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P-odd observables at the Υ peak

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Abstract

We study the γ -Z interference in the process $e^+e^- \rightarrow \Upsilon \rightarrow \tau^+\tau^-$ as a means to measure the neutral current coupling of the b-quark. The helicity amplitudes are calculated from resonant and background diagrams and the spin density matrix of the final state is discussed. The spin analyzer of the τ 's is illustrated with the decays $\pi\nu$ and $\rho\nu \rightarrow (\pi\pi)\nu$. With $10^8 \Upsilon$ a sensitivity to g_V^b of a few per cent could be reachable.

1 Introduction

A precise determination of the fermionic electroweak couplings can provide stronger hints on the nature of new physics at higher scales through the quantum corrections to the effective theory at lower energies. At present there is no solid experimental discrepancy with the predictions of the Standard Model [1], and this fact is used to set bounds on these new dynamics beyond the Standard Model.

In this context, the agreement between the measured $Z-f\bar{f}$ couplings and the Standard Model is very good. However, while the lepton couplings have been independently measured for the three families at LEP, for the quarks only the b and c events can be separated in the hadronic event sample and consequently their couplings measured exclusively. From the measurements of R_b and A_b [2] we can get the values of the vector and axial couplings of Z to $b\bar{b}$ [3]. To find the accuracy on the bottom- Z couplings obtainable from these measurements it is enough to use tree level expressions for R_b and A_b because higher order corrections will slightly shift the central values but will not modify the errors. Using the measurements quoted in [2], and defining the vector and axial couplings in the Standard Model as $g_V^b = -1/4 + \sin^2 \theta_W/3$ and $g_A^b = 1/4$, the accuracy on the bottom- Z couplings is: $\delta g_V^b = \pm 0.013$ and $\delta g_A^b = \pm 0.007$. It is important to notice that while the uncertainty in the g_A^b coupling is of 2.8%, the g_V^b measurement is worse, with an uncertainty of 7.5%.

For the light quarks this separation has not been achieved at LEP and so an exclusive determination of their couplings is not available. A study of the final state distributions at different meson facilities can provide independent measurements of these couplings. In a previous work [4], we showed that a Φ factory with polarized e^- beams can supply the information on the s-quark couplings.

We will show in this work, that a detailed study of the final state distributions of the decay products of the τ 's for $e^+e^- \rightarrow \Upsilon \rightarrow \tau^+\tau^-$ would provide valuable information on the vector $Z - b\bar{b}$ coupling, g_V^b .

To determine this coupling at energies well below the Z pole, we will have to measure the interference between γ and Z , where Z is coupled to a $b\bar{b}$ resonance. The $b\bar{b}$ mesons which can couple both to γ and Z are the Υ mesons. We are interested in a process in which the Υ is coupled to a Z either in its production or in its decay. This means we will study the decay of a polarized Υ or its weak decay.

For these reasons, we will analyze the leptonic decays of the Υ resonances, therefore we could use $\Upsilon(1S)$, $\Upsilon(2S)$ and $\Upsilon(3S)$, but not $\Upsilon(4S)$, that decays dominantly to $B\bar{B}$ where the information on the Υ polarization is lost. In this work we will concentrate on the $\Upsilon(1S)$ resonance, but everything would be similar for $\Upsilon(2S)$ and $\Upsilon(3S)$. We can see in Table 1 that the branching ratios of $\Upsilon(1S)$ to the three charged leptons are approximately 3%, but e^+e^- and $\mu^+\mu^-$ can not be used because their polarizations are not measurable in the detector through decay distributions. Obviously e^+e^- are stable particles and $\mu^+\mu^-$ at this energy do not decay inside the detector. As we will see explicitly later, all the relevant information on g_V^b , that comes from a P-odd $\gamma - Z$ interference, only appears at leading order in the polarization of the final leptons. This means that we are constrained to consider the decay $\Upsilon \rightarrow \tau^+\tau^-$ and to measure the τ polarizations.

Before entering a complete analysis of the τ observables, to estimate the sensitivity of this process to the vector coupling, let us first consider the τ^- longitudinal polarization, suggested in Ref. [5] in another context. We will make a reasonable approximation in order to get a simple and clean result: the resonant diagrams will dominate the process on the $\Upsilon(1S)$ peak. So, we consider only diagrams (2), (3) and (4) in Fig. 1. Under these conditions we get a longitudinal polarization from the parity violating interferences of the dominant amplitude (2) with the neutral current amplitudes (3) + (4).

$$P_{z'} = 2 \frac{8G_F}{\sqrt{2}} \frac{s}{4\pi\alpha} g_A \frac{g_V^b}{Q_b} \frac{(1 + \cos^2 \theta)|\vec{p}| p^0 + 2 \cos \theta (p^0)^2}{(1 + \cos^2 \theta)(p^0)^2 + \sin^2 \theta M_\tau^2} \quad (1)$$

where $g_A = 1/4$ is the axial coupling to the leptons, Q_b the charge of the b quark and $p^\mu = (p^0, \vec{p}(\theta))$ the four-momentum of the τ^- .

In this expression we can discover some interesting features of this observable,

- It is linear on g_V^b , the vector coupling which we want to determine, showing up together with the axial coupling of the Z to leptons.
- The magnitude of the τ -polarization on the Υ peak is set by the factor $8G_F M_\Upsilon^2 / (4\pi\alpha\sqrt{2}) \simeq 0.064$, that translates into $P_{z'}(\theta = 0) = 0.032$.
- It is independent of the hadronic structure of the resonance which cancels completely in the ratio.

All these properties will be modified when we include the non-resonant diagrams, but these new contributions will correct this result at the level of a few per cent, so this will be the dominant behaviour of our observables. These properties were already pointed out in Ref. [6], where it was shown that this polarization is enhanced in the vicinity of the resonance. Approximately at four amplitudes below the resonance one gets a polarization five times bigger, but the number of events decreases three orders of magnitude and so it is more efficient from the point of view of statistics to stay on the peak of the resonance.

The above example indicates that a more detailed analysis of the problem is worth. In the next section we calculate the complete $\tau^-\tau^+$ density matrix in this process which contains all the relevant information on g_V^b . In section 3 we study the hadronic decays of the τ to measure the τ density matrix, and from here we analyse, in section 4, the statistical accuracy that is possible to get in the measure of the vector coupling to the b quark in each channel.

2 $\tau^+\tau^-$ density matrix

The density matrix from $e^-(l_-, \xi_-)e^+(l_+, \xi_+) \rightarrow \tau^-(p_-, \lambda_-)\tau^+(p_+, \lambda_+)$, in terms of helicity amplitudes [7], and when our initial beams are unpolarized, Eq. (A.1), is given by,

$$\rho_{(\lambda_-, \lambda_+), (\lambda'_-, \lambda'_+)}^\tau = \sum_{\vec{\xi}} f_{(\lambda_-, \lambda_+), \vec{\xi}}(\theta) f_{(\lambda'_-, \lambda'_+), \vec{\xi}}^*(\theta) \quad (2)$$

where the angle (θ) is given by the direction of the τ^- relative to the initial e^- beam, with the x-z plane defined as the scattering plane.

Using reduced helicity amplitudes, $T_{\lambda\xi}^J$, and rotation matrices these helicity amplitudes are [7],

$$f_{\vec{\lambda}, \vec{\xi}}(\theta) = d_{\xi, \lambda}^J(\theta) T_{\vec{\lambda}\xi}^J \quad (3)$$

where $\lambda = \lambda_- - \lambda_+$, $\xi = \xi_- - \xi_+$ and $d_{\xi, \lambda}^J(\theta)$ are the reduced rotation matrices about the y-axis. If we neglect the electron mass, the total angular momentum of the process is always $J = 1$. Therefore we get Eq. (3) where we have re-defined our reduced helicity amplitudes with respect to Ref. [7], including in our definition several normalization factors irrelevant in our analysis [4].

Furthermore, helicity conservation in the electron vertex implies that ξ fixes completely $\vec{\xi} = (\xi_+, \xi_-)$. These reduced helicity amplitudes get contributions from diagrams (1) to (5) in Figure 1. For the dominant amplitudes contributing through interferences to P-odd and C-odd observables we can write

$$T_{\vec{\lambda},\xi}^- = T_{\vec{\lambda},\xi}^-(\gamma) + T_{\vec{\lambda},\xi}^-(\gamma Z_A) + T_{\vec{\lambda},\xi}^-(Z_A\gamma) + T_{\vec{\lambda},\xi}^-(Z_{A,A}) \quad (4)$$

where $T_{\vec{\lambda},\xi}^-(\gamma)$ accounts for the contribution of diagrams (1) and (2). $T_{\vec{\lambda},\xi}^-(\gamma Z_A)$ is the P-violating piece of diagram (3) plus the VA piece of diagram (5) and $T_{\vec{\lambda},\xi}^-(Z_A\gamma)$ the corresponding P-violating piece of diagram (4) plus the AV piece of diagram (5). Finally $T_{\vec{\lambda},\xi}^-(Z_{A,A})$ is the contribution of diagram (5) with axial couplings in the initial and final vertices. Notice that here we have not included the $T_{\vec{\lambda},\xi}^-(\gamma Z_V)$, $T_{\vec{\lambda},\xi}^-(Z\gamma_V)$ and $T_{\vec{\lambda},\xi}^-(Z_{V,V})$ pieces, because they are sub-dominant with respect to $T_{\vec{\lambda},\xi}^-(\gamma)$, both in the resonant and non-resonant components.

