CERN-TH.6978/93

INTRODUCTION TO CHIRAL PERTURBATION THEORY

A. Pich*†
Theory Division, CERN, CH-1211 Geneva 23

Abstract

An introduction to the basic ideas and methods of Chiral Perturbation Theory is presented. Several phenomenological applications of the effective Lagrangian technique to strong, electromagnetic and weak interactions are discussed.

Lectures given at the
V Mexican School of Particles and Fields
Guanajuato, México, December 1992

* On leave of absence from Departament de Física Teòrica, Universitat de València, and IFIC, Centre Mixte Universitat de València–CSIC, E-46100 Burjassot, València, Spain.
† Work supported in part by CICYT (Spain), under grant No. AEN90-0040.

CERN-TH.6978/93
August 1993

INTRODUCTION TO CHIRAL PERTURBATION THEORY

A. Pich*†

Theory Division, CERN, CH-1211 Geneva 23

Abstract

An introduction to the basic ideas and methods of Chiral Perturbation Theory is presented. Several phenomenological applications of the effective Lagrangian technique to strong, electromagnetic and weak interactions are discussed.

1 Effective Field Theories

Effective Field Theories (EFTs) are the appropriate theoretical tool to describe "low-energy" physics, where "low" is defined with respect to some energy scale Λ. What that means is that they only take explicitly into account the relevant degrees of freedom, i.e. those states with $m << \Lambda$, while the heavier excitations with $M >> \Lambda$ are integrated out from the action. One gets in this way a string of non-renormalizable interactions among the light states, which can be organized as an expansion in powers of energy/Λ. The information on the heavier degrees of freedom is then contained in the couplings of the resulting low-energy Lagrangian. Although EFTs contain an infinite number of terms, renormalizability is not an issue since, at a given order in the energy expansion, the low-energy theory is specified by a finite number of couplings; this allows for an order-by-order renormalization. Obviously, for this procedure to make sense, it is necessary that the spectrum of the fundamental theory contains a mass gap, separating the light and heavy states.

A simple example of EFT is provided by QED at very low energies, $\omega << m_e$, where ω denotes the photon energy. In this limit, one can describe the light-by-light scattering using an effective Lagrangian in terms of the electromagnetic field

*On leave of absence from Departament de Física Teòrica, Universitat de València, and IFIC, Centre Mixte Universitat de València–CSIC, E-46100 Burjassot, València, Spain.

†Work supported in part by CICYT (Spain), under grant No. AEN90-0040.

only. Gauge and Lorentz invariance constrain the possible structures present in the effective Lagrangian:

$$\mathcal{L}_{\rm eff} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{a}{m_e^4}\, (F^{\mu\nu} F_{\mu\nu})^2 + \frac{b}{m_e^4}\, F^{\mu\nu} F_{\nu\sigma} F^{\sigma\rho} F_{\rho\mu} + O(F^6/m_e^8). \qquad (1)$$

In the low-energy regime, all the information on the original QED dynamics is embodied in the values of the two low-energy couplings a and b. The values of these constants can be computed, by explicitly integrating out the electron field from the original QED generating functional (or equivalently, by computing the relevant light-by-light box diagrams). One then gets the well-known Euler-Heisenberg result [1]:

$$a = -\frac{\alpha^2}{36}, \qquad\qquad b = \frac{7\alpha^2}{90}. \qquad (2)$$

The important point to realize is that, even in the absence of an explicit computation of the couplings a and b, the Lagrangian (1) contains non-trivial information, which is a consequence of the imposed symmetries. The dominant contributions to the amplitudes for different low-energy photon reactions like $\gamma\gamma \to 2\gamma, 4\gamma, \ldots$ can be directly obtained from $\mathcal{L}_{\rm eff}$. Moreover, the order of magnitude of the constants a, b can also be easily estimated through a naïve counting of powers of the electromagnetic coupling and combinatorial and loop $[1/(16\pi^2)]$ factors.

The previous example is somehow academic, since perturbation theory in powers of α works extremely well in QED. However, the effective Lagrangian (1) would be valid even if the fine structure constant were big; the only difference would then be that we would not be able to perturbatively compute the couplings a and b.

We can mention two generic situations where EFTs become particularly useful:

- The underlying fundamental theory is unknown, but the symmetry properties of the light states can be used to build an effective Lagrangian. The low-energy couplings then parametrize the unknown new physics. A typical example are EFTs at the electroweak scale.

- Even if the underlying fundamental theory is known, sometimes it is not directly applicable in the low-energy region. For instance, due to confinement, the quark and gluon fields of QCD are not asymptotic states. Since we do not know how to solve QCD, we cannot derive the hadronic interactions directly from the original QCD Lagrangian. However, we do know the symmetry properties of the strong interactions; therefore, we can write an EFT in terms of the hadronic asymptotic states, and parametrize the unknown dynamical information in a few couplings.

The theoretical basis of EFTs can be formulated [2] as a "theorem"[1]: for a given set of asymptotic states, perturbation theory with the most general Lagrangian containing all terms allowed by the assumed symmetries will yield the

[1]Although this "theorem" is almost self-evident, it is only "proven" to the extent that no counter-examples are known.

most general S-matrix elements consistent with analyticity, perturbative unitarity and the assumed symmetries.

The purpose of these lectures is to give a pedagogical introduction to Chiral Perturbation Theory (ChPT), the low-energy effective field theory of the Standard Model. The chiral symmetry of the QCD Lagrangian is discussed in Section 2, and a toy low-energy model incorporating the right symmetry properties is studied in Section 3. The ChPT formalism is presented in Sections 4 and 5, where the lowest-order and next-to-leading-order terms in the chiral expansion are analysed. Section 6 contains a few selected phenomenological applications. The relation between the effective Lagrangian and the underlying fundamental QCD theory is studied in Section 7, which summarizes recent attempts to calculate the chiral couplings. The effective realization of the non-leptonic $\Delta S = 1$ interactions is described in Section 8, and a brief overview of the application of the chiral techniques to K decays is given in Sections 9, 10 and 11. Section 12 shows how ChPT can be used to work out the low-energy interactions of a possible light Higgs boson. Finally, Section 13 illustrates the use of the chiral techniques to describe the Goldstone dynamics associated with the Standard Model electroweak symmetry breaking. A few summarizing comments are collected in Section 14.

To prepare these lectures I have made extensive use of excellent reviews [3–10] and books [11–14] already existing in the literature. In many cases, I have sacrificed some rigour to simplify the presentation of the subject. A more careful discussion and further details can be found in those references.

2 Chiral Symmetry

In the absence of quark masses, the QCD Lagrangian [$q = \text{column}(u, d, \ldots)$]

$$\mathcal{L}^0_{QCD} = -\frac{1}{4}\text{Tr}(G_{\mu\nu}G^{\mu\nu}) + i\bar{q}_L\gamma^\mu D_\mu q_L + i\bar{q}_R\gamma^\mu D_\mu q_R \tag{3}$$

is invariant under independent global $G \equiv SU(N_f)_L \otimes SU(N_f)_R$ transformations[2] of the left- and right-handed quarks in flavour space:

$$q_L \xrightarrow{G} g_L\, q_L, \qquad q_R \xrightarrow{G} g_R\, q_R, \qquad g_{L,R} \in SU(N_f)_{L,R}. \tag{4}$$

The Noether currents associated with the chiral group G are [λ_a are Gell-Mann's matrices with $\text{Tr}(\lambda_a\lambda_b) = 2\delta_{ab}$]:

$$J^{a\mu}_X = \bar{q}_X\gamma^\mu \frac{\lambda_a}{2} q_X, \qquad (X = L, R; \quad a = 1, \ldots, 8). \tag{5}$$

[2] Actually, the Lagrangian (3) has a larger $U(N_f)_L \otimes U(N_f)_R$ global symmetry. However, the $U(1)_A$ part is broken by quantum effects [$U(1)_A$ anomaly], while the quark-number symmetry $U(1)_V$ is trivially realized in the meson sector. A discussion of the $U(1)_A$ part, within ChPT, is given in refs. [15, 16].

The corresponding Noether charges $Q^a_X = \int d^3x J^{a0}_X(x)$ satisfy the familiar commutation relations

$$[Q^a_X, Q^b_Y] = i\delta_{XY} f_{abc} Q^c_X, \tag{6}$$

which were the starting point of the Current Algebra methods of the sixties [17].

This chiral symmetry, which should be approximately good in the light quark sector (u,d,s), is however not seen in the hadronic spectrum. Although hadrons can be nicely classified in $SU(3)_V$ representations, degenerate multiplets with opposite parity do not exist. Moreover, the octet of pseudoscalar mesons happens to be much lighter than all the other hadronic states. To be consistent with this experimental fact, the ground state of the theory (the vacuum) should not be symmetric under the chiral group. The $SU(3)_L \otimes SU(3)_R$ symmetry spontaneously breaks down to $SU(3)_{L+R}$ and, according to Goldstone's theorem [18], an octet of pseudoscalar massless bosons appears in the theory.

More specifically, let us consider a Noether charge Q, and assume the existence of an operator O that satisfies

$$\langle 0|[Q, O]|0\rangle \neq 0; \tag{7}$$

this is clearly only possible if $Q|0\rangle \neq 0$. Goldstone's theorem then tells us that there exists a massless state $|G\rangle$ such that

$$\langle 0|J^0|G\rangle \, \langle G|O|0\rangle \neq 0. \tag{8}$$

The quantum numbers of the Goldstone boson are dictated by those of J^0 and O. The quantity in the left-hand side of Eq. (7) is called the order parameter of the spontaneous symmetry breakdown.

Since there are eight broken axial generators of the chiral group, $Q^a_A = Q^a_R - Q^a_L$, there should be eight pseudoscalar Goldstone states $|G^a\rangle$, which we can identify with the eight lightest hadronic states (π^+, π^-, π^0, η, K^+, K^-, K^0 and $\bar{K}^0$); their small masses being generated by the quark-mass matrix, which explicitly breaks the global symmetry of the QCD Lagrangian. The corresponding O^a must be pseudoscalar operators. The simplest possibility are $O^a = \bar{q}\gamma_5\lambda_a q$, which satisfy

$$\langle 0|[Q^a_A, \bar{q}\gamma_5\lambda_b q]|0\rangle = -\frac{1}{2}\,\langle 0|\bar{q}\{\lambda_a, \lambda_b\}q|0\rangle = -\frac{2}{3}\,\delta_{ab}\,\langle 0|\bar{q}q|0\rangle. \tag{9}$$

The quark condensate

$$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle \neq 0 \tag{10}$$

is then the natural-order parameter of Spontaneous Chiral Symmetry Breaking (SCSB).

The Goldstone nature of the pseudoscalar mesons implies strong constraints on their interactions, which can be most easily analysed on the basis of an effective Lagrangian.

3 A toy Lagrangian: the linear sigma model

The linear sigma model [19]

$$\begin{aligned} \mathcal{L}_\sigma &= \frac{1}{2}\left[\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec{\pi}\partial^\mu\vec{\pi}\right] - V(\sigma,\vec{\pi}), \\ V(\sigma,\vec{\pi}) &= \frac{\lambda}{4}\left(\sigma^2+\vec{\pi}^2-v^2\right)^2, \qquad (\lambda>0), \end{aligned} \tag{11}$$

provides a very simple example of SCSB. If $v^2<0$, the global symmetry $O(4)\sim SU(2)\otimes SU(2)$ is realized in the usual Wigner–Weyl way; one then has degenerate $\sigma,\vec{\pi}$ states with mass $m^2=-\lambda v^2$. However, for $v^2>0$, the potential $V(\sigma,\vec{\pi})$ has a family of minima occurring for all $\sigma,\vec{\pi}$ with $\sigma^2+\vec{\pi}^2=v^2$; these minima correspond to degenerate ground states, which transform into each other under chiral rotations. The symmetry is then realized à la Nambu–Goldstone, and three massless states appear, corresponding to the flat directions of $V(\sigma,\vec{\pi})$. Taking

$$\langle 0|\sigma|0\rangle = v, \qquad \langle 0|\vec{\pi}|0\rangle = 0, \tag{12}$$

and making the field redefinition $\hat{\sigma}=\sigma-v$, the Lagrangian takes the form

$$\mathcal{L}_\sigma = \frac{1}{2}\left[\partial_\mu\hat{\sigma}\partial^\mu\hat{\sigma} - 2\lambda v^2\hat{\sigma}^2 + \partial_\mu\vec{\pi}\partial^\mu\vec{\pi}\right] - \lambda v\hat{\sigma}\left(\hat{\sigma}^2+\vec{\pi}^2\right) - \frac{\lambda}{4}\left(\hat{\sigma}^2+\vec{\pi}^2\right)^2, \tag{13}$$

which shows that $\vec{\pi}$ corresponds to the three massless Goldstone modes, while the $\hat{\sigma}$ field acquires a mass $m_{\hat{\sigma}}^2=2\lambda v^2$.

To clarify the role of chiral symmetry on the Goldstone dynamics, it is useful to rewrite the sigma-model Lagrangian in a different way. Using the 2×2 matrix notation

$$\Sigma(x) \equiv \sigma(x) I + i\vec{\tau}\vec{\pi}, \tag{14}$$

the Lagrangian (11) takes the compact form

$$\mathcal{L}_\sigma = \frac{1}{4}\langle\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\rangle - \frac{\lambda}{16}\left(\langle\Sigma^\dagger\Sigma\rangle - 2v^2\right)^2, \tag{15}$$

where $\langle A\rangle$ denotes the trace of the matrix A. In this notation the Lagrangian is explicitly invariant under global chiral $G\equiv SU(2)_L\otimes SU(2)_R$ transformations:

$$\Sigma \xrightarrow{G} g_R\,\Sigma\,g_L^\dagger, \qquad g_{L,R}\in SU(2)_{L,R}. \tag{16}$$

We can now make the polar decomposition [5]

$$\begin{aligned} \Sigma(x) &= (v+S(x))\;U(\phi(x)), \\ U(\phi(x)) &= \exp\left(i\vec{\tau}\vec{\phi}(x)/v\right), \end{aligned} \tag{17}$$

in terms of a Hermitian scalar field S and pseudoscalar fields $\vec{\phi}$. These fields transform in a non-linear way under the chiral group:

$$S \xrightarrow{G} S, \qquad U(\phi) \xrightarrow{G} g_R\,U(\phi)\,g_L^\dagger. \tag{18}$$

The sigma-model Lagrangian then takes the form

$$\mathcal{L}_\sigma = \frac{v^2}{4}\left(1+\frac{S}{v}\right)^2 \langle \partial_\mu U^\dagger \partial^\mu U\rangle + \frac{1}{2}\left(\partial_\mu S \partial^\mu S - 2\lambda v^2 S^2\right) - \lambda v S^3 - \frac{\lambda}{4} S^4. \quad (19)$$

Equation (19) is very instructive:

- It shows explicitly that the Goldstone bosons have purely derivative couplings, as they should. This was not so obvious in Eq. (13). Of course one should get the same measurable amplitudes from both Lagrangians, which means that the original Lagrangian (13) gives rise to exact (and not very transparent) cancellations among different momentum-independent contributions.

