

Elastic s -wave $B\pi$, $D\pi$, DK and $K\pi$ Scattering from Lattice Calculations of Scalar Form Factors in Semileptonic Decays

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Abstract

We show how theoretical, principally lattice, calculations of the scalar form factors in semileptonic pseudoscalar-to-pseudoscalar decays can be used to extract information about the corresponding elastic s -wave scattering channels. We find values for the scattering lengths $m_\pi a = 0.179(17)(14)$, $0.26(26)$ and $0.29(4)$ for elastic s -wave isospin-1/2 $K\pi$, $B\pi$ and $D\pi$ channels respectively. We also determine phase shifts. For the DK channel we find hints that there is a bound state which can be identified with the recently discovered $D_{s0}^+(2317)$.

1 Introduction

The Omnès representation of form factors has been widely used in descriptions of kaon decays [1–4]. Usually, phase shift information or chiral perturbation theory $K\pi$ scattering amplitudes are taken as input to determine the scalar form factor in K_B decays. Here we explore what can be inferred about the s -wave isospin-1/2 $K\pi$ phase shift from theoretical lattice determinations of this form factor below $q_{\max}^2 = (m_K - m_\pi)^2$. Direct lattice determinations of scattering observables face serious difficulties [5, 6], but nonetheless phase shift information can be extracted from finite-volume effects in appropriate correlation functions [7, 8]. Results in the meson sector have appeared for $\pi\pi$ scattering in the isospin-2 channel (see [9] and references therein) and recently for $K\pi$ scattering in the isospin-3/2 channel [10]¹. Given this situation, we believe it is worthwhile to explore the alternative route proposed above. We also extend the discussion to encompass semileptonic decays of heavy-light pseudoscalar mesons, $H = B, D$, to pions and kaons, and the related elastic s -wave scattering reactions.

The form factor obtained from the Omnès representation becomes less sensitive to the details of the phase shift above threshold as the number of subtractions increases. This feature can be exploited in two ways. By using a large enough number of subtractions, one can determine the form factor without relying on any detailed knowledge of the phase shift apart from its value at threshold. Reference [11] used this to determine $|V_{ub}|$ from f_+ in semileptonic $B \rightarrow \pi$ decays, assuming only that the phase shift in the vector channel takes the value π at threshold. Conversely, using a small

¹Reference [10] uses NLO chiral perturbation theory to extract the phase shift for the isospin-1/2 channel in addition.

number of subtractions, the form factor may have a significant dependence on the phase shift. Hence one can use form factor information to learn about the phase shift itself. This is our main goal here.

We have taken published final results of lattice simulations to illustrate our procedure, with encouraging results. We note that integrating this procedure more closely into the analysis of lattice data could improve the quality of the results. This could be especially interesting given the growing body of unquenched lattice simulation data.

We will need a form for the phase shift as a function of the centre of mass energy, \sqrt{s} . For this we use a scattering matrix based on lowest-order chiral perturbation theory for the two-particle irreducible amplitudes, which satisfies unitarity. We describe our procedure for the case of elastic $K\pi$ scattering and subsequently apply it to $B\pi$ and $D\pi, DK$ scattering.

2 K_{l3} Decays and Elastic $K\pi$ Scattering

We need calculated values of the scalar form factor for K_{l3} decays. For this purpose we use two-flavour dynamical domain-wall fermion lattice results from [12]. This reference does not contain explicit values for the chirally-extrapolated form factors except at $q^2 = 0$, since the authors of [12] were focused on calculating $f_+(0)$ accurately in order to improve the determination of $|V_{us}|$. However it provides results for a range of quark masses which we have used to perform our own simple chiral extrapolation of f_0 . Our procedure ignores correlations (details are provided below) but we believe it leads to form factor inputs realistic enough for use in our exploratory study. One of our conclusions will be that accurate values of the chirally-extrapolated form factor at a wide range of q^2 values are very worthwhile.

2.1 Scalar Form Factor in K_{l3} Decays

The simulations in [12] were performed at a lattice spacing 0.12 fm with sea quark masses in the range $m_s/2$ to m_s , where m_s is the physical strange quark mass. First we used the fitted meson masses given in Table I of [12] to extract the physical light (up, down) and strange quark masses. For each quark mass combination and momentum channel ($\mathbf{p}_i \rightarrow \mathbf{p}_f$) we use the pole fit parameters given in Table III of [12] to determine $f_0(q^2(\mathbf{p}_i, \mathbf{p}_f, m_{ud}, m_s), m_{ud}, m_s)$. Subsequently, momentum-channel by momentum-channel, we do a linear fit in the quark masses and use this to extract the form factor at the physical ud and s masses and corresponding q^2 (computed using the continuum dispersion relation and physical neutral kaon and charged pion masses). We propagate errors by Monte Carlo through all the steps of the procedure assuming uncorrelated Gaussian-distributed inputs ([12] does not provide correlation information). The resulting form factor values are given in Table 1. The main result of [12] is $f_+(0) = f_0(0) = 0.968(9)(6)$ obtained by combining f_+ and f_0 information and using a more realistic chiral extrapolation. We will not use this point in our fits below since it depends on the same information as the points we do fit and also has different systematics. We will see below that our estimate for $f_0(0)$ is nevertheless compatible within errors, although this is not a target of our analysis.

2.2 Elastic $K\pi$ Scattering

Here we show how results for $f_0(q^2)$ in Table 1 can be used to extract information on the phase shift in elastic s -wave $K\pi$ scattering.

q^2 / GeV^2	$f_0(q^2)$
0.128	1.0013(9)
0.040	0.957(31)
-0.023	0.914(45)
-0.390	0.883(36)
-0.652	0.817(51)

Table 1 Input pairs $(q^2, f_0(q^2))$ obtained by the chiral extrapolation described in the text.

