2\alpha+\beta+\gamma-D)\Gamma(D-2\alpha-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma((D+1)/2-\alpha)} \left[k^0\right]^{D-2\alpha-\beta-\gamma} \tag{117}$$

Finally, we look at integrals of the type of I_2 but with tensor structures of p in their numerators. They are rather straightforward. For instance, consider the integral I_2 with a factor p^μ in its numerator, i.e. I_2^μ. It is obvious that only for $\mu = 0$ it is not zero. But we can rewrite p^0 as $(p^0 - a^0) + a^0$. The first piece gives rise to an integral I_2 with the index α decreased by one. The second piece, however, is trivialy a^0 times the integral I_2 itself. Therefore

$$I_2^\mu(\alpha,\beta;a) = g^{\mu 0}\left\{I_2(\alpha-1,\beta;a) + a^0 I_2(\alpha,\beta;a)\right\} \tag{118}$$

This method can be aplied in a judicious way to more complicated tensor structure in the numerator and, of course, to IG_2.

C.2 General one loop integral with a heavy propagator

The results of the previous part will allow us to obtain the value of the most important one loop integral that we often meet in both one and two loop calculations with static heavy quarks.

Consider

$$[I_3\,,\,I_3^\mu](\alpha,\beta,\gamma;k) \equiv \int \frac{d^Dp}{(2\pi)^D} \frac{[1\,,p^\mu]}{p^{2\alpha}(p-k)^{2\beta}p^{0\gamma}} \tag{119}$$

More complicated tensor structures in the numerator could also been treated. However, we do not consider them here so as not to go into somewhat tedious details that do not give us further information about the method of calculating integrals like (118). Notice that for $\gamma = 0$, I_3 reduces to a standard one loop integral without heavy propagators which expresion is well-known, (for a complete list, see [13]).

At first glance, the way to perform I_3 and I_3^μ is clear: a Feynman's parametrization of the two non-heavy propagators transforms (118) into IG_2 integrals, which can be done using Eq.(112). However, a technical problem arises: an usually very difficult parametric integral involving a hypergeometric function must be evaluated up to the order of ϵ desired (see Eq.(112)). In order to evaluate I_3 and I_3^μ for different values of the indices α, β and γ we will use both expansion of their propagators in terms of Gegenbauer polinomials and integration by parts. The idea is to use the algebraic relations, valid to all orders in ϵ, (88) to (95) of Appendix B which will allow us to get many different integrals from a few basic ones.

To begin with, consider the integral $I_3(1,1,1)$. The method of Gegenbauer polynomials, described in detail in Appendix A, is specially easy to apply in this case. In fact, it allows us to write

$$\begin{aligned}\int \frac{d^Dp}{(2\pi)^D}\frac{1}{p^2\,(p-k)^2\,p^0} &= \frac{i}{(4\pi)^2}\frac{2}{k}\sum_{j=0}^{\infty}(-)^j\frac{1}{(1+j)^2}C^1_{1+2j}(k^0/k) + O(\epsilon)\\ &\equiv \frac{i}{(4\pi)^2}\frac{2}{k}\Phi(k^0/k)\end{aligned} \tag{120}$$

Note that as $I_3(1,1,1)$ is finite, the results for $\alpha = 1-\epsilon$, $\beta = 1-\epsilon$ and $\gamma = 1-\epsilon$ must be equal to the righthand side of Eq.(119) up to order ϵ^0. Now, we would like to sum the Gegenbauer's serie Φ. Integration by parts gives us the solution in a very elegant way. On the one hand, differentiating $I_3(1,1,1)$ with respect to k^0 we get

$$\frac{1}{2}\frac{d}{dk^0}I_3(1,1,1) = I_3(1,2,0) - k^0\,I_3(1,2,1) \tag{121}$$

On the other hand, Eq.(88) in Appendix B allows us to write

$$2\epsilon\,I_3(1,1,1) = I_3(0,2,1) - k^2\,I_3(1,2,1) \tag{122}$$

Eliminating $I_3(1,2,1)$ between Eq.(120) and Eq.(121), we get up to order ϵ^0

$$\left[(1-x^2)\frac{d}{dx} - x\right]\Phi(x) = -\ln(-4x^2) \tag{123}$$

where $x = \hat{k}\cdot\hat{e}_0 = k^0/k$. The solution of this differential equation is

$$\Phi(x) = \frac{1}{\sqrt{1-x^2}}\left\{2\int_0^x dt\,\frac{\sin^{-1}t}{t} - \sin^{-1}(x)\,\ln(-4\,x^2)\right\} \tag{124}$$

