THE PERTURBATIVE QCD PREDICTION TO R_τ REVISITED

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ABSTRACT

The perturbative QCD prediction to the total hadronic width of the tau lepton is re-examined. A more convergent perturbative expansion is proposed, which is associated with a smaller renormalization-scheme dependence and better-defined higher-order uncertainties.

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The total τ hadronic width can be accurately calculated using analyticity and the operator product expansion [1–8]. The result, which is known to order $\alpha_s^3(m_\tau^2)$, turns out to be very sensitive to the value of the strong coupling constant [3]. Therefore, precise experimental measurements of the τ lifetime or its leptonic branching ratio can be used to infer a value of $\alpha_s(m_\tau^2)$. Moreover, non-perturbative contributions can be shown to be strongly suppressed, which allows for a reliable estimate of the theoretical uncertainties.

A detailed study of the τ hadronic width has already been done in ref. [8], where the value of $\alpha_s(m_\tau^2)$ implied by present data has been worked out. This analysis has shown that the final theoretical uncertainty is completely dominated by the uncalculated perturbative QCD corrections of order $\alpha_s^4(m_\tau^2)$. Therefore, the error in the determination of $\alpha_s(m_\tau^2)$ is small. Taking a conservative $\pm 130(\frac{\alpha_s(m_\tau^2)}{\pi})^4$ for the perturbative error, the resulting uncertainty on $\alpha_s(m_\tau^2)$ was estimated to be about 10% in ref. [8]. When the running coupling constant $\alpha_s(\mu^2)$ is evolved from the scale m_τ to higher energies the error scales roughly as $\alpha_s^2(\mu^2)$ and thus shrinks as μ increases. A modest precision of about 10% in $\alpha_s(m_\tau^2)$ then translates in a very precise determination of the QCD coupling at some higher-energy scale such as M_Z.

The purpose of this letter is to provide an improved QCD perturbative expansion of the total hadronic width of the tau. Within the framework of this revisited QCD prediction, the sensitivity of the extracted $\alpha_s(m_\tau^2)$ value to the unknown higher-order perturbative corrections and more generally the systematic error attached to the renormalization scheme ambiguity are reanalysed.

Following ref. [8], we normalize the hadronic τ decay width to the electronic one, i.e. we define the ratio

$$R_\tau \equiv \frac{\Gamma(\tau^- \to \nu_\tau \text{hadrons}(\gamma))}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma))}, \tag{1}$$

where (γ) represents possible additional photons or lepton pairs. R_τ can be written as a contour integral in the complex s-plane, along the circle $|s| = m_\tau^2$. For massless quarks, and neglecting the small non-perturbative and electroweak corrections, one has

$$R_\tau^{pert} = -6\pi i \oint_{|s|=m_\tau^2} \frac{ds}{s} \left(1 - 2\frac{s}{m_\tau^2} + 2\frac{s^3}{m_\tau^6} - \frac{s^4}{m_\tau^8} \right) D(s), \tag{2}$$

where the dynamical information is carried by the logarithmic derivative of the two-point correlation function of the vector (axial) current,

$$D(s) \equiv -s\frac{d}{ds}\Pi(s), \tag{3}$$

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which satisfies a homogeneous Renormalization Group Equation. Making use of this equation, the perturbative expansion of $D(s)$ in powers of the running coupling constant can be written in the form

$$D(s) \equiv \frac{1}{4\pi^2} \sum_{n=0} \tilde{K}_n(\xi) \, a^n(\xi^2 s) \,, \tag{4}$$

where $a = \dfrac{\alpha_s}{\pi}$, ξ is an arbitrary factor (of order unity) and

$$\begin{aligned} \tilde{K}_0(\xi) &= \tilde{K}_1(\xi) = 1, \\ \tilde{K}_2(\xi) &= K_2 - K_1 \beta_1 \log \xi, \\ \tilde{K}_3(\xi) &= K_3 - 2K_2 \beta_1 \log \xi + K_1 \left(\beta_1^2 \log^2 \xi - \beta_2 \log \xi \right), \end{aligned} \tag{5}$$

and similarly for the other $\tilde{K}_{n \geq 4}(\xi)$ functions. The $K_n = \tilde{K}_n(1)$ coefficients are known [9–11] to order α_s^3. For 3 flavours and in $\overline{MS}$, the K_n and β_n coefficients are[1]

$$\begin{aligned} K_0 = K_1 = 1; \qquad K_2 &= 1.6398; \qquad K_3(\overline{MS}) = 6.3711; \\ \beta_1 = -\frac{9}{2}; \qquad \beta_2 &= -8; \qquad \beta_3(\overline{MS}) = -\frac{3863}{192}. \end{aligned} \tag{6}$$

Inserting the expansion (4) in eq. (2), R_τ^{pert} can be expressed as

$$R_\tau^{pert} = 3 \sum_{n=0} \tilde{K}_n(\xi) A^{(n)}(a_\xi) \,, \tag{7}$$

where the functions

$$A^{(n)}(a_\xi) = \frac{1}{2\pi i} \oint_{|s|=m_\tau^2} \frac{ds}{s} \left(1 - 2\frac{s}{m_\tau^2} + 2\frac{s^3}{m_\tau^6} - \frac{s^4}{m_\tau^8} \right) a^n(-\xi^2 s) \tag{8}$$

are contour integrals in the complex plane which only depend on $a_\xi = \dfrac{\alpha_s(\xi^2 m_\tau^2)}{\pi}$. Note that, formally, the $A^{(n)}(a_\xi)$ functions obey the same renormalization-group equation as a^n:

$$\frac{\partial}{\partial \log \xi} A^{(n)}(a_\xi) = n \sum_{k=1} \beta_k A^{(n+k)}(a_\xi), \tag{9}$$

apart from the fact that now n is an index.