Taking into account the transformation properties under P of these amplitudes we get

$$\begin{aligned} T_{\vec{\lambda},\xi}^-(\gamma) &= T_{-\vec{\lambda},-\xi}^-(\gamma) = T_{\vec{\lambda},-\xi}^-(\gamma) = T_{-\vec{\lambda},\xi}^-(\gamma) \\ T_{\vec{\lambda},\xi}^-(\gamma Z_A) &= -T_{-\vec{\lambda},-\xi}^-(\gamma Z_A) = T_{\vec{\lambda},-\xi}^-(\gamma Z_A) = -T_{-\vec{\lambda},\xi}^-(\gamma Z_A) \\ T_{\vec{\lambda},\xi}^-(Z_A\gamma) &= -T_{-\vec{\lambda},-\xi}^-(Z_A\gamma) = -T_{\vec{\lambda},-\xi}^-(Z_A\gamma) = T_{-\vec{\lambda},\xi}^-(Z_A\gamma) \\ T_{\vec{\lambda},\xi}^-(Z_{AA}) &= T_{-\vec{\lambda},-\xi}^-(Z_{AA}) = -T_{\vec{\lambda},-\xi}^-(Z_{AA}) = -T_{-\vec{\lambda},\xi}^-(Z_{AA}) \end{aligned} \quad (5)$$

where the normalization of these reduced helicity amplitudes is such that

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= Tr(\rho^{out}) = 2 \sin^2 \theta |T_{(+,+),1}(\gamma)|^2 + (1 + \cos^2 \theta) |T_{(+,-),1}(\gamma)|^2 \\ &+ 4 \cos \theta Re\{T_{(+,-),1}(\gamma) T_{(+,-),1}^*(Z_{A,A})\} \end{aligned} \quad (6)$$

$$\sigma = \frac{16\pi}{3} (|T_{(+,+),1}(\gamma)|^2 + |T_{(+,-),1}(\gamma)|^2) \quad (7)$$

Notice that a C-odd forward-backward interference is generated by the axial couplings of the Z in both the electron and τ vertices.

We can calculate these reduced helicity amplitudes from the Feynman diagrams 1-5 of Fig. 1 following the method explained in App. B.

$$\begin{aligned}
KT_{(+,+),1}(\gamma) &= -i4\sqrt{2} \frac{e^2}{s} \left(1 + \frac{e^2}{s} Q_b^2 |F_\Upsilon|^2 P_\Upsilon\right) M_\tau \frac{\sqrt{s}}{2} \\
KT_{(+,-),1}(\gamma) &= -i8 \frac{e^2}{s} \left(1 + \frac{e^2}{s} Q_b^2 |F_\Upsilon|^2 P_\Upsilon\right) p^0 \frac{\sqrt{s}}{2} \\
KT_{(+,+),1}(\gamma Z_A) &= 0 \\
KT_{(+,-),1}(\gamma Z_A) &= -i8 \frac{8G_F}{\sqrt{2}} g_A \left(\frac{e^2}{s} Q_b g_V^b |F_\Upsilon|^2 P_\Upsilon - g_V\right) |\vec{p}| \frac{\sqrt{s}}{2} \\
KT_{(+,+),1}(Z_A \gamma) &= -i4\sqrt{2} \frac{8G_F}{\sqrt{2}} g_A \left(\frac{e^2}{s} Q_b g_V^b |F_\Upsilon|^2 P_\Upsilon - g_V\right) M_\tau \frac{\sqrt{s}}{2} \\
KT_{(+,-),1}(Z_A \gamma) &= -i8 \frac{8G_F}{\sqrt{2}} g_A \left(\frac{e^2}{s} Q_b g_V^b |F_\Upsilon|^2 P_\Upsilon - g_V\right) p^0 \frac{\sqrt{s}}{2} \\
KT_{(+,+),1}(Z_{AA}) &= 0 \\
KT_{(+,-),1}(Z_{AA}) &= i8 \frac{8G_F}{\sqrt{2}} g_A^2 |\vec{p}| \frac{\sqrt{s}}{2}
\end{aligned} \tag{8}$$

where K is only a constant which takes care of the different normalization of the helicity amplitudes and the Feynman amplitudes. $Q_b = -\frac{1}{3}$ is the charge of the b quark, $g_{V(A)}$ is the vector (axial) coupling to the leptons, P_Υ stands for the Breit-Wigner of the Υ ,

$$P_\Upsilon(s) = \frac{1}{(s - M_\Upsilon^2) + iM_\Upsilon \Gamma_\Upsilon} \tag{9}$$

and $F_\Upsilon(q^2)$ is a form factor defined as,

$$\langle \Upsilon(q, \omega) | \bar{\psi}_b(0) \gamma_\mu \psi_b(0) | 0 \rangle = F_\Upsilon(q^2) \varepsilon^*(\omega, q)_\mu \tag{10}$$

with $\varepsilon^*(\omega, q)_\mu$, the polarization four-vector. This form factor can be related to the amplitude for Υ to e^+e^- ,

$$\Gamma_e = \frac{1}{6\pi} Q_b^2 \frac{(4\pi\alpha)^2}{M_\Upsilon^4} |F_\Upsilon|^2 \frac{M_\Upsilon}{2} \tag{11}$$

Notice that all the hadronic uncertainties in our process will be included in this unique form factor. In Eq. (8) we can see the coupling g_V^b is only in the $T_{\lambda,\xi}^-(\gamma Z_A)$ and $T_{\lambda,\xi}^-(Z_A \gamma)$ amplitudes, so, as these amplitudes will contribute

to dominant order to the observables through interferences with $T_{\lambda,\xi}^-(\gamma)$, this means that only the P-odd observables will contain the information about the g_V^b coupling linearly. Then we are going to analyze the polarizations and the P-odd correlations.

The polarizations of τ^- are given as follows

$$\begin{aligned}
\frac{d\sigma}{d\Omega} P_{z'}^{(-)}(\theta) &= \rho_{(+,+),(+,+)} + \rho_{(+,-),(+,-)} - \rho_{(-,+),(-,+)} - \rho_{(-,-),(-,-)} \\
&= 2\text{Re}\{T_{(+,-),1}(\gamma)T_{(+,-),1}^*(\gamma Z_A)\}(1 + \cos^2 \theta) \\
&+ 4\text{Re}\{T_{(+,-),1}(\gamma)T_{(+,-),1}^*(Z_A\gamma)\} \cos \theta
\end{aligned} \tag{12}$$

$$\begin{aligned}
\frac{d\sigma}{d\Omega} P_{x'}^{(-)}(\theta) &= \rho_{(+,+),(-,+)} + \rho_{(+,-),(-,-)} + \rho_{(-,+),(+,+)} + \rho_{(-,-),(+,-)} \\
&= -2\sqrt{2}[\text{Re}\{T_{(+,+),1}(\gamma)T_{(+,-),1}^*(\gamma Z_A)\} \sin \theta \cos \theta \\
&+ (\text{Re}\{T_{(+,+),1}(\gamma)T_{(+,-),1}^*(Z_A\gamma)\} \\
&+ \text{Re}\{T_{(+,-),1}(\gamma)T_{(+,+),1}^*(Z_A\gamma)\}) \sin \theta]
\end{aligned} \tag{13}$$

$$\begin{aligned}
\frac{d\sigma}{d\Omega} P_{y'}^{(-)}(\theta) &= -i(\rho_{(+,+),(-,+)} + \rho_{(+,-),(-,-)} - \rho_{(-,+),(+,+)} - \rho_{(-,-),(+,-)}) \\
&= 2\sqrt{2}[\text{Im}\{T_{(+,+),1}(\gamma)T_{(+,-),1}^*(Z_{A,A})\} \sin \theta \\
&+ 2\text{Im}\{T_{(+,+),1}(\gamma)T_{(+,-),1}^*(\gamma)\} \sin \theta \cos \theta]
\end{aligned} \tag{14}$$

As we can see in these expressions, only $P_{z'}$ and $P_{x'}$ contain information on g_V^b , because both are P-odd, T-even observables, but $P_{y'}$ has no information, because it is P-even, T-odd. The T-odd observable $P_{y'}$ needs the interference between resonant and non-resonant amplitudes, that on the Υ peak are relatively imaginary. By the same argument, there is no interference between resonant and non-resonant pieces in the T-even observables, like $P_{z'}$ and $P_{x'}$, on the Υ peak. If we compare the longitudinal polarization, Eq. (12), with the result we obtained in Eq. (1) the difference is in the non-resonant terms proportional to g_V in Eq. (8). Then these new terms are suppressed on the Υ peak by a factor $g_V\alpha^2 Q_b/(9g_V^b b.r.(\Upsilon \rightarrow e^+e^-)^2) \approx 3.5 \times 10^{-4}$, and so our estimate in Eq. (1) is very good. This means basically, that this observable

will be, in the forward direction, $P_{z'}(\theta = 0) \simeq .185 \cdot g_V^b \simeq 0.032$. In Fig. 2, we can see a plot of this polarization as a function of θ .

On the other hand $P_{x'}$ contains similar reduced helicity amplitudes as $P_{z'}$, the only difference is that, as usual, transverse polarizations are suppressed by a mass insertion, that is a factor $M_\tau/p^0 \simeq .38$. Basically, apart from a different angular dependence, this is the reason that makes this observable less sensitive to g_V^b as we can see in Fig. 3. We get for instance $P_{x'}(\theta = \pi/2) \simeq -.063 \cdot g_V^b$, slightly worse than $P_{z'}$.

