

## Non-perturbative renormalization of lattice operators in coordinate space

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### Abstract

We present the first numerical implementation of a non-perturbative renormalization method for lattice operators, based on the study of correlation functions in coordinate space at short Euclidean distance. The method is applied to compute the renormalization constants of bilinear quark operators for the non-perturbative  $O(a)$ -improved Wilson action in the quenched approximation. The matching with perturbative schemes, such as  $\overline{\text{MS}}$ , is computed at the next-to-leading order in continuum perturbation theory. A feasibility study of this technique with Neuberger fermions is also presented.

# 1 Introduction

Correlation functions of quantum-field operators are computed non-perturbatively on the lattice by Monte Carlo simulations. From their long-distance behaviour, in QCD, non-perturbative features of the underlying theory, such as the hadron spectrum and matrix elements, can be extracted. On the other hand, their behaviour at short distance is expected to be controlled by perturbation theory and by the operator product expansion (OPE). As a result, relevant ultraviolet informations of the theory, such as the range of applicability of the OPE, the values of the Wilson coefficients and the renormalization constants of composite operators, can in principle be obtained by a comparison of the perturbative formulæ with the numerical results at short distance. This is the basic idea behind the non-perturbative renormalization techniques proposed in the last decade and widely used in present simulations. With the RI/MOM method, renormalization conditions are imposed on quark and gluon Green functions computed non-perturbatively in momentum space, in a fixed gauge, with given off-shell external states of large virtuality [1]-[10]. In the Schrödinger functional (SF) approach, the renormalization conditions are imposed in coordinate space at a given finite physical distance on suitable gauge invariant correlation functions in finite volume with SF boundary conditions [11]-[14]. The step-scaling technique [12] can then be used to convert the renormalization constants to their renormalization group invariant definitions. An advantage of these methods is that the conversion to more popular continuum schemes, such as the  $\overline{\text{MS}}$  scheme, can be implemented by performing a calculation only in continuum perturbation theory, by comparing renormalized correlation functions at short distances computed in dimensional regularization in the two schemes. In this way, more tedious calculations with lattice perturbation theory are completely avoided.

In this paper we present the first numerical implementation of a non-perturbative renormalization method based on the study of lattice correlation functions at short Euclidean distances in coordinate space [17]. We call this approach the “*X-space*” scheme. Preliminary results were presented in Refs. [18, 19]. We have computed numerically the two-point functions of all dimension three bilinear quark operators by discretizing the gluons *à la* Wilson and the fermions with the non-perturbatively  $O(a)$ -improved Wilson action. A feasibility study of this technique with Neuberger fermions is also presented. The multiplicatively renormalization constants of lattice bilinear operators are evaluated non-perturbatively by imposing renormalization conditions directly in  $x$ -space at short distance. The condition  $x_0 \gg a$ , where  $a$  is the lattice spacing and  $x_0$  is the renormalization point, has to be satisfied in order to keep discretization errors under control. On the other hand, the matching of the renormalization constants to the  $\overline{\text{MS}}$  scheme (or any other continuum scheme) can be computed in continuum perturbation theory when  $x_0 \ll \Lambda_{\text{QCD}}^{-1}$ . In this study we show that for  $a \lesssim 0.05$  fm (i.e.  $1/a \gtrsim 4$  GeV) it is possible to find a region on the same lattice in which perturbation theory can be applied and discretization effects are still under control. The existence of the window  $a \lesssim x_0 \lesssim \Lambda_{\text{QCD}}^{-1}$  requires however rather fine lattices, large volumes and therefore expensive simulations. Alternatively, one can appeal to a step-scaling technique analogous to the one proposed in Ref. [12]. The *X-space* renormalization method involves only gauge-invariant correlation functions among local operators at finite

physical distance, and can be easily applied to any fermion discretization. It can be very powerful for the evaluation of the renormalization constants of composite operators, such as the four-fermion operators relevant for the phenomenology of hadronic weak decays. The matching to the  $\overline{\text{MS}}$  scheme can be performed by using only continuum perturbation theory and the method is very simple to implement.

The paper is organized as follows: in sec. 2 we summarize the relevant formulæ in continuum perturbation theory and define the renormalization conditions which we will use in coordinate space; in sec. 3 we discuss the numerical results and in sec. 4 we present our conclusions.

## 2 X-space renormalization and perturbation theory

In this section we define the X-space renormalization scheme for bilinear quark operators and provide the perturbative expressions, at the next-to-leading order (NLO), needed to convert the results to any other continuum renormalization scheme, such as the  $\overline{\text{MS}}$  scheme<sup>1</sup>.

We consider the correlation functions of flavor non-singlet bilinear quark operators of the form

$$\langle O_\Gamma(x) O_\Gamma(0) \rangle, \quad (1)$$

where

$$O_\Gamma(x) = \bar{\psi}(x) \Gamma \psi(x) \quad (2)$$

with  $O_\Gamma = \{S, P, V_\mu, A_\mu, T_{\mu\nu}\}$  for  $\Gamma = \{1, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5, \frac{1}{2}[\gamma_\mu, \gamma_\nu]\}$  respectively and with flavor indices omitted.

