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TESTING QCD WITH TAU DECAYS

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ABSTRACT

The invariant-mass distribution of the hadronic final state in τ decay can be used for testing fundamental aspects of the strong interactions. Using standard QCD techniques, it is possible to compute certain weighted integrals of the hadronic spectrum. We work out some QCD predictions which will be useful for analysing the data. They provide a direct way to simultaneously measure $\alpha_s(m_\tau^2)$ and the parameters characterizing the non-perturbative dynamics, allowing for a better control of the theoretical errors in the determination of $\alpha_s(m_\tau^2)$.

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1. INTRODUCTION

The total τ hadronic width can be accurately calculated using analyticity and the operator product expansion [1–8]. The result, which is known to order $\alpha_s^3(m_\tau^2)$, turns out to be very sensitive to the value of the strong coupling constant. Therefore, precise experimental measurements of the τ lifetime or its leptonic branching ratio can be used to infer a value of $\alpha_s(m_\tau^2)$. Moreover, non-perturbative contributions can be shown [8] to be strongly suppressed, which allows for a reliable estimate of the theoretical uncertainties.

It is the inclusive nature of the total semi-hadronic decay rate that makes a rigorous theoretical calculation possible. Predictions can also be made for those semi-inclusive τ-decay widths associated with specific quark currents. One can separately compute the vector and axial-vector components of the τ hadronic width, and resolve these further into non-strange and strange contributions. This provides an independent way of extracting $\alpha_s(m_\tau^2)$, using the measured semi-inclusive τ-decay rates into an even or odd number of pions/kaons. Thus, the hadronic τ-decay data allow us to make a consistency check of the reliability of the theoretical analysis.

A detailed study of the τ hadronic width has already been done in ref. [8], where the value of $\alpha_s(m_\tau^2)$ implied by present data has been worked out. The purpose of the present paper is to spell out the additional information that can be obtained from the invariant-mass distribution of the hadronic final state in τ decay[1]. Although the distributions themselves cannot be predicted at present, it is possible to compute certain weighted integrals of the hadronic spectrum, using standard QCD techniques [5]. Generally speaking, the accuracy of these theoretical calculations can be much worse than the one of the total τ hadronic width, because non-perturbative effects then are not necessarily suppressed. In fact, choosing an appropriate weight function, non-perturbative effects can even be made to dominate the final result. But this is precisely what makes these integrals interesting: They can be used to measure the parameters characterizing the non-perturbative dynamics and therefore improve our understanding of QCD at long distances. In particular, they provide a direct way to experimentally measure the small non-perturbative contributions to the total τ hadronic width, allowing for a better control of the theoretical errors in the determination of $\alpha_s(m_\tau^2)$.

[1] Some QCD tests using hadronic τ-decay data have already been done in refs. [9].

It is convenient to normalize the hadronic τ-decay width to the electronic one, i.e. to define the ratio

$$R_\tau \equiv \frac{\Gamma(\tau^- \to \nu_\tau \text{hadrons}(\gamma))}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma))}\,, \tag{1.1}$$

where (γ) represents possible additional photons or lepton pairs. The theoretical analysis of R_τ involves the two-point correlation functions for the vector $V^\mu_{ij} = \bar{\psi}_j\gamma^\mu\psi_i$ and axial vector $A^\mu_{ij} = \bar{\psi}_j\gamma^\mu\gamma_5\psi_i$ colour singlet quark currents:

$$\Pi^{\mu\nu}_{ij,V}(q) \equiv i\int d^4x\, e^{iqx}\langle 0|T(V^\mu_{ij}(x)V^\nu_{ij}(0)^\dagger)|0\rangle\,, \tag{1.2a}$$

$$\Pi^{\mu\nu}_{ij,A}(q) \equiv i\int d^4x\, e^{iqx}\langle 0|T(A^\mu_{ij}(x)A^\nu_{ij}(0)^\dagger)|0\rangle\,. \tag{1.2b}$$

Here, the subscripts $i,j = u,d,s$ denote light-quark flavours. The vector (V) and axial vector (A) correlators in (1.2) admit the Lorentz decompositions

$$\Pi^{\mu\nu}_{ij,V/A}(q) = (-g^{\mu\nu}q^2 + q^\mu q^\nu)\,\Pi^{(J=1)}_{ij,V/A}(q^2) + q^\mu q^\nu\,\Pi^{(J=0)}_{ij,V/A}(q^2), \tag{1.3}$$

where the superscript (J) denotes the angular momentum, $J=1$ or $J=0$, in the hadronic rest frame.

The imaginary parts of the correlators $\Pi^{(J)}_{ij,V/A}(q^2)$ defined in (1.3) are proportional to the spectral functions for hadrons with the corresponding quantum numbers. The semi-hadronic decay rate of the τ can be written as an integral of these spectral functions over the invariant mass s of the final-state hadrons:

$$R_\tau = 12\pi S_{EW}\int_0^{m_\tau^2}\frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\text{Im}\Pi^{(0+1)}(s) - 2\frac{s}{m_\tau^2}\text{Im}\Pi^{(0)}(s)\right]\,, \tag{1.4}$$

where $S_{EW} \simeq 1.0194$ is an electroweak correction factor [10], which will be omitted in the following. The appropriate combinations of correlators are

$$\Pi^{(J)}(s) \equiv |V_{ud}|^2\left(\Pi^{(J)}_{ud,V}(s) + \Pi^{(J)}_{ud,A}(s)\right) + |V_{us}|^2\left(\Pi^{(J)}_{us,V}(s) + \Pi^{(J)}_{us,A}(s)\right). \tag{1.5}$$

Following ref. [8] we will decompose the different contributions to R_τ into three categories:

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}. \tag{1.6}$$

Here $R_{\tau,V}$ and $R_{\tau,A}$ denote the vector and axial-vector contributions in the Cabibbo-allowed sector; they correspond to the first two terms in the r.h.s. of eq. (1.5); $R_{\tau,S}$ contains the remaining Cabibbo-suppressed contributions.