We use a multiply-subtracted Omnès dispersion relation to express $f_0(q^2)$ for $q^2 < s_{\text{th}} \equiv (m_K + m_\pi)^2$ as [13]

$$f_0(q^2) = \left(\prod_{i=0}^n [f_0(s_i)]^{\alpha_i(q^2)} \right) \exp \left\{ I_\delta(q^2; s_0, \dots, s_n) \prod_{j=0}^n (q^2 - s_j) \right\}, \quad (1)$$

$$I_\delta(q^2; s_0, \dots, s_n) = \frac{1}{\pi} \int_{s_{\text{th}}}^{+\infty} \frac{ds}{(s - s_0) \cdots (s - s_n)} \frac{\delta(s)}{s - q^2}, \quad (2)$$

$$\alpha_i(s) \equiv \prod_{j=0, j \neq i}^n \frac{s - s_j}{s_i - s_j}, \quad \alpha_i(s_j) = \delta_{ij}, \quad \sum_{i=0}^n \alpha_i(s) = 1. \quad (3)$$

This representation requires as input the elastic $K\pi \rightarrow K\pi$ phase shift $\delta(s)$ in the isospin-1/2 scalar channel, plus the form factor values $\{f_0(s_i)\}$ at $n + 1$ positions $\{s_i\}$ below the $K\pi$ threshold.

The phase shift is obtained from the $K\pi$ scattering amplitude, T , using

$$T(s) = \frac{8\pi i s}{\lambda^{1/2}(s, m_K^2, m_\pi^2)} (e^{2i\delta(s)} - 1) \quad (4)$$

where s is the squared centre-of-mass energy and $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$ is the usual kinematic function. The (inverse) scattering amplitude, in the appropriate isospin and angular momentum channel, is found from [14, 15]

$$T^{-1}(s) = -\bar{I}_0(s) - \frac{1}{8\pi a \sqrt{s_{\text{th}}}} + \frac{1}{V(s)} - \frac{1}{V(s_{\text{th}})} \quad (5)$$

Here, V is the two-particle irreducible scattering amplitude, a is the scattering length and \bar{I}_0 is calculated from a one-loop bubble diagram². This description automatically implements elastic unitarity, which is necessary for the phase shift to be extracted from equation (4).

For the isospin-1/2 scalar $K\pi$ channel, we approximate V using lowest order chiral perturbation theory (ChPT) (see [17] for a compilation of tree-level and one-loop meson-meson amplitudes in ChPT):

$$V(s) \approx \frac{1}{4f_\pi^2} \left(m_K^2 + m_\pi^2 - \frac{5}{2}s + \frac{3}{2s}(m_K^2 - m_\pi^2)^2 \right), \quad (6)$$

where $f_\pi = 92.4 \text{ MeV}$.

²In the notation of reference [16], $\bar{I}_0(s) = T_G((m + M)^2) - T_G(s)$, where M and m are the masses of the two propagating particles

At least one subtraction is needed to make the phase-shift integral in equation (2) convergent. The smaller the number of subtractions, the larger the range of s over which knowledge of the phase shift is required. There is a balance to be achieved between the number of subtractions and the phase-shift information. Given a fixed number of subtractions, accurate values of the form factor at points covering a region of q^2 increases the knowledge of the phase shift that can be extracted.

We start by considering a single subtraction at $q^2 = 0$. Hence we fit $f_0(0)$ and the scattering length a to the input form factor values compiled in Table 1. From the chi-squared fit we obtain

$$f_0(0) = 0.950(10), \quad m_\pi a = 0.175(17). \quad (7)$$

with $\chi^2/\text{dof} = 0.3$ and a correlation coefficient -0.997 between the two fitted parameters. The scattering length agrees well with the experimental result 0.13–0.24 [18] and the one-loop $O(p^4)$ chiral perturbation theory result 0.17(2) [19] (both quoted in pion units). The form factor and phase shift are displayed in Figure 1. The calculated phase shift agrees remarkably well with the experimental data up to $\sqrt{s} = 1.2$ GeV. We remind the reader that this phase shift is obtained from the form factor via the Omnès relation: there is no fit to the phase shift data itself. Despite this apparent success, we should rule out this result. This is because we have fixed the upper limit in the Omnès phase integral of equation (2) at $s = (1.425 \text{ GeV})^2 = 5s_{\text{th}}$, where typically the integrand is around three times smaller than its maximum so that the unevaluated part of the integral may be sizeable. We have chosen this cutoff for the integration range because the experimental data shows the existence of additional structure above $s = (1.4 \text{ GeV})^2$ (see orange points [20] in lower plot of Figure 1), and furthermore, $K\eta$ coupled-channel dynamics may play a role at higher energies. These extra dynamical features are not captured by lowest order ChPT amplitude used here.

Two approaches to address this problem are: (i) use a more realistic model for the $K\pi$ scattering amplitude within a coupled-channel formalism, and use higher order ChPT to determine the two-particle irreducible amplitudes; (ii) use more subtractions to reduce the dependence on the phase shift at large s in the phase integral. The first approach may allow the determination of low-energy constants (LECs) appearing in higher orders in the chiral lagrangian. Although this is an attractive prospect, given the number and accuracy of the data points we have in the current analysis, we have not applied higher order ChPT. However, to estimate the possible effects on the determination of the scattering length, we have supplemented the leading order ChPT two-particle irreducible amplitude of equation (6) with an expression incorporating the exchange of vector and scalar resonances as given in [21]. This addition partially accounts for next-to-leading ChPT contributions. We have also checked that $K\eta$ coupled-channel dynamics does not significantly affect our results. We will note below the numerical effects on our analysis from both massive resonances and the $K\eta$ coupled channel.

Thus we have considered two subtractions at $q^2 = 0$ and $q^2 = -0.75 \text{ GeV}^2 \equiv q_1^2$ and fitted $f_0(0)$, $f_0(q_1^2)$ and the scattering length a . The chi-squared fit results are:

$$\begin{aligned} f_0(0) &= 0.946(24) \\ f_0(q_1^2) &= 0.832(59) \\ m_\pi a &= 0.187(95) \end{aligned} \quad \left(\begin{array}{ccc} 1 & -0.517 & -0.959 \\ & 1 & 0.738 \\ & & 1 \end{array} \right) \quad (8)$$

with $\chi^2/\text{dof} = 0.4$. The fitted form factor and resulting phase shift are illustrated in Figure 2.