Following [17], we impose non-perturbatively, in x-space and in the chiral limit, the renormalization conditions

$$\lim_{a \rightarrow 0} \langle O_\Gamma^X(x) O_\Gamma^X(0) \rangle \Big|_{x^2=x_0^2} = \langle O_\Gamma(x_0) O_\Gamma(0) \rangle_{\text{cont}}^{\text{free}} \quad (3)$$

where the renormalized operator is  $O_\Gamma^X(x, x_0) = Z_\Gamma^X(x_0) O_\Gamma(x)$  and  $x_0$  is the renormalization point. The renormalization condition (3) defines the X-space scheme. Note that  $x_0$  must satisfy the condition  $a \ll x_0 \ll \Lambda_{QCD}^{-1}$  to keep non-perturbative and discretization effects under control.

In order to illustrate the procedure and get the expressions needed to convert to the more popular continuum schemes, we have computed the correlation functions, at two loop in naïve dimensional regularization (NDR), in the massless case. The results, in Euclidean

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<sup>1</sup>All formulas presented in this section are obtained in the infinite-volume limit. When correlation functions in small volumes are considered, their perturbative expressions may need to be modified according to the boundary conditions used [15, 16]

space, read

$$\begin{aligned}
\langle S(x)S(0) \rangle &= \langle P(x)P(0) \rangle = \frac{N_c}{\pi^4 (x^2)^3} \left\{ 1 + \frac{2\alpha_s}{4\pi} \left( \frac{4}{\hat{\epsilon}} + \frac{4}{3} + 8\gamma_E - \frac{\gamma_S^{(0)}}{2} \ln\left(\frac{\mu^2 x^2}{4}\right) \right) \right\} \\
\langle V_\mu(x)V_\nu(0) \rangle &= \langle A_\mu(x)A_\nu(0) \rangle = -\frac{2N_c}{\pi^4 (x^2)^3} \left( \frac{1}{2}\delta_{\mu\nu} - \frac{x_\mu x_\nu}{x^2} \right) \left\{ 1 + 4\frac{\alpha_s}{4\pi} \right\} \\
\langle T_{\mu\nu}(x)T_{\rho\sigma}(0) \rangle &= -\frac{2N_c}{\pi^4 (x^2)^3} \left( \frac{1}{2}T_{\mu\nu\rho\sigma}^{(1)} - T_{\mu\nu\rho\sigma}^{(2)} \right) \\
&\quad \left\{ 1 + \frac{2\alpha_s}{4\pi} \left( -\frac{4}{3\hat{\epsilon}} + 4 - \frac{8}{3}\gamma_E - \frac{\gamma_T^{(0)}}{2} \ln\left(\frac{\mu^2 x^2}{4}\right) \right) \right\}
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
T_{\mu\nu\rho\sigma}^{(1)} &= \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} \\
T_{\mu\nu\rho\sigma}^{(2)} &= \frac{x_\mu x_\rho}{x^2}\delta_{\nu\sigma} - \frac{x_\mu x_\sigma}{x^2}\delta_{\nu\rho} - \frac{x_\nu x_\rho}{x^2}\delta_{\mu\sigma} + \frac{x_\nu x_\sigma}{x^2}\delta_{\mu\rho}.
\end{aligned} \tag{5}$$

In these expressions  $\alpha_s$  is the strong coupling constant,  $1/\hat{\epsilon} = 1/\epsilon - \ln(4\pi) - \gamma_E$  (we define  $d = 4 - 2\epsilon$  the space-time dimension) and the LO anomalous dimensions are  $\gamma_S^{(0)} = -8$ ,  $\gamma_T^{(0)} = 8/3$ , and  $\gamma_V^{(0)} = 0$ , the latter due to the conservation of the vector current. This conservation also determines the tensor structure of the vector current correlator. Note that, as expected, the leading short distance behaviour of the correlation functions in Eq. (4) is governed by  $(x^2)^{-3}$ . For the scalar and vector correlators the results in Eqs. (4) agree with previous computations [20, 21, 22].

By imposing the renormalization conditions (3) to the results in Eq. (4), we obtain the correlation functions for the renormalized operators in the X-space scheme:

$$\begin{aligned}
\langle S^X(x, x_0)S^X(0, x_0) \rangle &= \frac{N_c}{\pi^4 (x^2)^3} K_S^X(x, x_0) \\
\langle V_\mu^X(x, x_0)V_\nu^X(0, x_0) \rangle &= -\frac{2N_c}{\pi^4 (x^2)^3} \left( \frac{1}{2}\delta_{\mu\nu} - \frac{x_\mu x_\nu}{x^2} \right) K_V^X(x, x_0) \\
\langle T^X(x, x_0)_{\mu\nu}T^X(0, x_0)_{\rho\sigma} \rangle &= -\frac{2N_c}{\pi^4 (x^2)^3} \left( \frac{1}{2}T_{\mu\nu\rho\sigma}^{(1)} - T_{\mu\nu\rho\sigma}^{(2)} \right) K_T^X(x, x_0)
\end{aligned} \tag{6}$$

with

$$K_\Gamma^X(x, x_0) = 1 - \gamma_\Gamma^{(0)} \frac{\alpha_s}{4\pi} \ln\left(\frac{x^2}{x_0^2}\right). \tag{7}$$

