Meson Resonances at large N_C: Complex Poles vs Breit-Wigner Masses

J. Nieves

Instituto de Física Corpuscular (IFIC), Centro Mixto CSIC-Universidad de Valencia, Institutos de Investigación de Paterna, Aptd. 22085, E-46071 Valencia, Spain

E. Ruiz Arriola

Departamento de Física Atómica, Molecular y Nuclear, Universidad de Granada, E-18071 Granada, Spain.

Abstract

The rigorous quantum mechanical definition of a resonance requires determining the pole position in the second Riemann sheet of the analytically continued partial wave scattering amplitude in the complex Mandelstam s variable plane. For meson resonances we investigate the alternative Breit-Wigner (BW) definition within the large N_C expansion. By assuming that the pole position is $\mathcal{O}(N_C^0)$ and exploiting unitarity, we show that the BW determination of the resonance mass differs from the pole position by $\mathcal{O}(N_C^{-2})$ terms, which can be extracted from $\pi\pi$ scattering data. For the case of the $f_0(600)$ pole, the BW scalar mass is predicted to occur at ~ 700 MeV while the true value is located at ~ 800 MeV.

Key words: Chiral symmetry, Large N_C, Unitarization, Resonances, Scalar meson.
PACS: 12.39.Mk, 11.15.Pg, 12.39.Fe, 13.75.Lb

Meson resonances are key building blocks in intermediate energy hadronic physics (for a review see e.g. Ref. [1] and references therein). Most often they contribute as virtual intermediate states to physical processes. This poses the question on the suitable interpolating field since, when the resonance goes off-shell, a definition of the background becomes necessary and its non-elementary nature becomes evident (see e.g. [2,3]). The large N_C expansion of QCD [4,5] provides a handle on this problem since meson resonances with a $q\bar{q}$ component, dominant or sub-dominant for $N_C = 3$, become stable particles; their mass becomes a fixed number $m_R \sim N_C^0$ and their width is suppressed as $\Gamma_R \sim 1/N_C$ for a sufficient large number of colors. This justifies the usage of a tree level Lagrangian in terms of canonically quantized fields (see e.g. [6] and references therein); resonance widths appear naturally as decay rates of the classical stable particles or equivalently as a quantum self-energy correction to the resonance propagator. Depending on the numerical

⋆ Research Supported by DGI and FEDER funds, under contracts FIS2008-01143/FIS and the Spanish Consolider-Ingenio 2010 Programme CPAN (CSD2007-00042), by Junta de Andalucía under contract FQM0225. It is part of the European Community-Research Infrastructure Integrating Activity "Study of Strongly Interacting Matter" (acronym HadronPhysics2, Grant Agreement n. 227431) and of the EU Human Resources and Mobility Activity "FLAVIAnet" (contract number MRTN–CT–2006–035482) , under the Seventh Framework Programme of EU.

value of the mass, being above or below threshold, physical resonances turn into Feschbach resonances or bound states respectively (see e.g. [7]).

In general [4,5] one expects a series expansion of the complex pole position $s_R = m_R^2 - \mathrm{i}\,\Gamma_R m_R$, of the $S-$matrix in the Second Riemann Sheet (SRS),

$$s_R = s_R^{(0)} + \lambda s_R^{(1)} + \lambda^2 s_R^{(2)} + \dots \tag{1}$$

where $\lambda = 3/N_C$. The purpose of this note is to show that using the standard and well-known Breit-Wigner (BW) definition with a similar expansion

$$s_{\mathrm{BW}} = s_{\mathrm{BW}}^{(0)} + \lambda s_{\mathrm{BW}}^{(1)} + \lambda^2 s_{\mathrm{BW}}^{(2)} + \dots \tag{2}$$

one has that

$$s_{\mathrm{BW}} - \mathrm{Re}(s_{\mathrm{R}}) = \mathcal{O}(N_C^{-2})\,, \tag{3}$$

anticipating an improved convergence and also suggesting a model independent way of assessing the accuracy of the large N_C expansion.

