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B-parameters of the complete set of matrix elements of $\Delta B = 2$ operators from the lattice

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Abstract

We compute on the lattice the "bag" parameters of the five $\Delta B = 2$ operators of the supersymmetric basis, by combining their values determined in full QCD and in the static limit of HQET. The extrapolation of the QCD results from the accessible heavy-light meson masses to the B-meson mass is constrained by the static result. The matching of the corresponding results in HQET and in QCD is for the first time made at NLO accuracy in the $\overline{\mathrm{MS}}$(NDR) renormalization scheme. All results are obtained in the quenched approximation.

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1 Introduction

This paper is devoted to a combined analysis of the matrix elements of the complete set of $\Delta B = 2$ operators, which we computed on the lattice in both the static limit of the heavy quark effective theory (HQET) and in standard lattice QCD (with Wilson fermions). All five operators enter the phenomenological analyses of supersymmetric (SUSY) effects that might affect the Standard Model (SM) expectations for Δm_{B_d} and/or Δm_{B_s}. It is therefore convenient to work in the so-called SUSY basis of operators:

$$\begin{aligned}
O_1 &= \bar{b}^i \gamma_\mu (1-\gamma_5) q^i \, \bar{b}^j \gamma_\mu (1-\gamma_5) q^j \, , \\
O_2 &= \bar{b}^i (1-\gamma_5) q^i \, \bar{b}^j (1-\gamma_5) q^j \, , \\
O_3 &= \bar{b}^i (1-\gamma_5) q^j \, \bar{b}^j (1-\gamma_5) q^i \, , \\
O_4 &= \bar{b}^i (1-\gamma_5) q^i \, \bar{b}^j (1+\gamma_5) q^j \, , \\
O_5 &= \bar{b}^i (1-\gamma_5) q^j \, \bar{b}^j (1+\gamma_5) q^i \, ,
\end{aligned} \tag{1}$$

where the superscripts denote colour indices, and q stands for either d- or s- light quark flavour. The first of the above operators has been widely studied over the last decade, since it is crucial for the SM description of the $B^0 - \overline{B^0}$ mixing amplitude, whereas O_2 and O_3 were also recently studied because they are relevant for the SM estimates of the relative width difference in the neutral B-meson system, $(\Delta\Gamma/\Gamma)_{B_s}$ [1].

It is customary to parameterize the matrix elements of the operators (1) in terms of the so-called "bag"-parameters, which are introduced as a measure of the mismatch between the vacuum saturation approximation (VSA) and the actual value for each of the matrix elements, namely [2, 3]

$$\begin{aligned}
\langle \bar{B}^0_q | \hat{O}_1(\mu) | B^0_q \rangle &= \frac{8}{3} m^2_{B_q} f^2_{B_q} B_1(\mu) \, , \\
\langle \bar{B}^0_q | \hat{O}_2(\mu) | B^0_q \rangle &= -\frac{5}{3} \left(\frac{m_{B_q}}{m_b(\mu) + m_q(\mu)} \right)^2 m^2_{B_q} f^2_{B_q} B_2(\mu) \, , \\
\langle \bar{B}^0_q | \hat{O}_3(\mu) | B^0_q \rangle &= \frac{1}{3} \left(\frac{m_{B_q}}{m_b(\mu) + m_q(\mu)} \right)^2 m^2_{B_q} f^2_{B_q} B_3(\mu) \, , \\
\langle \bar{B}^0_q | \hat{O}_4(\mu) | B^0_q \rangle &= 2 \left(\frac{m_{B_q}}{m_b(\mu) + m_q(\mu)} \right)^2 m^2_{B_q} f^2_{B_q} B_4(\mu) \, , \\
\langle \bar{B}^0_q | \hat{O}_5(\mu) | B^0_q \rangle &= \frac{2}{3} \left(\frac{m_{B_q}}{m_b(\mu) + m_q(\mu)} \right)^2 m^2_{B_q} f^2_{B_q} B_5(\mu) \, .
\end{aligned} \tag{2}$$

The hat symbol denotes operators renormalized in some renormalization scheme at the renormalization scale μ. To determine the values of the "bag" parameters $B_{1-5}(\mu)$, we have performed a numerical simulation of (quenched) QCD on the lattice. While such a simulation can be made directly for the c-quark mass or somewhat heavier, present limitations of computational resources do not allow for a direct study of the b-quark. For this reason we work in the range of heavy-light pseudoscalar masses $m_P \in (1.7, 2.4)$ GeV, from which we have to extrapolate to the physical point, $m_{B_d} = 5.28$ GeV and/or $m_{B_s} = 5.37$ GeV. Although guided by the HQET scaling laws, this extrapolation is the dominant source of the systematic uncertainty in final results. To get around this problem we also computed the same matrix elements in the static limit of HQET on the lattice, and used them to constrain the extrapolations towards the physical point, $m_{B_{s/d}}$. It is technically challenging to combine results from two different theories, namely one should match the B-parameters obtained in QCD onto the HQET ones so that the heavy quark scaling laws can be safely used. A special care to this issue will be given in the body of this paper.

The main features of this work are:

- The B-parameters which appear in eq. (2) are computed using lattice QCD with Wilson fermions and are renormalized non-perturbatively in the (Landau) RI/MOM scheme. It is important to stress that we incorporated the recent proposal to remove the effects of the Goldstone boson contamination [4, 5];

- The B-parameters computed in the static limit of HQET on the lattice are matched onto the continuum $\overline{\rm MS}$(NDR) renormalization scheme. This matching has been made by using the one-loop (boosted) perturbative expressions [6, 7];

- The conversion of the operators computed in lattice QCD from RI/MOM to $\overline{\rm MS}$(NDR) scheme is made at NLO accuracy [8, 9]. Matching of the QCD operators onto the HQET ones has also been performed at NLO accuracy in a specified $\overline{\rm MS}$(NDR) scheme. In that procedure we use the recently computed 2-loop anomalous dimension matrices in HQET [10]. With this matching at hand, we were able to constrain the extrapolation, *i.e.* to interpolate to the physical b-quark mass;

- Final results are presented in the RI/MOM scheme and in the $\overline{\rm MS}$(NDR) scheme of ref. [9]. In addition, the parameters $B_{1,2,3}$ are also given in the $\overline{\rm MS}$(NDR) renormalization scheme of ref. [11]. [1]

The complete list of results can be found in table 1. Notice that we do not observe any $SU(3)$ breaking effect in the B-parameters, *i.e.*:

$$\left.\frac{B_i^{(s)}}{B_i^{(d)}}\right|_{i=1,\dots,5} = \Big\{0.99(2),\ 1.01(2),\ 1.01(3),\ 1.01(2),\ 1.01(3)\Big\}\ . \tag{3}$$

This paper is organized as follows: In sec. 2 we give the essential details of our lattice calculations and present the results as obtained for each heavy quark that we were able to access from our lattice and also in the static limit of the HQET. In sec. 3 we outline the general strategy to combine the results of the two theories. We then explicitly give all the necessary anomalous dimension matrices and present the results of the combined analysis for all the five B-parameters. In sec. 4 we discuss the systematic uncertainties which are included in the results given in table 1. We briefly conclude in sec. 5.

[1] Preliminary results were presented in ref. [12].

Scheme	RI/MOM	$\overline{\rm MS}$(NDR) [9]	$\overline{\rm MS}$(NDR) [11]
$B_1^{(d)}(m_b)$	$0.87(4)\ \binom{+5}{-4}$	$0.87(4)\ \binom{+5}{-4}$	$0.87(4)\ \binom{+5}{-4}$
$B_2^{(d)}(m_b)$	0.82(3)(4)	0.79(2)(4)	0.83(3)(4)
$B_3^{(d)}(m_b)$	1.02(6)(9)	0.92(6)(8)	0.90(6)(8)
$B_4^{(d)}(m_b)$	$1.16(3)\ \binom{+5}{-7}$	$1.15(3)\ \binom{+5}{-7}$	–
$B_5^{(d)}(m_b)$	$1.91(4)\ \binom{+22}{-7}$	$1.72(4)\ \binom{+20}{-6}$	–
$B_1^{(s)}(m_b)$	$0.86(2)\ \binom{+5}{-4}$	$0.87(2)\ \binom{+5}{-4}$	$0.87(2)\ \binom{+5}{-4}$
$B_2^{(s)}(m_b)$	0.83(2)(4)	0.80(1)(4)	0.84(2)(4)
$B_3^{(s)}(m_b)$	1.03(4)(9)	0.93(3)(8)	0.91(3)(8)
$B_4^{(s)}(m_b)$	$1.17(2)\ \binom{+5}{-7}$	$1.16(2)\ \binom{+5}{-7}$	–
$B_5^{(s)}(m_b)$	$1.94(3)\ \binom{+23}{-7}$	$1.75(3)\ \binom{+21}{-6}$	–

Table 1: *The main results of this paper: B-parameters defined in eq. (2), renormalized at $\mu = m_b = 4.6$ GeV and in three renormalization schemes:* RI/MOM, $\overline{\rm MS}$ *of ref. [9] and the* $\overline{\rm MS}$ *of ref. [11]. The results are obtained in the quenched approximation.*

2 Direct Lattice results

2.1 Computation in lattice QCD

In this subsection we recall the main elements of our lattice simulation, the details of which can be found in refs. [13, 14]. We work with a lattice of the size $24^3 \times 48$, at $\beta = 6.2$, and use the non-perturbatively improved Wilson action [15]. Note however that the 4-fermion operators, which are the main target of the present work, are not improved. Our data-set consists of 200 independent gauge field configurations. We work with 3 values of the heavy and 3 values of the light quark masses, corresponding to the Wilson hopping parameters: $\kappa_q \in \{0.1344, 0, 1349, 0.1352\}$, and $\kappa_Q \in \{0.125, 0, 122, 0.119\}$. The mass spectrum and the decay constants have already been discussed in our previous publications [13, 14, 16] and we immediately turn to the computation of the B-parameters.

The starting point is to compute the 2- and 3-point correlation functions

$$\mathcal{C}_{JJ}^{(2)}(t) = \langle \sum_{\vec{x}} J(\vec{x},t) J^\dagger(0)\rangle \ \xrightarrow{t \gg 0}\ \frac{\mathcal{Z}_J}{2\sinh M_J}\, e^{-M_J t}\ ,$$

$$\mathcal{C}_i^{(3)}(t_1,t_2) = \langle \sum_{\vec{x},\vec{y}} P_5(\vec{x},t_2)\hat{O}_i(\vec{0},0;\mu)P_5^\dagger(\vec{y},t_1)\rangle$$
$$\xrightarrow{T-t_2\gg 0} \frac{\sqrt{\mathcal{Z}_P}}{2\sinh M_P} e^{-M_P t_1} \cdot \langle P_q|\hat{O}_i(\mu)|P_q\rangle \cdot \frac{\sqrt{\mathcal{Z}_P}}{2\sinh M_P} e^{-M_P t_2} \ , \qquad (4)$$

where the operator $\hat{O}_i$ is placed at time equal to zero, the source of the pseudoscalar mesons ($P_5 = \bar{Q}\gamma_5 q$) is fixed at $t_1 = 16$, while the other source operator moves around the periodic lattice (of the size $T = 48$). At some $t_2 \equiv t$, which is sufficiently far from the first source and from the operator $\hat{O}_i$, the lowest lying heavy-light pseudoscalar meson, P_q, is isolated. J in the above equations stands for either P_5 or $A_0 = \bar{Q}\gamma_0\gamma_5 q$.