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[1] In ref. [8] the perturbative contribution to R_τ was parametrized in terms of the coefficients F_n, appearing in the expansion of the spectral function $\mathrm{Im}\Pi(s)$ in powers of $a(s)$. Both parametrizations are related by trivial factors: $K_2 = F_3$; $K_3 = F_4 + (\pi^2 \beta_1^2/12)$.

The running coupling $a(-\xi^2 s)$ in eq. (8) can be expanded in powers of a_ξ, with coefficients that are polynomials in $\log(-s/m_\tau^2)$. Doing so, the following perturbative expansion of the $A^{(n)}(a_\xi)$ functions is obtained:

$$
\begin{aligned}
A^{(0)}(a_\xi) &= 1, \\
A^{(1)}(a_\xi) &= a_\xi + \frac{\beta_1}{2} I_1\, a_\xi^2 + \left(\frac{\beta_2}{2} I_1 + \frac{\beta_1^2}{4} I_2 \right) a_\xi^3 \\
&\quad + \left(\frac{\beta_3}{2} I_1 + \frac{5}{8}\beta_1\beta_2 I_2 + \frac{\beta_1^3}{8} I_3 \right) a_\xi^4 + \mathcal{O}(a_\xi^5), \\
A^{(2)}(a_\xi) &= a_\xi^2 + \beta_1 I_1\, a_\xi^3 + \left(\beta_2 I_1 + \frac{3}{4}\beta_1^2 I_2 \right) a_\xi^4 + \mathcal{O}(a_\xi^5), \\
A^{(3)}(a_\xi) &= a_\xi^3 + \frac{3}{2}\beta_1 I_1\, a_\xi^4 + \mathcal{O}(a_\xi^5), \\
A^{(4)}(a_\xi) &= a_\xi^4 + \mathcal{O}(a_\xi^5),
\end{aligned}
\tag{10}
$$

which is regulated by the coefficients of the QCD β-function times the elementary integrals

$$
I_k = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1 - 2x + 2x^3 - x^4) \log^k(x). \tag{11}
$$

To order α_s^4, the needed integrals are

$$
I_1 = -\frac{19}{12} \simeq -1.58 \;, \quad I_2 = \frac{265}{72} - \frac{\pi^2}{3} \simeq 0.39 \;, \quad I_3 = -\frac{3355}{288} + \frac{19}{12}\pi^2 \simeq 3.98 \;. \tag{12}
$$

The perturbative expansion of R_τ in powers of $\alpha_s(\xi^2 m_\tau^2)$ takes the form

$$
R_\tau^{pert} = 3 \sum_{n=0} (\tilde{K}_n(\xi) + g_n(\xi))\, a_\xi^n, \tag{13}
$$

where the $g_n(\xi)$ coefficients depend on $\tilde{K}_{m<n}(\xi)$ and on $\beta_{m<n}$:

$$
\begin{aligned}
g_0(\xi) &= g_1(\xi) = 0, \\
g_2(\xi) &= \frac{\beta_1}{2} I_1 = 3.563, \\
g_3(\xi) &= \left(\tilde{K}_2(\xi)\beta_1 + \frac{\beta_2}{2} \right) I_1 + \frac{\beta_1^2}{4} I_2 \\
&= 19.99 \quad (\text{ for } \xi = 1), \\
g_4(\xi) &= \left(\tilde{K}_3(\xi)\frac{3\beta_1}{2} + \tilde{K}_2(\xi)\beta_2 + \frac{\beta_3}{2} \right) I_1 + \left(\tilde{K}_2(\xi)\frac{3\beta_1^2}{4} + \frac{5}{8}\beta_1\beta_2 \right) I_2 + \frac{\beta_1^3}{8} I_3 \\
&= 78.00 \quad (\text{ for } \xi = 1).
\end{aligned}
\tag{14}
$$

3

Since $a(m_\tau^2) \approx 0.1$, the value of $g_4(1)$ indicates that the $\mathcal{O}(\alpha_s^4(m_\tau^2))$ correction is at the few per cent level. One observes that the $g_n(1)$ contributions are larger than the direct $\tilde{K}_n(1)$ contributions. For instance, the bold guess value $\tilde{K}_4(1) \sim K_3(K_3/K_2) \approx 25$ is to be compared with $g_4(1) = 78$.