The information contained in the τ^+ polarizations is closely related to that of the τ^-

$$\begin{aligned} P_{z'}^{(+)} &= -P_{z'}^{(-)} \\ P_{x'}^{(+)} &= P_{x'}^{(-)} \\ P_{y'}^{(+)} &= -P_{y'}^{(-)} \end{aligned} \quad (15)$$

Finally, we can also consider the spin correlations between τ^+ and τ^- . We define these spin correlations as follows,

$$\frac{d\sigma}{d\Omega} \mathcal{C}_{ij}(\theta) = Tr(\sigma_i^{(-)} \sigma_j^{(+)} \rho^\tau) \quad (16)$$

With this definition is easy to see that the only P-odd observables will be the $\mathcal{C}_{x,y}$, $\mathcal{C}_{z,y}$, $\mathcal{C}_{y,x}$ and $\mathcal{C}_{y,z}$ correlations,

$$\begin{aligned} \frac{d\sigma}{d\Omega} \mathcal{C}_{zy}(\theta) &= i(-\rho_{(+, +), (+, -)} + \rho_{(-, +), (-, -)} + \rho_{(+, -), (+, +)} - \rho_{(-, -), (-, +)}) \\ &= -2\sqrt{2}[Im\{T_{(+, +), 1}(\gamma)T_{(+, -), 1}^*(\gamma Z_A)\} \sin \theta \cos \theta \\ &\quad + (Im\{T_{(+, +), 1}(\gamma)T_{(+, -), 1}^*(Z_A \gamma)\} \\ &\quad - Im\{T_{(+, -), 1}(\gamma)T_{(+, +), 1}^*(Z_A \gamma)\}) \sin \theta] \end{aligned} \quad (17)$$

$$\mathcal{C}_{yz}(\theta) = \mathcal{C}_{zy}(\theta) \quad (18)$$

$$\begin{aligned} \frac{d\sigma}{d\Omega} \mathcal{C}_{xy}(\theta) &= i(-\rho_{(+, +), (-, -)} - \rho_{(-, +), (+, -)} + \rho_{(+, -), (-, +)} + \rho_{(-, -), (+, +)}) \\ &= 2Im\{T_{(+, -), 1}(\gamma)T_{(+, -), 1}^*(\gamma Z_A)\} \sin^2 \theta \end{aligned} \quad (19)$$

$$\mathcal{C}_{yx}(\theta) = -\mathcal{C}_{xy}(\theta) \quad (20)$$

As we already pointed out and we can see explicitly in Eq. (12)-(20), all the relevant observables are associated to the amplitudes $T_{\tilde{\lambda},\xi}(\gamma Z_A)$ and $T_{\tilde{\lambda},\xi}(Z_A\gamma)$, and the relative strength of these observables to the dominant term is set by the factor $8G_F M_\Upsilon^2 / (4\pi\alpha\sqrt{2}) \simeq 0.064$. Unfortunately, the P-odd correlations are also T-odd which means they need an imaginary part. With our set of reduced helicity amplitudes, it becomes necessary to have an interference between a resonant and a non-resonant diagram to get an imaginary part. Therefore, these contributions have the same suppression that diagram (1) with respect to diagram (2), that is, $\alpha / (3 \text{ b.r.}(\Upsilon \rightarrow e^+e^-)) \simeq 1/10$. Notice also that the helicity structure of \mathcal{C}_{zy} is very similar to $P_{x'}$, and so it will have the same suppression factors, unlike \mathcal{C}_{xy} which is not helicity suppressed.

Then it is clear that our main observables will be the longitudinal polarizations of both τ 's, then $P_{x'}$ and finally \mathcal{C}_{xy} and \mathcal{C}_{zy} ordered from the most relevant to the least one.

In the next section we are going to connect these observables to measurable quantities, analyzing the angular distributions of the decay products of the two τ .

3 Decay of a polarized τ

The main τ decay channels are presented in Table 2. The purely leptonic decays have branching ratios of 17.65% for muons and 18.01% for electrons. Unfortunately, these decay modes have two neutrinos in the final state, which implies we can not reconstruct the τ direction. Then, their sensitivity to the τ polarization is small [8] compared with the hadronic decays.

So we will concentrate on the hadronic τ decays which have only one neutrino in the final state, which allows to reconstruct the τ direction if both τ decay hadronically [9]. These decays are $\tau^- \rightarrow \pi^- \nu_\tau$, with a branching ratio of 11.31%, $\tau^- \rightarrow \rho^- \nu_\tau$ which corresponds almost exactly to the two pions channel, with branching ratio 25.24%, and $\tau^- \rightarrow a_1^- \nu_\tau$ which is given by the sum of the three pion final states. In this work we will concentrate, as an example, in the decays $\tau^- \rightarrow \pi^- \nu_\tau$ and $\tau^- \rightarrow \rho^- \nu_\tau$, other τ decay

channels have been studied elsewhere, for instance $\tau \rightarrow a_1 \nu_\tau$ can be found in [10] for LEP physics,

3.1 $\tau^- \rightarrow \pi^- \nu_\tau$

This channel has been used for a long time to measure the τ polarization because of its good sensitivity. In this decay, we can easily get the differential decay width as,

$$\frac{1}{\Gamma} \frac{d\Gamma}{d\Omega} = \frac{1}{4\pi} [1 + \vec{P} \cdot \hat{k}(\Omega)] \quad (21)$$

where \hat{k} is the unit vector in the direction of the pion.

In Eq. (21) we can see that the polarization effects are not suppressed in the angular distribution of the decay pions. This is due to the fact that in this process there is only one reduced helicity amplitude, and so the complete angular distribution is necessarily proportional to this unique reduced amplitude. So this channel is specially sensitive to the τ polarization, and this has been the reason for its popularity as polarization analyzer in τ decays.

With these elements, following App. A, we can build a complete two steps angular distribution [4, 11] for the whole process, $e^- e^+ \rightarrow \tau^- \tau^+ \rightarrow (\pi^- \nu_\tau)(\pi^+ \bar{\nu}_\tau)$

$$\begin{aligned} \frac{d\sigma}{d\Omega d\Omega_+ d\Omega_-} &= \frac{d\sigma}{d\Omega} (1 + \vec{P}^{(-)} \cdot \hat{k}_- - \vec{P}^{(+)} \cdot \hat{k}_+ \\ &- C_{zz} \cos \theta_- \cos \theta_+ - C_{xx} \sin \theta_- \cos \phi_- \sin \theta_+ \cos \phi_+ \\ &- C_{yy} \sin \theta_- \sin \phi_- \sin \theta_+ \sin \phi_+ - C_{zx} \cos \theta_- \sin \theta_+ \cos \phi_+ \\ &- C_{xz} \sin \theta_- \cos \phi_- \cos \theta_+ - C_{zy} \cos \theta_- \sin \theta_+ \sin \phi_+ \\ &- C_{yz} \sin \theta_- \sin \phi_- \cos \theta_+ - C_{xy} \sin \theta_- \cos \phi_- \sin \theta_+ \sin \phi_+ \\ &- C_{yx} \sin \theta_- \sin \phi_- \sin \theta_+ \cos \phi_+) \end{aligned} \quad (22)$$

where (θ_\pm, ϕ_\pm) is the direction of the π^\pm in the rest frame of the τ^\pm and \hat{k}_\pm is the unit vector in this direction.

This will be the cross section we have to study to extract g_V^b from this channel. In section 4 we will analyze the sensitivity of the different observables we can construct from this cross section.

3.2 $\tau^- \rightarrow \rho^- \nu_\tau \rightarrow \pi^- \pi^0 \nu_\tau$

In the channel $\tau \rightarrow \rho \nu_\tau$ we have a spin 1 particle in the final state. This implies that we have two different helicity amplitudes in this decay, and so different combinations of these amplitudes enter in the polarized and unpolarized pieces,

$$\frac{1}{\Gamma_{\tau \rightarrow \rho \nu}} \frac{d\Gamma_{\tau \rightarrow \rho \nu}}{d\Omega} = \frac{1}{4\pi} [1 + \alpha_\rho \vec{P} \hat{k}(\Omega)] \quad (23)$$

where α_ρ is a ratio of reduced helicity amplitudes, which we can get in terms of the masses as,

$$\alpha_\rho = \frac{|T_{0,-1/2}|^2 - |T_{-1,-1/2}|^2}{|T_{0,-1/2}|^2 + |T_{-1,-1/2}|^2} = \frac{M_\tau^2 - 2M_\rho^2}{M_\tau^2 + 2M_\rho^2} = 0.456 \quad (24)$$

Then, we can see from here that in spite of its bigger statistics the sensitivity at this level is smaller than in $\tau \rightarrow \pi \nu_\tau$. However, as it was pointed out in [8], this situation can be improved if, in addition, we try to get some extra information on the ρ helicity. To do this, you have to include another step in this chain of decays, and analyze the decay $\rho^- \rightarrow \pi^- \pi^0$. The cross section for the whole process, $e^- e^+ \rightarrow \tau^- \tau^+ \rightarrow ((\pi^- \pi^0) \nu_\tau) (\rho^+ \bar{\nu}_\tau)$, can be written as,

$$\frac{d\sigma}{d\Omega d\Omega_1^- d\Omega_1^+ d\Omega_2} = \frac{d\sigma}{d\Omega} (e^+ e^- \rightarrow \tau^+ \tau^-) \frac{1}{\Gamma_\tau^2} \frac{d\Gamma_{\tau^- \tau^+ \rightarrow \rho^- \nu \rho^+ \bar{\nu}}}{d\Omega_1^- d\Omega_1^+} \frac{1}{\Gamma_\rho} \frac{d\Gamma_{\rho^- \rightarrow \pi^- \pi^0}}{d\Omega_2} \quad (25)$$

where the $\tau^- \tau^+$ decay amplitude to $\rho^- \nu \rho^+ \bar{\nu}$ is given below in Eq. (34), and we have added the last term which is the decay width of a polarized and aligned ρ into two pions. The expression for this decay width is

$$\frac{1}{\Gamma_{\rho \rightarrow \pi \pi}} \frac{d\Gamma_{\rho \rightarrow \pi \pi}}{d\Omega_2} = \frac{1}{4\pi} [1 - \sqrt{10} \sum_N \mathcal{D}_{N,0}^{(2)}(\phi_2, \theta_2, 0) t_{2,N}] \quad (26)$$

In this equation we can see that the ρ polarizations do not appear in the decay angular distribution because this is a strong decay and therefore P-conserving. The alignments, higher order multipole parameters, appear but the polarizations do not.

Now, we have to calculate the density matrix for a ρ coming from the decay of a polarized τ in its center of mass frame, and then apply the necessary Wigner rotation [12], as explained in App. A, to get the density matrix in a frame where the τ is moving.

This density matrix of a single ρ from the decay of one of the two τ s also contains information on the decay of the other τ if we study the correlations and do not integrate the second τ decay. So, as we are interested in the measure of correlations between the two τ s, we will study the ρ^- density matrix in a decay $\tau^- \tau^+ \rightarrow (\rho^- \nu_\tau)(\rho^+ \bar{\nu}_\tau)$ and in a decay $\tau^- \tau^+ \rightarrow (\rho^- \nu_\tau)(\pi^+ \bar{\nu}_\tau)$. Naturally this two different density matrices will coincide when we integrate completely the τ^+ decay products.