First we note that by performing two subtractions the value of the integrand in equation (2) at $s = (1.4 \text{ GeV})^2$, is now typically around six times smaller than its maximum, reducing the dependence on the phase shift at large s . The results presented above were obtained with a cutoff of $9s_{\text{th}} =$

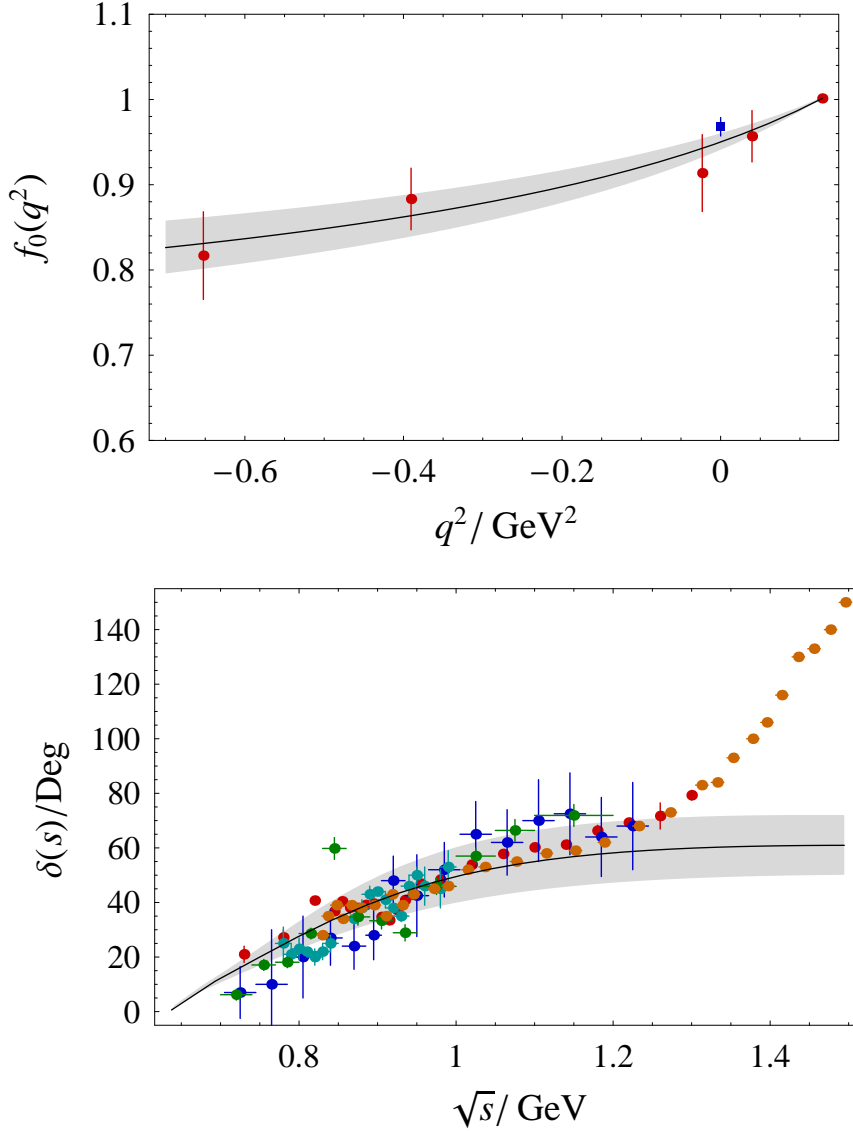


Figure 1 The upper plot shows the K_{13} form factor $f_0(q^2)$, with a 68% error band, obtained from a fit using a once-subtracted Omnès relation. Red points are the inputs from Table 1 and the blue square shows the result from [12] for $f_0(0)$ (not fitted). The lower plot shows the isospin-1/2 $K\pi$ s -wave phase shift with a 68% error band (grey). The phase shift plot also shows experimental data points from [22] (blue), [23] (red), [24] (cyan), [25] (green) and [20] (orange).

$(1.9 \text{ GeV})^2$ in the phase integral, where typically the integrand is around 1/30 of its maximum. We have checked that raising the cutoff to infinity leads to negligible changes in $f_0(0)$, less than 4 parts per thousand in $f_0(q_1^2)$ and around 2% in the scattering length. These variations are tiny compared to the statistical errors.

After the inclusion of a second subtraction, the central values of the fitted parameters and phase shift hardly change. However, the statistical errors on the fitted parameters have increased significantly and in consequence the derived phase shift has larger statistical uncertainty, as seen by comparing Figures 1 and 2. Nonetheless, this is one of the first determinations from a lattice calculation of the $K\pi$ scattering length, with an error comparable to the experimental one, and of the

