On norm attaining operators and multilinear maps

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I, Francisco J. Falcó Benavent, declare that this thesis titled, "On norm attaining operators and multilinear maps" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a degree of Doctor in Mathematics (University of Valencia) at this University.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.

Signed: ______________________________

Date: ______________________________

To Maika for all the cups of coffee during the long nights.
To Domingo, Manolo and Pilar for their dedication and wise advice.

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"If I have seen further it is by standing on the shoulders of giants."

— Isaac Newton

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Abstract

"A mathematician is a machine for converting coffee into theorems."

— Alfred Rényi

The main point of interest of this thesis is to study extensions of the Bishop-Phelps theorem and Bishop-Phelps-Bollobás theorem to different contexts. This thesis is divided into three chapters. In the first one we do a summary of the state of the art about norm attaining linear forms and we introduce the Bishop-Phelps and Bishop-Phelps-Bollobás Theorems.

The second chapter is devoted to the study of operator versions of Bishop-Phelps and Bishop-Phelps-Bollobás Theorems. In Section 2.2 we will study the extension of these results to the operator case from the point of view of attaining the numerical radius to conclude in Section 2.3.1 that the space L_1 satisfy the Bishop-Phelps-Bollobás Property for Numerical Radius. To finish, we will present the Lindenstrauss' result about norm attaining extensions of operator, which will be the motivation of our study from Section 3.2 to Section 3.6 in the next chapter.

In the third chapter, we extend the theory of norm attaining linear forms to the non-linear case. Focusing on the line of work initiated by Lindenstrauss, our main point of interest is to study whether the extensions of multilinear maps to the bidual are norm attaining, with special interest on multilinear forms over the space ℓ_1, see Sec-

tions 3.4 and 3.5. To finish, in Section 3.6 we will study the dependence of the Lindenstrauss-Bollobás Theorems introduced by Carando, Lassalle and Mazzitelli in [CLM12], Definition 3.6.1, and the n-linear version of Bishop-Phelps-Bollobás Theorem for spaces M-embedded or L-embedded in the bidual.

Resumen

"Un matemático es una máquina para convertir café en teoremas."

— Alfred Rényi

El principal punto de interés de esta tesis es el estudio de extensiones de los teoremas de Bishop-Phelps y Bishop-Phelps-Bollobás a diferentes contextos. Esta tesis se divide en tres capítulos. En el primero hacemos un repaso de la teoría de funcionales que alcanzan la norma. En este resumen introducimos el Teorema de Bishop-Phelps y el Teorema de Bishop-Phelps-Bollobás.

El segundo capítulo está dedicado al estudio de extensiones de los resultados de Bishop-Phelps y Bishop-Phelps-Bollobás al caso de operadores. En la sección 2.2 estudiaremos la extensión de estos resultados al caso de operadores desde el punto de vista de alcanzar el radio numérico, para concluir en la sección 2.3.1 que el espacio L_1 satisface la Propiedad de Bishop-Phelps-Bollobás para el Radio Numérico. Concluimos esta sección presentando el resultado de Lindenstrauss que establece que el conjunto de operadores cuya extensión al bidual alcanza la norma es denso. Este resultado es la motivación de nuestro estudio en las secciones 3.2 - 3.6 del próximo capítulo.

En el tercer capítulo, extendemos la teoría de formas lineales que alcanzan la norma al caso no lineal. Motivados por la línea de trabajo iniciada por Lindenstrauss, nuestro principal interés es estudiar el

comportamiento de las extensiones al bidual de funciones multilineales desde el punto de vista de alcanzar la norma. En particular nos centramos en el estudio de las extensiones de formas multilineales sobre el espacio ℓ_1, véanse las secciones 3.4 y 3.5. Para finalizar, en la sección 3.6 estudiaremos la relación entre los teoremas de Lindenstrauss-Bollobás introducidos por Carando, Lassalle y Mazzitelli en [CLM12], Definición 3.6.1, y la versión n-lineal del Teorema de Bishop-Phelps-Bollobás para espacios M-embedded o L-embedded en su bidual.

Resum

> *"Un matemàtic és una màquina per convertir cafè en teoremes."*
>
> — Alfred Rényi

El principal punt d'interés d'aquesta tesi és l'estudi d'extensions dels teoremes de Bishop-Phelps i Bishop-Phelps-Bollobás a diferents contextos. Aquesta tesi es divideix en tres capítols. En el primer fem un repàs de la teoria de formes lineals que alcancen la norma. En aquest resum introduïm el Teorema de Bishop-Phelps i el Teorema Bishop-Phelps-Bollobás.

El segon capítol està dedicat a l'estudi d'extensions dels resultats de Bishop-Phelps i Bishop-Phelps-Bollobás al cas d'operadors. A la secció 2.2 estudiarem l'extensió d'aquests resultats al cas d'operadors des del punt de vista d'alcançar el radi numèric, per a concloure a la secció 2.3.1 que l'espai L_1 verifica la Propietat de Bishop-Phelps-Bollobás per al Radi Numèric. Concluïm aquesta secció presentant el resultat de Lindenstrauss el cual estableix que el conjunt d'operadors verificant que la seva extensió al bidual alcança la norma és dens. Aquest resultat és la motivació del nostre estudi a les seccions 3.2 - 3.6 del pròxim capítol.

En el tercer capítol, extenem la teoria de formes lineals que alcancen la norma al cas no lineal. Motivats per la línia de treball iniciada per Lindenstrauss, el nostre principal interés és estudiar el comportament de les extensions al bidual de funcions multilineals des del punt de vis-

ta d'alcançar la norma. En particular ens centrem en l'estudi de les extensions de les formes multilineals sobre l'espai ℓ_1, com podeu comprovar a les seccions 3.4 y 3.5. Per finalitzar, a la secció 3.6 estudiarem la relació entre els teoremes de Lindenstrauss-Bollobás introduïts per Carando, Lassalle i Mazzitelli a [CLM12], Definició 3.6.1, i la versió n-lineal del Teorema de Bishop-Phelps-Bollobás per a espais M-embedded o L-embedded en el seu bidual.

Contents

Chapter 1

Norm attaining linear forms

"Begin at the beginning," the King said gravely, "and go on till you come to the end: then stop."
— Lewis Carroll, *Alice in Wonderland*

1.1 Introduction

This thesis is devoted to the study of norm attaining polynomials and multilinear maps on Banach spaces.

In this section we set forth the basic definitions and present the classical results of the norm attaining theory of functionals on Banach spaces and some of its important consequences. Also, here we present some of the classical results of the geometry of Banach spaces that are relevant for our purposes.

In this context, X denotes a real or complex Banach space and B_X, S_X denote respectively the closed unit ball and the unit sphere of the Banach space X. X^* and X^{**} stand for the topological dual and the bidual space respectively. As usual we denote the sign of a real or complex number by

$$sign(z) = \begin{cases} \dfrac{z}{|z|} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

First of all we recall that the definition of norm of a linear and continuous form f defined on X is

$$\|f\| = \sup_{x \in B_X} |f(x)|. \tag{1.1}$$

Naturally one can ask whether this supremum can be changed by a maximum, or equivalently whether there exists $x \in B_X$ with $f(x) = \|f\|$. If this is the case, we say that the linear form f attains its norm, or f is norm attaining. From a geometric point of view, a linear and continuous form f attains its norm means that the intersection of the affine hyperplane $\{x \in X : f(x) = 1\}$ and the closed unit ball of X is non-empty.

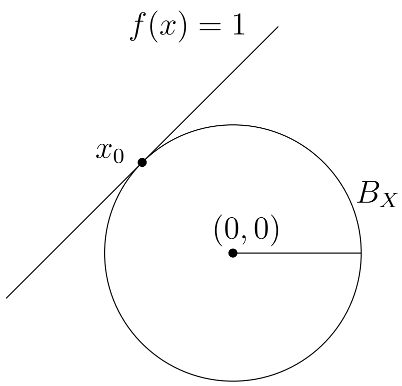

Figure 1.1 Norm attaining linear bounded form.

We will denote by $NA(X)$ the set of norm attaining linear and continuous forms on X.

Note that for the case where X is a finite dimensional space the Weierstrass Theorem and the Heine-Borel theorem tell us that every linear continuous form attains its norm. Therefore, from now on, we will consider only infinite dimensional Banach spaces, unless otherwise specified. In general we will be dealing with real Banach spaces, but some of the results can also be applied to complex Banach spaces. If this is the case we will indicate that the result holds for real and complex Banach spaces.

Definition 1.1.1 (Extreme point)**.** An extreme point of a convex set, C, is a point $x \in C$ with the property that if $x = \omega y + (1 - \omega)z$ with $y, z \in C$ and $\omega \in [0, 1]$, then $y = x$ and/or $z = x$.

In other words, an extreme point is a point that is not an interior point of any line segment lying entirely in C. Then, by the convexity of the closed unit ball, if for a Banach space X the set of extreme points of B_X is non-empty and the unit ball is the convex hull of its extreme points, we have that all the norm attaining forms of X attain their norm at extreme points. In particular, for dual Banach spaces X^*, by the Krein-Milman Theorem, B_{X^*} is the convex hull of its extreme points. Hence the norm of the functionals is the supremum of the modulus of the image of the functional over the extreme points.

Naturally the idea of attaining the norm can be generalized if we consider a locally convex space and we substitute the closed unit ball for any convex subset of the locally convex space. This leads us to the following definition.

Definition 1.1.2. (Klee, 1958) Let $C \subset E$ be a convex subset of a real locally convex space E: A point x_0 in the boundary of C is a support point if exists $f \in E^*$ such that $f(x_0) = \sup_{x \in C} f(x)$ (and then f is a support functional).

Klee, [Kle58], asked the following question,

Question 1.1.3. Must a bounded closed convex subset C of a Banach space necessarily have any support point?

In 1927, Hahn introduced the concept of reflexive space. Given a Banach space X over the field of real or complex numbers $\mathbb{K}$, there is a natural embedding of the space X in its bidual X^{**} given as follows: for any point $x \in X$ we define the bounded linear map δ_x from X^* to $\mathbb{K}$ by $\delta_x(f) = f(x)$ i.e.,

$$\begin{array}{rclcl} \delta : X & \hookrightarrow & X^{**} & & \\ x & \rightsquigarrow & \delta_x : \; X & \to & \mathbb{K} \\ & & f & \rightsquigarrow & \delta_x(f) = f(x). \end{array}$$

It is easy to see that this inclusion is in fact a linear isometry and a Banach space X is called reflexive if this isometry is an onto map. R. C. James proved the following result.

Theorem 1.1.4 (James). *A Banach space X is reflexive if and only if every linear form attains its norm.*

In terms of supporting functionals, as a consequence of the James Theorem, every linear and continuous form on a reflexive Banach space is a support functional of the closed unit ball of the space.

However if the space is not reflexive, as a consequence of the James Theorem, there are bounded linear forms that do not attain their norm. In fact, it is not hard to find specific examples of non norm attaining linear forms for spaces like the spaces of real or complex absolutely summable sequences, denoted by ℓ_1.

Example 1.1.5. Given the real or complex Banach space ℓ_1, the linear and continuous form

$$f\Big((x_i)_{i=1}^{\infty}\Big) = \sum_{i=1}^{\infty} \frac{i}{i+1} x_i$$

has norm one, but for all point $x \in B_{\ell_1}$, $|f(x)| < 1$. Hence f is not norm attaining.

The first positive result about norm attaining linear forms for general Banach spaces is obtained as a consequence of the Hahn-Banach Theorem.

Theorem 1.1.6 (Hahn-Banach, $\mathbb{R}$-version)**.** *Let Y be a subspace of a real linear space X, and let p be a positively homogeneous sublinear functional on X. If f is a linear functional on Y such that $f(x) \leq p(x)$ for every $x \in Y$, then there exists a linear functional $\tilde{f}$ on X such that $\tilde{f} = f$ on Y and $\tilde{f}(x) \leq p(x)$ for every $x \in X$.*

A function $p : X \mapsto \mathbb{R}$ on a vector space X is called a positively homogeneous sublinear functional if for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$ with $\alpha \geq 0$

$$p(\alpha x) = \alpha p(x) \text{ and } p(x+y) \leq p(x) + p(y).$$

Then, for any real Banach space X, if we fix a point $x_0 \in S_X$ and we consider $Y = span\{x_0\}$, the function

$$\begin{array}{rccc} f: & span\{x_0\} & \mapsto & \mathbb{R} \\ & \lambda x_0 & \rightsquigarrow & \lambda, \end{array}$$

is bounded by the norm function that is a positively homogeneous sublinear function. Therefore, there exists a linear function $\tilde{f} \in X^*$ with $\tilde{f}(x) \leq \|x\|$. Since the function $\tilde{f}$ is linear, we have $-\tilde{f}(x) = \tilde{f}(-x) \leq$

$\|-x\| = \|x\|$, hence $|\tilde{f}(x)| \leq \|x\|$. Therefore $\|\tilde{f}\| \leq 1 = |\tilde{f}(x_0)|$, hence the function f is norm attaining at the point x_0.

Corollary 1.1.7 (Hahn-Banach Separation Theorem, $\mathbb{R}$-version)**.** *Let X be a real Banach space and C and D two convex sets with interior of C, denoted by int(C), being non-empty. If int$(C) \cap D = \emptyset$, then there exists $f \in X^*$ and there exists a real number c such that $f(y) \geq c > f(x)$ for all y in D and for all x in int(C).*

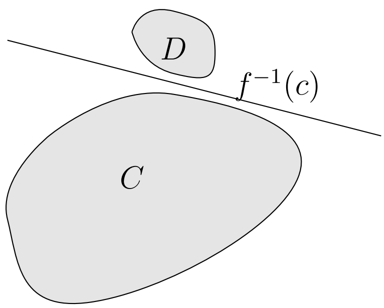

Figure 1.2 Hahn-Banach Separation Theorem.

An alternative proof of the existence of norm attaining functionals in real Banach spaces can be obtained using the Hahn-Banach Separation Theorem with the sets $C = B_X$ and $S = \{x\}$, where x is any point in S_X.

Now, we can extend this result to the complex case if we consider seminorms instead of positively homogeneous sublinear functionals.

Definition 1.1.8 (Seminorm)**.** Given a vector space X over the field of complex numbers $\mathbb{C}$. A seminorm is a function $p : X \mapsto \mathbb{R}$ such that for all $x, y \in X$ and for all $\alpha \in \mathbb{C}$

$$p(\alpha x) = |\alpha| p(x) \text{ and } p(x+y) \leq p(x) + p(y).$$

Theorem 1.1.9 (Hahn-Banach, $\mathbb{C}$-version)**.** *Let Y be a subspace of a complex linear space X, and let p be a seminorm on X. If f is a linear functional on Y such that $|f(x)| \leq p(x)$ for every $x \in Y$, then there exists a linear functional $\tilde{f}$ on X such that $\tilde{f} = f$ on Y and $|\tilde{f}(x)| \leq p(x)$ for every $x \in X$.*

The proof of the existence of norm attaining linear forms on complex Banach spaces is analogous to the ones in the case of real Banach spaces.

This answers in the affirmative the question asked by Klee when we restrict our attention to the unit ball of Banach spaces.

1.2 The Bishop-Phelps Theorem

James Theorem tells us that for a non-reflexive Banach space X the set of norm attaining linear bounded forms is not X^* and as a consequence of the Hahn-Banach Separation Theorem we know that this set is non-empty. So the following natural question araises

Question 1.2.1. How big is the set of norm attaining linear forms for an infinite dimensional Banach space?

In 1961 Bishop and Phelps proved what is today known as the Bishop-Phelps Theorem. Which is one of the most important results in functional analysis. The Bishop-Phelps Theorem states that every real or complex Banach space is subreflexive i.e. for every real or complex Banach space the set of linear and continuous functionals that attain their norm is norm-dense in its dual space.

Theorem 1.2.2 (Bishop-Phelps, [BP61])**.** *In any real or complex Banach space X, the linear functionals in X^* which attain their supremum on the unit ball of X are norm-dense in X^*.*

The proof relies on the use of Zorn's lemma on a certain partial ordering, defined by means of a convex cone, to get a point on the boundary of a specific set and then employing the Hahn-Banach Separation Theorem.

The same proof shows that the result is not only true for the unit ball of Banach spaces but also for any arbitrary bounded closed convex subset of a real Banach space, answering in the affirmative the question posted by Klee for real Banach spaces.

Using similar arguments Ekeland's Variational Principle and Brønsted-Rockafellar Principle can be proved.

Recall that if X is a Banach space and f an extended real-valued function on X, i.e.

$$f : X \mapsto \mathbb{R} \cup \{-\infty, +\infty\},$$

the effective domain of f is the set $\{x \in X : f(x) < +\infty\}$. The function is called proper if f is not identically $+\infty$ (i.e. the effective domain is not empty) and it never attains $-\infty$. We say that f is lower semicontinuous provided $\{x \in X : f(x) \leq r\}$ is closed in X for every $r \in R$, and a function f is convex if for every two points $x, y \in X$ and for every $t \in [0, 1]$ we have $f\Big(tx + (1-t)y\Big) \leq tf(x) + (1-t)f(y)$.

Theorem 1.2.3 (Ekeland's Variational Principle, [Phe93])**.** *Let $f : X \mapsto \mathbb{R} \cup \{-\infty, +\infty\}$ be a proper lower semicontinuous function that is bounded from below. Let $\epsilon > 0$ and suppose that at a given point x_0,*

$$f(x_0) \leq \inf_{x \in X} f(x) + \epsilon.$$

Then for any $\lambda > 0$, there exists z in the domain of f so that:

- $\lambda \|z - x_0\| \leq f(x_0) - f(z)$,

- $\|z - x_0\| \leq \epsilon/\lambda$,
- $\lambda\|x - z\| + f(x) > f(z)$, *whenever* $x \neq z$.

And as a consequence of the Ekeland's Variational Principle we can prove the Brønsted-Rockafellar Principle, which studies the differentiability properties of convex functions.

Definition 1.2.4 (Subdifferential)**.** Let f be a proper convex lower semicontinuous function on X, x an element of the domain of f and $\epsilon > 0$. For each ϵ define the ϵ-subdifferential $\partial_\epsilon f(x)$ by

$$\partial_\epsilon f(x) = \{x^* \in X : x^*(y) - x^*(x) \leq f(y) - f(x) + \epsilon \text{ for all } y \in X\}.$$

For each $\epsilon > 0$ and x in the domain of f, $\partial_\epsilon f(x)$ is non-empty, and $\partial_\epsilon f(x)$ is a weak-star closed set in X^*. As ϵ decreases, so does $\partial_\epsilon f(x)$. The intersection over ϵ of the nets $\partial_\epsilon f(x)$ is the subdifferential

$$\partial f(x) = \{x^* \in X : x^*(y) - x^*(x) \leq f(y) - f(x) \text{ for all } y \in X\}.$$

Theorem 1.2.5 (Brønsted-Rockafellar Principle, [Phe93])**.** *Assume that f is a convex proper lower semicontinuous function on X. Given x_0 in the domain of $f, \epsilon > 0, \lambda > 0$ and any $x_0^* \in \partial_\epsilon f(x_0)$, there exist vectors x in the domain of f and $x^* \in \partial_f(x)$ such that $\|x - x_0\| \leq \epsilon/\lambda$ and $\|x^* - x_0^*\| \leq \lambda$.*

Ekeland's Variational Principle and Brønsted-Rockafellar Principle can be used to prove the Bishop-Phelps Theorem. See for instance [FHH$^+$01] for a proof of Bishop-Phelps Theorem using Ekeland's Variational Principle and [Phe93] for a proof of the Theorem using Brønsted-Rockafellar Principle.

Naturally one can ask about the generalization of this result to the complex case. As Bishop and Phelps mentioned in their paper,

> "The theorem mentioned in the title will be proved for real Banach spaces; the result for complex spaces follows from this by considering the spaces over the real field and using the known isometry between complex functionals and the real functionals defined by their real parts."

Unfortunately in the complex case this argument only holds for balanced sets (like the unit ball of any Banach space) i.e., sets $C \subset X$ such that $\alpha C = C$ for all complex number α of modulus one.

In 1975 at a conference at Kent State University, Gilles Godefroy raised the question of whether there is a valid version of the Bishop-Phelps Theorem in complex spaces.

Question 1.2.6. Suppose that C is a bounded closed convex subset of a complex Banach space X. Must the functionals $f \in X^*$ which satisfy $\sup\{|\langle f, y\rangle| : y \in C\} = |\langle f, x\rangle|$ for some $x \in C$ be dense in X^*?

As we mentioned before, the proof of Bishop-Phelps Theorem answers this question in affirmative for balanced sets and Phelps proved in [Phe92] that this is also true for arbitrary bounded closed convex sets in Banach spaces having the Radon-Nikodým Property, see Definition 2.1.6. But it is not true in general for arbitrary convex sets in complex Banach spaces.

In fact, Lomonosov gave an example of a closed bounded convex set with no support points, answering in the negative the questions proposed by Klee and Godefroy.

For this, Lomonosov uses the algebra of analytic and bounded functions on the open unit disk $\mathbb{D}$, denoted by $\mathcal{H}^\infty(\mathbb{D})$ with the identity function E and endowed with the supremum norm, $\|f\| = \sup_{z\in\mathbb{D}} |f(z)|$, for all $f \in \mathcal{H}^\infty(\mathbb{D})$. Notice that every point $z \in \mathbb{D}$ defines a point evaluation function δ_z on $\mathcal{H}^\infty(\mathbb{D})$ by

$$\begin{array}{rccl} \delta_z : & \mathcal{H}^\infty(\mathbb{D}) & \hookrightarrow & \mathbb{C} \\ & f & \rightsquigarrow & \delta_z(f) = f(z). \end{array}$$

It is known that the space $\mathcal{H}^\infty(\mathbb{D})$ can be identified with the dual space of some Banach space X in such a way that every element δ_z is an element in X. Let S be the convex hull of the elements $\{\delta_z\}_{z\in\mathbb{D}}$. Then, Lomonosov proved the following result

Theorem 1.2.7 ([Lom00, Theorem 1]). *Suppose that the modulus of the functional $g \in \mathcal{H}^\infty(\mathbb{D})$ attains its maximum on the set S. Then there exists a complex number α such that $g = \alpha E$.*

Therefore the set of functionals attaining its maximum modulus at S is a 1-dimensional linear space. Hence, the set of functionals attaining its maximum modulus at S cannot be dense.

Coming back to the proof of the Bishop-Phelps Theorem, in some specific spaces the proof of the Bishop-Phelps Theorem can be done constructively by finding a norm dense set of norm attaining functionals. Now we present the case of the space ℓ_1 and the case of the space c_0 that will be of interest for our purposes. Here c_0 stands for the spaces of real or complex sequences converging to zero.

Proposition 1.2.8. *The set of norm attaining linear forms in ℓ_1 is dense in ℓ_1^*, and*

$$NA(\ell_1) = \{f \in \ell_1^* : \exists n_0 \in \mathbb{N} \text{ such that } f(e_{n_0}) = \|f\|\}.$$

Proof. Let denote by $\{e_k\}_{k=1}^\infty$ the canonical basis of ℓ_1. Given a linear and continuous form f on ℓ_1, we can identify f with a sequence of ℓ_∞, $\{f_n\}_{n=1}^\infty$. Without loss of generality we can assume that f has norm one. Then, given $\epsilon > 0$, there exists a natural number n_0 with $|f_{n_0}| > 1 - \epsilon$. Consider the new linear and continuous form g on ℓ_1 defined by

$$g_n = \begin{cases} f_n, & \text{if } n \neq n_0, \\ \dfrac{f_{n_0}}{|f_{n_0}|}, & \text{if } n = n_0. \end{cases}$$

Then, $\|f - g\| = |(f-g)(e_{n_0})| < \epsilon$ and $|g(e_{n_0})| = \left|\frac{f_{n_0}}{|f_{n_0}|}\right| = 1 = \|g\|$, so g is norm attaining.

For the second part, notice that if f is such that $f(e_{n_0}) = \|f\|$ for some natural number n_0, then f attains its norm at e_{n_0}. Hence $NA(\ell_1) \supseteq \{f \in \ell_1^* : \exists n_0 \in \mathbb{N} \text{ such that } f(e_{n_0}) = \|f\|\}$. On the other hand, if f attains its norm at a point $x \in B_{\ell_1}$, then $\|f\| = |\sum_{n=1}^{\infty} f(x_n e_n)| \leq \sum_{n=1}^{\infty} |f(x_n e_n)| \leq \|f\| \sum_{n=1}^{\infty} |x_n| \leq \|f\|$. Therefore, for any n_0 with $x_{n_0} \neq 0$ we have $\|f\| = \|f(e_{n_0})\|$, which concluded the proof. □

For the case of c_0 we can describe the set of norm attaining linear and continuous forms in the following way.

Proposition 1.2.9.

$$NA(c_0) = \{f \in c_0^* : \exists n_0 \in \mathbb{N} \textit{ such that } f(e_k) = 0 \textit{ if } k > n_0\}.$$

Proof. First we check that

$$NA(c_0) \subseteq \{f \in c_0^* : \exists n_0 \in \mathbb{N} \text{ such that } f(e_k) = 0 \text{ if } k > n_0\}.$$

Fix $f \in NA(c_0)$ of norm one. Then, there exists $x \in B_{c_0}$ such that $f(x) = 1$. We will check that for every natural number k with $|x(k)| < 1$ then $f(e_k) = 0$. Assume this is not the case. Then there exists k_0 with $|x(k_0)| < 1$ and $f(e_{k_0}) \neq 0$. Let

$$y(k) = \begin{cases} x(k) & \text{if } k \neq k_0 \\ sign(f(e_{k_0})) & \text{if } k = k_0. \end{cases}$$

Then since $x \in B_{c_o}$, $y \in B_{c_0}$ and $f(y) = f(x) + (sign(f(e_{k_0})) - x(k_0))f(e_{k_0}) > 1$ contrary to $\|f\| = 1$. Therefore $f(e_k) = 0$ for all natural numbers k with $|x(k)| < 1$ and since $x \in c_0$ there exists a natural number n_0 with $|x(k)| < 1$ for all $k > n_0$. Hence $f(e_k) = 0$ if $k > n_0$.