Following Appendix A, we can write the complete ρ^- density matrix from a decay $\tau^- \tau^+ \rightarrow (\rho^- \nu_\tau)(\rho^+ \bar{\nu}_\tau)$ as

$$\begin{aligned} \rho_{\mu-\mu'_-} &= \sum_{\lambda_+\lambda_-\lambda'_+\lambda'_-} \sum_{\nu_-\nu'_-\nu_+} f_{(\nu_+1/2)\lambda_+}^{(+)}(\Omega_+) d_{\mu-\nu_-}(\omega_-) f_{(\nu_--1/2)\lambda_-}^{(-)}(\Omega_-) \\ &\quad \rho_{(\lambda_-\lambda_+),(\lambda'_-\lambda'_+)}^{\tau} f_{(\nu_+1/2)\lambda'_+}^{(+)*}(\Omega_+) d_{\mu'_-\nu'_-}(\omega_-) f_{(\nu'_--1/2)\lambda'_-}^{(-)*}(\Omega_-) \end{aligned} \quad (27)$$

where we have used a reference system in LAB with the τ^- in the z-axis and the initial beams in the x-z plane, to simplify the expressions for the Wigner rotations.

We define an effective density matrix, $\bar{\rho}$ without the Wigner rotations that, if we integrated completely the Ω_+ variables, would correspond to the density matrix in the τ^- rest frame, which is

$$\rho_{\mu-\mu'_-} = d_{\mu-\nu_-}(\omega_-) \bar{\rho}_{\nu_-\nu'_-} d_{\mu'_-\nu'_-}(\omega_-) \quad (28)$$

The next step is to calculate this effective density matrix in terms of reduced helicity amplitudes. In a decay we define the helicity amplitudes as,

$$f_{\bar{\nu}\lambda}(\theta, \phi) = \sqrt{\frac{2j+1}{4\pi}} \mathcal{D}_{\lambda, \nu_1-\nu_2}^{j*}(\phi, \theta, 0) T_{\bar{\nu}}^j \quad (29)$$

With this definition and Eqs. (27) and (28), we get our effective density matrix. In this process, $\tau^- \rightarrow \rho^- \nu_\tau$, we have only two reduced helicity amplitudes if we take $M_\nu = 0$

$$\begin{aligned} KT_{-1,-1/2} &= i4G_F V_{ud} F_\rho(q^2) \sqrt{M_\tau k_2} \\ KT_{0,-1/2} &= i2\sqrt{2}G_F V_{ud} F_\rho(q^2) \sqrt{M_\tau k_2} \frac{M_\tau}{M_\rho} \end{aligned} \quad (30)$$

and also two amplitudes in the decay $\tau^+ \rightarrow \rho^+ \bar{\nu}_\tau$,

$$\begin{aligned} KT_{1,1/2} &= -i4G_F V_{ud} F_\rho(q^2) \sqrt{M_\tau k_2} \\ KT_{0,1/2} &= i2\sqrt{2}G_F V_{ud} F_\rho(q^2) \sqrt{M_\tau k_2} \frac{M_\tau}{M_\rho} \end{aligned} \quad (31)$$

where $F_\rho(q^2)$ is a form factor defined as,

$$\langle \rho(k, \omega) | T | \pi(\vec{k}_2) \pi(-\vec{k}_2) \rangle = F_\rho(q^2) \varepsilon^*(\omega, q)_\mu k_2^\mu \quad (32)$$

The rest of the notation is self-explanatory. The total amplitude $\rho \rightarrow \pi\pi$ is,

$$\Gamma_{\rho \rightarrow \pi\pi} = \frac{1}{2M_\rho} (|T_{-1,-1/2}|^2 + |T_{0,-1/2}|^2) \quad (33)$$

Following App. A, we now calculate the C.M. multipole parameters corresponding to the density matrix $\bar{\rho}$ in terms of these reduced amplitudes, taking into account that we do not integrate the direction of the second ρ .

$$\begin{aligned} \frac{d\Gamma_{\tau^- \tau^+}}{d\Omega_1^- d\Omega_1^+} &= Tr\{\rho\} = \rho_{-1,-1} + \rho_{0,0} \\ &= (|T_{-1,-1/2}|^2 + |T_{0,-1/2}|^2)(|T_{1,1/2}|^2 + |T_{0,1/2}|^2) \\ &\quad [1 + \alpha_\rho \bar{P} \cdot \hat{k}^- - \bar{\alpha} \bar{P} \cdot \hat{k}^+ - \alpha_\rho \bar{\alpha} (\mathcal{C}_{zz} \cos \theta_- \cos \theta_+ \\ &\quad + \mathcal{C}_{xx} \sin \theta_- \cos \phi_- \sin \theta_+ \cos \phi_+ + \mathcal{C}_{yy} \sin \theta_- \sin \phi_- \sin \theta_+ \sin \phi_+ \\ &\quad + \mathcal{C}_{zx} \cos \theta_- \sin \theta_+ \cos \phi_+ + \mathcal{C}_{xz} \sin \theta_- \cos \phi_- \cos \theta_+ \\ &\quad + \mathcal{C}_{zy} \cos \theta_- \sin \theta_+ \sin \phi_+ + \mathcal{C}_{yz} \sin \theta_- \sin \phi_- \cos \theta_+ \\ &\quad + \mathcal{C}_{xy} \sin \theta_- \cos \phi_- \sin \theta_+ \sin \phi_+ + \mathcal{C}_{yx} \sin \theta_- \sin \phi_- \sin \theta_+ \cos \phi_+)] \end{aligned} \quad (34)$$

here (θ_\pm, ϕ_\pm) is the direction of the ρ^\pm in the rest frame of the τ^\pm and \hat{k}_\pm is the unit vector in this direction. α_ρ was defined in Eq. (24) and $\bar{\alpha}$ is equal to α_ρ if the τ^+ decays to $\rho^+ \bar{\nu}_\tau$, but is equal to 1 if it decays to $\pi^+ \bar{\nu}_\tau$. This definitions hold for all the observables that follow.

$$\begin{aligned} Tr\{\rho\} \bar{t}_{20} &= \sqrt{\frac{1}{10}} (\rho_{-1,-1} - 2\rho_{0,0}) = \sqrt{\frac{1}{10}} (|T_{-1,-1/2}|^2 + |T_{0,-1/2}|^2) \\ &\quad (|T_{1,1/2}|^2 + |T_{0,1/2}|^2) [\gamma_\rho - \beta_\rho \bar{P}^- \cdot \hat{k}^- - \bar{\alpha} \gamma_\rho \bar{P}^+ \cdot \hat{k}^+] \end{aligned}$$

$$\begin{aligned}
& +\bar{\alpha}\beta_\rho(\mathcal{C}_{zz} \cos \theta_- \cos \theta_+ + \mathcal{C}_{xx} \sin \theta_- \cos \phi_- \sin \theta_+ \cos \phi_+ \\
& +\mathcal{C}_{yy} \sin \theta_- \sin \phi_- \sin \theta_+ \sin \phi_+ + \mathcal{C}_{zx} \cos \theta_- \sin \theta_+ \cos \phi_+ \\
& +\mathcal{C}_{xz} \sin \theta_- \cos \phi_- \cos \theta_+ + \mathcal{C}_{zy} \cos \theta_- \sin \theta_+ \sin \phi_+ \\
& +\mathcal{C}_{yz} \sin \theta_- \sin \phi_- \cos \theta_+ + \mathcal{C}_{xy} \sin \theta_- \cos \phi_- \sin \theta_+ \sin \phi_+ \\
& +\mathcal{C}_{yx} \sin \theta_- \sin \phi_- \sin \theta_+ \cos \phi_+)] \tag{35}
\end{aligned}$$

where we have introduced two new coefficients β_ρ and γ_ρ defined as

$$\gamma_\rho = \frac{|T_{-1,-1/2}|^2 - 2|T_{0,-1/2}|^2}{|T_{0,-1/2}|^2 + |T_{-1,-1/2}|^2} = \frac{2M_\rho^2 - 2M_\tau^2}{M_\tau^2 + 2M_\rho^2} = -1.18 \tag{36}$$

$$\beta_\rho = \frac{|T_{-1,-1/2}|^2 + 2|T_{0,-1/2}|^2}{|T_{0,-1/2}|^2 + |T_{-1,-1/2}|^2} = \frac{2M_\rho^2 + 2M_\tau^2}{M_\tau^2 + 2M_\rho^2} = 1.73 \tag{37}$$

we also need \bar{t}_{21} and \bar{t}_{22} ,

$$\begin{aligned}
Tr\{\rho\}\bar{t}_{21} &= \sqrt{\frac{3}{10}}\rho_{-1,0} = \sqrt{\frac{3}{10}}(|T_{-1,-1/2}|^2 + |T_{0,-1/2}|^2) \\
& (|T_{1,1/2}|^2 + |T_{0,1/2}|^2)[\delta_\rho(P_z^-(-\sin \theta_-) + P_x^-(\cos \theta_- \cos \phi_- - i \sin \phi_-) \\
& +P_y^-(\cos \theta_- \sin \phi_- + i \cos \phi_-)) - \delta_\rho\bar{\alpha}(\mathcal{C}_{zz}(-\sin \theta_-) \cos \theta_+ \\
& + \mathcal{C}_{xx}(\cos \theta_- \cos \phi_- - i \sin \phi_-) \sin \theta_+ \cos \phi_+ \\
& +\mathcal{C}_{yy}(\cos \theta_- \sin \phi_- + i \cos \phi_-) \sin \theta_+ \sin \phi_+ + \mathcal{C}_{zx}(-\sin \theta_-) \sin \theta_+ \cos \phi_+ \\
& +\mathcal{C}_{xz}(\cos \theta_- \cos \phi_- - i \sin \phi_-) \cos \theta_+ + \mathcal{C}_{zy}(-\sin \theta_-) \sin \theta_+ \sin \phi_+ \\
& +\mathcal{C}_{yz}(\cos \theta_- \sin \phi_- + i \cos \phi_-) \cos \theta_+ \\
& + \mathcal{C}_{xy}(\cos \theta_- \cos \phi_- - i \sin \phi_-) \sin \theta_+ \sin \phi_+ \\
& +\mathcal{C}_{yx}(\cos \theta_- \sin \phi_- + i \cos \phi_-) \sin \theta_+ \cos \phi_+)] \tag{38}
\end{aligned}$$

$$Tr\{\rho\}\bar{t}_{22} = \sqrt{\frac{3}{5}}\rho_{-1,1} = 0 \tag{39}$$

we have also introduced a new coefficient

$$\delta_\rho = \frac{T_{-1,-1/2}T_{0,-1/2}^*}{|T_{0,-1/2}|^2 + |T_{-1,-1/2}|^2} = \frac{\sqrt{2}M_\rho M_\tau}{M_\tau^2 + 2M_\rho^2} = 0.445 \tag{40}$$

In general, these multipole parameters can be complex, as \bar{t}_{21} , except the parameters with $L = 0$, which are always real.