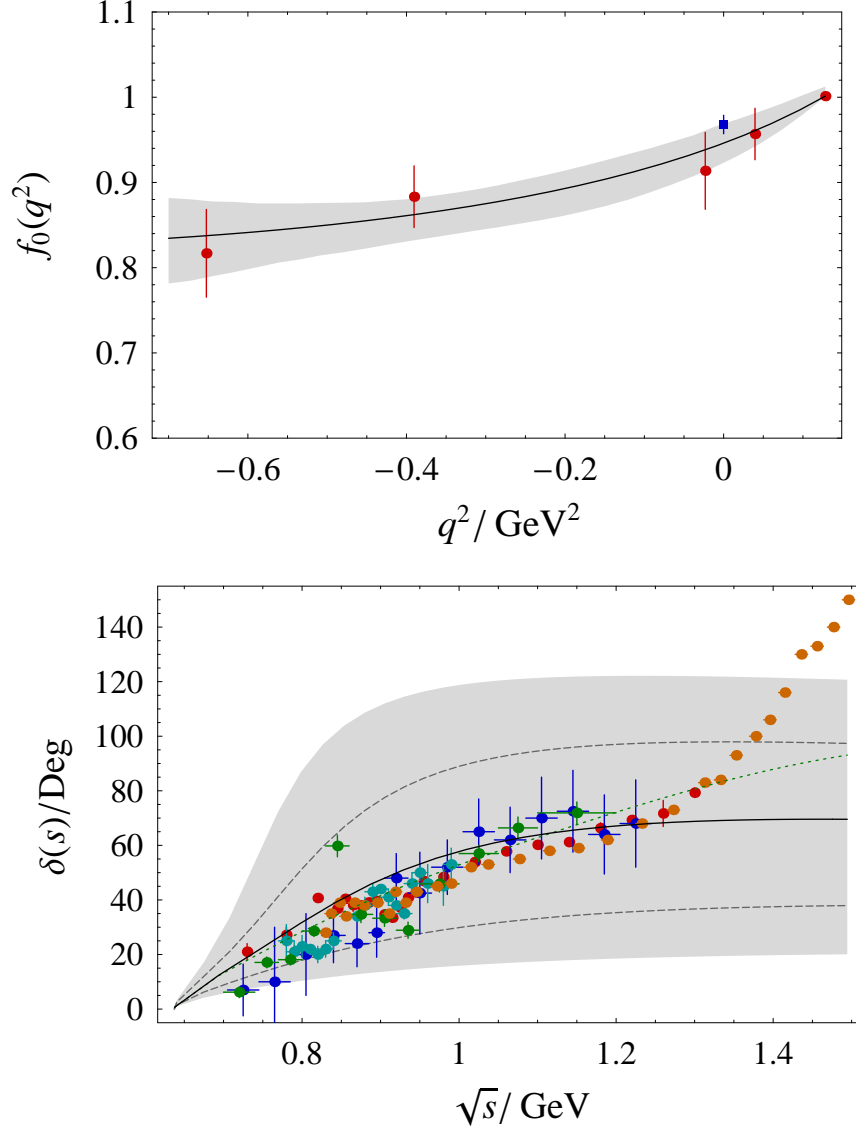


Figure 2 The upper plot shows the K_B form factor $f_0(q^2)$, with a 68% error band, obtained from a fit using a twice-subtracted Omnès relation. Red points are the inputs from Table 1 and the blue square shows the result from [12] for $f_0(0)$ (not fitted). The lower plot shows the isospin-1/2 $K\pi$ s -wave phase shift with a 68% error band (grey). The green dotted line shows the phase shift obtained with the inclusion of resonances (see text in paragraph preceding discussion of the $K\eta$ coupled channel). The dashed lines indicate the corresponding band obtained by artificially halving the error on the last four input form-factor points in Table 1. The phase shift plot also shows experimental data points as in Figure 1.

elastic phase shift. Future more accurate lattice simulations will improve the situation. For example, artificially halving the error on the last four input form-factor points in Table 1 leads to the error band indicated by the dashed lines for the phase shift in Figure 2 and gives a scattering length $m_\pi a = 0.187(48)$. This illustrates the balance between the statistical uncertainty of the outputs, the number of subtractions, knowledge of the s -dependence of the phase shift and the quality of the input form factor data.

Explicit massive resonance exchanges have been ignored in our model for the phase shift. Since the Omnès integration reaches s values where these contributions could be relevant, we try to estimate the associated uncertainties. Incorporating these exchanges will also allow us to estimate some next-to-leading ChPT effects [26, 27] on the scattering length. We obtain the resonance contribution to the isospin-1/2 $K\pi$ amplitude from equation (2.4) of reference [21], incorporating the exchange of the vector ρ and K^* resonances and also of the nonet scalar mesons with masses above 1 GeV. The vector coupling constant is fixed in the large- N_c limit [27], in good agreement with the lowest order results coming from the decay widths of the ρ and K^* mesons [26]. The scalar coupling constants and (common) masses are fixed to the values in equation (4.5) of [21], which lead to a reasonable description of the isospin-1/2 elastic $K\pi$ scattering amplitude up to around 1.4 GeV. When we supplement the leading order ChPT two-particle irreducible amplitude of equation (6) with this resonance contribution, we find no appreciable changes in the fitted values for f_0 and f_1 , while the scattering length increases by 6%. For the phase shift itself, the change is much smaller than the errors up to 1.3 GeV with some increase in the central line to match the data (see green dotted line in Figure 2). We will account for these effects in our systematic error for the scattering length in equation (11) below.

We have also examined $K\eta$ coupled channel effects. To do this we have modified equation (5) to become a 2×2 matrix equation in the coupled channel space,

$$T^{-1}(s) = -\bar{I}_0(s) - C + V^{-1}(s), \quad (9)$$

where the one-loop bubble integral and the coefficient matrix C are diagonal and we have approximated the V matrix using lowest order ChPT expressions for the $K\pi \rightarrow K\pi$, $K\pi \rightarrow K\eta$ and $K\eta \rightarrow K\eta$ amplitudes³ (see for instance [17]). In principle one should fit the two diagonal entries in the matrix C , but given our limited data and since we do not expect sizeable effects below $\sqrt{s} = 1.4$ GeV, we have fixed the entry corresponding to the $K\eta$ channel by demanding that $[T^{-1}(\mu)]_{K\eta, K\eta} = [V^{-1}(\mu)]_{K\eta, K\eta}$ for some scale $(m_K - m_\eta)^2 < \mu < (m_K + m_\eta)^2$ [28, 29]. The first entry in C is related to the $K\pi$ elastic scattering length, allowing us to fit $f_0(0)$, $f_0(q_1^2)$ and a as before. There are no appreciable changes in our fitted values for $f_0(0)$ and $f_0(q_1^2)$, while the scattering length is reduced by up to 5%. In the phase shift, one can see a cusp at the $K\eta$ threshold, but the change itself is small compared not only to the error bands in Figure 2 but also to those in Figure 1.

In our fits we observe almost complete anticorrelation of the fitted $f_0(0)$ and the scattering length, which means that one linear combination of these two parameters is redundant given the current accuracy of the input form factor information. This is not unexpected because the lowest order ChPT expressions for $f_0(0)$ and the scattering length are linearly related, depending only on $1/f_\pi^2$ (apart from masses). Of course, higher order corrections will not necessarily preserve this property: according to the Ademollo–Gatto theorem $f_0(0)$ does not have analytic terms from additional LECs in the $O(p^4)$ chiral lagrangian [30, 31], while such terms will affect the scattering length [31].

