Properties of the ρ and σ Mesons from Unitary Chiral Dynamics.

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The chiral limit of the ρ and σ masses and widths is discussed. We work within the inverse amplitude method to one loop in SU(2) ChPT and analyze the consequences that all chiral logarithms cancel out in the $\rho-$channel, while they do not cancel for the σ case, and how they strongly influence the properties of this latter resonance. Our results confirm and explain the different behavior of the σ and ρ poles for N_C not far from 3, but we extend the analysis to very large N_C, where the behavior of these two resonances is re-analyzed. We note that the rather natural requirement of consistency between resonance saturation and unitarization imposes useful constraints. By looking only at the $\rho-$channel, and within the single resonance approximation, we find that the masses of the first vector and scalar meson nonets, invoked in the single resonance approximation, turn out to be degenerated in the large N_C limit. On the contrary we show that, for sufficiently large N_C, the scalar meson evolution lies beyond the applicability reach of the one-loop inverse amplitude method and if the scalar channel is also incorporated in the analysis, it may lead, in some cases, to phenomenologically inconsistent results.

I. INTRODUCTION

The large N_C-limit of QCD [1, 2] makes quark-hadron duality manifest at the expense of introducing an infinite number of weakly interacting stable mesons and glueballs. While the corresponding counting rules are deduced at the quark-gluon level, internal consistency requires them to be valid also at the hadron level. Moreover, large N_C studies might clarify several features of the nuclear force, in particular, the role played by the ubiquitous scalar meson. This is an essential ingredient which contributes to the mid range nuclear attraction, which, with a mass of ~ 500 MeV, was originally proposed in the fifties [3] to provide saturation and binding in nuclei. During many years, there has been some arbitrariness on the "effective" scalar meson mass and coupling constant to the nucleon, partly stimulated by lack of other sources of information. For instance, in the very successful nucleon–nucleon charge dependent Bonn potential [4] any partial wave $^{2S+1}L_J$-channel is fitted with noticeably different scalar meson masses and couplings. The situation has steadily changed during the last decade, and the scalar meson has been finally resurrected [5], culminating with the inclusion of the 0^{++} resonance in the PDG [6] as the $f_0(600)$ resonance, also denoted as the σ. The σ-resonance is traditionally seen in $\pi\pi$ scattering, with a wide spread of values being displayed ranging from $400 - 1200$ MeV for the mass and a $600 - 1200$ MeV for the width [7]. These uncertainties have recently been sharpened by a benchmark prediction based on Roy equations and chiral symmetry [8] yielding the unprecedented accurate values

$$\sqrt{s_\sigma} = 441^{+16}_{-8} - \mathrm{i}\, 272^{+9}_{-12} \text{ MeV.} \tag{1}$$

Forward dispersion relations and Roy equations on the real axis have also been used by the Madrid group in Ref. [9] yielding a slightly heavier and narrower σ-resonance determination.

While the existence of this broad low lying state is by now out of question, the debate on the nature of the σ-meson is not completely over. Structures of the type tetraquark or glueball, etc. have been proposed (see e.g. Ref. [10] for a recent review and references therein).

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Lattice determinations of the lightest scalar mesons are challenging (for reviews see [11, 12]). It has been found that the mass of the lightest 0^{++} meson is suppressed relative to the mass of the 0^{++} glueball in quenched QCD at an equivalent lattice spacing [13]. In the quenched approximation it has been claimed [14], that $m_\sigma = 550$ MeV for pion masses as low as $m_\pi = 180$ MeV. On the other hand, a recent analysis of np scattering in the $^1S_0$ channel yields $m_\sigma = 510(10)$ MeV [15, 16], when uncorrelated 2π exchange is disregarded (see however Refs. [17, 18]).

Chiral Perturbation Theory (ChPT) for $\pi\pi$ scattering has been introduced in Refs. [19, 20]. We aim here to re-analyze the nature of the ChPT scalar resonance, stressing its differences and similarities with the ρ−meson. Most of the results will be obtained in the limit of massless pions, which will allow us to work with almost analytical equations, hence simplifying and enlightening the discussion. We will use a suitable generalization of the effective range expansion [19] such as the Inverse Amplitude Method [21, 22, 23, 24] to unitarize the one loop $\pi\pi$ ChPT amplitudes. The method has been successfully used in the past not only for $\pi\pi$ (coupled channel meson–meson in general) [25, 26, 27, 28, 29], but also for πN−scattering [30, 31]. Poles of the unitarized amplitudes in unphysical sheets provide masses and widths of the resonances, and the change of the position of these poles determines their properties for growing N_C. To carry out such a program requires extending the chiral amplitudes for an arbitrary number of colours. This is a delicate point because in any case one must insure that *all* possible leading N_C dependences should be contained in this unphysical $N_C > 3$ extrapolation. In our study all QCD N_C dependence appears through the Low Energy Constants (LEC's), which leading N_C behaviour could be obtained within the resonance saturation approach [32, 33]. We will further simplify the problem and we will work within the Single Resonance Approximation (SRA) scheme, where each infinite resonance sum is just approximated with the contribution from the lowest-lying meson nonet with the given quantum numbers. We will see how a rather natural requirement of consistency between resonance saturation and unitarization imposes useful constraints in the extreme $N_C \to \infty$ limit.

The previous works of Refs. [34, 35] use a similar framework, however with a different N_C scaling strategy; the $N_C = 3$ fitted results are simply re-scaled to the unphysical N_C values. One of the major, but crucial, differences of the present work with these other two is the use of the SRA to estimate the leading N_C behaviour of the LEC's. Though in the vicinity of $N_C = 3$ we find similar results, we will show that the requirement of consistency between resonance saturation and unitarization, for very large values of N_C, imposes useful constraints.