It is very important to notice that unlike α_ρ and δ_ρ , that are small, both γ_ρ and β_ρ are bigger than one and so they can enhance the information on the τ polarizations.

As we have already pointed out, the only difference between the decays $\tau^- \tau^+ \rightarrow (\rho^- \nu_\tau)(\rho^+ \bar{\nu}_\tau)$ and $\tau^- \tau^+ \rightarrow (\rho^- \nu_\tau)(\pi^+ \bar{\nu}_\tau)$ at this level is the value of the coefficient $\bar{\alpha}$

$$\begin{aligned}\tau^+ \rightarrow \rho^+ \bar{\nu}_\tau &\implies \bar{\alpha} = .456 \\ \tau^+ \rightarrow \pi^+ \bar{\nu}_\tau &\implies \bar{\alpha} = 1\end{aligned}\tag{41}$$

The final step to get the alignments appearing in Eq. (26) and Eq. (25) is to make the Wigner rotation, Eq. (A.13).

$$\begin{aligned}t_{20} = d_{0M}^2(\omega_-)\bar{t}_{2M} &= \bar{t}_{20}\left(\frac{3}{2}\cos^2\omega_- - \frac{1}{2}\right) \\ &+ \sqrt{6}\text{Re}\{\bar{t}_{21}\}\sin\omega_- \cos\omega_-\end{aligned}\tag{42}$$

$$\begin{aligned}t_{21} = d_{1M}^2(\omega_-)\bar{t}_{2M} &= -\bar{t}_{20}\cos\omega_- \sin\omega_- + i\text{Im}\{\bar{t}_{21}\}\cos\omega_- \\ &+ \text{Re}\{\bar{t}_{21}\}(\cos^2\omega_- - \sin^2\omega_-)\end{aligned}\tag{43}$$

$$\begin{aligned}t_{22} = d_{2M}^2(\omega_-)\bar{t}_{2M} &= \bar{t}_{20}\frac{\sqrt{6}}{4}\sin^2\omega_- - i\text{Im}\{\bar{t}_{21}\}\sin\omega_- \\ &- \text{Re}\{\bar{t}_{21}\}\cos\omega_- \sin\omega_-\end{aligned}\tag{44}$$

where ω is the Wigner rotation associated with the boost from the τ rest frame to the e^+e^- C.M. frame which transforms the ρ four-momentum, $k'_\rho = \Lambda k_\rho$. These rotations have the following expression

$$\sin\omega = \frac{M_\rho|\vec{p}_\tau|\sin\theta}{M_\tau|\vec{k}'_\rho|}, \quad \cos\omega = \frac{E_\rho E'_\rho M_\tau - ZM_\rho^2}{|\vec{k}_\rho||\vec{k}'_\rho|M_\tau}\tag{45}$$

with (Z, \vec{p}_τ) the τ four-momentum in the e^+e^- C.M. frame and θ the angle between the ρ and the direction of the boost in the τ rest frame.

Now we have the complete angular distributions of the different chains with final states:

- $(\pi^- \nu_\tau)(\pi^+ \bar{\nu}_\tau)$
- $(\pi^- \pi^0 \nu_\tau)(\pi^+ \bar{\nu}_\tau) + (\pi^+ \pi^0 \nu_\tau)(\pi^- \bar{\nu}_\tau)$
- $(\pi^- \pi^0 \nu_\tau)(\rho^+ \bar{\nu}_\tau) + (\pi^+ \pi^0 \nu_\tau)(\rho^- \bar{\nu}_\tau)$

Next we are going to find what is the statistical accuracy one can obtain in the measure of g_V^b

4 Statistical sensitivity

To estimate the obtainable precision in the measurement of g_V^b in a B-meson facility, we use the formalism of references [13, 14, 15]. In these references they call “ideal statistical error” of a parameter p which enters a function $f(x, y; p)$ to be determined experimentally, to the error obtained from a least-squares-fit to this function with N events. To obtain this error, we use that for large number of events, N , the likelihood function approaches a gaussian. Then, if the function $f(x, y; p)$ is normalized to one on the physical region, the ideal statistical error is given by

$$\sigma_p^2 = \frac{1}{N} \left[\int \left(\frac{\delta \ln f(x_1, \dots, x_n; p)}{\delta p} \right)^2 \cdot f(x_1, \dots, x_n; p) dx_1 \dots dx_n \right]^{-1} \quad (46)$$

The word “ideal” stands for the fact that we are not considering the efficiency of the detectors, effects of finite experimental resolution and we assume an ideal distribution of the N events according to $f(x, y; p)$.

In this case our function $f(x, y; p)$ will be the normalized cross section to the different channels. First we study the reaction $e^- e^+ \rightarrow \tau^- \tau^+ \rightarrow (\pi^- \nu_\tau)(\pi^+ \bar{\nu}_\tau)$. Our result is, in this channel,

$$\sigma_{g_V^b} = \frac{11.1}{\sqrt{N}} \quad (47)$$

where N is the number of $e^- e^+ \rightarrow \tau^- \tau^+ \rightarrow (\pi^- \nu_\tau)(\pi^+ \bar{\nu}_\tau)$ events. This means that in a B-meson facility with $10^8 \Upsilon(1S)$ produced per year one could get a sensitivity to g_V^b of $6 \cdot 10^{-2}$ only with this channel. In Eq. (22) is all the information available in the process, but not all these observables will be useful for our purposes. In particular the P-even, T-even correlations ($\mathcal{C}_{zz}, \mathcal{C}_{xx}, \mathcal{C}_{yy}, \mathcal{C}_{zx}$) get contributions from $|T_{\lambda, \xi}^+(\gamma)|^2$, so they are order 1 and

they have no information on g_V^b . On the other hand, the P-odd, T-odd correlations are not as sensitive to this parameter as the polarizations, then we can ask whether we can improve the sensitivity by integrating out some of the final state variables. From Eq. (22) we would like to eliminate the P-even correlations maintaining the polarizations and P-odd correlations, but unfortunately we can not achieve this result integrating out some final state variables. The only interesting possibility is to consider both τ decays independently, integrating out one of the two pion directions with respect to the τ . In this case we are considering the $e^-e^+ \rightarrow \tau^-\tau^+ \rightarrow ((\pi^-\nu_\tau)\tau^+ + \tau^-(\pi^+\bar{\nu}_\tau))$ events, where both τ are hadronically reconstructed. By doing this we increase considerably the number of events, because now we include also the events in which the second τ decays to ρ and a_1 . This is approximately a factor of 5 in the number of events for each tau, but we have two independent events for each Υ decay, then the number of events increases a factor of 10. The sensitivity one can get with this new decay distribution (we obtain it directly from (22) integrating out (θ_+, ϕ_+)) is

$$\sigma_{g_V^b} = \frac{14.6}{\sqrt{N}} \quad (48)$$

and now N is the number of $e^-e^+ \rightarrow \tau^-\tau^+ \rightarrow ((\pi^-\nu_\tau)\tau^+ + \tau^-(\pi^+\bar{\nu}_\tau))$ events. We can see that the difference between Eqs. (47) and (48) is roughly a factor $\sqrt{2}$, this is due to the fact that in Eq. (47) we included both the τ^+ and τ^- polarizations while in Eq. (48) we take into account only one of them. On the other hand, this means that the P-odd correlations do not improve the measurement of g_V^b in a significant way. Again with $10^8\Upsilon$ per year, one can get a sensitivity of $2.3 \cdot 10^{-2}$. Even more, the simplest polarization analyzer we can use is the energy of the pions. The energy of the pions in LAB is related with the angle in C.M. of the τ

$$E_\pi = \frac{E_\pi^* E_\tau + q k_\pi^* \cos \theta_-}{M_\tau} \quad (49)$$

where $(E_\pi^*, \vec{k}_\pi^*(\theta_-))$ is the four-momentum of the pion in the τ C.M., and (E_τ, \vec{q}) the four-momentum of the τ in LAB. Again from Eq. (22), if we integrate all the angular variables but θ_- and make this change of variable to E_π one gets a sensitivity of

$$\sigma_{g_V^b} = \frac{20.9}{\sqrt{N}} \quad (50)$$

And, in this way, we do not put any restriction on the second τ decay, then with $10^8\Upsilon$ per year one can get a sensitivity to g_V^b of $2.7 \cdot 10^{-2}$, simply using the $\pi\nu$ channel and measuring only the pion energy. From this point of view, it is evident that, if we consider only the $\pi\nu$ decay channel, this is the best strategy to measure g_V^b , because one can use all the $\tau \rightarrow \pi\nu$ events and is experimentally simpler.

We have also studied the channel $\tau \rightarrow \rho\nu$ as a polarization analyzer. Then if we apply Eq. (46) to the complete distribution $e^-e^+ \rightarrow \tau^-\tau^+ \rightarrow ((\pi^-\pi^0)\nu_\tau)(\rho^+\bar{\nu}_\tau) + ((\pi^+\pi^0)\nu_\tau)(\rho^-\bar{\nu}_\tau)$ given by Eq. (25) one gets

$$\sigma_{g_V^b} = \frac{12.4}{\sqrt{N}} \quad (51)$$

that is slightly worse than our result in the $e^-e^+ \rightarrow \tau^-\tau^+ \rightarrow (\pi^-\nu_\tau)(\pi^+\bar{\nu}_\tau)$ channel, but now we have more events because these channel has a bigger branching ratio. This translates in a B-meson facility with $10^8\Upsilon$ produced per year in a sensitivity to g_V^b of $2.1 \cdot 10^{-2}$. Other possibility is to use a combined channel as $e^-e^+ \rightarrow \tau^-\tau^+ \rightarrow (((\pi^-\pi^0)\nu_\tau)(\pi^+\bar{\nu}_\tau) + (\pi^-\nu_\tau)((\pi^+\pi^0)\bar{\nu}_\tau))$ where the sensitivity to the polarization of the $\tau^+ \rightarrow \pi^+\bar{\nu}_\tau$ is better than $\tau^+ \rightarrow \rho^+\bar{\nu}_\tau$ if we do not analyze the next decay $\rho^+ \rightarrow \pi^+\pi^0$. The result is,

$$\sigma_{g_V^b} = \frac{9.1}{\sqrt{N}} \quad (52)$$

and the number of events is similar to the previous case. The sensitivity to g_V^b that one can reach in this channel, with $10^8\Upsilon$, is $2.3 \cdot 10^{-2}$. As in the π channel we can increase the statistics by integrating one of the decays, although we lose some sensitivity. We integrate the direction of one of the ρ or equivalently the π direction in Eq. (25) and only require this τ to decay hadronically. Then we keep simply the decay of a τ to $\rho\nu$ and then the decay of ρ to $\pi\pi$. Then we get a sensitivity of,

$$\sigma_{g_V^b} = \frac{14.6}{\sqrt{N}} \quad (53)$$

but now the number of events has increased a factor of 10, because N is the number of $e^-e^+ \rightarrow \tau^-\tau^+ \rightarrow (((\pi^-\pi^0)\nu_\tau)\tau^+ + \tau^-((\pi^+\pi^0)\bar{\nu}_\tau))$ events. Again with $10^8\Upsilon$ per year, one gets a sensitivity of $1.6 \cdot 10^{-2}$.