³We use the Gell-Mann–Okubo relation to express the eta mass in terms of the pion and kaon masses.

We have redone our two-subtraction fit implementing a linear relation between $f_0(0)$ and $m_\pi a$ deduced from the one-subtraction fit results of equation (7), assuming that the correlation coefficient is exactly -1 instead of -0.997 . The new results are

$$f_0(q_1^2) = 0.827(32), \quad m_\pi a = 0.179(17) \quad (10)$$

with $f_0(0) = 0.948(10)$ and $\chi^2/\text{dof} = 0.3$, with correlation coefficient -0.991 . Once again this strong correlation could be used to reduce the number of fit parameters. However, $f_0(q_1^2)$ is less-reliably calculated in ChPT and therefore its relation to $m_\pi a$ less well-determined. Hence we proceed with these fit results.

The fitted form factor and derived phase shift are shown in Figure 3. Both the scattering length and derived phase shift below $\sqrt{s} = 1.4$ GeV are in remarkable agreement with experiment, with a 10% error for the scattering length and 10–20% for the phase shift. The scattering length also agrees with the recent lattice finite-volume effect result $0.1725^{+0.0029}_{-0.0157}$ [10] and with the one-loop ChPT result $0.17(2)$ [19]. Following the discussion above on massive resonance exchanges and $K\eta$ coupled channel effects, we conclude that they are covered by the statistical error bands in the phase shift, but we combine their effects in quadrature and include an 8% systematic error in our final result for the scattering length:

$$m_\pi a = 0.179(17)(14). \quad (11)$$

We expect that more precise, accurate form factor data will be more sensitive to higher-order ChPT corrections, making the χ^2 fully-dependent on both $f_0(0)$ and $m_\pi a$ and moving their correlation away from -1 . In that situation we will not need to implement the procedure above, since a three-parameter (two subtractions) fit should provide smaller errors than we saw in equation (8).

3 Semileptonic $H \rightarrow \pi, K$ Decays and s -wave Elastic Scattering Phase Shifts

3.1 $B\pi$ Scalar Form Factor and Phase Shifts

To discuss the $B\pi$ phase shifts we take over the formalism described above, with appropriate changes to the kinematics, considering a neutral B meson and charged pion. For the two-particle irreducible isospin-1/2 s -wave $B\pi$ scattering amplitude we use the leading contact term from the heavy meson chiral perturbation theory (HMChPT) lagrangian [32],

$$V(s) \approx \frac{1}{4f_\pi^2} \left(2(m_B^2 + m_\pi^2) - 3s + \frac{(m_B^2 - m_\pi^2)^2}{s} \right). \quad (12)$$

We have not included a contribution from the t -channel B^* -exchange diagram depending on the leading HMChPT $B^*B\pi$ interaction term, since this vanishes at s_{th} and has magnitude less than 1% of that from the expression above over a large range of s .

We take input scalar form factor values from the lattice QCD calculation by the HPQCD collaboration [33], assuming that the statistical errors, σ_i , are uncorrelated, while the quoted 9%, 3% and 1% systematic errors are combined to give an additional 10% fully-correlated error, ϵ_i , on each point. The input covariance matrix for the lattice data thus takes the form $C_{ij} = \sigma_i^2 \delta_{ij} + \epsilon_i \epsilon_j$. We also use the lightcone sumrule result for $f_0(0) = f_+(0)$ from [34]. These inputs are collected in Table 2.