It is possible to make the first $\tilde{K}_n(\xi) + g_n(\xi)$ coefficients smaller by taking a particular value of the renormalization scale. Owing to $g_n(1) > \tilde{K}_n(1)$, the obvious choice[2] is to take the value of ξ which reduces the g_n contribution, e.g. the one which satisfies $\tilde{K}_n(\xi) + g_n(\xi) \simeq K_n$. For $n \leq 2$ this requirement gives $\xi = e^{I_1/2}$ (i.e. it suggests to use the scale $\mu_0 = \xi m_\tau = 808$ MeV). In that case one gets $\tilde{K}_n(\xi) + g_n(\xi) = K_n$ for $n \leq 2$, and $\tilde{K}_3(\xi) + g_3(\xi) = K_3 - (\beta_1/2)^2(I_1^2 - I_2) = -4.34$. Hence, in addition to removing the $g_{n=2}$ contribution, this selection of scale flips the sign of the $\mathcal{O}(\alpha_s^3)$ term. However, this apparent improvement is misleading. The price to be paid to obtain this apparently faster convergence of the series is to have an almost twice bigger expansion parameter [$a(m_\tau^2) \approx 0.1$ implies $a(\mu_0^2) \approx 0.17$]. In fact it is shown below that the higher-order g_n contributions make the perturbative series *non-convergent* for this choice of scale. In contrast, with $\mu = m_\tau$ and for $a(m_\tau^2) = 0.1$, the expansion is convergent, but the contributions from the higher-order g_n coefficients are quite important. The reason of such uncomfortably large contributions stems from the complex integration along the circle $s = m_\tau^2 \exp(i\phi)$ ($\phi\epsilon[0, 2\pi]$) in eq. (8). When the running coupling $a(-\xi^2 s)$ is expanded in powers of a_ξ, one gets imaginary logarithms, $\log(-s/m_\tau^2) = i(\phi - \pi)$, which are large in some parts of the integration range. The radius of convergence of this expansion is actually quite small. To make the argument simpler, let us consider the β function to the one-loop approximation only. The perturbative expansion

$$a(-\xi^2 s) = \frac{a_\xi}{1 - \frac{\beta_1}{2}a_\xi \log\left(\frac{-s}{m_\tau^2}\right)} = a_\xi \sum_{n=0}^{\infty} \left(\frac{\beta_1}{2}a_\xi \log\left(\frac{-s}{m_\tau^2}\right)\right)^n, \qquad (15)$$

is convergent along the circle $|s| = m_\tau^2$, provided that $a_\xi < 2/(|\beta_1|\pi) = 0.14$. Therefore the series is (slowly) convergent for $a(m_\tau^2) \approx 0.1$ but it is non-convergent for $a(\mu_0^2) \approx 0.17$. A numerical analysis of the series involving the β_1, β_2, and β_3 coefficients shows that, at the three-loop level, an upper estimate for the convergence radius a_{conv} is

$$a_{conv} < 0.11 . \qquad (16)$$

2　After completion of this work, we received a paper by M. Luo and W.J. Marciano [12] where this value of μ_0 is in fact advocated.

The present level of accuracy achieved on the $\alpha_s(m_\tau^2)$ determination is fairly compatible with values above this convergence radius. The slow apparent convergence of the R_τ^{pert} expansion should not be attributed to the original K_n expansion of the dynamical two-point correlation function $D(s)$. It is the large $\log(s)$ range (i.e. $2\pi i$) over which $\alpha_s(s)$ is made to run, when calculating the $A^{(n)}(a_\xi)$ integrals, which produces this unwanted behaviour. Note[3] that there is no deep reason to stop the $A^{(n)}(a)$ integral expansions at $\mathcal{O}(\alpha_s^3)$. One can calculate the $A^{(n)}$ expansion to all orders in α_s, apart from the unknown $\beta_{n>3}$ contributions, which are likely to be negligible (see below). Even for $a(m_\tau^2)$ larger than the radius of convergence (16), the integrals $A^{(n)}(a)$ are well-defined functions that can be numerically computed, by using in eq. (8) the exact solution for $\alpha_s(s)$ obtained from the renormalization-group β-function equation. For illustration, the perturbative approximation to $\delta^{(0)} = (R_\tau^{pert} - 3)/3$ is shown in fig. 1 at the three-loop level (i.e. $\beta_{n>3} = 0$ and $\tilde{K}_{n>3}(1) = 0$), as a function of the order m where the expansion in powers of a has been truncated. The results plotted in fig. 1a correspond to $\xi = 1$ and $a_\xi = 0.1$. As m increases, the series converges to the exact result indicated by the horizontal line, but the difference is still sizeable for the $m = 3$ truncation that appears in the midst of a large initial oscillation. Fig. 1b shows the results obtained for $\xi m_\tau = \mu_0$ and $a_\xi = 0.17$ under the same assumptions; in this case, the non-convergent series makes very large oscillations around the exact result.

Thus a more appropriate approach is to use a $\tilde{K}_n$ expansion of R_τ^{pert} as in eq. (7), and to fully keep the known 3-loop-level calculation of the functions $A^{(n)}(a)$. The perturbative uncertainties will then be reduced to the corrections coming from the unknown $\beta_{n>3}$ and $K_{n>3}$ contributions, since the $g_n(\xi)$ contributions are properly resummed to all orders. To appreciate the size of the effect, Table 1 gives the exact results for $A^{(n)}(a)$ ($n = 1, 2, 3$) obtained at the one-, two- and three-loop approximations (i.e. $\beta_{n>1} = 0$, $\beta_{n>2} = 0$, and $\beta_{n>3} = 0$, respectively), together with the final value of $\delta^{(0)}$, the perturbative QCD correction to R_τ, for $a(m_\tau^2) = 0.1$. For comparison, the numbers coming from the truncated expressions at order $\alpha_s^3(m_\tau^2)$ are also given. Although the difference between the exact and truncated results represents a tiny 0.6% effect on R_τ, it produces a sizeable 4% shift on the value of $\delta^{(0)}$. The $\delta^{(0)}$ shift, which reflects into a corresponding shift in the experimental

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[3] A similar suggestion has been recently made in reference [13].

