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B–$\bar{B}$ Mixing in the HQET

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Abstract

We present a high statistics, quenched lattice calculation of the B-parameters B_{B_d} and B_{B_s}, computed at lowest order in the HQET. The results were obtained using a sample of 600 quenched gauge field configurations, generated by Monte Carlo simulation at $\beta = 6.0$ on a $24^3 \times 40$ lattice. For the light quarks the SW-Clover action was used; the propagator of the lattice HQET was also tree-level improved. Our best estimate of the renormalization scale independent B-parameter is $\hat{B}_{B_d} = 1.03 \pm 0.06 \pm 0.18$. $\hat{B}_{B_d}$ has been obtained by using "boosted" perturbation theory to calculate the renormalization constants which relate the matrix elements of the lattice operators to the corresponding amplitudes in the continuum. Due to the large statistics, the errors in the extraction of the matrix elements of the relevant bare operators are rather small. The main systematic error, corresponding to ± 0.18 in the above result, comes from the uncertainty in the evaluation of the renormalization constants, for which the one-loop corrections are rather large. The non-perturbative evaluation of these constants will help to reduce the final error. We also obtain $\hat{B}_{B_s}/\hat{B}_{B_d} = 1.01 \pm 0.01$ and $f^2_{B_s}\hat{B}_{B_s}/f^2_{B_d}\hat{B}_{B_d} = 1.38 \pm 0.07$.

1 Introduction

The study of B_d^0–$\bar{B}_d^0$ and B_s^0–$\bar{B}_s^0$ mixings may narrow the constraints on the elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix, which in turn serve to test the Standard Model, and to detect possible new physics beyond it [1]. The results of the phenomenological analyses which involve B^0–$\bar{B}^0$ mixings [1] strongly depend on two non-perturbative inputs: the value of the decay constant of the B-meson, f_B, and the value of the renormalization group invariant B-parameter, $\hat{B}_B$. These parameters enter in the theoretical predictions for x_d and x_s, to be defined below, through the mixing parameter $\xi_B \equiv f_B \sqrt{\hat{B}_B}$, ξ_B^2 corresponding, up to trivial factors, to the matrix elements of the relevant $\Delta B = 2$ operators. The B-meson decay constant has not been measured yet. However, several theoretical evaluations of f_B using numerical simulations of lattice QCD [2] and QCD sum rules [3] have been published to date. The B-parameter, which is equal to one in the vacuum saturation approximation, has also been estimated using the same non-perturbative methods [2, 3].

In this paper a high statistics lattice calculation of the B-parameter, at lowest order in the HQET, is presented. The relevant matrix elements have been computed on a statistical sample of 600 gauge field configurations. This has allowed us to study two- and three-point correlation functions at large time-distances with relatively small statistical errors, thus reducing the systematic errors due to the contamination of higher excited states. For the bare four-fermion operators, it was possible to isolate a good signal by using cube and double-cube smeared axial currents as the interpolating fields for the B-mesons, and to extract their matrix elements with small statistical errors. The major source of systematic error is presently given by the uncertainties in the evaluation of the renormalization constants of the different operators, which are known only at first order in perturbation theory. Indeed, as explained below, by renormalizing the operators in two different ways, which are equivalent up to $O(\alpha_s^2)$, we obtain either

$$\hat{B}_{B_d} = 1.21 \pm 0.08 \;\; \text{Method 1} \tag{1}$$

or

$$\hat{B}_{B_d} = 0.86 \pm 0.06 \;\; \text{Method 2}\,. \tag{2}$$

By combining eqs. (1) and (2), we derive our final result for this quantity

$$\hat{B}_{B_d} = 1.03 \pm 0.06 \pm 0.18\,, \tag{3}$$

where the last error is an estimate of the uncertainty due to the contribution of higher-order terms to the renormalization constants.

The ratio of B-parameters $\hat{B}_{B_s}/\hat{B}_{B_d}$, which may be obtained by studying the dependence of the B-parameter on the light quark mass, has also been considered. In

[1] The subcripts d or s referring to the light quark flavour will be written explicitely only when necessary for the discussion.

this ratio, most of the uncertainties due to higher-order terms cancel and we quote

$$\frac{\hat{B}_{B_s}}{\hat{B}_{B_d}} = 1.01 \pm 0.01 \,. \tag{4}$$

By using $f_{B_s}/f_{B_d} = 1.17 \pm 0.03$, as derived from the study of the two-point correlation functions on the same set of configurations, and $M_{B_s}/M_{B_d} = 1.017$ we then obtain

$$r_{sd} = \frac{f^2_{B_s} M^2_{B_s} \hat{B}_{B_s}}{f^2_{B_d} M^2_{B_d} \hat{B}_{B_d}} = 1.43 \pm 0.07 \,. \tag{5}$$

r_{sd} can also be obtained by taking directly the ratio of the matrix elements of the $\Delta B = 2$ operator, computed at the appropriate value of the quark masses. In this case we get

$$r_{sd} = \frac{f^2_{B_s} M^2_{B_s} \hat{B}_{B_s}}{f^2_{B_d} M^2_{B_d} \hat{B}_{B_d}} = 1.35 \pm 0.05 \,. \tag{6}$$

smaller than the result $r_{sd} = 1.54(13)$ obtained using propagating Wilson quarks instead of the HQET [4] [2]. The experimental values of M_{B_d} and M_{B_s} were taken from ref. [5].