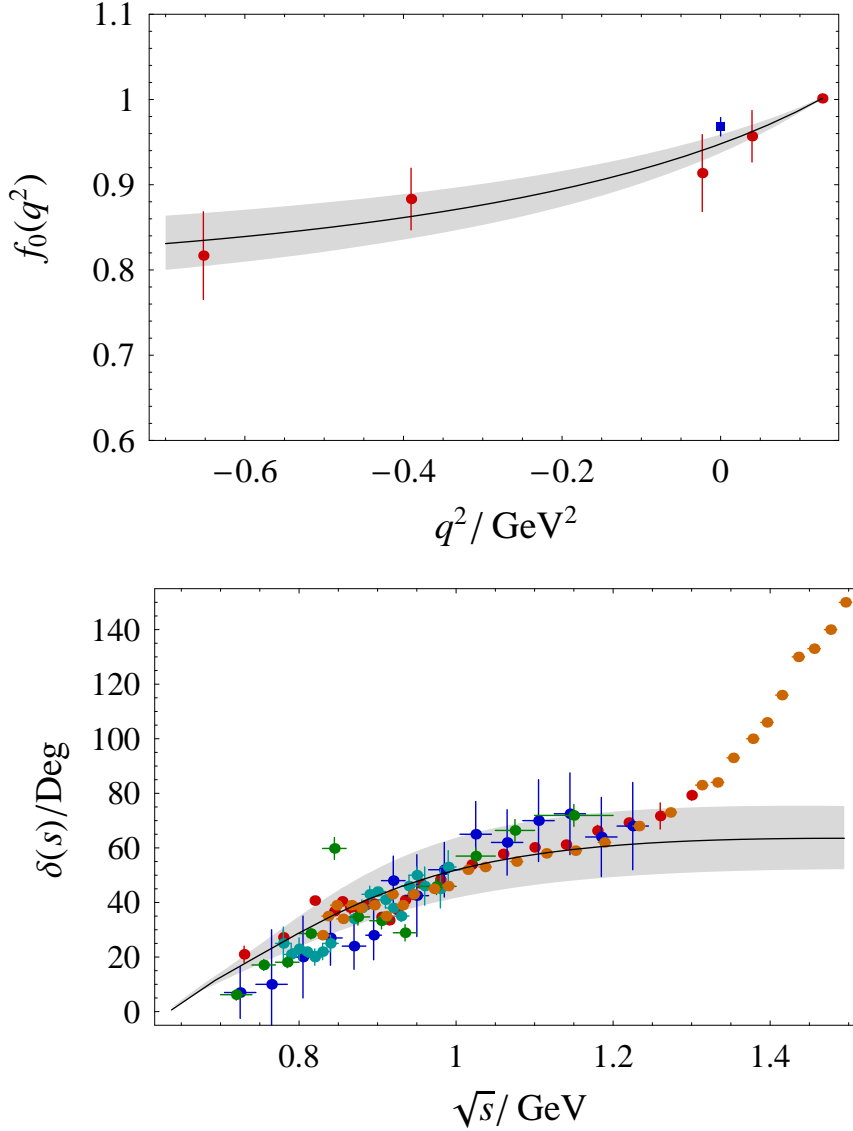


Figure 3 The upper plot shows the K_B form factor $f_0(q^2)$, with a 68% error band, obtained from a fit using a twice-subtracted Omnès relation, implementing a linear relation between $f_0(0)$ and the scattering length as described in the text. Red points are the inputs from Table 1 and the blue square shows the result from [12] for $f_0(0)$ (not fitted). The lower plot shows the isospin-1/2 $K\pi$ s -wave phase shift with a 68% error band (grey). The phase shift plot also shows experimental data points as in Figure 1.

Given the large mass of the B meson, the influence of inelastic single and multiple light meson production may be important within a few hundred MeV of threshold. To reduce the impact of these inelastic channels in the Omnès phase integral, we use two subtractions at $q^2 = 0$ and $q^2 = q_{\max}^2 = (m_B - m_\pi)^2$. Hence we have performed a three-parameter fit to $f_0(0)$, $f_0(q_{\max}^2)$ and the scattering length. The fit results are:

$$\begin{aligned}
 f_0(0) &= 0.258(31) \\
 f_0(q_{\max}^2) &= 1.17(24) \\
 m_\pi a &= 0.26(26)
 \end{aligned}
 \quad
 \begin{pmatrix}
 1 & 0.448 & 0.532 \\
 & 1 & 0.814 \\
 & & 1
 \end{pmatrix}
 \quad (13)$$

with $\chi^2/\text{dof} = 0.03$. We show the fitted form factor and the derived phase shift in Figure 4. We

	q^2 / GeV^2	$f_0(q^2)$
LCSR [34]	0	0.258 ± 0.031
HPQCD [33]	15.23	0.475 ± 0.026
	16.28	0.508 ± 0.025
	17.34	0.527 ± 0.025
	18.39	0.568 ± 0.024
	19.45	0.610 ± 0.024
	20.51	0.651 ± 0.025
	21.56	0.703 ± 0.026

Table 2 $B\pi$ scalar form factor inputs. A fully-correlated 10% systematic error should be added to the statistical error listed in the table for the HPQCD points.

have integrated up to $s = 5s_{\text{th}} \approx (12.1 \text{ GeV})^2$ where the integrand is typically one thousandth of its maximum value (reached at $s = (5.6 \text{ GeV})^2$ when the integral is evaluated for $q^2 = q_{\text{max}}^2/2$). We plot the phase shift up to $\sqrt{s} = 7.5 \text{ GeV}$ where the integrand is already 30 times smaller than its maximum value. We also observe that the fitted value for $f_0(q_{\text{max}}^2)$ agrees within errors with the heavy quark effective theory prediction in the soft-pion limit [35], $f_0(m_B^2) = f_B/f_\pi + \mathcal{O}(1/m_b^2) \approx 1.4(2)$ (using $f_B = 189(27) \text{ MeV}$ [36]).

We find that it is possible to determine the scattering length and the phase shift from current lattice QCD and sumrule form factor calculations, albeit with large errors. Moreover, we observe that our central phase-shift curve shows the existence of a resonance at $\sqrt{s} \approx 5.6 \text{ GeV}$ which may have some experimental support [37]. However, with the current level of errors in the form factor inputs we cannot give an upper bound for this resonance mass. The dashed lines in the phase shift plot show the effect of reducing the input errors to 1/4 of their current size, comparable to those for the K_B results. In this case we would be able to constrain the resonance mass, as can be seen from the figure. These reduced input errors would lead to a determination of the scattering length with 25% error. Having more input points, or indeed a functional form (as we use below for the $D\pi$ and DK cases), would of course also reduce the uncertainties.

3.2 $D\pi$ and DK Phase Shifts

To discuss the $D\pi$ phase shift we will use equation (12) with the obvious replacement $m_B \rightarrow m_D$. For the DK phase shift we project into the isospin zero channel, where the two-particle irreducible amplitude again takes the same form with the appropriate substitutions of masses and the replacement $f_\pi \rightarrow f_K \approx 110 \text{ MeV}$.

We take input scalar form factor values from the Fermilab-MILC-HPQCD lattice QCD calculation of reference [38]. The chiral extrapolation procedure adopted there leads to parameters for a Becirevic-Kaidalov (BK) [39] parameterisation of $f_0(q^2)$, and hence an explicit functional form is determined, rather than values at a set of q^2 points. For $f_0(q^2)$, the BK function is a simple pole form

$$f_0^{\text{BK}}(q^2) = \frac{F}{1 - q^2/(\beta m_{D_x}^2)}. \quad (14)$$

where $m_{D_x}^* = 2.010 \text{ GeV}, 2.112 \text{ GeV}$ for $D\pi, DK$ respectively. The BK parameters F and β are