- In the limit $\lambda >> 1$, the scalar field S becomes very heavy and can be integrated out from the Lagrangian. The linear sigma model then reduces to the familiar Lagrangian

$$\mathcal{L}_2 = \frac{v^2}{4}\langle \partial_\mu U^\dagger \partial^\mu U\rangle. \quad (20)$$

  As we will see in the next section, this is a universal model-independent interaction of the Goldstone bosons induced by SCSB. It is often claimed in the literature that the linear sigma model correctly describes the low-energy strong interactions. This is, however, quite a misleading statement. To the extent that one is only looking at the predictions coming from the model-independent lowest-order term (20), the comparison with experiment only tests the assumed pattern of SCSB.

- In order to be sensitive to the particular structure of the linear sigma model, one needs to test the model-dependent part involving the scalar field S. At low momenta ($p << M_S$), the dominant tree-level corrections originate from the exchange of an S particle, which generates the four-derivative term

$$\mathcal{L}_\sigma^4 = \frac{v^2}{8M_S^2}\langle \partial_\mu U^\dagger \partial^\mu U\rangle^2. \quad (21)$$

  It will be shown later that this kind of interaction does not agree with the experimental data. Therefore, the linear sigma model is not a phenomenologically viable EFT of QCD [20].

4 Effective Chiral Lagrangian at lowest order

We want to get an effective Lagrangian realization of QCD, at low energies, for the light-quark sector (u, d, s). Our basic assumption is the pattern of SCSB:

$$G \equiv SU(3)_L \otimes SU(3)_R \stackrel{SCSB}{\longrightarrow} H \equiv SU(3)_V. \quad (22)$$

The present understanding of the mechanism of SCSB is based in the dynamical generation of a non-zero vacuum expectation value of the scalar quark density, i.e. the vacuum condensate $v \equiv \langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle \neq 0$. The Goldstone bosons correspond to the zero-energy excitations over this vacuum condensate; their fields can be collected in a 3×3 unitary matrix $U(\phi)$,

$$\langle 0|\bar{q}_L^j q_R^i|0\rangle \;\longrightarrow\; \frac{v}{2}\, U^{ij}(\phi), \tag{23}$$

which parametrizes those excitations. A convenient parametrization is given by

$$U(\phi) \equiv \exp\left(i\sqrt{2}\Phi/f\right), \tag{24}$$

where

$$\Phi(x) \equiv \frac{\vec{\lambda}}{\sqrt{2}}\,\vec{\phi} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\,\eta_8}{\sqrt{6}} \end{pmatrix}. \tag{25}$$

The matrix $U(\phi)$ transforms linearly under the chiral group,

$$U(\phi) \xrightarrow{G} g_R\, U(\phi)\, g_L^\dagger, \tag{26}$$

but the induced transformation on the Goldstone fields $\vec{\phi}$ is highly non-linear.

Since there is a mass gap separating the pseudoscalar octet from the rest of the hadronic spectrum, we can build an EFT containing only the Goldstone modes. We should write the more general Lagrangian involving the matrix $U(\phi)$, which is consistent with chiral symmetry. Moreover, we can organize the Lagrangian in terms of increasing powers of momentum or, equivalently, in terms of an increasing number of derivatives (parity conservation requires an even number of derivatives):

$$\mathcal{L}_{\mathrm{eff}}(U) \;=\; \sum_n \mathcal{L}_{2n}. \tag{27}$$

In the low-energy domain we are interested in, the terms with a minimum number of derivatives will dominate.

Due to the unitarity of the U matrix, $UU^\dagger = I$, at least two derivatives are required to generate a non-trivial interaction. To lowest order, the effective chiral Lagrangian is uniquely given by the term

$$\mathcal{L}_2 = \frac{f^2}{4}\langle \partial_\mu U^\dagger \partial^\mu U\rangle. \tag{28}$$

This is exactly the structure (20), which we derived from the linear sigma model in the last section.

Expanding $U(\phi)$ in a power series in ϕ, one obtains the Goldstone's kinetic terms plus a tower of interactions involving an increasing number of pseudoscalars.

The requirement that the kinetic terms are properly normalized fixes the global coefficient $f^2/4$ in Eq. (28). All interactions among the Goldstones can then be predicted in terms of the single coupling f:

$$\mathcal{L}_2 = \frac{1}{2}\langle \partial_\mu\phi\partial^\mu\phi\rangle + \frac{1}{12f^2}\langle(\phi \overleftrightarrow{\partial}_\mu \phi)(\phi \overleftrightarrow{\partial}^\mu \phi)\rangle + O(\phi^6/f^4). \tag{29}$$

To compute the $\pi\pi$ scattering amplitude, for instance, is now a trivial perturbative exercise. One gets the well-known [21] Weinberg result $[t \equiv (p'_+ - p_+)^2]$

$$T(\pi^+\pi^0 \to \pi^+\pi^0) = \frac{t}{f^2}. \tag{30}$$

Similar results can be obtained for $\pi\pi \to 4\pi, 6\pi, 8\pi, \ldots$ It is the non-linearity of the effective Lagrangian that relates amplitudes with different numbers of Goldstone bosons, allowing for absolute predictions in terms of f.

The EFT technique becomes much more powerful if one introduces couplings to external classical fields. Let us consider an extended QCD Lagrangian, with quark couplings to external Hermitian matrix-valued fields v_μ, a_μ, s, p:

$$\mathcal{L}_{QCD} = \mathcal{L}^0_{QCD} + \bar{q}\gamma^\mu(v_\mu + \gamma_5 a_\mu)q - \bar{q}(s - i\gamma_5 p)q. \tag{31}$$

The external fields will allow us to compute the effective realization of general Green functions of quark currents in a very straightforward way. Moreover, they can be used to incorporate the electromagnetic and semileptonic weak interactions, and the explicit breaking of chiral symmetry through the quark masses:

$$\begin{aligned}
r_\mu &\equiv v_\mu + a_\mu = eQA_\mu + \ldots \\
\ell_\mu &\equiv v_\mu - a_\mu = eQA_\mu + \frac{e}{\sqrt{2}\sin\theta_W}(W^\dagger_\mu T_+ + \text{h.c.}) + \ldots \\
s &= \mathcal{M} + \ldots
\end{aligned} \tag{32}$$

Here, Q and $\mathcal{M}$ denote the quark-charge and quark-mass matrices, respectively,

$$Q = \frac{1}{3}\,\text{diag}(2, -1, -1), \qquad\qquad \mathcal{M} = \text{diag}(m_u, m_d, m_s), \tag{33}$$

and T_+ is a 3×3 matrix containing the relevant Cabibbo–Kobayashi–Maskawa factors

$$T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

Formally, the Lagrangian (31) is invariant under the following set of local $SU(3)_L \otimes SU(3)_R$ transformations:

$$\begin{aligned}
q_L &\longrightarrow g_L\, q_L, \\
q_R &\longrightarrow g_R\, q_R, \\
\ell_\mu &\longrightarrow g_L\, \ell_\mu\, g_L^\dagger + i g_L \partial_\mu g_L^\dagger, \\
r_\mu &\longrightarrow g_R\, r_\mu\, g_R^\dagger + i g_R \partial_\mu g_R^\dagger, \\
s + ip &\longrightarrow g_R\,(s + ip)\, g_L^\dagger.
\end{aligned} \tag{35}$$

We can use this formal symmetry to build a generalized effective Lagrangian for the Goldstone bosons, in the presence of external sources. Note that to respect the local invariance, the gauge fields v_μ, a_μ can only appear through the covariant derivative

$$D_\mu U = \partial_\mu U - ir_\mu U + iU\ell_\mu, \qquad D_\mu U^\dagger = \partial_\mu U^\dagger + iU^\dagger r_\mu - i\ell_\mu U^\dagger, \tag{36}$$

and through the field strength tensors

$$F_L^{\mu\nu} = \partial^\mu \ell^\nu - \partial^\nu \ell^\mu - i[\ell^\mu, \ell^\nu], \qquad F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu]. \tag{37}$$

At lowest order in momenta, the more general effective Lagrangian consistent with Lorentz invariance and with (local) chiral symmetry is of the form [15]

$$\mathcal{L}_2 = \frac{f^2}{4}\langle D_\mu U^\dagger D^\mu U \,+\, U^\dagger\chi \,+\, \chi^\dagger U\rangle, \tag{38}$$

where

$$\chi \,=\, 2B_0\,(s+ip), \tag{39}$$

and B_0 is a constant, which, like f, is not fixed by symmetry requirements alone.

Once special directions in flavour space, like the ones in Eq. (32), are selected for the external fields, chiral symmetry is of course explicitly broken. The important point is that (38) then breaks the symmetry in exactly the same way as the fundamental short-distance Lagrangian (31) does.

The power of the external field technique becomes obvious when computing the chiral Noether currents. Formally, the physical Green functions are obtained as functional derivatives of the generating functional $Z[v,a,s,p]$, defined via the path-integral formula

$$\exp\{iZ\} \,=\, \int \mathcal{D}q\,\mathcal{D}\bar{q}\,\mathcal{D}G_\mu\,\exp\left\{i\int d^4x\,\mathcal{L}_{QCD}\right\} \,=\, \int \mathcal{D}U(\phi)\,\exp\left\{i\int d^4x\,\mathcal{L}_{\rm eff}\right\}. \tag{40}$$

At lowest order in momenta, the generating functional reduces to the classical action $S_2 = \int d^4x\,\mathcal{L}_2$; therefore, the currents can be trivially computed by taking the appropriate derivatives with respect to the external fields:

$$\begin{aligned} J_L^\mu \doteq \frac{\delta S_2}{\delta \ell_\mu} \;&=\; \frac{i}{2}f^2 D_\mu U^\dagger U = \;\; \frac{f}{\sqrt{2}}D_\mu\phi - \frac{i}{2}\left(\phi \overleftrightarrow{D}^\mu \phi\right) + O(\phi^3/f), \\ J_R^\mu \doteq \frac{\delta S_2}{\delta r_\mu} \;&=\; \frac{i}{2}f^2 D_\mu U U^\dagger = -\frac{f}{\sqrt{2}}D_\mu\phi - \frac{i}{2}\left(\phi \overleftrightarrow{D}^\mu \phi\right) + O(\phi^3/f). \end{aligned} \tag{41}$$

The physical meaning of the chiral coupling f is now obvious; at $O(p^2)$, f equals the pion decay constant, $f = f_\pi = 93.2$ MeV, defined as

$$\langle 0|(J_A^\mu)^{12}|\pi^+\rangle \equiv i\sqrt{2}f_\pi p^\mu. \tag{42}$$

Similarly, by taking a derivative with respect to the external scalar source s, we learn that the constant B_0 is related to the quark condensate

$$\langle 0|\bar{q}^j q^i|0\rangle = -f^2 B_0 \delta^{ij}. \tag{43}$$

Taking $s = \mathcal{M}$ and $p = 0$, the χ term in Eq. (38) gives rise to a quadratic pseudoscalar-mass term plus additional interactions proportional to the quark masses. Expanding in powers of ϕ (and dropping an irrelevant constant), one has

$$\frac{f^2}{4} 2B_0 \langle \mathcal{M}(U + U^\dagger)\rangle = B_0 \left\{ -\langle \mathcal{M}\phi^2\rangle + \frac{1}{6f^2}\langle \mathcal{M}\phi^4\rangle + O(\phi^6/f^4) \right\}. \tag{44}$$

The explicit evaluation of the trace in the quadratic mass term provides the relation between the physical meson masses and the quark masses:

$$\begin{aligned} M^2_{\pi^\pm} &= 2\hat{m}B_0, \\ M^2_{\pi^0} &= 2\hat{m}B_0 - \varepsilon + O(\varepsilon^2), \\ M^2_{K^\pm} &= (m_u + m_s)B_0, \\ M^2_{K^0} &= (m_d + m_s)B_0, \\ M^2_{\eta_8} &= \frac{2}{3}(\hat{m} + 2m_s)B_0 + \varepsilon + O(\varepsilon^2), \end{aligned} \tag{45}$$

where[3]

$$\hat{m} = \frac{1}{2}(m_u + m_d), \qquad \varepsilon = \frac{B_0}{4}\frac{(m_u - m_d)^2}{(m_s - \hat{m})}. \tag{46}$$

Chiral symmetry relates the magnitude of the meson and quark masses to the size of the quark condensate. Using the result (43), one gets from the first equation in (45) the well-known Gell-Mann–Oakes–Renner relation [22]

$$f_\pi^2 M_\pi^2 = -\hat{m}\,\langle 0|\bar{u}u + \bar{d}d|0\rangle. \tag{47}$$

Taking out the common B_0 factor, Eqs. (45) imply the old Current Algebra mass ratios [22, 23],

$$\frac{M^2_{\pi^\pm}}{2\hat{m}} = \frac{M^2_{K^+}}{(m_u + m_s)} = \frac{M_{K^0}}{(m_d + m_s)} \approx \frac{3M^2_{\eta_8}}{(2\hat{m} + 4m_s)}, \tag{48}$$

and (up to $O(m_u - m_d)$ corrections) the Gell-Mann–Okubo mass relation [24],

$$3M^2_{\eta_8} = 4M^2_K - M^2_\pi. \tag{49}$$

---

[3]The $O(\varepsilon)$ corrections to $M^2_{\pi^0}$ and $M^2_{\eta_8}$ originate from a small mixing term between the π^0 and η_8 fields,

$$-B_0\langle \mathcal{M}\phi^2\rangle \longrightarrow -(B_0/\sqrt{3})\,(m_u - m_d)\,\pi^0\eta_8.$$

The diagonalization of the quadratic π^0, η_8 mass matrix, gives the mass eigenstates, $\pi^0 = \cos\delta\,\phi^3 + \sin\delta\,\phi^8$ and $\eta_8 = -\sin\delta\,\phi^3 + \cos\delta\,\phi^8$, where $\tan(2\delta) = \sqrt{3}(m_d - m_u)/\,(2(m_s - \hat{m}))$.

Note that the chiral Lagrangian automatically implies the successful quadratic Gell-Mann–Okubo mass relation, and not a linear one. Since $B_0 m_q \propto M_\phi^2$, the external field χ is counted as $O(p^2)$ in the chiral expansion.