Only a non-perturbative calculation of the renormalization constants, which relate the relevant lattice operators to the continuum one, can reduce the present uncertainty in the prediction of the B-parameters from the HQET. This computation is in progress and the results will be published elsewhere.

The plan of the paper is the following. In sec. 2 the relevant formulae for the definition of $\hat{B}_B$ are given; in sec. 3 the procedure followed to compute the B-parameter is presented; in sec. 4 the numerical calculations of the relevant matrix elements are described and the main results of this study are discussed; in the conclusions an outlook of possible developments of the calculations of this paper is given.

2 Master formulae for B^0–$\bar{B}^0$ mixings

B^0–$\bar{B}^0$ mixings are usually expressed as

$$x_{d,s} \equiv \frac{(\triangle M)_{B_{d,s}}}{\Gamma_{B_{d,s}}} \,, \tag{7}$$

where $(\triangle M)_{B_{d,s}}$ is the mass difference of the $B^0_{d,s}$–$\bar{B}^0_{d,s}$ system and $\Gamma_{B_{d,s}} = 1/\tau_{B_{d,s}}$ is the average total width. Using the effective $\Delta B = 2$ Hamiltonian discussed in ref. [6] one finds

$$x_{d,s} \,=\, \tau_{B_{d,s}} \frac{G_F^2}{6\pi^2} \eta_B \, M_{B_{d,s}} \, \xi^2_{B_{d,s}} \, M_W^2 \, S_0(x_t) \, |V_{t(d,s)}|^2 \,, \tag{8}$$

[2] In ref. [4], they also presented a result which includes the error due to the extrapolation to $a \to 0$. Since our result has been obtained only at $a^{-1} \sim 2$ GeV, we compare it with the result of ref. [4] obtained by fitting r_{sd} to a constant in a.

where $\eta_B \sim 0.55$, $S_0(x_t)$ is a function of the ratio $x_t = m_t^2/M_W^2$, m_t and M_W are the top quark and W masses respectively, G_F the Fermi constant, $V_{t(d,s)}$ the Cabibbo-Kobayashi-Maskawa matrix element and $\xi_{B_{d,s}} = f_{B_{d,s}} \sqrt{\hat{B}_{B_{d,s}}}$.

The renormalization group invariant parameter $\hat{B}_{B_{d,s}}$ is defined by

$$\hat{B}_{B_{d,s}} \;=\; \alpha_s(\mu^2)^{-6/23} \left[1 \,+\, \frac{\alpha_s(\mu^2)}{4\,\pi}\, J_5 \right] \; B_{B_{d,s}}(\mu) \;, \tag{9}$$

where

$$B_{B_{d,s}}(\mu) \;=\; \frac{\langle \bar{B}^0_{d,s} | \hat{O}_L(\mu) | B^0_{d,s} \rangle}{\frac{8}{3}\, f^2_{B_{d,s}}\, M^2_{B_{d,s}}} \tag{10}$$

μ is the operator renormalization scale and $f_{B_{d,s}}$ is the $B_{d,s}$-meson decay constant

$$\langle 0 | \bar{b}(0)\, \gamma_\mu\, \gamma_5\, q(0) | B(\vec{p}) \rangle \;=\; i\, p_\mu\, f_B \;. \tag{11}$$

$\hat{O}_L$ is the renormalized four–quark operator

$$\hat{O}_L \;=\; \left(\bar{b}(x)\, \gamma^\mu\, (1-\gamma_5)\, q(x) \right) \, \left(\bar{b}(x)\, \gamma_\mu\, (1-\gamma_5)\, q(x) \right) \;, \tag{12}$$

and $b(x)$ and $q(x)$ are the heavy and light quark fields respectively.

The factor J_5 can be written in terms of the one- and two-loop anomalous dimensions, $\gamma^{(0)}$ and $\gamma^{(1)}$, of the operator $\hat{O}_L$; if $\hat{O}_L(\mu)$ is renormalized in the $\overline{MS}$-scheme one gets [6]

$$\begin{aligned} \gamma^{(0)} &= 4 \;, \qquad \gamma^{(1)} = -7 + \frac{4}{9}\, n_f \;, \qquad \beta_0 = 11 - \frac{2}{3}\, n_f \;, \\ \beta_1 &= 102 - \frac{38}{3}\, n_f \;, \qquad J_{n_f} = \frac{\gamma^{(0)}\, \beta_1}{2\, \beta_0^2} - \frac{\gamma^{(1)}}{2\, \beta_0} \;. \end{aligned} \tag{13}$$

$\beta_{0,1}$ are the one- and two-loop coefficients of the β function and n_f the number of active light quark flavours. $n_f = 5$ for $\mu \geq m_b$. Notice that, strictly speaking, $\hat{B}_B$ is renormalization scale independent only up to the next-to-leading order.

Since m_b is larger than the current values of the lattice cut-off, in order to predict physical quantities relevant in B-physics a possibility is to use the lattice version of the HQET [7], which allows a systematic expansion in inverse powers of the heavy quark mass. It this case, to obtain the physical matrix elements of the effective Hamiltonian, one has to match appropriate lattice bare operators to the operator $\hat{O}_L(\mu)$, which is renormalized in the continuum "full" theory. The matching procedure is conventionally splitted in two separate steps [3]:

i) the matching of the continuum HQET to the full theory;

ii) the matching of the lattice HQET to its continuum counterpart.