After this analysis we can combine a series of independent measurements into a final value for g_V^b with an error given by

$$\sigma = \left(\sum_i \frac{1}{\sigma_i^2} \right)^{-1/2} \quad (54)$$

then we can combine the error obtained with the channels $e^-e^+ \rightarrow (\pi^-\nu_\tau)(\pi^+\bar{\nu}_\tau)$, $e^-e^+ \rightarrow (((\pi^-\pi^0)\nu_\tau)(\rho^+\bar{\nu}_\tau) + ((\rho^-\nu_\tau)((\pi^+\pi^0)\bar{\nu}_\tau))$ and $e^-e^+ \rightarrow (((\pi^-\pi^0)\nu_\tau)(\pi^+\bar{\nu}_\tau) + (\pi^-\nu_\tau)((\pi^+\pi^0)\bar{\nu}_\tau))$ and with $10^8\Upsilon$ one gets a sensitivity to g_V^b of $1.5 \cdot 10^{-2}$.

On the other hand, we can also combine the errors obtained in the $e^-e^+ \rightarrow (((\pi^-\pi^0)\nu_\tau)\tau^+ + \tau^-((\pi^+\pi^0)\bar{\nu}_\tau))$ and $e^-e^+ \rightarrow ((\pi^-\nu_\tau)\tau^+ + \tau^-(\pi^+\bar{\nu}_\tau))$ channels, and again with $10^8\Upsilon$ one gets a sensitivity of $1.3 \cdot 10^{-2}$.

Notice this result has been obtained with a sample of $10^8\Upsilon$, for a different number of Υ produced, the sensitivity to g_V^b would simply re-scale by a factor $\sqrt{10^8/N_\Upsilon}$.

5 Conclusions

In this work we have studied the possibilities of a high luminosity B-meson facility to measure with high precision the $Z - b\bar{b}$ vector coupling. At the energies of $\Upsilon(1S)$ we have used the $\tau^-\tau^+$ channel to determine this coupling through the τ polarizations. A complete analysis of the hadronic decay modes of the τ lepton has been done, with special attention to the $\tau^- \rightarrow \pi^-\nu_\tau$ and $\tau^- \rightarrow \rho^-\nu_\tau$ as polarization analyzers. We have built the complete correlated cross section with the decays of both τ s and from here we have calculated the ideal statistical errors obtainable in the measure of g_V^b . We have found that in a one year run in a B-meson facility with $10^8\Upsilon$ per year, one can get a sensitivity of $1.3 \cdot 10^{-2}$, comparable with the present precision in this coupling from the LEP/SLC measurements of R_b and A_b .

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APPENDIX A

In a decay, $A(\sigma) \rightarrow B(\lambda_b) + C(\lambda_c)$, we can obtain the density matrix, ρ^{out} , describing the complete final state of particles B and C in terms of helicity amplitudes and the initial density matrix in the following way,

$$\rho_{\vec{\lambda}, \vec{\lambda}'}^{out} = \sum_{\sigma\sigma'} f_{\vec{\lambda}, \sigma}(\Omega_1) \rho_{\sigma\sigma'}^{in}(\Omega) f_{\vec{\lambda}', \sigma'}^*(\Omega_1) \quad (\text{A.1})$$

where $\vec{\lambda} = (\lambda_b, \lambda_c)$.

It is very convenient to express our initial density matrix in a basis of irreducible tensor operators, $T_{L,M}$, with coefficients $t_{L,M}$, [4, 7],

$$\frac{\rho^{in}}{Tr(\rho^{in})} = \frac{1}{2j+1} \sum_{L,M}^{2j} (2L+1) t_{L,M}^{(a)*} T_{L,M}^{(a)} \quad (\text{A.2})$$

these coefficients, $t_{L,M}^{(a)}$, are the so-called multipole parameters,

$$Tr(\rho^{in}) t_{L,M}^{(a)}(\theta) = Tr(\rho^{in} T_{L,M}^{(a)}) = \sum_{\sigma, \sigma'} (\rho^{out})_{\sigma, \sigma'} C(1L1|\sigma M \sigma') \quad (\text{A.3})$$

where these $C(jLj|m'Mm)$ are Clebsch-Gordan coefficients.

For $L = 1$, we can relate, [7], the usual polarizations and the multipole parameters,

$$P_{x'} = -(t_{1,1} - t_{1,-1}) = -2 \text{Re}[t_{1,1}] \quad (\text{A.4})$$

$$P_{y'} = i(t_{1,1} + t_{1,-1}) = -2 \text{Im}[t_{1,1}] \quad (\text{A.5})$$

$$P_{z'} = \sqrt{2} t_{1,0} \quad (\text{A.6})$$

Using Eq. (A.2) to express the initial density matrix in the basis of irreducible tensors and replacing the helicity amplitudes in the C.M. frame of the decaying particle with Eq. (29), we get the final density matrix in terms of the multipole parameters of the A particle and the reduced helicity amplitudes,

$$\begin{aligned} \rho_{\vec{\lambda}, \vec{\lambda}'}^{out} &= Tr(\rho^{in}) \frac{\sqrt{2j+1}}{4\pi} \sum_{L,M} \sqrt{2L+1} (-)^{j-\lambda'} T_{\vec{\lambda}}^- T_{\vec{\lambda}'}^* \\ & t_{L,M}^{(a)*} C(j, j, L|\lambda', -\lambda, \lambda' - \lambda) \mathcal{D}_{M, \lambda - \lambda'}^{(L)*}(\Omega) \end{aligned} \quad (\text{A.7})$$

This density matrix contains all the information available in the process. For instance, if we want the angular distribution we just have to take the trace on $\vec{\lambda}$, and in the same way the density matrix for one of the final particles is obtained taking the trace on the helicities of the other particle.

To define a complete set of observables we generalize Eq. (A.2)

$$\frac{\rho^{out}}{Tr(\rho^{out})} = \frac{1}{(2j_1 + 1)(2j_2 + 1)} \sum_{L,M,L',M'}^{2j} (2L + 1)(2L' + 1) \mathcal{C}_{L,M,L',M'}^* T_{L,M}^{(b)} T_{L',M'}^{(c)} \quad (\text{A.8})$$

and then these generalized multipole parameters, $\mathcal{C}_{L,M,L',M'}$, include all the information on the density matrices of particles B and C, $t_{L,M}^{(b)} = \mathcal{C}_{L,M,0,0}$, $t_{L,M}^{(c)} = \mathcal{C}_{0,0,L,M}$ and additionally the correlations between them.

It is very important to notice here that Eq. (A.7) is only valid in the C.M. frame of the decaying particle. However, in general in the LAB frame the decaying particle will move with a momentum different from zero. So, we will be interested in the transformation properties of these density matrices under Lorentz boosts.

The transformation of an helicity amplitude under a boost from the C.M. frame to the LAB frame is just a rotation, the so-called Wigner rotation [12, 16, 17], that if we choose $\phi = 0$, is [7],

$$f_{\lambda_b \lambda_c \sigma}^{(CM)}(\theta, \phi = 0) = \sum_{\lambda'_b \lambda'_c} (-)^{\lambda'_c - \lambda_c} d_{\lambda'_b \lambda_b}^{j_b}(\omega_b) d_{\lambda'_c \lambda_c}^{j_c}(\omega_c) f_{\lambda'_b \lambda'_c \sigma}^{(LAB)}(\theta, \phi = 0) \quad (\text{A.9})$$

where this rotation, of angle ω_i , is given by [7],

$$\sin \omega_i = \frac{M_i \sinh \kappa \sin \theta}{|\vec{p}'_i|} \quad , \quad \cos \omega_i = \frac{E'_i E_i - \cosh \kappa M_i^2}{|\vec{p}_i| |\vec{p}'_i|} \quad (\text{A.10})$$

where κ is the parameter of the boost from the C.M. frame of the decaying particle to the LAB, related to the velocity by $v = \tanh \kappa$. M_i the mass of the particle and $(E_i, \vec{p}_i(\theta))$ its four-momentum in C.M. of the decaying particle, transformed under the boost as $p'_i = \Lambda p_i$, in a frame where the boost is along the z-axis. From Eq. (A.10) we can see that under a boost collinear to the particle three-momentum, $\theta = 0$, our states do not suffer any rotation.

The helicity amplitudes in Eq. (A.9) are functions of two variables. For instance, we could choose the invariant variables (s, t, u) , keeping the same

expression in any frame. Nevertheless, as we have seen in Eq. (29), these helicity amplitudes have a specially simple form in terms of C.M. variables. Then, we have used this freedom to express, both the C.M. and the LAB helicity amplitudes in Eq. (A.9) in terms of C.M. variables.