To extract the parameter $B_1(\mu)$, one computes

$$R_{B_1}(t) = \frac{\mathcal{C}_1^{(3)}(t_1,t;\mu)}{\frac{8}{3} Z_A^2\ \mathcal{C}_{AP}^{(2)}(t)\ \mathcal{C}_{AP}^{(2)}(t_1)} \xrightarrow{T-t\gg 0} \frac{\langle P_q|\hat{O}_1(\mu)|P_q\rangle}{\frac{8}{3}|\langle 0|\hat{A}_0|P_q\rangle|^2} \equiv B_1(\mu) \ , \qquad (5)$$

where the 2-points functions are used to eliminate the exponential terms from the 3-point functions (4) and also to divide out the $(8/3) f_P^2 m_P^2$ from eq. (2), thus accessing directly the wanted B-parameter. Similarly, to reach other four B-parameters we form the ratios

$$R_{B_i}(t) = \frac{\mathcal{C}_i^{(3)}(t_1,t;\mu)}{b_i\ Z_P^2(\mu)\ \mathcal{C}_{PP}^{(2)}(t)\ \mathcal{C}_{PP}^{(2)}(t_1)} \xrightarrow{T-t\gg 0} \frac{\langle P_q|\hat{O}_i(\mu)|P_q\rangle}{b_i\ |\langle 0|\hat{P}_5(\mu)|P_q\rangle|^2} \equiv B_i(\mu) \ , \qquad (6)$$

where $b_i \in \{-5/3, 1/3, 2, 2/3\}$ for $i \in \{2,3,4,5\}$, respectively.

As already discussed in our previous works [13, 14], the operators $\hat{O}_i(\mu)$ are renormalized non-perturbatively in the (Landau)RI/MOM scheme by using the method explained in detail in ref. [17] (see also references therein). The method is based on the possibility of computing amputated 4-quark vertices at sufficiently large quark virtualities with the operators inserted at zero momentum. The RI/MOM renormalization condition is merely imposed on various projections of the amputated Green functions, which then lead to a full set of 9 renormalization ($Z(\mu, g_0^2)$) and 16 subtraction ($\Delta(g_0^2)$) constants, where $g_0^2 = 6/\beta$ is the bare lattice coupling. The renormalized operators are given by

$$\hat{O}_i(\mu) \ = \ Z_{ij}(\mu, g_0^2)\ \left[O_j(g_0^2) \ + \ \Delta_{jk}(g_0^2) O_k(g_0^2)\right]_{k\neq j} \ . \qquad (7)$$

Since the computation of the off-shell quantities (4-quark vertices) is performed in the Landau gauge, the scheme is often referred to as the Landau RI/MOM scheme.

In practice, one computes all the renormalization and subtraction constants with specific (nonzero) values of the light quark masses and then extrapolates all Z's and Δ's to the chiral limit. It has been pointed out in refs. [4, 5] that such an extrapolation can be contaminated by the Goldstone boson (GB) contributions. The recipe to subtract these contributions away has been proposed and implemented in ref. [18] (see also appendix of this paper). We employed that prescription and recomputed the renormalization and subtraction constants. Their values, together with $Z_P(\mu)$ and Z_A which are needed in eqs. (5,6), are listed in appendix. This is a new feature of our computation which improves (corrects) our previous results, presented in refs. [13, 14].

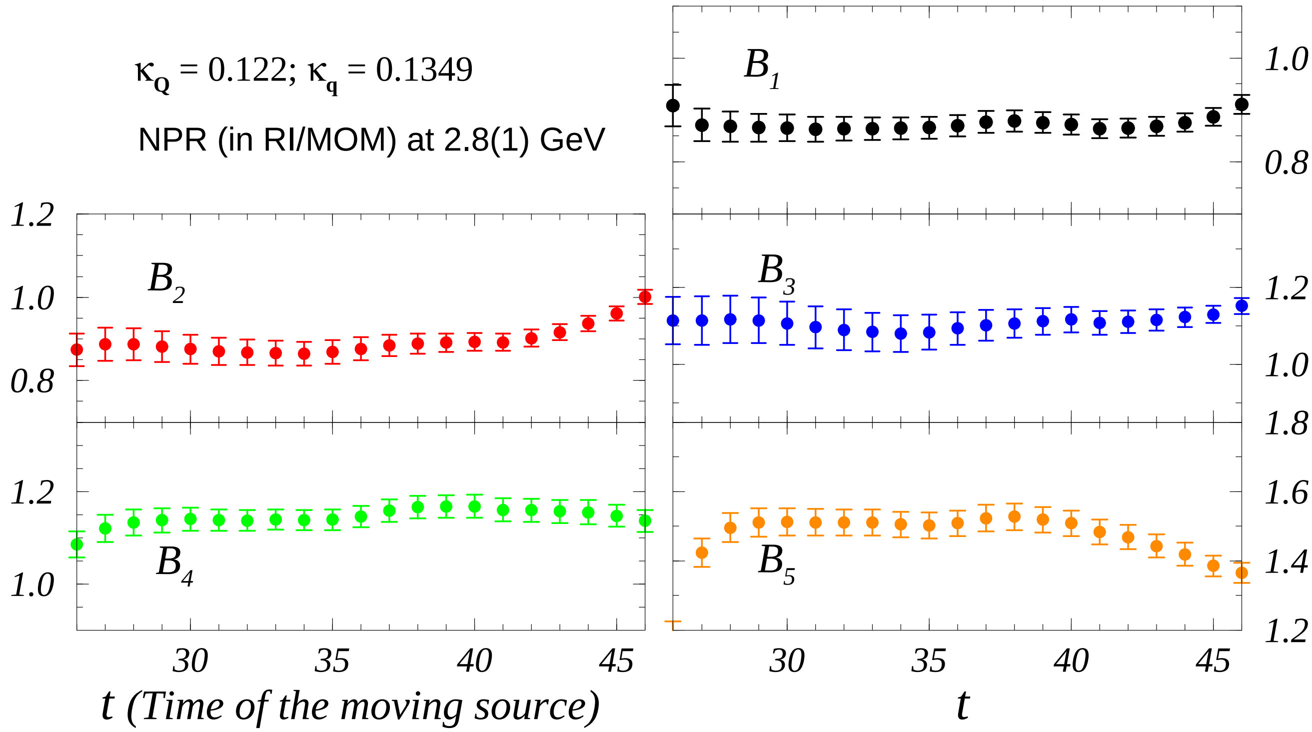

Figure 1: *Signals for the ratios $R_{B_i}(t)$, defined in eqs. (5) and (6), are shown for the combination: $\kappa_q = 0.1349$, $\kappa_Q = 0.122$. The operators are non-perturbatively renormalized (NPR) in the (Landau)RI/MOM scheme at $\mu \simeq 1/a = 2.8(1)$ GeV.*

In fig. 1, we illustrate the quality of the signals for the ratios $R_{B_{1-5}}(t)$ for a specific combination of heavy and light quarks. After inspecting the ratios for all 9 pairs of κ_Q–κ_q, we find that common stability plateaus are reached for

$$\begin{aligned} R_{B_1} &: \quad t \in [28, 35]\,; \\ R_{B_{2,3}} &: \quad t \in [30, 35]\,; \\ R_{B_{4,5}} &: \quad t \in [29, 35]\,. \end{aligned} \tag{8}$$

On each of these plateaus we fit the ratios to a constant and hence extract the parameters $B_{1-5}(\mu)$ for each combination of the heavy and the light quark masses. Every parameter is then linearly interpolated in the light quark mass to the *strange* and extrapolated to the *up/down* light quark mass, by using the (standard) lattice-plane method (see refs. [19] for details). The extrapolations are very smooth as it can be seen from fig. 2. In table 2, we present a detailed list of the results for all five B-parameters renormalized non-perturbatively at three different values of the renormalization scale μ in the (Landau)RI/MOM scheme. The results are given for the values of the heavy quark masses, which are in the region of the charm quark and slightly higher, and for the light quark interpolated to the strange and to the averaged up/down mass.

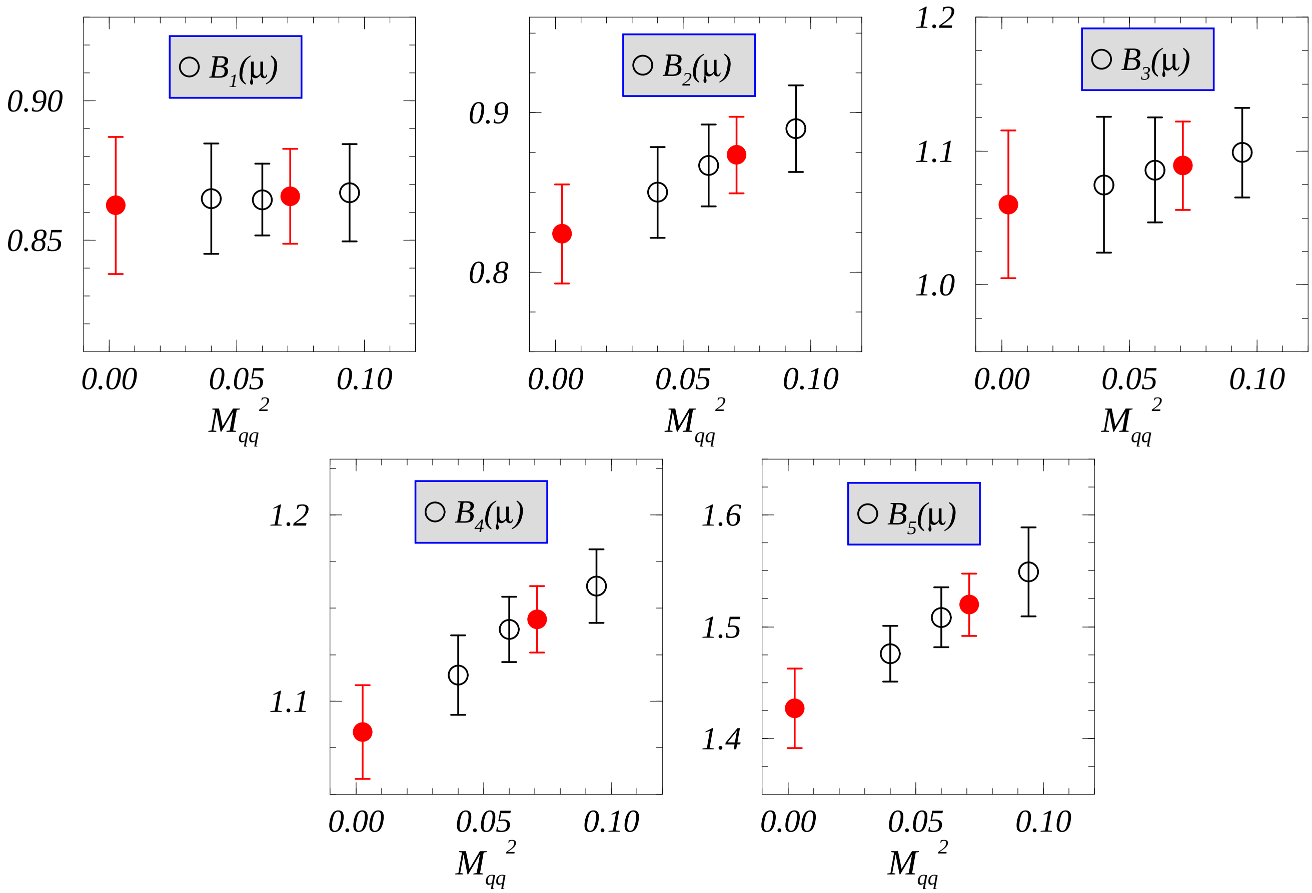

Figure 2: *Extrapolations to the light up/down and interpolations to the strange light quark mass are shown for all five B-parameters in the case of fixed heavy quark mass corresponding to* $\kappa_Q = 0.122$. *Empty symbols denote the B-parameters directly accessed in our simulation. Filled symbols correspond to the B parameters extrapolated to the physical* u/d *and* s *quark respectively. In the figure* $\mu a = 1.03$.