[3] As explained in the first of refs. [1] this is, however, unecessary.

The two-loop Next-to-Leading Order (NLO) anomalous dimension necessary for step i) has been computed using the $\overline{MS}$ dimensional regularization in ref. [8]. As noticed in ref. [9], the calculation was, however, not complete because some elements of the one-loop mixing matrix were missing. This calculation has been recently completed in refs. [10, 11]. The final result can be written as

$$\hat{O}_L(m_b) = C_L(\mu^2)\tilde{O}_L(\mu) + C_S(\mu^2)\tilde{O}_S(\mu) \; , \tag{14}$$

where $\hat{O}_L(m_b)$ is the operator in the full theory renormalized in the $\overline{MS}$-scheme at the a scale equal to b-quark mass ($\hat{O}_L(m_b) = \hat{O}_L(\mu = m_b)$) and

$$\begin{aligned} \tilde{O}_L &= 2\left(\bar{h}(x)\,\gamma^\mu\,(1-\gamma_5)\,q(x)\right)\left(\bar{h}^{(-)}(x)\,\gamma_\mu\,(1-\gamma_5)\,q(x)\right) \\ \tilde{O}_S &= 2\left(\bar{h}(x)\,(1-\gamma_5)\,q(x)\right)\left(\bar{h}^{(-)}(x)\,(1-\gamma_5)\,q(x)\right) \end{aligned} \tag{15}$$

are the operators of the effective theory, renormalized at the scale μ. The fields $\bar{h}$ and $\bar{h}^{(-)}$ create a heavy-quark or annihilate a heavy anti-quark state respectively. The coefficient functions are given by [10, 11]

$$\begin{aligned} C_L(\mu^2) &= \left(\frac{\alpha_s(m_b^2)}{\alpha_s(\mu^2)}\right)^{d_1}\left(1 + \frac{\alpha_s(\mu^2)-\alpha_s(m_b^2)}{4\pi}\,J\right)C_L(m_b^2) \\ &+ \left[\left(\frac{\alpha_s(m_b^2)}{\alpha_s(\mu^2)}\right)^{d_2} - \left(\frac{\alpha_s(m_b^2)}{\alpha_s(\mu^2)}\right)^{d_1}\right]\frac{\hat{\gamma}_{21}^{(0)}}{\hat{\gamma}_{22}^{(0)}-\hat{\gamma}_{11}^{(0)}}C_S(m_b^2)\;, \\ C_S(\mu^2) &= \left[\frac{\alpha_s(m_b^2)}{\alpha_s(\mu^2)}\right]^{d_2} C_S(m_b^2)\;, \end{aligned} \tag{16}$$

where, in näive dimensional regularization,

$$C_L(m_b^2) = 1 - 14\frac{\alpha_s(m_b^2)}{4\pi}\;, \qquad C_S(m_b^2) = -8\frac{\alpha_s(m_b^2)}{4\pi}\;, \tag{17}$$

with

$$d_i = \frac{\hat{\gamma}_{ii}^{(0)}}{2\beta_0} \qquad J = \beta_1\frac{d_1}{\beta_0} - \frac{\hat{\gamma}_{11}^{(1)}}{2\beta_0}\;. \tag{18}$$

The elements of the one-loop anomalous mixing matrix are given by

$$\hat{\gamma}_{11}^{(0)} = -8 \qquad \hat{\gamma}_{12}^{(0)} = 0\;, \qquad \hat{\gamma}_{21}^{(0)} = \frac{4}{3} \qquad \hat{\gamma}_{22}^{(0)} = -\frac{8}{3} \tag{19}$$

and

$$\hat{\gamma}_{11}^{(1)} = -\frac{808}{9} - \frac{52}{27}\pi^2 + \frac{64}{9}n_f\;. \tag{20}$$

We now consider step ii), i.e. the matching of the lattice to the continuum HQET. Owing to the breaking of chiral symmetry induced by the Wilson term, the discretized version of $\tilde{O}_L$, mixes with two new lattice operators as follows [12]–[14]

$$\begin{aligned} \tilde{O}_L(\mu) &= \left(1 + \frac{\alpha_s^L}{4\pi}[4 \ln(a^2\mu^2) + D_L]\right) O_L(a) \\ &+ \frac{\alpha_s^L}{4\pi} D_R\, O_R(a) + \frac{\alpha_s^L}{4\pi} D_N\, O_N(a) \end{aligned} \tag{21}$$

$$\tilde{O}_S(\mu) = \left(1 + \frac{\alpha_s^L}{4\pi}[\frac{4}{3} \ln(a^2\mu^2) + D_S]\right) O_S(a) + \dots \tag{22}$$

where $O_i(a)$ denote a bare lattice operator;

$$\begin{aligned} \frac{1}{2} O_R(a) &= (\bar{h}(x)\gamma^\mu (1+\gamma_5) q(x)) \left(\bar{h}^{(-)}(x)\gamma_\mu (1+\gamma_5) q(x)\right) ; \\ \frac{1}{2} O_N(a) &= (\bar{h}(x)\gamma^\mu (1-\gamma_5) q(x)) \left(\bar{h}^{(-)}(x)\gamma_\mu (1+\gamma_5) q(x)\right) \\ &+ (\bar{h}(x)\gamma^\mu (1+\gamma_5) q(x)) \left(\bar{h}^{(-)}(x)\gamma_\mu (1-\gamma_5) q(x)\right) \\ &+ 2\,(\bar{h}(x)(1-\gamma_5) q(x)) \left(\bar{h}^{(-)}(x)(1+\gamma_5) q(x)\right) \\ &+ 2\,(\bar{h}(x)(1+\gamma_5) q(x)) \left(\bar{h}^{(-)}(x)(1-\gamma_5) q(x)\right) ; \end{aligned} \tag{23}$$