In the  $\overline{\text{MS}}$  scheme, on the other hand, the correlation functions of the renormalized composite operators,  $O_\Gamma^{\overline{\text{MS}}}(x, \mu) = Z_\Gamma^{\overline{\text{MS}}}(\mu) O_\Gamma(x)$ , are obtained by subtracting the pole  $1/\hat{\epsilon}$  on the r.h.s of Eq. (4). The relations between the  $\overline{\text{MS}}$  and the X-space scheme are thus

the following

$$\begin{aligned}
K_S^{\overline{\text{MS}}}(x, \mu) &= \left\{ 1 + \frac{2\alpha_s}{4\pi} \left[ 4 \ln \left( \frac{\mu^2 x_0^2}{4} \right) + 8\gamma_E + \frac{4}{3} \right] \right\} K_S^{\text{X}}(x, x_0) \\
K_V^{\overline{\text{MS}}}(x, \mu) &= \left( 1 + 4 \frac{\alpha_s}{4\pi} \right) K_V^{\text{X}}(x, x_0) \\
K_T^{\overline{\text{MS}}}(x, \mu) &= \left\{ 1 + \frac{2\alpha_s}{4\pi} \left[ -\frac{4}{3} \ln \left( \frac{\mu^2 x_0^2}{4} \right) - \frac{8}{3}\gamma_E + 4 \right] \right\} K_T^{\text{X}}(x, x_0).
\end{aligned} \tag{8}$$

This also implies that the renormalization constants in the two schemes are related by

$$\begin{aligned}
\frac{Z_S^{\overline{\text{MS}}}(\mu)}{Z_S^{\text{X}}(x_0)} &= 1 + \frac{\alpha_s}{4\pi} \left[ 4 \ln \left( \frac{\mu^2 x_0^2}{4} \right) + 8\gamma_E + \frac{4}{3} \right], \\
\frac{Z_V^{\overline{\text{MS}}}(\mu)}{Z_V^{\text{X}}(x_0)} &= 1 + 2 \frac{\alpha_s}{4\pi}, \\
\frac{Z_T^{\overline{\text{MS}}}(\mu)}{Z_T^{\text{X}}(x_0)} &= 1 + \frac{\alpha_s}{4\pi} \left[ -\frac{4}{3} \ln \left( \frac{\mu^2 x_0^2}{4} \right) - \frac{8}{3}\gamma_E + 4 \right].
\end{aligned} \tag{9}$$

The renormalization condition (3) does not satisfy the vector and axial vector Ward Identities. At the NLO this can be easily seen by noticing that in the  $\overline{\text{MS}}$  scheme, which preserves them,  $K_V^{\overline{\text{MS}}}(x_0, \mu)$  has a finite term proportional to  $\alpha_s$ . In the X-space scheme this contribution is included in the renormalization constant and therefore  $Z_V^{\overline{\text{MS}}}/Z_V^{\text{X}} \neq 1$ , i.e. the Ward Identities are broken. They can be recovered by using continuum perturbation theory, which for the vector correlator is known up to four loops [21, 22], or non-perturbatively by matching the result for  $Z_V$  in the X-space scheme with the Ward Identity determination [23].

We conclude this section by recalling the expression for the renormalization group evolution of the renormalization constants at the NLO in  $\alpha_s$ :

$$Z_\Gamma^{\overline{\text{MS}}}(\mu') = \frac{c_\Gamma^{\overline{\text{MS}}}(\mu')}{c_\Gamma^{\overline{\text{MS}}}(\mu)} Z_\Gamma^{\overline{\text{MS}}}(\mu) \tag{10}$$

where

$$c_\Gamma^{\overline{\text{MS}}}(\mu) = \alpha_s(\mu)^{\frac{\gamma_\Gamma^{(0)}}{2\beta_0}} \left\{ 1 + \frac{\alpha_s}{4\pi} \left( \frac{\gamma_\Gamma^{(1)}}{2\beta_0} - \frac{\beta_1}{\beta_0} \frac{\gamma_\Gamma^{(0)}}{2\beta_0} \right) \right\} \tag{11}$$

with  $\alpha_s$  defined in  $\overline{\text{MS}}$  scheme. In the quenched theory, i.e. with  $N_f = 0$ , the first two coefficients of the expansion of the  $\beta$ -function are  $\beta_0 = 11$  and  $\beta_1 = 102$ , while the two-loop anomalous dimensions in the  $\overline{\text{MS}}$  scheme are  $\gamma_S^{(1)} = -404/3$ ,  $\gamma_V^{(1)} = 0$  and  $\gamma_T^{(1)} = 724/9$  [24, 25, 26]. In our study, the matching between X-space and  $\overline{\text{MS}}$  schemes has been performed at a scale  $\mu \sim 1/x_0$ . With this choice, logs in Eqs. (9) are small and need not to be resummed. In contrast, larger logs enter the evolution from the scale  $\mu \sim 1/x_0$  to the conventional scale  $\mu = 2$  GeV at which our final results are quoted. For this reason, the evolution function in Eq. (11) has been resummed by using the renormalization group equations at the NLO.