Although chiral symmetry alone cannot fix the absolute values of the quark masses, it gives information about quark-mass ratios. Neglecting the tiny $O(\varepsilon)$ effects, one gets the relations

$$\frac{m_d - m_u}{m_d + m_u} = \frac{(M_{K^0}^2 - M_{K^+}^2) - (M_{\pi^0}^2 - M_{\pi^+}^2)}{M_{\pi^0}^2} = 0.29, \qquad (50)$$

$$\frac{m_s - \hat{m}}{2\hat{m}} = \frac{M_{K^0}^2 - M_{\pi^0}^2}{M_{\pi^0}^2} = 12.6. \qquad (51)$$

In Eq. (50) we have subtracted the pion square-mass difference, to take into account the electromagnetic contribution to the pseudoscalar-meson self-energies; in the chiral limit ($m_u = m_d = m_s = 0$), this contribution is proportional to the square of the meson charge and it is the same for K^+ and π^+ [25]. The mass formulae (50) and (51) imply the quark ratios advocated by Weinberg [23]:

$$m_u : m_d : m_s = 0.55 : 1 : 20.3. \qquad (52)$$

Quark-mass corrections are therefore dominated by m_s, which is large compared with m_u, m_d. Notice that the difference $m_d - m_u$ is not small compared with the individual up- and down-quark masses; in spite of that, isospin turns out to be an extremely good symmetry, because isospin-breaking effects are governed by the small ratio $(m_d - m_u)/m_s$.

The ϕ^4 interactions in Eq. (44) introduce mass corrections to the $\pi\pi$ scattering amplitude (30),

$$T(\pi^+\pi^0 \to \pi^+\pi^0) = \frac{t - M_\pi^2}{f_\pi^2}, \qquad (53)$$

in perfect agreement with the Current Algebra result [21]. Since $f \approx f_\pi$ is fixed from pion decay, this result is now an absolute prediction of chiral symmetry!

The lowest-order chiral Lagrangian (38) encodes in a very compact way all the Current Algebra results obtained in the sixties [17]. The nice feature of the chiral approach is its elegant simplicity. Moreover, as we will see in the next section, the EFT method allows us to estimate higher-order corrections in a systematic way.

5 ChPT at $O(p^4)$

At next-to-leading order in momenta, $O(p^4)$, the computation of the generating functional $Z[v, a, s, p]$ involves three different ingredients:

1. The most general effective chiral Lagrangian of $O(p^4)$, $\mathcal{L}_4$, to be considered at tree level.

2. One-loop graphs associated with the lowest-order Lagrangian $\mathcal{L}_2$.
3. The Wess–Zumino–Witten functional [26] to account for the chiral anomaly [27, 28].

5.1 $O(p^4)$ Lagrangian

At $O(p^4)$, the most general[4] Lagrangian, invariant under parity, charge conjugation and the local chiral transformations (35), is given by [15]

$$\begin{aligned}\mathcal{L}_4 \; = \; & L_1 \langle D_\mu U^\dagger D^\mu U\rangle^2 \, + \, L_2 \langle D_\mu U^\dagger D_\nu U\rangle \langle D^\mu U^\dagger D^\nu U\rangle \\ & + L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle \, + \, L_4 \langle D_\mu U^\dagger D^\mu U\rangle \langle U^\dagger \chi + \chi^\dagger U\rangle \\ & + L_5 \langle D_\mu U^\dagger D^\mu U \left(U^\dagger \chi + \chi^\dagger U\right)\rangle \, + \, L_6 \langle U^\dagger \chi + \chi^\dagger U\rangle^2 \\ & + L_7 \langle U^\dagger \chi - \chi^\dagger U\rangle^2 \, + \, L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle \\ & - iL_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \, + \, L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle \\ & + H_1 \langle F_{R\mu\nu} F_R^{\mu\nu} + F_{L\mu\nu} F_L^{\mu\nu}\rangle \, + \, H_2 \langle \chi^\dagger \chi\rangle . \end{aligned} \qquad (54)$$

The terms proportional to H_1 and H_2 do not contain the pseudoscalar fields and are therefore not directly measurable. Thus, at $O(p^4)$ we need ten additional coupling constants L_i to determine the low-energy behaviour of the Green functions. These constants parametrize our ignorance about the details of the underlying QCD dynamics. In principle, all the chiral couplings are calculable functions of Λ_{QCD} and the heavy-quark masses. At the present time, however, our main source of information about these couplings is low-energy phenomenology.

5.2 Chiral loops

ChPT is a quantum field theory, perfectly defined through Eq. (40). As such, we must take into account quantum loops with Goldstone-boson propagators in the internal lines. The chiral loops generate non-polynomial contributions, with logarithms and threshold factors, as required by unitarity.

The loop integrals are homogeneous functions of the external momenta and the pseudoscalar masses occurring in the propagators. A simple dimensional counting shows that, for a general connected diagram with N_d vertices of $O(p^d)$ ($d = 2, 4, \ldots$), L loops and I internal lines, the overall chiral dimension is given by [2]

$$D = 2L + 2 + \sum_d N_d (d-2). \qquad (55)$$

Each loop adds two powers of momenta; this power suppression of loop diagrams is at the basis of low-energy expansions, such as ChPT. The leading $D = 2$

[4] Since we will only need $\mathcal{L}_4$ at tree level, the general expression of this Lagrangian has been simplified, using the $O(p^2)$ equations of motion obeyed by U. Moreover, a 3×3 matrix relation has been used to reduce the number of independent terms. For the two-flavour case, not all of these terms are independent [15, 20].

contributions are obtained with $L = 0$ and $d = 2$, i.e. only tree-level graphs with $\mathcal{L}_2$ insertions. At $O(p^4)$, we have tree-level contributions from $\mathcal{L}_4$ ($L = 0$, $d = 4$, $N_4 = 1$) and one-loop graphs with the lowest-order Lagrangian $\mathcal{L}_2$ ($L = 1$, $d = 2$).

ChPT is an expansion in powers of momenta over some typical hadronic scale, usually called the scale of chiral symmetry breaking Λ_χ. Since each chiral loop generates a geometrical factor $(4\pi)^{-2}$, plus a factor of $1/f^2$ to compensate the additional dimensions, one could expect [29] Λ_χ to be about $4\pi f_\pi \sim 1.2\,\mathrm{GeV}$.

The Goldstone loops are divergent and need to be renormalized. Although EFTs are non-renormalizable (i.e. an infinite number of counter-terms is required), order by order in the momentum expansion they define a perfectly renormalizable theory. If we use a regularization which preserves the symmetries of the Lagrangian, such as dimensional regularization, the counter-terms needed to renormalize the theory will be necessarily symmetric. Since by construction the full effective Lagrangian contains all terms permitted by the symmetry, the divergences can then be absorbed in a renormalization of the coupling constants occurring in the Lagrangian. At one loop (in $\mathcal{L}_2$), the ChPT divergences are $O(p^4)$ and are therefore renormalized by the low-energy couplings in Eq. (54):

$$L_i = L_i^r(\mu) + \Gamma_i \lambda, \qquad\qquad H_i = H_i^r(\mu) + \widetilde{\Gamma}_i \lambda, \tag{56}$$

where

$$\lambda = \frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} - \frac{1}{2}\left[\log(4\pi) + \Gamma'(1) + 1\right]\right\}. \tag{57}$$

The explicit calculation of the one-loop generating functional Z_4 [15] gives:

$$\begin{array}{llll} \Gamma_1 = \frac{3}{32}, & \Gamma_2 = \frac{3}{16}, & \Gamma_3 = 0\ , & \Gamma_4 = \frac{1}{8}, \\ \Gamma_5 = \frac{3}{8}\ , & \Gamma_6 = \frac{11}{144}, & \Gamma_7 = 0\ , & \Gamma_8 = \frac{5}{48}, \\ \Gamma_9 = \frac{1}{4}\ , & \Gamma_{10} = -\frac{1}{4}, & \widetilde{\Gamma}_1 = -\frac{1}{8}, & \widetilde{\Gamma}_2 = \frac{5}{24}. \end{array} \tag{58}$$

The renormalized couplings $L_i^r(\mu)$ depend on the arbitrary scale of dimensional regularization μ. This scale dependence is of course cancelled by that of the loop amplitude, in any physical, measurable quantity.

A typical $O(p^4)$ amplitude will then consist of a non-polynomial part, coming from the loop computation, plus a polynomial in momenta and pseudoscalar masses, which depends on the unknown constants L_i. The non-polynomial part (the so-called chiral logarithms) is completely predicted as a function of the lowest-order coupling f and the Goldstone masses. This chiral structure can be easily understood in terms of dispersion relations. Given the lowest-order Lagrangian $\mathcal{L}_2$, the non-trivial analytic behaviour associated with some physical intermediate state is calculable without the introduction of new arbitrary chiral coefficients. Analiticity then allows us to reconstruct the full amplitude, through a dispersive integral, up to a subtraction polynomial. ChPT generates (perturbatively) the correct dispersion integrals and organizes the subtraction polynomials in a derivative expansion.

5.3 The chiral anomaly

Although the QCD Lagrangian (31) is formally invariant under local chiral transformations, this is no longer true for the associated generating functional. The anomalies of the fermionic determinant break chiral symmetry at the quantum level [27, 28]. The anomalous change of the generating functional under an infinitesimal chiral transformation

$$g_{L,R} = 1 + i\alpha \mp i\beta + \ldots \tag{59}$$

is given by [28]:

$$\begin{aligned}
\delta Z[v,a,s,p] &= -\frac{N_C}{16\pi^2} \int d^4x \, \langle \beta(x)\, \Omega(x)\rangle, \qquad (60)\\
\Omega(x) &= \varepsilon^{\mu\nu\sigma\rho} \Big[v_{\mu\nu} v_{\sigma\rho} + \frac{4}{3}\, \nabla_\mu a_\nu \nabla_\sigma a_\rho + \frac{2}{3} i \, \{v_{\mu\nu}, a_\sigma a_\rho\} \\
&\qquad + \frac{8}{3} i \, a_\sigma v_{\mu\nu} a_\rho + \frac{4}{3}\, a_\mu a_\nu a_\sigma a_\rho \Big], \\
v_{\mu\nu} &= \partial_\mu v_\nu - \partial_\nu v_\mu - i\,[v_\mu, v_\nu], \qquad \nabla_\mu a_\nu = \partial_\mu a_\nu - i\,[v_\mu, a_\nu].
\end{aligned}$$

($N_C = 3$ is the number of colours, and $\varepsilon_{0123} = 1$.) This anomalous variation of Z is an $O(p^4)$ effect, in the chiral counting.

So far, we have been imposing chiral symmetry to construct the effective ChPT Lagrangian. Since chiral symmetry is explicitly violated by the anomaly at the fundamental QCD level, we need to add a functional Z_A with the property that its change under a chiral gauge transformation reproduces (60). Such a functional was constructed by Wess and Zumino [30], and reformulated in a nice geometrical way by Witten [31]. It has the explicit form:

$$\begin{aligned}
S[U,\ell,r]_{WZW} &= -\frac{iN_C}{240\pi^2} \int d\sigma^{ijklm} \left\langle \Sigma^L_i \Sigma^L_j \Sigma^L_k \Sigma^L_l \Sigma^L_m \right\rangle \\
&\quad - \frac{iN_C}{48\pi^2} \int d^4x \, \varepsilon_{\mu\nu\alpha\beta} \left(W(U,\ell,r)^{\mu\nu\alpha\beta} - W(\mathbf{1},\ell,r)^{\mu\nu\alpha\beta} \right), \qquad (61)\\
W(U,\ell,r)_{\mu\nu\alpha\beta} &= \Big\langle U\ell_\mu\ell_\nu\ell_\alpha U^\dagger r_\beta + \frac{1}{4} U\ell_\mu U^\dagger r_\nu U\ell_\alpha U^\dagger r_\beta + iU\partial_\mu\ell_\nu\ell_\alpha U^\dagger r_\beta \\
&\quad + i\partial_\mu r_\nu U\ell_\alpha U^\dagger r_\beta - i\Sigma^L_\mu \ell_\nu U^\dagger r_\alpha U\ell_\beta + \Sigma^L_\mu U^\dagger \partial_\nu r_\alpha U\ell_\beta \\
&\quad - \Sigma^L_\mu \Sigma^L_\nu U^\dagger r_\alpha U\ell_\beta + \Sigma^L_\mu \ell_\nu \partial_\alpha \ell_\beta + \Sigma^L_\mu \partial_\nu \ell_\alpha \ell_\beta \\
&\quad - i\, \Sigma^L_\mu \ell_\nu \ell_\alpha \ell_\beta + \frac{1}{2} \Sigma^L_\mu \ell_\nu \Sigma^L_\alpha \ell_\beta - i\Sigma^L_\mu \Sigma^L_\nu \Sigma^L_\alpha \ell_\beta \Big\rangle \\
&\quad - \; (L \leftrightarrow R)\,, \qquad (62)
\end{aligned}$$

where

$$\Sigma^L_\mu = U^\dagger \partial_\mu U, \qquad \Sigma^R_\mu = U \partial_\mu U^\dagger, \tag{63}$$

and $(L \leftrightarrow R)$ stands for the interchanges $U \leftrightarrow U^\dagger$, $\ell_\mu \leftrightarrow r_\mu$ and $\Sigma^L_\mu \leftrightarrow \Sigma^R_\mu$. The integration in the first term of Eq. (61) is over a five-dimensional manifold whose boundary is four-dimensional Minkowski space. The integrand is a surface term; therefore both the first and the second terms of S_{WZW} are $O(p^4)$, according to the chiral counting rules.

Since anomalies have a short-distance origin, their effect is completely calculable. The translation from the fundamental quark–gluon level to the effective chiral level is unaffected by hadronization problems. In spite of its considerable complexity, the anomalous action (61) has no free parameters.

The anomaly functional gives rise to interactions that break the intrinsic parity. It is responsible for the $\pi^0 \to 2\gamma$, $\eta \to 2\gamma$ decays, and the $\gamma 3\pi$, $\gamma\pi^+\pi^-\eta$ interactions. The five-dimensional surface term generates interactions among five or more Goldstone bosons.

6 Low-energy phenomenology at $O(p^4)$

At lowest order in momenta, the predictive power of the chiral Lagrangian was really impressive: with only two low-energy couplings, it was possible to describe all Green functions associated with the pseudoscalar-meson interactions. The symmetry constraints become less powerful at higher orders. Ten additional constants appear in the $\mathcal{L}_4$ Lagrangian, and many more would be needed at $O(p^6)$. Higher-order terms in the chiral expansion are much more sensitive to the non-trivial aspects of the underlying QCD dynamics.