As can see in Eq. (A.1), density matrices are a product of two helicity amplitudes, this means that using Eq. (A.9) we can get the transformation of the density matrix of particle B

$$\rho^{LAB} = d^{j_b}(\omega_b) \cdot \rho^{CM} \cdot d^{j_b T}(\omega_b) \quad (\text{A.11})$$

It is very interesting to see how the multipole parameters are affected by these rotations. We use Eq. (A.3) to express ρ^{CM} in terms of these multipole parameters

$$\begin{aligned} \rho_{\sigma\sigma'}^{LAB} &= \frac{1}{2^{j+1}} \sum_{LM} (2L+1) \sum_{\lambda\lambda'} d_{\sigma\lambda}^{(j)}(\omega_a) (jLj|\lambda'M\lambda) d_{\sigma'\lambda'}^{(j) T}(\omega_a) t_{LM}^{(b)*} \\ &= \frac{1}{2^{j+1}} \sum_{LM} (2L+1) (jLj|\sigma'\sigma - \sigma') d_{\sigma-\sigma'M}^{(L) T}(\omega) t_{LM}^{(b)*} \end{aligned} \quad (\text{A.12})$$

and comparing again with Eq. (A.3) that in LAB we get a new set of multipole parameters that are obtained simply applying the rotation to the C.M. ones.

$$t_{LM}^{(b)*} = \sum_{M'} d_{MM'}^L(\omega_b) t_{LM'}^{(b)*} \quad (\text{A.13})$$

This is all we need to obtain the multipole parameters and density matrices in the LAB frame.

APPENDIX B

In this appendix, we present the general method to calculate reduced helicity amplitudes, and we apply it to some examples in the $e^+e^- \rightarrow \tau^+\tau^-$ processes.

Reduced helicity amplitudes are easily calculable by means of Eq. (3) from the helicity amplitudes. So, our first step will be to obtain these helicity amplitudes from the Feynman amplitudes we can calculate from the diagrams with Feynman rules. Taking into account the normalization we have defined for our reduced helicity amplitudes in Eqs. (3) and (29) the difference between them and the Feynman amplitudes will just be a q^2 -dependent

phase space factor, irrelevant in all our observables. So, we can simply define:

$$M_{\sigma,\lambda_+,\lambda_-}(\theta) = K f_{\sigma,\vec{\lambda}}(\theta) \quad (\text{B.1})$$

with $M_{\sigma,\lambda_+,\lambda_-}(\theta)$ the Feynman amplitudes.

Now we will explicitly apply this procedure to the calculation of the reduced helicity amplitudes $T_{(\lambda,\lambda'),\xi}(\gamma)$ and $T_{(\lambda,\lambda'),\xi}(\gamma Z_A)$ corresponding to diagrams (1) + (2) and (3) + (5) in Figure 1.

The Kinematics in the C.M. frame of the e^+e^- system is defined by

$$\begin{aligned} l_-^\mu &= (E, 0, 0, |\vec{l}|) & q^\mu &= (l_- + l_+)^\mu \\ l_+^\mu &= (E, 0, 0, -|\vec{l}|) & k^\mu &= (k^0, |\vec{k}| \sin \theta, 0, |\vec{k}| \cos \theta) \end{aligned} \quad (\text{B.2})$$

where l_\pm^μ is the four-momentum of the e^\pm , whose helicities are $\xi_\pm = \pm 1/2$, k^μ the four-momentum of the τ^- and the helicities of the τ^\pm will be denoted as $\lambda_\pm = \pm 1/2$.

The Feynman amplitudes corresponding to these diagrams are

$$M_{\lambda_-, \lambda_+, \xi_+, \xi_-}^\gamma(\theta) = i \frac{e^2}{s} \left(1 + \frac{e^2}{s} Q_b^2 |F_\Upsilon(q^2)|^2 P_\Upsilon(q^2) \right) V_\nu^\tau(\lambda_-, \lambda_+, \theta) g_{\mu\nu} V_\mu^{e*}(\xi_-, \xi_+) \quad (\text{B.3})$$

$$\begin{aligned} M_{\lambda_-, \lambda_+, \xi_-, \xi_+}^{\gamma Z_A}(\theta) &= i \frac{8G_F}{\sqrt{2}} g_A \left(\frac{e^2}{s} Q_b g_V^b |F_\Upsilon(q^2)| P_\Upsilon(q^2) - g_V \right) \\ &A_\nu^\tau(\lambda_-, \lambda_+, \theta) g_{\mu\nu} V_\mu^{e*}(\xi_-, \xi_+) \end{aligned} \quad (\text{B.4})$$

where we have followed the notation of sections 2, and we have introduced the matrix elements of the leptonic currents,

$$\begin{aligned} V_l^\mu(\lambda_-, \lambda_+) &= \bar{u}(p_-, \lambda_-) \gamma^\mu v(p_+, \lambda_+) \\ A_l^\mu(\lambda_-, \lambda_+) &= \bar{u}(p_-, \lambda_-) \gamma^\mu \gamma_5 v(p_+, \lambda_+) \end{aligned} \quad (\text{B.5})$$

Now we need to obtain an explicit expression for these matrix elements. To do this, we follow the method of reference [18], which permits the calculation of these amplitudes using standard trace techniques. Then, the

complete results for the vector and axial currents, in the CM frame and with the momenta along the z -axis are,

$$\begin{aligned}
V_l^\mu(\lambda_-, \lambda_+ = \lambda_-) &= (0, 0, 0, -2M_l) \\
V_l^\mu(\lambda_-, \lambda_+ = -\lambda_-) &= (0, 4E_p\lambda_-, -2E_p i, 0) \\
A_l^\mu(\lambda_-, \lambda_+ = \lambda_-) &= (-4M_l\lambda_-, 0, 0, 0) \\
A_l^\mu(\lambda_-, \lambda_+ = -\lambda_-) &= (0, 2p, -4ip\lambda_-, 0)
\end{aligned} \tag{B.6}$$

With these results we have all the leptonic currents we need, because they are perfectly behaved Lorentz vectors or axial-vectors. Then, we just have to rotate them, if the momentum is in a different direction.

With all these elements, we simply use Eq. (B.1) and Eq. (3) with our Feynman amplitudes, Eqs. (B.3) and (B.4), to obtain the reduced helicity amplitudes. For instance, the expression for $M_{\lambda=(1/2,1/2),\bar{\xi}=(1/2,-1/2)}^\gamma(\theta)$ is,

$$\begin{aligned}
M_{(1/2,1/2),(1/2,-1/2)}^\gamma(\theta) &= i\frac{e^2}{s}(1 + \frac{e^2}{s}Q_b^2|F_\Upsilon(q^2)|^2P_\Upsilon(q^2)) \\
(0, -2M_\tau \sin \theta, 0, -2M_\tau \cos \theta) \cdot (0, -2\frac{\sqrt{s}}{2}, -2i\frac{\sqrt{s}}{2}, 0)^T &= \\
i\frac{e^2}{s}(1 + \frac{e^2}{s}Q_b^2|F_\Upsilon(q^2)|^2P_\Upsilon(q^2))4M_\tau\frac{\sqrt{s}}{2}(-\sqrt{2})\left(\frac{-\sin \theta}{\sqrt{2}}\right) & \tag{B.7}
\end{aligned}$$

where we have applied a rotation to the leptonic current of the τ with respect to Eq. (B.6) and we have taken the complex conjugate of Eq. (B.6) to obtain the electron current. The extra minus sign is due to the metric $g_{\mu\nu}$. In Eq. (B.7) we just have to remove the rotation matrix element $d_{10}(\theta)$, which is exactly the last term in this equation. This procedure has to be repeated with all the amplitudes and then, finally we get the following results

$$KT_{(+,+),1}(\gamma) = -i4\sqrt{2}\frac{e^2}{s}(1 + \frac{e^2}{s}Q_b^2|F_\Upsilon|^2P_\Upsilon)M_\tau\frac{\sqrt{s}}{2} \tag{B.8}$$

$$KT_{(+,-),1}(\gamma) = -i8\frac{e^2}{s}(1 + \frac{e^2}{s}Q_b^2|F_\Upsilon|^2P_\Upsilon)p^0\frac{\sqrt{s}}{2} \tag{B.9}$$

$$KT_{(+,+),1}(\gamma Z_A) = 0 \tag{B.10}$$

$$KT_{(+,-),1}(\gamma Z_A) = -i8 \frac{8G_F}{\sqrt{2}} g_A \left(\frac{e^2}{s} Q_b g_V^b |F_T|^2 P_T - g_V \right) |\vec{p}| \frac{\sqrt{s}}{2} \quad (\text{B.11})$$

In the same way, we can obtain all the reduced helicity amplitudes in Eq. (8).

References

- [1] R.M. Barnett *et al.*, *Phys. Rev.* **D54**, 1 (1996)
- [2] LEP collaborations, Electroweak Working Group and SLD Heavy Flavour Group preprint *CERN-PPE/96-183*
- [3] D. Schaile, *Precision tests of the Standard Electroweak Model*, ed. P. Langacker (World Scientific, 1995), p. 215
- [4] J. Bernabeu, F.J. Botella and O. Vives, *Nucl. Phys.* **B 472**, 659 (1996)
- [5] R. Koniuk, R. Leroux and N. Isgur, *Phys. Rev.* **D17**, 2915 (1978)
J. Bernabeu and P.Pascual, *Phys. Lett.* **B87**, 69 (1979)
- [6] J. Bernabeu and P. Pascual, *Nucl. Phys.* **B 172**, 93 (1980)
- [7] A.D. Martin and T.D. Spearman, *Elementary Particle Theory* (North-Holland , Amsterdam, 1970).
- [8] B.K. Bullock, K. Hagiwara and A.D. Martin, *Phys. Lett.* **B273**, 501 (1991)
- [9] J.H. Kühn, *Phys. Lett.* **B313**, 458 (1993)
- [10] A. Rouge, *Z. Phys* **C48**, 75 (1990)
- [11] C.A. Nelson, *Phys. Rev.* **D43**, 1465 (1991)
C.A. Nelson *et al.*, *Phys. Rev.* **D50**, 4544 (1994)
- [12] E.P. Wigner, *Ann. Math.* **40**, 194 (1939)
E.P. Wigner, *Rev. Mod. Phys.* **29**, 255 (1957)
H.P. Stapp, *Phys. Rev.* **103**, 425 (1957)
- [13] H. Cramer, *Mathematical Methods of Statistics*, Princeton University Press, 1946
- [14] P. Franzini, *The DAΦNE Physics Handbook*, eds. L. Maiani, G. Pancheri and N. Paver (INFN, Frascati, 1992), p.15
- [15] C.A. Nelson *Phys. Rev.* **D40**, 123 (1989)

- [16] G.C. Wick *Ann. Phys.* **18**, 65 (1962)
- [17] G. Barenboim, J. Bernabeu and O. Vives *Phys. Rev. Lett.* **77**, 3299 (1996)
- [18] H.W. Fearing and R.R. Silvar *Phys. Rev.* **D6**, 471 (1972).