Integrals of the type of I_3 with some of their indices greater than 1 can be obtained either differentiating $I_3(1,1,1)$ with respect to k^0 or using the relations derived by integration by parts (see Appendix B). The same method is valid for I_3^μ. For example,

$$\begin{aligned}I_3(1,2,1) &= \frac{1}{2k^0}\left\{2\,I_3(1,2,0) - \frac{d}{dk^0}I_3(1,1,1)\right\}\\ &= \frac{1}{k^2}\left\{I_3(0,2,1) - (D-4)\,I_3(1,1,1)\right\}\\ I_3(1-\epsilon,1,2) &= -\left\{2\,(1-\epsilon)\,I_3(2-\epsilon,1,0) + \frac{d}{dk^0}I_3(1-\epsilon,1,1)\right\}\\ I_3^\mu(1,1,1) &= \frac{1}{(D-3)}\left[I_3^\mu(0,2,1) - k^2\,I_3^\mu(1,2,1)\right]\end{aligned}$$

These integrals in turn can be used to generate other with greater exponents. When evaluating I_3^μ, we need the function $\Delta(x)$ defined by

$$k^2 \left[\frac{d}{dk^\mu} - \frac{k^\mu}{k^0}\frac{d}{dk^0}\right]\left[\frac{1}{k}\Phi\left(\frac{k^0}{k}\right)\right] \equiv \left[\frac{k^\mu}{k^0} - g^{\mu 0}\right]\left\{\ln\left(\frac{4k^{02}}{-k^2}\right) - \Delta\left(\frac{k^0}{k}\right)\right\} \qquad (125)$$

therefore

$$\Delta(x) = \frac{x}{1-x^2}\left[\Phi(x) - x\ln(-4x^2)\right] \qquad (126)$$

Another class of useful integrals are those with $\gamma = -2\epsilon$. Its basic integral is $I_3(1,1,-2\epsilon)$, which can be calculated up to order ϵ^0 using Eq.(112). In fact, in this case the hypergeometric function is $1+O(\epsilon^2)$ so that we have only to consider the factor 1 which gives rise to a very simple parametric integral. The result is

$$I_3(1,1,-2\epsilon) = -\frac{i}{(4\pi)^2}(-k^2)^\epsilon\left(\frac{-k^2}{4\pi}\right)^\epsilon \frac{1}{2\epsilon}[1+\epsilon(\gamma-4)]\,4^{-\epsilon} + O(\epsilon) \qquad (127)$$

It can also be computed expanding the propagators in Gegenbauer series (see Eq.(79) in Appendix A for details). Again, integration by parts allows us to obtain many other integrals with $\gamma = -2\epsilon$. For example,

$$I_3(2,1,-2\epsilon) = \frac{1}{k^2}\left\{(D-4)\,k^0 I_3(1,1,1-2\epsilon) - (D-3+2\epsilon)\,I_3(1,1,-2\epsilon)\right\} \qquad (128)$$

Finally, consider integrals with $\alpha = -\epsilon$ or $\beta = -\epsilon$. Starting from $I_3(-\epsilon,1,1)$, whatever integral of this type can be obtained. This integral has been calculated in Appendix A and some derived results are

$$\begin{aligned} I_3(-\epsilon,1,2) &= \frac{1}{2k^0}\left\{(D-4)\,k^2 I_3(1-\epsilon,1,1) - 2\,(D-3+\epsilon)\,I_3(-\epsilon,1,1)\right\} \\ I_3(1,-\epsilon,1) &= \frac{(D-4)}{2\,(D-3+\epsilon)}\left\{k^2 I_3(1,1-\epsilon,1) - I_3(0,1-\epsilon,1)\right\} \end{aligned} \qquad (129)$$

In Tables 2 and 3 we give the values up to order ϵ^0 of many integrals of the type of I_3 and I_3^μ which we have to know to evaluate two loop integrals.

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Figure captions

Figure 1: Diagrams which contribute (A) to the self–energy of a heavy quark and (B) to the heavy–light quark vertx with an insertion of the axial current.