With $p \lesssim M_K \, (M_\pi)$, we expect $O(p^4)$ corrections to the lowest-order amplitudes at the level of $p^2/\Lambda_\chi^2 \lesssim 20\% \, (2\%)$. We need to include those corrections if we aim to increase the accuracy of the ChPT predictions beyond this level. Although the number of free constants in $\mathcal{L}_4$ looks quite big, only a few of them contribute to a given observable. In the absence of external fields, for instance, the Lagrangian reduces to the first three terms; elastic $\pi\pi$ and πK scatterings are then sensitive to $L_{1,2,3}$. The two-derivative couplings $L_{4,5}$ generate mass corrections to the meson decay constants (and mass-dependent wave-function renormalizations). Pseudoscalar masses are affected by the non-derivative terms $L_{6,7,8}$; L_9 is mainly responsible for the charged-meson electromagnetic radius and L_{10}, finally, only contributes to amplitudes with at least two external vector or axial-vector fields, like the radiative semileptonic decay $\pi \to e\nu\gamma$.

Table 1, taken from ref. [32], summarizes the present status of the phenomenological determination [15,33] of the constants L_i. The quoted numbers correspond to the renormalized couplings, at a scale $\mu = M_\rho$. The values of these couplings at any other renormalization scale can be trivially obtained, through the logarithmic running implied by Eq. (56):

$$L_i^r(\mu_2) \, = \, L_i^r(\mu_1) \, + \, \frac{\Gamma_i}{(4\pi)^2} \, \log\left(\frac{\mu_1}{\mu_2}\right). \qquad (64)$$

Comparing the Lagrangians $\mathcal{L}_2$ and $\mathcal{L}_4$, one can make an estimate of the expected size of the couplings L_i in terms of the scale of SCSB. Taking $\Lambda_\chi \sim$

Table 1: Phenomenological values of the renormalized couplings $L_i^r(M_\rho)$. The last column shows the source used to extract this information.

i	$L_i^r(M_\rho) \times 10^3$	Source
1	0.7 ± 0.5	K_{e4}, $\pi\pi \to \pi\pi$
2	1.2 ± 0.4	K_{e4}, $\pi\pi \to \pi\pi$
3	-3.6 ± 1.3	K_{e4}, $\pi\pi \to \pi\pi$
4	-0.3 ± 0.5	Zweig rule
5	1.4 ± 0.5	$F_K : F_\pi$
6	-0.2 ± 0.3	Zweig rule
7	-0.4 ± 0.2	Gell-Mann–Okubo, L_5, L_8
8	0.9 ± 0.3	$M_{K^0} - M_{K^+}$, L_5, $(m_s - \hat{m}) : (m_d - m_u)$
9	6.9 ± 0.7	$\langle r^2 \rangle^\pi_{\mathrm{em}}$
10	-5.5 ± 0.7	$\pi \to e\nu\gamma$

$4\pi f_\pi \sim 1.2\,\mathrm{GeV}$, one would get

$$L_i \sim \frac{f_\pi^2/4}{\Lambda_\chi^2} \sim \frac{1}{4(4\pi)^2} \sim 2 \times 10^{-3}, \tag{65}$$

in reasonable agreement with the phenomenological values quoted in Table 1. This indicates a good convergence of the momentum expansion below the resonance region, i.e. $p < M_\rho$.

The chiral Lagrangian allows us to make a good book-keeping of phenomenological information with a few couplings. Once these couplings have been fixed, we can predict many other quantities. Moreover, the information contained in Table 1 is very useful to easily test different QCD-inspired models. Given any particular model aiming to correctly describe QCD at low energies, we no longer need to make an extensive phenomenological analysis of all the predictions of the model, in order to test its degree of reliability; we only need to calculate the predicted low-energy couplings, and compare them with the values in Table 1. For instance, if one integrates out the heavy scalar of the linear sigma model described in Section 3, the resulting Goldstone Lagrangian only contains the L_1 term [see Eq. (21)] at $O(p^4)$; obviously this is not a satisfactory approximation to the physical world[5].

An exhaustive description of the chiral phenomenology at $O(p^4)$ is beyond the scope of these lectures. Instead, I will just present a few examples to illustrate both the power and limitations of the ChPT techniques.

[5] A more detailed study of the renormalizable linear sigma model can be found in ref. [20]. The conclusion is that this model is clearly ruled out by the data.

6.1 Decay constants

In the isospin limit ($m_u = m_d = \hat{m}$), the $O(p^4)$ calculation of the meson-decay constants gives [15]:

$$
\begin{aligned}
f_\pi &= f\left\{1 - 2\mu_\pi - \mu_K + \frac{4M_\pi^2}{f^2}L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2}L_4^r(\mu)\right\}, \\
f_K &= f\left\{1 - \frac{3}{4}\mu_\pi - \frac{3}{2}\mu_K - \frac{3}{4}\mu_{\eta_8} + \frac{4M_K^2}{f^2}L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2}L_4^r(\mu)\right\}, \qquad (66) \\
f_{\eta_8} &= f\left\{1 - 3\mu_K + \frac{4M_{\eta_8}^2}{f^2}L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2}L_4^r(\mu)\right\},
\end{aligned}
$$

where

$$\mu_P \equiv \frac{M_P^2}{32\pi^2 f^2}\log\left(\frac{M_P^2}{\mu^2}\right). \qquad (67)$$

The result depends on two $O(p^4)$ couplings, L_4 and L_5. The L_4 term generates a universal shift of all meson-decay constants, $\delta f^2 = 16 L_4 B_0 \langle \mathcal{M} \rangle$, which can be eliminated taking ratios. From the experimental value [34]

$$\frac{f_K}{f_\pi} = 1.22 \pm 0.01, \qquad (68)$$

one can then fix $L_5(\mu)$; this gives the result quoted in Table 1. Moreover, one gets the absolute prediction [15]

$$\frac{f_{\eta_8}}{f_\pi} = 1.3 \pm 0.05. \qquad (69)$$

Taking into account isospin violations, one can also predict [15] a tiny difference between $f_{K^\pm}$ and f_{K^0}, proportional to $m_d - m_u$.

6.2 Electromagnetic form factors

At $O(p^2)$ the electromagnetic coupling of the Goldstone bosons is just the minimal one, obtained through the covariant derivative. The next-order corrections generate a momentum-dependent form factor

$$F^{\phi^\pm}(q^2) = 1 + \frac{1}{6}\langle r^2\rangle^{\phi^\pm} q^2 + \ldots\,; \qquad F^{\phi^0}(q^2) = \frac{1}{6}\langle r^2\rangle^{\phi^0} q^2 + \ldots \qquad (70)$$

The meson electromagnetic radius $\langle r^2\rangle^\phi$ gets local contributions from the L_9 term, plus logarithmic loop corrections [15]:

$$
\begin{aligned}
\langle r^2\rangle^{\pi^\pm} &= \frac{12 L_9^r(\mu)}{f^2} - \frac{1}{32\pi^2 f^2}\left\{2\log\left(\frac{M_\pi^2}{\mu^2}\right) + \log\left(\frac{M_K^2}{\mu^2}\right) + 3\right\}, \\
\langle r^2\rangle^{K^0} &= -\frac{1}{16\pi^2 f^2}\log\left(\frac{M_K}{M_\pi}\right), \qquad (71) \\
\langle r^2\rangle^{K^\pm} &= \langle r^2\rangle^{\pi^\pm} + \langle r^2\rangle^{K^0}.
\end{aligned}
$$

Since neutral bosons do not couple to the photon at tree level, $\langle r^2\rangle^{K^0}$ only gets a loop contribution, which is moreover finite (there cannot be any divergence because there exists no counter-term to renormalize it!). The predicted value, $\langle r^2\rangle^{K^0} = -0.04 \pm 0.03\,\mathrm{fm}^2$, is in perfect agreement with the experimental determination [35] $\langle r^2\rangle^{K^0} = -0.054 \pm 0.026\,\mathrm{fm}^2$.

The measured electromagnetic pion radius [36], $\langle r^2\rangle^{\pi^\pm} = 0.439 \pm 0.008\,\mathrm{fm}^2$, is used as input to estimate the coupling L_9. This observable provides a good example of the importance of higher-order local terms in the chiral expansion. If one tries to ignore the L_9 contribution, using instead some "physical" cut-off $p_{\max}$ to regularize the loops, one needs [37] $p_{\max} \sim 60\,\mathrm{GeV}$, in order to reproduce the experimental value; this is clearly nonsense. The pion charge radius is dominated by the $L_9^r(\mu)$ contribution, for any reasonnable value of μ.

The measured K^+ charge radius [38], $\langle r^2\rangle^{K^\pm} = 0.28 \pm 0.07\,\mathrm{fm}^2$, has a larger experimental uncertainty. Within present errors, it is in agreement with the parameter-free relation in Eq. (71).

6.3 K_{l3} decays

The semileptonic decays $K^+ \to \pi^0 l^+ \nu_l$ and $K^0 \to \pi^- l^+ \nu_l$ are governed by the corresponding hadronic matrix element of the vector current [$t \equiv (P_K - P_\pi)^2$],

$$\langle \pi | \bar{s}\gamma^\mu u | K \rangle = C_{K\pi} \left[(P_K + P_\pi)^\mu \; f_+^{K\pi}(t) \; + \; (P_K - P_\pi)^\mu \; f_-^{K\pi}(t) \right] , \qquad (72)$$

where $C_{K^+\pi^0} = 1/\sqrt{2}$, $C_{K^0\pi^-} = 1$. At lowest order, the two form factors reduce to trivial constants: $f_+^{K\pi}(t) = 1$ and $f_-^{K\pi}(t) = 0$. There is however a sizeable correction to $f_+^{K^+\pi^0}(t)$, due to $\pi^0\eta$ mixing, which is proportional to $(m_d - m_u)$,

$$f_+^{K^+\pi^0}(0) \; = \; 1 + \frac{3}{4}\, \frac{m_d - m_u}{m_s - \hat{m}} \; = \; 1.017. \qquad (73)$$

This number should be compared with the experimental ratio

$$\frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} \; = \; 1.028 \pm 0.010. \qquad (74)$$

The $O(p^4)$ corrections to $f_+^{K\pi}(0)$ can be expressed in a parameter-free manner in terms of the physical meson masses [15]. Including those contributions, one gets the more precise values

$$f_+^{K^0\pi^-}(0) = 0.977, \qquad\qquad \frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} = 1.022, \qquad (75)$$

which are in perfect agreement with the experimental result (74). The accurate ChPT calculation of these quantities allows us to extract [34] the most precise determination of the Kobayashi–Maskawa matrix element V_{us}:

$$|V_{us}| = 0.2196 \pm 0.0023. \qquad (76)$$

At $O(p^4)$, the form factors get momentum-dependent contributions. Since L_9 is the only unknown chiral coupling occurring in $f_+^{K\pi}(t)$ at this order, the slope λ_+ of this form factor can be fully predicted. Alternatively, we can use the measured slope [39],

$$\lambda_+ \equiv \frac{1}{6}\,\langle r^2\rangle^{K\pi}\, M_\pi^2 = 0.0300 \pm 0.0016, \tag{77}$$

as an input to get an independent determination of L_9. The value (77) corresponds [15] to $L_9^r(M_\rho) = (6.6 \pm 0.4) \times 10^{-3}$, in excellent agreement with the determination from the pion-charge radius, quoted in Table 1.

Instead of $f_-^{K\pi}(t)$, it is usual to parametrize the experimental results in terms of the so-called scalar form factor

$$f_0^{K\pi}(t) = f_+^{K\pi}(t) + \frac{t}{M_K^2 - M_\pi^2}\, f_-^{K\pi}(t). \tag{78}$$

The slope of this form factor is determined by the constant L_5, which in turn is fixed by f_K/f_π. One gets the result [15]:

$$\lambda_0 \equiv \frac{1}{6}\,\langle r^2\rangle_S^{K\pi}\, M_\pi^2 = 0.017 \pm 0.004. \tag{79}$$

The experimental situation concerning the value of this slope is far from clear; while an older high-statistics measurement [40], $\lambda_0 = 0.019 \pm 0.004$, confirmed the theoretical expectations, more recent experiments find higher values, which disagree with this result. Reference [41], for instance, report $\lambda_0 = 0.046 \pm 0.006$, which differs from (79) by more than 4 standard deviations. The Particle Data Group [39] quote a world average $\lambda_0 = 0.025 \pm 0.006$.

6.4 Meson masses

The relations (45) get modified at $O(p^4)$. The additional contributions depend on the low-energy constants L_4, L_5, L_6, L_7 and L_8. It is possible, however, to obtain one relation between the quark and meson masses, which does not contain any of the $O(p^4)$ couplings. The dimensionless ratios

$$Q_1 \equiv \frac{M_K^2}{M_\pi^2}, \qquad\qquad Q_2 \equiv \frac{(M_{K^0}^2 - M_{K^+}^2) - (M_{\pi^0}^2 - M_{\pi^+}^2)}{M_K^2 - M_\pi^2}, \tag{80}$$

get the same $O(p^4)$ correction [15]:

$$Q_1 = \frac{m_s + \hat{m}}{2\hat{m}}\,\{1 + \Delta_M\}, \qquad\qquad Q_2 = \frac{m_d - m_u}{m_s - \hat{m}}\,\{1 + \Delta_M\}, \tag{81}$$

where

$$\Delta_M = -\mu_\pi + \mu_{\eta_8} + \frac{8}{f^2}\,(M_K^2 - M_\pi^2)\,\left[2L_8^r(\mu) - L_5^r(\mu)\right]. \tag{82}$$

Therefore, at this order, the ratio Q_1/Q_2 is just given by the corresponding ratio of quark masses,

$$Q^2 \equiv \frac{Q_1}{Q_2} = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}. \tag{83}$$

To a good approximation, Eq. (83) can be written as an ellipse,

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1, \tag{84}$$

which constrains the quark-mass ratios. The observed values of the meson masses give $Q = 24$.

Obviously, the quark-mass ratios (52), obtained at $O(p^2)$, satisfy this elliptic constraint. At $O(p^4)$, however, it is not possible to make a separate determination of m_u/m_d and m_s/m_d without having additional information on some of the L_i couplings.

A useful quantity is the deviation of the Gell-Mann–Okubo relation,

$$\Delta_{\mathrm{GMO}} \equiv \frac{4M_K^2 - 3M_{\eta_8}^2 - M_\pi^2}{M_{\eta_8}^2 - M_\pi^2}. \tag{85}$$

Neglecting the mass difference $m_d - m_u$, one gets [15]

$$\begin{aligned} \Delta_{\mathrm{GMO}} \;=\;& \frac{-2\,(4M_K^2\mu_K - 3M_{\eta_8}^2\mu_{\eta_8} - M_\pi^2\mu_\pi)}{M_{\eta_8}^2 - M_\pi^2} \\ & - \frac{6}{f^2}\,(M_{\eta_8}^2 - M_\pi^2)\,\left[12L_7^r(\mu) + 6L_8^r(\mu) - L_5^r(\mu)\right]. \end{aligned} \tag{86}$$

Experimentally, correcting the masses for electromagnetic effects, $\Delta_{\mathrm{GMO}} = 0.21$. Since L_5 is already known, this allows the combination $2L_7 + L_8$ to be fixed .