2.2 Computation in the static limit of HQET

To avoid a confusion in notations, we consistently use "tilde" symbols over the operators and the B-parameters computed in HQET. Instead of the five operators that we listed in eq. (1), in HQET one deals with only four of them, namely

$$\begin{aligned}
\widetilde{O}_1 &= \bar{h}^i \gamma_\mu (1-\gamma_5) q^i \, \bar{h}^j \gamma_\mu (1-\gamma_5) q^j \,, \\
\widetilde{O}_2 &= \bar{h}^i (1-\gamma_5) q^i \, \bar{h}^j (1-\gamma_5) q^j \,, \\
\widetilde{O}_4 &= \bar{h}^i (1-\gamma_5) q^i \, \bar{h}^j (1+\gamma_5) q^j \,, \\
\widetilde{O}_5 &= \bar{h}^i (1-\gamma_5) q^j \, \bar{h}^j (1+\gamma_5) q^i \,,
\end{aligned} \tag{9}$$

	q=up/down			q=strange		
Scale (μ)	1.9(1) GeV	2.8(1) GeV	3.9(2) GeV	1.9(1) GeV	2.8(1) GeV	3.9(2) GeV
$B_1^{(Q_1)}(\mu)$	0.856(20)	0.841(20)	0.825(20)	0.866(15)	0.850(15)	0.835(15)
$B_2^{(Q_1)}(\mu)$	0.875(38)	0.817(31)	0.797(31)	0.904(27)	0.843(22)	0.822(22)
$B_3^{(Q_1)}(\mu)$	1.295(73)	1.072(51)	0.941(42)	1.284(48)	1.074(34)	0.947(28)
$B_4^{(Q_1)}(\mu)$	1.092(23)	1.107(22)	1.035(20)	1.112(16)	1.129(16)	1.056(15)
$B_5^{(Q_1)}(\mu)$	1.523(37)	1.322(29)	1.332(27)	1.606(30)	1.386(24)	1.392(22)
$B_1^{(Q_2)}(\mu)$	0.880(25)	0.862(24)	0.849(24)	0.883(17)	0.866(17)	0.852(17)
$B_2^{(Q_2)}(\mu)$	0.886(37)	0.824(31)	0.803(30)	0.940(26)	0.874(22)	0.851(22)
$B_3^{(Q_2)}(\mu)$	1.275(81)	1.060(55)	0.932(45)	1.294(48)	1.089(33)	0.963(28)
$B_4^{(Q_2)}(\mu)$	1.066(26)	1.083(25)	1.013(24)	1.124(17)	1.144(17)	1.070(16)
$B_5^{(Q_2)}(\mu)$	1.673(45)	1.427(36)	1.421(34)	1.784(35)	1.520(28)	1.511(26)
$B_1^{(Q_3)}(\mu)$	0.887(26)	0.869(25)	0.856(25)	0.886(16)	0.867(15)	0.855(15)
$B_2^{(Q_3)}(\mu)$	0.950(33)	0.881(27)	0.857(26)	0.962(23)	0.890(19)	0.869(19)
$B_3^{(Q_3)}(\mu)$	1.349(84)	1.123(54)	0.989(43)	1.288(51)	1.090(34)	0.966(27)
$B_4^{(Q_3)}(\mu)$	1.114(23)	1.136(21)	1.062(20)	1.119(14)	1.141(13)	1.068(12)
$B_5^{(Q_3)}(\mu)$	1.885(47)	1.592(35)	1.573(33)	1.925(32)	1.622(25)	1.601(23)

Table 2: *B-parameters, as defined in eq. (2), extracted from our non-perturbatively renormalized data at three values of the renormalization scale μ in the RI/MOM renormalization scheme. From top to bottom of the table, the bag parameters correspond to the heavy quark with $\kappa_{Q_1} = 0.125$, $\kappa_{Q_2} = 0.122$ and $\kappa_{Q_3} = 0.119$, respectively (they are separated by the horizontal lines).*

where h stands for the infinitely heavy (static) quark. In the HQET, the operator $\widetilde{O}_3$ is related to $\widetilde{O}_1$ and $\widetilde{O}_2$ by the equations of motion as

$$\widetilde{O}_3 = -\widetilde{O}_2 - \frac{1}{2}\widetilde{O}_1 \; . \qquad (10)$$

The computation of the first two operators has been explained in detail in refs. [6, 20]. The data-set consists of 600 configurations gathered on the $24^3 \times 40$ lattice at $\beta = 6.0$. The light quarks were simulated by using the tree-level improved Wilson action and three values of $\kappa_q \in \{0.1425, 1432, 1440\}$.

To reach the HQET parameters equivalent to the ones appearing in eq. (2), which we will call $\widetilde{B}_i$, one computes the following 2- and 3-point correlation functions:

$$\begin{aligned} \widetilde{\mathcal{C}}^{(2)}_{AA}(t) &= \langle \sum_{\vec{x}} \widetilde{A}_0(\vec{x},t)\widetilde{A}^\dagger_0(0)\rangle \xrightarrow{t\gg 0} \widetilde{\mathcal{Z}}_A e^{-\Delta\mathcal{E}t} , \\ \widetilde{\mathcal{C}}^{(3)}_{i}(t_1,t_2) &= \langle \sum_{\vec{x},\vec{y}} \widetilde{A}_0(\vec{x},t_2)\widetilde{O}_i(\vec{0},0;\mu)\widetilde{A}^\dagger_0(\vec{y},t_1)\rangle \\ &\xrightarrow{t\gg 0} \widetilde{\mathcal{Z}}_A \cdot \frac{\langle P_q|\widetilde{O}_i(\mu)|P_q\rangle}{2M_P} \cdot e^{-\Delta\mathcal{E}(t_1-t_2)} , \end{aligned} \tag{11}$$

where the source operators are the axial currents [2] which are extended by using the so-called cubic smearing procedure [21]. $\Delta\mathcal{E}$ in eq. (11), is the binding energy of the heavy meson P_q. Note also that in this case, $\sqrt{\widetilde{\mathcal{Z}}_A} = \langle 0|\widetilde{A}_0|P_q\rangle/\sqrt{2M_P}$. To get the desired B-parameters, we form the following ratios:

$$\widetilde{R}_{B_i}(t_1) = \frac{\widetilde{\mathcal{C}}^{(3)}_{i}(t_1,t_2;\mu)}{b_i\ Z_A^2\ \widetilde{\mathcal{C}}^{(2)}_{AA}(t_2)\ \widetilde{\mathcal{C}}^{(2)}_{AA}(t_1)} \xrightarrow{t_1\gg 0} \frac{\langle P_q|\widetilde{O}_i(\mu)|P_q\rangle}{b_i\ |\langle 0|\widetilde{A}_0|P_q\rangle|^2} \equiv \widetilde{B}_i(\mu) , \tag{12}$$

where $b_i \in \{8/3, -5/3, 2, 2/3\}$ for $i \in \{1,2,4,5\}$. In this case, the fixed time has been chosen to be $t_2 = 35$. In fig. 3, we show the quality of the signals for all four $\widetilde{R}_i(t_1)$. On the plateaus we fit to a constant and thus obtain the values of the corresponding parameters $\widetilde{B}_i(\mu)$. Our data is renormalized in the $\overline{\mathrm{MS}}$(NDR) scheme, after matching the lattice regularization scheme onto the $\overline{\mathrm{MS}}$(NDR) by using one-loop boosted perturbation theory, as explained in great detail in ref. [6]. The matching scale $\mu = q^*$ is varied between $2/a \le q^* \le \pi/a$, and the results are run to $\mu = m_b = 4.6$ GeV. The spread of values is assigned to the systematic uncertainty. As in the previous subsection, all 4 operators are linearly extrapolated (interpolated) in the the light quark mass to up/down (strange). The resulting values are listed in table 3. In that table errors are statistical only, obtained by using the standard jackknife procedure.

3 Extrapolation to the B-mesons

Armed with "raw" results obtained in full QCD (table 2) and in HQET (table 3), we now discuss the extrapolation of the QCD results to the physical B-meson mass. The aim of this section is to provide a consistent way to constrain that extrapolation by the static HQET results in order to reduce the systematic uncertainties.

A common wisdom is to follow the HQET scaling laws, according to which every B-parameter scales with the inverse heavy quark (meson) mass as a constant, and to extrapolate to the desired heavy meson mass. The (unknown) $1/m_P$ corrections are to be determined from the fit with our data. To use these scaling laws, however, one first need to

[2] In the static limit the axial current is identical to the pseudoscalar density.