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Table 1

Exact results for $A^{(n)}(a)$ $(n = 1, 2, 3)$ obtained at the one-, two- and three-loop approximations, together with the final value of $\delta^{(0)} \equiv (R_\tau^{pert} - 3)/3$, for $a(m_\tau^2) = 0.1$. For comparison, the numbers coming from the truncated expressions at order $\alpha_s^3(m_\tau^2)$ are also given.

Type of calculation	$A^{(1)}(a)$	$A^{(2)}(a)$	$A^{(3)}(a)$	$\delta^{(0)}$
$\beta_{n>1} = 0$	0.13247	0.01570	0.00170	0.1690
$\beta_{n>2} = 0$	0.13523	0.01575	0.00163	0.1714
$\beta_{n>3} = 0$	0.13540	0.01565	0.00160	0.1712
$\mathcal{O}(\alpha_s^3(m_\tau^2))$	0.14394	0.01713	0.00100	0.1784

$\alpha_s(m_\tau^2)$ determination, depends strongly on the coupling constant value; for $a(m_\tau^2) = 0.14$ the $\delta^{(0)}$ shift reaches the -20% level.

Notice that the difference between using the one- or two-loop approximation to the β-function is already quite small (1.4% effect on $\delta^{(0)}$), while the change induced by the three-loop corrections is completely negligible (0.1%). Therefore (unless the β-function has some unexpected pathological behaviour at higher orders), the error induced by the truncation of the β-function at third order should be smaller than 0.1% and therefore can be safely neglected. For the sake of illustration a sample of $a^{-n}A^{(n)}(a)$ functions, obtained through numerical integration, are represented on fig. 2a. One observes that $A^{(n)}(a) \ll a^n$ for large values of n or a; hence the $A^{(n)}$ expansion of R_τ^{pert} converges faster than the $D(s)$ expansion itself.

The only relevant source of perturbative uncertainty is then the unknown higher-order coefficients $K_{n>3}$. To obtain an estimate of the error induced on $\delta^{(0)}$, one can make the naïve guess $\Delta(\delta^{(0)}) \sim \pm(K_3/K_2)K_3 A^{(4)}(a)$, which for $a(m_\tau^2) = 0.1$ gives a small $\Delta(\delta^{(0)}) = \pm 0.004$ effect. The sensitivity of $\delta^{(0)}$ on K_4 can be appraised from fig. 2b where the QCD perturbative prediction is represented as a function of a for the three values $\tilde{K}_4(1) = +25, 0, -25$. In particular, one observes, for $a = a_* \simeq 0.19$, that the fourth-order term $\tilde{K}_4(\xi)A^{(4)}(a_*)$ cancels out (cf. fig. 2a) while the higher-order contributions $\tilde{K}_n(\xi)A^{(n)}(a_*)$ $(n > 4)$ are also reduced[4] $(A^{(n)}(a_*) \ll a_*^n$ for $n \gg 4)$.

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[4]   It follows that, for the particular choice of renormalization scale $\mu_* = \xi_* m_\tau$ which satisfies $\delta^{(0)}[exp] = \delta^{(0)}(a_{\xi_*} = a_*)$, the higher-order uncertainties are of order $\tilde{K}_5(\xi_*)A^{(5)}(a_*) = -10^{-4}\tilde{K}_5(\xi_*)$.

To estimate the sensitivity of the α_s determination on the choice of renormalization scheme, we consider the effect of changing the renormalization scale and changing the β_3 coefficient independently (a more involved analysis of this problem can be found in ref. [14]). We consider as an example an experiment that obtains $\delta^{(0)}[exp] = 0.2$. The $\alpha_s(M_z^2)$ determination extracted from this value as a function of the chosen renormalization scale $\mu = \xi m_\tau$ is shown in fig. 3a. The α_s evolution from the tau mass to the Z^0 mass is done following ref. [15], but changing the flavour by one unit at the $\mu^2 = 4m_q^2$ (charm and bottom) crossings rather than at the $\mu^2 = m_q^2$ crossing, in order to be consistent with the use of $n_f = 3$ at $\mu = m_\tau > m_c$. The full line [curve (1)] corresponds to the determination obtained using the $A^{(n)}(a_\xi)$ expansion of eq. (7) while the dotted line [curve (2)] corresponds[5] to the α_s determination using the a_ξ^n expansion of eq. (13). One observes that the μ-scale ambiguity is drastically reduced by the use of eq. (7). The resulting theoretical uncertainty, defined to be half the range spanned by varying μ from 1 GeV to 2.5 GeV, is ± 0.0009 using eq. (7), and ± 0.0035 using eq. (13). Hence, the μ-scale uncertainty attached to eq. (7) reaches a completely negligible level, owing to the actual experimental errors (typically ± 0.006). The shift between the two $\alpha_s(M_z^2)$ values obtained using eq. (7) and eq. (13) (0.003) is within the previously estimated theoretical uncertainties [8] (e.g. it is of the same size as the μ-scale ambiguity of eq. (13)). One remarks also that the Principle of Minimal Sensitivity (PMS) introduced in [16] points towards a value which is close to the tau mass ($\mu_{PMS} = 1.3$ GeV) for eq. (7), in contrast with the disturbingly small value obtained using eq. (13) ($\mu_{PMS} = 0.85$ GeV).