In order to determine the individual quark-mass ratios from Eqs. (81), we would need to fix the constant L_8. However, there is no way to find an observable that isolates this coupling. The reason is an accidental symmetry of the Lagrangian $\mathcal{L}_2 + \mathcal{L}_4$. The chiral Lagrangian remains invariant under the following simultaneous change [42] of the quark-mass matrix and some of the chiral couplings:

$$\begin{aligned} \mathcal{M}' \;&=\; \alpha\mathcal{M} + \beta(\mathcal{M}^\dagger)^{-1}\det\mathcal{M}, \qquad B_0' \;=\; B_0/\alpha, \\ L_6' \;&=\; L_6 - \zeta, \qquad L_7' \;=\; L_7 - \zeta, \qquad L_8' \;=\; L_8 + 2\zeta, \end{aligned} \tag{87}$$

where α and β are arbitrary constants, and $\zeta = \beta f^2/(32\alpha B_0)$. The only information on the quark-mass matrix $\mathcal{M}$ that we used to construct the effective Lagrangian was that it transforms as $\mathcal{M} \to g_R \mathcal{M} g_L^\dagger$. The matrix $\mathcal{M}'$ transforms in the same manner; therefore, symmetry alone does not allow us to distinguish between $\mathcal{M}$ and $\mathcal{M}'$. Since only the product $B_0\mathcal{M}$ appears in the Lagrangian, α merely changes the value of the constant B_0. The term proportional to β is a correction of $O(\mathcal{M}^2)$; when inserted in $\mathcal{L}_2$, it generates a contribution to $\mathcal{L}_4$, which is reabsorbed by the redefinition of the $O(p^4)$ couplings. All chiral predictions will be invariant under the transformation (87); therefore it is not possible to separately determine the values of the quark masses and the constants B_0, L_6, L_7 and L_8. We can only fix those combinations of chiral couplings and masses that remain invariant under (87).

Notice that (87) is certainly not a symmetry of the underlying QCD Lagrangian. The accidental symmetry arises in the effective theory because we are not making use of the explicit form of the QCD Lagrangian; only its symmetry properties under chiral rotations have been taken into account. Therefore, we can resolve the ambiguity by obtaining one additional information from outside the pseudoscalar-meson chiral Lagrangian framework. For instance, by analysing the isospin breaking in the baryon mass spectrum and the ρ-ω mixing [43], it is possible to fix the ratio $(m_s - \hat{m})/(m_d - m_u) = 43.7 \pm 2.7$. Inserting this number in Eq. (83), one gets [15]

$$\frac{m_s}{\hat{m}} = 25.7 \pm 2.6. \tag{88}$$

Moreover, one can now determine L_8 from Eqs. (81), and therefore fix L_7 with Eq. (86); one then gets the values quoted in Table 1. Other ways of resolving the ambiguity, by using different additional inputs [44, 45], lead to similar estimates of the quark-mass ratios and the low-energy couplings.

7 Information encoded in the chiral couplings

The effective theory takes explicitly into account the poles and cuts generated by the Goldstone bosons. Given the non-trivial analytic structure associated with those physical intermediate states, the full amplitudes are reconstructed up to a subtraction polynomial. Obviously, the subtraction constants L_i contain all the information on the heavy degrees of freedom, which do not appear in the low-energy Lagrangian.

It seems rather natural to expect that the lowest-mass resonances, such as ρ mesons, should have an important impact on the physics of the pseudoscalar bosons. In particular, the low-energy singularities due to the exchange of those resonances should generate sizeable contributions to the chiral couplings. This can be easily understood, making a Taylor expansion of the ρ propagator:

$$\frac{1}{p^2 - M_\rho^2} = \frac{-1}{M_\rho^2}\left\{1 + \frac{p^2}{M_\rho^2} + \ldots\right\}, \qquad (p^2 < M_\rho^2). \tag{89}$$

Below the ρ-mass scale, the singularity associated with the pole of the resonance propagator is replaced by the corresponding momentum expansion. The exchange of virtual ρ mesons should result in derivative Goldstone couplings proportional to powers of $1/M_\rho^2$.

It is well known, for instance, that the electromagnetic form factor of the charged pion is well reproduced by the vector-meson dominance (VMD) formula

$$F^{\pi^\pm}(t) \approx \frac{M_\rho^2}{M_\rho^2 - t}, \tag{90}$$

i.e. $\langle r^2\rangle^{\pi^\pm} \approx 6/M_\rho^2 = 0.4\,\mathrm{fm}^2$, to be compared with the measured value $\langle r^2\rangle^{\pi^\pm} = 0.439 \pm 0.008\,\mathrm{fm}^2$.

Writing a chiral-invariant $\rho\pi\pi$ interaction, with coupling G_V, one can compute the effect of a ρ-exchange diagram at low energies; the leading contribution [20,46] is a π^4 local interaction, with a coupling constant proportional to G_V^2/M_ρ^2. Since G_V can be directly measured from the $\rho \to 2\pi$ decay width, $|G_V| = 69$ MeV, the size of this contribution is fully predicted. Similarly, one can write a chiral invariant $\rho^0\gamma$ interaction, with coupling F_V; this coupling can be extracted from the $\rho^0 \to e^+e^-$ decay width, $|F_V| = 154$ MeV. The exchange of a ρ meson between the G_V and F_V vertices, generates a contribution to the electromagnetic form factor of the charged pion [46]:

$$\langle r^2\rangle^{\pi^\pm} = \frac{6F_V G_V}{f^2 M_\rho^2}. \tag{91}$$

From the success of the naïve VMD formula (90), one could expect $F_V G_V/f^2 \approx 1$, which is indeed approximately satisfied (one obtains 1.2 with the measured F_V and G_V values).

A systematic analysis of the role of resonances in the ChPT Lagrangian has been performed[6] in ref. [46]. One writes first a general chiral-invariant Lagrangian $\mathcal{L}(U,V,A,S,P)$, describing the couplings between meson resonances of the type V, A, S, P and the Goldstone bosons, at lowest-order in derivatives. The coupling constants of this Lagrangian are phenomenologically extracted from physics at the resonance-mass scale. One has then an effective chiral theory defined in the intermediate-energy region. Formally, the generating functional (40) is given in this theory by the path-integral formula

$$\exp\{iZ\} = \int \mathcal{D}U(\phi)\,\mathcal{D}V\,\mathcal{D}A\,\mathcal{D}S\,\mathcal{D}P\,\exp\left\{i\int d^4x\,\mathcal{L}(U,V,A,S,P)\right\}. \tag{92}$$

The integration of the resonance fields results in a low-energy theory with only Goldstones, i.e. the usual ChPT Lagrangian. At lowest-order this integration can be explicitly performed, expanding around the classical solution for the resonance fields. The resulting L_i couplings [46] are summarized in Table 2, which compares the different resonance-exchange contributions with the phenomenologically determined values of $L_i^r(M_\rho)$. For vector and axial-vector mesons only the $SU(3)$ octets contribute, whereas both octets and singlets are relevant in the case of scalar and pseudoscalar resonances.

[6] Related work can be found in ref. [47].

Table 2: V, A, S, S_1 and η_1 contributions to the coupling constants L_i^r in units of 10^{-3}. The last column shows the results obtained with the relations in Eq. (95).

i	$L_i^r(M_\rho)$	V	A	S	S_1	η_1	Total	Total$^{c)}$
1	0.7 ± 0.5	0.6	0	-0.2	$0.2^{b)}$	0	0.6	0.9
2	1.2 ± 0.4	1.2	0	0	0	0	1.2	1.8
3	-3.6 ± 1.3	-3.6	0	0.6	0	0	-3.0	-4.9
4	-0.3 ± 0.5	0	0	-0.5	$0.5^{b)}$	0	0.0	0.0
5	1.4 ± 0.5	0	0	$1.4^{a)}$	0	0	1.4	1.4
6	-0.2 ± 0.3	0	0	-0.3	$0.3^{b)}$	0	0.0	0.0
7	-0.4 ± 0.2	0	0	0	0	-0.3	-0.3	-0.3
8	0.9 ± 0.3	0	0	$0.9^{a)}$	0	0	0.9	0.9
9	6.9 ± 0.7	$6.9^{a)}$	0	0	0	0	6.9	7.3
10	-5.5 ± 0.7	-10.0	4.0	0	0	0	-6.0	-5.5

$^{a)}$ Input. $^{b)}$ Large-N_C estimate. $^{c)}$ With (95)

At lowest order, the most general interaction of the V octet to the Goldstone bosons contains two terms, corresponding to the couplings G_V and F_V described before. Due to the different parity, only one term with coupling F_A is present for the axial octet A. While V exchange generates contributions to L_1, L_2, L_3, L_9 and L_{10}, A exchange only contributes to L_{10} [46]:

$$L_1^V = \frac{G_V^2}{8M_V^2}, \qquad L_2^V = 2L_1^V, \qquad L_3^V = -6L_1^V, \tag{93}$$
$$L_9^V = \frac{F_V G_V}{2M_V^2}, \qquad L_{10}^{V+A} = -\frac{F_V^2}{4M_V^2} + \frac{F_A^2}{4M_A^2}.$$

To obtain the numbers in Table 2, the value of $L_9^r(M_\rho)$ has been fitted to determine $|G_V| = 53\,\mathrm{MeV}$; nevertheless, the qualitative conclusion would be the same with the $\rho \to 2\pi$ determination mentioned before. The axial parameters have been fixed using the old Weinberg sum rules [48]: $F_A^2 = F_V^2 - f_\pi^2 = (123\,\mathrm{MeV})^2$ and $M_A^2 = M_V^2 F_V^2/F_A^2 = (968\,\mathrm{MeV})^2$. The results shown in the table clearly establish a chiral version of vector (and axial-vector) meson dominance: whenever they can contribute at all, V and A exchange seem to completely dominate the relevant coupling constants.

There are different phenomenologically successful models in the literature for V and A resonances (tensor-field description [20, 46], massive Yang–Mills [49], hidden gauge formulations [50], etc.). It can be shown [51] that all models are equivalent (i.e. give the same contributions to the L_i), provided they incorporate the appropriate QCD constraints at high energies. Moreover, with additional QCD-inspired assumptions of high-energy behaviour, such as an unsubtracted dispersion relation for the pion electromagnetic form factor, all V and A couplings can be expressed in terms of f and M_V only [51]:

$$F_V = \sqrt{2} f_\pi, \qquad G_V = f_\pi/\sqrt{2}, \qquad F_A = f_\pi, \qquad M_A = \sqrt{2} M_V. \tag{94}$$

In that case, one has

$$L_1^V = L_2^V/2 = -L_3^V/6 = L_9^V/8 = -L_{10}^{V+A}/6 = f_\pi^2/(16M_V^2). \tag{95}$$

The last column in Table 2 shows the predicted numerical values of the L_i couplings, using the relations (95).

The analysis of scalar exchange is very similar [46]. Since the experimental information is quite scarce in the scalar sector, one needs to assume that the couplings L_5 and L_8 are due exclusively to scalar-octet exchange, to determine the scalar-octet couplings. The scalar-octet contributions to the other L_i ($i = 1,3,4,6$) are then fixed. Moreover, one can then predict $\Gamma(a_0 \to \eta\pi)$, in good agreement with experiment. The scalar-singlet-exchange contributions can be expressed in terms of the octet parameters using large-N_C arguments. For $N_C = \infty$, octet- and singlet-scalar exchange cancel in L_1, L_4 and L_6. Although the results in Table 2 cannot be considered as a proof for scalar dominance, they provide at least a convincing demonstration of its consistency.

Neglecting the higher-mass 0^- resonances, the only remaining meson-exchange is the one associated with the η_1, which generates a sizeable contribution to L_7 [15,46]. The magnitude of this contribution can be calculated from the quark-mass expansion of M_η^2. The result for L_7 is in close agreement with its phenomenological value.

The combined resonance contributions appear to saturate the L_i^r almost entirely [46]. Within the uncertainties of the approach, there is no need for invoking any additional contributions. Although the comparison has been made for $\mu = M_\rho$, a similar conclusion would apply for any value of μ in the low-lying resonance region between 0.5 and 1 GeV.

All chiral couplings are in principle calculable from QCD. They are functions of Λ_{QCD} and the heavy-quark masses m_c, m_b, m_t. Unfortunately, we are not able at present to make such a first-principle computation. Although the integral over the quark fields in Eq. (40) can be done explicitly, we do not know how to perform analytically the remaining integration over the gluon fields. A perturbative evaluation of the gluonic contribution would obviously fail in reproducing the correct dynamics of SCSB. A possible way out is to parametrize phenomenologically the SCSB and make a weak gluon-field expansion around the resulting physical vacuum.

The simplest parametrization [52] is obtained by adding to the QCD Lagrangian the term

$$\Delta\mathcal{L}_{\rm QCD} = -M_Q \left(\bar{q}_R U q_L + \bar{q}_L U^\dagger q_R\right), \tag{96}$$

Table 3: Leading-order ($\alpha_s = 0$) predictions for the L_i's, within the QCD-inspired model (96). The phenomenological values are shown in the second row for comparison. All numbers are given in units of 10^{-3}.

	L_1	L_2	L_3	L_9	L_{10}
$L_i^{th}(\alpha_s = 0)$	0.79	1.58	-3.17	6.33	-3.17
$L_i^r(M_\rho)$	0.7 ± 0.5	1.2 ± 0.4	-3.6 ± 1.3	6.9 ± 0.7	-5.5 ± 0.7

which serves to introduce the U field, and a mass parameter M_Q, which regulates the infra-red behaviour of the low-energy effective action. In the presence of this term the operator $\bar{q}q$ acquires a vacuum expectation value; therefore, (96) is an effective way to generate the order parameter due to SCSB. Making a chiral rotation of the quark fields, $Q_L = \xi q_L$, $Q_R = \xi^\dagger q_R$, with ξ chosen such that $U = \xi^2$, the interaction (96) reduces to a mass-term for the "dressed" quarks Q; the parameter M_Q can then be interpreted as a "constituent-quark mass".