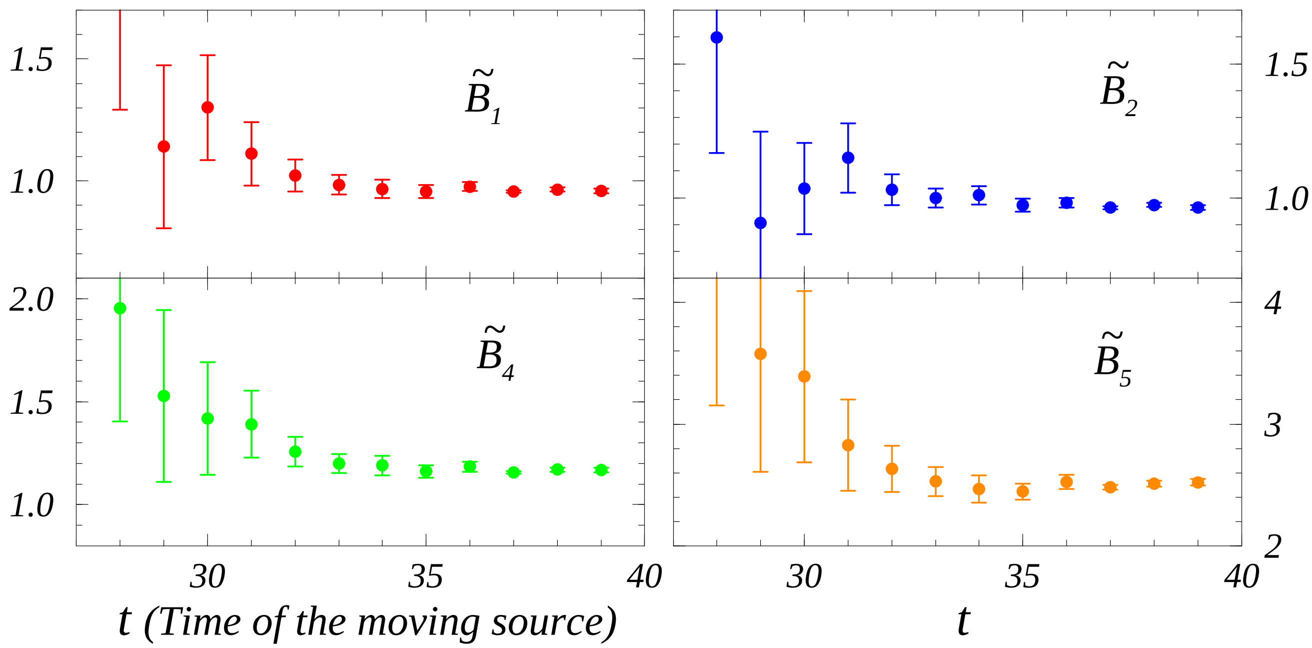

Figure 3: *Signals for the ratios* $\widetilde{R}_{B_i}(t)$ *defined in eqs. (12) from which the* $\widetilde{B}_i$ *parameters are extracted. The common plateaus are chosen for* $t_1 \equiv t \in [32, 36]$. *The plotted ratios correspond to* $\kappa_q = 0.1432$. *The operators are renormalized perturbatively and evolved to* $\mu = m_b$.

Light quark	*q=up/down*	*q=strange*
$\widetilde{B}_1(m_b)$	0.89(4)	0.89(4)
$\widetilde{B}_2(m_b)$	0.82(4)	0.83(3)
$\widetilde{B}_4(m_b)$	1.07(4)	1.07(4)
$\widetilde{B}_5(m_b)$	2.37(10)	2.40(10)

Table 3: $\tilde{B}$*-parameters, extracted from our HQET data by using the boosted perturbative 1-loop matching onto the continuum* $\overline{\mathrm{MS}}$*(NDR) renormalization scheme.*

relate the matrix elements of the QCD operators, $\langle O_i(\mu)\rangle$ from eq. (1), to the HQET ones, $\langle \widetilde{O}_i(\mu)\rangle$ of eq. (9) [24]. This matching is made in perturbation theory at some suitably chosen renormalization scale, for example $\mu = m_b$. The matching is crucial since the anomalous dimensions for these operators in the two theories (QCD and HQET) differ. Moreover, when dealing with the 4-fermion operators, it is important that the matching between the two theories is made at NLO accuracy because it is at this order that the scheme can be fully specified (leading order anomalous dimensions are universal). Before entering the details of that matching, we now outline the basic strategy that must be followed.

Matching of the QCD operators, renormalized at some high scale $\mu \gg m_P$, and the HQET ones, renormalized at some low scale $\mu' \ll m_P$, is made at $\mu = m_P$ by using the following expression [3]

$$\mathbf{W}^T_{QCD}[m_P,\mu]^{-1}\ \langle \vec{O}(\mu)\rangle_{m_P} \ = \ C(m_P)\ \mathbf{W}^T_{HQET}[m_P,\mu']^{-1}\ \langle \vec{\widetilde{O}}(\mu')\rangle + \mathcal{O}\left(\frac{1}{m_P}\right)\ +\ \dots \quad (13)$$

where $\mathbf{W}^T_{QCD}[\mu_2,\mu_1]^{-1}$ is the matrix encoding the full QCD evolution from a scale μ_1 to μ_2 of all five $\Delta B = 2$ operators which are, for convenience, collected in a five-component vector $\langle \vec{O}(\mu)\rangle_{m_P}$. Likewise for $\mathbf{W}^T_{HQET}[\mu_2,\mu_1]^{-1}$ in HQET. These matrices will be specified later on. We will be working in the $\overline{\rm MS}$(NDR) scheme in which the matrices of the anomalous dimension coefficients are known at NLO in both theories. Hence, the matching matrix, $C(m_P) = 1 + \sum_n c^{(n)}[\alpha_s(m_P)/4\pi]^n$, is also known at NLO, *i.e.* $c^{(1)}$ is completely determined [10].

On the HQET side, we also consider five operators, $\langle \vec{\widetilde{O}}(\mu)\rangle$, where we add the operator $\widetilde{O}_3$ by means of eq. (10). In this way the matching matrix $c^{(1)}$ is squared (5×5). We now put all the evolution expressions appearing in eq. (13) on its l.h.s.

$$\begin{aligned}
&\mathbf{W}^T_{HQET}[\mu',m_P]^{-1}C^{-1}(m_P)\mathbf{W}^T_{QCD}[m_P,\mu]^{-1}\langle \vec{O}(\mu)\rangle_{m_P} \ = \\
&\Big(\underbrace{\mathbf{W}^T_{QCD}[m_P,\mu]C(m_P)\mathbf{W}^T_{HQET}[m_P,\mu']^{-1}}_{\mathcal{M}_4[m_P,\mu,\mu']}\Big)^{-1}\langle \vec{O}(\mu)\rangle_{m_P} \ = \ \langle \vec{\widetilde{O}}(\mu')\rangle + \mathcal{O}\left(\frac{1}{m_P}\right) + \dots \quad (14)
\end{aligned}$$

so that the l.h.s. manifestly satisfies the HQET scaling laws which are the intrinsic property of the r.h.s. One proceeds similarly for the bilinear operators to define the matching constants $\mathcal{M}_2[m_P,\mu,\mu']$. In terms of B-parameters, eq. (14) then reads

$$\mathcal{M}_2^2[m_P,\mu,\mu']\ \Big(\mathbf{b}^{-1}\mathcal{M}_4^{-1}[m_P,\mu,\mu']\mathbf{b}\Big)\ \vec{B}(\mu) = \vec{\widetilde{B}}(\mu') + \mathcal{O}\left(\frac{1}{m_P}\right) + \dots \qquad (15)$$

where $\mathbf{b}$ is the diagonal matrix of the coefficients appearing in the definitions (2), *i.e.* $\mathbf{b} = \mathrm{diag}(8/3,-5/3,1/3,2,2/3)$, and by $\vec{B}(\mu)$ $(\vec{\widetilde{B}}(\mu'))$ we designated the vector column of our five B-parameters.

Based on the above discussion, a simple recipe can be applied to our data, namely evolve to the same $\mu = \mu'$ and create the quantity

$$\vec{\Phi}(m_P,\mu) = \mathcal{M}_2^2[m_P,\mu]\ \Big(\mathbf{b}^{-1}\mathcal{M}_4^{-1}[m_P,\mu]\mathbf{b}\Big)\cdot \vec{B}(\mu) \qquad (16)$$

which can be fit either freely as

$$\vec{\Phi}(m_P,\mu) = \vec{a}_0(\mu) + \frac{\vec{a}_1(\mu)}{m_P}\,, \qquad (17)$$

[3] Instead of matching at the point corresponding to the mass of the heavy quark, we choose to do it at the mass of the heavy-light meson, m_P.

where $\vec{a}_0(\mu)$ and $\vec{a}_1(\mu)$ are the fit parameters, or by constraining it by the static HQET results, *i.e.*

$$\vec{\Phi}(m_P,\mu) = \vec{a}'_0(\mu) + \frac{\vec{a}'_1(\mu)}{m_P} + \frac{\vec{a}'_2(\mu)}{m_P^2}\,, \tag{18}$$

where the coefficient $\vec{a}'_0(\mu)$ is constrained by the static value, $\vec{\tilde{B}}(\mu)$, so that one can probe the term $\mathcal{O}(1/m_P^2)$. As a result of these two procedures, we obtain the HQET values of the B-parameters, *i.e.* $\vec{\Phi}(m_{B_{s/d}},\mu)$, which are then to be matched back onto their QCD counterparts.

To keep the expressions as short as possible, we will now split the discussion into two pieces: we will first discuss the extrapolation of the first three B-parameters and then the last two. This can be done because all the matrices, $\mathbf{W}^T_{QCD}[\mu_2,\mu_1]$, $\mathbf{W}^T_{HQET}[\mu_2,\mu_1]$ and $C(\mu)$ are the block-matrices of the form $[3\times 3]\oplus[2\times 2]$.

3.1 Getting the physical results for $B^{\overline{\rm MS}}_{1,2,3}(m_b)$

At the leading order in perturbation theory, the anomalous dimensions for our B-parameters in the RI/MOM and $\overline{\rm MS}$ schemes are the same. This is not the case at NLO, and in order to proceed we need to convert our RI/MOM results from table 2 into the $\overline{\rm MS}$(NDR) scheme. It is crucial to specify the set of evanescent operators or the Dirac projectors used to renormalize the operators because only with this information at hand, the $\overline{\rm MS}$(NDR) scheme is unambiguously defined [25]. In this subsection, we will use the $\overline{\rm MS}$(NDR) scheme of ref. [11] (see eqs. (13-15) of their paper) in which the Wilson coefficients for the SM expression for the $(\Delta\Gamma/\Gamma)_{B_s}$ have been calculated at NLO. Therefore, the results for the B-parameters that will be presented in this subsection can be directly combined with the Wilson coefficients of ref. [11].