α_s^L is the lattice coupling constant and $D_L, \dots, D_R$ are constants which have been computed in one-loop lattice perturbation theory. The leading logarithmic correction to $\tilde{O}_S$ in eq. (22) is taken into account by renormalizing the operators at a scale $\mu = 1/a$; the term D_S and the ... represent finite terms of $O(\alpha_s)$. Since $\tilde{O}_S$ enters only at the NLO, they do not need to be computed, cf. eqs. (16) and (17). Two remarks are in order here:

- The value of the constants D_i depend on the light quark action used in the numerical simulation. In our case the tree-level improved SW-Clover action [15] was used and the operators were improved by rotating the light-quark propagators [16].
- In the calculation of the coefficients D_i, the heavy-quark field renormalization constant computed in ref. [17]–[19] was used. This is the definition which is consistent with improvement [19] and with the method used to extract the matrix elements of the operators $O_i(a)$, see also the discussion in refs. [20, 21].

The details of the numerical calculation of the coefficients D_i can be found in ref. [14]. Here, we only quote the final results:

$$D_L = -22.5 \qquad D_R = -5.4 \qquad D_N = -14.0 \tag{24}$$

Lattice perturbation theory behaves rather badly due to the presence of tadpole-like diagrams which are typical of the lattice case [4]. For logarithmically divergent operators, two remedies to this problem have been proposed in the recent years, namely boosted perturbation theory [23] or the non-perturbative renormalization of the operators on quark and gluon states, in a fixed gauge [24]. Only the first method can be used, since the non-perturbative calculation of the renormalization constants relevant in the present study has not been performed yet.

In eq. (21), for comparison with the recent calculation of the UKQCD collaboration [21], $\alpha_s^L = 6/4\pi\beta u_0^4$ has been used. u_0 is a measure of the average link variable, $u_0 = (8K_c)^{-1}$, with K_c the value of the Wilson hopping parameter K at which the pion mass vanishes. In the SW-Clover case, the numerical value of the "boosted" coupling given above is very close to the boosted coupling defined in terms of the elementary plaquette (see ref. [23] for the different definitions). We have also used $\alpha_s^L = \alpha_V(q^*)$, where $\alpha_V(q^*)$ was also introduced in ref. [23], and q^* is an appropriate scale, which can be extracted in one-loop perturbation theory. For q^*, the value computed for the heavy-light axial-vector current for the Wilson action in ref. [25] was used, $q^* a = 2.18$, with $a^{-1} = 2$ GeV, since the appropriate result for the present case is not known.

Putting the corrections of steps i) and ii) together, see eqs. (16), (21), (22) and (24), the renormalized operator $\hat{O}_L(m_b)$ can be written in terms of the bare lattice operators $O_i(a)$ as follows:

$$\hat{O}_L(m_b) = Z_{O_L} O_L(a) + Z_{O_R} O_R(a) + Z_{O_N} O_N(a) + Z_{O_S} O_S(a) . \tag{25}$$

The renormalization constants Z_{O_i} are obtained from products of the coefficients in eqs.(16), (21) and (22), which are computed independently and contain terms of $O(\alpha_s)$. Thus there are two different ways in which one can organize the final results, i.e. by including (M_1) or excluding (M_2) next-to-next-to-leading terms of $O(\alpha_s^2)$ (without logarithms). Since the corrections are large, different choices will result in B-parameters which differ by about 28%. With the present state of art this is an intrinsic systematic error that can be reduced only by incresing the order of the perturbative calculation in the continuum and applying the non-perturbative renormalization to the lattice operators.

In eq. (16), the Z_{O_i} have been evaluated by using $m_b = 5$ GeV, $\Lambda_{QCD}^{n_f=4} = 200$ MeV, $\mu = 2$ GeV $\sim 1/a$ and $n_f = 4$. At the NLO this choice of the parameters gives $\alpha_s(m_b^2) = 0.1842$, which is also the value used in eq. (9), since the matching between the full theory and the effective one is made at $\mu = m_b$.

In eq. (21) three possible values for α_s^L were used, and the mixing coefficients Z_{O_i} were computed with both options M_1 and M_2. The values of the Z_{O_i}, together with the corresponding values of α_s^L are given in table 1. In the same table we also give the renormalization constant of the heavy-light axial current, computed in ref. [26] at the NLO, which is needed for the calculation of the B-parameter, see secs. 3 and 4.

[4] The inaccuracy of lattice perturbation theory has been demonstrated explicitely in several cases where a non-perturbative determination of the renormalization constants was possible [22, 23].