The derivation of the low-energy effective chiral Lagrangian within this framework has been extensively discussed in ref. [52]. In the chiral and large-N_C limits, and including the leading gluonic contributions, one gets:

$$8L_1 = 4L_2 = L_9 = \frac{N_C}{48\pi^2}\left[1 + O\left(1/M_Q^6\right)\right], \tag{97}$$
$$L_3 = L_{10} = -\frac{N_C}{96\pi^2}\left[1 + \frac{\pi^2}{5N_C}\frac{\langle\frac{\alpha_s}{\pi}GG\rangle}{M_Q^4} + O\left(1/M_Q^6\right)\right].$$

Due to dimensional reasons, the leading contributions to the $O(p^4)$ couplings only depend on N_C and geometrical factors. It is remarkable that L_1, L_2 and L_9 do not get any gluonic correction at this order; this result is independent of the way SCSB has been parametrized (M_Q can be taken to be infinite). Table 3 compares the predictions obtained with only the leading term in Eqs. (97) (i.e. neglecting the gluonic correction) with the phenomenological determination of the L_i couplings. The numerical agreement is quite impressive; both the order of magnitude and the sign are correctly reproduced (notice that this is just a free-quark result!). Moreover, the gluonic corrections shift the values of L_3 and L_{10} in the right direction, making them more negative.

The results (97) obey almost all the short-distance relations (95). Comparing the predictions for $L_{1,2,9}$ in the VMD approach of Eq. (95) with the QCD-inspired ones in Eq. (97), one gets a quite good estimate of the ρ mass:

$$M_V = 2\sqrt{2}\pi f = 830\,\text{MeV}. \tag{98}$$

Is it quite easy to prove that the interaction (96) is equivalent to the mean-field approximation of the Nambu–Jona–Lasinio (NJL) model, where SCSB is triggered by four-quark operators. It has been conjectured recently [53] that integrating out the quark and gluon fields of QCD, down to some intermediate scale Λ_χ, gives rise to an extended NJL Lagrangian. By introducing collective fields (to be identified later with the Goldstone fields and S, V, A resonances) the model can be transformed into a Lagrangian bilinear in the quark fields, which can therefore be integrated out. One then gets an effective Lagrangian, describing the couplings of the pseudoscalar bosons to vector, axial-vector and scalar resonances. Extending the analysis beyond the mean-field approximation, ref. [53] obtains predictions for 20 measurable quantities, including the L_i's, in terms of only 4 parameters. The quality of the fits is quite impressive. Since the model contains all resonances that are known to saturate the L_i couplings, it is not surprising that one gets an improvement of the mean-field-approximation results, specially for the constants L_5 and L_8, which are sensitive to scalar exchange. What is more important, this analysis clarifies a potential problem of double-counting: in certain limits the model approches either the pure quark-loop predictions of Eqs. (97) or the VMD results (95), but in general it interpolates between these two cases.

8 $\Delta S = 1$ non-leptonic weak interactions

The Standard Model predicts strangeness-changing transitions with $\Delta S = 1$ via W-exchange between two weak charged currents. At low energies ($E << M_W$), the heavy fields W, Z, t, b, c can be integrated out. Using standard operator-product-expansion techniques, the $\Delta S = 1$ weak interactions are described by an effective Hamiltonian [54]

$$\mathcal{H}_{\text{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_i C_i(\mu)\, Q_i \; + \; \text{h.c.}\,, \tag{99}$$

which is a sum of local four-quark operators, constructed with the light (u, d, s) quark fields only,

$$\begin{aligned}
Q_1 &\equiv 4\,(\bar{s}_L\gamma^\mu d_L)\,(\bar{u}_L\gamma_\mu u_L), & Q_2 &\equiv 4\,(\bar{s}_L\gamma^\mu u_L)\,(\bar{u}_L\gamma_\mu d_L), \\
Q_3 &\equiv 4\,(\bar{s}_L\gamma^\mu d_L) \sum_{q=u,d,s} (\bar{q}_L\gamma_\mu q_L), & Q_4 &\equiv 4 \sum_{q=u,d,s} (\bar{s}_L\gamma^\mu q_L)\,(\bar{q}_L\gamma_\mu d_L), \\
Q_5 &\equiv 4\,(\bar{s}_L\gamma^\mu d_L) \sum_{q=u,d,s} (\bar{q}_R\gamma_\mu q_R), & Q_6 &\equiv -8 \sum_{q=u,d,s} (\bar{s}_L q_R)\,(\bar{q}_R d_L),
\end{aligned} \tag{100}$$

modulated by Wilson coefficients $C_i(\mu)$, which are functions of the heavy W, t, b, c masses and an overall renormalization scale μ. Only five of these operators are independent, since $Q_4 = -Q_1 + Q_2 + Q_3$. From the point of view of chiral $SU(3)_L \otimes SU(3)_R$ and isospin quantum numbers, $Q_- \equiv Q_2 - Q_1$ and Q_i $(i = 3, 4, 5, 6)$ transform as $(8_L, 1_R)$ and induce $|\Delta I| = 1/2$ transitions, while $Q_1 + 2/3 Q_2 - 1/3 Q_3$ transforms like $(27_L, 1_R)$ and induces both $|\Delta I| = 1/2$ and $|\Delta I| = 3/2$ transitions.

In the absence of strong interactions, $C_2(\mu) = 1$ and all other Wilson coefficients vanish. The Standard Electroweak Model then gives rise to $|\Delta I| = 1/2$

and $|\Delta I| = 3/2$ amplitudes of nearly equal size, while experimentally the ratio between the two amplitudes is a factor of 20. To solve this large discrepancy, QCD effects should be enormous. The leading α_s corrections indeed give, for μ-values around 1 GeV, an enhancement by a factor of 2 to 3 of the Q_- Wilson coefficient with respect to the $Q_+ \equiv Q_2 + Q_1$ one. Moreover, the gluonic exchanges generate the additional $|\Delta I| = 1/2$ operators Q_i $(i = 3, 4, 5, 6)$. Nevertheless, this by itself is not enough to explain the experimentally observed rates, without simultaneously appealing to a further enhancement in the hadronic matrix elements of at least some of the isospin 1/2 four-quark operators. The computation of hadronic matrix elements at the K-mass scale is however a very difficult non-perturbative problem.

The effect of $\Delta S = 1$ non-leptonic weak interactions can be incorporated in the low-energy chiral theory, as a perturbation to the strong effective Lagrangian $\mathcal{L}_{\rm eff}(U)$. At lowest order in the number of derivatives, the most general effective bosonic Lagrangian, with the same $SU(3)_L \otimes SU(3)_R$ transformation properties as the four-quark Hamiltonian in Eqs. (99) and (100), contains two terms[7]:

$$\mathcal{L}_2^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left\{ g_8 \langle \lambda L_\mu L^\mu \rangle + g_{27} \left(L_{\mu 23} L_{11}^\mu + \frac{2}{3} L_{\mu 21} L_{13}^\mu \right) + \text{h.c.} \right\}, \quad (101)$$

where

$$\lambda = (\lambda_6 - i\lambda_7)/2, \qquad\qquad L_\mu = i f^2 U^\dagger D_\mu U. \quad (102)$$

The chiral couplings g_8 and g_{27} measure the strength of the two parts in the effective Hamiltonian (99) transforming as $(8_L, 1_R)$ and $(27_L, 1_R)$, respectively, under chiral rotations. Their values can be extracted from $K \to 2\pi$ decays [55]:

$$|g_8| \approx 5.1, \qquad\qquad g_{27}/g_8 \approx 1/18. \quad (103)$$

The huge difference between these two couplings shows the enhancement of the octet $|\Delta I| = 1/2$ transitions.

Using the effective Lagrangian (101), the calculation of hadronic weak transitions becomes a straightforward perturbative problem. The highly non-trivial QCD dynamics has been parametrized in terms of the two chiral couplings. Of course, the interesting problem that remains to be solved is to compute g_8 and g_{27} from the underlying QCD theory, and therefore to gain a dynamical understanding of the so-called $|\Delta I| = 1/2$ rule. Although this is a very difficult task, considerable progress has been achieved recently. Applying the QCD-inspired model of Eq. (96) to the weak sector, a quite successful estimate of these two couplings has been obtained [56]. A very detailed description of this calculation, and a comparison with other approaches, can be found in ref. [56].

Once the couplings g_8 and g_{27} have been phenomenologically fixed to the values in Eq. (103), other decays like $K \to 3\pi$ or $K \to 2\pi\gamma$ can be easily predicted

[7] One can build an additional octet term with the external χ field, $\langle \lambda \left(U^\dagger \chi + \chi^\dagger U \right) \rangle$; however, this term does not contribute to on-shell amplitudes.

at $O(p^2)$. As in the strong sector, one reproduces in this way the successful soft-pion relations of Current Algebra. However, the data are already accurate enough for the next-order corrections to be sizeable. Moreover, many transitions do not occur at $O(p^2)$. For instance, due to a mismatch between the minimum number of powers of momenta required by gauge invariance and the powers of momenta that the lowest-order effective Lagrangian can provide, the amplitude for any non-leptonic radiative K-decay with at most one pion in the final state ($K \to \gamma\gamma, K \to \gamma l^+ l^-, K \to \pi\gamma\gamma, K \to \pi l^+ l^-$, ...) vanishes to lowest order in ChPT [57–59]. These decays are then sensitive to the non-trivial quantum field theory aspects of ChPT.

Unfortunately, at $O(p^4)$ there is a very large number of possible terms, satisfying the appropriate $(8_L, 1_R)$ and $(27_L, 1_R)$ transformation properties [60]. Using the $O(p^2)$ equations of motion obeyed by U to reduce the number of terms, 35 independent structures (plus 2 contact terms involving external fields only) remain in the octet sector alone [60, 61]. Restricting the attention to those terms that contribute to non-leptonic amplitudes where the only external gauge fields are photons, still leaves 22 relevant octet terms [62]. Clearly, the predictive power of a completely general chiral analysis, using only symmetry constraints, is rather limited. Nevertheless, as we are going to see in the next sections, it is still possible to make predictions.

Due to the complicated interplay of electroweak and strong interactions, the low-energy constants of the weak non-leptonic chiral Lagrangian encode a much richer information than in the pure strong sector. These chiral couplings contain both long- and short-distance contributions, and some of them (like g_8) have in addition a CP-violating imaginary part. Genuine short-distance physics, such as the electroweak penguin operators, have their corresponding effective realization in the chiral Lagrangian. Moreover, there are four $O(p^4)$ terms containing an $\varepsilon_{\mu\nu\alpha\beta}$ tensor, which get a direct (probably dominant) contribution from the chiral anomaly [63, 64].

In recent years, there have been several attempts to estimate these low-energy couplings using different approximations, such as factorization [56, 65], weak-deformation model [66], effective-action approach [56, 67], or resonance exchange [62, 68]. Although more work in this direction is certainly needed, a qualitative picture of the size of the different couplings is already emerging.

9 $K \to 2\pi, 3\pi$ decays

Imposing isospin and Bose symmetries, and keeping terms up to $O(p^4)$, a general parametrization of the $K \to 3\pi$ amplitudes involves ten measurable parameters [69], α_i, β_i, ζ_i, ξ_i, γ_3 and ξ'_3, where $i = 1, 3$ refers to the $\Delta I = 1/2, 3/2$ pieces. At $O(p^2)$, the quadratic slope parameters ζ_i, ξ_i and ξ'_3 vanish; therefore the lowest-order Lagrangian (101) predicts five $K \to 3\pi$ parameters in terms of the two couplings g_8 and g_{27}, extracted from $K \to 2\pi$. These predictions give the right qualitative pattern, but there are sizeable differences with the measured parameters. Moreover, non-zero values for some of the slope parameters have been clearly established experimentally.

The agreement is substantially improved at $O(p^4)$ [70]. In spite of the large number of unknown couplings in the general effective $\Delta S = 1$ Lagrangian, only 7 combinations of these weak chiral constants are relevant for describing the $K \to 2\pi$ and $K \to 3\pi$ amplitudes [71]. Therefore, one has 7 parameters for 12 observables, which results in 5 relations. The extent to which these relations are satisfied provides a non-trivial test of chiral symmetry at the four-derivative level. The results of such a test [71] are shown in Table 4, where the 5 conditions have been formulated as predictions for the 5 slope parameters. The comparison is very successful for the two $\Delta I = 1/2$ parameters. The data are not good enough to say anything conclusive about the other three $\Delta I = 3/2$ predictions; moreover, the possible discrepancy in the value of ξ_3 is not very significative, because this parameter is expected to be rather sensitive to electromagnetic effects, which have been omitted in the analysis.

Table 4: Predicted and measured values of the quadratic slope parameters in the $K \to 3\pi$ amplitudes [71]. All values are given in units of 10^{-8}.

Parameter	Experimental value	Prediction
ζ_1	-0.47 ± 0.15	-0.47 ± 0.18
ξ_1	-1.51 ± 0.30	-1.58 ± 0.19
ζ_3	-0.21 ± 0.08	-0.011 ± 0.006
ξ_3	-0.12 ± 0.17	0.092 ± 0.030
ξ_3'	-0.21 ± 0.51	-0.033 ± 0.077

The $O(p^4)$ analysis of these decays has also clarified the role of long-distance effects ($\pi\pi$ rescattering) in the dynamical enhancement of $\Delta I = 1/2$ amplitudes. The $O(p^4)$ corrections give indeed a sizeable constructive contribution, which results [70] in a fitted value for $|g_8|$ that is about 30% smaller than the lowest-order determination (103). While this certainly goes in the right direction, it also shows that the bulk of the enhancement mechanism comes from a different source.

10 Radiative K Decays

Owing to the constraints of electromagnetic gauge invariance, radiative K decays with at most one pion in the final state do not occur at $O(p^2)$ [57–59]. Moreover, only a few terms of the $O(p^4)$ Lagrangian are relevant for these kinds of processess

[57–59]:

$$\mathcal{L}_4^{\Delta S=1,\mathrm{em}} \doteq -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \, g_8 \Big\{ -\frac{ie}{f^2} F^{\mu\nu} \, \{ w_1 \langle Q\lambda L_\mu L_\nu \rangle + w_2 \langle Q L_\mu \lambda L_\nu \rangle \} + e^2 f^2 w_4 F^{\mu\nu} F_{\mu\nu} \langle \lambda Q U^\dagger Q U \rangle + \mathrm{h.c.} \Big\}. \quad (104)$$

The small number of unknown chiral couplings allows us to derive useful relations among different processes and to obtain definite predictions. Moreover, the absence of a tree-level $O(p^2)$ contribution makes the final results very sensitive to the loop structure of the amplitudes.