Table captions

Table 1: Special values of the integral $I_2(\alpha, \beta; k)$ in terms of

$$\frac{1}{\epsilon_0} \equiv \frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\left(\frac{4k^{0\,2}}{\mu^2}\right)$$

Table 2: Special values of the integral $I_3(\alpha, \beta, \gamma; k)$ in terms of $1/\epsilon_0$, defined in Table 1, and

$$\frac{1}{\not\epsilon} \equiv \frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\left(\frac{-k^2}{\mu^2}\right)$$

Table 3: Special values of the integral

$$\frac{(4\pi)^2}{i} I_3^{\mu}(\alpha, \beta, \gamma; k) \equiv g^{\mu 0} \mathcal{A} + \left[\frac{k^{\mu}}{k^0} - g^{\mu 0}\right] \mathcal{B}$$

Table 4: Renormalized values of the diagrams for the heavy quark self–energy.

Table 5: Renormalized values of the diagrams for the axial current vertex.

Table 6: $\triangle Z_f$ for different values of N_f, Λ_{QCD} and a^{-1}.

α	β	$\frac{(4\pi)^2}{i} (k^0)^{\alpha+2\beta-4} I_2(\alpha,\beta;k)$
1	1	$\frac{2}{\epsilon_0} - 4$
2	1	$\frac{2}{\epsilon_0}$
3	1	2
1	2	$-\frac{1}{\epsilon_0}$
2	2	$\frac{1}{\epsilon_0} - 2$
3	2	$-\frac{1}{\epsilon_0} + 3$
1	$1-\epsilon$	$\frac{1}{\epsilon_0} - 4 + \ln\left(\frac{-4\,k^{0\,2}}{\mu^2}\right)$
2	$1-\epsilon$	$\frac{1}{\epsilon_0} + \ln\left(\frac{-4\,k^{0\,2}}{\mu^2}\right)$
1	$2-\epsilon$	$-\frac{1}{2\epsilon_0} - \frac{1}{2} - \frac{1}{2}\ln\left(\frac{-4\,k^{0\,2}}{\mu^2}\right)$
2	$2-\epsilon$	$\frac{1}{2\epsilon_0} - \frac{3}{2} + \frac{1}{2}\ln\left(\frac{-4\,k^{0\,2}}{\mu^2}\right)$
$1-2\epsilon$	1	$\frac{1}{\epsilon_0} - 4 + \ln\left(\frac{k^{0\,2}}{\mu^2}\right)$
$1-2\epsilon$	2	$-\frac{1}{\epsilon_0} - \ln\left(\frac{k^{0\,2}}{\mu^2}\right)$
$2-2\epsilon$	1	$\frac{1}{\epsilon_0} + 2 + \ln\left(\frac{k^{0\,2}}{\mu^2}\right)$
$2-2\epsilon$	2	$\frac{1}{\epsilon_0} - 2 + \ln\left(\frac{k^{0\,2}}{\mu^2}\right)$
-2ϵ	1	-1
-2ϵ	2	$\frac{1}{2\epsilon_0} + \frac{1}{2}\ln\left(\frac{k^{0\,2}}{\mu^2}\right)$

Table 1: Integral I_2 up to order ϵ^0

α	β	γ	k-Factor	$\frac{(4\pi)^2}{i} \times k - Factor \times I_3(\alpha,\beta,\gamma;k;0)$
1	1	1	$\sqrt{k^2}$	$\Phi\left(\frac{k^0}{\sqrt{k^2}}\right)$
1	2	1	$k^0\, k^2$	$\frac{1}{\epsilon_0}$
2	1	1	$\frac{k^4}{k^0}$	$\frac{2}{\not\epsilon} - 2\ln\left(\frac{4\, k^{0\,2}}{-k^2}\right)$
1	1	2	k^2	$-\frac{2}{\not\epsilon} + 2\ln\left(\frac{4\, k^{0\,2}}{-k^2}\right)$
1	1	$-2\,\epsilon$	1	$-\frac{1}{2\,\not\epsilon} + 2 - \frac{1}{2}\ln\left(\frac{-k^2}{4\mu^2}\right)$
1	2	$-2\,\epsilon$	k^2	$\frac{1}{\epsilon_0} + \ln\left(\frac{-k^2}{4\mu^2}\right)$
2	1	$-2\,\epsilon$	k^2	$\frac{1}{2\,\not\epsilon} + \frac{1}{2}\ln\left(\frac{-k^2}{4\mu^2}\right)$
$-\epsilon$	1	1	$\frac{1}{k^0}$	$-\frac{1}{\epsilon_0} + 3 - \ln\left(\frac{4\, k^{0\,2}}{\mu^2}\right)$
$-\epsilon$	2	1	k^0	$\frac{1}{\not\epsilon} + \ln\left(\frac{4\, k^{0\,2}}{\mu^2}\right)$
$-\epsilon$	1	2	1	$\frac{1}{\epsilon_0} + \ln\left(\frac{4\, k^{0\,2}}{\mu^2}\right)$
1	$-\epsilon$	1	$\frac{1}{k^0}$	1
2	$-\epsilon$	1	$\frac{k^2}{k^0}$	-2
1	$-\epsilon$	2	1	$-\frac{1}{\not\epsilon} - \ln\left(\frac{k^2}{\mu^2}\right)$