The other inclusion holds because the space $(\mathbb{R}^{n_0}, \|\cdot\|_1)$ is finite dimensional, hence its unit ball is compact. Therefore every element of the dual of $(\mathbb{R}^{n_0}, \|\cdot\|_1)$ is norm attaining, hence every element f of ℓ_1 with all but a finite number of coordinates zero is norm attaining. □

Corollary 1.2.10. *The set of norm attaining linear forms in c_0 is dense in c_0^*.*

Remark 1.2.11. *Notice $NA(c_0)$ is an infinite dimensional non-closed vector space.*

Naturally one can ask if the set of norm attaining linear forms contains always a vectorial space structure inside. For instance, does the set of norm attaining linear forms contain a subspace of finite dimension greater than one?

In recent years a lot of work has been done about whether the set of norm attaining linear forms is big in the sense that has a vectorial space structure. Special attention has been paid to study whether the set of norm attaining linear forms is lineable or even spaceable. We will discuss this particular scenario in Section 3.5.

Definition 1.2.12. A subset M of a topological vector space E is said to be lineable (respectively spaceable) in E if $M \cup \{0\}$ contains an infinite dimensional space (respectively infinite dimensional closed space).

Bandyopadhyay and Godefroy, [BG06], motivated by proximality questions, have investigated the spaceability properties of the norm attaining functional on a Banach space X using isometric duality theory.

To be more specific they show that when a non-reflexive space X enjoys the Radon-Nikodým Property, see Definition 2.1.6, the set of norm attaining linear forms is not a linear space. On the other hand, they also show that for a Banach space whose dual unit ball is weak-star-sequentially compact, there exists an equivalent norm $|\cdot|$ of X such that the set of norm attaining linear forms of $(X, |\cdot|)$ is spaceable if and only if there exists an infinite dimensional quotient space of X which is isomorphic to a dual space.

Using Definition 1.2.12 we have by Remark 1.2.11 that $NA(c_0)$ is lineable and by the following proposition that the set of norm attaining linear forms of any dual space is spaceable.

Proposition 1.2.13. *For every infinite dimensional Banach space X with predual X_* the set of norm attaining linear forms is spaceable.*

Proof. Since X is an infinite dimensional Banach space, the predual X_* is an infinite dimensional Banach space too. Then, by the Hahn-Banach Separation Theorem for every $x_* \in S_{X_*}$ there exists $x \in S_X$ such that $x(x_*) = \|x\| = 1$, hence the natural inclusion $X_* \subseteq X^*$ give us $X_* \subseteq NA(X)$. □

As a consequence of Proposition 1.2.13 and Proposition 1.2.9 we get an elementary proof of the non-existence of the predual of c_0.

Corollary 1.2.14. *The Banach space c_0 does not have predual.*

Proof. By Proposition 1.2.13 we only need to show that $NA(c_0)$ does not contain an infinite dimensional Banach space. Assume $NA(c_0)$ contains an infinite dimensional Banach space Y. Hence there exist a sequence $\{x^*_{n_k}\}_{k=0}^{\infty} \subset Y$ such that $x^*_{n_k}(e_{n_k}) \neq 0$, $x^*_{n_k}(e_j) = 0$ for $j > n_k$, $\|x^*_{n_k}\| = 1$ and $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence. Define $\alpha_{n_1} = \frac{1}{4}$ and $\alpha_{n_k} := \frac{1}{2^{2k}} \min\{x^*_{n_i}(e_j) \neq 0 : 1 \leq i \leq k-1, 1 \leq j \leq n_k\}$, and consider the sequence $x_r = \sum_{k=1}^{r} \alpha_{n_k} x_{n_k}^*$. Then, the sequence $\{x_r\}_{r=1}^{\infty}$ converges to the point $x = \sum_{k=1}^{\infty} \alpha_{n_k} x_{n_k}^* \in c_0$, so if Y where closed, x would be an element of Y. But by the construction, x does not have finite support since $|x(e_{n_r})| \geq (\frac{1}{2^{2r}} - \sum_{k=r+1}^{\infty} \frac{1}{2^{2k}})|x_{n_r}(e_{n_r})| > 0$ for all natural number r. Therefore $x \notin Y$ which is a contradiction with Y being closed. $\square$

1.3 The Bishop-Phelps-Bollobás Theorem

In 1970, motivated by problems related to numerical radius, Bollobás made a refinement of the Bishop-Phelps Theorem. Bollobás gave a quantitative version of Theorem 1.2.2 as follows:

Theorem 1.3.1 (Bishop-Phelps-Bollobás, [Bol70])**.** *Given $\frac{1}{2} > \epsilon > 0$, if $x_0 \in X$ and $f \in X^*$ with $\|x_0\| = \|f^*\| = 1$ are such that*

$$|1 - f(x_0)| < \frac{\epsilon^2}{2},$$

then there are $y_0 \in X$ and $g \in X^*$ such that

$$\|y_0\| = \|g\| = g(y_0) = 1, \quad \|y_0 - x_0\| < \epsilon + \epsilon^2 \text{ and } \|f - g\| < \epsilon.$$

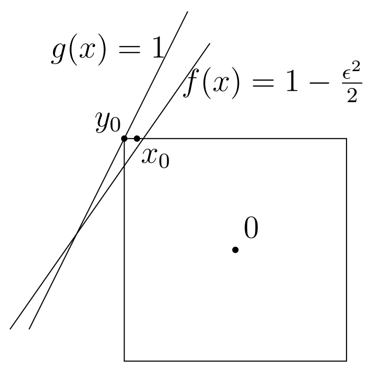

Figure 1.3 Bishop-Phelps-Bollobás Theorem.

With this result, Bollobás started the study of simultaneously approximating both operators and the points at which they almost attain their norms.

A refinement of this result can be obtained using the Brønsted-Rockafellar Principle for the indicator function $f = \delta_{B_X}$ defined by $\delta_{B_X}(x) = 0$ if x is in B_X and $\delta_{B_X} = \infty$ otherwise, and using ϵ^2 instead of ϵ and $\epsilon = \lambda$ in the theorem.

Theorem 1.3.2. *Given $\frac{1}{2} > \epsilon > 0$, if $x_0 \in X$ and $f \in X^*$ with $\|x_0\| = \|f^*\| = 1$ are such that*

$$|1 - f(x_0)| < \frac{\epsilon^2}{2}$$

then there are $y_0 \in X$ and $g \in X^*$ such that

$$\|y_0\| = \|g\| = g(y_0) = 1, \quad \|y_0 - x_0\| < \epsilon \;\; \text{and} \;\; \|f - g\| < \epsilon.$$

See [Koz14] for the details of the proof.

Notice that these estimations are optimal in the sense that we can not lower the bounds obtained.

Theorem 1.3.3 (Bollobás, [Bol70])**.** *For any $0 < \epsilon < 1$ there exist a Banach space X, a point $x \in S_X$ and a functional $f \in S_{X^*}$ such that $f(x) = 1 - (\epsilon^2/2)$ but if $y \in S_X, g \in S_{X^*}$ are such that $g(y) = 1$ then either $\|f - g\| > \epsilon$ or $\|x - y\| > \epsilon$.*

Chapter 2

Norm attaining operators

> *"Math is like love – a simple idea but it can get complicated."*
>
> — R. Drabek

Once the Bishop-Phelps and Bishop-Phelps-Bollobás results have been established, it is natural to wonder when we can get a more general version of these theorems. Here we study the vector valued versions of these results and the Lindenstrauss' result.

2.1 Norm attaining operators

If X and Y are Banach spaces we will denote by $\mathcal{L}(X;Y)$ the space of all linear and continuous operators from X into Y endowed with its natural norm $\|T\| = \sup_{x \in B_X}\{\|T(x)\|\}$. In the particular case of $Y = X$ we will write $\mathcal{L}(X)$ for $\mathcal{L}(X;X)$.

At the end of their paper, Bishop and Phelps raised the question of extending their result to the operators case;

"A possible generalization of this theorem remains open:

> Suppose E and F are Banach spaces, and let $\mathcal{L}(E;F)$ be the Banach space of all continuous linear transformations from E into F, with the usual norm. For which E and F are those T such that $\|T\| = \|T(x)\|$ (for some x in E, $\|x\| = 1$) dense in $\mathcal{L}(E;F)$?"

This question was answered in the negative two years later by Lindenstrauss in [Lin63], where he proved that for certain Banach spaces X and Y the subset of norm attaining operators from X into Y is not norm dense in the space of all continuous and linear operators $\mathcal{L}(X;Y)$. In particular he showed that for a specific renorming $|||\cdot|||$ of c_0, the identity map from $(c_0, \|\cdot\|_\infty)$ to $(c_0, |||\cdot|||)$ cannot be approximated by norm attaining operators.

Nevertheless, there are also remarkable situations in which a Bishop-Phelps Theorem for operators does hold, such as when the domain space has the Radon-Nikodým Property, see Definition 2.1.6, [Bou77], or as a particular case of Lindenstrauss' result if the space X is reflexive, see Theorem 2.4.1 below. For the last case, we have that the points where the set of operators whose extension to the bidual attain their norm in Lindenstrauss Theorem are points of X. Therefore the set of operators whose extension attain the norm is the same as the set of norm attaining operators. Thus a Bishop-Phelps Theorem for operators holds when the domain space is reflexive.

Motivated by the study of these cases, Lindenstrauss introduced two properties on a Banach space, called A and B, as follows:

Definition 2.1.1 (Property A)**.** A Banach space X has Property A if the set of norm attaining operators from X to Y is norm dense in $\mathcal{L}(X;Y)$ for every Banach space Y.

Some elementary examples of spaces with Property A are finite dimensional spaces (because of the compactness of the unit ball), reflexive spaces and spaces with the Radon-Nikodým Property. In fact by Theorem 2.1.7, a Banach space has the Radon-Nikodým Property if and only if it has the Property A for every equivalent norm. But for instance c_0 and $L_1(\mu)$ for μ a non-atomic measure are examples of spaces failing Property A.

Definition 2.1.2 (Property B)**.** A Banach space Y has Property B if the set of norm attaining operators from X to Y is norm dense in $\mathcal{L}(X;Y)$ for every Banach space X.

The Bishop-Phelps Theorem tells us that the scalar field of the real or complex numbers has the Property B.

Motivated by the study of Property A, Schachermayer, [Sch83a], introduced the Property α as a sufficient condition for a Banach space to have Property A and in the same way Lindenstrauss introduced Property β a sufficient condition for a Banach space to have Property B.

Definition 2.1.3 (Property α)**.** A Banach space Y has Property α if there is a subset $\{(y_i, y_i^*) : i \in I\} \subset S_Y \times S_{Y^*}$ such that

- $y_i^*(y_i) = 1$ for every $i \in I$,
- there is a constant $0 \leq \rho < 1$ such that $|y_j^*(y_i)| \leq \rho$ for every $i, j \in I, i \neq j$,
- The set of points $\{y_i : i \in I\}$ is a 1-norming set in Y^* i.e. $\|y^*\| = \sup\{|y^*(y_i)| : i \in I\}$ for every $y^* \in Y^*$.

Definition 2.1.4 (Property β)**.** A Banach space Y has Property β if there is a subset $\{(y_i, y_i^*) : i \in I\} \subset S_Y \times S_{Y^*}$ such that

- $y_i^*(y_i) = 1$ for every $i \in I$,
- there is a constant $0 \leq \rho < 1$ such that $|y_j^*(y_i)| \leq \rho$ for every $i, j \in I, i \neq j$,

- The set of functionals $\{y_i^* : i \in I\}$ is a 1-norming set in Y i.e. $\|y\| = \sup\{|y_i^*(y)| : i \in I\}$ for every $y \in Y$.

Notice that the only difference between Property α and Property β is the last condition, but also both properties are related in the sense that a Banach space has Property α if and only if its dual has Property β.

An example of a space having Property α is ℓ_1 and some examples of spaces having Property β are c_0 and its bidual ℓ_∞. For the finite dimensional case we have that both properties are equivalent and they are satisfied if and only if the unit ball of the space is a polyhedron i.e. the finite intersection of closed semispaces.

Coming back to the question posted by Klee, we can try to characterize the Banach spaces that can answer this question in the affirmative i.e., the Banach spaces that have the Bishop-Phelps Property.

Definition 2.1.5 (Bishop-Phelps Property, [Bou77])**.** Given a Banach space X and a subset B of X, we say that B has the Bishop-Phelps Property if for every Banach space Y and for every operator $T \in \mathcal{L}(X;Y)$, there exists a sequence of operators $\{T_n\}_{n=1}^{\infty} \subset \mathcal{L}(X;Y)$ converging to T in the operator norm, such that, every operator T_n attains its supremum on B.

We say that a Banach space X has the Bishop-Phelps Property if every absolutely convex, bounded, closed non-empty subset of X has the Bishop-Phelps Property.

This definition was made by Bourgain, [Bou77], who used the Radon-Nikodým Property to characterize the spaces that have the Bishop-Phelps Property.

Definition 2.1.6 (Radon-Nikodým Property)**.** A Banach space X has the Radon-Nikodým Property provided for every measure space (Ω, Σ, μ)

with $\mu(\Omega) < \infty$, and every μ-continuous measure $F : \Sigma \mapsto X$ of finite variation, there exists a Bochner integrable function $f : \Omega \mapsto X$ such that $F(E) = \int_E f \, d\mu$ for every $E \in \Sigma$.

Theorem 2.1.7 (Bourgain, [Bou77])**.** *Given a real Banach space X, the following are equivalent*

- *X has the Bishop-Phelps Property,*
- *X has the Radon-Nikodým Property.*

2.2 The Bishop-Phelps-Bollobás Property for Operators

Since Lindenstrauss showed that we cannot expect an operator version of Bishop-Phelps result for all pair of Banach spaces X and Y we cannot expect either that a generalization of the Bishop-Phelps-Bollobás result to the operator case will hold in general.

Motivated by this idea, in 2008, Acosta, Aron, García and Maestre introduced the following property to characterize the pairs of spaces that satisfy the operator version of Bishop-Phelps-Bollobás Theorem.

Definition 2.2.1 (*BPBp* ,[AAGM08])**.** Let X and Y be real or complex Banach spaces. We say that the pair (X, Y) satisfies the *Bishop-Phelps-Bollobás Property for operators*, *BPBp* for short, (or that the Bishop-Phelps-Bollobás Theorem holds for all bounded operators from X to Y) if given $\epsilon > 0$, there are $\delta(\epsilon) > 0$ and $\beta(\epsilon) > 0$ with $\lim_{t \to 0} \beta(t) = 0$ such that for all $T \in S_{\mathcal{L}(X,Y)}$, if $x \in S_X$ with $\|T(x)\| > 1 - \delta(\epsilon)$, then there exist a point $y \in S_X$ and an operator $G \in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions:

- $\|G(y)\| = 1$,

- $\|y - x\| < \beta(\epsilon)$,
- $\|G - T\| < \epsilon$.

In [AAGM08] the authors showed that a necessary and sufficient condition on Y for the pair (l_1, Y) to satisfy the $BPBp$ is for Y to have the Approximating Hyperplane Series Property ($AHSP$).

Definition 2.2.2 ($AHSP$, [AAGM08])**.** A real Banach space X is said to have the $AHSP$ if for every $\epsilon > 0$ there exists $0 < \gamma < \epsilon$ such that for every sequence $(x_k) \subseteq S_X$ and for every convex series $\sum_{k=1}^{\infty} \alpha_k$ with

$$\|\sum_{k=1}^{\infty} \alpha_k x_k\| > 1 - \gamma,$$

there exist a subset $A \subseteq \mathbb{N}$, a subset $\{y_k : k \in A\} \subseteq S_X$, and a certain $g \in S_{X^*}$ satisfying:

- $\sum_{k \in A} \alpha_k > 1 - \epsilon$,
- $\|x_k - y_k\| < \epsilon$ for all $k \in A$,
- $g(y_k) = 1$ for all $k \in A$.

And more recently Aron, Choi, Kim, Lee and Martín, in [ACK$^+$14], study a Bishop-Phelps-Bollobás version of Lindenstrauss properties A and B using universal Bishop-Phelps-Bollobás spaces.

Definition 2.2.3 (Universal BPB spaces, [ACK$^+$14])**.** Let X and Y be Banach spaces. We say that X is a universal BPB domain space if for every Banach space Z, the pair (X, Z) has the BPBp with the function $\beta(\epsilon) = \epsilon$ for all $\epsilon \in (0, 1)$. We say that Y is a universal BPB range space if for every Banach space Z, the pair (Z, Y) has the BPBp with the function $\beta(\epsilon) = \epsilon$ for all $\epsilon \in (0, 1)$.

For universal BPB domains X the authors show that there exists a universal function $\delta_X(\epsilon)$ such that for every Y, the pair (X, Y) has the BPBp with this function.

Theorem 2.2.4 (Aron-Choi-Kim-Lee-Martín, [ACK+14])**.** *If X is a universal BPB domain space, then there is a function $\delta_X : (0,1) \mapsto \mathbb{R}^+$ such that for every Banach space Y, (X, Y) has the BPBp with δ_X. In other words, for every Y, $\delta(X, Y) > \delta_X$.*

And a similar result is obtained when working with range spaces as can be seen in the following theorem.

Theorem 2.2.5 (Aron-Choi-Kim-Lee-Martín, [ACK+14])**.** *If Y is a universal BPB range space, then there is a function $\delta_Y : (0,1) \mapsto \mathbb{R}^+$ such that for every Banach space X, (X, Y) has the BPBp with δ_Y . In other words, for every X, $\delta(X, Y) > \delta_Y$.*

In 2012, Aron, Choi, García and Maestre, [ACGM11], showed that an extension of the Bishop-Phelps-Bollobás Theorem holds for all bounded linear operators from $L_1(\mu)$ into $L_\infty[0,1]$, where μ is a σ-finite measure. The same year Choi and Kim, [CK11], motivated by the characterization of the $BPBp$ in terms of the $AHSP$ for ℓ_1, tried to extend this characterization to the space L_1. They showed that if the pair $(L_1(\mu), Y)$ has the $BPBp$ then Y has the $AHSP$, and if Y has the Radon-Nikodým Property then the $AHSP$ is also a sufficient condition. However the $AHSP$ on Y is not a sufficient condition for the pair $(L_1(\mu), Y)$ to have the $BPBp$, as Schachermayer showed using the space of continuous functions on the interval $[0,1]$, [Sch83b], and the Radon-Nikodým Property for Y is not always necessary as can be shown by using the space L_∞ and the result of Aron, Choi, García and Maestre, [ACGM11].

2.3 The Bishop-Phelps-Bollobás Property for Numerical Radius

As a particular case, we can consider the situation where the Banach spaces X and Y are the same i.e., when we consider linear and continuous operator from a Banach space X into itself.

Given an operator $T \in \mathcal{L}(X)$, the numerical radius of T is defined by $\nu(T) = \sup\{|f(T(x))| : (x, f) \in \Pi(X)\}$ where $\Pi(X) := \{(x, x^*) : x \in X, x^* \in X^*, \|x\| = \|x^*\| = x^*(x) = 1\}$. The pairs of elements $(x, f) \in \Pi(X)$ are usually called states.

Motivated by the study of norm attaining operators initiated by Lindenstrauss, Sims asked in his PhD, [Sim72], whether the set of numerical radius attaining operators is dense in the space of all continuous linear operators on a Banach space. Twenty years later, in 1992, Payá, [Pay92] gave a counterexample to this question using the renorming of c_0 of the example of Lindenstrauss (as suggested by Cardassi in [Car85c]). Also, another counterexample was given the same year by Acosta, Aguirre and Payá in [AAP92] using the space $X = \ell_2 \oplus_\infty G$, where G is a Gowers' space.

Nonetheless, the study of denseness of numerical radius attaining operators has been investigated in parallel to the study of norm attaining operators and many positive results have been found in this direction. Berg and Sims [BS84] gave a positive answer for uniformly convex spaces and Cardassi showed that the answer is positive for $\ell_1, c_o, C(K)$ (where K is metrizable and compact), $L_1(\mu)$ and uniformly smooth spaces [Car85c, Car85b, Car85a].

Notice that the numerical radius of a Banach space X is a continuous seminorm on X bounded by the natural norm on $\mathcal{L}(X)$. In particular we say that the Banach space X has numerical index 1 if $\|T\| = \nu(T)$

for all operators $T \in \mathcal{L}(X)$ i.e., when the value of the norm and the value of the numerical radius coincide for all operators T.

Some examples of Banach spaces with numerical index 1 are:

Example 2.3.1 ([KMP06])**.**

- Every Banach space X, with extreme points in B_X, such that $|x^*(x)| = 1$ for every extreme point x of B_X and every extreme point x^* of B_{X^*} (in fact this is also a sufficient condition in the finite dimensional case),
- Every Banach space with a subset $C \subseteq S_X$ such that $\overline{co}(C) = B_X$ and $|x^*(c)| = 1$ for every extreme point x^* of B_{X^*} and every point $c \in C$,
- Every Banach space X whose dual has a norming set C such that $|x^{**}(c^*)| = 1$ for every extreme point x^{**} of $B_{X^{**}}$ and avery $c^* \in C$.

We say that $T \in \mathcal{L}(X)$ attains its numerical radius if there exists $(x, f) \in \Pi(X)$ such that $|f(T(x))| = \nu(T) = \|T\|$.

Using this notation, we can reformulate the Bishop-Phelps-Bollobás Theorem, roughly speaking, asserting that any ordered pair that *"almost belongs"* to $\Pi(X)$ can be approximated in the product norm by elements of $\Pi(X)$. And we can define the Bishop-Phelps-Bollobás Property for numerical radius as follows.

Definition 2.3.2 (*BPBp-ν*, [GK13])**.** A Banach space X is said to have the *Bishop-Phelps-Bollobás Property for numerical radius*, *BPBp-ν* for short, if given $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that for all $T \in \mathcal{L}(X)$ of norm one, if $(x, x^*) \in \Pi(X)$ is such that $|x^*(T(x))| > 1 - \delta(\epsilon)$, then there exist $G \in \mathcal{L}(X)$, with $\nu(G) = 1$ and a pair $(y, y^*) \in \Pi(X)$ such that

$$\|T - G\| \leq \epsilon, \quad \|x - y\| \leq \epsilon, \|x^* - y^*\| \leq \epsilon \text{ and } |y^*(G(y))| = 1.$$

In [GK13] Guirao and Kozhushkina study the Bishop-Phelps-Bollobás Property for numerical radius, which has as its main point of interest the natural extension of Bishop-Phelps-Bollobás' result to the numerical radius on Banach spaces of numerical index 1. The authors use constructive versions of Bollobás Theorem to prove that the space $\ell_1(\mathbb{C})$ has the *BPBp-ν* and dualizing the constructions they also get the same result for $c_0(\mathbb{C})$. Our aim now is to show that the Banach space of Lebesgue real-valued integrable functions over the real line, that we will denote by L_1, has the Bishop-Phelps-Bollobás Property for numerical radius.

2.3.1 The Bishop-Phelps-Bollobás Property for Numerical Radius on L_1

In [AAGM08], Acosta, Aron, García and Maestre proved that L_1 has the *AHSP*. But, we know that L_1 does not have the Radon-Nikodým Property, and so we can not apply the results of Choi and Kim to obtain that the pair (L_1, L_1) has the *BPBp*. However in [CKLM14], Choi, Kim, Lee and Martín have proved that the pair (L_1, L_1) has the *BPBp*. An alternative proof of this result for $L_1(\mathbb{R})$ can be done by modifying the proof presented in Theorem 2.3.7.

Even though Choi, Kim, Lee and Martín have shown that the pair (L_1, L_1) has the *BPBp*, there is no known relation between the pair (X, X) having the *BPBp* and the space X having the *BPBp-ν*.

Our main point of interest in this section is to prove that the space L_1 has the *BPBp-ν*, Theorem 2.3.7. But, before we give the proof of the Theorem we need some necessary technical lemmas.

Lemma 2.3.3. *Let $A \subseteq \mathbb{R}$ be a measurable set and A^c its complement. The operator R from $\mathcal{L}(L_1)$ to $\mathcal{L}(L_1)$ defined, for every element $T \in \mathcal{L}(L_1)$, by $R(T)(f) = T(f\chi_A - f\chi_{A^c})\chi_A - T(f\chi_A - f\chi_{A^c})\chi_{A^c}$ is an* *isometry, i.e.* $\|T\| = \|R(T)\|$ *for all operators* $T \in \mathcal{L}(L_1)$*.*

Also for every point $x \in L_1$ *and every linear form* $f \in L_\infty$*, if we denote by* $r(f) = f\chi_A - f\chi_{A^c}$ *and by* $r(x) = x\chi_A - x\chi_{A^c}$*, then*

$$< r(x), r(f) >=< x, f >= \int_{\mathbb{R}} x(t)f(t)dt$$

and

$$< R(T)(r(x)), r(f) >=< T(x), f >= \int_{\mathbb{R}} (T(x)(t))f(t)dt.$$

Proof. First of all notice that for every $x \in L_1$, $\|x\| = \|r(x)\|$ and $r(r(x)) = x$, and for every $f \in L_\infty$, $\|f\| = \|r(f)\|$ and $r(r(f)) = f$ hence the operators $x \mapsto r(x)$ on L_1 and $f \mapsto r(f)$ on L_∞ are onto isometries. Also

$$R(T)(x) = T(x\chi_A - x\chi_{A^c})\chi_A - T(x\chi_A - x\chi_{A^c})\chi_{A^c}.$$

Moreover,

$$T(r(x)) = T(x\chi_A - x\chi_{A^c}) = T(x\chi_A - x\chi_{A^c})\chi_A + T(x\chi_A - x\chi_{A^c})\chi_{A^c}.$$

Therefore $\|R(T)(x)\| = \|T(r(x))\|$, and so R is an isometry.

For the second part, notice that

$$\begin{aligned}
&< r(x), r(f) > = < x\chi_A - x\chi_{A^c}, f\chi_A - f\chi_{A^c} > \\
&= < x\chi_A, f\chi_A > - < x\chi_{A^c}, f\chi_A > - < x\chi_A, f\chi_{A^c} > + < x\chi_{A^c}, f\chi_{A^c} > \\
&= < x\chi_A, f\chi_A > + < x\chi_{A^c}, f\chi_{A^c} > \\
&= < x, f > .
\end{aligned}$$

Since $R(T)(r(x)) = R(T)(x\chi_A - x\chi_{A^c}) = T(x\chi_A + x\chi_{A^c})\chi_A - T(x\chi_A +$

$x\chi_{A^c})\chi_{A^c} = T(x)\chi_A - T(x)\chi_{A^c}$, we get that

$$\begin{aligned}< R(T)(r(x)), r(f) > &= < R(T)(r(x)), f\chi_A > - < R(T)(r(x)), f\chi_{A^c} > \\ &= < T(x)\chi_A - T(x)\chi_{A^c}, f\chi_A > - < T(x)\chi_A - T(x)\chi_{A^c}, f\chi_{A^c} > \\ &= < T(x)\chi_A, f\chi_A > - < -T(x)\chi_{A^c}, f\chi_{A^c} > \\ &= < T(x), f > .\end{aligned}$$

□

The next lemma can be easily proved for ℓ_1 using the fact that ℓ_1 has an atomic measure. Now we want to prove this result for the space L_1 that has a strictly non-atomic measure.