Large N_C scaling away from the physical $N_C = 3$ value, but relatively close to it, has been applied to chiral unitarized $\pi\pi$ amplitudes in Refs. [8,9] as a method to learn on the nature of meson resonances and on the induced N_C dependence of the corresponding pole masses and widths. We have recently shown [10] that in some cases, as for instance that of the $f_0(600)$ resonance, there is a lack of predictive power on the true N_C behaviour of the pole in the $N_C \to \infty$ limit, which could only be fixed by fine-tuning the parameters to unrealistically precise values. Two loop unitarized calculations are, in addition, beset by large uncertainties [11]. Though meaningful consequences can be drawn by studying the behaviour of the resonance in the vicinity of $N_C = 3$, we do not share the view [12] that one can reliably follow the N_C trajectory far from the real world ($N_C = 3$), extrapolating from calculations which are phenomenologically successful at $N_C = 3$, mainly because large uncertainties are built in. In addition to spurious $1/N_C$ corrections, the amplitude may not contain all possible leading N_C terms which are relevant at the resonance energies when N_C grows. We believe instead that more robust results might be achieved by examining observables which are *parametrically* suppressed by $1/N_C^2$, rather than just by $1/N_C$ corrections, but keeping always $N_C = 3$. This is in fact the way how the large N_C expansion has traditionally proven to be most powerful [13,14].

Let us consider for definiteness elastic $\pi\pi$ scattering in a given isospin–angular momentum sector denoted as (T, J), and let us also neglect coupled channel effects. The $S-$matrix is defined as

$$S_{\mathrm{TJ}}(s) = e^{2\mathrm{i}\delta_{TJ}(s)} = 1 - 2\mathrm{i}\ \rho(s) t_{\mathrm{TJ}}(s)\,,\ s \ge 4m^2 \tag{4}$$

with s the total $\pi\pi$ center of mass energy, $\delta_{TJ}(s)$ the phase shift, $t_{\mathrm{TJ}}(s)$ the scattering amplitude, m the pion mass and

$$\rho(s) = \frac{1}{16\pi}\sqrt{1 - \frac{4m^2}{s}}\,,\ s \ge 4m^2 \tag{5}$$

the phase space in our particular normalization. For simplicity we will drop the partial wave channel (T, J) indices in what follows. Using Eq. (4) we deduce

$$\tan\left[\delta(s) - \frac{\pi}{2}\right] = \frac{\mathrm{Re}\ t^{-1}(s)}{\rho(s)}\,,\ s \ge 4m^2\,. \tag{6}$$

Let us write the large N_C expansion of the partial wave amplitude

$$t(s) = \lambda t_1(s) + \lambda^2 t_2(s) + \lambda^3 t_3(s) + \dots \tag{7}$$

where the $t_n(s)$ are taken as N_C independent. From two-particle unitarity, which we write in the inverse amplitude form as

$$t(s)^{-1} = \mathrm{Re}\, t(s)^{-1} + \mathrm{i}\ \rho(s)\,, \tag{8}$$

we get the constraints

$$\mathrm{Im}\, t_1(s) = 0\,, \tag{9}$$
$$\mathrm{Im}\, t_2(s) = -\rho(s)\, t_1^2(s)\,, \tag{10}$$
$$\mathrm{Im}\, t_3(s) = -2\rho(s) t_1(s)\, \mathrm{Re} t_2(s)\,, \tag{11}$$

and so on. Note that the leading N_C amplitude is real in the elastic scattering region, as expected from a tree level $\pi\pi$ amplitude [4,5]. Of course, this does not preclude the appearance of the left cut discontinuity which occurs due to particle exchange in the t and u channels. Clearly any pole, s_0, occurring for the leading N_C and *real* amplitude will be either real or occurs in complex conjugated pairs. The latter is excluded as this would violate causality. If $s_0 < 4m^2$ it corresponds to a bound state while for $s_0 > 4m^2$ it can be associated to a Feschbach resonance.