Options		Z_{O_L}	Z_{O_R}	Z_{O_N}	Z_{O_S}	Z_A
α_s^L Standard	M_1	.7949	-.0317	-.0822	-.1229	.9035
.0796	M_2	.7651	-.0394	-.1021	-.1229	.8989
α_s^L Boosted u_0	M_1	.6850	-.0581	-.1506	-.1229	.7932
.1458	M_2	.6286	-.0722	-.1871	-.1229	.7847
α_s^L Boosted V	M_1	.6283	-.0717	-.1859	-.1229	.7363
.1800	M_2	.5581	-.0891	-.2310	-.1229	.7257

Table 1: *Renormalization constants for different choices of the lattice coupling constants and options, see the text.*

Two observations are necessary at this point.

- In ref. [21] they have used the option M_2, without, however, including the next-to-leading corrections of step i). Moreover, they took for α_s the value given by its leading logarithmic expression. This is not consistent at the next-to-leading order and introduces a further (and easily avoidable) systematic effect of $\sim 10\%$ in the final result.
- Since, in absence of a better control of perturbation theory, the uncertainties are so large, it is not worth at this point to vary the value of Λ_{QCD}, or to worry about "quenching", or other more subtle questions.

3 Computation of B_B on the lattice.

In order to obtain B_B, we compute the following two- and three-point correlation functions

$$C^{RR'}(t) = \sum_{\vec{x}} \langle 0|A_0^R(\vec{x},t)\, A_0^{R'\,\dagger}(\vec{0},0)|0\rangle \ , \qquad (26)$$

$$C^{RR'}_{O_i}(t_1,t_2) = \sum_{\vec{x}_1,\vec{x}_2} \langle 0|A_0^R(\vec{x}_1,t_1)\, O_i(\vec{0},0)\, A_0^{R'\,\dagger}(\vec{x}_2,t_2)|0\rangle \ , \qquad (27)$$

where i =L,R,N,S denotes one of the operators on the r.h.s. of eq. (25). The labels R, R' =L,S,D correspond to different interpolation operators used for the B-mesons

$$\begin{aligned} A_\mu^L(x) &= \bar{h}(x)\,\gamma_\mu\,\gamma_5\,q(x) \ , \\ A_\mu^S(x) &= \sum_i \bar{h}(x_i)\,\gamma_\mu\,\gamma_5\,q(x) \ , & (28) \\ A_\mu^D(x) &= \sum_{i,j} \bar{h}(x_i)\,\gamma_\mu\,\gamma_5\,q(x_j) \ , & (29) \end{aligned}$$

i.e. the local (L), the cube-smeared (S) and the double cube-smeared (D) axial currents [27]. Correlators involving smeared sources were computed in the Coulomb gauge. The last two operators were proven to be effective for isolating the lightest B-meson state.

At large time distances the correlation functions above behave as

$$C^{RR'}(t) \longrightarrow Z^R\, Z^{R'}\, e^{-\triangle E\, t} \ , \tag{30}$$

and

$$C^{RR'}_{O_i}(t_1,t_2) \longrightarrow Z^R\, Z^{R'}\, \frac{\langle \bar{B}^0 | O_i(a) | B^0 \rangle}{2\, M_B} e^{-\triangle E\, (-t_1+t_2)} \ . \tag{31}$$

In the above equations

$$Z^R \ = \ \frac{1}{\sqrt{2\, M_B}}\, \langle 0 | A^R_0(\vec{0},0) | B \rangle \ , \tag{32}$$

and $\triangle E$ is the binding energy of the B-meson. To extract the matrix elements, one takes the ratio

$$R^{RR'}_{O_i}(t_1,t_2) \ = \ \frac{C^{RR'}_{O_i}(-t_1,t_2)}{\frac{8}{3}\, C^{RL}(-t_1)\, C^{R'L}(t_2)} \ , \tag{33}$$

which at large time distances behaves as

$$R^{RR'}_{O_i}(t_1,t_2) \longrightarrow B^{RR'}_{O_i} \ \equiv \ \frac{\langle \bar{B}^0 | O_i(a) | B^0 \rangle}{\frac{8}{3}\, M_B^2\, f_B^2\, Z_A^{-2}} \ , \tag{34}$$

where Z_A is the renormalization constant for the axial current.

The physical value of the B-parameter is then given by

$$B^{RR'}_B(m_b) \ = \sum_{i=L,R,N,S} Z_{O_i}\, Z_A^{-2}\, B^{RR'}_{O_i} \ . \tag{35}$$

We call this method the ratio method for determining the B-parameter. One may also consider the ratio

$$R^{RR'}_B(t_1,t_2) \ = \sum_{i=L,R,N,S} Z_{O_i}\, \frac{C^{RR'}_{O_i}(-t_1,t_2)}{\frac{8}{3}\, C^{RL}(-t_1)\, C^{R'L}(t_2)} \ , \tag{36}$$

which at large time distances behaves as

$$R^{RR'}_B(t_1,t_2) \longrightarrow Z_A^2\, B^{RR'}_B(m_b) \equiv \frac{\langle \bar{B}^0 | \hat{O}_L(m_b) | B^0 \rangle}{\frac{8}{3}\, M_B^2\, f_B^2\, Z_A^{-2}} \ . \tag{37}$$

We call this method the combined-ratio method.

Both the ratio and the combined-ratio method should give the same value for B_B. This is a check of our numerical results.