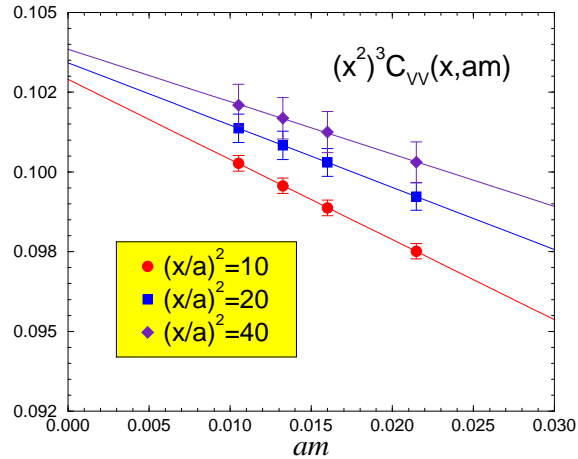


Figure 1: *The vector correlator  $C_{VV}(x)$  for different values of  $x^2$  as a function of the quark mass. The line represents the result of the linear extrapolation to the chiral limit.*

### 3 Numerical results

In this section we provide the numerical details of our computation and present the results obtained non-perturbatively for the renormalization constants of the bilinear quark operators.

We generated a sample of 180 gauge configurations in the quenched approximation with the standard SU(3) Wilson gluonic action at  $\beta = 6.45$  ( $a \sim 0.048$  fm) and  $V = 32^3 \times 70$ . For these configurations we evaluated fermion propagators with the non-perturbatively  $O(a)$ -improved Wilson action for hopping parameter values  $\kappa = 0.1349, 0.1351, 0.1352, 0.1353$ . We computed the two-point flavor non-singlet correlation functions

$$\begin{aligned}
 C_{SS}(x) &= \langle S(x)S(0) \rangle \quad , & C_{PP}(x) &= \langle P(x)P(0) \rangle \quad , \\
 C_{VV}(x) &= \sum_{\mu} \langle V_{\mu}(x)V_{\mu}(0) \rangle \quad , & C_{AA}(x) &= \sum_{\mu} \langle A_{\mu}(x)A_{\mu}(0) \rangle \quad , \\
 C_{TT}(x) &= \sum_{\mu, \nu, \rho} \left( \frac{1}{6} \delta_{\mu\nu} - \frac{1}{3} \frac{x_{\mu}x_{\nu}}{x^2} \right) \langle T_{\mu\rho}(x)T_{\nu\rho}(0) \rangle
 \end{aligned} \tag{12}$$

of local bilinear operators in the standard way and estimated the statistical errors by a jackknife procedure. These functions have been averaged over points which are equivalent under hypercubic rotations. In the range of  $x^2$  we have studied, our data show a mild mass dependence, and a linear or quadratic extrapolations to the chiral limit give compatible results within the errors. In the following we will show the results linearly extrapolated to the chiral limit. An example of this extrapolation, in the case of the vector correlator, is illustrated in Fig. 1.

In Fig. 2 (left) we show the correlation function  $C_{VV}(x)$  as a function of  $x^2$  and the corresponding one parameter fit to  $A(x^2)^{-3}$ . The naïve expected behaviour is clearly satisfied, even if a large scattering of the points due to lattice artifacts is visible, particularly

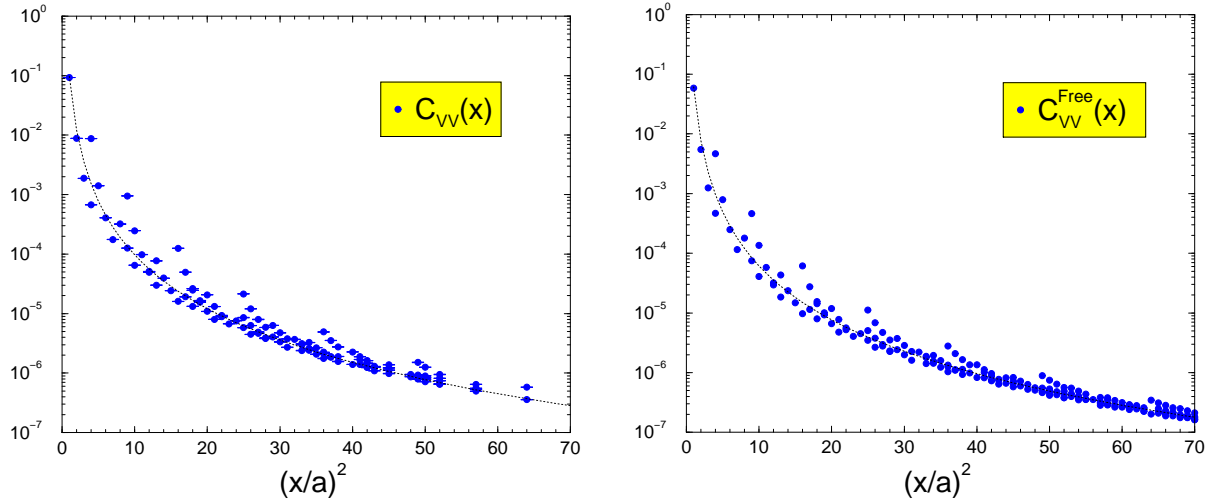


Figure 2:  $C_{VV}(x)$  in the interacting (left) and in the free (right) theory. In the first case, the dotted curve is a one parameter fit showing the  $(x^2)^{-3}$  behaviour. In the free case the curve represents the prediction of the free theory in the continuum limit