To analytically continue the scattering amplitude to the complex Mandelstam $s-$plane, we remind that above threshold, elastic unitarity fixes the imaginary part of the inverse of the $t-$matrix, which is then determined as the boundary value in the upper lip of the unitarity cut,

$$\begin{aligned} t^{-1}(s + \mathrm{i}\ \epsilon) &= \mathrm{Re}\ t^{-1}(s) + \mathrm{i}\ \mathcal{R}(s + \mathrm{i}\ \epsilon), \\ \mathcal{R}(s + \mathrm{i}\ \epsilon) &\equiv \rho(s) \geq 0, \quad s \geq 4m^2\,. \end{aligned} \tag{12}$$

Resonances manifest as poles in the fourth quadrant of the SRS of the $t-$matrix. The $t-$matrix in the First Riemann Sheet (FRS), t_{I}, is defined in the complex plane by means of an analytical continuation of its boundary value in Eq. (12) at the upper lip of the unitarity cut. The $t-$matrix in the SRS (t_{II}) is related to t_{I}, thanks to $\mathcal{R}(s + \mathrm{i}\ \epsilon) = -\mathcal{R}(s - \mathrm{i}\ \epsilon)$, by [15]

$$t_{II}^{-1}(z) = t_I^{-1}(z) - 2\mathrm{i}\ \mathcal{R}(z), \quad z \in \mathbb{C}\,, \tag{13}$$

which implements continuity through the unitarity right cut, and the requirement that there are only two Riemann sheets associated to this cut,

$$t_{II}^{-1}(s \mp \mathrm{i}\ \epsilon) = t_I^{-1}(s \pm \mathrm{i}\ \epsilon), \quad s \geq 4m^2\,. \tag{14}$$

Let $s_R = m_R^2 - \mathrm{i}\ m_R \Gamma_R$, the position of the pole associated to the resonance R. By definition s_R, it is solution of the equation $t_{II}^{-1}(s_R) = 0$, which can be expressed as

$$\mathrm{Re}\ t_I^{-1}(s_R) = -\mathrm{i}\ \mathcal{R}(s_R^*)\,, \tag{15}$$

where we have used that $\mathcal{R}(s_R) = -\mathcal{R}(s_R^*)$ [1] .

In the large N_C limit, Re t_I^{-1} and $\mathcal{R}$ scale as $\mathcal{O}(N_C)$ and $\mathcal{O}(N_C^0)$, respectively, and thus one easily finds that m_R and Γ_R do scale as $\mathcal{O}(N_C^0)$ and $\mathcal{O}(N_C^{-1})$, respectively as we now show. Indeed, the resonance pole position, s_R, satisfies Eq. (15), and propose N_C expansions of the type (for simplicity, we will drop out the sub-index I, associated to the FRS)

$$s_R = x_R + \frac{y_R}{N_C} + \mathcal{O}(N_C^{-2})\,, \tag{16}$$
$$\mathrm{Re}\ t^{-1} = \left(\mathrm{Re}\ t^{-1}\right)_{(1)} + \left(\mathrm{Re}\ t^{-1}\right)_{(0)} + \mathcal{O}(N_C^{-1})\,, \tag{17}$$

where we have used that any pole generated by the re-summation of diagrams must necessarily scale as $\mathcal{O}(N_C^0)$ for a sufficiently large number of colors and that Re t^{-1} scales as $\mathcal{O}(N_C)$ (we use an obvious notation in the N_C expansion of Re t^{-1}, where $\left(\mathrm{Re}\ t^{-1}\right)_{(j)}$ scales as $\mathcal{O}(N_C^j)$). The large N_C expansion of Eq. (15) reads

[1] We are being abusive regarding notation. Here Re $t_I^{-1}(z)$ is an analytical function which has not right cut and it does correspond to the real part of a function *only* when $z = s + \mathrm{i}\ \epsilon$.

$$\underbrace{\left(\mathrm{Re}\ t^{-1}\right)_{(1)}(x_R)}_{\mathcal{O}(N_C)} + \underbrace{\frac{y_R}{N_C}\left[\left(\mathrm{Re}\ t^{-1}\right)_{(1)}\right]'(x_R) + \left(\mathrm{Re}\ t^{-1}\right)_{(0)}(x_R)}_{\mathcal{O}(N_C^0)} + \mathcal{O}(N_C^{-1}) = \underbrace{-\mathrm{i}\ \rho(x_R)}_{\mathcal{O}(N_C^0)} + \mathcal{O}(N_C^{-1})\,, \quad (18)$$