Lemma 2.3.4. *Given a pair $(x, f) \in \Pi(L_1)$ with $f(t) \geq 0$ for all $t \in \mathbb{R}$, let $A = \{t \in \mathbb{R} : x(t) > 0\}$. Then $\mu(\{t \in \mathbb{R} : x(t) < 0\}) = 0$ and $\mu(\{t \in A : f(t) < 1\}) = 0$. Also, for every point $y \in L_1$ if $\{t \in \mathbb{R} : y(t) > 0\} \subseteq A$ and $\mu(\{t \in \mathbb{R} : y(t) < 0\}) = 0$, then $< y, f >= \|y\|$.*

Proof. Consider x, f and A as in the lemma. If $\mu(\{t \in \mathbb{R} : x(t) < 0\}) > 0$ we have

$$\begin{aligned}1 &=< x, f > \\ &= \int_{\mathbb{R}} x(t)f(t)dt \leq \int_A x(t)dt + \int_{A^c} x(t)f(t)dt \\ &\leq \int_A x(t)dt < \|x\| = 1,\end{aligned}$$

which is a contradiction. Hence $\mu(\{t \in \mathbb{R} : x(t) < 0\}) = 0$.

Now, if $\mu(\{t \in A : f(t) < 1\}) > 0$, then there exists $1 > \epsilon > 0$ with $A_\epsilon = \{t \in A : f(t) < 1 - \epsilon\}$ and $\|x\chi_{A_\epsilon}\| > 0$. Then

$$\begin{aligned} 1 &=< x, f > \\ &= \int_{\mathbb{R}} x(t)f(t)dt \\ &\leq \int_{A_\epsilon^c} x(t)dt + \int_{A_\epsilon} x(t)(1-\epsilon)dt \\ &< \|x\| = 1 \end{aligned}$$

which is again a contradiction. So $\mu(\{t \in A : f(t) < 1\}) = 0$.

To finish assume $y \in L_1$ with $\{t \in \mathbb{R} : y(t) > 0\} \subseteq A$ and $\mu(\{t \in \mathbb{R} : y(t) < 0\}) = 0$. Then since $f(t) = 1$ for almost all t in A,

$$< y, f >= \int_{\mathbb{R}} y(t)f(t)dt = \int_A y(t)f(t)dt = \int_A y(t)dt = \|y\|.$$

□

Before presenting our main result of this section we need to prove the last technical lemma that will be used to modify the operator in Theorem 2.3.7.

Lemma 2.3.5. *Given two measurable sets I and S and an operator $T \in \mathcal{L}(L_1)$, for any finite number of pairwise disjoint measurable sets $I_1, \ldots, I_j$ of finite measure, with $I = \bigcup_{i=1}^j I_i$*

$$(T(\chi_I)\chi_{\{t\in\mathbb{R}:T(\chi_I)(t)>0\}\cap S})(t) \leq \sum_{i=1}^{j}(T(\chi_{I_i})\chi_{\{t\in\mathbb{R}:T(\chi_{I_i})(t)>0\}\cap S})(t)$$

almost everywhere. Also $\|\sum_{i=1}^j T(\chi_{I_i})\chi_{\{t\in\mathbb{R}:T(\chi_{I_i})(t)>0\}\cap S}\| \leq \|T\|\|\chi_I\|$.

Proof. Since $T(\chi_I) = \sum_{i=1}^j T(\chi_{I_i})$ there is no loss of generality in assuming that equality also holds for the measurable functions after taking representatives of the equivalence class, i.e., we assume that $T(\chi_I)(t) = \sum_{i=1}^j T(\chi_{I_i})(t)$ for all real numbers t in S.

Notice that for $i = 1, \ldots, j$

$$(T(\chi_{I_i})\chi_S)(t) \leq T(\chi_{I_i})\chi_{\{t\in\mathbb{R}:T(\chi_{I_i})(t)>0\}\cap S}(t).$$

Hence, if $T(\chi_I)(t) \leq 0$, by the linearity of T the required inequality holds and if $T(\chi_I)(t) > 0$,

$$\begin{aligned} T(\chi_I)\chi_{\{t\in\mathbb{R}:T(\chi_I)(t)>0\}\cap S})(t) &= \sum_{i=1}^{j}(T(\chi_{I_i})\chi_S)(t) \\ &\leq \sum_{i=1}^{j} T(\chi_{I_i})\chi_{\{t\in\mathbb{R}:T(\chi_{I_i})(t)>0\}\cap S}(t). \end{aligned}$$

Notice that

$$\begin{aligned} \| \sum_{i=1}^{j} T(\chi_{I_i})\chi_{\{t\in\mathbb{R}:T(\chi_{I_i})(t)>0\}\cap S}\| &= \sum_{i=1}^{j} \int_{\{t\in\mathbb{R}\,:\,T(\chi_{I_i})(t)>0\}\cap S} T(\chi_{I_i})(t)dt \\ \leq \sum_{i=1}^{j} \Big(\int_{\{t\in\mathbb{R}\,:\,T(\chi_{I_i})(t)>0\}\cap S} T(\chi_{I_i})(t)dt &+ \int_{\{t\in\mathbb{R}\,:\,T(\chi_{I_i})(t)<0\}\cap S} |T(\chi_{I_i})(t)|dt \Big) \\ = \sum_{i=1}^{j} \|T(\chi_{I_i})\chi_S\| \leq \sum_{i=1}^{j} \|T(\chi_{I_i})\| &\leq \sum_{i=1}^{j} \|T\|\|\chi_{I_i}\| = \|T\|\|\chi_I\|. \end{aligned}$$

Therefore, $\| \sum_{i=1}^{j} T(\chi_{I_i})\chi_{\{t\in\mathbb{R}:T(\chi_{I_i})(t)>0\}\cap S}\| \leq \|T\|\|\chi_I\|$. □

Remark 2.3.6. *Let us call by $\Big\{\Delta_n := \{[\frac{z}{2^n}, \frac{z+1}{2^n}) : z \in \mathbb{Z}\}\Big\}_n$ the dyadic partitions of the real line into segments of length $\frac{1}{2^n}$. Notice that the set of simple functions whose measurable sets are dyadic sets is dense in the set of simple functions. Hence they are norm dense in the Banach space of Lebesgue integrable functions. In particular, for every measurable set B and any integrable function $x \in L_1$ there exists a sequence of simple integrable functions $\{x_k\}_{k=1}^{\infty}$ approximating $x\chi_B$ in norm with $x_k = \sum_{i=1}^{N} \alpha_i \chi_{D_i\cap B}$ and D_i dyadic segments.*

Theorem 2.3.7. *The Banach space L_1 satisfies the BPBp-ν.*

Proof. Consider $x \in B_{L_1}$, $f \in B_{L_1^*} = B_{L_\infty}$ with $\|x\| = \|f\| = 1$ and $\langle x, f\rangle = 1$. Consider also an operator $T \in \mathcal{L}(L_1)$, $\|T\| = 1$ and assume

$$\langle T(x), f\rangle > 1 - \delta/2 \tag{2.1}$$

with $0 < \delta < 1/4$.

From now on, we fix a representant of the equivalence classes of $x \in L_1$ and $f \in L_\infty$. That is, two measurable functions in the equivalence classes and we denote these representants in the same way as x and f.

First we do the proof assuming $f(t) \geq 0$ for all real number t.

Since the norm of f is $\|f\| = esssup_{t\in\mathbb{R}}|f(t)|$, we can assume that $0 \leq f(t) \leq 1$ for all real number t. Let

$$B = \{t \in \mathbb{R} : f(t) \geq 1 - \delta^{1/4}\}. \tag{2.2}$$

Consider the measurable set $S := \{t \in \mathbb{R} : x(t) > 0\}$. Then by Lemma 2.3.4 and the fact that $\langle f, x\rangle = 1$ we have that $\mu(S) > 0$. Also, by Remark 2.3.6 there exists a function $z \in L_1$ of norm one with $\|x - z\| \leq \delta/2$, where $z = \sum_{i=1}^N \alpha_i \chi_{D_i\cap S}$ with D_i being mutually disjoint dyadic segments such that $\mu(D_i \cap S) > 0$ and $\alpha_i > 0$ for $i = 1, \dots, N$. Therefore

$$\langle T(z), f\rangle \geq \langle T(x), f\rangle - \|x - z\| > 1 - \delta.$$

By Lemma 2.3.4 f attains its norm at z, i.e., $(z, f) \in \Pi(L_1)$. Let

$$D_0 = \bigcup_{i=1}^N D_i \cap S \qquad \text{and} \qquad g = \chi_B + f\chi_{B^c}. \tag{2.3}$$

It is easy to see that

$$\|f - g\| \leq \delta^{1/4}. \tag{2.4}$$

Now, for every natural number $i = 1, \ldots, N$ and for $n \in \mathbb{N}$, consider

$$F_{n,i} := \{I \in \Delta_n : \langle T(\chi_{I \cap D_i \cap S}), f\rangle \leq (1 - \sqrt{\delta})\|\chi_{I \cap D_i \cap S}\|\}. \tag{2.5}$$

Put $D^i = (D_i \cap S) \setminus \Big(\cup_{n \in \mathbb{N}} \cup_{I \in F_{n,i}} I \Big)$ which is measurable. Let $D = \bigcup_{i=1}^N D^i$. Then, by the construction of D, $D_0 \setminus D$ is a union of a sequence of disjoint measurable sets $\{I_k\}_{k=1}^{\infty}$ where I_k is of the form $R \cap D_i \cap S$ with R a dyadic set in some Δ_n and D_i one of the sets that appear in the definition of z. By (2.5) the set I_k is such that $\langle T(\chi_{I_k}), f\rangle \leq (1 - \sqrt{\delta})\|\chi_{I_k}\|$. Hence using the Monotone Convergence Theorem we obtain

$$\begin{aligned}
1 - \delta < \langle T(z), f\rangle &= \langle T(\sum_{i=1}^N \alpha_i \sum_{k=1}^{\infty} \chi_{I_k} + z\chi_D), f\rangle \\
&= \sum_{i=1}^N \alpha_i \sum_{k=1}^{\infty} \langle T(\chi_{I_k}), f\rangle + \langle T(z\chi_D), f\rangle \quad \text{by linearity of } T \text{ and } f, \\
&\leq (1 - \sqrt{\delta}) \sum_{i=1}^N \alpha_i \sum_{k=1}^{\infty} \|\chi_{I_k}\| + \langle T(z\chi_D), f\rangle \\
&= (1 - \sqrt{\delta})\|z\chi_{D_0 \setminus D}\| + \langle T(z\chi_D), f\rangle \quad \text{by definition of } z, \\
&\leq (1 - \sqrt{\delta})\|z\chi_{D_0 \setminus D}\| + \|z\chi_D\| \quad \text{because } T \text{ and } f \text{ are of norm one}, \\
&= 1 - \sqrt{\delta}\|z\chi_{D_0 \setminus D}\|.
\end{aligned}$$

Therefore $\|z\chi_{D_0 \setminus D}\| < \sqrt{\delta}$. Hence

$$\|z\chi_D\| = \|z\| - \|z\chi_{D_0 \setminus D}\| > 1 - \sqrt{\delta} > 0.$$

Since $\|z\chi_D\| > 0$ define

$$y = \frac{z\chi_D}{\|z\chi_D\|}. \tag{2.6}$$

Then y has norm one, $\{t \in \mathbb{R} : y(t) > 0\} \subseteq D_0$ and $y(t) \geq 0$

for all real number t. By Lemma 2.3.4, considering the set D_0 and the state $(z, f) \in \Pi(L_1)$ we obtain that $(y, f) \in \Pi(L_1)$. Also, by Lemma 2.3.4, $\mu(\{t \in D_0 : f(t) < 1\}) = 0$ so we have $f(t) = 1$ for almost all t in D_0, hence $f(t) = 1$ for almost all t in D. Therefore $1 =< y, f >= \int_{\mathbb{R}} y(t)f(t)dt = \int_D y(t)f(t)dt = \int_D y(t) = \int_D y(t)g(t)dt = \int_{\mathbb{R}} y(t)g(t)dt =< y, g >$ hence $(y, g) \in \Pi(L_1)$.

Let us see that z and y are close.

$$\begin{aligned} \|z - y\| &= \|z - \frac{z\chi_D}{\|z\chi_D\|}\| \\ &\leq \|z - z\chi_D\| + \|z\chi_D - \frac{z\chi_D}{\|z\chi_D\|}\| \\ &= \|z - z\chi_D\| + 1 - \|z\chi_D\| \\ &< 2\sqrt{\delta}. \end{aligned}$$

Hence $\|x - y\| < \delta + 2\sqrt{\delta}$.

Now we modify the operator T on the set D.

For simplicity denote by $W_I = \{t \in \mathbb{R} : T(\chi_I \cap D)(t) > 0\}$ for every dyadic set I in Δ_n for some natural number n. For each such dyadic set I in Δ_n, define the sequence of integrable functions $\{h_k^I\}_k$ by

$$h_k^I = \sum_{i=1}^{2^{k-1}} T(\chi_{I_i \cap D})\chi_{W_{I_i} \cap B} \tag{2.7}$$

where the sets $\{I_1, \ldots, I_{2^{k-1}}\}$ are the disjoint dyadic sets of Δ_{n+k-1} whose union is I.

Then by Lemma 2.3.5 we have a sequence of positive increasing functions almost everywhere, so as a consequence of the Monotone Convergence Theorem the integrable sequence of functions $\{h_k^I\}_{k=1}^{\infty}$ converges to an integrable function h^I. Notice that for every natural number k, $\|h_k^I\| \leq \|\chi_I\|$ hence $\|h^I\| \leq \|\chi_I\|$, and $\|h_k^I\| = \int_{\mathbb{R}} h_k^I(t)dt = \int_B h_k^I(t)dt =$

$\int_B h_k^I(t)g(t)dt = \langle h_k^I, g\rangle$. Therefore

$$\|h_k^I\| = \langle h_k^I, g\rangle \text{ and } \|h^I\| = \langle h^I, g\rangle. \tag{2.8}$$

Also, since the sequence of functions $\{h_k^I\}_{k=1}^{\infty}$ is positive and increasing almost everywhere, by (2.8) we have, for all $k \in \mathbb{N}$,

$$\begin{aligned}\|h^I - h_k^I\| &= \int_{\mathbb{R}} |(h^I - h_k^I)(t)|dt \\ &= \int_{\mathbb{R}} (h^I - h_k^I)(t)dt \\ &= \int_{\mathbb{R}} h^I(t)dt - \int_{\mathbb{R}} h_k^I(t)dt \\ &= \int_{\mathbb{R}} |h^I(t)|dt - \int_{\mathbb{R}} |h_k^I(t)|dt \\ &= \|h^I\| - \|h_k^I\| \\ &= \langle h^I, g\rangle - \langle h_k^I, g\rangle \\ &= \langle h^I - h_k^I, g\rangle.\end{aligned}$$

That is

$$\|h^I - h_k^I\| = \langle h^I - h_k^I, g\rangle. \tag{2.9}$$

By (2.5) we have that if $\mu(I \cap D) > 0$ then $\langle T(\chi_{I\cap D}), f\rangle > (1 - \sqrt{\delta})\|\chi_{I\cap D}\|$, but

$$\begin{aligned}\langle T(\chi_{I\cap D}), f\rangle &= \int_{\mathbb{R}} T(\chi_{I\cap D})(t)f(t)dt \leq \int_{W_I} T(\chi_{I\cap D})(t)f(t)dt \\ &= \langle T(\chi_{I\cap D})\chi_{W_I}, f\rangle\end{aligned}$$

so

$$\langle T(\chi_{I\cap D})\chi_{W_I}, f\rangle > (1 - \sqrt{\delta})\|\chi_{I\cap D}\|. \tag{2.10}$$

Therefore

$(1-\sqrt{\delta})\|\chi_{I\cap D}\| \leq \langle T(\chi_{I\cap D})\chi_{W_I}, f\rangle$

$$
\begin{aligned}
&= \int_{W_I \cap B} T(\chi_{I\cap D})(t) f(t)dt + \int_{W_I \cap B^c} T(\chi_{I\cap D})(t)\chi_{W_I}(t) f(t)dt \\
&= \int_{W_I \cap B} T(\chi_{I\cap D})(t) f(t)dt + \int_{\left(W_I\cap B\right)^c} T(\chi_{I\cap D})(t) f(t)dt \\
&\leq \int_{W_I \cap B} T(\chi_{I\cap D})(t)dt + \int_{\left(W_I\cap B\right)^c} (1-\delta^{1/4})|T(\chi_{I\cap D})(t)|dt \ \text{ (by (2.2))} \\
&= \|T(\chi_{I\cap D})\chi_{W_I\cap B}\| + (1-\delta^{1/4})\|T(\chi_{I\cap D})\chi_{\left(W_I\cap B\right)^c}\| \\
&= \|T(\chi_{I\cap D})\| - \delta^{1/4}\|T(\chi_{I\cap D})\chi_{\left(W_I\cap B\right)^c}\| \\
&\leq \|\chi_{I\cap D}\| - \delta^{1/4}\|T(\chi_{I\cap D})\chi_{\left(W_I\cap B\right)^c}\|.
\end{aligned}
$$

Therefore $\delta^{1/4}\|T(\chi_{I\cap D})\chi_{\left(W_I\cap B\right)^c}\| \leq \sqrt{\delta}\|\chi_{I\cap D}\|$, and for any dyadic interval I

$$
\|T(\chi_{I\cap D})\chi_{\left(W_I\cap B\right)^c}\| \leq \delta^{1/4}\|\chi_{I\cap D}\|. \tag{2.11}
$$

Define now the operator G on the simple functions whose measurable sets are dyadic as follows:

$$
G(\sum_{i=1}^{j} \beta_i \chi_{I_i}) = \sum_{i=1}^{j} \beta_i \Big(T(\chi_{I_i\cap D^c}) + h^{I_i} + (\|\chi_{I_i\cap D}\| - \|h^{I_i}\|)y\Big). \tag{2.12}
$$

Notice that $\sum_{i=1}^{j} \beta_i T(\chi_{I_i\cap D^c})$ is well defined because T is linear. Also, for every dyadic set $I \in \Delta_n$, if R, Q are two disjoint dyadic sets in Δ_{n+1} whose union is I, by the construction of the sequences $\{h_k^I\}_{k=1}^{\infty}$, $\{h_k^R\}_{k=1}^{\infty}$ and $\{h_k^Q\}_{k=1}^{\infty}$ we have for all $k > 1$, $h_k^I = h_{k-1}^R + h_{k-1}^Q$. Hence $h^I = h^R + h^Q$ and since $\|h^I\| = \langle h^I, g\rangle$, $\|h^R\| = \langle h^R, g\rangle$ and $\|h^Q\| = \langle h^Q, g\rangle$,

$$
\begin{aligned}
\|\chi_{I\cap D}\| - \|h^I\| &= (\|\chi_{R\cap D}\| + \|\chi_{Q\cap D}\|) - \langle h^I, g\rangle \qquad \text{(by (2.8))} \\
&= (\|\chi_{R\cap D}\| + \|\chi_{Q\cap D}\|) - \langle h^R + h^Q, g\rangle \\
&= (\|\chi_{R\cap D}\| + \|\chi_{Q\cap D}\|) - (\langle h^R, g\rangle + \langle h^Q, g\rangle)
\end{aligned}
$$

$$= (\|\chi_{R\cap D}\| + \|\chi_{Q\cap D}\|) - (\|h^R\| + \|h^Q\|) \qquad \text{(by (2.8))}$$
$$= (\|\chi_{R\cap D}\| - \|h^R\|) + (\|\chi_{Q\cap D}\| - \|h^Q\|).$$

By using induction, $\sum_{i=1}^{N} \beta_i\Big(h^{I_i} + (\|\chi_{I_i\cap D}\| - \|h^{I_i}\|)y\Big)$ is well defined. Hence G is well defined and linear. To finish, by density of the simple functions whose measurable sets are dyadic we can extend the operator G to L_1.

Now let us compute the norm of G and the distance between T and G. For this, it enough to compute the norm and the distance over dyadic sets I.

$$\begin{aligned}\|G(\chi_I)\| &\leq \|T(\chi_{I\cap D^c})\| + \|h^I\| + \Big\|(\|\chi_{I\cap D}\| - \|h^I\|)y\Big\| \\ &\leq \|\chi_{I\cap D^c}\| + \|\chi_{I\cap D}\| = \|\chi_I\|.\end{aligned}$$

Therefore $\|G\| \leq 1$. On the other hand, it is easy to check that for any function s in L_1,

$$\langle G(s), g\rangle = \langle T(s\chi_{D^c}), g\rangle + \int_D s(t)dt.$$

For this, by Remark 2.3.6 let's consider a sequence of simple functions $s_k = \sum_{i=1}^{N_k} \beta_i\chi_{I_i}$ were N_k is a natural number and I_i are some dyadic segments, with $\|s - s_k\| \to 0$ when k goes to infinity. Then,

$$\begin{aligned}\langle G(s), g\rangle &= \lim_{k\to\infty} \langle G(s_k), g\rangle \\ &= \lim_{k\to\infty} \langle G(\sum_{i=1}^{N_k} \beta_i\chi_{I_i}), g\rangle \\ &= \lim_{k\to\infty} \langle \sum_{i=1}^{N_k} \beta_i\Big(T(\chi_{I_i\cap D^c}) + h^{I_i} + (\|\chi_{I_i\cap D}\| - \|h^{I_i}\|)y\Big), g\rangle \\ &= \lim_{k\to\infty} \langle \sum_{i=1}^{N_k} \beta_i T(\chi_{I_i\cap D^c}), g\rangle + \langle \sum_{i=1}^{N_k} \beta_i\Big(h^{I_i} + (\|\chi_{I_i\cap D}\| - \|h^{I_i}\|)y\Big), g\rangle\end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \langle T(\sum_{i=1}^{N_k} \beta_i \chi_{I_i\cap D^c}), g\rangle + \sum_{i=1}^{N_k} \beta_i \|\chi_{I_i\cap D}\| \\
&= \lim_{k\to\infty} \langle T(\sum_{i=1}^{N_k} \beta_i \chi_{I_i\cap D^c}), g\rangle + \sum_{i=1}^{N_k} \beta_i \int_{\mathbb{R}} \chi_{I_i\cap D}(t)dt \\
&= \lim_{k\to\infty} \langle T(\sum_{i=1}^{N_k} \beta_i \chi_{I_i\cap D^c}), g\rangle + \int_{\mathbb{R}} \sum_{i=1}^{N_k} \beta_i \chi_{I_i\cap D}(t)dt \\
&= \langle T(s\chi_{D^c}), g\rangle + \int_D s(t)dt
\end{aligned}
$$

Therefore for any dyadic set I with $\mu(I\cap D) > 0$, we have $\langle G(\chi_{I\cap D}), g\rangle = \|\chi_{I\cap D}\|$, hence G has norm one.

Also,

$$
\begin{aligned}
\|T(\chi_i) - G(\chi_I)\| &\leq \|T(\chi_{I\cap D}) - h^I - (\|\chi_{I\cap D}\| - \|h^I\|)y\| \\
&\leq \|T(\chi_{I\cap D}) - h_1^I\| + \|h^I - h_1^I\| + (\|\chi_{I\cap D}\| - \|h^I\|) \\
&\leq \delta^{1/4}\|\chi_{I\cap D}\| + \|h^I - h_1^I\| + (\|\chi_{I\cap D}\| - \|h^I\|) \qquad \text{(by (2.11))} \\
&= \delta^{1/4}\|\chi_{I\cap D}\| + \langle h^I - h_1^I, g\rangle + (\|\chi_{I\cap D}\| - \|h^I\|) \qquad \text{(by (2.9))} \\
&= \delta^{1/4}\|\chi_{I\cap D}\| + \langle h^I, g\rangle - \langle h_1^I, g\rangle + (\|\chi_{I\cap D}\| - \|h^I\|) \\
&= \delta^{1/4}\|\chi_{I\cap D}\| + \|h^I\| - \|h_1^I\| + \|\chi_{I\cap D}\| - \|h^I\| \qquad \text{(by (2.8))}
\end{aligned}
$$

$$
\begin{aligned}
&= \delta^{1/4}\|\chi_{I\cap D}\| + \|\chi_{I\cap D}\| - \|h_1^I\| \\
&= \delta^{1/4}\|\chi_{I\cap D}\| + \|\chi_{I\cap D}\| - \|T(\chi_{I\cap D})\| + \|T(\chi_{I\cap D})\chi_{\left(W_I\cap B\right)^c}\| \\
&\leq 2\delta^{1/4}\|\chi_{I\cap D}\| + \|\chi_{I\cap D}\| - \|T(\chi_{I\cap D})\| \\
&\leq 2\delta^{1/4}\|\chi_{I\cap D}\| + \|\chi_{I\cap D}\| - \langle T(\chi_{I\cap D}), f\rangle \\
&\leq 2\delta^{1/4}\|\chi_{I\cap D}\| + \sqrt{\delta}\|\chi_{I\cap D}\|, \qquad \text{(by (2.5))}
\end{aligned}
$$

so $\|T - G\| \leq 3\delta^{1/4}$.

To conclude it is easy to check that

$$
\begin{aligned}
\langle G(y), g\rangle &= \frac{1}{\|z\chi_D\|}\langle G(\sum_{i=1}^{N}\alpha_i\chi_{D_i\cap D}), g\rangle \\
&= \frac{1}{\|z\chi_D\|}\sum_{i=1}^{N}\alpha_i\langle G(\chi_{D_i\cap D}), g\rangle \\
&= \frac{1}{\|z\chi_D\|}\sum_{i=1}^{N}\alpha_i(\|h^{D_i}\| + \|\chi_{D_i\cap D}\| - \|h^{D_i}\|) \\
&= \frac{1}{\|z\chi_D\|}\sum_{i=1}^{N}\alpha_i\|\chi_{D_i\cap D}\| = \frac{\|z\chi_D\|}{\|z\chi_D\|} = 1.
\end{aligned}
$$

which proves the theorem in the case $f(t) \geq 0$ for all real numbers t.