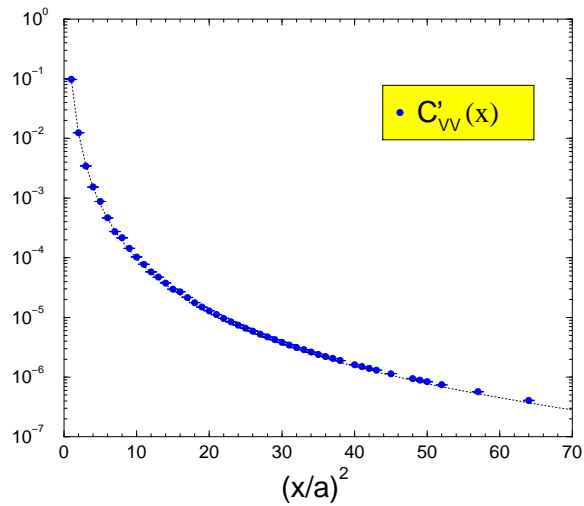


Figure 3: The corrected vector correlator  $C'_{VV}(x)$ . The dotted curve is a one parameter fit showing the  $(x^2)^{-3}$  behaviour.

in the short distance region. Lorentz invariance requires the correlator in the continuum limit to be a function of  $x^2$  only. The lattice data presented in Fig. 2 (left) show instead that the results for  $C_{VV}(x)$  computed at points which correspond to the same value of  $x^2$  are often quite separated. To clarify the origin of these effects, we studied the correlators in the free theory as a function of volume, lattice spacing and quark masses. The free theory prediction for the lattice correlation function  $C_{VV}(x)$ , in infinite volume and in the chiral limit, is shown as an example in Fig. 2 (right). The spread of the data observed in the interacting case turns out to be well reproduced in the free theory, at fixed volume and lattice spacing. For values of  $x^2$  in the perturbative region, finite volume effects are found to be negligible in the range of masses we use. On the other hand, the spread of the data in the free case is considerably reduced by decreasing the lattice spacing. This suggests that the dominant contributions due to lattice artifacts comes from discretization effects.

In order to reduce discretization effects in the interacting case, we define “corrected” correlation functions

$$C'_{\Gamma\Gamma}(x) = \frac{C_{\Gamma\Gamma}(x)}{\Delta_{\Gamma\Gamma}(x)} \quad (13)$$

where  $\Delta_{\Gamma\Gamma}(x)$  is the ratio of free correlator on the lattice over the continuum one, computed in infinite volume and in the chiral limit,

$$\Delta_{\Gamma\Gamma}(x) = \frac{\langle O_{\Gamma}(x)O_{\Gamma}(0) \rangle_{\text{lat}}^{\text{free}}}{\langle O_{\Gamma}(x)O_{\Gamma}(0) \rangle_{\text{cont}}^{\text{free}}}. \quad (14)$$

By construction,  $\Delta_{\Gamma\Gamma}(x)$  is equal to unit up to discretization effects. The results for the corrected function  $C'_{VV}(x)$  are shown in Fig. 3. These results, as well as those used in the following analysis, have been also averaged over points which correspond to the same  $x^2$ . The comparison between  $C_{VV}(x)$  and  $C'_{VV}(x)$  shows that, once tree-level discretization effects are removed, the spread of the data is greatly reduced. This analysis further supports the interpretation that the spread in the interacting theory is due to discretization effects. Similar conclusions apply also to other correlators.

Once the correlators have been corrected at tree level with the factor that attempt to reduce discretization effects at the leading order, they still suffer for remaining discretization errors, at  $\mathcal{O}(g^2 a^2)$  and higher. Their effects is shown in Fig. 4, where we plot the corrected correlation functions  $C'_{\Gamma\Gamma}(x)$  normalized to their continuum counterparts in the free theory,

$$R_{\Gamma\Gamma}(x) = \frac{C'_{\Gamma\Gamma}(x)}{C_{\Gamma\Gamma}(x)_{\text{cont}}^{\text{free}}} \quad (15)$$

in the case of the vector current and the scalar density operators. Note that, particularly in the scalar case, the remaining lattice artifacts are still larger than statistical errors. A further reduction of these effects could be obtained by computing the  $\mathcal{O}(g^2 a^2)$  terms in lattice perturbation theory.