II. ONE LOOP $\pi\pi$ CHPT AMPLITUDES IN THE CHIRAL LIMIT

After projecting out in isospin and angular momentum (IJ), the scalar–isoscalar and vector-isovector $\pi\pi$ amplitudes to one loop accuracy and in the chiral limit read [19, 20] (in the centre of mass frame):

$$T^{00}(s) = \underbrace{-\frac{s}{f^2}}_{T_2^{00}} \underbrace{-\frac{s^2}{576\pi^2 f^4}\left\{28\bar{l}_2 + 22\bar{l}_1 + 51 - 14\ln(s/m^2) - 36\ln(-s/m^2)\right\}}_{T_4^{00}} \qquad (2)$$

$$T^{11}(s) = \underbrace{-\frac{s}{6f^2}}_{T_2^{11}} \underbrace{-\frac{s^2}{576\pi^2 f^4}\left\{2\bar{l}_2 - 2\bar{l}_1 + \frac{2}{3} + \ln(s/m^2) - \ln(-s/m^2)\right\}}_{T_4^{11}} \qquad (3)$$

where s is the square of the total energy of the two pions, f is the pion decay constant in the chiral limit (~ 88 MeV), m is the pion mass (this apparent dependence on m is fictitious, as we will see below) and T_2^{IJ} and T_4^{IJ} are the tree level and one loop amplitudes, and the normalization of the total amplitude is fixed by its relation with the elastic phase shifts

$$T^{IJ} = -8\pi\sqrt{s}\left(\frac{e^{2i\delta^{IJ}} - 1}{2ip}\right) \qquad (4)$$

with p the centre of mass pion momentum, $\sqrt{s}/2$ for massless pions. The logarithm in the above equations is defined as ($z \in \mathbb{C}$)

$$\ln z = \ln|z| + i\,\mathrm{Arg}(z), \quad \mathrm{Arg}(z) \in [-\pi, \pi[\qquad (5)$$

In Eqs. (2) and (3) the last logarithm ($\ln(-s/m^2)$, with m the pion mass) produces the unitarity right hand cut and it accounts for perturbative two particle elastic unitarity

$$\mathrm{Im}T_4^{IJ}(s + i0^+) = -\frac{|T_2^{IJ}(s)|^2}{16\pi} + \mathcal{O}(1/f^4), \quad s > 0 \qquad (6)$$

while the first logarithm in these two equations provides the left hand cut required by crossing symmetry, and it leads to complex amplitudes for $s < 0$. Finally, $\bar{l}_{1,2}$ are scale independent LEC's. Up to a numerical factor, the quantity $\bar{l}_i$ is the value of the renormalized coupling constant $l_i^r(\mu)$ at the scale $\mu = m$,

$$\bar{l}_1 = 96\pi^2 l_1^r(\mu) - \ln(m^2/\mu^2), \quad \bar{l}_2 = 48\pi^2 l_2^r(\mu) - \ln(m^2/\mu^2) \tag{7}$$

The LEC's $\bar{l}_i$ do not exist in the chiral limit, $m \to 0$, but contain a chiral logarithm with unit coefficient, i.e., in the chiral limit $\bar{l}_i$ tend to infinity like $-\ln m$. Note however, all dependence on the pion mass m cancels out in Eqs. (2) and (3), as expected, since T^{00} and T^{11} are well defined in the chiral limit $m \to 0$. The one loop amplitudes depend on a certain scale μ through the renormalized LEC's $l_i^r(\mu)$ and the right (unitarity) and left hand cut logs $\ln(-s/\mu^2)$ and $\ln(s/\mu^2)$, respectively.

Already at this level we note here, the remarkable difference between the $\sigma-$ and $\rho-$meson channels which becomes obvious from the effective range expansion in the early work of Lehmann [19]. For $s > 0$, and besides the imaginary part, left and right hand cut logs cancel out in the $\rho-$meson channel, while these logs add up in the scalar–isoscalar sector. Moreover, at a scale of about 770 MeV, the contribution of the logs is comparable in size to that of the renormalized LEC's l_i^r, being thus the $\sigma-$channel dynamics strongly influenced by these logarithms stemming from the chiral loops.

Away from the chiral limit, some logs survive in the $\rho-$channel as well, but their contribution is suppressed by powers of the pion of mass, i.e. terms of the type $m^2 \ln(\pm s/m^2)$ (see for instance the appendix B of Ref. [36], where analytical expressions for the left hand cut integrals can be found).

Finally, we recall here the relation among the SU(2)×SU(2) and SU(3)×SU(3) LEC's [37],

$$l_1^r(\mu) = 4L_1^r(\mu) + 2L_3 - \frac{\nu_K}{24}, \quad l_2^r(\mu) = 4L_2^r(\mu) - \frac{\nu_K}{12} \tag{8}$$

with $32\pi^2\nu_K = \left(\ln(\bar{m}_K^2/\mu^2) + 1\right)$, with $\bar{m}_K \sim 468$ MeV the kaon mass in the $m \to 0$ limit, which permits re-write the amplitudes in terms of $L_{1,2,3}^r(\mu)$. The scale dependence of the SU(3)×SU(3) LEC's reads

$$L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{(4\pi)^2} \ln(\mu_1/\mu_2), \quad 2\Gamma_1 = \Gamma_2 = \frac{3}{16}, \Gamma_3 = 0. \tag{9}$$

III.   ONE LOOP IAM ρ AND σ POLES

Any unitarization method resums a perturbative expansion of the scattering amplitude in such way that two body elastic unitarity

$$\text{Im}\left(\frac{1}{T^{IJ}}\right) = \frac{p}{8\pi\sqrt{s}}, \quad s > 4m^2 \tag{10}$$

is implemented exactly. Let us pay attention to the one loop IAM, in which the $T^{IJ}-$matrix is approximated by [21, 22, 23, 24]

$$T^{IJ}(s) = \frac{(T_2^{IJ})^2(s)}{T_2^{IJ}(s) - T_4^{IJ}(s)}, \tag{11}$$

which perturbatively reproduces ChPT to one loop[1]. Resonances correspond to poles in the fourth quadrant of the Second Riemann Sheet (SRS), defined by continuity on the upper lip of the right unitarity cut with the physical First Riemann Sheet (FRS), $T_{\text{SRS}}(s \pm i0^+) = T_{\text{FRS}}(s \mp i0^+)$. Thus within this scheme, we find that mass and width of the resonance $(s_R = m_R^2 - im_R\Gamma_R)$ are determined from the zeros of the denominator of Eq. (11) in the SRS,

$$T_2^{IJ}(s_R) = T_4^{IJ}(s_R)\Big|_{\text{SRS}}. \tag{12}$$

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[1] Some problems associated to the exact position of the Adler's zeros and the reliability of the IAM predictions for scalar waves have been recently discussed in Ref. [29].