For the general case, consider the measurable set $A = \{t \in \mathbb{R} : f(t) > 0\}$. Then by Lemma 2.3.3 applied to the set A, we have $r(f)(t) \geq 0$ for all real number t, and $r(f)$, $r(x)$ and $R(T)$ satisfy the conditions of $\langle r(x), r(f)\rangle \in \Pi(L_1)$, $\|R(T)\| = 1$ and $\langle R(T)(r(x)), r(f)\rangle > 1 - \delta/2$. By the previous case we can find $(y, g) \in \Pi(L_1)$ and $G \in \mathcal{L}(L_1)$ such that $\|y - r(x)\| \leq \delta + 2\sqrt{\delta}$, $\|g - r(f)\| \leq \delta^{1/4}$, $\|G - R(T)\| \leq 3\delta^{1/4}$ and $\|G\| = \langle G(y), g\rangle = 1$. Now by Lemma 2.3.3 again we obtain that $(r(y), R(G)) \in \Pi(L_1)$ and $R(G) \in \mathcal{L}(L_1)$ are such that $\|r(y) - x\| \leq \delta + 2\sqrt{\delta}$, $\|R(G) - f\| \leq \delta^{1/4}$, $\|R(G) - T\| \leq 3\delta^{1/4}$ and $\|R(G)\| = \langle R(G)(r(y)), R(G)\rangle = 1$ which concludes the proof.

□

Remark 2.3.8. *These results can be extended to operators in $\mathcal{L}(L_1(\mathbb{R}^n))$, by using the dyadic partitions of the space $\mathbb{R}^n$ on cubes defined by $\Delta_n := \{\prod_{i=1}^{n}[\frac{z_i}{2^n}, \frac{z_i+1}{2^n}) : z_i \in \mathbb{Z}, i = 1, \ldots, n\}$. In general, for every finite dimensional real Banach space $\mathbb{R}^n$, the space $L_1(\mathbb{R}^n)$ has the Bishop-Phelps-Bollobás Property for numerical radius. Clearly the same kind of argument proves that the space $L_1[0,1]$ and actually $L_1(R)$ for any n-interval $R = \Pi_{i=1}^{n}[a_i, b_i]$ (of positive measure) in R^n has the BPBp-ν.*

2.4 Norm attaining extension to the bidual

With the counterexample provided by Lindenstrauss the hope of a generalization of the Bishop-Phelps result to the operator case vanishes. However, the author proved a general result for denseness of norm attaining operators using the adjoint operator. Given an operator T between two Banach spaces X and Y, the adjoint of T is defined from Y^* into X^* by $T^*(y^*)(x) = y^*(T(x))$, for all $x \in X, y^* \in Y^*$.

Theorem 2.4.1 (Lindenstrauss, [Lin63])**.** *Given two Banach spaces X and Y, the set of operators whose second adjoint attain their norm is norm-dense in $\mathcal{L}(X;Y)$.*

After this celebrated theorem several authors have tried to extend this result in many directions. For example in [CLM12], Carando, Lassalle, and Mazzitelli have studied the existence of the polynomial version of Lindenstrauss result under certain conditions and the possible existence of a quantitative version based on Bollobás result, as we will see in Section 3.6. Also, a multilinear approach to Lindenstrauss result, by using Arens extensions, has been studied during the last years, as we will see in Section 3.2. In particular, Acosta, García, and Maestre, [AGM06], extend this result to the multilinear case proving that the multilinear forms whose Arens extensions are norm attaining are norm dense, see Theorem 3.2.4.

Chapter 3

Norm attaining multilinear forms

"When you have mastered numbers, you will in fact no longer be reading numbers, any more than you read words when reading books. You will be reading meanings."

— W. E. B. Du Bois

In this chapter we extend the theory of norm attaining linear forms to the non-linear case. Our main point of interest is to study when the extensions of multilinear maps to the bidual are norm attaining, with special interest on multilinear forms on the space ℓ_1. To finish we will study the relation of the Lindenstrauss-Bollobás Theorems introduced by Carando, Lassalle and Mazzitelli in [CLM12], see Definition 3.6.1, and the n-linear version of Bishop-Phelps-Bollobás Theorem for spaces M-embedded or L-embedded in the bidual.

3.1 Norm attaining multilinear forms

Recall that the space of continuous linear operators $\mathcal{L}(X;Y^*)$ can be isometrically identified with the space of continuous bilinear forms $\mathcal{L}(X,Y)$ by the relation $T \in \mathcal{L}(X;Y^*)$ if and only if $A_T \in \mathcal{L}(X,Y)$ where $A_T(x,y) := \Big(T(x)\Big)(y)$, for all $x \in X$ and all $y \in Y$.

This naturally leads us to think about bilinear versions of Bishop-Phelps, Bishop-Phelps-Bollobás and Lindenstrauss' results and in a more general framework about multilinear versions of these results.

In general we say that a multilinear form $A \in \mathcal{L}(X_1, \dots, X_n; Y)$ attains its norm if there exists an n-tuple $(x_1^0, \dots, x_n^0) \in B_{X_1} \times \cdots \times B_{X_n}$ such that

$$\|A(x_1^0, \dots, x_n^0)\|_Y = \sup_{\substack{x_i \in B_{X_i} \\ i=1,\dots,n}} \|A(x_1, \dots, x_n)\|_Y = \|A\|_Y.$$

Therefore we can naturally set the questions:

Question 3.1.1. For what spaces $X_1, \dots, X_n$ is the set of norm attaining n-linear forms of $\mathcal{L}(X_1, \dots, X_n)$ dense in $\mathcal{L}(X_1, \dots, X_n)$?

We say that the spaces $X_1, \dots, X_n$ satisfy the n-linear version of Bollobás result if given $\epsilon > 0$ there are $\eta(\epsilon) > 0$ and $\beta(\epsilon) > 0$ with $\lim_{\epsilon \to 0} \beta(\epsilon) = 0$ such that for all $A \in S_{\mathcal{L}(X_1,\dots,X_n)}$, if $(x_1, \dots, x_n) \in S_{X_1} \times \cdots \times S_{X_n}$ is such that $|A(x_1, \dots, x_n)| > 1 - \eta(\epsilon)$, then there exist an n-tuple $(y_1, \dots, y_n) \in S_{X_1} \times \cdots \times S_{X_n}$ and the exist $B \in S_{\mathcal{L}(X_1,\dots,X_n)}$ that satisfy the following conditions:

$$\|B(y_1, \dots, y_n)\| = 1, \quad \|x_i - y_i\| < \beta(\epsilon) \text{ for } i = 1, \dots, n \text{ and } \|A - B\| < \epsilon.$$

Question 3.1.2. For what spaces $X_1, \dots, X_n$ does there exist an n-linear version of Bollobás result?

The first positive result in this direction appeared in [AFW95] where Aron, Finet and Werner showed that the Radon-Nikodým Property is a sufficient condition for the denseness of norm attaining multilinear forms. To be more specific.

Theorem 3.1.3. *If X is a Banach space with the Radon-Nikodým Property, then the set of norm attaining forms of $\mathcal{L}(^nX)$ is norm dense in $\mathcal{L}(^nX)$, for every natural number $n \geq 2$.*

In the particular case of $X = \ell_1$, we provide a simpler and constructive proof, generalizing Proposition 1.2.8.

Theorem 3.1.4. *For every natural number $n \geq 2$ the set of norm attaining forms of $\mathcal{L}(^n\ell_1)$ is norm dense in $\mathcal{L}(^n\ell_1)$.*

Proof. Fix $A \in \mathcal{L}(^n\ell_1), \|A\| = 1$.

$$\begin{aligned}
\|A\| &= \sup_{\substack{\|x_i\|=1\\ i=1,\dots,n}} |A(x_1,\dots,x_n)| \\
&= \sup_{\substack{\|x_i\|=1\\ i=1,\dots,n}} \Big| \sum_{t_1,\dots,t_n\in\mathbb{N}} \Big(\prod_{i=1}^n x_i\big(t_i\big)\Big) A(e_{t_1},\dots,e_{t_n})\Big| \\
&\leq \sup_{\substack{\|x_i\|=1\\ i=1,\dots,n}} \Big\{ \sum_{t_1,\dots,t_n\in\mathbb{N}} \Big|\Big(\prod_{i=1}^n x_i\big(t_i\big)\Big) A(e_{t_1},\dots,e_{t_n})\Big|\Big\} \\
&\leq \sup_{\substack{\|x_i\|=1\\ i=1,\dots,n}} \Big\{ \sup_{k_1,\dots,k_n\in\mathbb{N}} \Big\{|A(e_{k_1},\dots,e_{k_n})|\Big\} \sum_{t_1,\dots,t_n\in\mathbb{N}} \Big|\prod_{i=1}^n x_i\big(t_i\big)\Big|\Big\} \\
&= \sup_{\substack{\|x_i\|=1\\ i=1,\dots,n}} \Big\{ \sup_{k_1,\dots,k_n\in\mathbb{N}} \Big\{|A(e_{k_1},\dots,e_{k_n})|\Big\} \prod_{i=1}^n \|x_i\|\Big\} \\
&= \sup_{k_1,\dots,k_n\in\mathbb{N}} \Big\{|A(e_{k_1},\dots,e_{k_n})|\Big\} \leq \|A\|.
\end{aligned}$$

Hence $\|A\| = \sup_{k_1,\dots,k_n\in\mathbb{N}} \Big\{|A(e_{k_1},\dots,e_{k_n})|\Big\}$.

Fix $1 > \epsilon > 0$ and consider $k'_1, \ldots, k'_n$ such that $|A(e_{k'_1}, \ldots, e_{k'_n})| > 1 - \epsilon$. Then define

$$B(e_{k_1}, \ldots, e_{k_n}) = \begin{cases} sign(A(e_{k_1}, \ldots, e_{k_n})) & \text{if } (k_1, \ldots, k_n) = (k'_1, \ldots, k'_n), \\ A(e_{k_1}, \ldots, e_{k_n}) & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} \|A - B\| = &\sup_{k_1,\ldots,k_n \in \mathbb{N}} |(A-B)(e_{k_1}, \ldots, e_{k_n})| \\ =&|(A-B)(e_{k'_1}, \ldots, e_{k'_n})| \\ =&|A(e_{k'_1}, \ldots, e_{k'_n}) - sign\Big(A(e_{k'_1}, \ldots, e_{k'_n})\Big)| \\ =&1 - |A(e_{k'_1}, \ldots, e_{k'_n})| < \epsilon. \end{aligned}$$

To finish, we are going to show that B is norm attaining.

$$\begin{aligned} \|B\| &= \sup_{k_1,\ldots,k_n \in \mathbb{N}} \Big\{|B(e_{k_1}, \ldots, e_{k_n})|\Big\} \\ &= \max\Big\{|B(e_{k'_1}, \ldots, e_{k'_n})|, \sup_{(k_1,\ldots,k_n)\neq(k'_1,\cdots,k'_n)} |B(e_{k_1}, \ldots, e_{k_n})|\Big\} \\ &= \max\Big\{\Big|\Big(sign(A(e_{k'_1}, \ldots, e_{k'_n}))\Big)\Big|, \sup_{(k_1,\ldots,k_n)\neq(k'_1,\cdots,k'_n)} |A(e_{k_1}, \ldots, e_{k_n})|\Big\} \\ &= \max\Big\{1, \sup_{(k_1,\ldots,k_n)\neq(k'_1,\cdots,k'_n)} |A(e_{k_1}, \ldots, e_{k_n})|\Big\} \\ &= 1 = |B(e_{k'_1}, \ldots, e_{k'_n})|. \end{aligned}$$

Hence B is norm attaining at the point $(e_{k'_1}, \cdots, e_{k'_n})$.

□

However, a general result for multilinear mappings cannot be expected. A first counterexample was given in [AAP96] for bilinear forms. To be more specific they show that the same Gowers' space G used to show that there is no Bishop-Phelps Theorem for numerical radius in

[AAP92] can be used to exhibit an example of a Banach space whose set of norm attaining bilinear forms is not dense. A few years later, Jiménez Sevilla and Payá in [JP98] showed a stronger result. For every natural number n, using the canonical predual of a suitable Lorentz sequence space, the authors gave an example of a Banach space X such that X satisfies the n-linear version of Bishop-Phelps result but X does not satisfy the $n+1$-linear version of Bishop-Phelps result.

Definition 3.1.5 (Lorentz sequence space)**.** By admissible sequence we shall mean a decreasing sequence $\omega = \{\omega_n\}_{n=1}^{\infty}$ of positive numbers such that $\omega_1 = 1$ and $\omega \in c_0 \setminus \ell_1$. For $1 \leq p < \infty$, we define the Lorentz sequence space as the Banach space of all sequences of (real or complex) scalars $x = \{x_n\}_{n=1}^{\infty}$ for which

$$\|x\| =: \sup_{\pi} \left(\sum_{n} |x_{\pi(n)}|^p \omega_n \right)^{\frac{1}{p}} < \infty,$$

where π ranges over all possible permutations of the integers.

Lorentz sequence space is denoted by $d(\omega, p)$.

Recall that for $p > 1$ the Lorentz sequence space is reflexive, and hence it has the Radon-Nikodým Property, so we will be interested in the case of $p = 1$. Also, for this case we know that the predual of $d(\omega, 1)$ is denoted by $d_*(\omega, 1)$ and it is defined by

$$d_*(\omega, 1) = \left\{ a \in c_0 : \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \bar{x}_k}{\sum_{k=1}^{n} \omega_k} = 0 \right\},$$

where $(\bar{x}_n)_n$ is the decreasing rearrangement of $\{|a_n|\}$. The norm of $x \in d_*(\omega, 1)$ is defined by

$$\|x\| = \sup_{n} \frac{\sum_{k=1}^{n} \bar{x}_k}{\sum_{k=1}^{n} \omega_k}.$$

The second question was studied by Choi and Song in [CS09] where they show that the Bishop-Phelps-Bollobás Theorem fails for n-linear forms on $\ell_1 \times \cdots \times \ell_1$. In particular, for the bilinear case, they show that given the bilinear form A defined over the canonical basis of ℓ_1 by $A(e_i, e_j) = 1 - \delta_{ij}$, for every $\eta > 0$ there exists a point $x_\eta \in S_{\ell_1}$ such that $A(x_\eta, x_\eta) > 1 - \eta$, but if $0 < \epsilon < 1$ and B is a norm attaining bilinear form at (y_1, y_2) with $\|A - B\| < 1$ then $\|x_\eta - y_1\| \geq 1/2$ or $\|x_\eta - y_2\| \geq 1/2$. Therefore the Banach space ℓ_1 does not satisfy the n-linear version of the Bishop-Phelps-Bollobás Theorem even though ℓ_1 has the Radon-Nikodým Property and it satisfies the n-linear version of Bishop-Phelps Theorem for every natural number n bigger or equal than two.

Notice also that the Banach space ℓ_∞ has the AHSP. Hence the pair (l_1, Y) satisfies the Bishop-Phelps-Bollobás Property for operators. One way of seeing that ℓ_∞ has the AHSP is to use that ℓ_∞ satisfies Property β and the fact that Property β is a sufficient condition for the AHSP. However, as Choi and Song proved, [CS09], the pair $\ell_1 \times \ell_1$ does not satisfy the Bishop-Phelps-Bollobas Property for bilinear maps. Therefore, even though we have an isometric isomorphism between the space $\mathcal{L}(X, Y)$ and the space $\mathcal{L}(X; Y^*)$, it is not enough to know that the Bishop-Phelps-Bollobás Property for operators holds for the pair $(X; Y^*)$ in order to ensure that the pair (X, Y) satisfies the Bishop-Phelps-Bollobás Property for bilinear forms. However if the Bishop-Phelps-Bollobás Property for bilinear forms holds for the pair (X, Y), then the Bishop-Phelps-Bollobás Property for operators holds for the pair $(X; Y^*)$.

Therefore in general we can not expect to get multilinear versions of Bollobás result using multilinear versions of Bishop-Phelps result or operator versions of Bollobás result.

3.2 Arens extensions and norm attaining multilinear forms

Now, it is clear that the multilinear version of Bishop-Phelps and Bishop-Phelps-Bollobás Theorems do not hold in general, but since Lindenstrauss result is true for any pair of Banach spaces X, Y, we can ask whether or not this result still works for bilinear maps defined on $X \times Y$ or more generally for n-linear maps. For this we need first a tool to extend multilinear maps.

In 1951, Arens [Are51] found a natural way to extend a continuous bilinear mapping $A : X_1 \times X_2 \mapsto Y$ to a continuous bilinear mapping from $X_1^{**} \times X_2^{**}$ into Y^{**}. His method consists in applying three times the operation defined as

$$\begin{array}{rccl} A^t : & Y^* \times X_1 & \mapsto & X_2^* \\ & (y^*, x_1) & \rightsquigarrow & A^t(y^*, x_1)(x_2) = y^*(A(x_1, x_2)), \end{array}$$

for $x_1 \in X_1, x_2 \in X_2$ and $y^* \in Y^*$. The first extension is defined as $A^{ttt} : X^{**} \times Y^{**} \mapsto Z^{**}$ and the second one is A^{TtttT}, where $B^T(x_1, x_2) = B(x_2, x_1)$ for any bilinear mapping B. These extensions, that are in general different, are known as Arens products. This procedure was generalized by Aron and Berner [AB78] to arbitrary multilinear mappings.

For our purposes we will use an alternative approach due to Davie and Gamelin [DG89]. The key of this approach is Goldstine's theorem as it is based on limits in the *weak-star topology*, denoted by $w(X^{**}, X^*)$. Consider Σ_n the group of all permutations of the set $\{1, \dots, n\}$. Given $\sigma \in \Sigma_n$ they defined the extension A_σ associated to σ of an n-linear form A defined on $X_1 \times \cdots \times X_n$, by

$$A_\sigma(x_1^{**}, \dots, x_n^{**}) = \lim_{d_{\sigma(1)}} \cdots \lim_{d_{\sigma(n)}} A(x_{d_1}, \dots, x_{d_n}),$$

where $\{x_{d_i}\}_{d_i}$ is a bounded net in X_i ($\|x_{d_i}\| \leq \|x_i^{**}\|$, for all d_i) $w(X^*, X)$ convergent to $x_i^{**} \in X_i^{**}$, for $i = 1, \ldots, n$. The mapping A_σ is called an *Arens extension* of A and there are $n!$ Arens extensions that may be different from each other. In fact, in Proposition 3.2.3 we will see a specific example of an n-linear form whose Arens extensions are pairwise different. When convenient, we shall write $A_{\sigma(1),\ldots,\sigma(n)}$ instead of A_σ. In particular, for $n = 2$, $A_{Id} = A_{1,2} = A^{ttt}$ and $A_{2,1} = A^{TtttT}$, where Id is the identity permutation of the set $\{1, 2\}$.

Note that the use of the $w(X^*, X)$ topology does not generally allow the use of sequences in the above limits. However we will show in Theorem 3.3.2 that in the study of norm attaining multilinear forms one can reduce such iterated limits to sequential ones.

Even though all the Arens extensions of a multilinear form A have the same norm as A, it is worth mentioning that in [AGM03] the following example is given of a bilinear mapping such that only one of its Arens extensions attains its norm.

Example 3.2.1 ([AGM03, Example 2])**.** The bilinear form $A \in \mathcal{L}(^2\ell_1)$ defined by

$$A(x_1, x_2) := \sum_{t_1=1}^{\infty} x_1\big(t_1\big)\Big(\sum_{t_2=1}^{t_1} \frac{t_2}{t_2+1} x_2\big(t_2\big)\Big),$$

is such that neither A nor $A_{Id} = A_{1,2}$ is norm attaining, but $A_{2,1}$ is norm attaining.

This example leads us to believe that the extensions of a bilinear form may have different behaviors from the point of view of attaining their norms, and it is the core of our study in this section.

To start, we can consider the natural generalization of the Example 3.2.1.

Example 3.2.2. Let $A \in \mathcal{L}(^n\ell_1)$ be defined by

$$A(x_1,\ldots,x_n) = \sum_{k_1=1}^{\infty} x_1(k_1)\Big(\sum_{k_2=1}^{k_1} \frac{k_2}{k_2+1} x_2(k_2)(\cdots(\sum_{k_n=1}^{k_{n-1}} \frac{k_n}{k_n+1} x_n(k_n))\cdots)\Big).$$

Then $A_{n,n-1,\ldots,1}$ is norm attaining but neither A nor any other Arens extension of A is norm attaining.

Proof. Clearly $\|A\| \leq 1$ and since

$$A(e_{k_1},\ldots,e_{k_n}) = \prod_{i=2}^{n} \frac{k_i}{k_i+1} \qquad \text{if} \qquad k_1 \geq k_2 \geq \ldots \geq k_n, \tag{3.1}$$

we have that A has norm one. It is also easy to see that A does not attain its norm.

We first prove that the extension $A_{n,n-1,\ldots,1}$ is norm attaining.

By (3.1), $\lim_{k_1} \ldots \lim_{k_n} A(e_{k_1},\ldots,e_{k_n}) = 1$. Therefore, if $x^{**} \in \ell_1^{**}$ is a ω^*-cluster point of the sequence $\{e_k\}_{k=1}^{\infty}$, then

$$\begin{aligned} A_{n,n-1,\ldots,1}(x^{**},\ldots,x^{**}) &= \lim_{d_n} \cdots \lim_{d_1} A(e_{d_1},\ldots,e_{d_n}) \\ &= \lim_{k_1} \ldots \lim_{k_n} A(e_{k_1},\ldots,e_{k_n}) = 1 \end{aligned}$$

for a suitable subnet $\{e_d\}_d$ of the sequence $\{e_k\}_{k=1}^{\infty}$. So the extension $A_{n,n-1,\ldots,1}$ attains its norm at $(x^{**},\ldots,x^{**})$.

Now, we are going to show that the other extensions do not attain their norms. Consider $\sigma \in \Sigma_n, \sigma \neq Id$. Assume that A_σ attains its norm. Then, there exists an n-tuple $(x_1^{**},\ldots,x_n^{**})$ in the closed unit ball of $\ell_1^{**} \times \cdots \times \ell_1^{**}$ such that $A_\sigma(x_1^{**},\ldots,x_n^{**}) = 1$. As $\sigma \neq Id$ there exist $1 \leq r < s \leq n$ with $\sigma(r) > \sigma(s)$.

We may assume without loss of generality $x_{d_i}(k) \geq 0$, for all i, d_i, k. Indeed, for each $i = 1,\ldots,n$, consider a net $\{x_{d_i}\}_{d_i}$ in the closed unit ball of ℓ_1, ω^*-convergent to x_i^{**}. Let $\widehat{x_i^{**}}$ be a ω^*-cluster point of the net $\left\{\{|x_{d_i}(k)|\}_{k=1}^{\infty}\right\}_{d_i}$. Hence there exists a subnet, that we denote in the same way, ω^*-convergent to $\widehat{x_i^{**}}$. In this case, $\lim_{d_{\sigma(i)}} x_{d_{\sigma(i)}} = x_{\sigma(i)}^{**}$, $i = 1, \ldots, n$, also. Therefore,

$$
\begin{aligned}
1 = A_\sigma(x_1^{**}, \ldots, x_n^{**}) &= \lim_{d_{\sigma(1)}} \cdots \lim_{d_{\sigma(n)}} A(x_{d_1}, \ldots, x_{d_n}) \\
&= \lim_{d_{\sigma(1)}} \cdots \lim_{d_{\sigma(n)}} \sum_{t_1=1}^{\infty} x_{d_1}\Big(t_1\Big)\Bigg(\cdots\Big(\sum_{t_n=1}^{t_{n-1}} \frac{t_n}{t_n+1} x_{d_n}\Big(t_n\Big)\Big)\cdots\Bigg) \\
&\le \lim_{d_{\sigma(1)}} \cdots \lim_{d_{\sigma(n)}} \sum_{t_1=1}^{\infty} \Big|x_{d_1}\Big(t_1\Big)\Big|\Bigg(\cdots\Big(\sum_{t_n=1}^{t_{n-1}} \frac{t_n}{t_n+1} \Big|x_{d_n}\Big(t_n\Big)\Big|\Big)\cdots\Bigg) \\
&\le \lim_{d_{\sigma(1)}} \cdots \lim_{d_{\sigma(n)}} \sum_{t_{\sigma(s)}=1}^{\infty} \Big|x_{d_{\sigma(s)}}\Big(t_{\sigma(s)}\Big)\Big| \times \\
&\qquad\qquad \Big(\sum_{t_{\sigma(r)}=1}^{t_{\sigma(s)}} \frac{t_{\sigma(r)}}{t_{\sigma(r)}+1}\Big|x_{d_{\sigma(r)}}\Big(t_{\sigma(r)}\Big)\Big|\Big) \prod_{k\neq i,j} \|x_{d_{\sigma(k)})}\| \\
&\le \lim_{d_{\sigma(r)}} \lim_{d_{\sigma(s)}} \sum_{t_{\sigma(s)}=1}^{\infty} \Big|x_{d_{\sigma(s)}}(t_{\sigma(s)})\Big|\Big(\sum_{t_{\sigma(r)}=1}^{t_{\sigma(s)}} \frac{t_{\sigma(r)}}{t_{\sigma(r)}+1}\Big|x_{d_{\sigma(r)}}\Big(t_{\sigma(r)}\Big)\Big|\Big) \\
&= A_{1,2}(\widehat{x_{d(\sigma(r))}^{**}}, \widehat{x_{d(\sigma(s))}^{**}}).
\end{aligned}
$$

But this is a contradiction because $\widehat{x_{d(\sigma(r))}^{**}}, \widehat{x_{d(\sigma(s))}^{**}}$ belongs to the unit ball of ℓ_1^{**} and the extension $A_{2,1}$ of the Example 3.2.1 is not norm attaining.

□

Using the idea of Example 3.2.1, we can produce an example of an n-linear map such that all the Arens extensions of this map are pairwise different. For $r \neq s$, define the affine subspace of $\ell_1 \times \cdots \times \ell_1$

$$
X_{r,s} := \left\{ (x_1, \ldots, x_n) \in \ell_1 \times \cdots \times \ell_1 : \begin{array}{c} x_i = e_1 \text{ if } i \neq r, s \\ x_r(1) = x_s(1) = 0 \end{array} \right\}.
$$

Then, for two fixed different permutations $\sigma, \eta \in \Sigma_n$ we can define $i_0 := \max\{i : \sigma(i) \neq \eta(i), 1 \leq i \leq n\}$ and consider the affine subspace $X_{\sigma(i_0),\eta(i_0)}$.