Table 2: Integral I_3 up to order ϵ^0

α	β	γ	$k-Factor$	$k-Factor \times A$	$k-Factor$	$k-Factor \times B$
1	1	1	1	$-\frac{1}{\not\epsilon} + 2$	1	$+\Delta$
1	2	1	k^2	$\frac{1}{\not\epsilon}$	k^2	$\frac{1}{\epsilon_0} - \Delta$
2	1	1	k^2	$\frac{1}{\not\epsilon}$	k^2	$+\Delta$
1	1	-2ϵ	$\frac{1}{k^0}$	$-\frac{1}{4\not\epsilon} + \frac{1}{2} - \frac{1}{4}\ln\left(\frac{-k^2}{4\mu^2}\right)$	$\frac{1}{k^0}$	$-\frac{1}{4\not\epsilon} + 1 - \frac{1}{4}\ln\left(\frac{-k^2}{4\mu^2}\right)$
1	2	-2ϵ	$\frac{k^2}{k^0}$	$\frac{1}{\not\epsilon} - 1 + \ln\left(\frac{k^{02}}{\mu^2}\right)$	$\frac{k^2}{k^0}$	$\frac{1}{\not\epsilon} - 1 + \ln\left(\frac{k^{02}}{\mu^2}\right)$
2	1	-2ϵ	$\frac{k^2}{k^0}$	1	$\frac{k^2}{k^0}$	1
1	$-\epsilon$	1	$\frac{1}{k^2}$	$-\frac{1}{4}$	$\frac{1}{k^{02}}$	$\frac{1}{2}$
1	$-\epsilon$	2	$\frac{1}{k^0}$	1	$\frac{1}{k^0}$	-1
2	$-\epsilon$	1	1	$\frac{1}{2\not\epsilon} - \frac{1}{2} + \frac{1}{2}\ln\left(\frac{k^2}{\mu^2}\right)$	0	
$-\epsilon$	1	1	$\frac{1}{k^2}$	$-\frac{1}{4}$	$\frac{1}{k^{02}}$	$-\frac{1}{\epsilon_0} + \frac{5}{2} - \ln\left(\frac{4k^{02}}{\mu^2}\right)$
$-\epsilon$	2	1	1	$\frac{1}{2\not\epsilon} - \frac{1}{2} + \frac{1}{2}\ln\left(\frac{k^2}{\mu^2}\right)$	1	$\frac{1}{\epsilon_0} + \ln\left(\frac{-k^2}{\mu^2}\right)$
$-\epsilon$	1	2	$\frac{1}{k^0}$	$-\frac{1}{\epsilon_0} + 3 - \ln\left(\frac{4k^{02}}{\mu^2}\right)$	$\frac{1}{k^0}$	$\frac{1}{\epsilon_0} + 1 + \ln\left(\frac{4k^{02}}{\mu^2}\right)$

Table 4: Integral I_3^μ up to order ϵ^0

SELF-ENERGY DIAGRAM	VALUE[a]		
	COLOR	$1/\epsilon^2$	$1/\epsilon$
$A.1$	$\frac{C_2^2(R)}{4}$	$\frac{1}{2}$	1
$A.2$	$\frac{C_2(R)}{4N}$	$\frac{1}{2}$	$\frac{1}{2}$
$A.3$	$\frac{C_2(R)\,C_2(G)}{32}$	0	0
$A.4$	$\frac{C_2(R)\,T(R)}{8}\,N_f$	-1	$-\frac{4}{3}$
$A.5$	$\frac{C_2(R)\,C_2(G)}{32}$	0	1
$A.6$	$\frac{C_2(R)\,C_2(G)}{32}$	5	$\frac{23}{6}$