The above condition leads to simple equations for the ρ and σ resonances:

$$s_\rho = \frac{288\pi^2 f^2}{3\mathrm{i}\pi + 2 + 6\bar{l}_2 - 6\bar{l}_1} = \frac{96\pi^2 f^2}{\mathrm{i}\pi + 2/3 - 384\pi^2 \left(2L_1^r(\mu) - L_2^r(\mu) + L_3\right)} \tag{13}$$

$$s_\sigma = \frac{288\pi^2 f^2}{18\mathrm{i}\pi + 25\mathrm{i}\arctan\left(\frac{\Gamma_\sigma}{m_\sigma}\right) - 25\ln|s_\sigma/m^2| + 51/2 + 14\bar{l}_2 + 11\bar{l}_1}$$

$$= \frac{288\pi^2 f^2}{18\mathrm{i}\pi + 25\mathrm{i}\arctan\left(\frac{\Gamma_\sigma}{m_\sigma}\right) - 25\ln|s_\sigma/\mu^2| + 51/2 + 192\pi^2\left(22L_1^r(\mu) + 14L_2^r(\mu) + 11L_3 - \frac{25}{48}\nu_K\right)} \tag{14}$$

where to compute the amplitude in the fourth quadrant of the SRS at $s = s_R$, we have used:

$$\ln(-s_R) = \ln\left(-m_R^2 + \mathrm{i}m_R\Gamma_R\right) = \ln|s_R| + \mathrm{i}\left[\pi - \arctan\left(\frac{\Gamma_R}{m_R}\right)\right] - 2\pi\mathrm{i} \tag{15}$$

$$\ln(s_R) = \ln\left(m_R^2 - \mathrm{i}m_R\Gamma_R\right) = \ln|s_R| - \mathrm{i}\arctan\left(\frac{\Gamma_R}{m_R}\right) \tag{16}$$

We note that in Eqs. (13) and (14), large chiral logarithms of the pion mass do not appear, which guaranties mild pion mass dependences of the $\rho-$ and $\sigma-$masses and widths [38]. This is highly desirable to better control the needed chiral extrapolation of lattice QCD calculations.

From Eq. (13), we trivially find

$$m_\rho^2 = 48\pi^2 f^2 \frac{\bar{l}_2 - \bar{l}_1 + 1/3}{(\bar{l}_2 - \bar{l}_1 + 1/3)^2 + \pi^2/4}, \quad \Gamma_\rho = \frac{\pi}{2}\frac{m_\rho}{\bar{l}_2 - \bar{l}_1 + 1/3} \tag{17}$$

(we express the $\rho-$mass and width in terms of the combination $\bar{l}_2 - \bar{l}_1 = -192\pi^2\left(2L_1^r(\mu) - L_2^r(\mu) + L_3\right)$, because in this difference the logarithm of the pion mass cancels out). The above expressions give reasonable estimates of the pole position of the $\rho-$resonance. Using, for instance, at the scale $\mu = m_\rho \sim 770$ MeV, the set of LEC's determined in Ref. [39] from an $\mathcal{O}(p^6)$ study of the $K_{\ell 4}$ decays,

$$10^3\, L_1^r(m_\rho) = 0.52 \pm 0.23, \quad 10^3\, L_2^r(m_\rho) = 0.72 \pm 0.24, \quad 10^3\, L_3 = -2.70 \pm 0.99 \tag{18}$$

and taking into account the strong correlations among the L_i^r, we estimate [40] $\bar{l}_1 = 0.3 \pm 1.2$ and $\bar{l}_2 = 4.77 \pm 0.45$, with a linear correlation coefficient $r(\bar{l}_1, \bar{l}_2) = -0.69$, that leads[2] to $m_\rho = 830^{+120}_{-90}$ MeV and $\Gamma_\rho = 270^{+190}_{-90}$ MeV. Similar results ($m_\rho \sim 815$ MeV and $\Gamma_\rho \sim 254$ MeV) are obtained by using $\bar{l}_1 = -0.4 \pm 0.6$ and $\bar{l}_2 = 4.3 \pm 0.1$ obtained from an $\mathcal{O}(p^6)$ Roy equation analysis of $\pi\pi$ scattering [41].

Since the analysis carried out here involves only $\mathcal{O}(p^4)$ amplitudes, one might think, it would be more appropriate to use values for the LEC's determined from $\mathcal{O}(p^4)$ accuracy studies. If we had used for the central values of $L_{1,2,3}^r(m_\rho)$, the results from the $\mathcal{O}(p^4)$ fit of Ref. [39],

$$10^3\, L_1^r(m_\rho) = 0.46, \quad 10^3\, L_2^r(m_\rho) = 1.49, \quad 10^3\, L_3 = -3.18 \tag{19}$$

while keeping the same errors and correlations as in Eq. (18), we would have found $\bar{l}_1 = -0.9 \pm 1.2$, $\bar{l}_2 = 6.23 \pm 0.45$ with $r(\bar{l}_1, \bar{l}_2) = -0.69$. Those values will lead to $m_\rho = 690^{+80}_{-60}$ MeV and $\Gamma_\rho = 150^{+60}_{-40}$ MeV, in better agreement with data. Of course, there will be finite pion mass corrections, though we expect them to be quite small for the ρ mass, while its width will be reduced by about 25% [38]. Note that if we fix $(\bar{l}_2 - \bar{l}_1 + 1/3)$ to $(1.25 \times (\Gamma_\rho/m_\rho)_{\rm exp} \times 2/\pi)^{-1} \sim 6.45$, as deduced from the second relation of Eq. (17) and the hypothesis that the Γ_ρ/m_ρ ratio increases by about 25% for massless pions, the first relation of Eq. (17) leads to $m_\rho \approx 8.33f$, in excellent agreement with the experimental value.