Proposition 3.2.3. *Let $B \in \mathcal{L}(^n\ell_1)$ defined by*

$$B(x_1, x_2, \ldots, x_n) := \sum_{r=2}^{n} \sum_{j=1}^{r-1} (\prod_{\substack{i=1 \\ i \neq r,j}}^{n} x_i(1)) A(\{x_j(t)\}_{t=2}^{\infty}, \{x_r(t)\}_{t=2}^{\infty})$$

where A is the bilinear form of the Example 3.2.1. If we consider two different permutations $\sigma, \eta \in \Sigma_n$, with $\sigma(i_0) > \eta(i_0)$ where $i_0 := \max\{i : \sigma(i) \neq \eta(i), 1 \leq i \leq n\}$, then the Arens extension B_σ attains its supremum on the ω^*-closure of the set $X_{\sigma(i_0),\eta(i_0)}$ defined as above, whereas B_η does not attain its supremum on the ω^*-closure of the set $X_{\sigma(i_0),\eta(i_0)}$.

Furthermore, the Arens extensions of B associated to these permutations are different, hence all the extensions of B are pairwise different.

Proof. Fix two permutations $\sigma, \eta \in \Sigma_n$ such that $\sigma(i_0) > \eta(i_0)$. First we show that $|B_\sigma|$ does not attain the supremum at a w^*-cluster point of $X_{\sigma(i_0),\eta(i_0)}$.

Notice that $x_{\sigma(i_0)}(1) = x_{\eta(i_0)}(1) = 0$ and $x_d(1) = 1$ for all d different from $\sigma(i_0), \eta(i_0)$ whenever $(x_1, \ldots, x_n)$ belongs to $X_{\sigma(i_0),\eta(i_0)}$. Then, on the subset $X_{\sigma(i_0),\eta(i_0)}$, $B(x_1, \ldots, x_n) = A\Big(\{x_{\eta(i_0)}(t)\}_{t=2}^{\infty}, \{x_{\sigma(i_0)}(t)\}_{t=2}^{\infty}\Big)$ so

$$\sup_{(x_1,\ldots,x_n) \in X_{\sigma(i_0),\eta(i_0)}} |B(x_1, \ldots, x_n)| = 1.$$

Let $(x_1^{**}, \ldots, x_n^{**})$ be an element in the ω^*-closure of $X_{\sigma(i_0),\eta(i_0)}$ and consider nets $\{x_{d_i}\}_{d_i \in D_i}$, $i = 1, \ldots, n$, ω^*-convergent to x_i^{**} respectively, such that $(x_{d_1}, \ldots, x_{d_n}) \in X_{\sigma(i_0),\eta(i_0)}$ for all $d_1, \ldots, d_n$. Then $B(x_{d_1}, \ldots, x_{d_n}) = A(\{x_{d_{\eta(i_0)}}(t)\}_{t=2}^{\infty}, \{x_{d_{\sigma(i_0)}}(t)\}_{t=2}^{\infty})$ in $X_{\sigma(i_0),\eta(i_0)}$.

Therefore,

$$\begin{aligned} B_\sigma(x_1^{**},\dots,x_n^{**}) &= \lim_{d_{\sigma(1)}}\cdots\lim_{d_{\sigma(n)}} B(x_{d_1},\dots,x_{d_n}) \\ &= \lim_{d_{\sigma(1)}}\cdots\lim_{d_{\sigma(n)}} A\Big(\{x_{d_{\eta(i_0)}}(t)\}_{t=2}^{\infty},\{x_{d_{\sigma(i_0)}}(t)\}_{t=2}^{\infty}\Big) \\ &= \lim_{d_{\eta(i_0)}}\lim_{d_{\sigma(i_0)}} A\Big(\{x_{d_{\eta(i_0)}}(t)\}_{t=2}^{\infty},\{x_{d_{\sigma(i_0)}}(t)\}_{t=2}^{\infty}\Big) \\ &= A_{1,2}\Big(\widehat{x_{\eta(i_0)}^{**}},\widehat{x_{\sigma(i_0)}^{**}}\Big) \end{aligned}$$

where $\widehat{x_{\eta(i_0)}^{**}}$, $\widehat{x_{\sigma(i_0)}^{**}}$ are the ω^*-limit of the nets $\Big\{\{x_{d_{\eta(i_0)}}(t)\}_{t=2}^{\infty}\Big\}_{d_{\eta(i_0)}}$, $\Big\{\{x_{d_{\sigma(i_0)}}(t)\}_{t=2}^{\infty}\Big\}_{d_{\sigma(i_0)}}$ respectively. This happens since the backward shift defined from ℓ_1 into ℓ_1 as $L(\{x(t)\}_{t=1}^{\infty}) = \{x(t)\}_{t=2}^{\infty}$ is $\|\cdot\|$-$\|\cdot\|$-continuous, so L is ω-ω-continuous and the canonical extension $\widehat{L}$ defined from ℓ_1^{**} into ℓ_1^{**} is ω^*-ω^*-continuous. As $\{x_{d_{\eta(i_0)}}\}_{d_{\eta(i_0)}}$ and $\{x_{d_{\sigma(i_0)}}\}_{d_{\sigma(i_0)}}$ are ω^*-convergent to $x_{d_{\eta(i_0)}}^{**}$ and $x_{d_{\sigma(i_0)}}^{**}$ respectively, then $L(\{x_{d_{\eta(i_0)}}\}_{d_{\eta(i_0)}})$ and $L(\{x_{d_{\sigma(i_0)}}\}_{d_{\sigma(i_0)}})$ are ω^*-convergent to some points $\widehat{x_{\eta(i_0)}^{**}}$ and $\widehat{x_{\sigma(i_0)}^{**}}$ respectively.

By Example 3.2.1, $A_{1,2}$ does not attain its norm, hence $|B_\sigma|$ does not attains its supremum on the ω^*-closure of $X_{\sigma(i_0),\eta(i_0)}$.

Similar calculations show that $B_\eta(x_1^{**},\dots,x_n^{**}) = A_{2,1}(\widehat{x_{\eta(i_0)}^{**}},\widehat{x_{\sigma(i_0)}^{**}})$. By Example 3.2.1, $A_{2,1}$ attains its norm, hence $|B_\eta|$ attains its supremum on the ω^*-closure of $X_{\sigma(i_0),\eta(i_0)}$.

□

Based on Lindenstrauss' result and making use of the Arens extensions to the second duals, Acosta [Aco98] proved a Lindenstrauss type result for bilinear forms whose third Arens transpose attains its norm. Afterwards, in [AGM03] the denseness of bilinear forms whose Arens extensions to the biduals attain their norms at the same point was established.

The generalization of Lindenstrauss' result to n-linear vector-valued mappings was finally obtained in [AGM06] in its strongest form.

Theorem 3.2.4 ([AGM06, Theorem 2.1])**.** *Let X_i be Banach spaces $(1 \leq i \leq n)$. Then the set of n-linear forms on $X_1 \times \cdots \times X_n$ such that all their Arens extensions to $X_1^{**} \times \cdots \times X_n^{**}$ attain their norms at the same n-tuple is dense in the space $\mathcal{L}(X_1, \ldots, X_n)$.*

The asymmetry between the Arens extensions of Proposition 3.2.3 reveals the importance of the stronger condition of attaining their norms simultaneously in Theorem 3.2.4.

3.3 Sequences characterization of norm attaining Arens extensions

We now present some general results on norm attaining Arens extensions of multilinear forms on general Banach spaces.

It is well known that, under the first axiom of separability, nets can be replaced with sequences, which turns out to be an advantage when dealing with limits. Our first result is just a lemma that will clarify how to pass from nets to sequences in the context of several indices that will be helpful in the context of multilinear mappings.

Lemma 3.3.1. *Let $n \in \mathbb{N}$. For each $j = 1, \ldots, n$ let D_j be an infinite directed set. Consider a family $\{a_{\alpha_1,\ldots,\alpha_n}\}_{(\alpha_1,\ldots,\alpha_n)\in D_1\times\cdots\times D_n}$ of real or complex numbers. If the iterated limit $a := \lim_{\alpha_1\in D_1} \cdots \lim_{\alpha_n\in D_n} a_{\alpha_1,\ldots,\alpha_n}$ is finite then there exist strictly increasing sequences $\{\alpha_j(m)\}_{m=1}^{\infty}$ in D_j, $1 \leq j \leq n$, such that*

$$\lim_{m_1\to\infty} \cdots \lim_{m_n\to\infty} a_{\alpha_1(m_1),\ldots,\alpha_n(m_n)} = a.$$

Proof. We proceed by induction on n. For $n = 1$, since $\lim_{\alpha_1 \in D_1} a_{\alpha_1} = a$, for each $k \in \mathbb{N}$ there exists $\alpha_1(k) \in D_1$ such that $|a_{\alpha_1} - a| < \frac{1}{k}$ for all $\alpha_1 \geq \alpha_1(k)$. Besides, by the condition on D_1, we can choose the sequence $\{\alpha_1(k)\}_{k\in\mathbb{N}}$ strictly increasing.

Assume that the result is true for $n-1$ and let us prove it for n. So, if we assume that $a = \lim_{\alpha_1 \in D_1} \cdots \lim_{\alpha_n \in D_n} a_{\alpha_1,\ldots,\alpha_n}$ is finite, define

$$b_{\alpha_1,\ldots,\alpha_{n-1}} := \lim_{\alpha_n \in D_n} a_{\alpha_1,\ldots,\alpha_{n-1},\alpha_n}.$$

By the assumption before applied to the family of numbers

$$\{b_{\alpha_1,\ldots,\alpha_{n-1}}\}_{(\alpha_1,\ldots,\alpha_{n-1})\in D_1\times\cdots\times D_{n-1}},$$

for each $j = 1, \ldots, n-1$ there exists a strictly increasing sequence $\{\alpha_j(m_j)\}_{m_j\in\mathbb{N}}$ such that

$$a = \lim_{m_1\to\infty} \cdots \lim_{m_{n-1}\to\infty} b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})}.$$

Let us construct the sequence $\{\alpha_n(k)\}_{k\in\mathbb{N}}$ by induction on k.

Since $b_{\alpha_1(1),\ldots,\alpha_{n-1}(1)} = \lim_{\alpha_n\in D_n} a_{\alpha_1(1),\ldots,\alpha_{n-1}(1),\alpha_n}$, there exists $\alpha_n(1) \in D_n$ such that

$$|b_{\alpha_1(1),\ldots,\alpha_{n-1}(1)} - a_{\alpha_1(1),\ldots,\alpha_{n-1}(1),\alpha_n}| < 1$$

for all $\alpha_n \geq \alpha_n(1)$. Assume that we have found $\alpha_n(1), \ldots, \alpha_n(k-1) \in D_n$ with $\alpha_n(1) < \ldots < \alpha_n(k-1)$ and such that $|b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})} - a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n}| < \frac{1}{l}$ for all $\alpha_n \geq \alpha_n(l)$, all $1 \leq m_1, \ldots, m_{n-1} \leq l$ and all $l = 1, \ldots, k-1$.

Fix $1 \leq m_1, \ldots, m_{n-1} \leq k$. Since

$$b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})} = \lim_{\alpha_n\in D_n} a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n},$$

there exists $\alpha_n(m_1,\ldots,m_{n-1}) \in D_n$, with $\alpha_n(m_1,\ldots,m_{n-1}) \geq \alpha_n(k-1)$, such that

$$|b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})} - a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n}| < \frac{1}{k}$$

for all $\alpha_n \geq \alpha_n(m_1,\ldots,m_{n-1})$. Take $\alpha_n(k) > \alpha_n(m_1,\ldots,m_{n-1})$ for all $1 \leq m_1,\ldots,m_{n-1} \leq k$. Then

$$|b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})} - a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n}| < \frac{1}{k}$$

whenever $\alpha_n \geq \alpha_n(k)$. Hence the limit $\lim_{k\to\infty} a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n(k)}$ exists and is equal to $b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})}$. Now,

$$\begin{aligned} a &= \lim_{m_1\to\infty}\cdots\lim_{m_{n-1}\to\infty} b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})} \\ &= \lim_{m_1\to\infty}\cdots\lim_{m_{n-1}\to\infty}\lim_{m_n\to\infty} a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n(m_n)} \end{aligned}$$

and the proof is complete. □

Theorem 3.3.2. *Let $X_1,\ldots,X_n$ be infinite dimensional Banach spaces, $C \in \mathcal{L}(X_1,\ldots,X_n)$ and $\sigma \in \Sigma_n$. If the extension C_σ attains its norm, then there exist sequences $\{x^1_{m_1}\}_{m_1=1}^\infty,\ldots,\{x^n_{m_n}\}_{m_n=1}^\infty$ with each $x^k_{m_k} \in B_{X_k}$, $m_k \in \mathbb{N}$ and $k=1,\ldots,n$, such that*

$$\lim_{m_{\sigma(1)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty}|C(x^1_{m_1},\ldots,x^n_{m_n})| = \|C\|.$$

Proof. For simplicity we assume that $\sigma = Id$. Let $(x_1^{**},\ldots,x_n^{**})$ be a point in $B_{X_1^{**}}\times\cdots\times B_{X_n^{**}}$ where C_σ attains its norm. Let $K = \{k : x_k^{**} \in X^{**}\setminus X\}$. By density, each x_k^{**} is the weak-star limit of a net $\{x^k_{\alpha_k}\}_{\alpha_k\in D_k}$ in B_{X_k}, $k \in K$. For $k \notin K$, set $D_k = \mathbb{N}$ and $x^k_{\alpha_k} := x_k^{**} \in X$ for all $\alpha_k \in D_k$. Then

$$\|C\| = \|C_\sigma\| = |C_\sigma(x_1^{**}, \dots, x_n^{**})| = \lim_{\alpha_1 \in D_1} \cdots \lim_{\alpha_n \in D_n} |C(x^1_{\alpha_1}, \dots, x^n_{\alpha_n})|.$$

By Lemma 3.3.1 applied to $a_{\alpha_1,\dots,\alpha_n} := |C(x^1_{\alpha_1}, \dots, x^n_{\alpha_n})|$, we obtain the desired sequences $\{x^k_{m_k}\}_{m_k=1}^\infty$, for every $1 \leq k \leq n$. □

Given a Banach space X, a sequence $\{x_n\}_{n=1}^\infty$ in X such that $\{x_n\}_{n=1}^\infty$ is a basis for its closed linear span is called a *basic sequence* in X. It is easy to show that every infinite-dimensional Banach space has a basic sequence; that is, every Banach space X contains a closed, infinite-dimensional subspace Y with a basis. For this we need the definition of cofinal set. A subset D of A is said to be cofinal if for every $a \in A$, there exists some $d \in D$ such that $a \leq d$.

Proposition 3.3.3. *Let X be a non reflexive infinite dimensional Banach space, and let $\{x_n\}_{n=1}^\infty$ be a basic sequence. Then, any non zero weak-star cluster point of $\{x_n\}_{n=1}^\infty$ belongs to $X^{**} \setminus \overline{\{x_i\}_{i=1}^\infty}$.*

Proof. Let Z be the closed linear span of $\{x_n\}_{n=1}^\infty$ and let $\{x_n^*\}_{n=1}^\infty$ be the orthogonal functionals in Z^* associated to $\{x_n\}_{n=1}^\infty$. By the Hahn-Banach Extension Theorem, we can consider each x_n^* in X^*.

Let $x^{**} \in X^{**}$ be a non zero cluster point of $\{x_n\}_{n=1}^\infty$, and let $\{x_d\}_{d\in D}$ be a subnet of $\{x_n\}_{n=1}^\infty$ weak-star converging to x^{**}.

We first prove that x^{**} is none of the vectors x_n. Assume this is not the case, that is $x^{**} = x_{n_0}$ for some n_0. Since $\{x_d\}_{d\in D}$ weak-star converges to x^{**}, the net $\{\langle x_d, x_{n_0}^*\rangle\}_{d\in D}$ converges to $\langle x^{**}, x_{n_0}^*\rangle = 1$. Then, there is $\tilde{d} \in D$ such that

$$|\langle x_d, x_{n_0}^*\rangle| > \frac{1}{2} \tag{3.2}$$

for all $d \geq \tilde{d}$. Since D is cofinal, there is $d_1 \in D$ such that $d_1 \geq \tilde{d}$ and $d_1 \geq n_0 + 1 > n_0$. By the biorthogonality of $\{x_n^*\}_{n=1}^\infty$ it follows that $\langle x_{d_1}, x_{n_0}^*\rangle = 0$, which contradicts (3.2).

We prove now that $x^{**} \notin Z$. Let us assume that $x^{**} \in Z$. Then there is a unique sequence of scalars $\{a_n\}_{n=1}^{\infty}$ so that $x^{**} = \sum_{n=1}^{\infty} a_n x_n$. Let $\epsilon > 0$ and take $n_1 := 1$. Since $\{\langle x_d, x_1^* \rangle\}_{d \in D}$ converges to $\langle x^{**}, x_1^* \rangle = a_1$, there is $\tilde{\tilde{d}} \in D$ so that $|\langle x_d, x_1^* \rangle - a_1| < \epsilon$ for all $d \geq \tilde{\tilde{d}}$. Since D is cofinal, there is $\tilde{d}_1 \in D$ such that $\tilde{d}_1 \geq 2$. Let $\tilde{d}_2 \geq \tilde{d}_1, \tilde{\tilde{d}}$. Then $n_2 := \tilde{d}_2 \geq \tilde{d}_1 > 1 = n_1$. Therefore $\langle x_{\tilde{d}_2}, x_1^* \rangle = \langle x_{n_2}, x_1^* \rangle = 0$ and $|\langle x_{\tilde{d}_2}, x_1^* \rangle - a_1| < \epsilon$. Hence, $|a_1| < \epsilon$. This shows that $a_1 = 0$. Reiterating this process we can prove that $a_n = 0$ for all $n \in \mathbb{N}$, which contradicts that $x^{**} \neq 0$.

□

Now, we can use Proposition 3.3.3 to characterize the multilinear forms that attain their norm at points strictly in the bidual, in the case that the spaces $X_1, \ldots, X_n$ admit a normalized Schauder basis. Recall that a *Schauder basis* of a Banach space X is a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of X such that for every element $x \in X$ there exists a unique sequence $\{\alpha_n\}_{n=1}^{\infty}$ of scalars in $\mathbb{K}$ so that $x = \sum_{n=0}^{\infty} \alpha_n x_n$, where the convergence is understood with respect to the norm topology, i.e., $\lim_{m\to\infty} \|x - \sum_{n=0}^{m} \alpha_n x_n\|_X = 0$.

Theorem 3.3.4. *Let $n \in \mathbb{N}$. For each $1 \leq j \leq n$ let X_j be a Banach space with a normalized Schauder basis $\{x_n^j\}_{n=1}^{\infty}$. Let $C \in \mathcal{L}(X_1, \ldots, X_n)$ and $\sigma \in \Sigma_n$. If there exist strictly increasing sequences of natural numbers $\{k(j, m_j)\}_{m_j=1}^{\infty}$, $j = 1, \ldots, n$, such that*

$$\lim_{m_{\sigma(1)}\to\infty} \cdots \lim_{m_{\sigma(n)}\to\infty} |C(x_{k(1,m_1)}^1, \ldots, x_{k(n,m_n)}^n)| = \|C\|$$

then C_σ attains its norm at a point in $(B_{X_1^{**}} \setminus X_1) \times \cdots \times (B_{X_n^{**}} \setminus X_n)$.

Proof. Consider any $1 \leq j \leq n$. Let x_j^{**} be a cluster point of the subsequence $\{x_{k(j,m_j)}^j\}_{m_j=1}^{\infty}$ and hence of the sequence $\{x_n^j\}_{n=1}^{\infty}$. As the Schauder basis is normalized, $x_j^{**} \in B_{X_j^{**}}$ and by Proposition 3.3.3 $x_j^{**} \notin$

X_j. Let $\{x_{k(j,d_j)}\}_{d_j \in D_j}$ be a subnet of $\{x_{k(j,m_j)}\}_{m_j=1}^{\infty}$ that weak-star converges to x_j^{**}. Then

$$\begin{aligned} \|C_\sigma\| = \|C\| &= \lim_{m_{\sigma(1)}\to\infty} \cdots \lim_{m_{\sigma(n)}\to\infty} |C(x^1_{k(1,m_1)}, \dots, x^n_{k(n,m_n)})| \\ &= \lim_{d_{\sigma(1)}\in D_{\sigma(1)}} \cdots \lim_{d_{\sigma(n)}\in D_{\sigma(n)}} |C(x^1_{k(1,d_1)}, \dots, x^n_{k(n,d_n)})| \\ &= |C_\sigma(x_1^{**}, \dots, x_n^{**})|. \end{aligned}$$

□

3.4 Norm attaining extensions of multilinear forms on ℓ_1

Our aim now is to show that, when working with the space ℓ_1, one can strengthen the results in Section 3.3. But before, let us recall some well known facts about ℓ_1 that we need to use later. The first is that, since ℓ_1^{**} is the third dual of c_0, then ℓ_1 is a complemented subspace of ℓ_1^{**}. Actually, $\ell_1^{**} = c_0^* \oplus c_0^\perp = \ell_1 \oplus c_0^\perp$, where a linear form belongs to $c_0^\perp$ if and only if it vanishes on c_0. Moreover, ℓ_1^{**} is an 1-sum of ℓ_1 and $c_0^\perp$ ([HWW93, p.158]) i.e. if we denote by $\pi : \ell_1^{**} \to \ell_1$ the projection of ℓ_1^{**} onto ℓ_1, we have that $\|x^{**}\| = \|\pi(x^{**})\| + \|x^\perp\|$ for every x^{**} in ℓ_1^{**}, where $x^\perp = x^{**} - \pi(x^{**})$. If A is in $\mathcal{L}(^n\ell_1)$, since the unit ball of ℓ_1 is the convex hull of its extremal points, we have, as in the linear case, that

$$\|A\| = \sup_{k_1,\dots,k_n\in\mathbb{N}} |A(e_{k_1}, \dots, e_{k_n})|.$$

In particular, it is easy to see that

$$NA(^n\ell_1) = \{f \in \mathcal{L}(^n\ell_1) : \exists (k_1, \dots, k_n) \in \mathbb{N}^n \text{ such that } f(e_{k_1}, \dots, e_{k_n}) = \|f\|\},$$

where $NA(^n\ell_1)$ stands for the set of norm attaining n-linear form on $\ell_1 \times \overset{(n)}{\cdots} \times \ell_1$.

We have seen that, even if the norm of an extension of a multilinear functional is attained in points of the bidual, we can deal with sequential limits of points in the unit ball of the space, Theorem 3.3.2. We now prove that, when dealing with bilinear forms defined on $\ell_1 \times \ell_1$, sequences in the unit ball of ℓ_1 can be replaced by subsequences of the canonical basis of ℓ_1, and so a full characterization works.

To get this characterization we need to first prove a few lemmas. The first one asserts that if a multilinear form A on ℓ_1 attains its norm at a point of $B_{\ell_1^{**}} \setminus \ell_1 \times \cdots \times B_{\ell_1^{**}} \setminus \ell_1$, then this point can be chosen in $c_0^\perp \times \cdots \times c_0^\perp$.

Lemma 3.4.1. *Let $A \in \mathcal{L}(^n\ell_1^{**})$ with $\|A\| = 1$, $x_1^{**}, \ldots, x_n^{**} \in B_{\ell_1^{**}} \setminus \ell_1$ and $x_i^\perp = x_i^{**} - \pi(x_i^{**})$, $i = 1, \ldots, n$. If A attains its norms at $(x_1^{**}, \ldots, x_n^{**})$ then A attains its norm at $(\frac{x_1^\perp}{\|x_1^\perp\|}, \ldots, \frac{x_n^\perp}{\|x_n^\perp\|})$ too.*

Proof. Let us prove it first for $n = 1$, that is, for A being linear. If we assume that $|A(x_1^\perp)| < \|x_1^\perp\|$ then for some $\varepsilon \in \mathbb{K}$ with $|\varepsilon| = 1$

$$1 = A(\varepsilon x_1^{**}) = A(\varepsilon x_1^\perp) + A(\varepsilon \pi(x_1^{**})) < \|x_1^\perp\| + \|\pi(x_1^{**})\| = \|x_1^{**}\| = 1$$

which is a contradiction.

Assume now that A is bilinear. The associated linear mapping $A_1(y) := A(y, x_2^{**})$, $y \in \ell_1^{**}$, attains its norm at $x_1^{**} \in B_{\ell_1^{**}} \setminus \ell_1$ and so, by the linear case, A_1 attains its norm at $\frac{x_1^\perp}{\|x_1^\perp\|}$. Now, if we consider the other associated linear mapping $A_2(y) := A(\frac{x_1^\perp}{\|x_1^\perp\|}, y)$, $y \in \ell_1^{**}$, attains its norm at x_2^{**} and so at $\frac{x_2^\perp}{\|x_2^\perp\|}$, that is, $|A(\frac{x_1^\perp}{\|x_1^\perp\|}, \frac{x_2^\perp}{\|x_2^\perp\|})| = 1$.

An easy induction yields the general case. □

Lemma 3.4.2. *Let M and N be subsets of $\mathbb{N}$, $0 < \beta < 1$ and for each $n \in \mathbb{N}$ let $a_n \geq 0$ be such that $\sum_{n=1}^\infty a_n = 1$. If $\sum_{t \in M} a_t + \sum_{t \in N} a_t > 2 - \beta$ then $\sum_{t \in M \cap N} a_t > 1 - \beta$.*

Proof. Since

$$1 = \sum_{n=1}^{\infty} a_n \geq \sum_{t \in M \setminus N} a_t + \sum_{t \in N \setminus M} a_t + \sum_{t \in M \cap N} a_t$$

it follows that

$$\sum_{t \in M \setminus N} a_t + \sum_{t \in N \setminus M} a_t \leq 1 - \sum_{t \in M \cap N} a_t.$$

Combining this with the hypothesis we finally get that

$$\begin{aligned} 2 - \beta &< \sum_{t \in M \setminus N} a_t + \sum_{t \in M \cap N} a_t + \sum_{t \in N \setminus M} a_t + \sum_{t \in M \cap N} a_t \\ &\leq 2 \sum_{t \in M \cap N} a_t - \sum_{t \in M \cap N} a_t + 1 = \sum_{t \in M \cap N} a_t + 1. \end{aligned}$$

□

Theorem 3.4.3. *Given a bilinear form $A \in \mathcal{L}(^2\ell_1)$ of norm one, the following are equivalent,*

(a) $\lim\limits_i \lim\limits_j |A(e_{m_i}, e_{n_j})| = 1$ *for some strictly increasing sequences of natural numbers $\{m_i\}_{i=1}^{\infty}$ and $\{n_j\}_{j=1}^{\infty}$,*

(b) There exist $x_1^{**}, x_2^{**} \in \ell_1^{**} \setminus \ell_1$ of norm one such that $|A_{Id}(x_1^{**}, x_2^{**})| = 1$.