In principle the hadronic spectral functions should be calculable within QCD. However, since they are sensitive to the non-perturbative dynamics which binds quarks into hadrons, we are still very far away from being able to do that, specially in the low-s region. Nevertheless, weighted integrals of these spectral functions can be calculated systematically, by exploiting the analytic properties of the correlators $\Pi^{(J)}_{ij,V/A}(s)$ which, for any arbitrary weight function $W(s)$ without singularities in the region $|s| \le s_0$, imply

$$\int_0^{s_0} ds\, W(s)\, \mathrm{Im}\Pi^{(J)}_{ij,V/A}(s) \;=\; \frac{i}{2} \oint_{|s|=s_0} ds\, W(s)\, \Pi^{(J)}_{ij,V/A}(s). \tag{1.7}$$

Eq. (1.7) relates the weighted integral of the spectral function along the physical cut with a contour integral in the complex plane running counter-clockwise around the circle $|s| = s_0$. Thus, in order to study these weighted integrals, one only needs to know the correlators for complex s of order s_0.

For s_0 values not too small, one can assume the validity of the short-distance Operator Product Expansion (OPE) to hold [11] . One can then organize the perturbative and non-perturbative contributions to the correlators into an expansion in powers of $1/s$,

$$\Pi^{(J)}_{ij,V/A}(s) = \sum_{D=0,2,4,\ldots} \frac{1}{(-s)^{D/2}} \sum_{\dim\mathcal{O}=D} \mathcal{C}^{(J)}_{ij,V/A}(s,\mu) \langle \mathcal{O}(\mu)\rangle \ , \tag{1.8}$$

where the inner sum is over local gauge-invariant scalar operators of dimension D. The parameter μ in (1.8) is an arbitrary factorization scale which separates long-distance non-perturbative effects, which are absorbed into the vacuum matrix elements $\langle \mathcal{O}(\mu)\rangle$, from short-distance effects, which are incorporated into the Wilson coefficients $\mathcal{C}^{(J)}_{ij,V/A}(s,\mu)$. The $D = 0$ term (unit operator) corresponds to the pure perturbative contributions, neglecting quark masses. The leading quark-mass corrections generate the $D = 2$ term. The first dynamical operators involving non-perturbative physics appear at $D = 4$.

An updated review of the present status of the OPE for vector and axial vector correlators can be found in ref. [8], where all known information on their Wilson coefficients is given. We will use those results to work out the QCD predictions for certain weighted integrals, which we find particularly well suited to be fitted to the data. Inserting the OPE (1.8) into (1.7) and evaluating the integration along the circle, the weighted integrals can be expressed as an expansion in powers of $1/s_0$, with coefficients depending only logarithmically on s_0.

In Section 2 we discuss spectral moments ($W(s) = s^k$), which allow a simple theoretical study to be made. Weighted integrals of the directly measured hadronic-mass distribution are considered in Section 3; although formally more involved, these integrals are more suitable for performing an experimental analysis. A discussion of the results is finally given in Section 4, where we evaluate the potential sensitivity of a combined fit ot these weighted distributions.

2. SPECTRAL MOMENTS

From the invariant-mass distributions of the hadronic final state in τ decay it is possible to extract the corresponding spectral functions. One can then define integrals of the type [5,12,13] ($k \geq 0$):

$$\mathcal{M}^{(J)}_{ij,V/A}(s_0,k) \equiv 4\pi(k+1)\int_0^{s_0}\frac{ds}{s_0}\left(\frac{s}{s_0}\right)^k \mathrm{Im}\Pi^{(J)}_{ij,V/A}(s)\,. \tag{2.1}$$

From the theoretical point of view, these integrals are nice objects to study because they separate the different power corrections in the OPE in a very clean way [13]. In the chiral limit, and neglecting the small $\alpha_s(s)$ dependence of the ($D \neq 0$) Wilson coefficients $\mathcal{C}^{(J)}_{ij,V/A}(s,\mu)$, one gets

$$\mathcal{M}^{(0+1)}_{ij,V/A}(s_0,k) = \mathcal{F}^{(k)}[a(s_0)] + 4\pi^2(-1)^k\frac{k+1}{s_0^{k+1}}\sum_{\mathrm{dim}\mathcal{O}=2k+2}\mathcal{C}^{(0+1)}_{ij,V/A}(\mu)\langle\mathcal{O}(\mu)\rangle\;, \tag{2.2}$$

where

$$\mathcal{F}^{(k)}[a(s_0)] = \sum_{n=0} K_n\,\frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\,(1-x^{k+1})\,a^n(-xs_0) \equiv \sum_{n=0} K_n\, I_n^{k+1}[a(s_0)] \tag{2.3}$$

contains the pure perturbative contribution. Here, $a(s) \equiv \frac{\alpha_s(s)}{\pi}$, and the coefficients K_n are defined by the perturbative expansion of the logarithmic derivative of the relevant correlator function:

$$D(s) \equiv -s\frac{d}{ds}\Pi^{(0+1)}_{ij,V/A}(s)\Big|_{pert} = \frac{1}{4\pi^2}\sum_{n=0}K_n\,a(-s)^n\;, \tag{2.4}$$

which is known [14–16] to $\mathcal{O}(a^3)$: $K_0 = K_1 = 1$, $K_2 = 1.6398$, $K_3(\overline{\mathrm{MS}}) = 6.3711$.