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[2] For narrow resonances, their mass and width are related to the pole position as $\sqrt{s_R} \approx m_R - \mathrm{i}\,\Gamma_R/2$, and it is also usual to use this notation for broader resonances. The use of $\sqrt{s_R}$ to determine the resonance mass and width leads to heavier and narrower states. As a matter of example, with the parameters of Eq. (18), we find $\sqrt{s_\rho} = \left(840^{+140}_{-100} - \mathrm{i}\,133^{+89}_{-43}\right)$ MeV. Variations are much more drastic for the case of the $\sigma-$resonance, because it is significantly broader than the $\rho-$meson.

Let us now pay attention to the case of the $\sigma-$resonance. Solving numerically Eq. (14), with the set of parameters of Eqs. (18) and (19), we find

$$\sqrt{s_\sigma} = \left(401^{+12}_{-16} - i\, 277^{+23}_{-26}\right) \text{ MeV}, \quad \text{LEC's from } \mathcal{O}(p^6)\ K_{\ell 4}\ [\text{Eq. (18)}] \tag{20}$$

$$\sqrt{s_\sigma} = \left(410^{+8}_{-13} - i\, 257^{+25}_{-27}\right) \text{ MeV}, \quad \text{LEC's from } \mathcal{O}(p^4)\ K_{\ell 4}\ [\text{Eq. (19)}] \tag{21}$$

which are in fair agreement with the benchmark determination based on Roy equation and chiral symmetry [8] given in Eq. (1). Finite pion mass corrections produce a moderate enhancement (around 10%) of the resonance mass, while its width is reduced by a similar amount [38].

In sharp contrast with the case of the ρ meson, the properties of the $\sigma-$meson ($f_0(600)$) turn out to be strongly influenced by the chiral logarithm $\ln\left|s_\sigma/\mu^2\right|$. Actually, there exist large cancellations in the combination of LEC's $192\pi^2\left(22L_1^r(\mu) + 14L_2^r(\mu) + 11L_3 - \frac{25}{48}\nu_K\right)$ appearing in Eq. (14), and this contribution plays a role much less important than in the case of the $\rho-$meson, for which the LEC's and the discontinuity through the unitarity cut determine mostly its properties. Indeed, if for the $\sigma-$resonance, the LEC's contribution is neglected, one will find $\sqrt{s_\sigma} = 417 - i\, 236$ MeV. The comparison of this latter result with those displayed in Eqs. (20) and (21) shows that the bulk of the dynamics of the $\sigma-$resonance is not determined[3] by the LEC's, but rather by unitarity ($18 i\pi$), the constant term $51/2$ and the chiral logarithm $-25\ln\left(s_\sigma/\mu^2\right)$. This chiral log, due to both the left and right hand cut contributions, favours smaller (larger) values of the Γ_σ/m_σ ratio for $|s_\sigma|^{\frac{1}{2}}$ smaller (larger) than the renormalization scale μ. When this contribution is neglected, we find with the set of parameters of Eq. (18) [Eq. (19)], that the Γ_σ/m_σ ratio comes out to be around a factor 2.2 [1.6] greater than if the chiral log was considered. Actually, the pole exists as long as the real part of the denominator in Eq. (14) remains positive, and it imposes a constraint to $\left|s_\sigma/\mu^2\right|$

$$\frac{\Gamma_\sigma}{m_\sigma} = \frac{18\pi + 25\arctan\left(\frac{\Gamma_\sigma}{m_\sigma}\right)}{192\pi^2\left(22L_1^r(\mu) + 14L_2^r(\mu) + 11L_3 - \frac{25}{48}\nu_K\right) + \frac{51}{2} - 25\ln\left|s_\sigma/\mu^2\right|} > 0 \tag{22}$$

$$\Rightarrow\ 25\ln\left|s_\sigma/\mu^2\right| < 192\pi^2\left(22L_1^r(\mu) + 14L_2^r(\mu) + 11L_3 - \frac{25}{48}\nu_K\right) + \frac{51}{2} \tag{23}$$

After this discussion, it is easy to understand the nomenclature commonly used in the literature of *dynamically generated* referred to the $\sigma-$resonance. Actually, it is possible to describe the scalar channel with the leading order plus a cutoff (or another regularization parameter) playing the role of some combination of higher order parameters, while for the case of the $\rho-$resonance the $\mathcal{O}(p^4)$ LEC's are needed [26, 36, 42, 43].

IV.   LARGE N_C LIMIT OF THE ONE LOOP IAM ρ AND σ POLES

The large N_C extension of the σ and ρ pole positions obtained in the previous section is straightforward. We should just consider that the pion weak decay constant scales as $\mathcal{O}(\sqrt{N_C})$, while the LEC's $L_{1,2,3}$ behave as $\mathcal{O}(N_C)$, with $L_2 - 2L_1 = \mathcal{O}(N_C^0)$ [1, 2]. The chiral logs, as well as the renormalization scale dependence of the LEC's [Eq. (9)] are subleading in $1/N_C$. Thus, we trivially find for $N_C \gg 3$ and massless pions

$$\widehat{m}_\rho^2 = -\frac{\widehat{f}^2}{4\widehat{L}_3}, \qquad\qquad \widehat{\Gamma}_\rho = \frac{\widehat{m}_\rho^3}{96\pi\widehat{f}^2} \tag{24}$$