4 Numerical results.

As explained in the previous section, the determination of B_B requires the computation of the two- and three-point correlation functions (26) and (27). The SW-Clover fermion action [15] for the light quarks was used, in the quenched approximation. The tree-level improved [13, 14] propagators of the heavy quarks were computed in the static limit. Our results are based on a set of 600 gauge field configurations, computed on a lattice of size $24^3 \times 40$ at $\beta = 6.0$. The calculations were performed at three values of the masses of the light quarks, corresponding to $K = 0.1425, 0.1432$ and 0.1440. This allows to extrapolate the results to the chiral limit. All the errors have been computed with the jacknife method by decimating 30 configurations at a time.

The procedure to measure B_B is standard. At fixed t_1 (t_2), we study the behaviour of the ratios (33) and (36) as a function of t_2 (t_1), searching for a plateau in t_2 (t_1). $B_B(m_b)$ is defined by the weighted average of the data points in the plateau region, if this exits. We will take as our best determination of $B_B(m_b)$, the value evaluated in a time interval where the ratios appear to be independent of both t_1 and t_2. Notice that, contrary to the UKQCD collaboration, we never compute the matrix elements of parity-odd terms of the different operators in eqs. (15) and (23), because they are zero by parity. For this reason the matrix element of O_L and O_R are equal and in the following only the results for O_L will be given.

In order to improve the isolation of the lightest meson state at short time-distances, the ratios (33) and (36) using single and double cubic smeared axial currents were computed. The sizes of the cubes used in our simulation are $L_s = 5, 7$ and 9. From our previous studies, $L_s = 7$ and 9 were shown to give a good isolation of the lightest state for $t/a > 4 - 5$ or $t/a < 36 - 35$ [28]. For this reason, even though a plateau seems to set in at much ealier times, see fig. 1, the B-parameters were extracted from the ratios (33) and (36) at fixed $t_2 = 34$–36 and for $t_1 \geq 4$.

In fig. 1, in order to display the quality of our data, the results for R_{O_L}, R_{O_S} and R_{O_N} and $R_B = B_B$ as a function of t_1/a, at t_2 fixed are shown in a specific case. With our large statistics, we are able to observe the plateaux over large time-distances. Thus we were able to fit the ratios up to $t_1/a = 7$–9. On the basis of our previous experience, this makes us confident that the lightest meson state has been isolated.

We have fitted the ratios (33) and (36) to a constant, for several time-intervals. As an example, in table 2 the value of $B_B(m_b)$, at several values of the light-quark Wilson parameter K, is given. Since the results are almost independent of the value of the light quark mass, they can be safely extrapolated. The extrapolations to the chiral limit ($K_c = 0.14543(1)$) and to the mass of the strange quark ($K_s = 0.14367(6)$) are also presented in table 2.

Finally, we discuss the dependence of our results on the smearing size. The value of all the matrix elements obtained with different smearing sizes are very close. For example, in the same case as that considered in table 2, the values of $B_B(m_b)$ in the chiral limit are 0.78(2), 0.76(2) and 0.74(3) for $L_s = 5, 7, 9$, respectively. Within the statistical errors, a systematic shift of the results with the smearing size cannot

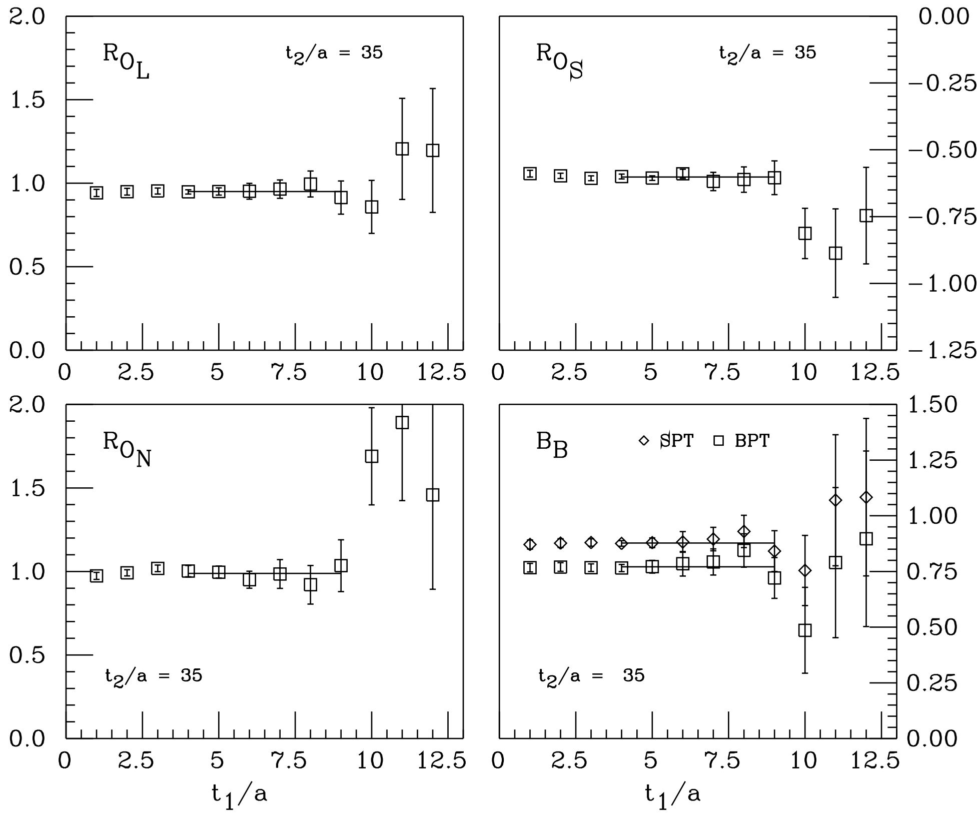

Figure 1: *The ratios R_{O_L}, R_{O_S}, R_{O_N} and B_B from the combined-ratio method are shown as a function of the time t_1, at $t_2/a = 35$. They were computed at $K = 0.1432$, by using A_μ^D, eq. (29), with $L_s = 7$. The labels SPT and BPT refer to standard and boosted perturbation with the V-coupling respectively. The lines show the time-interval of the fits.*