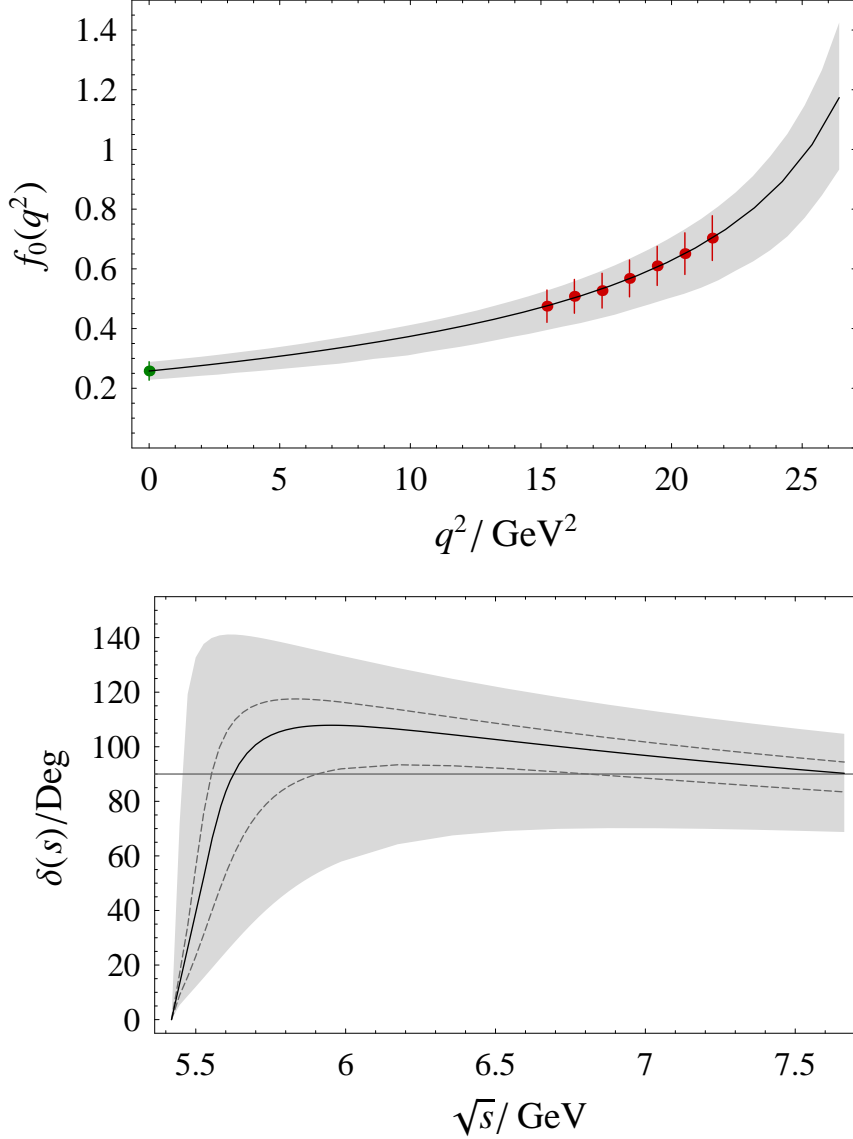


Figure 4 $B\pi$ isospin-1/2 scalar form factor and phase shift, together with 68% confidence level bounds (grey bands). The points on the form factor plot are the inputs given in Table 2. The dashed curves on the phase shift plot show the effect on the statistical uncertainty of reducing the input errors to 1/4 of their current value. The intercept of the phase shift with the horizontal line at 90° indicates the position of a resonance.

compiled in Table I of [38] and repeated here:

$$\begin{aligned}
 F_{D\pi} &= 0.64(3), & \beta_{D\pi} &= 1.41(6). \\
 F_{DK} &= 0.73(3), & \beta_{DK} &= 1.31(7).
 \end{aligned}
 \tag{15}$$

The errors above are statistical. We have added in quadrature a further 10% error to the F parameter to account for the systematic uncertainty for the form factors quoted in [38].

The fitting procedure we use here is as follows. We assume that F and β are uncorrelated⁴ Gaussian-distributed variables. We perform a Monte Carlo procedure by generating an ensem-

⁴Reference [38] does not provide correlation information for the fitted parameters.

ble of $\{F, \beta\}$ pairs. For each pair, we determine two Omnès subtraction parameters $f_1 \equiv f_0(0)$, $f_2 \equiv f_0(q_{\max}^2)$ and the scattering length a by minimising

$$\int_{-0.5 \text{ GeV}^2}^{q_{\text{upper}}^2} dq^2 |f_0^{\text{Omnès}}(q^2; a, f_1, f_2) - f_0^{\text{BK}}(q^2; F, \beta)|^2 \quad (16)$$

where $f_0^{\text{Omnès}}$ is easily obtained from equation (1). For $D\pi$ we take $q_{\text{upper}}^2 = 2 \text{ GeV}^2$, while for DK , $q_{\text{upper}}^2 = (m_D - m_K)^2$ (these choices correspond to the q^2 ranges shown in Figure (3) of [38]). This produces a three dimensional distribution for f_1 , f_2 and a .

For $D\pi$ we determine the following central values and correlation matrix:

$$\begin{aligned} f_0(0) &= 0.64(7) \\ f_0(q_{\max}^2) &= 1.38(17) \\ m_\pi a &= 0.29(4) \end{aligned} \quad \begin{pmatrix} 1 & 0.9 & 0.0 \\ & 1 & 0.4 \\ & & 1 \end{pmatrix} \quad (17)$$

Since we are fitting a three-parameter function to input form-factors determined by two parameters, we find that the correlation matrix has determinant compatible with zero. This feature could be avoided by using the Omnès parameterisation throughout the analysis.

We show our fitted form factor and derived phase shift in Figure 5. We have integrated the phase shift in the Omnès integral up to $s = 5s_{\text{th}} \approx (4.5 \text{ GeV})^2$ where the integrand is typically one 250th of its maximum value (reached at $s = (2.2 \text{ GeV})^2$ when the integral is evaluated for $q^2 = q_{\max}^2/2$). We plot the phase shift up to $\sqrt{s} = 3.5 \text{ GeV}$ where the integrand is already 40 times smaller than its maximum value.

The fitted form factor is indistinguishable by eye from the input BK curve. Since the Omnès expression for the form factor is founded only on very general properties, we observe that the BK parameterisation could be replaced by the Omnès form throughout an analysis of lattice data, removing the need for a fit like the one done above. We remark that the scattering length can be determined with a small error and differs from the value $m_\pi a = 0.18$ found from lowest order HMChPT. We believe that the small errors result from fitting a functional form rather than a small set of points. We predict the existence of an $I = 1/2$ s -wave resonance at $2.2(1) \text{ GeV}$.