$$\widehat{m}_\sigma^2 = \frac{3\widehat{f}^2}{50\widehat{L}_2 + 22\widehat{L}_3}, \qquad \widehat{\Gamma}_\sigma = \frac{\widehat{m}_\sigma^3}{16\pi\widehat{f}^2} \tag{25}$$

where the hat over a symbol $(\widehat{O})$ implies its value in the $N_C \gg 3$ limit. Although we expect that the N_C behavior close to the physical value $N_C = 3$ of the σ is non $q\bar{q}$ due to the chiral logs, for a sufficiently large N_C, the above equations, provided that $-\widehat{L}_3$ and $(25\widehat{L}_2 + 11\widehat{L}_3)$ are positive quantities, show that for both resonances, the mass scales as $\mathcal{O}(N_C^0)$, while the width decreases as $1/N_C$. Thus, in this N_C regime both resonances would follow a $q\bar{q}$ pattern in the nomenclature of Refs. [34, 35]. Nevertheless, the very large N_C pole in (25) could be located at a rather different position from that of $N_C = 3$, as we will discuss below. This large N_C pole might then be interpreted

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[3] On the contrary, for the case of the $\rho-$meson neglecting the LEC's contribution leads to unrealistic results: $s_\rho = 288\pi^2 f^2/(2 + 3i\pi))$, which gives $m_\rho = 689$ MeV and $\Gamma_\rho = 3245$ MeV, or equivalently $\sqrt{s_\rho} = 1175 - i\, 951$ MeV.

as a sub-dominant $q\bar{q}$ component of the σ resonance. From a sufficiently large value of N_C on, such component may become dominant, and beyond that N_C the associated pole would behave as a $q\bar{q}$ state, although the original state only had a small admixture of $q\bar{q}$ [44]. Something similar was found at two loops in [35], but we are showing here that it is possible also at one loop. On the other hand, if either $\widehat{L}_3$ or $(25\widehat{L}_2 + 11\widehat{L}_3)$ were exactly zero, the pole position (s_R) of the associated resonance would grow with N_C as $\widehat{f}^2$, for sufficiently large N_C.

In any case, we face here the fundamental problem of determining the values of f, L_2 and L_3 for unphysical $N_C \neq 3$ values. The separation between the large N_C leading and subleading parts of the measured L_i is not possible. In general, one has the scale independent combination

$$L_i^r(m_\rho) = L_i^r(\mu) + \frac{\Gamma_i}{(4\pi)^2} \ln(\mu/m_\rho) = A_i N_C + B_i \qquad (26)$$

where A_i and B_i are N_C independent. Note that only the $N_C = 3$ combination is experimentally accessible. However a meaningful extension of the chiral amplitudes to an arbitrary number of colors requires some knowledge of the different A_i and B_i coefficients, and of course of those appearing in a similar decomposition of the pion decay constant. This is of particular importance in the scalar channel due to the large cancellation[4] existing in the combination $(25L_2^r(m_\rho) + 11L_3)$.

In Ref. [34], the prescription of scaling $f \to f\sqrt{N_C/3}$, and $L_i^r(m_\rho) \to L_i^r(m_\rho)(N_C/3)$ for $i = 2, 3$ is adopted. There, $2L_1^r(m_\rho) - L_2^r(m_\rho)$ is kept constant and an uncertainty on the scale $\mu = 0.5-1$ GeV is also taken into account. However, as N_C starts significantly deviating from the physical value $N_C = 3$, such prescription might not be accurate enough. For example, let us consider the case of the $\sigma-$resonance. As mentioned above, there exists a large cancellation in the LEC's contribution to the scalar channel, and it plays a minor role for $N_C = 3$. As a result, not only the absolute value, but also the sign of $(25L_2^r(m_\rho) + 11L_3)$ is subject to sizable uncertainties. Depending on the set of values used, one finds different signs for this combination of LEC's, which induces totally different behaviours when the number of colors grows. Thus, with the $\mathcal{O}(p^6)$ values of Eq. (18), this combination is negative, while it turns out to be positive when the $\mathcal{O}(p^4)$ set of Eq. (19) is considered[5].

- When $(25L_2^r(m_\rho) + 11L_3)$ is negative, the real part of the denominator of Eq. (14) approaches zero for increasing N_C, which implies that m_σ also approaches zero, while the width grows even faster that m_σ^{-1}. From a given value of N_C on, such that the real part of the denominator of Eq. (14) becomes negative, Eq. (14) does not admit solution in the fourth quadrant. In this scenario, the $\sigma-$resonance disappears, in the $N_C \gg 3$ limit, from this quadrant of the SRS, and the pole appears in the third quadrant[6], though $\sqrt{s_\sigma}$ still lies in the fourth quadrant. This is precisely the variable used in Refs. [34, 35]. To illustrate this point, in Fig. 1 we show results for the $\sigma-$pole, as a function of N_C. We have fixed $L_{1,2,3}^r(m_\rho)$ to the values labeled as IAM in Table I of Ref. [34]. From $N_C = 11$ on, the pole moves to the third quadrant, while $\sqrt{s_\sigma}$ is still placed in the fourth quadrant. These massless pion $\sqrt{s_\sigma}$ results compare nicely to those displayed in the middle panels of Fig. 2 of Ref. [34], indicating that neglecting pion mass and coupled channel effects constitutes also a good approximation to analyze the large N_C behaviour of this resonance. The behaviour at small values of N_C close to 3 support the conclusions of Refs. [34, 35] on the non-dominant $q\bar{q}$ nature of the σ in the real world, while the results for larger values of N_C would indicate that there does not exist a subdominant $q\bar{q}$ component in the σ wave function. Nevertheless, these predictions are subject to sizable uncertainties beyond let us say $N_C = 10$, since $|\sqrt{s_\sigma}|$ is already 0.9 GeV for $N_C = 10$, and it reaches values close to 1.35 GeV for $N_C = 30$. The applicability of the one-loop IAM is, at least, doubtful for these large values of $|\sqrt{s}|$. Although the one loop IAM amplitude incorporates contributions to all orders in the chiral expansion (increasing powers of $1/f^{2n}$), it only accurately accounts for those needed to exactly restore two-body elastic unitarity. Therefore, a more precise knowledge of the leading N_C terms of the $\mathcal{O}(p^6)$, $\mathcal{O}(p^8)$, etc... amplitudes seems to be required. In this context the two loop IAM analysis carried out in Ref. [35] is better founded.