10.1 $K_S \to \gamma\gamma$

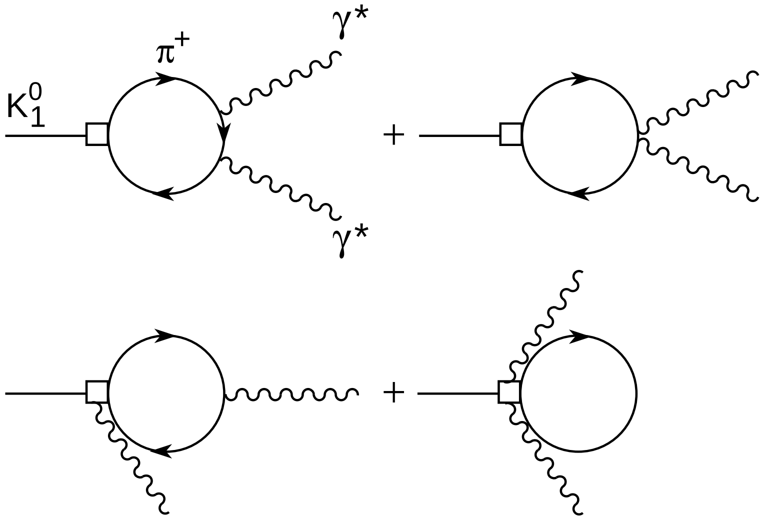

Figure 1: Feynman diagrams for $K_1^0 \to \gamma^* \gamma^*$.

The symmetry constraints do not allow any direct tree-level $K_1^0\gamma\gamma$ coupling at $O(p^4)$ ($K_{1,2}^0$ refer to the CP-even and CP-odd eigenstates, respectively). This decay proceeds then through one loop of charged pions as shown in Fig. 1 (there are similar diagrams with charged kaons in the loop, but their sum gives a zero contribution to the decay amplitude). Moreover, since there are no possible counter-terms to renormalize divergences, the one-loop amplitude is necessarily finite. Although each of the four diagrams in Fig. 1 is quadratically divergent, these divergences cancel in the sum. The resulting prediction [72] is in very good agreement with the experimental measurement [73]

$$Br(K_S \to \gamma\gamma) = \left\{ \begin{array}{cc} 2.0 \times 10^{-6} & \text{(theory)} \\ (2.4 \pm 1.2) \times 10^{-6} & \text{(experiment)} \end{array} \right. . \quad (105)$$

10.2 $K_{L,S} \to \mu^+\mu^-$

There are well-known short-distance contributions (electroweak penguins and box diagrams) to the decay $K_L \to \mu^+\mu^-$. However, this transition is dominated by long-distance physics. The main contribution proceeds through a two-photon intermediate state: $K_2^0 \to \gamma^*\gamma^* \to \mu^+\mu^-$. Contrary to $K_1^0 \to \gamma\gamma$, the prediction for the $K_2^0 \to \gamma\gamma$ decay is very uncertain, because the first non-zero contribution occurs[8] at $O(p^6)$. That makes very difficult any attempt to predict the $K_L \to \mu^+\mu^-$ amplitude.

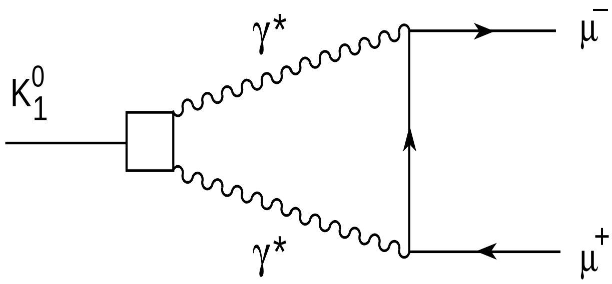

Figure 2: Feynman diagram for the $K_1^0 \to \mu^+\mu^-$ decay. The $K_1^0\gamma^*\gamma^*$ vertex is generated through the one-loop diagrams shown in Fig. 1

The situation is completely different for the K_S decay. A straightforward chiral analysis [74] shows that, at lowest order in momenta, the only allowed tree-level $K^0\mu^+\mu^-$ coupling corresponds to the CP-odd state K_2^0. Therefore, the $K_1^0 \to \mu^+\mu^-$ transition can only be generated by a finite non-local loop contribution. The two-loop calculation has been performed recently [74], with the result:

$$\frac{\Gamma(K_S \to \mu^+\mu^-)}{\Gamma(K_S \to \gamma\gamma)} = 1.9 \times 10^{-6}, \qquad \frac{\Gamma(K_S \to e^+e^-)}{\Gamma(K_S \to \gamma\gamma)} = 7.9 \times 10^{-9}, \qquad (106)$$

well below the present experimental upper limits [39]. Although, in view of the smallness of the predicted ratios, this calculation seems quite academic, it has important implications for CP-violation studies.

The longitudinal muon polarization $\mathcal{P}_L$ in the decay $K_L \to \mu^+\mu^-$ is an interesting measure of CP violation. As for every CP-violating observable in the neutral kaon system, there are in general two different kinds of contributions to $\mathcal{P}_L$: indirect CP violation through the small K_1^0 admixture of the K_L (ε effect), and direct CP violation in the $K_2^0 \to \mu^+\mu^-$ decay amplitude.

[8] At $O(p^4)$, this decay proceeds through a tree-level $K_2^0 \to \pi^0, \eta$ transition, followed by $\pi^0, \eta \to \gamma\gamma$ vertices. Because of the Gell-Mann–Okubo relation, the sum of the π^0 and η contributions cancels exactly to lowest order. The decay amplitude is then very sensitive to $SU(3)$ breaking.

In the Standard Model, the direct-CP-violating amplitude is induced by Higgs exchange with an effective one-loop flavour-changing $\bar{s}dH$ coupling [75]. The present lower bound [76] on the Higgs mass $m_H > 60$ GeV (95% C.L.), implies [75, 77] a conservative upper limit $|\mathcal{P}_{L,\mathrm{Direct}}| < 10^{-4}$. A much larger value $\mathcal{P}_L \sim O(10^{-2})$ appears quite naturally in various extensions of the Standard Model [78]. It is worth emphasizing that $\mathcal{P}_L$ is especially sensitive to the presence of light scalars with CP-violating Yukawa couplings. Thus, $\mathcal{P}_L$ seems to be a good signature to look for new physics beyond the Standard Model; for this to be the case, however, it is very important to have a good quantitative understanding of the Standard Model prediction to allow us to infer, from a measurement of $\mathcal{P}_L$, the existence of a new CP-violation mechanism.

The chiral calculation of the $K_1^0 \to \mu^+\mu^-$ amplitude allows us to make a reliable estimate[9] of the contribution to $\mathcal{P}_L$ due to K^0-$\bar{K}^0$ mixing [74]:

$$1.9 < |\mathcal{P}_{L,\varepsilon}| \times 10^3 \left(\frac{2 \times 10^{-6}}{Br(K_S \to \gamma\gamma)} \right)^{1/2} < 2.5. \tag{107}$$

Taking into account the present experimental errors in $Br(K_S \to \gamma\gamma)$ and the inherent theoretical uncertainties due to uncalculated higher-order corrections, one can conclude that experimental indications for $|\mathcal{P}_L| > 5 \times 10^{-3}$ would constitute clear evidence for additional mechanisms of CP violation beyond the Standard Model.

10.3 $K_L \to \pi^0\gamma\gamma$

Assuming CP conservation, the most general form of the amplitude for $K_2^0 \to \pi^0\gamma\gamma$ depends on two independent invariant amplitudes A and B [59],

$$\begin{aligned}
\mathcal{A}[K_L(p_K) \to \pi^0(p_0)\gamma(q_1)\gamma(q_2)] &= \\
\epsilon_\mu(q_1)\,\epsilon_\nu(q_2) \Bigg\{ \frac{A(y,z)}{M_K^2} \left(q_2^\mu q_1^\nu - q_1 \cdot q_2\, g^{\mu\nu} \right) &+ \frac{2B(y,z)}{M_K^4} \Big(p_K \cdot q_1\, q_2^\mu p_K^\nu \\
+ p_K \cdot q_2\, q_1^\nu p_K^\mu - p_K^\mu p_K^\nu\, q_1 \cdot q_2 &- p_K \cdot q_1\, p_K \cdot q_2\, g^{\mu\nu} \Big) \Bigg\},
\end{aligned} \tag{108}$$

where $y \equiv |p_K \cdot (q_1 - q_2)|/M_K^2$ and $z = (q_1 + q_2)^2/M_K^2$.

Only the amplitude $A(y,z)$ is non-vanishing to lowest non-trivial order, $O(p^4)$, in ChPT. Again, the symmetry constraints do not allow any tree-level contribution from $O(p^4)$ terms in the Lagrangian. The $A(y,z)$ amplitude is therefore determined by a finite-loop calculation [58]. The relevant Feynman diagrams are analogous to the ones in Fig. 1, but with an additional π^0 line emerging from the weak vertex; charged kaon loops also give a small contribution in this case. Due to

[9] Taking only the absorptive parts of the $K_{1,2} \to \mu^+\mu^-$ amplitudes into account, a value $|\mathcal{P}_{L,\varepsilon}| \approx 7 \times 10^{-4}$ was estimated previously [79]. However, this is only one out of four contributions to $\mathcal{P}_L$ [74], which could all interfere constructively with unknown magnitudes.

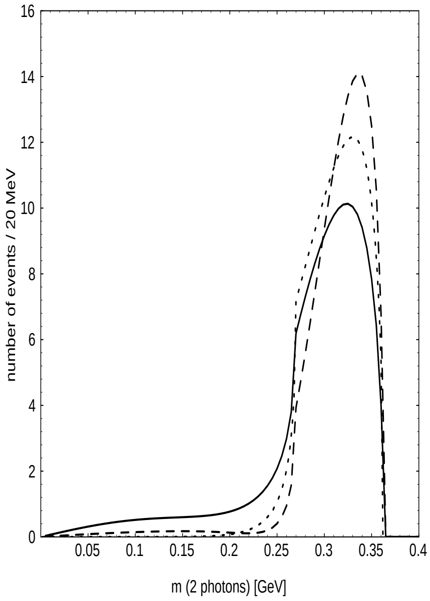

Figure 3: 2γ-invariant-mass distribution for $K_L \to \pi^0\gamma\gamma$: $O(p^4)$ (dotted curve), $O(p^6)$ with $a_V = 0$ (dashed curve), $O(p^6)$ with $a_V = -0.9$ (full curve). The spectrum is normalized to the 50 unambiguous events of NA31 (without acceptance corrections).

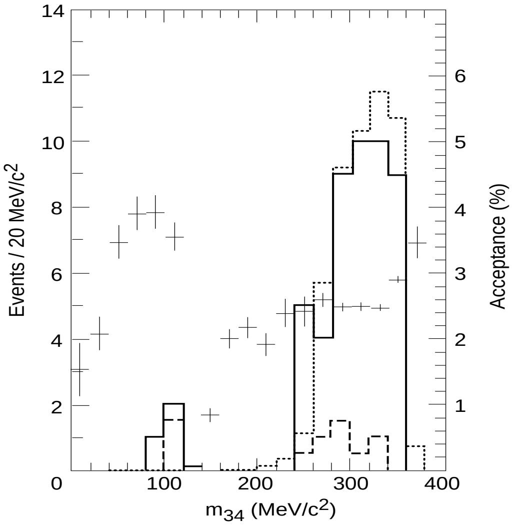

Figure 4: Measured [81] 2γ-invariant-mass distribution for $K_L \to \pi^0\gamma\gamma$ (solid line). The dashed line shows the estimated background. The experimental acceptance is given by the crosses. The dotted line simulates the $O(p^4)$ ChPT prediction.

the large absorptive $\pi^+\pi^-$ contribution, the spectrum in the invariant mass of the two photons is predicted [58, 80] to have a very characteristic behaviour (dotted line in Fig. 3), peaked at high values of $m_{\gamma\gamma}$. The agreement with the measured two-photon distribution [81], shown in Fig. 4, is remarkably good. However, the $O(p^4)$ prediction [58, 80] for the rate, $Br(K_L \to \pi^0\gamma\gamma) = 0.67 \times 10^{-6}$, is smaller than the experimental value [81, 82]:

$$Br(K_L \to \pi^0\gamma\gamma) = \begin{cases} (1.7 \pm 0.3) \times 10^{-6} & \text{NA31 [81]}, \\ (2.2 \pm 1.0) \times 10^{-6} & \text{E731 [82]}. \end{cases} \tag{109}$$

Since the effect of the amplitude $B(y, z)$ first appears at $O(p^6)$, one could worry about the size of the next-order corrections. In fact, a naïve VMD estimate through the decay chain $K_L \to \pi^0, \eta, \eta' \to V\gamma \to \pi^0\gamma\gamma$ [83] results in a sizeable contribution to $B(y, z)$ [66],

$$\begin{aligned} A(y,z)|_{\rm VMD} &= a_V \frac{G_8 M_K^2 \alpha}{\pi} \left(3 - z + \frac{M_\pi^2}{M_K^2}\right), \\ B(y,z)|_{\rm VMD} &= -2 a_V \frac{G_8 M_K^2 \alpha}{\pi}, \end{aligned} \tag{110}$$

with $a_V \approx 0.32$. However, this type of calculation predicts a photon spectrum peaked at low values of $m_{\gamma\gamma}$, in strong disagreement with experiment. As first emphasized in ref. [66], there are also so-called direct weak contributions associated with V exchange, which cannot be written as a strong VMD amplitude with an external weak transition. Model-dependent estimates of this direct contribution [66] suggest a strong cancellation with the naïve vector-meson-exchange effect, i.e. $|a_V| < 0.32$; but the final result is unfortunately quite uncertain.

A detailed calculation of the most important $O(p^6)$ corrections has been performed recently [84]. In addition to the VMD contribution, the unitarity corrections associated with the two-pion intermediate state (i.e. $K_L \to \pi^0\pi^+\pi^- \to \pi^0\gamma\gamma$) have been included[10]. Figure 3 shows the resulting photon spectrum for $a_V = 0$ (dashed curve) and $a_V = -0.9$ (full curve). The predicted branching ratio is:

$$BR(K_L \to \pi^0\gamma\gamma) = \begin{cases} 0.67 \times 10^{-6}, & O(p^4), \\ 0.83 \times 10^{-6}, & O(p^6),\, a_V = 0, \\ 1.60 \times 10^{-6}, & O(p^6),\, a_V = -0.9. \end{cases} \qquad (111)$$

The unitarity corrections by themselves raise the rate only moderately. Moreover, they produce an even more pronounced peaking of the spectrum at large $m_{\gamma\gamma}$, which tends to ruin the success of the $O(p^4)$ prediction. The addition of the V exchange contribution restores again the agreement. Both the experimental rate and the spectrum can be simultaneously reproduced with $a_V = -0.9$.