At Leading Order (LO), we find

$$\left(\mathrm{Re}\ t^{-1}\right)_{(1)}(x_R) = 0\,. \quad (19)$$

This forces x_R to be real and guaranties that m_R scales as $\mathcal{O}(N_C^0)$ in the $N_C \gg 3$ limit. At Next-to-Leading-Order (NLO), we have

$$-\frac{\mathrm{Im}\ y_R}{N_C} = \rho(x_R)\ \frac{1}{\frac{d}{ds}\left(\mathrm{Re}\ t^{-1}\right)_{(1)}(s)}\Bigg|_{s=x_R} \quad (20)$$

$$\frac{\mathrm{Re}\ y_R}{N_C}\left[\left(\mathrm{Re}\ t^{-1}\right)_{(1)}\right]'(x_R) = -\left(\mathrm{Re}\ t^{-1}\right)_{(0)}(x_R)\,. \quad (21)$$

Unitarity fixes the sign the of the imaginary part, showing that for large, but finite N_C, the real pole comes from the 4th quadrant. The first of the above equations ensures that the resonance width, Γ_R, scales as $\mathcal{O}(N_C^{-1})$, for very large values of N_C.

Now, we could rewrite Eq. (15), with accuracy $\mathcal{O}(N_C^{-2})$, as

$$\begin{aligned}\mathrm{Re}\ t^{-1}(s_R) &= \mathrm{Re}\ t^{-1}(m_R^2) - \mathrm{i}\ m_R\Gamma_R\left[\mathrm{Re}\ t^{-1}\right]'(m_R^2) - \frac{m_R^2\Gamma_R^2}{2}\left[\mathrm{Re}\ t^{-1}\right]''(m_R^2) + \mathcal{O}(N_C^{-2})\\ &= -\mathrm{i}\ \rho(m_R^2) + m_R\Gamma_R\rho'(m_R^2) + \mathcal{O}(N_C^{-2})\,.\end{aligned} \quad (22)$$

Thus, we find that

$$\mathrm{Re}\ t^{-1}(m_R^2) = \underbrace{m_R\Gamma_R\left\{\rho' + \frac{m_R\Gamma_R}{2}\left[\mathrm{Re}\ t^{-1}\right]''\right\}\Bigg|_{s=m_R^2}}_{\mathcal{O}(N_C^{-1})} + \mathcal{O}(N_C^{-3}) \quad (23)$$

$$m_R\Gamma_R = \underbrace{\frac{\rho}{\left[\mathrm{Re}\ t^{-1}\right]'}\Bigg|_{s=m_R^2}}_{\mathcal{O}(N_C^{-1})} + \mathcal{O}(N_C^{-3})\,. \quad (24)$$

Thus, at the resonance pole mass Re t^{-1} scales as $\mathcal{O}(N_C^{-1})$ instead of $\mathcal{O}(N_C)$. The reason is that the pole is moving, as we will show below, at speed $1/N_C^2$ towards the real axis. This is the first theorem of this work. In principle, the derivatives of Re t^{-1} at $s = m_R^2$ do still grow linearly with N_C. On the other hand, since $\tan x = x + \mathcal{O}(x^3)$, we also find

$$\delta(m_R^2) = \frac{\pi}{2} + \underbrace{\delta'(m_R^2)\ \frac{\left[\rho^2\left[\mathrm{Re}\ t^{-1}\right]'\right]'}{2\left(\left[\mathrm{Re}\ t^{-1}\right]'\right)^3}\Bigg|_{s=m_R^2}}_{\mathcal{O}(N_C^{-1})} + \mathcal{O}(N_C^{-3}) \quad (25)$$

where we have used that

$$\delta'(m_R^2) = \left[\frac{\mathrm{Re}\ t^{-1}}{\rho}\right]'\frac{1}{1+\left(\frac{\mathrm{Re}\ t^{-1}}{\rho}\right)^2}\Bigg|_{s=m_R^2} = \underbrace{\frac{\left[\mathrm{Re}\ t^{-1}\right]'}{\rho}\Bigg|_{s=m_R^2}}_{\mathcal{O}(N_C)} + \mathcal{O}(N_C^{-1})\,. \quad (26)$$