The conversion of the operators $O^{\rm RI/MOM}_{1,2,3}(\mu)$ and $P^{\rm RI/MOM}_5(\mu)$ to the $\overline{\rm MS}$ scheme is provided by the following expressions

$$\begin{pmatrix}\langle O_1(\mu)\rangle\\ \langle O_2(\mu)\rangle\\ \langle O_3(\mu)\rangle\end{pmatrix}^{\overline{\rm MS}} = \left[\,\mathbb{I} \;+\; r_{123}\frac{\alpha_s(\mu)}{4\pi}\,\right]\begin{pmatrix}\langle O_1(\mu)\rangle\\ \langle O_2(\mu)\rangle\\ \langle O_3(\mu)\rangle\end{pmatrix}^{\rm RI/MOM}\,,$$

$$\langle P_5(\mu)\rangle^{\overline{\rm MS}} = \left(1 + r_P\frac{\alpha_s(\mu)}{4\pi}\right)\langle P_5(\mu)\rangle^{\rm RI/MOM}\,, \tag{19}$$

where the NLO matching coefficients are given by [8, 9, 10]

$$r_P = \frac{16}{3}\,,\qquad r_{123} = \frac{1}{9}\begin{pmatrix} -42\;+\;72\log 2 & 0 & 0\\ 0 & 61+44\log 2 & -7+28\log 2\\ 0 & -25+28\log 2 & -29+44\log 2\end{pmatrix}. \tag{20}$$

To get the central values, we will convert the results from table 2, obtained at $\mu = 2.8(1)$ GeV and run them to $\mu = m_b = 4.6$ GeV [22]. It should be noted that the matching RI/MOM $\to$

$\overline{\rm MS}$(NDR), made in ref. [14], was incorrect because the $\overline{\rm MS}$ scheme was not the one of ref. [11], but rather the one of ref. [9]. Although the numerical differences are very small, the physical results presented in ref. [14] are not fully consistent because the matrix elements matched onto the $\overline{\rm MS}$ scheme of ref. [9] were combined with the Wilson coefficients of ref. [11]. The correct physical results were presented in ref. [23].

The evolution from the scale μ to m_b, in this $\overline{\rm MS}$ scheme, is described by [11]

$$\begin{pmatrix} O_1(m_b) \\ O_2(m_b) \\ O_3(m_b) \end{pmatrix}^{\overline{\rm MS}} = \mathbf{W}^T_{QCD}[m_b,\mu]^{-1} \begin{pmatrix} O_1(\mu) \\ O_2(\mu) \\ O_3(\mu) \end{pmatrix}^{\overline{\rm MS}} \tag{21}$$

where the operator $\mathbf{W}_{QCD}[m_b,\mu] = M(m_b)U(m_b,\mu)M^{-1}(\mu)$ contains the information on the evolution obtained at the leading $(U(\mu,m_b))$ and the NLO $(M(\mu))$ in perturbation theory. For our purpose, it is convenient to write the evolution matrix in the following form

$$\mathbf{W}_{QCD}[m_b,\mu] = w(m_b)w^{-1}(\mu)\,, \tag{22}$$

where

$$w(\mu) \;=\; M(\mu)\;\alpha_s(\mu)^{-\gamma_0^T/2\beta_0}\;, \tag{23}$$

and $\beta_0 = 11 - 2n_F/3$. The scheme independent, one-loop anomalous dimension matrix is

$$\gamma_0 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -28/3 & 4/3 \\ 0 & 16/3 & 32/3 \end{pmatrix}\,,$$

whereas the NLO contribution

$$M(\mu) \;=\; \mathbb{I} \;+\; J_{123}^{\overline{\rm MS}}\,\frac{\alpha_s(\mu)}{4\pi} \tag{24}$$

is encoded in the matrix

$$J_{123}^{\overline{\rm MS}} \;=\; \begin{pmatrix} \frac{485}{242} & 0 & 0 \\ 0 & -\frac{4592}{9075} & -\frac{19083}{3025} \\ 0 & \frac{1612}{3025} & -\frac{36233}{9075} \end{pmatrix}. \tag{25}$$

We have set $n_F = 0$, since our lattice results are obtained in the quenched approximation.

The evolution of the pseudoscalar density is given by

$$\langle P_5(m_b)\rangle^{\overline{\rm MS}} = \left(\frac{\alpha_s(\mu)}{\alpha_s(m_b)}\right)^{-\gamma_P/2\beta_0}\left[\,1+\frac{\alpha_s(\mu)-\alpha_s(m_b)}{4\pi}J_P^{\overline{\rm MS}}\right]\,\langle P_5(\mu)\rangle^{\overline{\rm MS}}\,, \tag{26}$$

where $\gamma_P = -8$ and $J_P^{\overline{\rm MS}} = 998/363$, for $n_F = 0$.

	q=up/down			q=strange		
κ_Q	0.125	0.122	0.119	0.125	0.122	0.119
m_P [GeV]	1.75(9)	2.02(10)	2.26(12)	1.85(8)	2.12(9)	2.36(10)
$B_1^{\overline{\rm MS}}(m_b)$	0.891(21)	0.913(26)	0.920(27)	0.900(16)	0.916(18)	0.919(16)
$B_2^{\overline{\rm MS}}(m_b)$	0.745(29)	0.771(29)	0.824(26)	0.788(21)	0.817(21)	0.833(18)
$B_3^{\overline{\rm MS}}(m_b)$	0.896(41)	0.889(43)	0.943(41)	0.902(27)	0.918(26)	0.921(26)

Table 4: *B-parameters, extracted from our lattice QCD data after conversion to the* $\overline{\rm MS}$*(NDR) scheme of ref. [11] at* $\mu = 2.8(1){\rm GeV}$ *and running to* $\mu = m_b$.

With all of the above formulae at hand, we convert our B-parameters from table. 2 to the $\overline{\rm MS}$ scheme (at $\mu = 2.8(1)$ GeV), run them to $\mu = m_b$, and list their values in table 4.

The next step is to match the B-parameters from table 4 from QCD to the HQET where we can use the heavy quark scaling laws to extrapolate each B-parameter to the physical point, *i.e.* to $m_{B_{d/s}}$. The crucial ingredient that enters the matrix $\mathcal{M}_4^{-1}[m_P, m_b] \equiv \mathcal{M}_4^{-1}[m_P, m_b, m_b]$ of eq. (14), and hence the quantity $\vec{\Phi}(m_P, m_b)$ in eq. (16), is the matching matrix $C(m_P)$ which relates the QCD operators, computed in the $\overline{\rm MS}$(NDR) scheme of ref. [11], onto the HQET ones computed in the $\overline{\rm MS}$(NDR) scheme of ref. [6] (and vice versa). In ref. [10] it has been shown that (at NLO) this matrix has the following form:

$$C_{123}(m_P) = \mathbb{I} + c_{123}^{(1)} \frac{\alpha_s(m_P)}{4\pi} \; ; \qquad c_{123}^{(1)} = \begin{pmatrix} -14 & -8 & 0 \\ 0 & 61/12 & -13/4 \\ 0 & -77/12 & -121/12 \end{pmatrix} , \tag{27}$$

where we introduced the index "123" since we consider only these operators here.

The last piece of information needed to construct $\mathcal{M}_4^{-1}[m_P, m_b]$ of eq. (14), is the evolution operator in the HQET at NLO. In a notation analogous to eq. (22) we have

$$\langle \vec{\widetilde{O}}(m_b) \rangle = \mathbf{W}_{HQET}^T[m_b, \mu]^{-1} \langle \vec{\widetilde{O}}(\mu) \rangle$$

$$\mathbf{W}_{HQET}[m_b, \mu] = \widetilde{w}(m_b) \widetilde{w}^{-1}(\mu) \, , \tag{28}$$

where

$$\widetilde{w}(\mu) \;=\; \widetilde{M}(\mu) \; \alpha_s(\mu)^{-\widetilde{\gamma}_0^T/2\beta_0} \; . \tag{29}$$

In this case [10]

$$\widetilde{\gamma}_0 = \; -\frac{8}{3} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} ,$$

$$\widetilde{M}(\mu) \;=\; \mathbb{I} \;+\; \widetilde{J}_{123}^{\overline{\rm MS}}\,\frac{\alpha_s(\mu)}{4\pi}\;, \tag{30}$$

$$\widetilde{J}_{123}^{\overline{\rm MS}} = \begin{pmatrix} -\dfrac{317}{1089}+\dfrac{68\pi^2}{297} & -\dfrac{2639}{8712}+\dfrac{\pi^2}{99} & 0 \\ 0 & -\dfrac{3181}{4356}+\dfrac{74\pi^2}{297} & \dfrac{1913}{4356}-\dfrac{2\pi^2}{99} \\ 0 & \dfrac{2639}{4356}-\dfrac{2\pi^2}{99} & -\dfrac{3907}{4356}+\dfrac{74\pi^2}{297} \end{pmatrix}\,. \tag{31}$$

where, as before, we have set $n_F = 0$.

To obtain the quantity $\vec{\Phi}(m_P, m_b)$ of eq. (16), one also needs $\mathcal{M}_2^{-1}[m_P, m_b]$. At NLO this information can be extracted from ref. [26]. The matching of the axial current and of the pseudoscalar density is given by

$$\begin{aligned} A_0 \;&=\; \left[\,1\;-\;\frac{8}{3}\frac{\alpha_s(\mu)}{4\pi}\,\right]\;\widetilde{A}_0(\mu)\;, \\ P_5(\mu) \;&=\; \left[\,1\;+\;\frac{8}{3}\frac{\alpha_s(\mu)}{4\pi}\right]\;\widetilde{A}_0(\mu)\;. \end{aligned} \tag{32}$$

while the expression for the running of the axial current (for $n_F = 0$) is

$$\widetilde{A}_0(m_b) = \left(\frac{\alpha_s(\mu)}{\alpha_s(m_b)}\right)^{-\widetilde{\gamma}_A/2\beta_0}\;\left[1-\left(\frac{439}{1089}-\frac{28\pi^2}{297}\right)\frac{\alpha_s(\mu)-\alpha_s(m_b)}{4\pi}\right]\widetilde{A}_0(\mu)\;, \tag{33}$$

where the leading order anomalous dimension is given by $\widetilde{\gamma}_A = -4$.

By combining all of the above ingredients, we form the quantities $\vec{\Phi}_{1,2,3}(m_P, m_b)$, use eq. (17), and extrapolate to $m_{B_{d/s}}$. The result of that extrapolation is:

$$\vec{\Phi}_{1,2,3}(m_{B_d}, m_b) = \begin{pmatrix} 1.038(91) \\ 1.055(64) \\ 1.117(94) \end{pmatrix}\;, \qquad \vec{\Phi}_{1,2,3}(m_{B_s}, m_b) = \begin{pmatrix} 1.009(36) \\ 1.045(24) \\ 1.070(48) \end{pmatrix}\;, \tag{34}$$

where we wrote separately the results of the extrapolation of our data with the light quark extrapolated to d (left), and those with the light quark interpolated to s (right). By including the $\widetilde{B}$-parameters from table 3, we fit our data to eq. (18), from which we get

$$\vec{\Phi}_{1,2,3}(m_{B_d}, m_b) = \begin{pmatrix} 0.976(42) \\ 0.943(28) \\ 0.845(52) \end{pmatrix}\;, \qquad \vec{\Phi}_{1,2,3}(m_{B_s}, m_b) = \begin{pmatrix} 0.967(21) \\ 0.955(17) \\ 0.857(31) \end{pmatrix}\;. \tag{35}$$

These two extrapolations, for each component of the vector $\vec{\Phi}_{1,2,3}(m_P, m_b)$, are illustrated in fig. 4 for the case of the light d-quark. Results of that extrapolation are the B-parameters in HQET, which we then match back onto their QCD values to get our final results for B-parameters. These numbers are presented in table 5.