$\mathcal{F}^{(k)}[a(s_0)]$ can be simply expanded in powers of $a(s_0)$; one gets

$$\begin{aligned}\mathcal{F}^{(k)}[a(s_0)] =& 1 + a(s_0) + \left[K_2 - \frac{\beta_1}{2(k+1)}\right]a^2(s_0)\\ &+\left[K_3 - \frac{1}{k+1}\left(K_2\beta_1+\frac{\beta_2}{2}\right)+\frac{\beta_1^2}{4}\left(\frac{2}{(k+1)^2}-\frac{\pi^2}{3}\right)\right]a^3(s_0)+\mathcal{O}(a^4).\end{aligned} \tag{2.5}$$

However, the large $\log(s)$ range (i.e. $2\pi i$) over which $\alpha_s(s)$ is made to run when calculating the integrals along the unit circle in eq. (2.3), gives rise to larger expansion coefficients than in eq. (2.4). The effect of higher-order corrections then appears more sizeable[2], specially for small values of k. The slow apparent convergence of the expansion in powers of $a(s_0)$ should not be attributed to the original K_n expansion of the dynamical two-point correlation function $D(s)$. Note that there is no deep reason to stop the integral expansions to $\mathcal{O}(a^3)$. One can calculate them to all orders in α_s, up to the unknown $\beta_{n>3}$ contributions. In other words, the integrals $I_n^{k+1}[a(s_0)]$ in eq. (2.3) are well-defined functions which can be numerically computed, by using for $\alpha_s(s)$ the exact solution of the renormalization-group β-function equation [17]. We checked that the difference between using the one- or two-loop approximation to the β function is already quite small, while the change induced by the three-loop corrections is completely negligible ($\sim 0.1\%$). The final perturbative result is then very stable, and the error induced by the truncation of the β function at third order can be safely neglected. Equation (2.3) then provides a much better expansion of $\mathcal{F}^{(k)}[a(s_0)]$, which appears to converge faster than the $D(s)$ expansion.

As shown in eq. (2.2), the k moment of the spectral function isolates the contributions of dimension $D = 2k + 2$ in the OPE. Choosing different values of k, it is then possible to study the different terms in the OPE. This nice property is no longer true when the $\alpha_s(s)$ dependence of the $(D \neq 0)$ Wilson coefficients is taken into account. This is however a small $\mathcal{O}(\alpha_s^2)$ effect, which can be taken into account in a combined fit of the different moments.

3. WEIGHTED DISTRIBUTIONS

The spectral moments discussed in the previous section allow a simple analysis to be made. However, on the theoretical side, they suffer from two drawbacks. On the one hand, when expressed in the form of an integration along a close contour in the complex plane, these moments may receive contributions from the region near the real axis, where the use of the OPE is not justified; in the case of R_τ, these contributions are suppressed by the kinematical factor $(1 - s/m_\tau^2)^2$. On the other hand, the dependence of the $\mathcal{M}^{(J)}_{ij,V/A}(s_0, k)$ moments on s_0 is not well defined, in the sense that their derivatives with respect to this

[2] In fact, the radius of convergence of the α_s expansion in eq. (2.5) is quite small [17]: for $a(s_0)$ values slightly larger than 0.11 the perturbative expansion of $\mathcal{F}^{(0)}[a(s_0)]$ in powers of $a(s_0)$ becomes non-convergent.

variable involve $\mathrm{Im}\Pi(s_0)$, which, for $s_0 \sim \mathcal{O}(1\ \mathrm{GeV}^2)$ on the real axis, is not a quantity within the reach of perturbative QCD, even supplemented by the OPE expansion.

In addition, on the experimental side, these spectral moments suffer also from two drawbacks. On the one hand, what is directly measured is the hadronic invariant-mass squared distribution $\frac{dR_\tau}{ds}$, which, because of the same $(1 - s/m_\tau^2)^2$ factor, is statistically very limited at the end point of the spectrum. On the other hand, because of the finite experimental resolution on s, the raw distribution extends above the kinematical limit $s = m_\tau^2$. This raw distribution must therefore be corrected in order to obtain detector-effect free moments to be compared with the QCD predictions. Such a correction stage in the analysis implies systematic effects which are going to affect mostly the tail of the distribution. It is therefore ill-advised to use moments built upon $\mathrm{Im}\Pi(s) \propto [(1-s/m_\tau^2)^2(1+2s/m_\tau^2)]^{-1} \times \frac{dR_\tau}{ds}$, since they enhance dramatically the contribution of the statistically- and systematically-limited tail of the s-distribution.