Proof. $(a) \Rightarrow (b)$ is a consequence of Theorem 3.3.4.
$(b) \Rightarrow (a)$: Let $\{x_{d_1}\}_{d_1 \in D_1}$ and $\{x_{d_2}\}_{d_2 \in D_2}$ be nets in the unit ball of ℓ_1 weak-star-convergent to the points x_1^{**} and x_2^{**} respectively. Notice that ℓ_1 is an L-summand space in its bidual so $\|x_s^{**}\| = \|\pi(x_s^{**})\| + \|x_s^{**} - \pi(x_s^{**})\|$, for $s = 1, 2$, where π is the projection from ℓ_1^{**} onto ℓ_1. For each $n \in \mathbb{N}$ let π_n denote the projection from ℓ_1^{**} onto $\ell_1 \times \overset{(n)}{\cdots} \times \ell_1$. Note that π_n is weak-star continuous.

By Lemma 3.4.1 we can assume that $\pi(x_1^{**}) = \pi(x_2^{**}) = 0$. Consider the linear form $A_{Id}(\cdot, x_2^{**})$ on ℓ_1 with norm one defined by $A_{Id}(x, x_2^{**}) = \lim_{d_2} A(x, x_{d_2})$ for all x in ℓ_1, whenever $\{x_{d_2}\}_{d_2 \in D_2}$ is a net in the unit ball of ℓ_1 weak-star convergent to x_2^{**}.

Let us see that there exists a strictly increasing sequence of natural numbers $\{m_i\}_{i=1}^{\infty}$ with $\lim_i |A_{Id}(e_{m_i}, x_2^{**})| = 1$. If this is not the case, then there exists $\epsilon > 0$ and there exists a natural number r with $|A_{Id}(e_k, x_2^{**})| \leq 1 - \epsilon$ for all $k > r$. Let $\{x_{d_1}\}_{d_1 \in D_1}$ be a net in the unit ball of ℓ_1 weak-star convergent to x_1^{**}. Since $\pi(x_1^{**}) = 0$ and $\pi_r(x_{d_1})$ converges to $\pi_r(x_1^{**}) = 0$ then $\{x_{d_1} - \pi_r(x_{d_1})\}_{d_1 \in D_1}$ weak-star converges to x_1^{**}. Moreover, $\|x_{d_1} - \pi_r(x_{d_1})\| \leq \|x_{d_1}\| \leq 1$ and so, by replacing x_{d_1} with $x_{d_1} - \pi_r(x_{d_1})$, we can assume that $\pi_r(x_{d_1}) = 0$, i.e. $x_{d_1}(t) = 0$ for all $t = 1, \ldots, r$.

Therefore for all $d_1 \in D_1$

$$
\begin{aligned}
|A_{Id}(x_{d_1}, x_2^{**})| &= |\sum_{t=1}^{\infty} x_{d_1}(t) A_{Id}(e_t, x_2^{**})| \\
&\leq \sum_{t=1}^{\infty} |x_{d_1}(t)| \; |A_{Id}(e_t, x_2^{**})| \\
&\leq \sum_{t=r+1}^{\infty} |x_{d_1}(t)| \; |A_{Id}(e_t, x_2^{**})| \\
&\leq 1 - \epsilon,
\end{aligned}
$$

contradicting that $|\lim_{d_1} A_{Id}(x_{d_1}, x_2^{**})| = |A_{Id}(x_1^{**}, x_2^{**})| = 1$.

Without loss of generality assume that for all $i \in \mathbb{N}$

$$
1 - |A_{Id}(e_{m_i}, x_2^{**})| \leq 2^{-(2i+2)}. \tag{3.3}
$$

By using induction, let us find a strictly increasing sequence of natural numbers $\{n_j\}_{j=1}^{\infty}$ such that $|A(e_{m_i}, e_{n_j})| \geq 1 - 2^{-i}$ for all $1 \leq i \leq j$.

Let $\{x_{d_2}\}_{d_2 \in D_2}$ be a net in the unit ball of ℓ_1 weak-star convergent to

x_2^{**}. Since $|\lim_{d_2} A_{Id}(e_{m_1}, x_{d_2})| = |A_{Id}(e_{m_1}, x_2^{**})| > 1 - 2^{-4}$, there exists d_o in D_2 with $|A(e_{m_1}, x_{d_0})| > 2^{-1}$. Then

$$\begin{aligned} 2^{-1} < |A(e_{m_1}, x_{d_0})| &\leq \sum_{t\in\mathbb{N}} |x_{d_0}(t)|\ |A(e_{m_1}, e_t)| \\ &\leq \sup_{t\in\mathbb{N}}\{|A(e_{m_1}, e_t)|\} \sum_{t\in\mathbb{N}} |x_{d_0}(t)| \\ &\leq \sup_{t\in\mathbb{N}}\{|A(e_{m_1}, e_t)|\}. \end{aligned}$$

Let n_1 be a natural number with $|A_{Id}(e_{m_1}, e_{n_1})| > 2^{-1}$. Now, assume we have found $n_1 < \cdots < n_r$ with $|A(e_{m_i}, e_{n_j})| > 1 - 2^{-i}$ for $1 \leq i \leq j \leq r$ and let us find n_{r+1}. Considering that $\pi(x_2^{**}) = 0$, by replacing x_{d_2} with $x_{d_2} - \pi_{n_r}(x_{d_2})$, we can assume that $\pi_{n_r}(x_{d_2}) = 0$, i.e. $x_{d_2}(t) = 0$ for all $t = 1, \ldots, n_r$ and all $d_2 \in D_2$.

By (3.3), consider x_0 an element of the net $\{x_{d_2}\}_{d_2\in D_2}$ such that

$$|A(e_{m_i}, x_0)| \geq 1 - 2^{-2i} \text{ for } i = 1, \ldots, r+1. \tag{3.4}$$

For each $i = 1, \ldots, r+1$ define the set

$$T_i := \{t \in \mathbb{N} : t > n_r, |A_{Id}(e_{m_i}, e_t)| \geq 1 - 2^{-i}\}. \tag{3.5}$$

Therefore, for every $i = 1, \ldots, r+1$,

$$\begin{aligned} 1 - 2^{-2i} &\leq |A(e_{m_i}), x_0)| \\ &\leq \sum_{t\in T_i} |x_0(t)||A(e_{m_i}, e_t)| + \sum_{t\notin T_i} |x_0(t)||A(e_{m_i}, e_t)| \\ &\leq \sum_{t\in T_i} |x_0(t)| + (1 - 2^{-i}) \sum_{t\notin T_i} |x_0(t)| \\ &\leq \sum_{t\in T_i} |x_0(t)| + (1 - 2^{-i})\Big(1 - \sum_{t\in T_i} |x_0(t)|\Big) \\ &= (1 - 2^{-i}) + 2^{-i} \sum_{t\in T_i} |x_0(t)|, \end{aligned}$$

where in the first inequality we have used (3.4). Thus $2^{-i}\sum_{t\in T_i}|x_0(t)| \geq 2^{-i} - 2^{-2i}$ and so

$$\sum_{t\in T_i}|x_0(t)| \geq 1 - 2^{-i}. \tag{3.6}$$

We use now finite induction and Lemma 3.4.2 to see that $\bigcap_{i=1}^{r+1} T_i \neq \emptyset$. Indeed, by (3.6)

$$\sum_{t\in T_1}|x_0(t)| + \sum_{t\in T_2}|x_0(t)| > 2 - \left(\frac{1}{2} + \frac{1}{2^2}\right),$$

and Lemma 3.4.2 yields that $\sum_{t\in T_1\cap T_2}|x_0(t)| > 1 - \left(\frac{1}{2} + \frac{1}{2^2}\right)$. If for some $1 \leq l < r+1$ we assume that

$$\sum_{t\in\cap_{j=1}^{l} T_j}|x_0(t)| > 1 - \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^l}\right),$$

then

$$\sum_{t\in\cap_{j=1}^{l} T_j}|x_0(t)| + \sum_{t\in T_{l+1}}|x_0(t)| > 2 - \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^l} + \frac{1}{2^{l+1}}\right).$$

Once more, Lemma 3.4.2 yields that

$$\sum_{t\in\cap_{j=1}^{l+1} T_j}|x_0(t)| > 1 - \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{l+1}}\right).$$

Therefore, we can conclude that

$$\sum_{t\in\cap_{j=1}^{r+1} T_j}|x_0(t)| > 1 - \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{r+1}}\right),$$

and so $\bigcap_{j=1}^{r+1} T_j \neq \emptyset$. Let $n_{r+1} := \min\left(\bigcap_{j=1}^{r+1} T_j\right)$. Note that $n_{r+1} > n_r$.

From (3.5) it follows that

$$|A(e_{m_i}, e_{n_{r+1}})| \geq 1 - 2^{-i} \tag{3.7}$$

for all $i = 1, \ldots, r+1$.

By Equation (3.7)

$$1 \geq \liminf_j |A(e_{m_i}, e_{n_j})| \geq 1 - 2^{-i} \ .$$

Then

$$\lim_i \liminf_j |A(e_{m_i}, e_{n_j})| = 1. \tag{3.8}$$

To finish the proof, we show that the lim inf can be replaced with lim just by choosing a suitable subsequence of $\{e_{n_j}\}_{j=1}^{\infty}$.

Let us proceed once more by induction. By (3.7), $|A(e_{m_1}, e_{n_j})| \geq 1 - \frac{1}{2}$ for all $j \geq 1$. Then, there exists a subsequence $\{e_{n_{j_k}}\}_{k=1}^{\infty}$ of $\{e_{n_j}\}_{j=1}^{\infty}$ such that $\lim_k |A(e_{m_1}, e_{n_{j_k}})|$ exists and is great than or equal to $1 - \frac{1}{2}$. To make the notation clear, we write $n(1,k) := n_{j_k}$ and so,

$$\lim_k |A(e_{m_1}, e_{n(1,k)})| \geq 1 - \frac{1}{2}.$$

Assume that we have a chain of sequences $\{e_{n(1,j)}\}_{j=1}^{\infty}, \ldots, \{e_{n(p,j)}\}_{j=1}^{\infty}$ each of them being a subsequence of the previous one, such that

$$\lim_j |A(e_{m_i}, e_{n(i,j)})| \geq 1 - \frac{1}{2^i},$$

for all $i = 1, \ldots, p$. Let us construct a subsequence $\{e_{n(p+1,j)}\}_{j=1}^{\infty}$ of $\{e_{n(p,j)}\}_{j=1}^{\infty}$ such that $|A(e_{m_{p+1}}, e_{n(p+1,j)})| \geq 1 - \frac{1}{2^{p+1}}$ for all $j \in \mathbb{N}$. Indeed, since $|A(e_{m_{p+1}}, e_{n(p,j)}| \geq 1 - \frac{1}{2^{p+1}}$ for all $j \geq p+1$, there exists a subsequence $\{e_{n(p,j)_l}\}_{l=1}^{\infty}$ of $\{e_{n(p,j)}\}_{j=1}^{\infty}$ such that $\lim_l |A(e_{m_{p+1}}, e_{n(p,j)_l})|$ exists and is great than or equal to $1 - \frac{1}{2^{p+1}}$. We write $n(p+1,l) := n(p,j)_l$

and so

$$\lim_l |A(e_{m_{p+1}}, e_{n(p+1,l)})| \geq 1 - \frac{1}{2^{p+1}}.$$

So, we have countably many sequences $\{e_{n(1,j)}\}_{j=1}^\infty, \{e_{n(2,j)}\}_{j=1}^\infty, \ldots$, each of them being a subsequence of the previous one, such that

$$\lim_j |A(e_{m_i}, e_{n(i,j)})| \geq 1 - \frac{1}{2^i},$$

for all $i = 1, 2, \ldots$. The diagonal sequence $\{e_{n(j,j)}\}_{j=1}^\infty$ is the one we are looking for. Note that $\{e_{n(j,j)}\}_{j=i}^\infty$ is a subsequence of $\{e_{n(i,j)}\}_{j=1}^\infty$ and then $\lim_j |A(e_{m_i}, e_{n(j,j)})|$ exists and

$$\lim_j |A(e_{m_i}, e_{n(j,j)})| \geq 1 - \frac{1}{2^i},$$

for all $i \in \mathbb{N}$.

Therefore, we have found sequences $\{e_{m_i}\}_{i=1}^\infty$ and $\{e_{n(j,j)}\}_{j=1}^\infty$, with $\{m_i\}_{i=1}^\infty$ and $\{n(j,j)\}_{j=1}^\infty$ strictly increasing, for which there exists

$$\lim_i \lim_j |A(e_{m_i}, e_{n(j,j)})| = 1.$$

This concludes the case with $\pi(x_1^{**}) = \pi(x_2^{**}) = 0$.

If $\pi(x_1^{**}) \neq 0$ or $\pi(x_2^{**}) \neq 0$, then $y_1^{**} = x_1^{**} - \pi(x_1^{**})$ and $y_2^{**} = x_2^{**} - \pi(x_2^{**})$ are non zero points of $\ell_1^{**} \setminus \ell_1$ with $|A(\frac{y_1^{**}}{\|y_1^{**}\|}, \frac{y_2^{**}}{\|y_2^{**}\|})| = 1$, and the former case gives us the desired result.

□

Corollary 3.4.4. *Given a bilinear form $A \in \mathcal{L}(^2\ell_1)$ of norm one and $\sigma \in \Sigma_2$, the following are equivalent,*

(a) $\lim\limits_{(i,\sigma(1),\sigma)} \lim\limits_{(j,\sigma(2),\sigma)} |A(e_{m(i,1,\sigma)}, e_{m(j,2,\sigma)})| = 1$ *for some strictly increasing sequences of natural numbers* $\big(m(i,1,\sigma)\big)_{i=1}^\infty$ *and* $\big(m(j,2,\sigma)\big)_{j=1}^\infty$,

(b) There exist $x_{1,\sigma}^{**}, x_{2,\sigma}^{**} \in \ell_1^{**}\backslash\ell_1$ *of norm one such that* $|A_\sigma(x_{1,\sigma}^{**}, x_{2,\sigma}^{**})| = 1$.

Remark 3.4.5. *We do not know if Theorem 3.4.3 is valid for n-linear mappings with $n > 2$. Our conjecture is the following: Let $n \in \mathbb{N}$, $A \in \mathcal{L}(^n\ell_1)$ and $\sigma \in \Sigma_n$. If A_σ attains its norm on ℓ_1^{**} but only in n-tuples that belong to $(B_{\ell_1^{**}} \setminus \ell_1)^n$, then there exist increasing sequences of natural numbers $\{k(j,m_j)\}_{m_j=1}^\infty$, $j = 1, \dots, n$, such that*

$$\lim_{m_{\sigma(1)}\to\infty} \cdots \lim_{m_{\sigma(n)}\to\infty} |A(e_{k(1,m_1)}, \dots, e_{k(n,m_n)})| = \|A\|$$

Next we give the following lemma.

Lemma 3.4.6. *Given a sequence of n-tuples $\{(k_1(h), \dots, k_n(h))\}_{h=1}^\infty$ in $\mathbb{N}^n$ such that each $\{k_j(h)\}_{h=1}^\infty$ is strictly increasing, $j = 1, \dots, n$, define the n-linear mapping $A : \ell_1^n \longrightarrow \mathbb{R}$ by*

$$A(e_{k_1}, \dots, e_{k_n}) = \begin{cases} \left(\frac{k_1(h)}{k_1(h)+1}\right)^n & \text{if } k_i = k_i(h),\ i = 1, \dots, n \text{ for some } h \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that $\|A\| = 1$ and there is no permutation σ such that A_σ attains its norm (at any n-tuple of $B_{\ell_1^{**}} \times \cdots \times B_{\ell_1^{**}}$).

Proof. Note first that, for arbitrary $x_i := \sum_{k=1}^\infty a_{k,i} e_k \in B_{\ell_1}$, $i = 1, \dots, n$, if we fix $1 \le j \le n$ then

$$\begin{aligned} |A(x_1, \cdots, x_n)| &\le \|x_1\| \dots \|x_{j-1}\| \cdot \|x_{j+1}\| \cdots \|x_n\| \sum_{h=1}^\infty \frac{k_1(h)}{k_1(h)+1} |a_{k_j(h),j}| \\ &\le \sum_{h=1}^\infty \frac{k_1(h)}{k_1(h)+1} |a_{k_j(h),j}|. \end{aligned}$$

Thus, for any $\sigma \in \Sigma_n$, A_σ, any $x_1^{**}, \dots, x_{k-j}^{**}, x_{j+1}^{**}, \dots, x_n^{**} \in B_{\ell_1^{**}}$ and any $x_j \in B_{\ell_1}$, by taking nets in B_{ℓ_1} weak-star convergent if necessary, we get

$$|A_\sigma(x_1^{**},\dots,x_{j-1}^{**},x_j,x_{j+1}^{**},\dots,x_n^{**})| \leq \sum_{h=1}^{\infty} \frac{k_1(h)}{k_1(h)+1}|a_{k_j(h),j}| < 1. \quad (3.9)$$

Hence if there exist a permutation $\sigma \in \Sigma_n$ and $x_1^{**},\dots,x_n^{**}$ in $B_{\ell_1^{**}}$ such that $\|x_j^{**}\| = 1$ for every j and

$$|A_\sigma(x_1^{**},\dots,x_n^{**})| = 1,$$

we have that $x_1^{**},\dots,x_n^{**} \in B_{\ell_1^{**}} \setminus B_{\ell_1}$. Moreover, by Lemma 3.4.1 it can also be assumed that x_j^{**} belongs to $c_0^{\perp}$ for $j = 1,\dots,n$. Finally, by making a rearrangement of coordinates, if necessary, we can assume that σ is the identity permutation.

We define $B : \ell_1 \times \ell_1 \to \mathbb{R}$ by $B(x,y) = A_{Id}(x,y,x_3^{**},\dots,x_n^{**})$. Clearly

$$|B_{Id}(x_1^{**},x_2^{**})| = |A_{Id}(x_1^{**},\dots,x_n^{**})| = 1.$$

By Theorem 3.4.3, there exist two sequences (e_{n_j}) and (e_{m_l}) such that

$$\lim_{j\to\infty}\lim_{l\to\infty}|B(e_{n_j},e_{m_l})| = 1.$$

Thus there exist j, l such that

$$|B(e_{n_j},e_{m_l})| > \frac{1}{2}.$$

But, there exists h_0 such that $n_j < k_1(h)$ and $m_l < k_2(h)$ for every $h \geq h_0$ and we get that

$$(n_j, m_l, k_3,\dots,k_n) \notin \{(k_1(h),\dots,k_n(h)) : h \geq h_0\},$$

for every $k_3,\dots,k_n \in \mathbb{N}$ with $k_3 > k_3(h_0),\dots,k_n > k_n(h_0)$. Now consider a net $\{x_{d_j}\}_{d_j \in D_j}$ in B_{ℓ_1} weak-star convergent to x_j^{**} for $j = 3, \dots, n$. Since x_j^{**} belongs to $c_0^{\perp}$, as in the proof of Theorem 3.4.3, we can assume additionally that for every $d_j \in D_j$, the first $k_j(h_0)$-components of x_{d_j} are 0. Hence

$$A(e_{n_j}, e_{m_l}, e_{k_3}, \dots, e_{k_{n-1}}, x_{d_n}) = 0,$$

for every $d_n \in D_n$. Hence

$$A_{Id}(e_{n_j}, e_{m_l}, e_{k_3}, \dots, e_{k_{n-1}}, x_n^{**}) = 0,$$

for every $k_3 > k_3(h_0), \dots, k_n > k_n(h_0)$. By induction we obtain the contradiction

$$B(e_{n_j}, e_{m_l}) = A_{Id}(e_{n_j}, e_{m_l}, x_3^{**}, \dots, x_n^{**}) = 0.$$

□

Theorem 3.4.7. *Given a subset $P \subseteq \Sigma_n$, there exists an n-linear form $A(P) \in \mathcal{L}(^n\ell_1)$ with $\|A(P)\| = 1$ such that $A(P)_\sigma$ is norm attaining if and only if $\sigma \in P$.*

Proof. The proof will be divided into two cases.

If P is the empty set, consider $A(P) \in \mathcal{L}(^n\ell_1)$

$$A(P)(e_{k_1}, e_{k_2}, \dots, e_{k_n}) = \begin{cases} \left(\frac{k_1}{k_1+1}\right)^n & \text{if } k_1 = k_2 = \cdots = k_n, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 3.4.6, $A(P)$ does not attain its norm at any point of the unit ball of ℓ_1^{**}.

If P is not empty, consider

$$A(P)(e_{k_1}, e_{k_2}, \dots, e_{k_n}) = \begin{cases} \prod_{i=1}^n \frac{k_i}{k_i+1} & \text{if } \exists \sigma \in P, k_{\sigma(1)} \leq \cdots \leq k_{\sigma(n)}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\|A(P)\| = 1$. A similar argument to the one given in (3.9) shows that for any $\sigma \in \Sigma_n$, A_σ does not attain its norm at any n-tuple in $B_{\ell_1^{**}} \times \cdots \times B_{\ell_1^{**}}$ with at least a coordinate j belonging to B_{ℓ_1}. If $\sigma \in P$ then

$$\lim_{k_{\sigma(1)}\to\infty} \cdots \lim_{k_{\sigma(n)}\to\infty} A(P)(e_{k_1}, \ldots, e_{k_n}) = 1.$$

Hence, considering x^{**} a weak-star cluster point of the sequence $\{e_k\}_{k=1}^{\infty}$ we obtain

$$A(P)_\sigma(x^{**}, \ldots, x^{**}) = 1.$$

Thus, $A(P)_\sigma$ is norm attaining.

Now we see that $A(P)_\sigma$ does not attain its norm whenever σ is not in P. For simplicity we will assume that σ is the identity permutation. Let us assume that $A(P)_{Id}$ does attain its norm at $(x_1^{**}, \ldots, x_n^{**}) \in B_{\ell_1^{**}} \times \cdots \times B_{\ell_1^{**}}$. By the above observation, x_i^{**} is a point in $B_{\ell_1^{**}} \setminus \ell_1$ for $i = 1 \ldots, n$. By Lemma 3.4.1 we can assume that $\pi(x_i^{**}) = 0$ for $i = 1, \ldots, n$. Let $\{x_{d_i}\}_{d_i \in D_i}$ be nets in the unit sphere of ℓ_1 weak-star convergent to x_i^{**}, for $i = 1, \ldots, n$.

Let $l_0 = 0$. Since $|A(P)_{Id}(x_1^{**}, \ldots, x_n^{**})| = 1$ there exists $d_1^0 \in D_1$ with

$$|A(P)_{Id}(x_{d_1^0}, x_2^{**}, \ldots, x_n^{**})| > 1 - 2^{-n}. \tag{3.10}$$

Let l_1 be such that $\|\pi_{l_1}(x_{d_1^0})\| > 1/2$. Now, using (3.10) and since $\pi(x_2^{**}) = 0$ we can find $d_2 \in D_2$ and a natural number l_2 with

$$|A(P)_{Id}(x_{d_1^0}, x_{d_2^0}, x_3^{**}, \ldots, x_n^{**})| > 1 - 2^{-n}$$

and $\|\pi_{l_2}(x_{d_2^0})\| - \|\pi_{l_1}(x_{d_2^0})\| > 1/2$. In general, by using finite induction over i, we can find $d_i^0 \in D_i$ and a natural number l_i such that, for $i = 2, \ldots, n$, $|A(P)_{Id}(x_{d_1^0}, \ldots, x_{d_i^0}, x_{i+1}^{**}, \ldots, x_n^{**})| > 1 - 2^{-n}$ and $\|\pi_{l_i}(x_{d_i^0})\| -$

$\|\pi_{l_{i-1}}(x_{d_i^0})\| > 1/2$.

But then, if we let $C := \{(t_1, \ldots, t_n) \in \mathbb{N}^n : l_{i-1} < t_i \leq l_i \text{ for } i = 1, \ldots n\}$, since Id is not in P, we have $A(P)(e_{t_1}, \ldots, e_{t_n}) = 0$ for all $(t_1, \ldots, t_n) \in C$. Therefore,

$$\begin{aligned}
1 - 2^{-n} &< |A(P)(x_{d_1^0}, \ldots, x_{d_n^0})| \\
&\leq \sum_{(t_1,\ldots,t_n)} \prod_{i=1}^{n} |x_{d_i^0}(t_i)| A(P)(e_{t_1}, \ldots, e_{t_n}) \\
&= \sum_{(t_1,\ldots,t_n)\notin C} \prod_{i=1}^{n} |x_{d_i^0}(t_i)| A(P)(e_{t_1}, \ldots, e_{t_n}) \\
&< \sum_{(t_1,\ldots,t_n)\notin C} \prod_{i=1}^{n} |x_{d_i^0}(t_i)| \\
&\leq 1 - \sum_{(t_1,\ldots,t_n)\in C} \prod_{i=1}^{n} |x_{d_i^0}(t_i)| \\
&= 1 - \prod_{i=1}^{n} \Big(\|\pi_{k_i}(x_{d_i^0})\| - \|\pi_{k_{i-1}}(x_{d_i^0})\| \Big) \\
&< 1 - 2^{-n}
\end{aligned}$$

which is a contradiction. Hence $A(P)_{Id}$ does not attain its norm. □

Theorem 3.4.8. *Let $A \in \mathcal{L}(^n\ell_1)$ of norm one such that, for every $\epsilon > 0$ and every $\sigma \in \Sigma_n$, there exist subsequences $\{e_{k(i,m_i,\sigma)}\}_{m_i=1}^{\infty}$ (that depend on σ and ϵ) of the sequence $\{e_k\}_{k=1}^{\infty}$ so that*

$$\lim_{m_{\sigma(1)}\to\infty} \cdots \lim_{m_{\sigma(n)}\to\infty} |A(e_{k(1,m_1,\sigma)}, \ldots, e_{k(n,m_n,\sigma)})| > 1 - \epsilon.$$

Then for every $\epsilon > 0$ and each subset $P \subseteq \Sigma_n$, there exists $A(P,\epsilon) \in \mathcal{L}(^n\ell_1)$ with $\|A(P,\epsilon)\| = 1$ such that $\|A(P,\epsilon) - A\| \leq \epsilon$, and $A(P,\epsilon)_\sigma$ is norm attaining if and only if $\sigma \in P$.

