

$K \rightarrow \pi\pi$ matrix elements beyond the leading-order chiral expansion *

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We propose an approach for calculating $K \rightarrow \pi\pi$ decays to the next-to-leading order in chiral expansion. A detailed numerical study of this approach is being performed.

1. Introduction

Controlling the effects of final state interactions (FSI) is one of the main barriers towards a high-precision theoretical prediction for $K \rightarrow \pi\pi$ decays. This is particularly difficult for lattice QCD because of the analytic continuation to Euclidean space. The finite-volume techniques developed in Refs. [1,2] can exactly take into account the FSI effects, but their numerical implementation is very demanding. In the foreseeable future, the most practical approach, which is also reliable and systematically improvable, to the study of non-leptonic kaon decays remains the combination of lattice QCD and chiral perturbation theory (χ PT).

Apart from a study of the CP-conserving, $\Delta I = 3/2$, $K \rightarrow \pi\pi$ decay in Ref. [3], all the existing lattice results for matrix elements of the form $\langle \pi\pi | Q | K \rangle$ [4] are obtained by simulating matrix elements of the kind $\langle \pi | Q | K \rangle$, then using lowest-order χ PT [5] to relate them to the desired matrix elements. In this procedure, the effects of higher-order chiral corrections due to FSI, which could be very large, are completely missing³.

Here we present the status of an on-going project in which all the relevant matrix elements for ϵ'/ϵ and the $\Delta I = 1/2$ rule are being calculated by computing matrix elements of the kind $\langle \pi\pi | Q | K \rangle$, at some “unphysical” kinematics, on the lattice, and then using the chiral expansion to next-to-leading order (NLO) to obtain them at the “physical” kinematics. In this talk, we focus⁴ on the study of the chiral behaviour of $\Delta I = 3/2$, $K \rightarrow \pi\pi$ decay amplitudes associated with the operators

$$\begin{aligned} Q_4 &= (\bar{s}_\alpha d_\alpha)_L (\bar{u}_\beta u_\beta - \bar{d}_\beta d_\beta)_L + (\bar{s}_\alpha u_\alpha)_L (\bar{u}_\beta d_\beta)_L, \\ Q_7 &= \frac{3}{2} (\bar{s}_\alpha d_\alpha)_L \sum_{q=u,d,s,c} e_q (\bar{q}_\beta q_\beta)_R, \\ Q_8 &= \frac{3}{2} (\bar{s}_\alpha d_\beta)_L \sum_{q=u,d,s,c} e_q (\bar{q}_\beta q_\alpha)_R, \end{aligned} \quad (1)$$

where α, β are colour indices and e_q is the electric charge of q . $(\bar{\psi}_1 \psi_2)_{L,R}$ means $\bar{\psi}_1 \gamma_\mu (1 \mp \gamma_5) \psi_2$.

2. Choice of “unphysical” kinematics

The matrix elements $\langle \pi^+ \pi^0 | Q_i | K^+ \rangle$ ($i = 4, 7$ and 8) are computed in the unphysical kinematics such that K^+ and one of the pions are always at rest, while the other pion might carry non-zero spatial momentum. We denote these matrix

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³In Ref. [3], the decay amplitude is also obtained at the precision of leading-order chiral expansion.

⁴For details of other aspects of this work, please refer to Ref. [6].

elements by

$$\begin{aligned} \langle \pi^+(\vec{p}_\pi)\pi^0(\vec{0})|Q_i|K^+\rangle_{\text{unphys}} &, \quad \text{if } \pi^0 \text{ is at rest,} \\ \langle \pi^+(\vec{0})\pi^0(\vec{p}_\pi)|Q_i|K^+\rangle_{\text{unphys}} &, \quad \text{if } \pi^+ \text{ is at rest.} \end{aligned}$$

Notice that the spatial momentum \vec{p}_π in the above matrix elements might be zero as well. In this case, both pions are at rest. The correlators used to extract these matrix elements are discussed in Ref. [6].

A technical difficulty arising at this stage is that the final states $|\pi^+(\vec{p}_\pi)\pi^0(\vec{0})\rangle$ and $|\pi^+(\vec{0})\pi^0(\vec{p}_\pi)\rangle$ are not purely $I = 2$, because Bose-Einstein statistics does not rule out the $I = 1$ components. In order to eliminate these components, we take the symmetric combination

$$\begin{aligned} &\langle \pi^+\pi^0|Q_i|K^+\rangle_{\text{unphys}}^{\text{sym}} \\ &= \frac{1}{2} \left(\langle \pi^+(\vec{p}_\pi)\pi^0(\vec{0})|Q_i|K^+\rangle_{\text{unphys}} \right. \\ &\quad \left. + \langle \pi^+(\vec{0})\pi^0(\vec{p}_\pi)|Q_i|K^+\rangle_{\text{unphys}} \right). \end{aligned} \quad (2)$$