The above considerations tend to favour the use of integrals of the type $(k, l \geq 0)$

$$R_\tau^{kl}(s_0) \equiv \int_0^{s_0} ds \left(1 - \frac{s}{s_0}\right)^k \left(\frac{s}{m_\tau^2}\right)^l \frac{dR_\tau}{ds}. \tag{3.1}$$

Here the factor $(1 - s/s_0)^k$ supplements $(1 - s/m_\tau^2)^2$ for $s_0 \neq m_\tau^2$, in order to squeeze the integrand at the crossing of the positive real-axis and, therefore, it improves the reliability of the theoretical analysis through the OPE. This factor implies, moreover, that the determination of α_s and the non-perturbative parameters, through a simultaneous fit of different R_τ^{kl} moments, ought to be stable with respect to changes in s_0; this is because their first k derivatives with respect to s_0 do not involve $\mathrm{Im}\Pi(s_0)$ directly. For $s_0 \simeq m_\tau^2$, the same $(1 - s/s_0)^k$ factor, which is no longer needed from the theoretical point of view, reduces the contribution from the tail of the distribution, which is badly defined experimentally. Of course, the precisions of the experimental R_τ^{kl} measurements are going to worsen when k and/or l grow, but this can be accounted for, together with the strong correlations between the various measurements (cf. section 4).

Using the decomposition (1.6), we can analogously define the corresponding weighted distributions $R_{\tau,V}^{kl}(s_0)$, $R_{\tau,A}^{kl}(s_0)$ and $R_{\tau,S}^{kl}(s_0)$, involving the measured semi-inclusive τ decays into an even/odd number of pions, and an odd number of kaons respectively. These three moments obviously add up to (3.1)

$$R_\tau^{kl}(s_0) = R_{\tau,V}^{kl}(s_0) + R_{\tau,A}^{kl}(s_0) + R_{\tau,S}^{kl}(s_0). \tag{3.2}$$

The theoretical expressions for the R_τ^{kl} moments, although straightforward to obtain, look rather cumbersome. We will organize the results in the form:

$$R_\tau^{kl}(s_0) = \left(|V_{ud}|^2 + |V_{us}|^2\right) \mathcal{P}^{kl}(s_0) + \sum_{D=2,4,\ldots} \Delta^{kl}(s_0, D), \tag{3.3a}$$

$$R_{\tau,V}^{kl}(s_0) = |V_{ud}|^2 \left\{ \frac{1}{2}\mathcal{P}^{kl}(s_0) + \sum_{D=2,4,\ldots} \Delta_{ud,V}^{kl}(s_0, D) \right\}, \tag{3.3b}$$

$$R_{\tau,A}^{kl}(s_0) = |V_{ud}|^2 \left\{ \frac{1}{2}\mathcal{P}^{kl}(s_0) + \sum_{D=2,4,\ldots} \Delta_{ud,A}^{kl}(s_0, D) \right\}, \tag{3.3c}$$

$$R_{\tau,S}^{kl}(s_0) = |V_{us}|^2 \left\{ \mathcal{P}^{kl}(s_0) + \sum_{D=2,4,\ldots} \Delta_{us}^{kl}(s_0, D) \right\}, \tag{3.3d}$$

where

$$\Delta_{ij}^{kl}(s_0, D) = \Delta_{ij,V}^{kl}(s_0, D) + \Delta_{ij,A}^{kl}(s_0, D), \tag{3.4a}$$

$$\Delta^{kl}(s_0, D) = |V_{ud}|^2 \Delta_{ud}^{kl}(s_0, D) + |V_{us}|^2 \Delta_{us}^{kl}(s_0, D). \tag{3.4b}$$

The function $\mathcal{P}^{kl}(s_0)$ stands for the purely perturbative part, neglecting quark masses, which is the same for all the components of $R_\tau^{kl}(s_0)$. The inverse-power corrections of dimension D are collected in the terms $\Delta_{ij,V/A}^{kl}(s_0, D)$.

It is convenient to use the binomial expansion $(1 - x/x_0)^k = \sum_m C_m^k(x_0) x^m$, where $x \equiv s/m_\tau^2$, $x_0 \equiv s_0/m_\tau^2$ and

$$C_m^k(x_0) = \begin{cases} (-1)^m x_0^{-m} \binom{k}{m} & \text{if } k \geq m \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

The results can then be written in a compact way in terms of the functions

$$\begin{aligned} h_m^k(x_0) &\equiv C_m^k(x_0) - 3C_{m-2}^k(x_0) + 2C_{m-3}^k(x_0), \\ j_m^k(x_0) &\equiv C_m^k(x_0) - 2C_{m-1}^k(x_0) + C_{m-2}^k(x_0). \end{aligned} \tag{3.6}$$

3.1. Perturbative Contribution

Using the perturbative expansion (2.4), the function $\mathcal{P}^{kl}(s_0)$ can be expressed as

$$\mathcal{P}^{kl}(s_0) = 3 \sum_{n=0} K_n \, \mathcal{A}_n^{kl}[a, x_0], \tag{3.7}$$

where

$$\mathcal{A}_n^{kl}[a, x_0] \;=\; 2\sum_{r>0} \frac{x_0^r}{r}\, h_{r-l-1}^k(x_0)\, I_n^r[a(s_0)], \tag{3.8}$$

are contour integrals [see eq. (2.3)], which only depend on $x_0 \equiv s_0/m_\tau^2$ and on the value of the running coupling constant at the scale s_0.