Proof. Consider the n-linear form

$$B(x_1,\dots,x_n)=\sum_{k_1,\dots,k_n\in\mathbb{N}} B(e_{k_1},\dots,e_{k_n})\prod_{i=1}^{n} x_i\Big(k_i\Big)$$

for $x_1,\dots,x_n\in\ell_1$, where

$$B(e_{k_1},\dots,e_{k_n})=\begin{cases}A(e_{k_1},\dots,e_{k_n}) & \text{if } 1-\frac{\epsilon}{2}\geq |A(e_{k_1},\dots,e_{k_n})|,\\ (1-\frac{\epsilon}{2})sign(A(e_{k_1},\dots,e_{k_n})), & \text{if } |A(e_{k_1},\dots,e_{k_n})|>1-\frac{\epsilon}{2}.\end{cases}$$

We have $\|B\|\leq 1-\frac{\epsilon}{2}$.

For a fixed a non empty subset of permutations P, consider the n-linear form $A(P)$ from Theorem 3.4.7 and define the n-linear form $A(P,\epsilon)$ as follows.

$$\begin{aligned}A(P,\epsilon)(e_{k_1},\dots,e_{k_n}) &:= B(e_{k_1},\dots,e_{k_n})\\ &\qquad + sign\Big(B(e_{k_1},\dots,e_{k_n})\Big)\frac{\epsilon}{2}A(P)(e_{k_1},\dots,e_{k_n})\end{aligned}$$

if $k_1=k(1,m_1,\sigma),\dots,\ k_n=k(n,m_n,\sigma)$, for some $\sigma\in P$, with $m_{\sigma(1)}\leq\cdots\leq m_{\sigma(n)}$ and $B(e_{k_1},\dots,e_{k_n})$ otherwise.

Clearly,

$$\|A(P,\epsilon)\|\leq\|B\|+\frac{\epsilon}{2}\|A(P)\|\leq 1-\frac{\epsilon}{2}+\frac{\epsilon}{2}=1.$$

Hence $\|A(P,\epsilon)\|\leq 1$.

By hypothesis, for each $\sigma\in\Sigma_n$, there exist sequences $\{e_{k(i,m_i,\sigma)}\}_{m_i=1}^{\infty}$, with the property that $\{k(i,m_i,\sigma)\}_{m_i=1}^{\infty}$ is strictly increasing, such that

$$\lim_{m_{\sigma(1)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty}|A(e_{k(1,m_1,\sigma)},\dots,e_{k(n,m_n,\sigma)})|>1-\frac{\epsilon}{2}. \qquad (3.11)$$

From (3.11) there exists $m^0_{\sigma(1)}$ such that for every $m_{\sigma(1)} \geq m^0_{\sigma(1)}$

$$\lim_{m_{\sigma(2)}\to\infty} \cdots \lim_{m_{\sigma(n)}\to\infty} |A(e_{k(1,m_1,\sigma)}, \ldots, e_{k(n,m_n,\sigma)})| > 1 - \frac{\epsilon}{2}.$$

Taking $m_{\sigma(1)} \geq m^0_{\sigma(1)}$, there is $m^0_{\sigma(2)}$ that depends on $m_{\sigma(1)}$, such that for every $m_{\sigma(2)} \geq m^0_{\sigma(2)}$

$$\lim_{m_{\sigma(3)}\to\infty} \cdots \lim_{m_{\sigma(n)}\to\infty} |A(e_{k(1,m_1,\sigma)}, \ldots, e_{k(n,m_n,\sigma)})| > 1 - \frac{\epsilon}{2}.$$

Repeating this process and assuming that we have fixed natural numbers $m_{\sigma(1)}, m_{\sigma(1)}, \ldots, m_{\sigma(n-1)}$ with $m_{\sigma(1)} \geq m^0_{\sigma(1)}$, $m_{\sigma(2)} \geq m^0_{\sigma(2)}, \ldots,$ $m_{\sigma(n-1)} \geq m^0_{\sigma(n-1)}$, where $m^0_{\sigma(n-1)}$ depends on $m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(n-2)}$, so that $\lim_{m_{\sigma(n)}} |A(e_{k(1,m_1,\sigma)}, \ldots, e_{k(n,m_n,\sigma)})| > 1 - \frac{\epsilon}{2}$, we can find $m^0_{\sigma(n)}$ that depends on $m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(n-1)}$, such that for every $m_{\sigma(n)} \geq m^0_{\sigma(n)}$

$$|A(e_{k(1,m_1,\sigma)}, \ldots, e_{k(n,m_n,\sigma)})| > 1 - \frac{\epsilon}{2}.$$

Then,

$$|B(e_{k(1,m_1,\sigma)}, \ldots, e_{k(n,m_n,\sigma)})| = 1 - \frac{\epsilon}{2}, \tag{3.12}$$

and then, $\|B\| = 1 - \frac{\epsilon}{2}$. Moreover, given $\sigma \in P$ and $\delta > 0$ we can take $m_{\sigma(1)} \leq \cdots \leq m_{\sigma(n)}$ big enough so that $A(P)(e_{k(1,m_1,\sigma)}, \ldots, e_{k(n,m_n,\sigma)}) \geq 1 - \delta$. Hence,

$$|A(P,\epsilon)(e_{k(1,m_1,\sigma)}, \ldots, e_{k(n,m_n,\sigma)})| \geq 1 - \frac{\epsilon}{2} + \frac{\epsilon}{2}(1-\delta)$$

and so, $\|A(P,\epsilon)\| \geq 1 - \frac{\epsilon}{2} + \frac{\epsilon}{2}(1-\delta)$. Thus, $\|A(P,\epsilon)\| = 1$.

Notice that $|A(P,\epsilon)(x_1,\ldots,x_n) - B(x_1,\ldots,x_n)| \leq \frac{\epsilon}{2}$, for all $x_1, \ldots, x_n$ in ℓ_1, hence

$$\|A(P,\epsilon) - A\| \leq \|A(P,\epsilon) - B\| + \|B - A\| \leq \epsilon.$$

Now we show that $A(P,\epsilon)_\sigma$ is norm attaining if and only if $\sigma \in P$.

Let $\sigma \notin P$ and assume that there is $(x_1^{**},\dots,x_n^{**}) \in B_{\ell_1^{**}} \times \cdots \times B_{\ell_1^{**}}$ such that $A(P,\epsilon)_\sigma(x_1^{**},\dots,x_n^{**}) = 1$. Since $A(P)_\sigma(x_1^{**},\dots,x_n^{**}) < 1$ we have $1 < |B_\sigma(x_1^{**},\dots,x_n^{**})| + \frac{\epsilon}{2}$. Hence $|B_\sigma(x_1^{**},\dots,x_n^{**})| > 1 - \frac{\epsilon}{2}$, which is impossible.

Take now $\sigma \in P$. From (3.12) we have for $m_{\sigma(1)} \le \cdots \le m_{\sigma(n)}$

$$|A(P,\epsilon)(e_{k(1,m_1,\sigma)},\dots,e_{k(n,m_n,\sigma)})| = 1 - \frac{\epsilon}{2} + \frac{\epsilon}{2}\prod_{i=1}^{n}\frac{m_i}{m_i+1}.$$

Hence

$$\begin{aligned}&\lim_{m_{\sigma(1)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty}|A(P,\epsilon)(e_{k(1,m_1,\sigma)},\dots,e_{k(n,m_n,\sigma)})|\\ &= \lim_{m_{\sigma(1)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty} 1 - \frac{\epsilon}{2} + \frac{\epsilon}{2}\prod_{i=1}^{n}\frac{m_i}{m_i+1} = 1,\end{aligned}$$

so $A(P,\epsilon)_\sigma$ is norm attaining at a point $(x_1^{**},\dots,x_n^{**})$, where each x_j^{**} is a weak-star cluster point of the sequence $\{e_{k(j,m_j,\sigma)}\}_{m_j=1}^{\infty}$, $j = 1,\dots,n$.

If $P = \emptyset$, for every h, by taking $\varepsilon = \frac{1}{h}$, the process above gives the existence of a sequence of n-tuples $\{(k_1(h),\dots,k_n(h))\}_{h=1}^{\infty}$ in $\mathbb{N}^n$ such that each $\{k_j(h)\}_{h=1}^{\infty}$ is strictly increasing, $j = 1,\dots,n$, and

$$|A(e_{k_1(h)},\dots,e_{k_n(h))})| > 1 - \frac{1}{h}.$$

We let

$$C(e_{k_1},\dots,e_{k_n}) = \begin{cases} \left(\frac{k_1(h)}{k_1(h)+1}\right)^n & \text{if } k_i = k_i(h),\ i = 1,\dots,n \text{ some } h \in \mathbb{N},\\ 0 & \text{otherwise,}\end{cases}$$

and define $A(P,\epsilon)$ at $(e_{k_1},\dots,e_{k_n})$ as

$$A(P,\epsilon)(e_{k_1},\dots,e_{k_n}) := B(e_{k_1},\dots,e_{k_n}) \\ + sign\Big(B(e_{k_1},\dots,e_{k_n})\Big)\frac{\epsilon}{2}C(e_{k_1},\dots,e_{k_n}).$$

Notice that as before $\|A(P,\epsilon)\| \leq \|B\| + \frac{\epsilon}{2}\|C\| = 1$, and

$$|A(P,\epsilon)(e_{k_1(h)},\dots,e_{k_n(h)})| = |B(e_{k_1(h)},\dots,e_{k_n(h)})| + \frac{\varepsilon}{2}\Big(\frac{k_1(h)}{k_1(h)+1}\Big)^n \\ = 1 - \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\Big(\frac{k_1(h)}{k_1(h)+1}\Big)^n.$$

Then, for every h such that $\frac{1}{h} < \frac{\varepsilon}{2}$, we obtain that $\|A(P,\epsilon)\| = 1$. On the other hand, since C does not attain the norm at any point of $B_{\ell_1^{**}}$, by Lemma 3.4.6, neither can $A(P,\epsilon)$. To conclude the proof, notice $\|A(P,\epsilon) - A\| \leq \|A(P,\epsilon) - B\| + \|B - A\| \leq \epsilon/2 + \epsilon/2 = \epsilon$.

□

Remark 3.4.9. *If the conjecture in Remark 3.4.5 were true, we could get the following result. Let $A \in \mathcal{L}(^n\ell_1)$ of norm one. The following are equivalent:*

1. *For every $\epsilon > 0$ and every $\sigma \in \Sigma_n$, there exist subsequences $\{e_{k(i,m_i)}\}_{m_i=1}^{\infty}$ (that depend on σ and ϵ) of the sequence $\{e_k\}_{k=1}^{\infty}$ so that*

$$\lim_{m_{\sigma(1)}\to\infty} \cdots \lim_{m_{\sigma(n)}\to\infty} |A(e_{k(1,m_1)},\dots,e_{k(n,m_n)})| > 1 - \epsilon.$$

2. *For every $\epsilon > 0$ and each subset $P \subseteq \Sigma_n$, there exists $A(P,\epsilon) \in \mathcal{L}(^n\ell_1)$ with $\|A(P,\epsilon)\| = 1$ such that $\|A(P,\epsilon) - A\| \leq \epsilon$, and $A(P,\epsilon)_\sigma$ is norm attaining on $\ell_1^{**} \times \overset{(n)}{\cdots} \times \ell_1^{**}$ if and only if $\sigma \in P$.*

3.5 Spaceability of multilinear norm attaining Arens extensions

Even though we have seen in Proposition 3.2.3 that the Arens extensions of a multilinear form can be pairwise different, we could think that the examples used in Example 3.2.2 and Theorem 3.4.7 are only extremal cases and in most of the situations there is no need in differentiating between attaining the norm only some of the extensions or all of them, for multilinear forms on ℓ_1. However as we will see now, this is not the case.

Lemma 3.5.1. *For every natural number n, consider n sequences of non-negative numbers $\{x_i(t)\}_{t=1}^{\infty}$, $i = 1, \dots, n$ with $\sum_{t=1}^{\infty} x_i(t) \leq 1$. If*

$$\sum_{t=1}^{\infty} x_1(t) \cdots x_n(t) > \delta$$

for some $1 > \delta > 3/4$, then there exists only one natural number m_0 with $x_1(m_0), \dots, x_n(m_0) > \delta$.

Proof. First we will prove that there exists m_1 such that $x_1(m_1) > \delta$. Assume this is not the case. Then $x_1(t) \leq \delta$ for all $t \in \mathbb{N}$. Then,

$$\begin{aligned}
\delta &< \sum_{t=1}^{\infty} x_1(t) \cdots x_n(t) \\
&\leq \delta \sum_{t=1}^{\infty} x_2(t) \cdots x_n(t) \\
&\leq \delta \sum_{t_2,\dots,t_n=1}^{\infty} x_2(t_2) \cdots x_n(t_n) \\
&= \delta \Big(\sum_{t_2=1}^{\infty} x_2(t_2) \Big) \cdots \Big(\sum_{t_n=1}^{\infty} x_n(t_n) \Big) \\
&\leq \delta,
\end{aligned}$$

which is a contradiction. Therefore there exists m_1 with $x_1(m_1) > \delta$.

We can repeat the same proof to see that, for $i = 1, \dots n$, there exists m_i such that $x_i(m_i) > \delta$. It only remains to see that $m_1 = \dots = m_n$ and take $m_0 = m_1$.

Assume that $m_i \neq m_j$ for some $1 \leq i, j \leq n$, $i \neq j$. Then,

$$\begin{aligned}
\delta &< \sum_{t=1}^{\infty} x_1(t) \cdots x_n(t) \\
&\leq \sum_{t=1}^{\infty} x_i(t) x_j(t) \\
&= x_i(m_i)x_j(m_i) + x_i(m_j)x_j(m_j) + \sum_{t \in \mathbb{N}, t \neq m_i, m_j} x_i(t)x_j(t) \\
&\leq x_i(m_i)1/4 + 1/4 x_j(m_j) + \sum_{t_i, t_j \in \mathbb{N}, t_i, t_j \neq m_i, m_j} x_i(t_i)x_j(t_j) \\
&\leq 1/2 + \Big(\sum_{t_i \in \mathbb{N}, t_i \neq m_i, m_j} x_i(t_i) \Big) \Big(\sum_{t_j \in \mathbb{N}, t_j \neq m_i, m_j} x_j(t_j) \Big) \\
&< 1/2 + 1/16 < 3/4 < \delta
\end{aligned}$$

which is a contradiction.

Notice that since $x_1(m_0), \dots, x_n(m_0) > \delta > 3/4$ and $\sum_{t=1}^{\infty} x_i(t) \leq 1, i = 1, \dots, n$, we have that m_0 is unique. □

Theorem 3.5.2. *For every set $P \subseteq \Sigma_n$, there exists an infinite dimensional Banach space $Y \subset \mathcal{L}(^n\ell_1)$ such that for all $B \in Y \setminus \{0\}$, B_σ is norm attaining if and only if $\sigma \in P$.*

Proof. Let's fix a set $P \subseteq \Sigma_n$, and consider a disjoint partition of the natural numbers into an infinite number of infinite sets $\{\mathbb{N}_m\}_{m=1}^{\infty}$ i.e. $\bigcup_m \mathbb{N}_m = \mathbb{N}$ and $\mathbb{N}_m \bigcap \mathbb{N}_{m'} = \emptyset$ iff $m \neq m'$, with $\mathbb{N}_m$ being infinite for $m = 1, 2, \dots$.

The sets $\mathbb{N}_m$ are naturally ordered by the order defined on the natural numbers, so we can assume that $\mathbb{N}_m = \{(m, t)\}_{t=1}^{\infty}$ with $(m, t) < (m, k)$

iff $t < k$.

Let

$$A(m)(e_{k_1}, \ldots, e_{k_n}) = \begin{cases} A(P)(e_{t_1}, \ldots, e_{t_n}) \text{ if } k_i = (m, t_i) \in \mathbb{N}_m, i = 1, \ldots, n, \\ 0 \qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

where $A(P)$ is the n-linear form of Theorem 3.4.7.

Let Y be the vector space defined by

$$Y := \{\sum_{m \in \mathbb{N}} \lambda_m A(m) : \lambda_m \in \mathbb{K}, \lim_{m \mapsto \infty} \lambda_m = 0\}.$$

For every B in Y we have

$$\begin{aligned} \|B\| &= \sup_{x_1, \ldots, x_n \in B_{\ell_1}} |B(x_1, \ldots, x_n)| \\ &= \sup_{k_1, \ldots, k_n} |B(e_{k_1}, \ldots, e_{k_n})| \\ &= \sup_{m \in \mathbb{N}} \sup_{k_1, \ldots, k_n \in \mathbb{N}} |\lambda_m A(m)(e_{k_1}, \ldots, e_{k_n})| \end{aligned}$$

$$\begin{aligned} &= \sup_{m \in \mathbb{N}} |\lambda_m| \|A(P)\| = \sup_{m \in \mathbb{N}} |\lambda_m| \\ &= \max_{m \in \mathbb{N}} |\lambda_m|. \end{aligned}$$

Now, we prove that for all B in $Y \setminus \{0\}$, B_σ is norm attaining if and only if $\sigma \in P$. Let fix $B \in Y \setminus \{0\}$ of norm one.

First we prove that B_σ is norm attaining for $\sigma \in P$. Let fix σ in P.

Since B has norm one, there exists m_0 with $\|B\| = |\lambda_{m_0}|$. Then

$$\lim_{(m_0,k_{\sigma(1)})\to\infty} \cdots \lim_{(m_0,k_{\sigma(n)})\to\infty} B(e_{(m_0,k_1)},\ldots,e_{(m_0,k_n)}) = \lim_{k_{\sigma(1)}\to\infty} \cdots \lim_{k_{\sigma(n)}\to\infty} A(P)(e_{k_1},\ldots,e_{k_n}) = 1,$$

since $\sigma \in P$.

Hence, considering a weak-star cluster point x^{**} of the sequence $\{e_{(m_0,k)}\}_{k=1}^{\infty}$, we obtain

$$B_\sigma(x^{**},\ldots,x^{**}) = 1.$$

Thus, B_σ is norm attaining.

Now we see that if $\sigma \notin P$, then B_σ is not norm attaining. Fix $\sigma \notin P$ and assume that B_σ attains its norm. Then, there exists $(x_1^{**},\ldots,x_n^{**}) \in B_{\ell_1^{**}} \times \cdots \times B_{\ell_1^{**}}$ with $B_\sigma(x_1^{**},\ldots,x_n^{**}) = 1$. Let $\{x_{d_i}\}_{d_i\in D_i}$ be nets in B_{ℓ_1} weak-star convergent to x_i^{**} for $i = 1,\ldots,n$. Therefore

$$\lim_{d_{\sigma(1)}} \ldots \lim_{d_{\sigma(n)}} B(x_{d_1},\ldots,x_{d_n}) = B_\sigma(x_1^{**},\ldots,x_n^{**}) = 1.$$

Fix $1 > \delta > 3/4$. Then, there exists $\alpha_1 \in D_1$ with

$$\lim_{d_{\sigma(2)}} \ldots \lim_{d_{\sigma(n)}} B(x_{d_1},x_{d_2},\ldots,x_{d_n}) > \delta$$

for all $d_{\sigma(1)} > \alpha_1$. For fixed $d_{\sigma(1)} > \alpha_1$, there exists $\alpha_{1,2} \in D_2$ with

$$\lim_{d_{\sigma(3)}} \ldots \lim_{d_{\sigma(n)}} B(x_{d_1},x_{d_2},x_{d_3},\ldots,x_{d_n}) > \delta$$

for all $d_{\sigma(2)} > \alpha_{1,2}$. Fix $d_{\sigma(2)}$ with $d_{\sigma(2)} > \alpha_{1,2}$.

In this way, for fixed $d_{\sigma(1)},\ldots,d_{\sigma(i)}$, for $1 \le i < n-1$, there exists

$\alpha_{1,\ldots,i+1}$ with

$$\lim_{d_{\sigma(i+1)}} \ldots \lim_{d_{\sigma(n)}} B(x_{d_1}, x_{d_2}, \ldots, x_{d_n}) > \delta$$

for all $d_{\sigma(i+1)} > \alpha_{1,\ldots,i+1}$. Fix $d_{\sigma(i+1)}$ with $d_{\sigma(i+1)} > \alpha_{1,\ldots,i+1}$. After $n-1$ steps, for fixed $d_{\sigma(1)}, \ldots, d_{\sigma(n-1)}$ there exists $\alpha_{1,\ldots,n}$ with

$$B(x_{d_1}, x_{d_2}, \ldots, x_{d_n}) > \delta$$

for all $d_{\sigma(n)} > \alpha_{1,\ldots,n}$. Fix $d_{\sigma(n)}$ with $d_{\sigma(n)} > \alpha_{1,\ldots,n}$.

Then,

$$B(x_{d_1}, \ldots, x_{d_n}) > \delta.$$

For every $m \in \mathbb{N}$ define $\pi_m : \ell_1 \mapsto \ell_1$ by $\Big(\pi_m(x)\Big)(t) = x(t)$ if $t \in \mathbb{N}_m$ and $\Big(\pi_m(x)\Big)(t) = 0$ if $t \notin \mathbb{N}_m$.

Notice

$$\begin{aligned}
\delta < B(x_{d_1}, \ldots, x_{d_n}) &= B(\sum_{m_1 \in \mathbb{N}} \pi_{m_1}(x_{d_1}), \ldots, \sum_{m_n \in \mathbb{N}} \pi_{m_n}(x_{d_n})) \\
&= \sum_{m_1, \ldots, m_n \in \mathbb{N}} B(\pi_{m_1}(x_{d_1}), \ldots, \pi_{m_n}(x_{d_n})) \\
&= \sum_{m \in \mathbb{N}} B(\pi_m(x_{d_1}), \ldots, \pi_m(x_{d_n})) \\
&= \sum_{m \in \mathbb{N}} A(m)(\pi_m(x_{d_1}), \ldots, \pi_m(x_{d_n})) \\
&\leq \sum_{m \in \mathbb{N}} \|\pi_m(x_{d_1})\| \cdots \|\pi_m(x_{d_n})\|
\end{aligned}$$

Therefore, by Lemma 3.5.1, there exists m_0 with $\|\pi_{m_0}(x_{d_1})\|, \ldots, \|\pi_{m_0}(x_{d_n})\| > \delta$.

Now, if we consider another $\tilde{d}_{\sigma(n)} > \alpha_{1,\ldots,n}$, by the same argument, there exists $\tilde{m}_0$ with $\|\pi_{\tilde{m}_0}(x_{d_1})\|, \ldots, \|\pi_{\tilde{m}_0}(x_{\tilde{d}_n})\| > \delta$. But, since $1 \geq \|x_{d_{\sigma(1)}}\| = \sum_{m \in \mathbb{N}} \|\pi_m(x_{d_{\sigma(1)}})\|$, and $\|\pi_{m_0}(x_{d_{\sigma(1)}})\|, \|\pi_{\tilde{m}_0}(x_{d_{\sigma(1)}})\| > \delta > 3/4$ we have that $\widetilde{m}_0 = m_0$.

Therefore, for fixed $d_{\sigma(1)}, \ldots, d_{\sigma(n-1)}$, there exists m_0 with

$$\|\pi_{m_0}(x_{d_{\sigma(1)}})\|, \ldots, \|\pi_{m_0}(x_{d_{\sigma(n)}})\| > \delta$$

for all $d_{\sigma(n)} \in D_{\sigma(n)}$ with $d_{\sigma(n)} > \alpha_{1,\ldots,n}$.

Now, fix $d_{\sigma(1)}, \ldots, d_{\sigma(n-2)}$, and consider $\tilde{d}_{\sigma(n-1)} \in D_{\sigma(n-1)}$ with $\tilde{d}_{\sigma(n-1)} > \alpha_{1,\ldots,n-1}$. Then, there exists $\tilde{\alpha}_{1,\ldots,n}$ with

$$B(x_{d_1}, \ldots, x_{\tilde{d}_{\sigma(n-1)}}, \ldots, x_{d_n}) > \delta$$

for all $d_{\sigma(n)} > \tilde{\alpha}_{1,\ldots,n}, \alpha_{1,\ldots,n}$.

Fix $d_{\sigma(n)} > \alpha_{1,\ldots,n}, \tilde{\alpha}_{1,\ldots,n}$. Arguing as before, we find $\tilde{m}_0$ with

$$\|\pi_{\tilde{m}_0}(x_{d_{\sigma(1)}})\|, \ldots, \|\pi_{\tilde{m}_0}(x_{\tilde{d}_{\sigma(n-1)}})\|, \|\pi_{\tilde{m}_0}(x_{d_{\sigma(n)}}))\| > \delta$$

for all $d_{\sigma(n)} \in D_{\sigma(n)}$ with $d_{\sigma(n)} > \tilde{\alpha}_{1,\ldots,n}, \alpha_{1,\ldots,n}$. But, as before $1 \geq \|x_{d_{\sigma(n)}}\| = \sum_{m \in \mathbb{N}} \|\pi_m(x_{d_{\sigma(n)}})\|$, and $\|\pi_{m_0}(x_{d_{\sigma(n)}})\|, \|\pi_{\tilde{m}_0}(x_{d_{\sigma(n)}})\| > \delta > 3/4$ hence $\widetilde{m_0} = m_0$.

We will do one more case for the sake of completeness. Fix $d_{\sigma(1)}, \ldots, d_{\sigma(n-3)}$, and consider $\tilde{d}_{\sigma(n-2)} \in D_{\sigma(n-2)}$ with $\tilde{d}_{\sigma(n-2)} > \alpha_{1,\ldots,n-2}$. Then, there exists $\tilde{\alpha}_{1,\ldots,n-1}$ with

$$\lim_{d_{\sigma(n)}} B(x_{d_1}, \ldots, x_{\tilde{d}_{\sigma(n-2)}}, \ldots, x_{\tilde{d}_{\sigma(n-1)}}, \ldots, x_{d_n}) > \delta$$

for all $\tilde{d}_{\sigma(n-1)} > \tilde{\alpha}_{1,\ldots,n-1}, \alpha_{1,\ldots,n-1}$. Then, for fixed $\tilde{d}_{\sigma(n-1)}$ there exists $\tilde{\alpha}_{1,\ldots,n}$ with

$$B(x_{d_1}, \ldots, x_{\tilde{d}_{\sigma(n-2)}}, \ldots, x_{\tilde{d}_{\sigma(n-1)}}, \ldots, x_{\tilde{d}_{\sigma(n)}}, \ldots, x_{d_n}) > \delta$$

for all $\tilde{d}_{\sigma(n)} > \tilde{\alpha}_{1,\ldots,n}, \alpha_{1,\ldots,n}$.