In order to extract the renormalization constants, the previous analysis suggests to implement the renormalization condition defined in Eq. (3) directly to the corrected correlation functions  $C'_{\Gamma\Gamma}(x)$ . This is equivalent to impose

$$\left. \frac{\langle O_{\Gamma}^X(x)O_{\Gamma}^X(0) \rangle}{\langle O_{\Gamma}(x)O_{\Gamma}(0) \rangle_{\text{lat}}^{\text{free}}} \right|_{x^2=x_0^2} = 1. \quad (16)$$



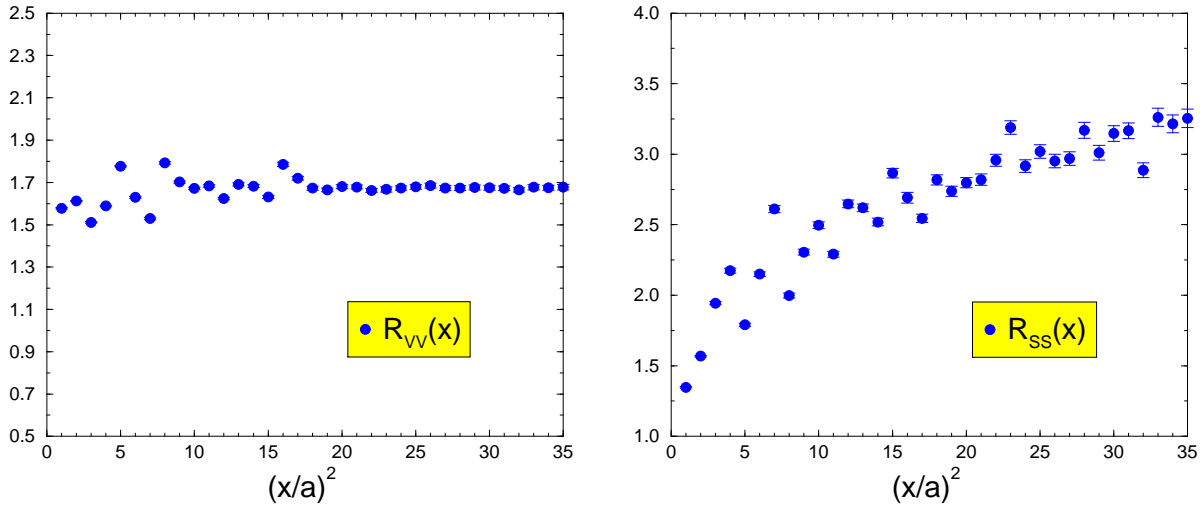


Figure 4: The ratios  $R_{VV}(x)$  (left) and  $R_{SS}(x)$  (right) defined in Eq. (15).

	$Z_V$	$Z_A$	$Z_T$
This work	0.801(2)(18)(6)	0.833(2)(27)(6)	0.895(2)(21)
RI-MOM	0.803(3)	0.833(3)	0.898(6)
SF	0.808(1)	0.825(8)	--

	$Z_P/Z_S$	$Z_S$	$Z_P$
This work	0.888(2)(8)	0.702(4)(27)(23)	0.624(3)(19)(21)
RI-MOM	0.897(4)	0.679(8)	0.609(8)
SF	0.912(9)	--	0.61(1)

Table 1: Wilson results for the renormalization constants in the  $\overline{MS}$  scheme at  $\mu = 2$  GeV, and comparison with other non-perturbative techniques: RI-MOM [9] and Schrödinger Functional [13, 14, 28].

Since  $O_\Gamma^X(x) = Z_\Gamma^X(x_0)O_\Gamma(x)$ , the above condition implies

$$Z_\Gamma^X(x_0) = 1/\sqrt{R_{\Gamma\Gamma}(x_0)}. \quad (17)$$

The renormalization constants  $Z_V$ ,  $Z_A$ ,  $Z_S$ ,  $Z_P$ ,  $Z_T$  in the X-space scheme and the ratio  $Z_P/Z_S$  are shown in Fig. 5 as a function of  $(x_0/a)^2$ . The scattering is reduced for  $(x_0/a)^2 \geq 8$ , signaling that discretization effects are moderate above this point. For  $(x_0/a)^2 \leq 21$ , which corresponds to  $1/x_0 \gtrsim 0.9$  GeV, the dependence of the renormalization constants on the renormalization scale is compatible with the NLO prediction of perturbation theory, indicated by solid lines in Fig. 5. In particular, the ratio  $Z_P/Z_S$  is expected to be independent of the renormalization scale and we see that in the range  $(x_0/a)^2 = [9, 21]$  a reasonable good plateau is observed. In this window we extract the values of all renormalization constants by fitting the corresponding correlation functions with the perturbative formulæ given in sec. 2. The results, translated to the  $\overline{MS}$  scheme at the scale  $\mu = 2$  GeV are presented in Tab. 1. The first quoted error is statistical and it is by far the smallest one. The second is an estimate of the uncertainty coming from the spread of