If the running coupling $a(-xm_\tau^2)$ is expanded in powers of $a(\mu^2)$, one gets a perturbative expansion of $\mathcal{A}_n^{kl}[a, x_0]$ which is regulated by the coefficients of the QCD β function times elementary logarithmic integrals in the complex plane [17]. Taking $\mu^2 = s_0$, the resulting expansion for $\mathcal{P}^{kl}(s_0)$ takes the form

$$\mathcal{P}^{kl}(s_0) \;=\; 3r_\tau^{kl}(x_0) \sum_{n=0} \left(K_n + g_n^{kl}(x_0)\right) a^n(s_0), \tag{3.9}$$

where $r_\tau^{kl}(x_0)$ is the parton-level prediction,

$$r_\tau^{kl}(x_0) \;=\; 2\sum_{r>0} \frac{x_0^r}{r}\, h_{r-l-1}^k(x_0), \tag{3.10}$$

and the coefficients $g_n^{kl}(x_0)$ depend on $K_{m<n}$ and $\beta_{m<n}$. To order $a^3(s_0)$, one has

$$\begin{aligned}
g_0^{kl}(x_0) &= g_1^{kl}(x_0) \;=0, \\
g_2^{kl}(x_0) &= -\,\beta_1\, H_1^{kl}(x_0), \\
g_3^{kl}(x_0) &= -\,\frac{\pi^2}{12}\beta_1^2 - (2\beta_1 K_2 + \beta_2)\; H_1^{kl}(x_0) + \beta_1^2\, H_2^{kl}(x_0), \\
g_4^{kl}(x_0) &= -\,\frac{\pi^2}{4}\left(\beta_1^2 K_2 + \frac{5}{6}\beta_1\beta_2\right) - \left(3\beta_1 K_3 + 2\beta_2 K_2 + \beta_3 - \beta_1^3\frac{\pi^2}{4}\right)\; H_1^{kl}(x_0) \\
&\quad + \left(3\beta_1^2 K_2 + \frac{5}{2}\beta_1\beta_2\right)\; H_2^{kl}(x_0) - \frac{3}{2}\beta_1^3\, H_3^{kl}(x_0),
\end{aligned} \tag{3.11}$$

where

$$H_m^{kl}(x_0) \;=\; \frac{1}{r_\tau^{kl}} \sum_{r>0} \frac{x_0^r}{r^{m+1}}\, h_{r-l-1}^k(x_0). \tag{3.12}$$

A sample of numerical values are given in table 1. Since $a(m_\tau^2) \approx 0.1$, the $g_4^{kl}(x_0)$ values indicate that the $\mathcal{O}(\alpha_s^4)$ corrections are at the few per cent level. One observes that in general the $g_n^{kl}(x_0)$ contributions are larger than the direct K_n contributions. For example, the bold guess $K_4 \sim K_3(K_3/K_2) \approx 25$ is to be compared with the $g_4^{kl}(1) \sim 100$ values.

The reason of these large kl-dependent contributions has already been mentioned in the previous section. These large contributions can be resummed, in order to keep the

Table 1

Parton level prediction $r_\tau^{kl}(x_0)$ and first $g_n^{kl}(x_0)$ coefficients, for $x_0 = 1$ and different values of k and l

$k,\ l$	$r_\tau^{kl}(1)$	$g_2^{kl}(1)$	$g_3^{kl}(1)$	$g_4^{kl}(1)$
0, 0	1.000	3.56	20.0	78.0
0, 1	0.300	2.14	0.51	−116.
0, 2	0.133	1.58	−5.30	−147.
0, 3	0.071	1.26	−8.14	−157.
1, 0	0.700	4.17	28.3	161.
1, 1	0.167	2.59	5.16	−90.6
1, 2	0.062	1.94	−2.03	−136.
1, 3	0.029	1.57	−5.60	−151.
2, 0	0.533	4.67	35.6	240.
2, 1	0.105	2.97	9.40	−63.8
2, 2	0.033	2.26	1.04	−123.
2, 3	0.013	1.84	−3.19	−144.

well-behaved K_n expansion, by computing the functions $\mathcal{A}_n^{kl}[a, x_0]$ numerically, using in eq. (3.8) the exact solution for $\alpha_s(s)$ obtained from the renormalization-group β-function equation.

3.2. Leading Quark-Mass Corrections

The up and down quark-mass corrections are negligible (unless one considers very low s_0 values, where in any case the OPE is no longer valid). The correction from the strange quark mass must be taken into account to analyse the $R_{\tau,S}^{kl}$ moments; it is however unessential, because of the Cabibbo suppression, in the total R_τ^{kl} moments. For completeness we give here the resulting formulae:

$$\Delta_{ij,V/A}^{kl}(s_0, 2) = 18 \sum_{r\geq 0} j_{r-l}^{k}(x_0) \left\{ -\frac{m_i^2(s_0) + m_j^2(s_0)}{m_\tau^2} S_r(s_0) \pm \frac{m_i(s_0) m_j(s_0)}{m_\tau^2} Q_r(s_0) \right\} \tag{3.13}$$

$$S_0(s_0) = \frac{1+x_0}{2} + \frac{7+11x_0}{6} a(s_0) + \mathcal{O}(a^2),$$
$$S_{r\geq 0}(s_0) = \frac{x_0^r}{r+1} \left\{ \frac{x_0}{2} - \left[1 - \frac{5}{6}x_0 + \frac{1+r(1-x_0)}{r(r+1)} \right] a(s_0) \right\} + \mathcal{O}(a^2), \tag{3.14}$$
$$Q_r(s_0) = \frac{x_0^{r+1}}{r+1} \left[1 + \left(\frac{17}{3} + \frac{2}{r+1} \right) a(s_0) \right] - \delta_{r,0} \frac{1}{3} a(s_0) + \mathcal{O}(a^2).$$