Similarly, the $\overline{MS}$ $\alpha_s(M_z^2)$ value obtained when changing the Renormalization Scheme through the β_3 coefficient, according to ref. [16]:

$$
\begin{aligned}
\tilde{K}_3^{RS}(1) &= \tilde{K}_3^{\overline{MS}}(1) - C \ , \\
\beta_3^{RS} &= \beta_3^{\overline{MS}} + C\beta_1 \ , \\
\alpha_s^{RS}(m_\tau^2) &\simeq \alpha_s^{\overline{MS}}(m_\tau^2) \left[1 + C \left(\frac{\alpha_s^{\overline{MS}}(m_\tau^2)}{\pi} \right)^2 \right] \ ,
\end{aligned}
\tag{17}
$$

is shown in fig. 3b for the two expansions, as a function of $\beta_3^{RS}/\beta_3^{\overline{MS}}$. In that case also the uncertainty is significantly reduced using eq. (7). The resulting theoretical uncertainty

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[5]   The fall-off of the curves at small values of μ is due to the fact that with such small scales the assumed $\delta^{(0)}$ value cannot be obtained exactly (i.e. the experimental χ^2 would sharply increase in that region).

using eq. (7), defined to be half the range spanned by varying β^{RS} from 0 to $2\beta^{\overline{MS}}$, is ± 0.0005; with these purely conventional error definitions, it is comparable but smaller than the μ-scale error.

Another means of estimating the theoretical uncertainty is to consider the effect of the missing $\tilde{K}_4(1)$ coefficient on the α_s determination. Figure 4a represents the $\alpha_s(M_z^2)$ determination extracted from $\delta^{(0)}[exp] = 0.2$, as a function of μ, using the $A^{(n)}$ expansion to order $n = 4$ with $\tilde{K}_4(1) = -25$ [curve (1)], $\tilde{K}_4(1) = 0$ [curve (2)], $\tilde{K}_4(1) = +25$ [curve (3)] and to order $n = 3$ [curve (4), the same as curve (1) of fig. 3a]. Note that the four curves cross exactly at the scale $\mu_* = \xi_* m_\tau$, where the a_{ξ_*} value which is given by the $n = 3$ determination yields $A^{(4)}(a_{\xi_*}) = 0$ (cf. fig. 2a) and that, by construction, curves (4) ($\tilde{K}_4(\xi) = 0$) and (2) ($\tilde{K}_4(1) = 0$) cross again at $\mu = m_\tau$ ($\xi = 1$). Using the $n = 4$ prediction, one observes that the μ-scale ambiguity is almost totally removed. Thus, the only remnant source of theoretical uncertainty, in that case, comes from $\tilde{K}_4(1)$. Taking half the largest range spanned by varying K_4 from -25 to $+25$ as a measure of this uncertainty, one obtains ± 0.0008, which is of a similar size as the μ-scale error. Figure 4b is the same as fig. 4a, but using the a^n expansion. Again, the fall-off of the curves at small μ values reflects the fact that $\delta^{(0)} = 0.2$ cannot be obtained with too small scales. One observes that, even at order $n = 4$, the μ dependence is reduced, but not removed, when using the a^n expansion; therefore the estimation of the theoretical uncertainty must account for it.

To summarize, we have shown that the standard QCD perturbative prediction to the total hadronic width of the tau lepton leads to a non-convergent expansion for $\alpha_s > 0.11\pi$. The lack of convergence is not connected with the dynamical two-point correlation function, but is due to the large $\log(s)$ range over which α_s is made to run in the course of the calculations. The revised expression we propose makes use of the known coefficients of the Renormalization Group Equation to resum the non-convergent part of the series to all orders in α_s. In addition, it has been shown that, using this approach, the Renormalization Scheme dependence is strongly reduced with respect to the standard one and that the higher-order uncertainties are better defined. Within the framework of this revisited QCD prediction, the uncertainties attached to the experimental α_s determination derived from R_τ are presently dominated by experimental errors. The theoretical uncertainties due to unknown higher-order contributions and Renormalization Scheme ambiguities have been estimated to be at the level of $\sigma[\alpha_s(M_z^2)] \sim 0.001$. A complementary analysis, using the hadronic final state invariant-mass-squared distribution, will be presented in a forthcoming

publication [17]. Combined with the R_τ measurement, this more complete analysis allows for the simultaneous determination of α_s and of the relevant non-perturbative terms, thus removing most of the theoretical uncertainties attached to the non-perturbative contributions.

9

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Figure captions

• Figures 1 : convergence of the α_s expansion.

$\delta^{(0)}$, the QCD perturbative correction to R_τ, as a function of m, the order to which the α_s perturbative expansion is stopped ($\beta_{n>3} = 0$ and $\tilde{K}_{n>3}(1) = 0$). Figures 1a and 1b correspond to $\xi = 1$, $a_\xi = 0.1$ and $\xi = e^{I_1/2}$, $a_\xi = 0.17$ respectively. In the latter case the coefficients $\tilde{K}_n(\xi) \neq 0$ are accounted for.