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[4] For instance, the $\mathcal{O}(p^4)$ values of Eq. (19) leads to $|(25L_2^r(m_\rho) + 11L_3)/(11L_3)| = 0.06^{+0.29}_{-0.03}$.

[5] Even more worrying, if we made use of $L_2 - 2L_1 = \mathcal{O}(N_C^0)$ and replaced $(25L_2^r(m_\rho) + 11L_3)$ by $(50L_1^r(m_\rho) + 11L_3)$, within the same accuracy in the N_C expansion, this LEC's combination would come out now negative with the set of parameters of Eq. (19), and substantially closer to zero than before, when the parameters of Eq. (18) are used.

[6] To compute the SRS amplitude at $s = -a - i\,b$, with $a > 0, b > 0$ (third quadrant), Eqs. (15) and (16) should be replaced by

$$\ln(-s) = \ln|s| + i \arctan\left(\frac{b}{a}\right) - 2\pi i \qquad (27)$$

$$\ln(s) = \ln|s| + i \arctan\left(\frac{b}{a}\right) - i\pi \qquad (28)$$

- On the contrary, when the combination $(25L_2^r(m_\rho) + 11L_3)$ is positive, the large N_C limit of the σ−meson is qualitatively identical to that of the ρ−meson, with $\widehat{m}_\sigma \sim \mathcal{O}(N_C^0)$ and $\widehat{\Gamma}_\sigma \sim \mathcal{O}(N_C^{-1})$. There will exist a $q\bar{q}$ component in the σ meson that would become dominant in the $N_C \gg 3$ limit.

  However, if it happens that for $N_C = 3$, the magnitude $192\pi^2(25L_2^r(m_\rho)+11L_3)$, though positive, is close to zero and significantly smaller than $51/2$, there exists a transient region of low and intermediate values of N_C where the expected scaling rules are not satisfied and the chiral log $-25\ln\left(s_\sigma/\mu^2\right)$ induces non trivial and unexpected dependences on N_C. This indicates that the $q\bar{q}$ component of the σ is sub-dominant when $N_C = 3$. Once N_C is sufficiently large, the real part becomes rather constant, while the imaginary part starts decreasing, as predicted by Eq. (25). However, this asymptotic value for the mass would be out of the range of applicability of one loop ChPT, and the caveats mentioned above on the need of some detailed leading N_C $\mathcal{O}(p^6)$, $\mathcal{O}(p^8)$,.. input will apply also here.

Thus, the $N_C \gg 3$ behaviour of the σ−meson within the IAM method to one loop depends critically on the sign and size of a parameter combination which value cannot be pinned down reliably with the needed accuracy. This crucial role played by the critical LEC's combination $(25L_2^r(m_\rho)+11L_3)$ to determine the $N_C \to \infty$ limit of the σ−resonance was firstly pointed out in [45]. Here, we provide in addition an error analysis, address the relevance of identifying the leading N_C term in the decomposition of Eq. (26), analyze the relation between σ and ρ channels (see below also) and discuss the transition to a region where the pole is located in Re $s < 0$ half-plane[7]. The above discussion clearly illustrates the need of having some reliable and independent insight[8] into the N_C leading and sub-leading terms of Eq. (26). To this end, we will next make use of the resonance saturation approach [32, 33].

V. LEC'S: RESONANCE SATURATION APPROACH AND LARGE N_C LIMIT

In the resonance saturation approach one writes down a Lagrangian including the resonance fields and integrate them out [32, 33], yielding values for the LEC's at some scale not too far away from the resonance region. It is
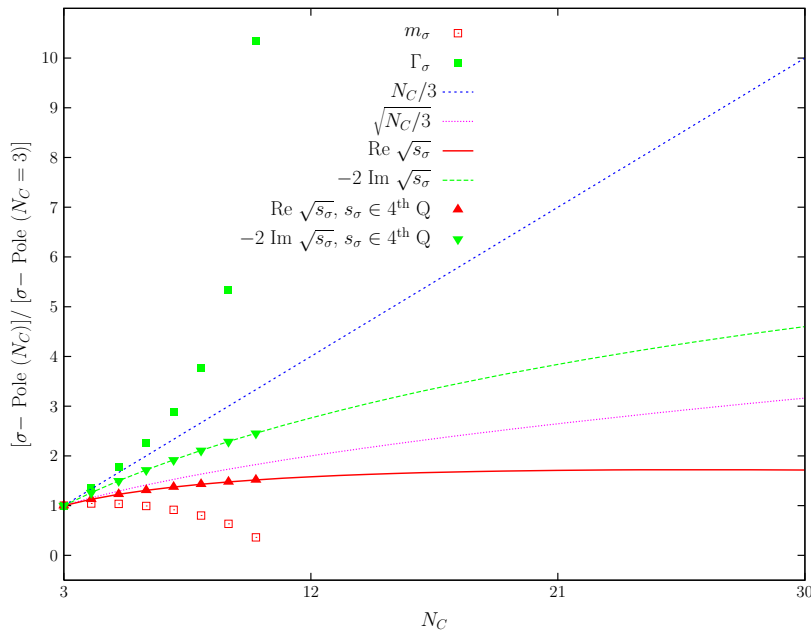

FIG. 1: σ−pole results, as a function of the number of colors, obtained with $L_{1,2,3}^r(m_\rho)$ fixed to the values labeled as IAM in Table I of Ref. [34] and extended to arbitrary N_C as in that reference. Up to $N_C = 10$ we solve Eq. (14), beyond this number of colors, the pole lies in the third quadrant (we solve Eq. (14) with the modifications mentioned in footnote 6). All results are normalized by the $N_C = 3$ results: $m_\sigma = 328.1$ MeV, $\Gamma_\sigma = 629.6$ MeV and $\sqrt{s_\sigma} = (412.6 - \mathrm{i}\, 250.3)$ MeV.