3.3. Non-Perturbative Contributions

In the chiral limit, and neglecting the small logarithmic dependence of the Wilson coefficients $\mathcal{C}_{ij,V/A}^{(J)}(s,\mu)$ on s, the contribution from dimension D operators to the R_τ^{kl} moments are found to be

$$\Delta_{ij,V/A}^{kl}(s_0, D) = (-1)^{\frac{D-2}{2}} 12\pi^2 \frac{h_{D/2-l-1}^k(x_0)}{m_\tau^D} \sum_{\dim\mathcal{O}=D} \mathcal{C}_{ij,V/A}^{(0+1)}(s,\mu) \langle \mathcal{O}(\mu) \rangle. \tag{3.15}$$

When the logarithmic dependence of the Wilson coefficients on s is taken into account, the factors $\Delta_{ij,V/A}^{kl}(s_0, D)$ get additional corrections, but they are suppressed by two powers of $\alpha_s(s_0)$. The effect of non-zero quark masses is also very small. We will only consider these corrections for the $D=4$ term. Using the results of ref. [8], one finds in this case

$$\begin{aligned} \Delta_{ij,V/A}^{kl}(s_0, 4) \; = \; -\frac{\pi^2}{m_\tau^4} \Big\{ & T_{V/A}^1 \, C_{1-l}^k(x_0) \; + \; T_{V/A}^2 \, C_{-l}^k(x_0) \\ & + \sum_{r\neq 0} \frac{x_0^r}{r} \left[T_{V/A}^3 \, h_{r-l+1}^k(x_0) \; + \; T_{V/A}^4 \, j_{r-l}^k(x_0) \right] \Big\}, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} T_{V/A}^1 = & \left(1 - \frac{11}{18} a(s_0)\right) \langle \frac{\alpha_s}{\pi} GG \rangle + 12 \left(1 - a(s_0) - \frac{13}{3} a^2(s_0)\right) \langle m_i \bar\psi_i \psi_i + m_j \bar\psi_j \psi_j \rangle \\ & \pm 16 a(s_0) \left(1 + \frac{59}{8} a(s_0)\right) \langle m_j \bar\psi_i \psi_i + m_i \bar\psi_j \psi_j \rangle \\ & + \frac{16}{9} a(s_0) \left[1 - \left(\frac{257}{72} - 9\zeta(3)\right) a(s_0)\right] \sum_k \langle m_k \bar\psi_k \psi_k \rangle \\ & + \frac{3}{7\pi^2} \Big\{ 42 m_i^2(s_0) m_j^2(s_0) + \left[-12 a^{-1}(s_0) + 7\right] \left[m_i^4(s_0) + m_j^4(s_0)\right] \\ & \quad \mp 16 m_i(s_0) m_j(s_0) \left[m_i^2(s_0) + m_j^2(s_0)\right] - \sum_k m_k^4(s_0) \Big\}, \end{aligned} \tag{3.17a}$$

$$
\begin{aligned}
T^2_{V/A} = & -24\left\langle (m_i \mp m_j)(\bar\psi_i\psi_i \mp \bar\psi_j\psi_j)\right\rangle \\
& + \frac{3}{7\pi^2}\left[24a^{-1}(s_0) - 11\right]\left[m_i(s_0) \mp m_j(s_0)\right]\left[m_i^3(s_0) \mp m_j^3(s_0)\right] \\
& \pm \frac{18}{\pi^2}\left[m_i(s_0) \mp m_j(s_0)\right]^2 m_i(s_0) m_j(s_0),
\end{aligned}
\qquad (3.17b)
$$

$$
\begin{aligned}
T^3_{V/A} = & a^2(s_0)\left\{\frac{11}{8}\left\langle \frac{\alpha_s}{\pi} GG\right\rangle + 27\left\langle m_i\bar\psi_i\psi_i + m_j\bar\psi_j\psi_j\right\rangle \right. \\
& \left. \mp 36\left\langle m_j\bar\psi_i\psi_i + m_i\bar\psi_j\psi_j\right\rangle - 4\sum_k \langle m_k\bar\psi_k\psi_k\rangle \right\} \\
& + \frac{9}{\pi^2}\left[m_i^4(s_0) + m_j^4(s_0)\right],
\end{aligned}
\qquad (3.17c)
$$

$$
T^4_{V/A} = -\frac{18}{\pi^2}\left[m_i(s_0) \mp m_j(s_0)\right]\left[m_i^3(s_0) \mp m_j^3(s_0)\right]. \qquad (3.17d)
$$

4. DISCUSSION

By using the formulae given in the previous section, a combined fit of different $R_\tau^{kl}(s_0)$ moments should result in experimental values for $\alpha_s(m_\tau^2)$ and for the coefficients of the inverse-power corrections in the OPE. Contrary to what happens with the $\mathcal{M}^{(J)}_{ij,V/A}(s_0,k)$ moments discussed in Section 2, the $R_\tau^{kl}(s_0)$ distributions mix power corrections of different dimensionality. As shown in eq. (3.15), the leading $[\mathcal{O}(\alpha_s)]$ non-perturbative corrections are regulated by the factor $h^k_{D/2-l-1}(x_0)$; thus $R_\tau^{kl}(s_0)$ gets contributions from operators of dimension D in the ranges

$$
2(l+1+m) \le D \le 2(k+l+1+m) \qquad (4.1)
$$

where $m = 0, 2, 3$.