• Figures 2 : Sensitivity to the higher-order contributions

Figure 2a represents the $a^{-n}A^{(n)}(a)$ functions for $n = 1, 2, 3, 4, 5$ and 10. One observes that $A^{(3)}(a) = 0$ for a particular a_* value and that $A^{(n)}(a) \ll a^n$ for large n or a values. Figure 2b shows the $\delta^{(0)}(a)$ function for the three values $\tilde{K}_4(1) = +25, 0, -25$.

• Figures 3 : Renormalization Scheme uncertainties.

One considers an hypothetical experiment having measured $\delta^{(0)}[exp] = 0.2$ from which is extracted a determination of the $\overline{MS}$ value of $\alpha_s(M_z^2)$. Figure 3a represents the $\alpha_s(M_z^2)$ determination as a function of the chosen renormalization scale $\mu = \xi m_\tau$. Figure 3b shows the $\alpha_s(M_z^2)$ determination as a function of the second renormalization scheme dependent quantity : $\beta_3^{RS}/\beta_3^{\overline{MS}}$. In both figures, the curves (1) and (2) are obtained using the $A^{(n)}(a_\xi)$ and a_ξ^n expansions, respectively.

• Figure 4 : Overall theoretical uncertainties.

Figure 4a represents the $\alpha_s(M_z^2)$ determination extracted from $\delta^{(0)}[exp] = 0.2$ using the $A^{(n)}$ expansion