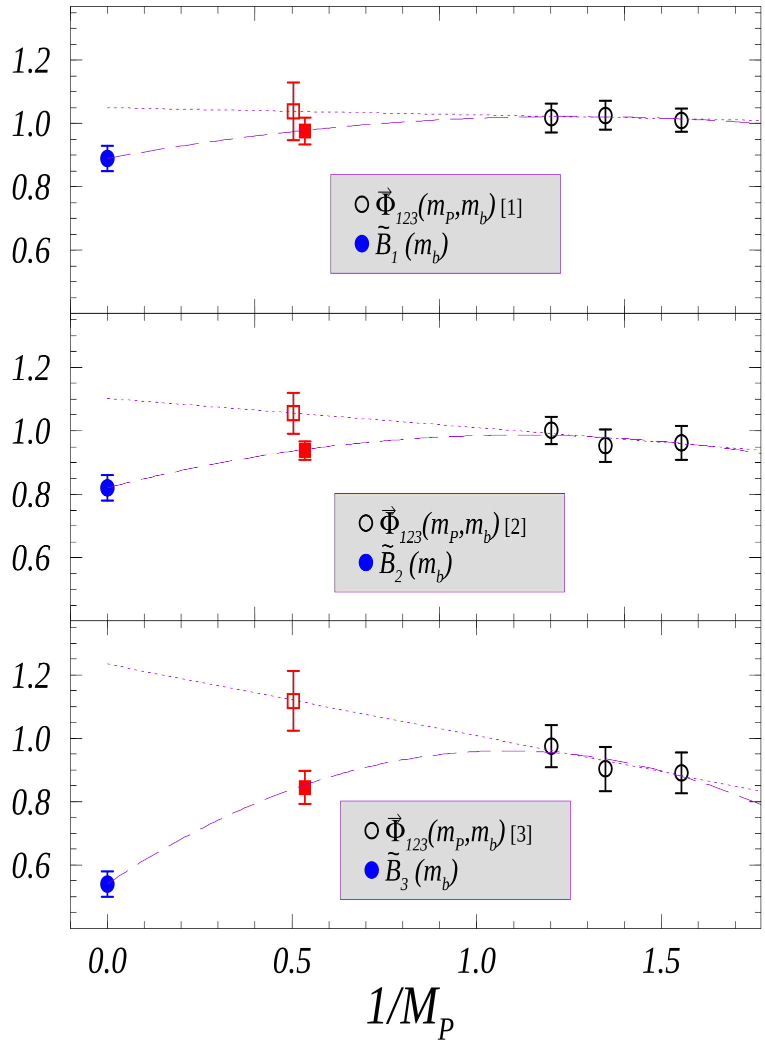

Figure 4: *Extrapolation to the physical B_d meson mass (squared symbols) in the inverse heavy meson mass. The unconstrained linear extrapolation for each of the components of the vector $\vec{\Phi}_{1,2,3}(m_P, m_b)$ from our data (empty circles) to $\vec{\Phi}_{1,2,3}(m_{B_d}, m_b)$ (empty square) is depicted by the dotted line. The result of the constrained extrapolation (filled squares) by the static HQET bag-parameters (filled circles) is marked by the dashed line. i^{th} component of the vector is marked by $[i]$.*

An important issue to be mentioned is the treatment of the statistical errors when constraining by the static HQET results because the three points obtained in QCD are correlated among themselves and are uncorrelated from the one obtained in HQET. Although it may look trivial, we prefer to mention how these errors have been treated in this work. [4] For each jack of our QCD data we form the so-called augmented χ_a^2 by including the HQET

[4]For a clear discussion about the treatment of the statistical errors in such a situation, see ref. [27].

value $\tilde{B}_j$ and its error $\tilde{\sigma}_j$ as:

$$(\chi_a^2)_j \; = \; \sum_i \left[\frac{\Phi(m_{P_i})[j] - (a_0')_j - \dfrac{(a_1')_j}{m_{P_i}} - \dfrac{(a_2')_j}{m_{P_i}^2}}{\sigma_j[\Phi(m_{P_i})]} \right]^2 + \left(\frac{\tilde{B}_j - (a_0')_j}{\tilde{\sigma}_j} \right)^2 , \tag{36}$$

where "$[j]$" denotes the j^{th} component of the vector $\vec{\Phi}(m_{P_i})$, the error of which is $\sigma_j[\Phi(m_{P_i})]$ for the i^{th} value of our three heavy-light meson masses, m_{P_i}. By minimizing $(\chi_a^2)_j$, we find the values of the parameters $(a_{0,1,2}')_j$, where $(a_0')_j$ is constrained by the prior knowledge of the static result. In this way we get the value for $\Phi(m_{B_{d/s}})[j]$, for each jack of our QCD data. The final error, $\sigma_j(\Phi(m_{B_{d/s}}))$, that we quoted in eq. (35), is obtained as a standard error over all jacks (N_{JK})

$$\sigma_j(\Phi(m_{B_{d/s}})) = \sqrt{ \frac{N_{JK}-1}{N_{JK}} \left[\sum_{k=1}^{N_{JK}} \left(\Phi_k(m_{B_{d/s}})[j] \right)^2 - \frac{1}{N_{JK}} \left(\sum_{k=1}^{N_{JK}} \Phi_k(m_{B_{d/s}})[j] \right)^2 \right] } \; . \tag{37}$$

	unconstrained		*constrained*	
Light quark	d	s	d	s
$B_1^{\overline{\rm MS}}(m_b)$	0.938(81)	0.905(32)	0.875(37)	0.867(18)
$B_2^{\overline{\rm MS}}(m_b)$	0.923(32)	0.915(30)	0.826(25)	0.836(15)
$B_3^{\overline{\rm MS}}(m_b)$	1.192(101)	1.141(52)	0.901(56)	0.914(33)

Table 5: *Final results for the first three B-parameters defined in eq. (2), in the $\overline{\rm MS}$(NDR) scheme of ref. [11] at $\mu = m_b$.*

From table 5 we see that by extrapolating the B-parameters from the range of masses accessed from our lattice to the $m_{B_{d/s}}$, without including the static HQET results, we always overshoot the ones that are obtained by including the static HQET values. This is especially pronounced for the parameter $B_3(m_b)$. At this point it is not clear whether this is the real physical effect, or it is due to the lattice artefacts: *e.g.* our heavier mesons may be more subject to $\mathcal{O}(a)$ effects, our HQET data are only perturbatively renormalized etc. Further research is needed to clarify this issue.

3.2 Physical results for $B_{4,5}^{\overline{\rm MS}}(m_b)$

As in the previous section, we first convert our directly computed $B_{4,5}^{\rm RI/MOM} \to B_{4,5}^{\overline{\rm MS}}$. We choose the $\overline{\rm MS}$(NDR) scheme of ref. [9], according to which

$$\begin{pmatrix} \langle O_4(\mu)\rangle \\ \langle O_5(\mu)\rangle \end{pmatrix}^{\overline{\rm MS}} = \left[\mathbb{I} \; + \; r_{45} \frac{\alpha_s(\mu)}{4\pi} \right] \begin{pmatrix} \langle O_4(\mu)\rangle \\ \langle O_5(\mu)\rangle \end{pmatrix}^{\rm RI/MOM} , \tag{38}$$

with the NLO matching coefficient given by [9]

$$r_{45} = -\frac{2}{3} \begin{pmatrix} -17 + \log 2 & 3(1 - \log 2) \\ -3(1 + \log 2) & 1 + \log 2 \end{pmatrix} . \tag{39}$$

As in the previous subsection, we convert the results from table 2 at $\mu = 2.8(1)$ GeV and evolve them in the $\overline{\rm MS}$ scheme to the scale $\mu = m_b = 4.6$ GeV. The evolution is described by an equation analogous to eq. (21) in which the one-loop anomalous dimension now reads [9]

$$\gamma_0 = \begin{pmatrix} -16 & 0 \\ -6 & 2 \end{pmatrix} ,$$

whereas the NLO part ($n_F = 0$) is

$$M(\mu) \;=\; \mathbb{I} \;+\; J_{45}^{\overline{\rm MS}} \,\frac{\alpha_s(\mu)}{4\pi} \,, \qquad J_{45}^{\overline{\rm MS}} \;=\; \begin{pmatrix} \frac{24379}{5808} & \frac{5895}{1936} \\ \frac{45}{16} & -\frac{5807}{5808} \end{pmatrix} . \tag{40}$$

Our $B_{4,5}$-parameters, in the $\overline{\rm MS}$ scheme and at $\mu = m_b$, are given in table 6. The matching

	q=up/down			q=strange		
κ_Q	0.125	0.122	0.119	0.125	0.122	0.119
m_P [GeV]	1.75(9)	2.02(10)	2.26(12)	1.85(8)	2.12(9)	2.36(10)
$B_4^{\overline{\rm MS}}(m_b)$	1.098(22)	1.074(25)	1.126(21)	1.119(16)	1.134(17)	1.131(13)
$B_5^{\overline{\rm MS}}(m_b)$	1.235(25)	1.308(31)	1.442(30)	1.288(20)	1.390(24)	1.466(21)

Table 6: *B-parameters, extracted from our lattice QCD data and converted to the $\overline{\rm MS}$(NDR) scheme of ref. [9].*

onto the corresponding operators in HQET is made through [10]

$$C_{45}(m_P) = \mathbb{I} + c_{45}^{(1)} \,\frac{\alpha_s(m_P)}{4\pi} \,, \qquad c_{45}^{(1)} = \frac{1}{2} \begin{pmatrix} 17 & -11 \\ 7 & -21 \end{pmatrix} . \tag{41}$$

As for the evolution of these operators in HQET, the matrix of the leading order anomalous dimension coefficients is

$$\widetilde{\gamma}_0 = \;-\; \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix} ,$$

while the NLO contribution ($n_F = 0$) reads [10]

$$\widetilde{M}(\mu) \;=\; \mathbb{I} \;+\; \widetilde{J}_{45}^{\overline{\rm MS}} \,\frac{\alpha_s(\mu)}{4\pi} \,, \qquad \widetilde{J}_{45}^{\overline{\rm MS}} = \begin{pmatrix} -\dfrac{833}{726} + \dfrac{74\pi^2}{297} & \dfrac{1987}{2178} - \dfrac{2\pi^2}{99} \\ \dfrac{1987}{2178} - \dfrac{2\pi^2}{99} & -\dfrac{833}{726} + \dfrac{74\pi^2}{297} \end{pmatrix} . \tag{42}$$

We use all the above formulae to create the quantities $\vec{\Phi}_{4,5}(m_P, m_b)$ (16) and extrapolate in the heavy meson mass by using eq. (17). The results are

$$\vec{\Phi}_{4,5}(m_{B_d}, m_b) = \begin{pmatrix} 1.189(70) \\ 2.144(108) \end{pmatrix} , \qquad \vec{\Phi}_{4,5}(m_{B_s}, m_b) = \begin{pmatrix} 1.184(25) \\ 2.158(60) \end{pmatrix} . \tag{43}$$