On the theoretical side, the value $k = 0$ is particularly attractive, for $s_0 = m_\tau^2$, since only $D = 2(l+1)$, $D = 2(l+3)$ and $D = 2(l+4)$ contribute in that case. Note that there is no contribution with $D = 2(l+2)$. Therefore, $R_\tau^{0l}(s_0)$ is mainly sensitive to the non-perturbative effects of dimension $D = 2(l+1)$, the next dimensionality contribution being suppressed by four additional powers of s_0. In particular, since there are no gauge-invariant operators of dimension $D = 2$, only the $D = 6$ and $D = 8$ operators contribute to R_τ^{00}. One recovers the conclusion that the leading non-perturbative terms of dimension $D = 4$ are absent in the unweighted integral $R_\tau^{00}(m_\tau^2) = R_\tau$. Hence, non-perturbative corrections are tiny for the total τ hadronic width, which is thus mainly sensitive to the perturbative contributions and, therefore, to the value of α_s [8]. This is however no longer true for the moments R_τ^{kl}, since $h_{1-l}^k(x_0) \neq 0$ in general. In particular, $R_\tau^{01}(s_0)$ tests the $D = 4$ contributions to the OPE, $R_\tau^{02}(s_0)$ the terms with $D = 6$, and so on.

Since the perturbative contribution $\mathcal{P}^{kl}(s_0)$ is the same for all the components of $R_\tau^{kl}(s_0)$, one can further test the non-perturbative dynamics by taking differences where perturbative effects cancel:

$$R_{\tau,V}^{kl}(s_0) - R_{\tau,A}^{kl}(s_0) \;=\; |V_{ud}|^2 \sum_{D=2,4,\dots} \left\{ \Delta_{ud,V}^{kl}(s_0,D) - \Delta_{ud,A}^{kl}(s_0,D) \right\}. \qquad (4.2)$$

Again, taking $k = 0$ in eq. (4.2), the terms of dimension $D = 2(l+1)$ are cleanly separated.

The relative contribution of the perturbative term can also be reduced by taking ratios of moments. This is because the leading $\mathcal{O}(\alpha_s)$ corrections cancel out in the ratios, since $K_1 + g_1^{kl}(x_0) = 1 \quad \forall k,l$. Thus, the perturbative corrections to the normalized moments $D_\tau^{kl} \equiv R_\tau^{kl}/R_\tau^{00}$ are $\mathcal{O}(\alpha_s^2)$. The corrections are exactly known up to order α_s^4, because the $\mathcal{O}(\alpha_s^4)$ coefficient does not depend on K_4. For example, the α_s expansion of the purely perturbative prediction for D_τ^{10} reads

$$D_\tau^{10} \;=\; 0.7(1 + 0.61\ a^2 + 7.74\ a^3 + 72.1\ a^4). \qquad (4.3)$$

Note that the α_s correction remains sizeable ($a = 0.1$ implies a 2% correction on D_τ^{10}). While the total τ hadronic width is mainly sensitive to the perturbative effects ($a = 0.1$ implies a 20% correction on R_τ), the shape of the hadronic-mass distribution (and therefore the normalized moments D_τ^{kl}) is also regulated by non-perturbative dynamics. But, as is further demonstrated below, the D_τ^{kl} moments still depend in a very significant way on α_s.

For the sake of illustration, to evaluate the potential sensitivity of a combined fit involving weighted integrals, we consider a hypothetical experiment having measured a set of R_τ^{kl} moments, including R_τ itself. For simplicity, we assume that no attempt is made to disentangle the vector/axial-vector or Cabibbo-suppressed contributions. Since $R_\tau = R_\tau^{00}$ is the overall normalization of the s distribution, only the shape of the latter provides additional information with respect to R_τ. Thus the combined fit depends only on R_τ and on the normalized moments D_τ^{kl}.

The covariance matrix which describes the precision of the measurements can be expressed in terms of moments. For R_τ, which is almost not correlated to the moments, one gets

$$\sigma[R_\tau] \simeq R_\tau \sqrt{\frac{1 + R_\tau/2}{N_h}}, \qquad (4.4)$$

where N_h is the total number of observed hadronic decays, and, for the D_τ^{kl} moments,

$$\sigma^2[D_\tau^{kl}, D_\tau^{k'l'}] = \frac{1}{N_h^{\rm eff}}(D_\tau^{k+k',l+l'} - D_\tau^{kl} D_\tau^{k'l'}), \tag{4.5}$$

where $N_h^{\rm eff} < N_h$ is the effective number of hadronic decays used to calculate the weighted integrals. Both N_h and $N_h^{\rm eff}$ are supposed to be reduced with respect of the "actually" used number of events in order to account for systematic effects. In the following, we use the conservative values $N_h = 5000$ and $N_h^{\rm eff} = 1000$, which are already below the data samples accumulated by each of the LEP experiments. It should be pointed out that the systematical uncertainties are not expected to be overwhelming, since the needed D_τ^{kl} moments do not probe details of the s distribution, but only its gross features (e.g. the average s value).

The D_τ^{kl} moments are highly correlated quantities. For example, the correlation coefficient between the moments D_τ^{13} and D_τ^{14} is already $\rho_{13;14} = 0.97$. This implies, in particular, that there is very little information to gain by including D_τ^{14} in a fit already using D_τ^{13}.