[a](It must be multiplied by $i\,(\alpha/\pi)^2\,\delta_{\alpha\beta}\,k^0\,\gamma^0$)

Table 5: Renormalized two loop diagrams for the heavy quark self–energy

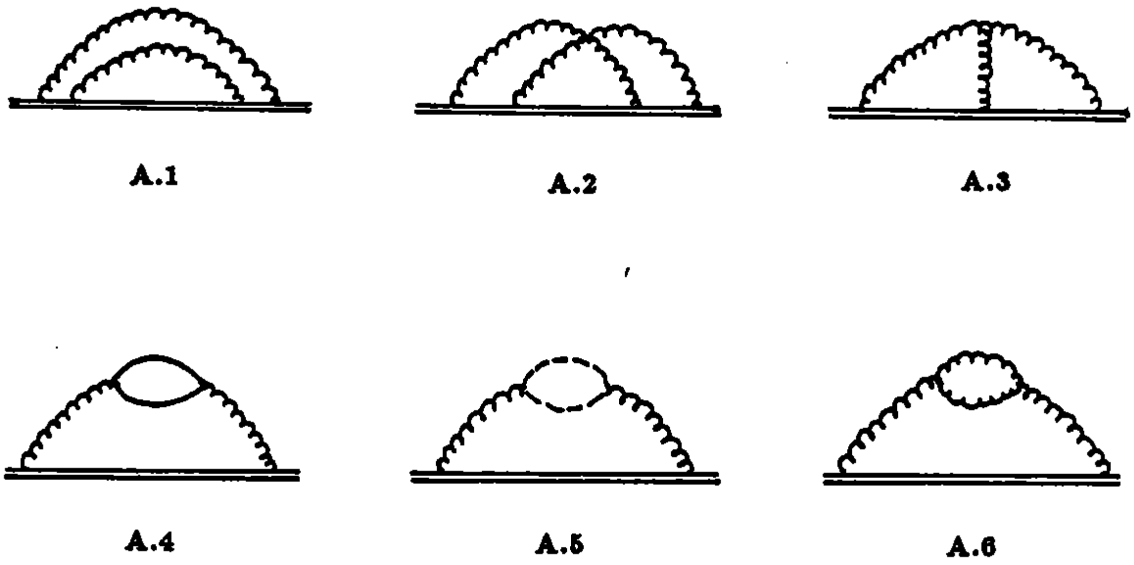
VERTEX DIAGRAM	VALUE[a]		
	COLOR	$1/\epsilon^2$	$1/\epsilon$
$B.1$	$\frac{C_2^2(R)}{4^2}$	$-\frac{1}{2}$	$\xi(2)$
$B.2$	$\frac{C_2^2(R)}{4^2}$	-1	-2
$B.3$	$\frac{C_2^2(R)}{4^2}$	$\frac{1}{2}$	0
$B.4$	$\frac{C_2(R)}{32N}$	-1	-2
$B.5$	$\frac{C_2(R)}{32N}$	$\frac{1}{2}$	0
$B.6$	$\frac{C_2(R)}{32N}$	0	$-\xi(2)-1$
$B.7$	$\frac{3\,C_2(R)\,C_2(G)}{64}$	0	$1-4\xi(2)$
$B.8$	$\frac{3\,C_2(R)\,C_2(G)}{64}$	-1	$-\frac{1}{3}$
$B.9$	$\frac{3\,C_2(R)\,C_2(G)}{64}$	0	0

[a](It must be multiplied by $(\alpha/\pi)^2\,\delta_{\alpha\beta}$)