After incorporating the values for $\widetilde{B}_{4,5}(m_b)$ given in table 3, the fit to eq. (18), gives

$$\vec{\Phi}_{4,5}(m_{B_d}, m_b) = \begin{pmatrix} 1.112(27) \\ 2.072(49) \end{pmatrix} , \qquad \vec{\Phi}_{4,5}(m_{B_s}, m_b) = \begin{pmatrix} 1.121(12) \\ 2.101(35) \end{pmatrix} . \tag{44}$$

The two ways of reaching the physically relevant results are shown in fig. 5. As in the previous subsection, for the final results we need to match back onto the QCD. These values are presented in table 7. From tabs. 5 and 7 we conclude that, for all the B-parameters,

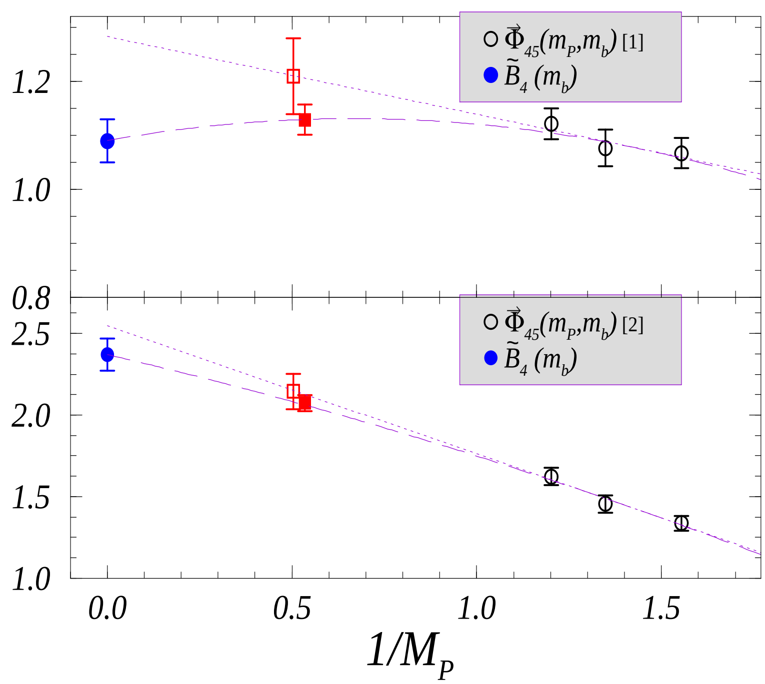

Figure 5: *The same as fig. 4 but for* $\vec{\Phi}_{4,5}(m_P, m_b)$.

the extrapolations in which we do not include the static results lead to higher values.

4 Systematic uncertainties

The central values of this work are the ones obtained by combining the static HQET and QCD results, which are given in tabs. 5 and 7.

We now need to attribute systematic errors to these results. We do not discuss the errors due to the use of the quenched approximation and refer to our results as quenched.

	unconstrained		*constrained*	
Light quark	*d*	*s*	*d*	*s*
$B_4^{\overline{\rm MS}}(m_b)$	1.228(72)	1.223(26)	1.148(28)	1.157(13)
$B_5^{\overline{\rm MS}}(m_b)$	1.784(90)	1.798(50)	1.724(41)	1.750(29)

Table 7: *Final results for the last two B-parameters defined in eq. (2), in the* $\overline{\rm MS}$*(NDR) scheme of ref. [9] at* $\mu = m_b$.

The results of ref. [28], however, are quite encouraging in that they indicate that the values of the parameters $B_{1,2,3}(m_b)$ remain practically unchanged after switching from $n_F = 0$ to $n_F = 2$.

- Systematic uncertainties present in the static results were estimated to be in the range of $(3 \div 4)\%$ for all the $\widetilde{B}$-parameters. This error is almost entirely due to the choice of the renormalization point q^* at which we used the boosted perturbative expressions. We varied $1/a \leq q^* \leq \pi/a$, and then evolved the resulting (continuum) B-parameters from $\mu = q^*$ to $\mu = m_b$. The spread of values $\widetilde{B}(m_b)$ with respect to the central one (obtained from $q^* = 2.6/a$), has been assigned to the systematic error.

- The QCD values are obtained after the non-perturbative renormalization in the RI/MOM scheme at $\mu = 2.8(1)$ GeV. We repeated the whole procedure described in the previous section, but starting from our results obtained at $\mu = 1.9(1)$ GeV and at $\mu = 3.9(2)$ GeV. The final results get modified as follows:
  - $\Delta B_1/B_1^{\rm central} < \pm 1\%$;
  - $\Delta B_2/B_2^{\rm central} \simeq \pm 1\%$;
  - $\Delta B_3/B_3^{\rm central} \simeq \pm 8\%$;
  - $\Delta B_4/B_4^{\rm central} \simeq -3\%$;
  - $\Delta B_5/B_5^{\rm central} \simeq +11\%$.

- Interpolation/extrapolation in the light quark mass is made linearly. For the average up/down quark mass we need to account for the possibility of the quadratic term in this extrapolation. As it can be seen from fig. 2, the extrapolations are smooth for all the bag parameters. If we include a quadratic term in the extrapolation, we obtain results which are fully compatible with the ones presented here. This is true for both QCD and static HQET B-parameters.

- The value $a^{-1}(m_{K^*}) = 2.72(13)$ GeV, has been used throughout the paper. Another option would be to use the kaon decay constant, from which we obtain $a^{-1}(f_K) = 2.69(16)$ GeV. Being completely consistent with $a^{-1}(m_{K^*})$, this choice affects our final results by only $+1\%$.

- Even though we use the improved action, our operators are not improved. Therefore, our results for the bag parameters suffer from $\mathcal{O}(a)$ discretization errors. The hope is that these errors cancel in the ratios (5,6) from which the bag parameters are actually extracted. A conservative estimate on the size of these uncertainties can be obtained if we improve the axial current in the ratio R_1, as $A_0(t) \to A_0(t) + c_A(P_5(t+1) - P_5(t-1))/2$, with the known value for the parameter $c_A = -0.04$ [29]. From this exercise we conclude the further increase in our final results for B_1 by $\sim 4\%$.

  As for the HQET values, we checked that our values for the $\widetilde{B}_i$ (obtained at $\beta = 6.0$) are indistinguishable from the ones that can be extracted from the UKQCD data at $\beta = 6.2$ [34]. That gives us more confidence that the $\mathcal{O}(a)$ effects in the static HQET data are indeed small.

- We used the two-loop running coupling $\alpha_s(\mu)$ by taking $\Lambda_{\rm QCD}^{(n_F=0)} = 0.25$ GeV. We tried to vary $\Lambda_{\rm QCD}^{(n_F=0)}$ by 10% (which covers all the presently available lattice estimates [30]), and see that the final results vary in the range of $\pm 1.5\%$.

We now write our results in a fully explicit form as:

$$\begin{aligned}
B_1^{(d)\overline{\rm MS}}(m_b) &= 0.87(4)(3)(0)\left({}^{+4}_{-2}\right)\ , & B_1^{(s)\overline{\rm MS}}(m_b) &= 0.87(2)(3)(0)\left({}^{+4}_{-2}\right)\ ,\\
B_2^{(d)\overline{\rm MS}}(m_b) &= 0.83(3)(3)(1)(2)\ , & B_2^{(s)\overline{\rm MS}}(m_b) &= 0.84(2)(3)(1)(2)\ ,\\
B_3^{(d)\overline{\rm MS}}(m_b) &= 0.90(6)(3)(7)(2)\ , & B_3^{(s)\overline{\rm MS}}(m_b) &= 0.91(3)(3)(7)(2)\ ,\\
B_4^{(d)\overline{\rm MS}}(m_b) &= 1.15(3)(4)\left({}^{+0}_{-4}\right)(3)\ , & B_4^{(s)\overline{\rm MS}}(m_b) &= 1.16(2)(4)\left({}^{+0}_{-4}\right)(3)\ ,\\
B_5^{(d)\overline{\rm MS}}(m_b) &= 1.72(4)(5)\left({}^{+19}_{-00}\right)(3)\ , & B_5^{(s)\overline{\rm MS}}(m_b) &= 1.75(3)(5)\left({}^{+20}_{-00}\right)(3)\ ,
\end{aligned} \tag{45}$$

where, besides the first statistical errors, the following sources of the systematic uncertainty are being written out respectively: systematics of the calculation in the static limit of HQET, the error in the renormalization of B-parameters computed in QCD, combined error due to the variation of a^{-1} and of $\Lambda_{\rm QCD}^{(n_F=0)}$ (and also due to the improvement of the axial current in the case of B_1). After adding all systematic errors in squares we arrive at the complete set of results already given in table 1.

To be able to fully reconstruct the numbers that we presented in table 1, we also need to provide the reader with the formulae allowing the conversion of the parameters $B_2(m_b)$ and $B_3(m_b)$ from the $\overline{\rm MS}$(NDR) scheme of ref. [11] to the one of ref. [9]. This is achieved by using the following formula

$$\begin{pmatrix} \langle O_2(\mu)\rangle \\ \langle O_3(\mu)\rangle \end{pmatrix}^{\overline{\rm MS}\ [9]} = \left[\mathbb{I} + \frac{\alpha_s(\mu)}{12\,\pi}\begin{pmatrix} -11 & 1 \\ 1 & 5 \end{pmatrix}\right]\begin{pmatrix} \langle O_2(\mu)\rangle \\ \langle O_3(\mu)\rangle \end{pmatrix}^{\overline{\rm MS}\ [11]}\ , \tag{46}$$

which we obtained after rotating the operators $Q_{1,2}^{SLL}(\mu)_{\overline{\rm MS}}$ of ref. [9] to the SUSY basis (1).

5 Concluding remarks

In this paper we computed the B-parameters for all five $\Delta B = 2$ operators. The extrapolation of the results obtained directly in lattice QCD in the region of masses $m_P \sim 2$ GeV to the physically interesting mass $m_{B_{d/s}}$, has been constrained by using the static HQET result. The matching QCD $\leftrightarrow$ HQET and running in each of the two theories have been made by the consistent use of the perturbative expressions known at NLO. The final results are presented in three renormalization schemes (see table 1).

Our results can be improved in many ways. We combined the results of the QCD lattice simulations performed at $\beta = 6.2$ with the HQET ones obtained at $\beta = 6.0$. Naturally, a good strategy would be to do the computation at the same value of β in both theories, to vary the value of β (*i.e.* of the lattice spacing) and attempt extrapolating to the continuum limit. All numbers are obtained in the quenched approximation ($n_F = 0$). An investigation of the sea quark effects on our quenched values by repeating the analysis performed in this paper with $n_F = 2$, would be very welcome.