¿From the above equations we see that $\delta'(m_R^2)$ grows linearly with N_C, while $\delta(m_R^2)$ reaches the value $\pi/2$, up to corrections of the order $\mathcal{O}(N_C^{-1})$. This latter result constitutes our second theorem. More importantly, from Eq. (25) it is trivial to find a value of s for which the phase shift differs of $\pi/2$ in terms suppressed by three powers of the number of colors. This is to say

$$\delta(s_{\rm BW}) = \frac{\pi}{2} + \mathcal{O}(N_C^{-3})\,, \tag{27}$$

where

$$s_{\rm BW} = m_R^2 - \underbrace{\frac{\delta(m_R^2)-\pi/2}{\delta'(m_R^2)}}_{\mathcal{O}(N_C^{-2})}\,, \tag{28}$$

$$= m_R^2 - \underbrace{\left.\frac{\left[\rho^2\left[{\rm Re}\, t^{-1}\right]'\right]'}{2\left(\left[{\rm Re}\, t^{-1}\right]'\right)^3}\right|_{s=m_R^2}}_{\mathcal{O}(N_C^{-2})} + \mathcal{O}(N_C^{-4})\,. \tag{29}$$

Thus, we see that the existence of a pole in the SRS guaranties that there exists a value of $s_{\rm BW}$, which can naturally be identified with the BW position, where the phase shift is $\pi/2$, up to $\mathcal{O}(N_C^{-3})$ corrections. The BW mass, $\sqrt{s_{\rm BW}}$, differs from the pole mass, m_R, in $\mathcal{O}(N_C^{-2})$ terms, which can be computed thanks to Eq. (28). Note that the above relation has been deduced under the assumption of a finite large N_C limit of the resonance pole position. Our Eq. (28) is nothing but the first iteration in Newton's method for solving the BW condition, $\delta(s) = \pi/2$, starting from the resonance mass as the initial guess. The meaning is just that large N_C implies that a straight line extrapolation of the phase shift from the resonance should give a good estimate for the BW mass.

Ideally one would like to test Eq. (28) directly from data, but the existing uncertainties in the resonance pole m_R, as well as in the derivatives of the amplitude forced us to use instead a suitable parameterization [16,17]. The fact that the corrections are largely suppressed at large N_C provides some confidence on the accuracy of the result. Using the conformal mapping parameterizations of Ref. [16] for the isoscalar–scalar $\pi\pi$ phase shift [2]

$$\rho(s)\cot\delta_{00}(s) = \frac{m^2}{s-m^2/2}\left[\frac{m}{\sqrt{s}} + B_0 + B_1 w + B_2 w^2\right]\,, \tag{30}$$

where the conformal mapping is

$$w(s) = \frac{\sqrt{s}-\sqrt{4m_K^2-s}}{\sqrt{s}+\sqrt{4m_K^2-s}}\,. \tag{31}$$

We take three representative sets discussed in Ref. [16] and compiled for completeness in Table 1. We confirm that the resulting complex pole position slightly overshoots the Roy equation value $\sqrt{s_\sigma} = 441^{+16}_{-8} - {\rm i}\, 272^{+9}_{-12}$MeV [18]. Quoted errors in both the true BW position $\delta_{00}(m^2_{\sigma,\,\rm BW}) = \pi/2$ and the large-N_C *predicted* BW position using Eq. (28) just reflect uncertainties in the input parameters B_0, B_1 and B_2 as well as the induced complex pole. Actually, for Sets A and C we find a 100 MeV wide stability plateau of the predicted value from Eq. (28) around the pole mass. The discrepancy is compatible with the expected $1/N_C^4$ correction of the BW value, given the fact that Γ_σ is large. While a serious attempt to evaluate this correction would require a much more reliable parameterization and better data (higher order derivatives of the phase shift enter) it is surprising that despite its large width, our Eq. (28) may accommodate the large shift from the

[2] We use here the ghost-full version and the Adler zero located at the lowest order ChPT $s_A = m^2/2$ [16]. Our results show little dependence on this choice.

	Set A	Set B	Set C
B_0	3.57(17)	7.63(23)	4.3(3)
B_1	$-24.3(5)$	$-23.2(6)$	$-26.7(6)$
B_2	$-6.3(1.3)$	$-23.0(1.4)$	$-14.1(1.4)$
$\sqrt{s_\sigma}$ [MeV]	466(4)−i 232 (3)	477(7)−i 322(6)	476(6) −i 255(4)
$m_{\sigma,\ R}$ [MeV]	404(5)	350(10)	401(8)
$m_{\sigma,\ \mathrm{BW}}$ [MeV]	803 (4)	865 (4)	807 (5)
$m_\sigma\|_{\mathrm{BW},N_C\gg3}$ [MeV]	657 (4)	726 (5)	678 (6)