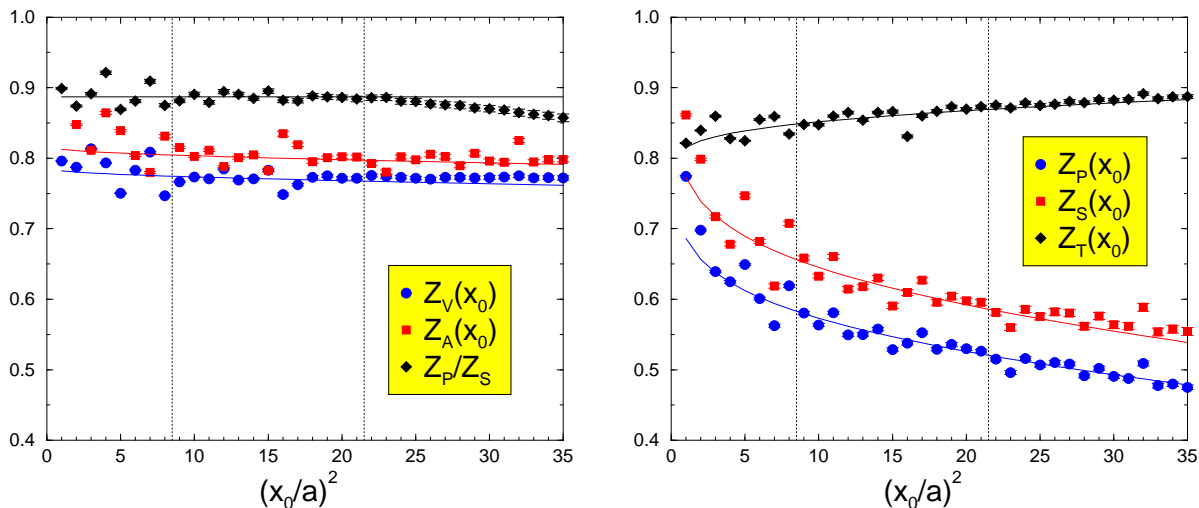


Figure 5: Renormalization constants in the X-space scheme as a function of the renormalization scale  $x_0$ . The solid line indicate the scale dependence predicted by NLO perturbation theory. The dashed lines show the boundaries of the fitting window .

the points within the fitting window. This error could be further reduced by going to finer lattices or by evaluating the  $\mathcal{O}(g^2 a^2)$  terms in lattice perturbation theory. The third error in Tab. 1 is an estimate of the systematics due to higher orders in continuum perturbation theory, obtained by varying the renormalization scale  $\mu$  in the perturbative expressions of Eq. (9) in the range  $1 \leq \mu x_0 \leq 2$ . This uncertainty can be reduced by performing a N<sup>2</sup>LO computation in perturbation theory and/or by implementing the step scaling technique<sup>2</sup> proposed in Ref. [12]. The latter would require simulations at several lattice spacings and goes beyond the scope of this exploratory study.

In order to investigate the applicability of the X-space renormalization method to different discretizations of the fermionic action, we also performed a feasibility study by using Neuberger fermions. We used 80 configurations generated with the same gluonic action at  $\beta = 6.0$  ( $a \sim 0.093$  fm) and  $V = 16^3 \times 32$  which were retrieved from the repository at the ‘‘Gauge Connection’’ [27]. For these configurations, overlap propagators at bare masses  $ma = 0.040, 0.055, 0.070, 0.085, 0.100$  have been evaluated, see Refs. [7, 8] for details. The results have been quadratically extrapolated to the chiral limit, as suggested by perturbation theory. The results for the renormalization constants of the vector and axial-vector currents are shown in Fig. 6 as a function of  $(x_0/a)^2$ . Since in this case the lattice spacing is larger than in the case of Wilson fermions, the window contains at most three points, which makes it difficult a reliable comparison with perturbation theory. Nevertheless the data plotted in Fig. 6 show a smooth behaviour compatible with the chiral properties of Neuberger fermions, and the value of  $Z_V = Z_A$  for  $3 \leq (x_0/a)^2 \leq 5$ , corrected for the matching factor in Eq. (9), is compatible with the Ward identity estimate obtained

<sup>2</sup>It is interesting to note that, since the running of the operators with the renormalization scale is scheme dependent but regularization independent, it is possible to implement the step scaling technique by using any discretization of the fermionic action. The results, extrapolated to the continuum limit, can then be used to evolve the renormalization constants computed non-perturbatively also with a different discretization.

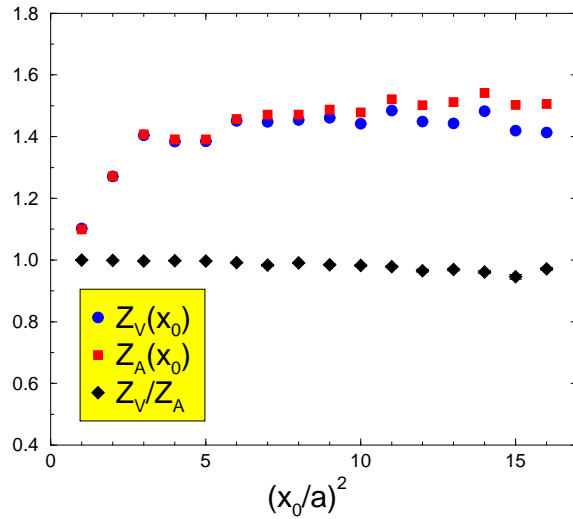


Figure 6: Renormalization constants  $Z_V$ ,  $Z_A$  and the ratio  $Z_V/Z_A$  for Neuberger fermions as a function of  $x_0$ .

in Ref. [7, 8],  $Z_A = 1.55(4)$ . In this case, the implementation of a step scaling technique would have allowed us to impose the renormalization conditions at shorter distances, where perturbation theory is reliable but discretization effects remain negligible.