Figure Captions

Figure 1: Resonant and non-resonant diagrams for the process $e^+e^- \rightarrow \tau^+\tau^-$ at the Υ region.

Figure 2: Longitudinal τ^- polarization, $P_{z'}(\theta)$

Figure 3: Transverse τ^- polarization, $P_{x'}(\theta)$

Table Captions

Table 1: Dominant $\Upsilon(1S)$ decay channels.

Table 2: Dominant τ decay channels.

Table 1

| $\Upsilon(1S) \quad I^G(J^{PC}) = 0^-(1^{--})$ | |
|--|--------------------------------|
| Mass $M_\Upsilon = 9460.37 \pm 0.21 \text{ MeV}$ | |
| Width $\Gamma = 52.5 \pm 1.8 \text{ KeV}$ | |
| Decay modes | Fraction Γ_i/Γ |
| $\tau^+\tau^-$ | $(2.97 \pm 0.35)\%$ |
| $\mu^+\mu^-$ | $(2.52 \pm 0.17)\%$ |
| e^+e^- | $(2.48 \pm 0.07)\%$ |
| $J/\psi(1S) \text{ anything}$ | $(1.1 \pm 0.4) \times 10^{-3}$ |
| $\gamma 2h^+2h^-$ | $(7.0 \pm 1.5) \times 10^{-4}$ |
| $\gamma 3h^+3h^-$ | $(5.4 \pm 2.0) \times 10^{-4}$ |
| $\gamma 4h^+4h^-$ | $(7.4 \pm 3.5) \times 10^{-4}$ |

Table 2

| $\tau \quad J = \frac{1}{2}$ | |
|--|----------------------------|
| Mass $M_\tau = 1777.00_{-0.27}^{+0.30} \text{ MeV}$ | |
| Mean life $\tau = (291.0 \pm 1.5) \times 10^{-15} \text{ s}$ | |
| Decay modes | Fraction Γ_i/Γ |
| $\mu^-\bar{\nu}_\mu\nu_\tau$ | $(17.35 \pm 0.10)\%$ |
| $e^-\bar{\nu}_e\nu_\tau$ | $(17.83 \pm 0.08)\%$ |
| $\pi^-\nu_\tau$ | $(11.31 \pm 0.15)\%$ |
| $\pi^-\pi^0\nu_\tau$ | $(25.24 \pm 0.16)\%$ |
| $h^-\pi^0\nu_\tau$ | $(9.50 \pm 0.14)\%$ |
| $h^-h^-\nu_\tau$ | $(9.80 \pm 0.10)\%$ |

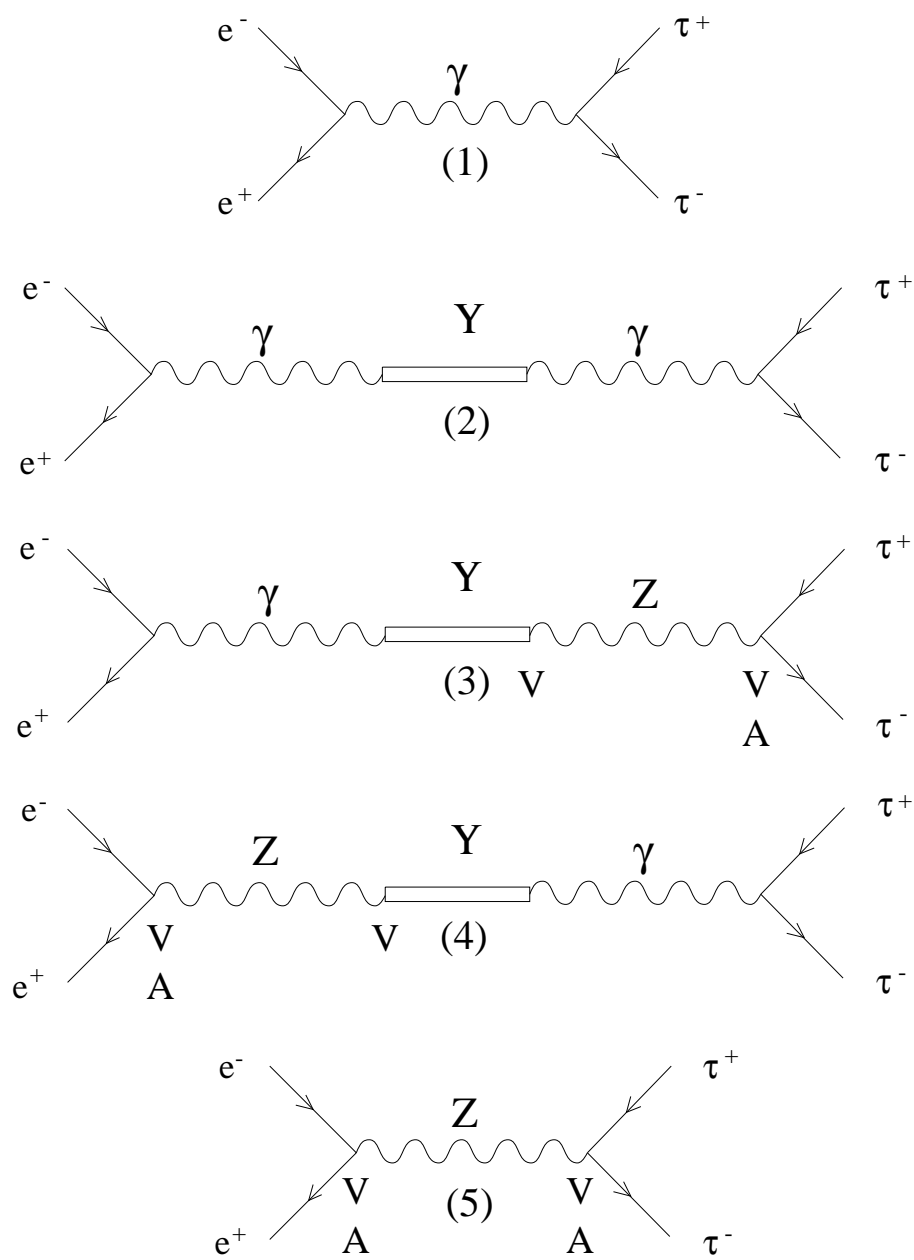


Fig. 1

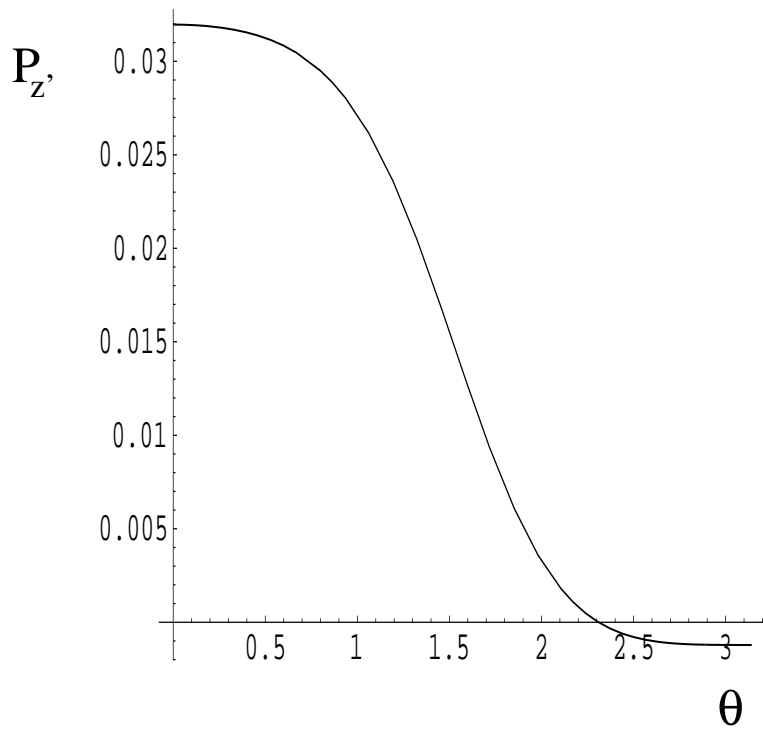


Fig. 2

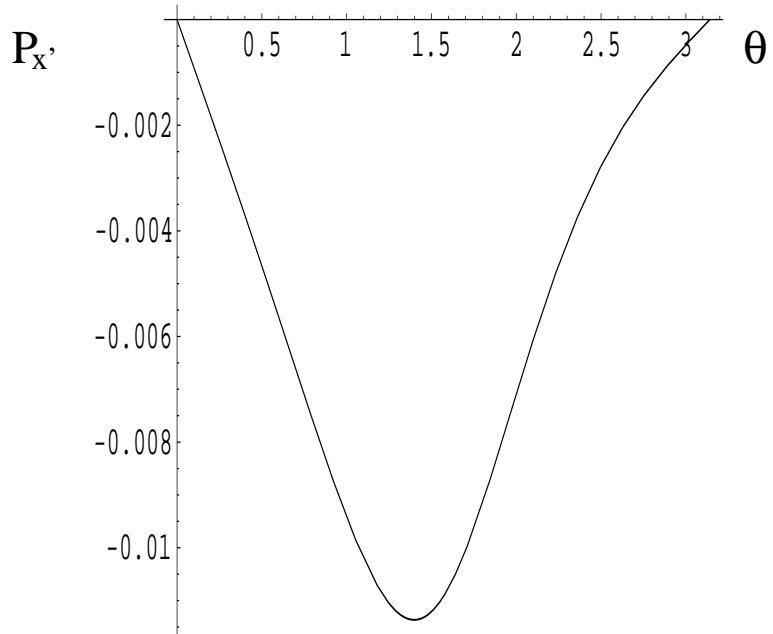


Fig. 3