3. Chiral expansion at NLO

In order to relate $\langle \pi^+\pi^0|Q_i|K^+\rangle_{\text{unphys}}^{\text{sym}}$ to their counter parts at physical kinematics, denoted as $\langle \pi^+\pi^0|Q_i|K^+\rangle_{\text{phys}}$, we resort to NLO chiral expansion. Under the chiral $SU(3)_L \otimes SU(3)_R$ transformation, Q_4 is in the (27,1) representation and $Q_{7,8}$ are in the (8,8) representation. Therefore the leading term in the chiral expansion for Q_4 ($Q_{7,8}$) is of $\mathcal{O}(p^2)$ ($\mathcal{O}(p^0)$). Chirally expanding Q_4 ($Q_{7,8}$) to $\mathcal{O}(p^4)$ ($\mathcal{O}(p^2)$) requires the knowledge of counterterm operators in the relevant chiral representation at this order, as well as one-loop calculations in χ PT using the leading-order operators.

There is only one χ PT representative for operators in (27,1) ((8,8) representation at $\mathcal{O}(p^2)$ ($\mathcal{O}(p^0)$). At $\mathcal{O}(p^4)$, there are thirty four χ PT representatives for (27,1) operators, labelled as $\mathcal{O}_i^{(27)}$ ($i = 1, 2, \dots, 34$) in Ref. [7]. For the purpose of this work, we only need $\mathcal{O}_2^{(27)}$, $\mathcal{O}_4^{(27)}$, $\mathcal{O}_5^{(27)}$, $\mathcal{O}_7^{(27)}$, $\mathcal{O}_{22}^{(27)}$ and $\mathcal{O}_{24}^{(27)}$ ⁵. The seven $\mathcal{O}(p^2)$ chiral representatives for (8,8) operators can be found in Eq.

⁵Notice that $\mathcal{O}_{22}^{(27)}$ and $\mathcal{O}_{24}^{(27)}$ are not included in Ref. [8]. $\mathcal{O}_{24}^{(27)}$ also contributes to $\langle \pi|Q|K \rangle$.

(9) of Ref. [9]. Here we label them as $\mathcal{O}_i^{(8,8)}$ ($i = 1, 2, \dots, 7$).

The one-loop χ PT calculations at physical kinematics, in infinite volume, are reported in Ref. [10] for (27,1), and Ref. [9] for (8,8). In our work, the chiral logarithms at the chosen unphysical kinematics are much more complicated than in these references, because of the energy-momentum injection at the operator. The resulting amplitudes depend upon three kinematical variables, \bar{M}_K , \bar{M}_π (kaon and pion masses in the simulation) and \bar{E}_π (energy of the pion carrying a non-zero spatial momentum in the simulation), while they only depend upon M_K and M_π (physical kaon and pion masses) at the physical kinematics. Furthermore, our numerical simulations are performed in a finite volume in the quenched approximation. This requires one-loop calculations in finite-volume quenched χ PT (q χ PT). So far, we have only finished the one-loop (unquenched) χ PT calculations in infinite volume as an exercise. These calculations have been performed in a general way, such that the four-momenta of K^+ , π^+ and π^0 are used as the kinematical variables, and energy-momentum conservation is only implemented at the $\mathcal{O}(p^2)$ strong vertices. The desired kinematics is imposed at the end of the calculations.

The results for the NLO chiral expansion for the amplitudes of interest are

$$\begin{aligned} \mathcal{M}_{\text{phys}}^{(27)} &= \frac{-6\sqrt{2}}{f_K f_\pi^2} \times \left\{ \alpha M_K^2 - \alpha M_\pi^2 \right. \\ &+ (\beta_4 - \beta_5 + 4\beta_7 + 2\beta_{22})M_K^4 \\ &+ (-4\beta_2 + 3\beta_4 + \beta_5 - 2\beta_7 \\ &\quad \left. - 2\beta_{22} + 16\beta_{24})M_K^2 M_\pi^2 \right. \\ &+ (4\beta_2 - 4\beta_4 - 2\beta_7 - 16\beta_{24})M_\pi^4 \left. \right\} \\ &+ \frac{\alpha}{f_K f_\pi^2} \times (\text{one loop})_{\text{phys}}^{(27)}, \\ \mathcal{M}_{\text{phys}}^{(8,8)} &= \frac{2\sqrt{2}}{f_K f_\pi^2} \times \left\{ \gamma \right. \\ &+ [-(\delta_2 + \delta_3) + 2(\delta_4 + \delta_5) + 4\delta_6]M_K^2 \\ &+ [(\delta_1 + \delta_2) + 4(\delta_4 + \delta_5) + 2\delta_6]M_\pi^2 \left. \right\} \end{aligned}$$

$$+ \frac{\gamma}{f_K f_\pi^2} (\text{one loop})_{\text{phys}}^{(8,8)}, \quad (3)$$