Since we are only concerned with the resolution power of such an experiment, we can assume that its measurements coincide with the exact $[\mathcal{O}(K_3)]$ predictions obtained by choosing, for example, $\alpha_s(m_\tau^2) = 0.34$ and by using for the calculations of the non-perturbative contributions the numerical values quoted in [8] ; in particular,

$$\left\langle \frac{\alpha_s}{\pi} GG \right\rangle = (0.02 \pm 0.01)\,\mathrm{GeV}^4\,, \qquad \text{and} \qquad O(6) = (0.002 \pm 0.001)\,\mathrm{GeV}^6, \tag{4.6}$$

where $O(D=6) = \sum_{\dim\mathcal{O}=6} \mathcal{C}_{ij,V+A}^{(0+1)}(s,\mu)\langle\mathcal{O}(\mu)\rangle$. Neglecting the $D \geq 8$ terms, and using the K_n expansion of eq. (3.7) (i.e. the numerical evaluation of the $\mathcal{A}_n^{kl}$ functions), these numerical values yield $R_\tau = 3.58$ with an experimental error $\sigma[R_\tau] = 0.085$, and, for the set of the first four $D_\tau^{1,l=0\to3}$ moments, the values quoted in table 2.

Table 2

Expected measurements and their precisions for $N_h = 5000$ and $N_h^{\rm eff} = 1000$

$k,\ l$	$D_\tau^{kl}(1)$	$\sigma[D_\tau^{kl}]$
1, 0	0.7232	0.0072
1, 1	0.1479	0.0035
1, 2	0.0612	0.0015
1, 3	0.0272	0.0010

Taking for granted the above non-perturbative constants, the experiment would obtain $\alpha_s = 0.34 \pm 0.04$ from R_τ alone and $\alpha_s = 0.34 \pm 0.03$ from the four D_τ^{1l} moments. By itself, such an agreement between the two α_s determinations, if observed, would provide a remarkable confirmation of both the applicability of the perturbative QCD expansion at this small energy scale, and the validity of the OPE applied in a context where non-perturbative effects are present. Note that, although the effective statistics is assumed to be five times smaller for the moment determinations, the precision achieved from the shape of the s distribution is nevertheless better than the one derived from R_τ. This is because there are several correlated but more accurately measured quantities entering into the moment fit. Although the QCD correction to the D_τ^{kl} moments is much smaller than the one to R_τ [cf. eq. (4.3)], their individual contribution to the fit is of a similar importance. This can be appreciated by considering their α_s expansion. For example, the precision on α_s achieved by the R_τ measurement and the sole D_τ^{10} measurement can be estimated, using table 1 with eq. (4.3) and table 2, to be

$$\begin{aligned} \sigma[\alpha_s] &\simeq \left(\frac{\mathrm{d}R_\tau}{\mathrm{d}\alpha_s}\right)^{-1} \sigma[R_\tau] \simeq 0.03 \ , \\ \sigma[\alpha_s] &\simeq \left(\frac{\mathrm{d}D_\tau^{10}}{\mathrm{d}\alpha_s}\right)^{-1} \sigma[D_\tau^{10}] \simeq 0.04 \ , \end{aligned} \tag{4.7}$$

respectively.

Dropping the knowledge of the non-perturbative constants and performing a simultaneous four-parameter fit to the five quantities, the experiment would obtain $\alpha_s = 0.34 \pm 0.042$, $\langle \frac{\alpha_s}{\pi} GG \rangle = (0.02 \pm 0.015)$ GeV^4, $O(6) = (0.002 \pm 0.002)$ GeV^6 and $O(8) = (0. \pm 0.002)$ GeV^8 (the non-perturbative term that is the most correlated with the α_s determination is the gluon condensate, for which $\rho = -0.72$). Therefore such an experiment would be in a position to measure simultaneously $\alpha_s(m_\tau^2)$ and the non-perturbative condensates [$O(8)$ included] to a level of precision competitive with their presently available determinations, while keeping for α_s an accuracy comparable to the one achieved when using R_τ alone, but assuming the values of the non-perturbative contributions to be known.

The above-quoted uncertainties can of course be improved by using a much larger data sample. If enough statistics is accumulated, one could perform other interesting tests [5,9] by disentangling the vector, axial-vector and strange-quark components of R_τ and of the weighted integrals. The behaviour of the OPE itself and its validity range could be tested by performing the analysis at different values of s_0 and by a complementary study of D_τ^{k0} moments, with increasing values of k. One would expect the reliability of the power expansion to deteriorate when going down to small s_0 values or for large k values. It would be interesting to see where the short-distance OPE analysis becomes meaningless. The absence of a sizeable non-perturbative contribution of dimension 2 (there are no gauge- and Lorentz-invariant operators with $D = 2$ in the OPE) could also be checked experimentally: for example, by extracting α_s from a subset of R^{kl} moments with $l \neq 0$, since such moments receive no contributions from the $O(2)$ term. However, allowing for an extra $O(2)$ term implies a considerable loss of precision on the α_s measurement.

It is remarkable that the decay properties of the third-generation lepton not only allow for a measurement of the strong coupling constant at a rather low mass-scale, but, in addition, provide a direct experimental way of bounding the size of the non-perturbative contributions. Performing a combined fit such as the one advocated in this paper, the theoretical sources of errors in the α_s determination from the τ decays are essentially reduced to the perturbative ones, which were shown to be very small in a previous publication [17]. The reliability of the α_s value derived from the τ-decay analysis can be reinforced (or disproved) by the comparison of the values obtained using the total width of the τ lepton and the s distribution.

Obviously, in order to perform such a test, rather good experimental data are needed to control the systematic uncertainties. Hence, a not so large ($N_h^{\rm eff} \sim 1000$) but clean sample of τ-decay events is required. In particular, owing to the semi-inclusive nature of these distributions, good identification of neutral particles is mandatory. The modern high-statistics experiments have already reached the needed accuracy to allow a meaningful test to be done. Larger samples of events collected at future τ-factory machines will certainly improve the sensitivity of the analysis, allowing us to extract the rich QCD-information contained in the τ decays.

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