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[7] Note that in this case, the path integral for the resonance field would not be well defined.

[8] Though desirable, the knowledge of such a decomposition for f is less relevant, since it will not affect the existence or not of the σ state in the $N_C \gg 3$ limit.

common practice to adopt $\mu = m_\rho$ as a reasonable choice. The generalization of this approach to the large N_C limit requires, in addition to including infinitely many resonances, the use of short distance constraints [46], which are conditions stemming from the analysis of Green's functions in QCD at high momentum. In the Single Resonance Approximation (SRA), each infinite resonance sum is approximated with the contribution from the first meson nonet with the given quantum numbers. This is meaningful at low energies where the contributions from higher–mass states are suppressed by their corresponding propagators. Within SRA one finds (see Refs. [32, 33] for notation),

$$[L_1^r(m_\rho)]^{\mathrm{SRA}} = \frac{G_V^2}{8M_V^2} - \frac{c_d^2}{6M_S^2} + \frac{\tilde{c}_d^2}{2M_{S_1}^2} \tag{29}$$

$$[L_2^r(m_\rho)]^{\mathrm{SRA}} = \frac{G_V^2}{4M_V^2} \tag{30}$$

$$L_3^{\mathrm{SRA}} = -\frac{3G_V^2}{4M_V^2} + \frac{c_d^2}{2M_S^2} \tag{31}$$

where M_{S_1} and M_S are the singlet and octet scalar masses, respectively and M_V that of the nonet of vector mesons.

In the $N_C \gg 3$ limit, octet and singlet mesons become degenerate and thus $M_{S_1} = M_S$, while G_V, c_d and $\tilde{c}_d$ are all $\mathcal{O}(N_C^{\frac{1}{2}})$. The large N_C condition $L_2 - 2L_1 = \mathcal{O}(N_C^0)$ can then be achieved by taking $c_d^2 = 3\tilde{c}_d^2$, while the short distance constraints can be used to determine the resonance couplings at leading order in the N_C expansion [32, 33, 46],

$$\sqrt{2}G_V = 2c_d = f \tag{32}$$

All together this allows to estimate the leading N_C terms ($A_i's$ coefficients in Eq. (26)) of $L_2^r(m_\rho)$ and L_3 LEC's [46],

$$[L_2^r(m_\rho)]^{\mathrm{SRA}} = \frac{f^2}{8M_V^2} + \mathcal{O}(N_C^0), \qquad L_3^{\mathrm{SRA}} = -\frac{3f^2}{8M_V^2} + \frac{f^2}{8M_S^2} + \mathcal{O}(N_C^0) \tag{33}$$

The subleading N_C corrections to the SRA predictions of the LEC's are difficult to estimate in a model independent way (see however [47, 48]). The last column of Table 2 in Ref. [46] shows[9] the estimates for $L_2^r(m_\rho)$ and L_3 obtained with $M_V = 0.77$ GeV and $M_S = 1.0$ GeV, and those, by simply scaling with $N_C/3$, can be used to determine $\widehat{L}_{2,3}$,

$$10^3 \, \widehat{L}_2^{\mathrm{SRA}} = 1.8 \times \frac{N_C}{3}, \qquad 10^3 \, \widehat{L}_3^{\mathrm{SRA}} = -4.3 \times \frac{N_C}{3} \tag{34}$$

The above value for L_3^{SRA} provides an estimate for the $\rho-$mass, in the $N_C \gg 3$ limit, of around 700 MeV, while its width decreases as $1/N_C$ [Eq. (24)]. This result is in good agreement with the findings of Ref. [34], also obtained within the one loop IAM scheme, but including both finite pion mass and coupled channel effects. The results for the $\rho-$resonance are robust, and since the mass is moderately small, we expect they would not be much affected by leading N_C contributions showing up at order $\mathcal{O}(p^6)$ or higher, as confirmed by the two loop results of Ref. [35]. Under these circumstances, if we identify $\widehat{m}_\rho$ with M_V (mass of the nonet of vector mesons that is introduced in the SRA), and require consistency between Eqs. (24) and (33), we find

$$L_3^{\mathrm{SRA}} = -\frac{3f^2}{8M_V^2} + \frac{f^2}{8M_S^2} + \mathcal{O}(N_C^0) = -\frac{f^2}{4M_V^2}\left(1 + \mathcal{O}(m^2/M_V^2)\right) + \mathcal{O}(N_C^0) \tag{35}$$

where we have used that $\widehat{f}^2/\widehat{L}_3 = f^2/L_3 + \mathcal{O}(1/N_C)$. From the above equation, it trivially follows

$$M_S = M_V + \mathcal{O}(1/N_C) + \mathcal{O}(mN_C^0) \tag{36}$$

which is based on the relation $G_V = \sqrt{2}c_d + \mathcal{O}(N_C^0)$ [Eq. (33)] and the assumed scheme, namely SRA–one–loop IAM. It is worth noting that the large N_c identity of scalar and vector meson masses, Eq. (36), has also been derived within the context of mended symmetries [49] as well as chiral quark models [50]. Computing finite pion mass corrections to Eq. (36) is straightforward. Indeed, in the large N_C limit, the only relevant pion mass corrections to the amplitude are

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[9] Note that in this reference, f is fixed to 92 MeV.

those proportional to $\bar{l}_1 - \bar{l}_2$ and $\bar{l}_4$ (see for instance Eq. (B7) in Ref. [36]). Taking into account that $\bar{l}_4$ is determined by the combination $2L_4^r(\mu) + L_5^r(\mu)$ and that L_5 dominates in the $N_C \gg 3$ limit, one easily finds

$$\widehat{m}_\rho^2 = -\frac{\widehat{f}^2}{4\widehat{L}_3} \left(1 - \frac{8m^2}{\widehat{f}^2} \widehat{L}_5 \right) \tag{37}$$

and by using [46],

$$[L_5^r(m_\rho)]^{\mathrm{SRA}} = \frac{f^2}{4M_S^2} + \mathcal{O}(N_C^0), \tag{38}$$

the SRA–one–loop IAM consistency requirement now leads to

$$M_S^2 = M_V^2 - 4m^2 + \mathcal{O}(1/N_C) \tag{39}$$

which constitutes one of the main results of this work.