for the physical kinematics, and

$$\begin{aligned} (\mathcal{M}^{(27)})_{\text{unphys}}^{\text{sym}} &= \frac{-6\sqrt{2}}{f_K f_\pi^2} \times \left\{ \right. \\ &\alpha (\bar{E}_\pi \bar{M}_\pi + \frac{\bar{E}_\pi \bar{M}_K}{2} + \frac{\bar{M}_\pi \bar{M}_K}{2}) \\ &+ 4\beta_2 \bar{M}_\pi^4 + (4\beta_4 + 2\beta_7) \bar{E}_\pi \bar{M}_\pi^3 \\ &+ (\beta_4 - \beta_5 + \beta_7) \bar{M}_\pi^3 \bar{M}_K \\ &+ (\beta_4 - \beta_5 + \beta_7 + 2\beta_{22}) \bar{E}_\pi \bar{M}_\pi^2 \bar{M}_K \\ &+ (-4\beta_2 + 8\beta_{24}) \bar{M}_\pi^2 \bar{M}_K^2 \\ &+ (-2\beta_5 + 4\beta_7 + 4\beta_{22}) \bar{E}_\pi \bar{M}_\pi \bar{M}_K^2 \\ &+ (\beta_4 + 2\beta_7) (\bar{M}_\pi + \bar{E}_\pi) \bar{M}_K^3 \\ &+ (-16\beta_{24}) \bar{E}_\pi^2 \bar{M}_\pi^2 + 2\beta_{22} \bar{E}_\pi^2 \bar{M}_\pi \bar{M}_K \\ &\left. + 8\beta_{24} \bar{E}_\pi^2 \bar{M}_K^2 \right\} \\ &+ \frac{\alpha}{f_K f_\pi^2} \times [(\text{one loop})^{(27)}]_{\text{unphys}}^{\text{sym}}, \\ (\mathcal{M}^{(8,8)})_{\text{unphys}}^{\text{sym}} &= \frac{2\sqrt{2}}{f_K f_\pi^2} \times \left\{ \right. \\ &+ [4(\delta_4 + \delta_5) + 2\delta_6] \bar{M}_\pi^2 + [-(\delta_1 + \delta_2)] \bar{E}_\pi \bar{M}_\pi \\ &+ \left[\frac{1}{2}(\delta_1 + \delta_2) - (\delta_2 + \delta_3) \right] (\bar{M}_\pi + \bar{E}_\pi) \bar{M}_K \\ &+ [2(\delta_4 + \delta_5) + 4\delta_6] \bar{M}_K^2 \left. \right\} \\ &+ \frac{\gamma}{f_K f_\pi^2} [(\text{one loop})^{(8,8)}]_{\text{unphys}}^{\text{sym}}, \quad (4) \end{aligned}$$

for the unphysical kinematics described in Sec. 2. Here α (γ) is the coupling constant accompanying the $\mathcal{O}(p^2)$ ($\mathcal{O}(p^0)$) operator, and β_i (δ_i) are the coupling constants associated with $\mathcal{O}(p^4)$ ($\mathcal{O}(p^2)$) counterterm operators for (27,1) ((8,8)) chiral representation. We are able to reproduce the $(\text{one loop})_{\text{phys}}^{(27)}$ and $(\text{one loop})_{\text{phys}}^{(8,8)}$ results in Refs. [10] and [9]. We have also checked that all the amplitudes in Eqs. (3) and (4) are independent of the renormalisation scale μ_χ in χ PT with the same μ_χ dependence in each NLO coupling.

By having enough data points for the amplitudes in Eq. (4) at different values of \bar{E}_π , \bar{M}_π and \bar{M}_K , one can determine the coupling constants up to NLO chiral expansion and construct the physical amplitudes in Eq. (3). We are currently

performing a detailed numerical study (clover action, $\beta = 6.0$) of this strategy, as well as the one-loop calculations in finite-volume $q\chi$ PT. Figure 1 shows an example of our numerical work.

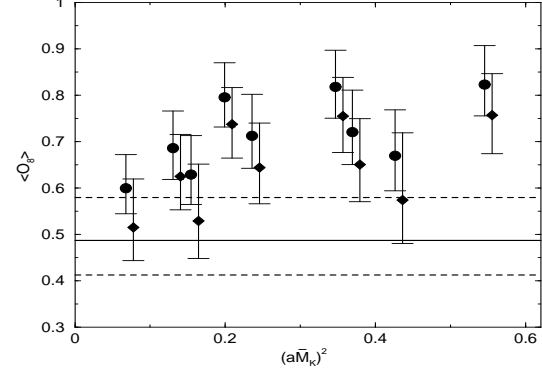


Figure 1. Bare lattice $\langle \pi\pi(I=2)|Q_8|K \rangle$, in lattice units, plotted against $(a\bar{M}_K)^2$. Circles are the data points in which both pions are at rest. Diamonds are the data points in which one of the pions carries non-zero spatial momentum. Points with the same $(a\bar{M}_K)^2$ are shifted for clarity. The solid line is the matrix element in the chiral limit, obtained via a fit to Eq. (4) using only data points where all the energies are below 1 GeV and setting chiral logs to zero. The dashed lines are the associated statistical errors.

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