Fix $\tilde{d}_{\sigma(n)} > \alpha_{1,\dots,n}, \tilde{\alpha}_{1,\dots,n}$. Arguing as before, we find $\tilde{m}_0$ with

$$\|\pi_{\tilde{m}_0}(x_{d_{\sigma(1)}})\|, \dots, \|\pi_{\tilde{m}_0}(x_{\tilde{d}_{\sigma(n-2)}})\|, \|\pi_{\tilde{m}_0}(x_{\tilde{d}_{\sigma(n-1)}})\|, \|\pi_{\tilde{m}_0}(x_{d_{\sigma(n)}}))\| > \delta$$

for all $d_{\sigma(n)} \in D_{\sigma(n)}$ with $d_{\sigma(n)} > \tilde{\alpha}_{1,\dots,n}$. But, as before $1 \geq \|x_{d_{\sigma(n)}}\| = \sum_{m\in\mathbb{N}} \|\pi_m(x_{d_{\sigma(n)}})\|$, and $\|\pi_m(x_{d_{\sigma(n)}})\|, \|\pi_{m_0}(x_{\tilde{d}_{\sigma(n)}})\| > \delta > 3/4$, and hence $\widetilde{m_0} = m_0$.

Therefore, if $d_{\sigma(1)}, \dots, d_{\sigma(n-2)}$ are fixed,

$$\|\pi_{m_0}(x_{d_1})\|, \dots, \|\pi_{m_0}(x_{d_n})\| > \delta$$

for every $d_{\sigma(n-1)} \in D_{\sigma(n-1)}$ with $d_{\sigma(n-1)} > \alpha_{1,\dots,n-1}$, and every $d_n \in D_n$ with $d_{\sigma(n)} > \alpha_{1,\dots,n}$, where $\alpha_{1,\dots,n}$ depends of $d_{\sigma(1)}, \dots, d_{\sigma(n-1)}$.

Notice that the same argument can be repeated to get that

$$\|\pi_{m_0}(x_{d_1})\|, \dots, \|\pi_{m_0}(x_{d_n})\| > \delta$$

for every $d_{\sigma(1)} \in D_{\sigma(1)}$ with $d_{\sigma(1)} > \alpha_1$, $d_{\sigma(2)} \in D_{\sigma(2)}$ with $d_{\sigma(2)} > \alpha_{1,2}$, …, $d_{\sigma(n)} \in D_{\sigma(n)}$ with $d_{\sigma(n)} > \alpha_{1,\dots,n}$, where $\alpha_{1,\dots,i}$ depends of $d_{\sigma(1)}, \dots, d_{\sigma(i-1)}$ for $i = 2, \dots, n$.

Since this holds for every δ with $3/4 < \delta < 1$, we have that

$$\lim_{d_i} \|\pi_{m_0}(x_{d_i}) - x_{d_i}\| = 0,$$

and hence $\{\pi_{m_0}(x_{d_i})\}_{d_i\in D_i}$ converges weak-star to x_i^{**}, $i = 1, \dots, n$.

Now, consider the map $\pi : \ell_1 \mapsto \ell_1$ defined by $\big(\pi(x)\big)(t) = x(m_0, t)$. Since π is $\|\cdot\|$-$\|\cdot\|$-continuous, π is ω-ω-continuous and the canonical extension $\widehat{\pi}$ defined from ℓ_1^{**} into ℓ_1^{**} is ω^*-ω^*-continuous. Therefore, as $\{x_{d_i}\}_{d_i\in D_i}$ is ω^*-convergent to x_i^{**} we have that $\{\pi(x_{d_i})\}_{d_i\in D_i}$ is ω^*-convergent to $\widehat{\pi}(x_i^{**})$ and since π has norm one, $\widehat{\pi}(x_i^{**}) \in B_{\ell_1^{**}}$.

Therefore

$$\begin{aligned}A_\sigma(\hat{\pi}(x_1^{**}),\dots,\hat{\pi}(x_n^{**})) &= \lim_{d_{\sigma(1)}}\dots\lim_{d_{\sigma(n)}} A(\pi(x_{d_1}),\dots,\pi(x_{d_n}))\\ &= \lim_{d_{\sigma(1)}}\dots\lim_{d_{\sigma(n)}} A(m_0)(x_{d_1},\dots,x_{d_n})\\ &= \lim_{d_{\sigma(1)}}\dots\lim_{d_{\sigma(n)}} B(x_{d_1},\dots,x_{d_n})\\ &= B_\sigma(x_1^{**},\dots,x_n^{**}) = 1,\end{aligned}$$

which is a contradiction, since A_σ is not norm attaining for $\sigma \notin P$. Therefore B_σ is not norm attaining for $\sigma \notin P$.

It only remains to see that Y is a Banach space. It is obvious that Y is an infinite dimensional vector space, so we only need to prove that Y is closed. Consider $\{B_r = \sum_{m=1}^{\infty} \lambda_{m,r} A(m)\}_{r=1}^{\infty}$, a Cauchy sequence. Then, we have $\|B_{r_1} - B_{r_2}\| = \sup_{m\in\mathbb{N}} |\lambda_{m,r_1} - \lambda_{m,r_2}|$, hence $\{\lambda_{m,r}\}_{r=1}^{\infty}$ is a Cauchy sequence too, when we fix m. Therefore we can define $\lambda_m = \lim_{r\mapsto\infty} \lambda_{m,r}$. If we let $B = \sum_{m\in\mathbb{N}} \lambda_m A(m)$ it is easy to check that $\lim_r B_r = B \in Y$.

In fact Y is isometrically isomorphic to c_0.

□

3.6 The Lindenstrauss-Bollobás Theorems

Motivated by the scalar-valued 2-homogeneous polynomial Lindenstrauss Theorem and the BPB Property for operators Carando, Lassalle and Mazzitelli, [CLM12], introduced the Lindenstrauss-Bollobás Theorems.

Definition 3.6.1 (Lindenstrauss-Bollobás Theorems)**.** We say that the Lindenstrauss-Bollobas Theorem holds for $\mathcal{L}(X_1, ..., X_n; Y)$, if given $\epsilon > 0$, there are $\delta(\epsilon) > 0$ and $\beta(\epsilon) > 0$ with $\lim_{t\to 0} \beta(t) = 0$ such that for all $A \in \mathcal{L}(X_1, ..., X_n; Y)$ of norm 1, and $x_j \in S_{X_j}$, $j = 1, ..., n$, with

$A(x_1,...,x_n) > 1-\delta(\epsilon)$, there exist $B \in S_{L(X_1,...,X_n;Y)}$ and $y_j \in S_{X_j^{**}}$, $j = 1,...,n$, satisfying that all the Arens extensions of B attain their norm at $(y_1,...,y_n)$,

$$\|y_j - x_j\| < \beta(\epsilon) \text{ for } j = 1,...,n, \quad \textit{and} \quad \|A - B\| < \epsilon.$$

The authors gave several examples of Banach spaces not satisfying the Lindenstrauss-Bollobás Theorem, using preduals of Lorentz sequence spaces. Here we show that M-embedded and L-embedded spaces in its bidual satisfy the Lindenstrauss-Bollobás Theorem for n-linear forms if and only if the Banach space satisfy the n-linear version of Bishop-Phelps-Bollobás Theorem.

Definition 3.6.2. Let X be a real or complex Banach space.

- A linear projection P is called an M-projection if

  $$\|x\| = \max\{\|P(x)\|, \|x - P(x)\|\} \text{ for all } x \in X.$$

  A linear projection P is called an L-projection if $\|x\| = \|P(x)\| + \|x - P(x)\|$ for all $x \in X$.

- A closed subspace $J \subset X$ is called an M-summand if it is the range of an M-projection. A closed subspace $J \subset X$ is called an L-summand if it is the range of an L-projection.

- A closed subspace $J \subset X$ is called an M-ideal if $J^\perp$ is an L-summand in X^*.

Definition 3.6.3 (M-embedded)**.** A Banach space X is called an M-embedded space if X is an M-ideal in X^{**}.

Example 3.6.4. Some examples of M-embedded spaces are:

- c_0 (more generally $c_0(\Gamma)$).
- The canonical predual $d(\omega,1)_*$ of the Lorentz sequence space $d(\omega,1)$,
- The canonical predual $(L^{p,1})_*$ of the Lorentz function space $L^{p,1}$,
- K(H) if H is a Hilbert space (the set of compact operators acting on H).

Theorem 3.6.5. *Given a Banach space X which is M-embedded in its bidual, X satisfies the Lindenstrauss-Bollobás Theorem for n-linear forms if and only if X satisfies the n-linear version of the Bishop-Phelps-Bollobás Theorem.*

Proof. The if condition is obvious. For the other implication consider $1/2 > \beta(\epsilon) > 0$. By the hypothesis there exist $\beta(\epsilon)$ and $\delta(\epsilon)$ satisfying the Lindenstrauss-Bollobás Theorem.

Fix an n-linear form $A \in \mathcal{L}(^nX)$, $\|A\| = 1$, and consider an n-tuple $(x_1, \dots, x_n) \in B_X \times \cdots \times B_X$, such that $|A(x_1, \dots, x_n)| > \delta(\epsilon)$. By the hypothesis there exists $B \in \mathcal{L}(^nX)$ and there exist points $x_i^{**} \in B_{X^{**}}$, $i = 1, \dots, n$, such that:

- $\|(x_1, \dots, x_n) - (x_1^{**}, \dots, x_n^{**})\|_{X^{**}} \leq \beta(\epsilon)$,
- $\|B\| = \|B_\sigma^{**}\| = |B_\sigma^{**}(x_1^{**}, \dots, x_n^{**})| = 1$ for all $\sigma \in \Sigma_n$,
- $\|B - A\| \leq \epsilon$.

For every $i = 1, \dots, n$,

$$1 = \|x_i^{**}\| = \max\{\|x_i^{**} - \pi(x_i^{**})\|_{X^{**}}, \|\pi(x_i^{**})\|_X\}.$$

Therefore, since the projection map from X_i^{**} to X_i is a non-expansive map, we have

$$\begin{aligned}\|x_i^{**} - \pi(x_i^{**})\|_{X^{**}} &= \|x_i^{**} - \pi(x_i^{**})\| \\ &\leq \|x_i^{**} - x_i\| + \|x_i - \pi(x_i^{**})\| \\ &\leq \beta(\epsilon) + \beta(\epsilon) < 1.\end{aligned}$$

Hence $1 = \|x_i^{**}\| = \|\pi(x_i^{**})\|$.

For every $1 \leq i \leq n$ define $y_i \in B_X$ as $y_i = \pi(x_i^{**})$. Since $\|y_i\| = \|\pi(x_i^{**})\| = 1$ we have $y_i \in S_X$ and $\|y_i - x_i\| = \|\pi(x_i^{**}) - x_i\| \leq \|x_i^{**} - x_i\| \leq \beta(\epsilon)$. To finish, we only need to verify that B attains its norm at $(y_1, \dots, y_n)$. To prove this we will show that $B(x_1^{**}, \dots, x_{i-1}^{**}, x_i^{**} - \pi(x_i^{**}), x_{i+1}^{**}, \dots, x_n^{**}) = 0$ for all $i = 1, \dots, n$. If this were not the case, we would have $|B(x_1^{**}, \dots, x_{i-1}^{**}, x_i^{**} - \pi(x_i^{**}), x_{i+1}^{**}, \dots, x_n^{**})| > 0$ and $\beta(\epsilon) \geq \|x_i - x_i^{**}\| \geq \|x_i^{**} - \pi(x_i^{**})\|$ for some $i \in \{1, \dots, n\}$. Let β be a complex number of modulus one such that $sign\Big(\beta B(x_1^{**}, \dots, x_{i-1}^{**}, x_i^{**} - \pi(x_i^{**}), x_{i+1}^{**}, \dots, x_n^{**})\Big) = sign B(x_1^{**}, \dots, x_n^{**})$ and denote by $\alpha = \beta\big(1 - \|x_i^{**} - \pi(x_i^{**})\|\big)$. Then $|\alpha| > 0$.

If we consider

$$z_i = x_i^{**} + \alpha(x_i^{**} - \pi(x_i^{**})),$$

z_i has norm one because X is M-embedded in its bidual and

$$\begin{aligned}|B(x_1^{**}, \dots, x_{i-1}^{**}, z_i, x_{i+1}^{**}, \dots, x_n^{**})| &= |B(x_1^{**}, \dots, x_{i-1}^{**}, x_i^{**}, x_{i+1}^{**}, \dots, x_n^{**}) \\ &\qquad + \alpha B(x_1^{**}, \dots, x_i^{**} - \pi(x_i^{**}), \dots, x_n^{**})| \\ &= |B(x_1^{**}, \dots, x_n^{**})| + |\alpha B(x_1^{**}, \dots, x_i^{**} - \pi(x_i^{**}), \dots, x_n^{**})| \\ &> |B(x_1^{**}, \dots, x_n^{**})| = 1.\end{aligned}$$

But this is a contradiction since B has norm one. Hence the proof is complete. $\square$

As a particular case, since the canonical predual $d(\omega,1)_*$ of the Lorentz sequence space $d(\omega,1)$ is an M-embedded space, we can use Theorem 3.6.5 and the result of Jiménez Sevilla and Payá, [JP98][Theorem 2.6], to prove the following result of [CLM12]

Proposition 3.6.6. *Let ω be an admissible sequence in ℓ_r, for some $1 < r < \infty$. There is no Lindenstrauss-Bollobás theorem for $\mathcal{L}(^n d_*(\omega,1))$, if $n \geq r$.*

Definition 3.6.7 (L-summand)**.** An Banach space X is called an L-embedded space if X is an L-summand in X^{**}.

The following is a well known proposition about the relation between M-embedded and L-embedded spaces

Proposition 3.6.8 ([HWW93, Corollary 1.3, page 102])**.** *If X is an M-embedded space, then X^* is an L-embedded space.*

Therefore the duals of the spaces showed in Example 3.6.4 are examples of L-embedded spaces.

Theorem 3.6.9. *Given a Banach space X which is L-embedded in its bidual, X satisfies the Lindenstrauss-Bollobás Theorem for n-linear forms if and only if X satisfies the n-linear version of the Bishop-Phelps-Bollobás Theorem.*

Proof. The if condition is obvious. As before, for the other implication consider $1/2 > \epsilon > 0$. By the hypothesis there exist $\beta(\epsilon)$ and $\delta(\epsilon)$ satisfying the Lindenstrauss-Bollobás Theorem.

Fix an n-linear form $A \in \mathcal{L}(^n X)$, $\|A\| = 1$, and consider an n-tuple $(x_1, \dots, x_n) \in B_X \times \cdots \times B_X$, such that $|A(x_1, \dots, x_n)| > 1 - \delta(\epsilon)$. By the hypothesis there exists $B \in \mathcal{L}(^n X)$ and there exist points $x_i^{**} \in B_{X^{**}}$, $i = 1, \dots, n$, such that:

- $\|(x_1,\dots,x_n)-(x_1^{**},\dots,x_n^{**})\|_{X^{**}} \le \beta(\epsilon)$,
- $\|B\| = \|B_\sigma^{**}\| = |B_\sigma^{**}(x_1^{**},\dots,x_n^{**})| = 1$ for all $\sigma \in \Sigma_n$,
- $\|B-A\| \le \epsilon$.

For every $i = 1,\dots,n$,

$$1 = \|x_i^{**}\| = \|(x_i^{**}) - \pi(x_i^{**})\|_{X^{**}} + \|\pi(x_i^{**})\|_X$$

so $\pi(x_i^{**}) \neq 0$ and $|B(\frac{\pi(x_1^{**})}{\|\pi(x_1^{**})\|},\dots,\frac{\pi(x_n^{**})}{\|\pi(x_n^{**})\|})| = 1$.

Now we only have to check that $\|\pi(x_i^{**}) - x_i\|_X \le 2\epsilon$, but

$$\begin{aligned}
\epsilon &\ge \|x_i^{**} - x_i\|_{X^{**}} \\
&= \|(x_i^{**} - x_i) - \pi(x_i^{**} - x_i)\|_{X^{**}} + \|\pi(x_i^{**} - x_i)\|_X \\
&\ge \|\pi(x_i^{**} - x_i)\|_X = \|\pi(x_i^{**}) - x_i\|_X \\
&\ge \|x_i\| - \|\pi(x_i^{**})\| \\
&= 1 - \|\pi(x_i^{**})\|
\end{aligned}$$

so $\|\pi(x_i^{**})\| \ge 1 - \epsilon$.

Therefore

$$\begin{aligned}
\|\frac{\pi(x_i^{**})}{\|\pi(x_i^{**})\|} - x_i\| &\le \|\frac{\pi(x_i^{**})}{\|\pi(x_i^{**})\|} - \pi(x_i^{**})\| + \|\pi(x_i^{**}) - x_i\| \\
&\le \epsilon + \|x_i^{**} - x_i\| \\
&\le 2\epsilon.
\end{aligned}$$

□

As a consequence of Theorem 3.6.9 and the fact that $\mathcal{L}({}^n\ell_1)$ does not satisfy the multilinear Bishop-Phelps Theorem, we get that $\mathcal{L}({}^n\ell_1)$ does not satisfy the Lindenstrauss-Bollobás Theorem for any natural number n.

Derived Works

"A writer is a person for whom writing is more difficult than it is for other people."

— Thomas Mann

Some of the results included in this dissertation have been submitted and accepted for publication.

The first paper,

> J. Falcó. The Bishop-Phelps-Bollobás Property for numerical radius on L_1. *J. Math. Anal. Appl.*, 414(1):125–133, 2014,

contains the results showed in Section 2.3.1. After the publication of this work some recent developments have been done in this area. In particular, Kim, Lee, and Martín, [KLM14a] have studied the Bishop-Phelps-Bollobás Property for numerical radius finding sufficient conditions for Banach spaces to ensure the BPBp-ν. Among other results, they show that $L_1(\mu)$-spaces have this property for every measure μ and that every infinite-dimensional separable Banach space can be renormed to fail the BPBp-ν, showing that the Radon-Nikodým Property or even reflexivity is not a sufficient condition on X to get BPBp-ν.

The second paper,

> J. Falcó, D. García, M. Maestre, and P. Rueda. Norm Attaining Arens Extensions on ℓ_1. *Abstr. Appl. Anal.*, pages Art. ID 315641, 10, 2014,

contains some of the results presented in Chapter 3. To be more specific this publication covers the results presented in Section 3.3 and Section 3.4 with special emphasis on the behavior of the norm attaining multilinear Arens extensions on ℓ_1.

Further research directions

"Nothing in life is to be feared, it is only to be understood. Now is the time to understand more, so that we may fear less."

— Marie Curie

Even though the area of study of norm attaining has been deeply studied since Bishop and Phelps proved their theorem, there are still many open questions. One elementary question whose answer is still not known is

Question. Does $\mathbb{R}^2$ have the Property B?

We know that $\mathbb{R}^2$ has properties B for some specific norms. For instance if the unit ball of $\mathbb{R}^2$ is a polyhedron, we have that the space has Property β and as a consequence Property B. But the situation is not clear for other norms like the Euclidean norm.

Acosta, Aron, García, and Maestre, [AAGM08], used the AHSP to characterize the Banach spaces Y such that the pair (ℓ_1, Y) has the $BPBp$. In the same paper, the authors initiated the study of the $BPBp$ when the domain space is c_0. They were able to get estimations of the constants appearing in the $BPBp$ for the pairs (ℓ_∞^n, Y) when Y is a uniformly convex Banach space.

Theorem. *Let Y be a uniformly convex Banach space with modulus of convexity $\delta(\epsilon)$. Let $n \in \mathbb{N}, 0 < \epsilon < 1, 0 < \epsilon < \epsilon'$ with $\epsilon + \frac{\epsilon'}{\epsilon^{1/3}} <$*

$\min\{\delta(\epsilon), 2/3(\epsilon + \epsilon^{2/3})\}$. *For any* $x_0 \in B_{\ell_\infty^n}$ *and* $T \in S_{\mathcal{L}(\ell_\infty^n, Y)}$ *such that* $T(x_0) > 1 - \epsilon$, *there exist* $z_0 \in B_{\ell_\infty^n}$ *and* $V \in S_{\mathcal{L}(\ell_\infty^n, Y)}$ *such that*

$$V(z_0) = 1, \;\; \|z_0 - x_0\| < \epsilon^{1/4} + \epsilon^{1/3}, \;\; \|V - T\| \leq \epsilon + 6n(\sqrt{\epsilon} + \epsilon^{1/6}) + (\epsilon' + \frac{\epsilon'}{\epsilon^{1/3}}).$$

However, these estimations depend on the dimension n, and hence they cannot be used to get a general result about c_0 or ℓ_∞.

More recently, Kim [Kim13] removed the dependence of the dimensions in this situations and proved that if Y is a uniformly convex Banach space, the the pair (c_0, Y) has the Bishop-Phelps-Bollobás Property, [Kim13, Corollary 2.6]. Also, Kim proved the following result:

Theorem ([Kim13, Theorem 2.7])**.** *Let* X *be the real Banach space* c_0 *and let* Y *be a real strictly convex space. Then* (X, Y) *has the BPBp if and only if* Y *is uniformly convex. In particular, if the pair* (ℓ_∞, Y) *has the BPBP,then* Y *is uniformly convex.*

And as a consequence

Corollary ([Kim13, Corollary 2.9])**.** *The Bishop-Phelps-Bollobás theorem holds for bilinear forms on* $c_0 \times \ell_p$ *for* $1 < p < \infty$.

But, the first natural question when studying extensions of Bollobás result on the space c_0 is still open,

Question. Does $c_0 \times c_0$ satisfies a version of the Bishop-Phelps-Bollobás Theorem?

And a full characterization is still not known,

Question. Does there exists a geometric property like the AHSP such that the couple (c_0, Y) has the $BPBp$ if and only if Y satisfies this property?

Or more generally,

Question. Can we find a characterization of the pairs of spaces (X, Y) with the $BPBp$?

Even though the $BPBp$-ν is a more recent topic than the $BPBp$ lots of attention have been paid to its study, see for instance [GK13, KLM14b, AGR14]. However we are far away from a full characterization of the spaces with the $BPBp$-ν. Therefore it would be useful to find more spaces that satisfy $BPBp$-ν in order to look for a full characterization

Question. What spaces of numerical index one have the $BPBp$-ν?

In the non-linear case, following the line of research started by Lindenstrauss, we have presented in Chapter 3 several new results, but some of them have only been proved for bilinear maps. Are they true in general? The key point to extend the theory to higher dimension is the validity of Theorem 3.4.3 for n-linear maps, but the techniques we used to prove the result do not work for n-linear maps with $n \geq 3$.

Question. Is Theorem 3.4.3 valid for n-linear mappings with $n \geq 3$?

If we could answer this question in the affirmative, this would allow us to extend the results obtained in Section 3.4 in the way we explained in Remark 3.4.9.

Proposition 3.2.3, Corollary 3.4.4, Theorem 3.4.7 and Theorem 3.5.2 show the pathological behavior of the norm attaining Arens extensions on ℓ_1. Naturally, we can wonder about whether this is a specific property of the geometry of the space ℓ_1 or if we can consider different spaces. The proof of these results rely on the geometric properties of the space ℓ_1 so a natural candidate to consider is the Lorentz sequence space because of the similarity of the geometry of this space and the space ℓ_1.

Question. Do the results of Section 3.4 and Section 3.5 hold if we consider the Lorentz sequence space $d(\omega, 1)$ with $\omega \in c_0 \setminus \ell_1$ in place of ℓ_1?

In Section 3.6 we have seen that in many situations the Lindenstrauss-Bollobás Theorem is equivalent to the n-linear version of the Bishop-Phelps-Bollobás Theorem but we don't know whether or not this is always true.

Question. For every Banach space, X, does the Lindenstrauss-Bollobás Theorem for n-linear forms hold if and only if the space X satisfy the n-linear version of Bishop-Phelps-Bollobás Theorem?

Looking at a different direction of research, the topic of attaining the norm has been extended to different areas like symmetric bilinear forms or polynomials. For instance it is known that under certain conditions the set of norm attaining symmetric n-linear form is dense in the set of symmetric n-linear forms, see [CK96]. And the same result holds for n-homogeneous polynomials if we assume that the space has the Radon-Nikodým Property, see [AFW95]. Also, for 2-homogeneous polynomials Aron, García, and Maestre, [AGM03], proved a version of Lindenstrauss Theorem as follows:

Theorem (Aron-García-Maestre, [AGM03])**.** *For every Banach space X, the set of all 2-homogeneous polynomials on X whose extension to X^{**} is norm attaining is dense in the set of 2-homogeneous polynomials.*

And for n-linear polynomials, we have the general result if the Banach space X is separable and has the Approximation Property i.e, if X is separable and every compact operator is a limit of finite rank operators.

Theorem (Carando-Lassalle-Mazzitelli, [CLM12]). *Let X, Y be Banach spaces. Suppose that X is separable and has the Approximation Property. Then, the set of all polynomials from X to Y whose Aron-Berner extension attain their norm is dense in the set of all polynomials from X to Y.*

However, the questions about whether there is a general Lindenstrauss Theorem for symmetric bilinear forms and/or n-homogeneous polynomials like Theorem 3.2.4 remains open.

Question.

- Is the set of symmetric n-linear forms, whose extension to $X^{**} \times \cdots \times X^{**}$ is norm attaining, dense in the space of symmetric n-linear forms on X? Or, in particular, is the set of symmetric bilinear forms, whose extension to $X^{**} \times X^{**}$ is norm attaining, dense in the space symmetric bilinear forms on X?
- Is the set of n-homogeneous polynomials whose canonical extension to X^{**} is norm attaining dense in the set of n-homogeneous polynomials on X for $n \geq 3$?

To finish, Choi and Song in [CS09] show that the Bishop-Phelps-Bollobás Theorem fails for n-linear forms on $\ell_1 \times \cdots \times \ell_1$. However, it is not know if the same situation happens for n-homogeneous polynomials,

Question. Does the Bishop-Phelps-Bollobás Theorem hold for n-homogeneous polynomials on ℓ_1?

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