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Appendix: Non-perturbative calculation of the renormalization and subtraction constants in the (Landau)RI/MOM scheme

In this appendix we give the numerical values for the resulting matrices of the renormalization and subtraction constants which are obtained in the (Landau)RI/MOM scheme by using the method of refs. [17, 18]. These are computed in the following basis of operators:

$$
\begin{aligned}
Q_1 &= \overline{q}^i\gamma_\mu(1-\gamma_5)q^i\ \overline{q}^j\gamma_\mu(1-\gamma_5)q^j\ ,\\
Q_2 &= \overline{q}^i\gamma_\mu(1-\gamma_5)q^i\ \overline{q}^j\gamma_\mu(1+\gamma_5)q^j\ ,\\
Q_3 &= \overline{q}^i(1+\gamma_5)q^i\ \overline{q}^j(1-\gamma_5)q^j\ ,\\
Q_4 &= \overline{q}^i(1-\gamma_5)q^i\ \overline{q}^j(1-\gamma_5)q^j\ ,\\
Q_5 &= \frac{1}{2}\overline{q}^i\sigma_{\mu\nu}(1-\gamma_5)q^i\ \overline{q}^j\sigma_{\mu\nu}(1-\gamma_5)q^j \quad (\mu>\nu)\ ,
\end{aligned}
\tag{47}
$$

which are equivalent to those appearing in eq. (1) (after setting $\bar{b}\to\bar{q}$):

$$
\begin{aligned}
&Q_1 = O_1\ , \qquad Q_2 = -2O_5\ ,\\
&Q_3 = O_4\ , \qquad Q_4 = O_2\ ,\\
&Q_5 = O_2 + 2O_3\ .
\end{aligned}
\tag{48}
$$

The difference between these and the results for the renormalization constants presented in refs. [13, 14] is that the present renormalization and subtraction constants are not polluted by the Goldstone boson contributions. To eliminate those, we applied the recipe of ref. [18]. Since that paper has not been released yet, we briefly explain the main steps here.

- ⊙ Starting from the 4-quark Green functions computed in the Landau gauge, with all momenta in the external legs equal, $G_i(p) = \langle q(p)\ \bar{q}(p)\ Q_i\ \bar{q}(p)\ q(p)\rangle$, one constructs the amputated ones as

$$
\Lambda_i(p) = \left(\prod_{k=1}^{4} S^{-1}(p)\right) G_i(p)\ ,
\tag{49}
$$

  where $S^{-1}(p)$ stands for the inverse quark propagator.

- ⊙ The amputated Green functions are projected onto various Dirac structures as

$$
\Big(\Gamma_i(p)\Big)_j = \mathrm{Tr}\Big[\Lambda_i(p)P_j\Big]\ ,
\tag{50}
$$

  where P_j are suitable projectors satisfying the orthogonality relation

$$
\Big(\Gamma_i^{(0)}(p)\Big)_j = \mathrm{Tr}\Big[\Lambda_i^{(0)}(p)P_j\Big] = \delta_{ij}\ ,
\tag{51}
$$

  where $\Lambda_i^{(0)}(p)$ stands for the tree level amputated Green functions. The explicit expressions for the projectors P_j can be found in ref. [17] (eq. (37)).

⊙ Eq. (51) is turned into the RI/MOM renormalization condition as

$$\Big(\hat{\Gamma}_i(p/\mu)\Big)_j\Big|_{p^2=\mu^2} = \delta_{ij}\,, \tag{52}$$

where the renormalized amputated Green function, $\hat{\Gamma}_i(p/\mu)$, is expressed as

$$\hat{\Gamma}_i(p/\mu) = \Gamma_k(p)\Big(\delta_{kj} + \Delta_{kj}(g_0^2)\Big) Z_{ji}(\mu, g_0^2)\,, \tag{53}$$

up to an overall wave function renormalization, which is trivial to compute after imposing the vector Ward identity on the quark propagator. The condition (52) is applicable for virtualities $\Lambda^2_{\rm QCD} \ll p^2 \ll (\pi/a)^2$.

⊙ Thus, for each of the operators from the basis (47), one obtains 5 equations from which the subtraction ($\Delta(g_0^2)$) and renormalization ($Z(\mu; g_0^2)$) constants are computed. In matrix form, the final result writes

$$\vec{Q}(\mu) \ = \ Z(\mu; g_0^2)\ \Big[\mathbb{I} + \Delta(g_0^2)\Big]\ \vec{Q}(g_0^2)\ . \tag{54}$$

The structure of the Z-matrix is determined by the chirality, *i.e.* $Q_1(\mu)$ does not mix with any other operator, [5] $Q_{2,3}(\mu)$ mix with each other but not with the other operators. The same goes for the $Q_{4,5}(\mu)$ operators. The remaining elements of the 5×5 matrix are filled by the subtraction constants $\Delta_{ij}(g_0^2)$.

⊙ In practice, however, the above procedure is implemented by computing $\hat{\Gamma}_i(p, \kappa_q)$ at several values of the (light) quark mass (*i.e.* various κ_q), followed by the extrapolations of each $\Delta_{ij}(g_0^2)$ and $Z_{ij}(\mu, g_0^2)$ to the chiral limit. This extrapolation can be dangerous because the operators are inserted at zero momentum (all external legs in the Green function have the same momentum), and the coupling to the Goldstone boson contaminates the short distance behaviour (which we are interested in). In particular, we find that for the vertices of the structure $\gamma_5 \otimes \gamma_5$ this coupling is indeed large. In addition, via projections (50), it may pollute the extraction of the renormalization and subtraction constants for the other operators. Therefore, for each projected amputated four-quark Green function, one should subtract the Goldstone contribution. For the parity even operators (which are the ones that we consider in this paper), this Goldstone contribution can appear as a pole, but also as a double pole, *i.e.*:

$$\Big(\Gamma_i(p;\kappa_q)\Big)_j \equiv \Gamma_{ij}(p;\kappa_q) = \alpha_{ij}(p)\ +\ \frac{\beta_{ij}(p)}{m_P^2}\ +\ \frac{\gamma_{ij}(p)}{m_P^4}\ +\ \delta_{ij}\ m_P^2. \tag{55}$$

Note that we also added a term $\delta\ m_P^2$, to account for the the linear dependence in the quark mass ($m_P^2 \propto m_q$). [6] A judicious way to subtract the Goldstone contributions,

[5] In other words $Z_{11}(\mu) \neq 0$, whereas $Z_{12} = Z_{13} = Z_{14} = Z_{15} = 0$.

[6] The linear dependence in the quark mass arises after the cancellation of the quadratic quark mass term against the Goldstone pole contribution. More detailed discussion will be presented in ref. [18].

and thus to reach the term $\alpha_{ij}(p)$, is to consider the following combination for the fit with the data [18]

$$\frac{m_{P_1}^2 \Gamma_{ij}(p;\kappa_{q_1}) - m_{P_2}^2 \Gamma_{ij}(p;\kappa_{q2})}{m_{P_1}^2 - m_{P_2}^2} \;=\; \alpha_{ij}(p) \;-\; \frac{\gamma_{ij}(p)}{m_{P_1}^2 m_{P_2}^2} \;+\; \delta_{ij}(p)\Big(m_{P_1}^2 + m_{P_2}^2\Big) \quad (56)$$

by which the pole-like contribution is automatically eliminated. This is to be performed for each value of p and accounting for all the mass combinations. [7] The resulting $\alpha_{ij}(p)$ is thus the chiral value of the projection $\Gamma_{ij}(p)$ free from the Goldstone boson contamination.

The procedure sketched above has been applied at three values of the renormalization scale: $a\mu = 0.71$, 1.03 and 1.41. In physical units, these values correspond to $\mu =$ 1.9(1) GeV, 2.8(1) GeV and 3.9(2) GeV, respectively. The complete list of results for the subtraction ($\Delta_{ij}(a)$) and renormalization constants ($Z_{ij}(\mu a)$) is presented in table 8.

As for the renormalization constants for the bilinear quark operators, which are necessary to compute the ratios (5) and (6), we use the following values [31]:

$$\begin{aligned} Z_P^{\mathrm{RI/MOM}}\Big(1.9(1)\ \mathrm{GeV}\Big) &= 0.510(9)\,; \\ Z_P^{\mathrm{RI/MOM}}\Big(2.8(1)\ \mathrm{GeV}\Big) &= 0.575(7)\,; \\ Z_P^{\mathrm{RI/MOM}}\Big(3.9(2)\ \mathrm{GeV}\Big) &= 0.630(6)\,. \end{aligned} \quad (57)$$

In addition, in the same study, we obtained $Z_A = 0.814(4)$. Notice that we used the method of ref. [32], to avoid the large Goldstone boson contribution to the value of $Z_P^{\mathrm{RI/MOM}}$ [4]. As for the constant Z_A, our value is consistent with the findings of other lattice groups [29].

[7] Besides the light masses already mentioned in the previous section, for the discussion of the renormalization constants we also had the results for $\kappa_q = 0.1333$ at our disposal. In this way, we could make 6 combinations and therefore a safe extrapolation to the chiral limit.

Scale (μ)	1.9(1) GeV	2.8(1) GeV	3.9(2) GeV
$Z_{11}(\mu)$	0.663(6)	0.645(9)	0.644(13)
Δ_{12}	-0.072(3)	-0.069(11)	-0.063(10)
Δ_{13}	-0.015(2)	-0.011(4)	-0.014(2)
Δ_{14}	0.020(2)	0.021(7)	0.015(6)
Δ_{15}	0.011(3)	0.006(1)	0.005(1)
$Z_{22}(\mu)$	0.723(5)	0.691(7)	0.683(5)
$Z_{23}(\mu)$	0.315(3)	0.257(7)	0.202(9)
Δ_{21}	-0.052(2)	-0.055(9)	-0.050(9)
Δ_{24}	-0.250(7)	-0.169(20)	-0.168(19)
Δ_{25}	0.013(2)	0.014(2)	0.015(1)
$Z_{32}(\mu)$	0.023(1)	0.022(1)	0.021(3)
$Z_{33}(\mu)$	0.322(12)	0.392(20)	0.467(13)
Δ_{31}	0.018(1)	0.018(4)	0.014(3)
Δ_{34}	0.351(10)	0.233(34)	0.220(31)
Δ_{35}	-0.008(1)	-0.007(3)	-0.005(1)
$Z_{44}(\mu)$	0.414(9)	0.473(17)	0.534(13)
$Z_{45}(\mu)$	-0.017(2)	-0.015(2)	-0.015(4)
Δ_{41}	0.008(1)	0.008(3)	0.005(1)
Δ_{42}	0.009(1)	0.001(2)	0.001(1)
Δ_{43}	0.208(4)	0.143(16)	0.144(15)
$Z_{54}(\mu)$	-0.307(4)	-0.233(7)	-0.176(7)
$Z_{55}(\mu)$	0.914(9)	0.814(16)	0.761(18)
Δ_{51}	0.009(1)	0.008(2)	0.005(1)
Δ_{52}	0.009(1)	0.005(3)	0.008(1)
Δ_{53}	0.121(1)	0.084(6)	0.089(8)

Table 8: *The values of the renormalization $Z_{ij}(\mu)$ and subtraction constants Δ_{ij} computed non-perturbatively in the (Landau) RI/MOM scheme at three different values of the renormalization scale μ at $\beta = 6.2$.*

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