## 4 Conclusions

We have studied the correlation functions of two fermion bilinear operators in coordinate space at short Euclidean distances. A good statistical signal can be obtained with a small number of configurations. The spread of the data computed at different lattice points with the same  $x^2$ , indicates that discretization errors can be large. This spread has been greatly reduced by normalizing the correlation functions with the analogous ones computed in the free theory at finite lattice spacing. A straightforward application of these results is a determination of the renormalization constants of the composite operators. Even if with larger uncertainties, the values of the renormalization constants which we have obtained are in good agreement with previous non-perturbative determinations. This technique can be easily applied to any fermion discretization, as shown in this paper, and it involves only gauge-invariant correlation functions among local operators at finite Euclidean distance. The matching with more popular renormalization schemes, such as the  $\overline{\text{MS}}$  scheme, is easy because it requires calculations only performed in continuum perturbation theory. Therefore in the future, after more accurate studies of the systematic, the X-space method could become a powerful technique to renormalize the four-fermion operators relevant in hadronic weak decays.

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## References

- [1] G. Martinelli, C. Pittori, C. T. Sachrajda, M. Testa and A. Vladikas, Nucl. Phys. B **445** (1995) 81, [hep-lat/9411010].
- [2] V. Gimenez, L. Giusti, F. Rapuano and M. Talevi, Nucl. Phys. B **531** (1998) 429, [hep-lat/9806006].
- [3] M. Gockeler *et al.*, Nucl. Phys. B **544** (1999) 699, [hep-lat/9807044].
- [4] A. Donini, V. Gimenez, G. Martinelli, M. Talevi and A. Vladikas, Eur. Phys. J. C **10** (1999) 121, [hep-lat/9902030].
- [5] L. Giusti and A. Vladikas, Phys. Lett. B **488** (2000) 303, [hep-lat/0005026].
- [6] T. Blum *et al.*, Phys. Rev. D **66** (2002) 014504, [hep-lat/0102005].
- [7] L. Giusti, C. Hoelbling and C. Rebbi, Phys. Rev. D **64** (2001) 114508 [Erratum-ibid. D **65** (2002) 079903], [hep-lat/0108007].
- [8] L. Giusti, C. Hoelbling and C. Rebbi, Nucl. Phys. Proc. Suppl. **106** (2002) 739, [hep-lat/0110184].
- [9] D. Becirevic, V. Gimenez, V. Lubicz, G. Martinelli, M. Papinutto and J. Reyes, arXiv:hep-lat/0401033.
- [10] C. Gatttringer, M. Gockeler, P. Huber and C. B. Lang, arXiv:hep-lat/0404006.
- [11] M. Luscher, R. Narayanan, P. Weisz and U. Wolff, Nucl. Phys. B **384** (1992) 168, [hep-lat/9207009].
- [12] M. Luscher, P. Weisz and U. Wolff, Nucl. Phys. B **359** (1991) 221.
- [13] M. Luscher, S. Sint, R. Sommer and H. Wittig, Nucl. Phys. B **491** (1997) 344, [hep-lat/9611015].
- [14] S. Capitani, M. Luscher, R. Sommer and H. Wittig [ALPHA Collaboration], Nucl. Phys. B **544** (1999) 669, [hep-lat/9810063].

- [15] M. Luscher, Phys. Lett. B **118** (1982) 391.
- [16] M. Luscher, Nucl. Phys. B **219** (1983) 233.
- [17] G. Martinelli et al., Phys. Lett. B **411** (1997) 141, [hep-lat/9705018].
- [18] D. Becirevic et al., Nucl. Phys. Proc. Suppl. **119** (2003) 442, [hep-lat/0209168].
- [19] V. Gimenez *et al.*, arXiv:hep-lat/0309150.
- [20] K. G. Chetyrkin, Phys. Lett. B **390** (1997) 309, [arXiv:hep-ph/9608318].
- [21] S. G. Gorishnii, A. L. Kataev and S. A. Larin, Phys. Lett. B **259** (1991) 144.
- [22] L. R. Surguladze and M. A. Samuel, Phys. Rev. Lett. **66** (1991) 560, [Erratum-ibid. **66** (1991) 2416].
- [23] M. Bochicchio, L. Maiani, G. Martinelli, G. C. Rossi and M. Testa, Nucl. Phys. B **262** (1985) 331.
- [24] K. G. Chetyrkin, Phys. Lett. B **404** (1997) 161, [hep-ph/9703278].
- [25] J. A. M. Vermaseren, S. A. Larin and T. van Ritbergen, Phys. Lett. B **405** (1997) 327, [hep-ph/9703284].
- [26] J. A. Gracey, Phys. Lett. B **488** (2000) 175, [hep-ph/0007171].
- [27] We retrieved the first 80 gauge configurations OSU\_Q60a produced by G. Kilcup, D. Pekurovsky, L. Venkataraman, Nucl. Phys. (Proc. Suppl.) **53** (1997) 345 from the repository at the “Gauge Connection” (<http://qcd.nersc.gov/>).
- [28] M. Guagnelli, R. Petronzio, J. Rolf, S. Sint, R. Sommer and U. Wolff [ALPHA Collaboration], Nucl. Phys. B **595** (2001) 44, [hep-lat/0009021].