Table 1
Large N_C Predicted Breit-Wigner Resonances for the isoscalar-scalar channel $(T,J)=(0,0)$ in $\pi\pi$ scattering using the large N_C formula $m_\sigma^2|_{\mathrm{BW,largeN_C}} = m_{\sigma,R}^2 - (\delta_{00}(m_{\sigma,R}^2)-\pi/2)/\delta_{00}'(m_{\sigma,R}^2)$ compared to the true BW result, $\delta_{00}(m_{\sigma,\ \mathrm{BW}}^2)=\pi/2$ where $s_\sigma = m_{\sigma,R}^2 - i m_{\sigma,R}\Gamma_{\sigma,R}$ represents the pole of the $S-$matrix in the SRS, $1/S_{\mathrm{II}}(s_\sigma)=0$. We use the parameterization for the $\delta_{00}(s)$ phase shift of Ref. [16] (see Eq. (30) in the main text).

pole to the BW mass by appealing large N_C arguments which tacitly assume the expansion of Eq. (1) and thus a small width approximation. Of course, the final answer regarding how the scalar meson mass scales with N_C can only be given by performing dynamical lattice QCD calculations with variable N_C (see e.g. Ref. [19] for a review).

Determining s_R from s_{BW} or viceversa are in principle equivalent procedures, but low energy based approximations such as unitarized ChPT [20,21] are expected to work better when predicting s_{BW} from s_R since $\sqrt{|s_R|}\sim 0.5$ GeV and $\sqrt{s_{\mathrm{BW}}}\sim 0.8$ GeV. Actually, if we take the analytical one loop partial wave amplitudes given in Ref. [22], unitarize with the IAM method [23,24,25,26] and use $\bar l_1=-0.4(6)$, $\bar l_2 = 4.3(1)$, $\bar l_3 = 2.9(2.4)$, $\bar l_4 = 4.4(2)$, from the analysis of Roy equations within ChPT [27] [3], we find a reasonable good description of the phase shift at low energy that leads to a rather good value of $\sqrt{s_\sigma} = 410(10) - \mathrm{i}\ 270(10)$ MeV. However, discrepancies with data become important as the energy increases. and the phase shift never takes the value $\delta_{00}(s)=\pi/2$. Despite these deficiencies, the large N_C formula, Eq. (28), provides still a reasonable value for the Breit-Wigner mass $m_\sigma|_{\mathrm{BW},N_C\gg3} = 600(10)$ MeV. The difference of this value to the estimate of Table 1 is consistent with the corresponding values of the phase shifts since at $\sqrt{s}=500$ MeV one has $\delta_{00}=45.7(6)^{\mathrm{o}}, 39.1(6)^0$ and $43.4(9)^{\mathrm{o}}$ for Sets A,B and C respectively whereas one finds a significant smaller value $\delta_{00}=34.7(5)^{\mathrm{o}}$ for the chiral IAM unitarized case with $\bar l_{1,2,3,4}$ from Ref. [27]. We stress that the phase shift in the chiral unitary representation itself never passes through 90^0. This result reinforces the advocated picture; while in terms of the chiral representation the pole and the BW masses are far apart, within the large N_C framework they are connected, as they approach to each other at speed $\mathcal{O}(1/N_C^2)$. Note, however, that in practice we *never* depart from the physical $N_C=3$ value.

We summarize our results. We have analyzed the connection between the pole mass and the Breit-Wigner mass of the $\pi\pi$ scattering amplitude within the large N_C expansion. We have shown that assuming that both masses are $\mathcal{O}(N_C^0)$ the difference is $\mathcal{O}(N_C^{-2})$ *parametrically* suppressed and computable numerically from the data. This allows to *predict* the BW mass from the pole mass successfully even in the hostile case of the rather wide $f_0(600)$ resonance. Thus, while the pole and Breit-Wigner masses are far apart numerically they turn out to be connected within the large N_C approximation. That would indicate the presence of a $q\bar q$ component in the $\sigma-$wave function. Such component, likely sub-dominant in the real world $N_C=3$ [9], would ensure for a sufficiently large number of colours, the N_C-behaviour ($m_\sigma\sim N_C^0$ and $\Gamma_\sigma\sim 1/N_C$) of the σ pole parameters that has allowed us to relate pole and BW masses.

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