	$B_B(m_b)$		
K	$t_2 = 36$, $t_1 = 4$–8	$t_2 = 35$, $t_1 = 4$–9	$t_2 = 34$, $t_1 = 4$–6
0.1425	0.796(15)	0.774(19)	0.79(3)
0.1432	0.792(16)	0.771(19)	0.79(3)
0.1440	0.787(18)	0.77(2)	0.79(3)
0.14543	0.78(2)	0.76(2)	0.79(4)
0.14367	0.79(2)	0.77(2)	0.79(3)

Table 2: *Values of $B_B(m_b)$ obtained as explained in the text, by using A_μ^D, eq. (29), with $L_s = 7$. In order to obtain the renormalized operators, the boosted V-coupling has been used.*

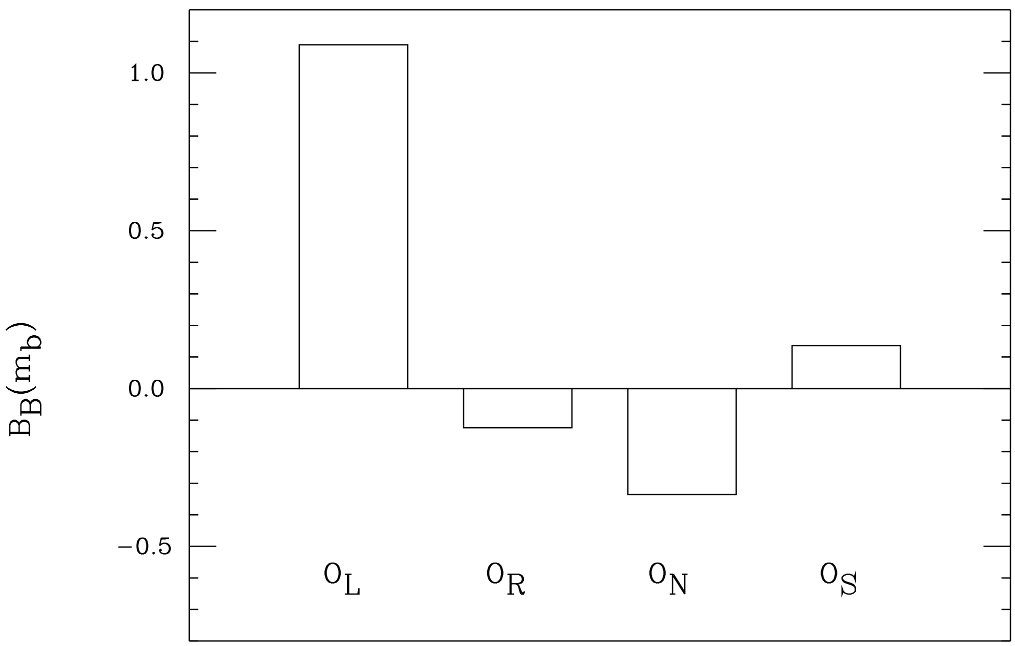

Figure 2: *Comparison of the contributions of the operators O_L, O_R, O_N and O_S to the B-parameter $B_{B_d}(m_b)$. The renormalized operator has been computed with the V-coupling, using the option M_1.*

be appreciated. Thus we added in quadrature the differences of the central values obtained with different smearings as an error. In fig. 2, in order to show the numerical importance of the different terms, the contributions of the operators $O_{L,R,N,S}$ to the B-parameter are shown. As expected, the operator O_L gives the largest contribution; the correction given by the three other operators is of the order of 20% and cannot be neglected.

We now give the final results by considering first the ratio method. For the different operators the results are

$$B_{O_L} = B_{O_R} = (0.94 \pm 0.02 \pm 0.04) \; , \qquad B_{O_N} = (0.98 \pm 0.05 \pm 0.04) \; ,$$
$$B_{O_S} = -(0.60 \pm 0.01 \pm 0.03) \; , \qquad (38)$$

where, as usual, the first error is the statistical one and the second is the systematic one, which takes into account of differences coming from different choices of the interval of the fit and of the smearing size.

By using in eq. (25) the numerical values of the renormalization constants from table 1, the values of B_{B_d} given in table 3 were obtained. The results from the ratio or the combined-ratio methods are indistinguishable. Notice, however, the large differences coming from different choices of (equivalent) sets of perturbative renormalization

Options		Ratio method	Combined-ratio method
α_s^L Standard	M_1	0.87(4)	0.87(5)
0.0796	M_2	0.81(4)	0.81(5)
α_s^L Boosted u_0	M_1	0.82(5)	0.81(5)
0.1458	M_2	0.67(5)	0.67(4)
α_s^L Boosted V	M_1	0.77(6)	0.76(5)
0.1800	M_2	0.55(6)	0.54(4)

Table 3: *Values of $B_{B_d}(m_b)$ for different choices of the lattice coupling constants and options, see the text.*

constants. From the results of the table, by using eq. (9), it is straightforward to derive the renormalization group invariant $\hat{B}_{B_d}$.