The situation, however, is totally different in the scalar sector. From the estimates given in Eq. (34), we find that the combination $(25\widehat{L}_2^{\mathrm{SRA}} + 11\widehat{L}_3^{\mathrm{SRA}})$ turns out to be small compared to the log contribution and negative. Thus, for N_C sufficient large the σ−pole disappears from the fourth quadrant of the SRS, and one finds a similar behaviour to that depicted in Fig. 1. This might hint at a definite non $q\bar{q}$ nature of the σ−resonance, as suggested in Ref. [34], since there is no trace even of the existence of a sub-dominant $q\bar{q}$ component. Finite pion mass corrections can be easily taken into account, but they do not modify the discussion. Note however, once more, the large cancellation that occurs in this combination of LEC's, since $|(25\widehat{L}_2^{\mathrm{SRA}} + 11\widehat{L}_3^{\mathrm{SRA}})/(11\widehat{L}_3^{\mathrm{SRA}})| = 0.05$. As a consequence, it is difficult to draw any robust conclusion on the nature (existence or not of the $q\bar{q}$ component) of the lightest spin–isospin scalar meson in the $N_C \to \infty$ limit. Actually, a small variation of the short distance constraint relations of Eq. (32), increasing slightly the ratio c_d/G_V, or approximating M_S by $M_V \sim 770$ MeV, as suggested by Eq. (36), would reduce $|L_3^{\mathrm{SRA}}|$ and lead to positive values for the LEC's combination $(25\widehat{L}_2^{\mathrm{SRA}} + 11\widehat{L}_3^{\mathrm{SRA}})$.

Under these circumstances, we will consider a different scenario. Let us assume that $(25\widehat{L}_2^{\mathrm{SRA}} + 11\widehat{L}_3^{\mathrm{SRA}})$ is positive, leading then to a stable σ−resonance in the $N_C \gg 3$ limit. Its mass then could be identified to that of the first nonet of scalar mesons introduced in the SRA. If we take $M_S = \widehat{m}_\sigma$ and $M_V = \widehat{m}_\rho$, and requiring consistency, between Eqs. (30–31) and the mass of the σ− and ρ−mesons in the large N_C limit, given in Eqs. (24–25), we find

$$M_S = 2M_V + \mathcal{O}(1/N_C), \qquad c_d = f + \mathcal{O}(1/N_C) \tag{40}$$

for massless pions. The second condition ($c_d \sim f$) is phenomenologically strongly disfavored [46], while the first one contradicts the more robust result of Eq. (36) (or (39)) based only in the vector channel. There are different ways to circumvent this apparent contradiction: not identifying $M_S = \widehat{m}_\sigma$ and assuming that, in the large N_C limit, the σ meson becomes heavier than the scalar meson nonet, invoked in the SRA, or correcting the SRA estimates of the LEC's by considering, for instance, contributions of tensor resonances [51], etc.... Nevertheless, one should bear in mind that for this large value of the mass ($\sim 2M_V$), IAM results, based on the first two orders of the chiral expansion, should not be very reliable because of the limited control on the leading N_C terms appearing beyond $1/f^4$.

VI. CONCLUSIONS

We have started looking at the chiral limit of the one loop SU(2) ChPT amplitudes. We have shown how in the chiral limit, a major source of distinction between the σ− and ρ− channels is due to the role played by chiral logarithms; while for the σ−channel the logs add up into sizeable contributions, in the ρ−channel they cancel exactly. Next, we have used the IAM to unitarize the amplitudes and have looked for poles in the SRS of the amplitudes. We have found a fair description of the established properties of the σ− and ρ−resonances, showing the little effect of finite pion mass and coupled channel corrections.

The properties of these resonances for growing number of colors have been also discussed. Our results confirm and explain the different behavior, in agreement with Refs. [34, 35], of the ρ and σ for N_C not far from 3, but, when extending the analysis to the large N_C limit, no robust conclusion of the σ pole behavior can be inferred from the one loop IAM only. This is due to the large cancellation existing in the combination of LEC's that govern the scalar channel; there exists a critical value for a combination of LEC's which cannot be pinned down with the needed accuracy at *any* value of N_C.

Finally, we have discussed further constraints deduced by requiring consistency between resonance saturation and unitarization. By looking only at the ρ−channel, we have found that the masses of the first vector and scalar meson

nonets, invoked in the SRA, turn out to be degenerated in the large N_C limit (see Eq. (36) or Eq. (39) that incorporates finite pion mass corrections). The two loop calculation of Ref. [35] supports this latter result. If we look at the right top panel of Fig. 1 of this reference, we observe that above the $N_C = 6 - 10$ region, the $\sigma-$mass becomes rather constant, while its width rapidly decreases. Moreover, this large N_C asymptotic value of the mass is around two or three times bigger than Re $\sqrt{s_\sigma}\big|_{N_C=3}$, and thus it lies in the 1 GeV region, quite close then to the $\rho-$mass.

Within the restricted IAM unitarization approach assumed here, we would find a scenario where the σ resonance would become degenerate with the ρ and stable, for a sufficiently large number of colors. This complies to the consequences of mended symmetries [49] as well as with chiral quark model calculations [50]. However, the nature of the $\sigma-$resonance in the real world ($N_C = 3$) would be totally different to that of the $\rho-$meson, being it mostly governed by chiral logarithms stemming from unitarity and crossing symmetry, justifying the widely accepted nature of the σ as a dynamically generated meson.

Acknowledgments

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