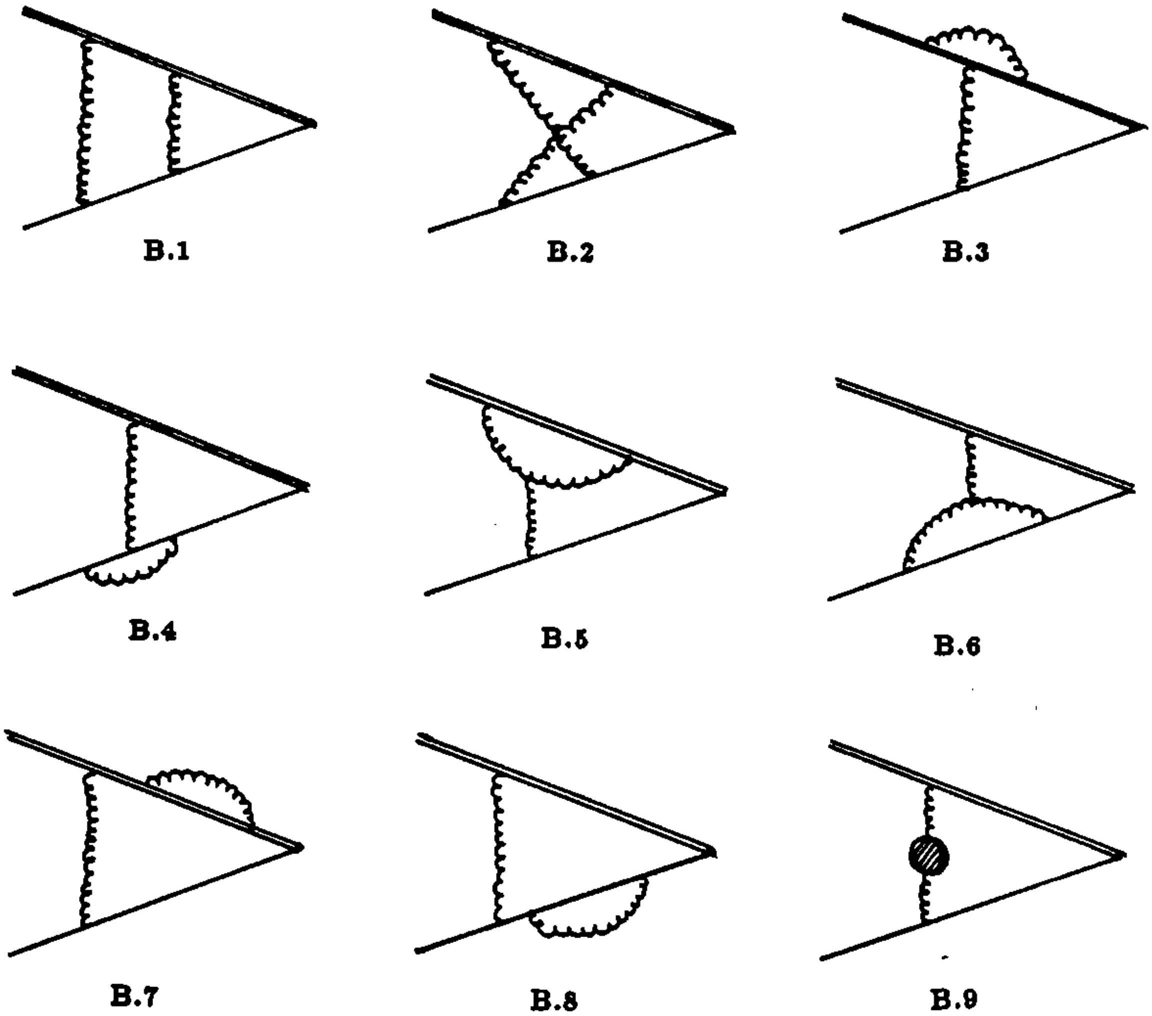
Table 6: Renormalized two loop diagrams for the heavy–light quark vertex

$\triangle Z_f$	$\Lambda_{QCD} = 100$ Gev			$\Lambda_{QCD} = 200$ Gev		
N_f	$a^{-1} = 2$ Gev	$a^{-1} = 4$ Gev	$a^{-1} = 6$ Gev	$a^{-1} = 2$ Gev	$a^{-1} = 4$ Gev	$a^{-1} = 6$ Gev
2	0.012	0.002	−0.003	0.019	0.026	−0.004
3	0.014	0.002	−0.003	0.022	0.003	−0.005
4	0.017	0.002	−0.004	0.027	0.004	−0.005
5	0.020	0.003	−0.005	0.032	0.004	−0.007
6	0.026	0.004	−0.006	0.041	0.006	−0.008

Table 7: $\triangle Z_f$ for various values of N_f and Λ_{QCD}



Two loop diagrams for the heavy quark selfenergy



Two loop diagrams for the axial current vertex

FIGURE 1