When we consider the DK channel we find that for almost all of our Monte Carlo trials, the fitted value for the scattering length is huge, effectively infinite. This tells us that $\text{Re}T^{-1}(s_{\text{th}}) = 0$ as can be seen from equation (5). Hence there should be a resonance at threshold, $(m_D + m_K)^2 = (2.36 \text{ GeV})^2$. This can be understood by noting the existence of a 0^+ state, $D_{s_0}^+(2317)$, discovered by Babar [40], which is likely an isoscalar [41]. Neglecting isospin-violating decays to $D_s^+ \pi^0$, this state could be considered as an isoscalar s -wave DK bound state. In this case, following Levinson's theorem [42], the phase shift close to threshold has the form $\pi + pa + \dots$, where p is the centre-of-mass three-momentum. Using three parameter fits (two subtractions and a) we find that the scattering length is effectively zero in 70% of our Monte Carlo trials. Given this, we assume that the phase shift is π over the range where the integrand of the phase-shift integral is significant and obtain an excellent two-parameter fit (two subtractions):

$$\begin{aligned} f_0(0) &= 0.73(9) \\ f_0(q_{\max}^2) &= 1.08(12) \end{aligned} \quad (18)$$

with correlation coefficient 0.975.

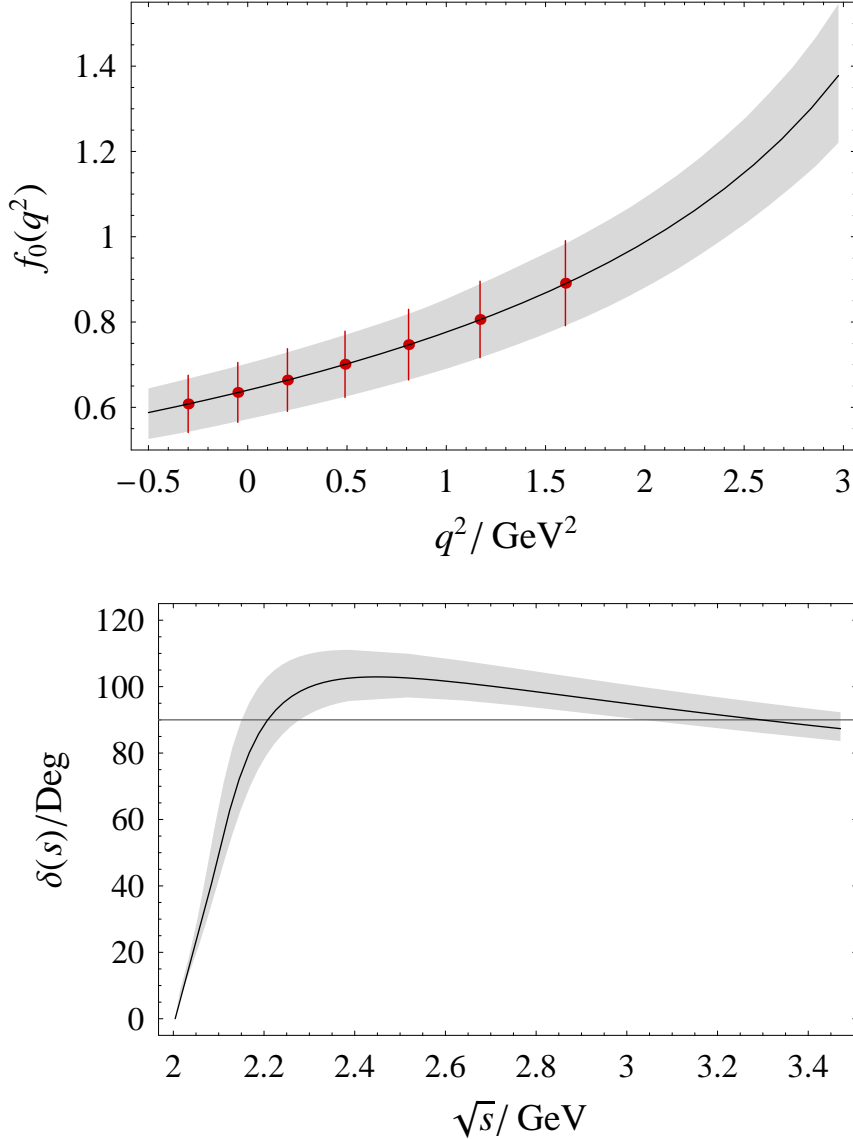


Figure 5 $D\pi$ isospin-1/2 scalar form factor and phase shift, together with 68% confidence level bounds. The points on the form factor plot are read off from Figure 3 of [38], but with error bars expanded to include a 10% systematic error, and are displayed to show the good agreement with our fit. The intercept of the phase shift with the horizontal line at 90° indicates the position of a resonance.

4 Conclusions

We have shown how existing theoretical, principally lattice, calculations of the scalar form factors in semileptonic pseudoscalar-to-pseudoscalar decays can be used to extract information about the phase shifts in the corresponding elastic s -wave scattering channels. The Omnès expression for the form factor rests on general principles of analyticity and unitarity. We remark that it provides a model-independent functional form that can be used in analysing lattice data, replacing more phenomenological parameterisations. Using the Omnès expression throughout would allow correlations in the lattice data to be taken into account.

From K_{l3} decays we have determined the elastic $K\pi$ scattering length with an uncertainty of around 12% and find a reasonable description of phase-shift data up to 1.3 GeV. Improved form-factor data

could potentially be used to learn information about low-energy constants (LECs) in the $O(p^4)$ chiral lagrangian.

For $B\pi$ the extracted scattering length and phase shift suffer from large uncertainties. We found hints of a resonance around 5.6 GeV. Reduced errors on the input form factor points, or increasing the number of points, could enable the position of the resonance to be established.

For $D\pi$ and DK we took advantage of functional forms for the $f_0(q^2)$ arising from lattice simulations. For $D\pi$ we were able to extract the scattering length with 13% statistical error and found a phase shift showing the existence of a resonance at around 2.2 GeV. The scattering length so determined is around 70% higher than that predicted by lowest order HMChPT. For DK we found hints that there is a bound state which could be identified with the $D_{s0}^+(2317)$.

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