In table 4, our results for B_B are compared to other determinations. In the table $\hat{B}^{(1)}_{B_d}$ and $\hat{B}^{(2)}_{B_d}$ refers to the renormalization group invariant B-parameters computed from $B_{B_d}(m_b)$ using the leading or next-to-leading formulae respectively. The leading formula is obtained from eq. (9) by dropping the term proportional to J_5 and by using the one-loop expression for α_s (at one loop $\alpha_s(m_b^2) = 0.2342$). As for the results of the other groups, irrespectively of the fact that they used leading or next-to-leading formulae to compute $B_{B_d}(m_b)$, we evolved their value from the scale used in the original paper to $m_b = 5$ GeV using the one loop evolution equations. For this reason some of the numbers in the table may differ from those quoted in the corresponding references.

Our results are in good agreement with those of UKQCD [21], when a similar recipe for the renormalization constants is used [5]. The UKQCD number reported in table 4 should be compared with the value obtained with the M_2 recipe and the u_0 coupling constant, see table 3. With the exception of the result of ref. [33], which was obtained using the Wilson action for light quarks and is surprisingly large, the static value is smaller than other determinations of the same quantity obtained in the "full" theory, i.e. by extrapolating in the heavy quark mass m_Q results obtained for $m_Q \leq m_b$ [30]–[32]. A large value was also obtained by the QCD sum rule calculation of ref. [29].

¿From the study of the B-parameter as a function of the light-quark mass, we also get

$$\frac{B_{B_s}}{B_{B_d}} = \frac{\hat{B}_{B_s}}{\hat{B}_{B_d}} = 1.01 \pm 0.01 \ . \tag{39}$$

Notice that this ratio is almost independent of choice of the different possible options in the calculation of the renormalization constants, see sec. 2.

Using the same set of gauge field configurations, the following ratio has also been

[5] We recall that there are some differences because they have ommitted the NLO corrections and used by mistake the constants of ref. [13] instead of ref. [14]. These differences are, however, numerically unimportant.

$\hat{B}^{(1)}_{B_d}$	$\hat{B}^{(2)}_{B_d}$	$B_{B_d}(m_b)$	Authors	Ref.	Remarks
1.45(22)	1.58(24)	0.99(15)	Narison *et al.*	[29]	QSSR
1.29(7)	1.40(7)	0.88(5)	ELC	[30]	Extrapolated Wilson $\beta = 6.4$
1.30(9)	1.42(10)	0.89(7)	Soni *et al.*	[31]	Extrapolated Wilson $\beta = 5.7 - 6.3$
1.31(7)	1.42(7)	0.895(47)	JLQCD	[32]	Extrapolated Wilson $\beta = 6.1$
1.23(9)	1.34(10)	0.840(60)	JLQCD	[32]	Extrapolated Wilson $\beta = 6.3$
1.01(6)	1.10(6)	0.69(4)	UKQCD	[21]	Static Clover $\beta = 6.2$
1.42(6)	1.54(6)	0.97(4)	Christensen *et al.*	[33]	Static Wilson $\beta = 6.0$
1.11(7)	1.21(8)	0.76(5)	APE	This work	Static Clover $\beta = 6.0$ M_1
0.79(6)	0.86(6)	0.54(4)	APE	This work	Static Clover $\beta = 6.0$ M_2

Table 4: *Values of the B-parameter as determined by previous studies, and from the present one, are presented for comparison.* $\hat{B}^{(1)}_{B_d}$ *and* $\hat{B}^{(2)}_{B_d}$ *refers to the renormalization group invariant B-parameters computed from* $B_{B_d}(m_b)$ *using the leading or next-to-leading formulae respectively. The results of this work are those obtained by using the "Boosted V"* α_s^L*.*

measured (see ref. [28] for details)

$$\frac{f_{B_s}}{f_{B_d}} = 1.17 \pm 0.03 \tag{40}$$

Combining eqs. (39) and (40), one gets

$$\frac{f^2_{B_s} B_{B_s}}{f^2_{B_d} B_{B_d}} = 1.38 \pm 0.07 \ . \tag{41}$$

By using eq. (8) and the above result, one obtains

$$\frac{x_s}{x_d} = (1.45 \pm 0.13) \frac{|V_{ts}|^2}{|V_{td}|^2} \ , \tag{42}$$

where, for the masses and lifetimes of the B-mesons, we have used the values given in ref. [5].

5 Conclusions

In this work we have computed the B-parameter of the B-meson in the static limit, on 600 gauge field configurations with an improved action. Given the large set of configurations, our statistical errors are rather small. The large statistics allows also a reduction of the systematic errors coming from the imperfect isolation of the ground state. Errors coming from the extrapolation to the chiral limit are also quite small. Moreover, for this quantity, the error coming from the quenched approximation was estimated to be negligible [34]. Thus the most important source of systematic error in our results is the determination of the renormalization constants, which are known so far only in one-loop lattice perturbation theory. In order to improve the accuracy of the lattice predictions for the B parameter in the static theory, a non-perturbative computation of these constants is clearly necessary.

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