to order $n = 4$ with $\tilde{K}_4(1) = -25$ (curve (1)), $\tilde{K}_4(1) = 0$ (curve (2)), $\tilde{K}_4(1) = +25$ (curve (3)) and to order $n = 3$ (curve (4)). The four curves cross exactly at the scale $\mu_* = \xi_* m_\tau$. Figure 4b is the same as figure 4a, but using the a^n expansion.
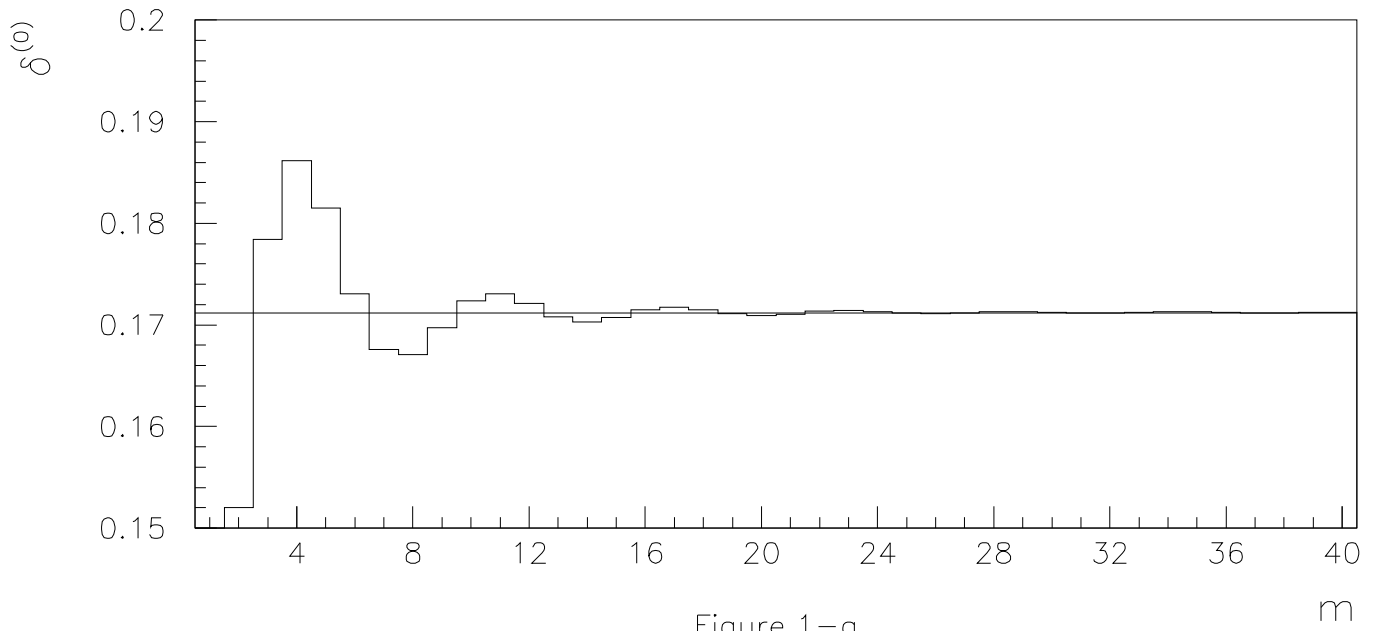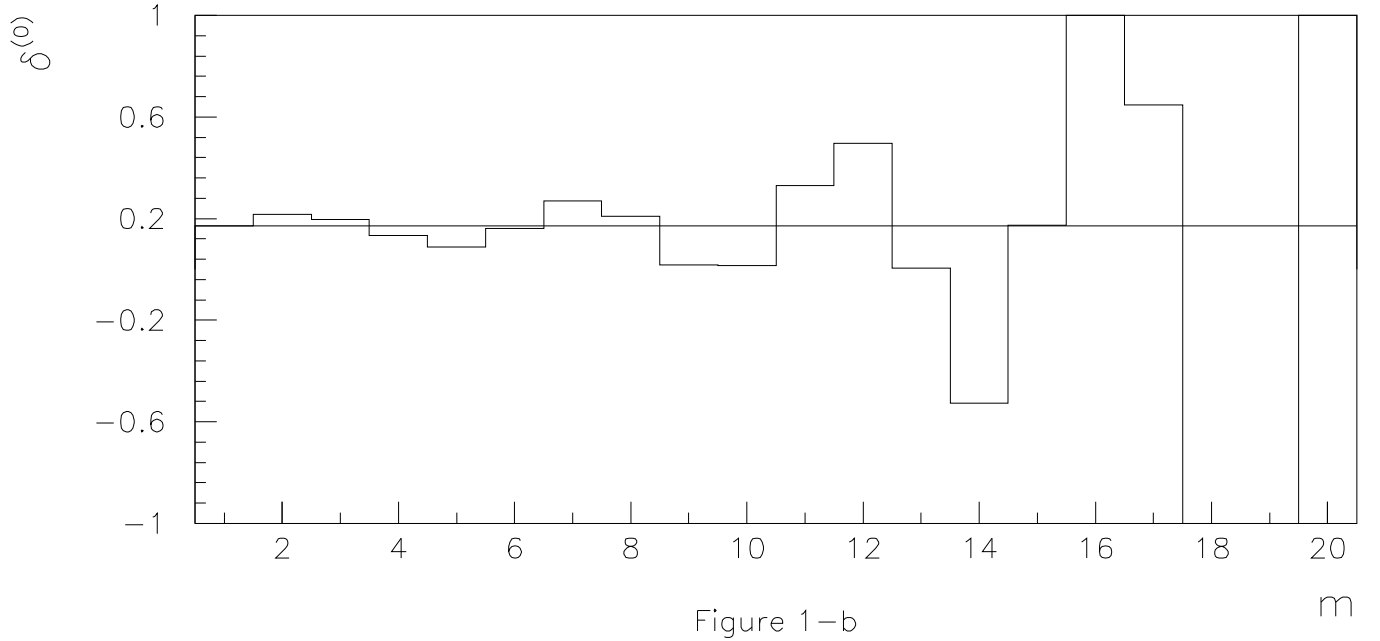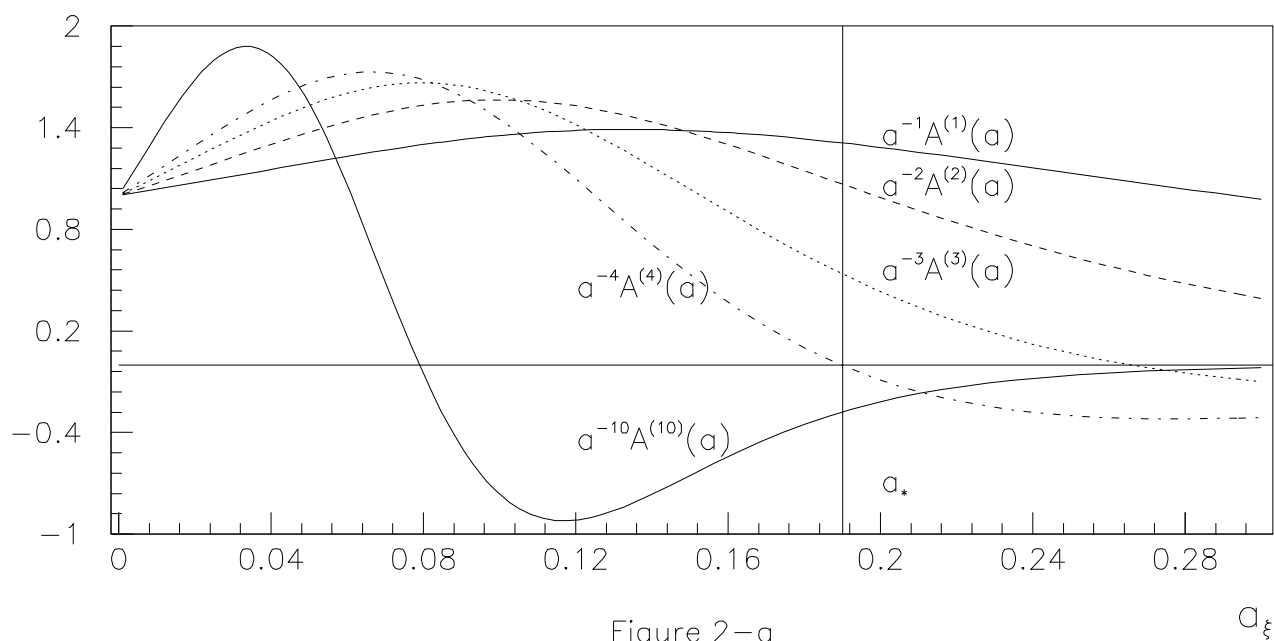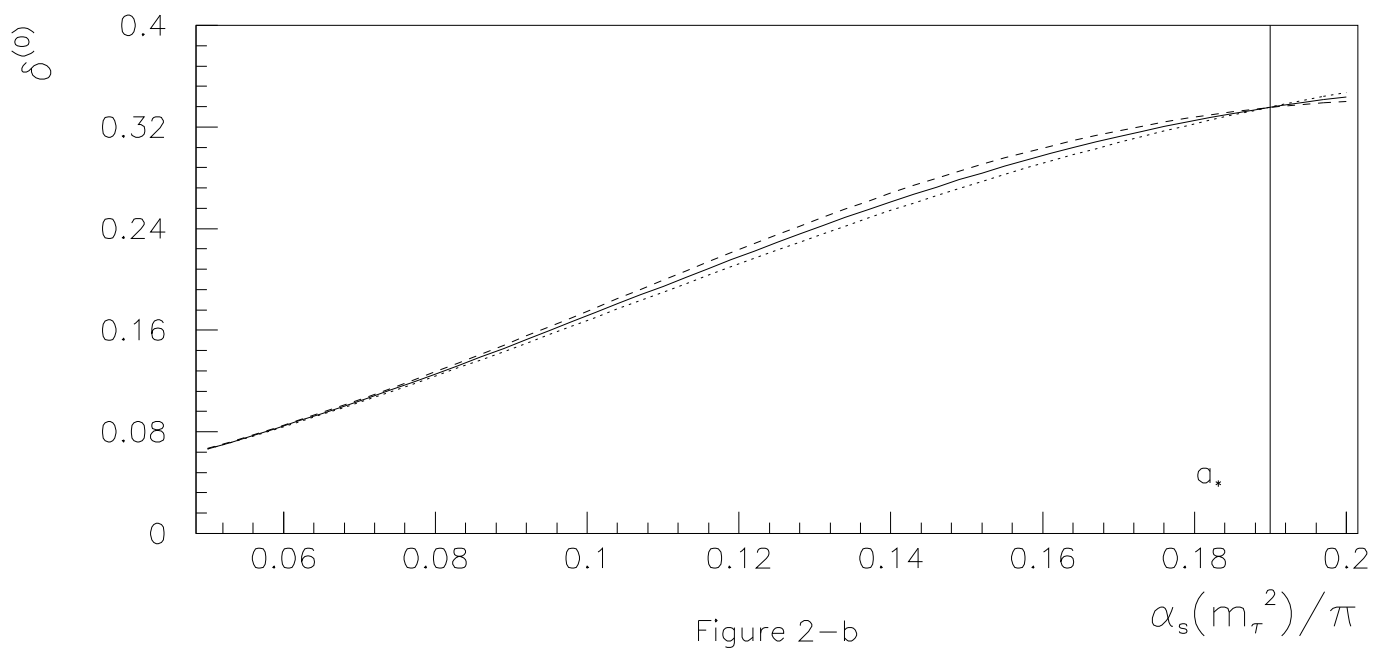
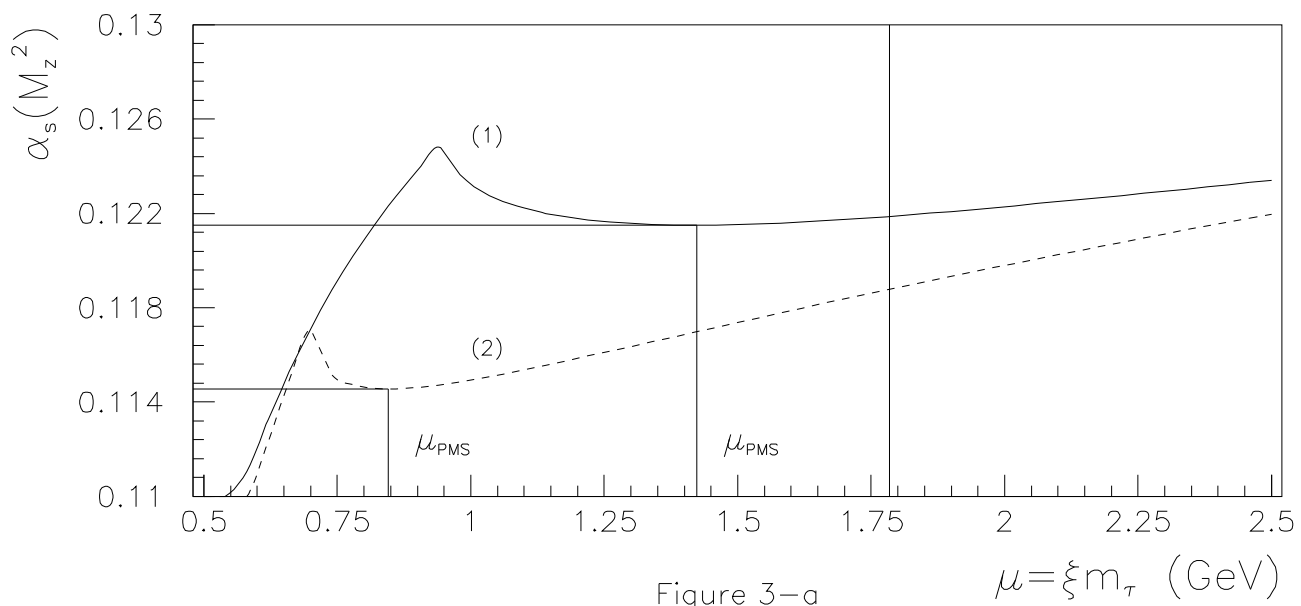Figure 1−a

Figure 1−b

Figure 2−a

Figure 2−b

Figure 3-a

Figure 3-b

Figure 4-a

Figure 4-b