10.4 $K \to \pi l^+ l^-$

In contrast to the previous processes, the $O(p^4)$ calculation of $K^+ \to \pi^+ l^+ l^-$ and $K_S \to \pi^0 l^+ l^-$ involves a divergent loop, which is renormalized by the $O(p^4)$ Lagrangian. The decay amplitudes can then be written [57] as the sum of a calculable loop contribution plus an unknown combination of chiral couplings,

$$\begin{aligned} w_+ &= -\frac{1}{3}(4\pi)^2[w_1^r + 2w_2^r - 12L_9^r] - \frac{1}{3}\log\left(M_K M_\pi/\mu^2\right), \qquad (112) \\ w_S &= -\frac{1}{3}(4\pi)^2[w_1^r - w_2^r] - \frac{1}{3}\log\left(M_K^2/\mu^2\right), \end{aligned}$$

where w_+, w_S refer to the decay of the K^+ and K_S respectively. These constants are expected to be of order 1 by naïve power-counting arguments. The logarithms have been included to compensate the renormalization-scale dependence of the chiral couplings, so that w_+, w_S are observable quantities. If the final amplitudes are required to transform as octets, then $w_2 = 4L_9$, implying $w_S = w_+ + \log(M_\pi/M_K)/3$ [57]. It should be emphasized that this relation goes beyond the usual requirement of chiral invariance.

[10] The charged-pion loop has also been computed in ref. [85].

The measured $K^+ \to \pi^+ e^+ e^-$ decay rate determines two possible solutions for w_+ [57]. The same parameter w_+ regulates [57] the shape of the invariant-mass distribution of the final lepton pair. A fit to the recent BNL E777 data [86] gives

$$w_+ = 0.89^{+0.24}_{-0.14}, \tag{113}$$

solving the previous two-fold ambiguity in favour of the positive solution, as expected from model-dependent theoretical estimates [66]. Once w_+ has been fixed, one can make predictions [57] for the rates and Dalitz-plot distributions of the related modes $K^+ \to \pi^+ \mu^+ \mu^-$, $K_S \to \pi^0 e^+ e^-$ and $K_S \to \pi^0 \mu^+ \mu^-$.

The rare decay $K_L \to \pi^0 e^+ e^-$ is an interesting process in looking for new CP-violating signatures. If CP were an exact symmetry, only the CP-even state K_1^0 could decay via one-photon emission, while the decay of the CP-odd state K_2^0 would proceed through a two-photon intermediate state and, therefore, its decay amplitude would be suppressed by an additional power of α. When CP-violation is taken into account, however, an $O(\alpha)$ $K_L \to \pi^0 e^+ e^-$ decay amplitude is induced, both through the small K_1^0 component of the K_L (ε effect) and through direct CP-violation in the $K_2^0 \to \pi^0 e^+ e^-$ transition. The electromagnetic suppression of the CP-conserving amplitude then makes it plausible that this decay is dominated by the CP-violating contributions.

The short-distance analysis of the product of weak and electromagnetic currents allows a reliable estimate of the direct CP-violating $K_2^0 \to \pi^0 e^+ e^-$ amplitude. The corresponding branching ratio induced by this amplitude has been estimated [87] to be around

$$Br(K_L \to \pi^0 e^+ e^-)\Big|_{\mathrm{Direct}} \simeq 5 \times 10^{-12}, \tag{114}$$

the exact number depending on the values of m_t and the quark-mixing angles.

The indirect CP-violating amplitude induced by the K_1^0 component of the K_L is given by the $K_S \to \pi^0 e^+ e^-$ amplitude times the CP-mixing parameter ε. Using the octet relation between w_+ and w_S, the determination of the parameter ω_+ in Eq. (113) implies

$$Br(K_L \to \pi^0 e^+ e^-)\Big|_{\mathrm{Indirect}} \leq 1.6 \times 10^{-12}. \tag{115}$$

Comparing this value with the one in Eq. (114), we see that the interesting direct CP-violating contribution is expected to be bigger than the indirect one. This is very different from the situation in $K \to \pi\pi$, where the contribution due to mixing completely dominates.

The present experimental upper bound [88] (90% C.L.)

$$Br(K_L \to \pi^0 e^+ e^-)\Big|_{\mathrm{Exp}} < 5.5 \times 10^{-9}, \tag{116}$$

is still far away from the expected Standard Model signal, but the prospects for getting the needed sensitivity of around 10^{-12} in the next few years are rather encouraging. In order to be able to interpret a future experimental measurement of this decay as a CP-violating signature, it is first necessary, however, to pin down the actual size of the two-photon exchange CP-conserving amplitude.

Using the computed $K_L \to \pi^0\gamma\gamma$ amplitude, one can estimate the two-photon exchange contribution to $K_L \to \pi^0 e^+e^-$, by taking the absorptive part due to the two-photon discontinuity as an educated guess of the actual size of the complete amplitude. At $O(p^4)$, the $K_L \to \pi^0 e^+e^-$ decay amplitude is strongly suppressed (it is proportional to m_e), owing to the helicity structure of the $A(y,z)$ term [59]:

$$Br(K_L \to \pi^0\gamma^*\gamma^* \to \pi^0 e^+e^-)\Big|_{O(p^4)} \sim 5 \times 10^{-15}. \tag{117}$$

This helicity suppression is, however, no longer true at the next order in the chiral expansion. The $O(p^6)$ estimate of the amplitude $B(y,z)$ [84] gives rise to

$$Br(K_L \to \pi^0\gamma^*\gamma^* \to \pi^0 e^+e^-)\Big|_{O(p^6)} \sim \begin{cases} 0.3 \times 10^{-12}, & a_V = 0, \\ 1.8 \times 10^{-12}, & a_V = -0.9. \end{cases} \tag{118}$$

Although the rate increases of course with $|a_V|$, there is some destructive interference between the unitarity corrections of $O(p^6)$ and the V-exchange contribution (for $a_V = -0.9$). In order to get a more accurate estimate, it would be necessary to make a careful fit to the $K_L \to \pi^0\gamma\gamma$ data, taking the experimental acceptance into account, to extract the actual value of a_V.

11 The chiral anomaly in non-leptonic K decays

The chiral anomaly also appears in the non-leptonic weak interactions. A systematic study of all non-leptonic K decays where the anomaly contributes at leading order, $O(p^4)$, has been performed recently [63]. Only radiative K decays are sensitive to the anomaly in the non-leptonic sector.

The manifestations of the anomaly can be grouped in two different classes of anomalous amplitudes: reducible and direct contributions. The reducible amplitudes arise from the contraction of meson lines between a weak $\Delta S = 1$ vertex and the Wess-Zumino-Witten functional (61). In the octet limit, all reducible anomalous amplitudes of $O(p^4)$ can be predicted in terms of the coupling g_8. The direct anomalous contributions are generated through the contraction of the W boson field between a strong Green function on one side and the Wess–Zumino–Witten functional on the other. Their computation is not straightforward, because of the presence of strongly interacting fields on both sides of the W. Nevertheless, due to the non-renormalization theorem of the chiral anomaly [89], the bosonized form of the direct anomalous amplitudes can be fully predicted [64]. In spite of its anomalous origin, this contribution is chiral-invariant. The anomaly turns out to contribute to all possible octet terms of $\mathcal{L}_4^{\Delta S=1}$ proportional to the $\varepsilon_{\mu\nu\alpha\beta}$ tensor. Unfortunately, the coefficients of these terms get also non-factorizable contributions of non-anomalous origin, which cannot be computed in a model-independent way. Therefore, the final predictions can only be parametrized in terms of four dimensionless chiral couplings, which are expected to be positive and of order one.

The most frequent "anomalous" decays $K_L \to \pi^+\pi^-\gamma$ and $K^+ \to \pi^+\pi^0\gamma$ share the remarkable feature that the normally dominant bremsstrahlung amplitude is strongly suppressed, making the experimental verification of the anomalous amplitude substantially easier. This suppression has different origins: $K^+ \to \pi^+\pi^0$ proceeds through the small 27-plet part of the non-leptonic weak interactions, whereas $K_L \to \pi^+\pi^-$ is CP-violating. The remaining non-leptonic K decays with direct anomalous contributions are either suppressed by phase space [$K^+ \to \pi^+\pi^0\pi^0\gamma(\gamma)$, $K^+ \to \pi^+\pi^+\pi^-\gamma(\gamma)$, $K_L \to \pi^+\pi^-\pi^0\gamma$, $K_S \to \pi^+\pi^-\pi^0\gamma(\gamma)$] or by the presence of an extra photon in the final state [$K^+ \to \pi^+\pi^0\gamma\gamma$, $K_L \to \pi^+\pi^-\gamma\gamma$]. A detailed phenomenological analysis of these decays can be found in ref. [63].

12 Interactions of a light Higgs

The hadronic couplings of a light Higgs particle are fixed by low-energy theorems [90–93], which relate the $\phi \to \phi' h^0$ transition with a zero-momentum Higgs to the corresponding $\phi \to \phi'$ coupling. Although, within the Standard Model, the possibility of a light Higgs boson is already excluded [76], an extended scalar sector with additional degrees of freedom could easily avoid the present experimental limits, leaving the question of a light Higgs open to any speculation.

The quark–Higgs interaction can be written down in the general form

$$\mathcal{L}_{h^0\bar{q}q} = -\frac{h^0}{u} \left\{ k_d \, \bar{d} M_d d + k_u \, \bar{u} M_u u \right\} , \tag{119}$$

where $u = (\sqrt{2}G_F)^{-1/2} \approx 246\,\mathrm{GeV}$, M_u and M_d are the diagonal mass matrices for up- and down-type quarks respectively, and the couplings k_u and k_d depend on the model considered. In the Standard Model, $k_u = k_d = 1$, while in the usual two-Higgs-doublet models (without tree-level flavour-changing neutral currents) $k_d = k_u = \cos\alpha / \sin\beta$ (model I) or $k_d = -\sin\alpha/\cos\beta$, $k_u = \cos\alpha/\sin\beta$ (model II), where α and β are functions of the parameters of the scalar potential.

The couplings of h^0 to the octet of pseudoscalar mesons can be easily worked out, using ChPT techniques. The Yukawa interactions of the light-quark flavours can be trivially incorporated through the external scalar field s, together with the light-quark-mass matrix $\mathcal{M}$:

$$s = \mathcal{M} \left\{ 1 + \frac{h^0}{u} (k_d A + k_u B) \right\} , \tag{120}$$

where $A \equiv \mathrm{diag}(0,1,1)$ and $B \equiv \mathrm{diag}(1,0,0)$. It remains to compute the contribution from the heavy flavours c, b, t. Their Yukawa interactions induce a Higgs–gluon coupling through heavy-quark loops,

$$\mathcal{L}_{h^0GG} = \frac{\alpha_s}{12\pi} \left(n_d k_d + n_u k_u \right) \frac{h^0}{u} G^a_{\mu\nu} G^{\mu\nu}_a . \tag{121}$$

Here, $n_d = 1$ and $n_u = 2$ are the number of heavy quarks of type down and up respectively. The operator $G^a_{\mu\nu}G^{\mu\nu}_a$ can be related to the trace of the energy-momentum tensor; in the three light-flavour theory, one has

$$\Theta^\mu_\mu = -\frac{b\alpha_s}{8\pi} G^a_{\mu\nu}G^{\mu\nu}_a + \bar{q}\mathcal{M}q, \tag{122}$$

where $b = 9$ is the first coefficient of the QCD β-function. To obtain the low-energy representation of $\mathcal{L}_{h^0GG}$ it therefore suffices to replace Θ^μ_μ and $\bar{q}\mathcal{M}q$ by their corresponding expressions in the effective chiral Lagrangian theory. One gets [90–92],

$$\mathcal{L}^{\rm eff}_{h^0GG} = \xi \frac{h^0}{u}\frac{f^2}{2} \left\{\langle D_\mu U^\dagger D^\mu U\rangle + 3B_0\langle U^\dagger \mathcal{M} + \mathcal{M}U\rangle\right\}. \tag{123}$$

The information on the heavy quarks, which survives in the low-energy limit, is contained in the coefficient $\xi \equiv 2(n_d k_d + n_u k_u)/(3b) = 2(k_d + 2k_u)/27$.

Using the chiral formalism, the present experimental constraints on a very light neutral scalar have been investigated in refs. [92] and [93], in the context of two-Higgs-doublet models. A Higgs in the mass range $2m_\mu < m_{h^0} < 2M_\pi$ can be excluded (within model II), analysing the decay $\eta \to \pi^0 h^0$ [92]. A more general analysis [93], using the light-Higgs production channels $Z \to Z^* h^0$, $\eta' \to \eta h^0$, $\eta \to \pi^0 h^0$ and $\pi \to e\nu h^0$, allows us to exclude a large area in the parameter space (α, β, m_{h^0}) of both models (I and II) for $m_{h^0} < 2m_\mu$.

13 Effective theory at the electroweak scale

In spite of the spectacular success of the Standard Model, we still do not really understand the dynamics underlying the electroweak symmetry breaking $SU(2)_L \otimes U(1)_Y \to U(1)_{\rm em}$. The Higgs mechanism provides a renormalizable way to generate the W and Z masses and, therefore, their longitudinal degrees of freedom. However, an experimental verification of this mechanism is still lacking.

The scalar sector of the Standard Model Lagrangian can be written in the form

$$\mathcal{L}(\Phi) = \frac{1}{2}\langle D^\mu \Sigma^\dagger D_\mu \Sigma\rangle - \frac{\lambda}{16}\left(\langle \Sigma^\dagger\Sigma\rangle - u^2\right)^2, \tag{124}$$

where

$$\Sigma \equiv \begin{pmatrix} \Phi^0 & \Phi^+ \\ \Phi^- & \Phi^{0*} \end{pmatrix} \tag{125}$$

and $D_\mu\Sigma$ is the usual gauge-covariant derivative

$$D_\mu\Sigma \equiv \partial_\mu\Sigma + ig\frac{\vec{\tau}}{2}\,\vec{W}_\mu\,\Sigma - ig'\Sigma\frac{\tau_3}{2}B_\mu. \tag{126}$$

In the limit where the coupling g' is neglected, $\mathcal{L}(\Phi)$ is invariant under global $G \equiv SU(2)_L \otimes SU(2)_C$ transformations,

$$\Sigma \xrightarrow{G} g_L \, \Sigma \, g_C^\dagger , \qquad\qquad g_{L,C} \in SU(2)_{L,C} \tag{127}$$

($SU(2)_C$ is the so-called custodial-symmetry group). The symmetry properties of $\mathcal{L}(\Phi)$ are very similar to the ones of the linear-sigma-model Lagrangian (15). Performing an analogous polar decomposition [see Eqs. (17)],

$$\begin{aligned} \Sigma(x) &= \frac{1}{\sqrt{2}} \left(u + H(x)\right) \, U(\phi(x)), \\ U(\phi(x)) &= \exp\left(i\vec{\tau}\vec{\phi}(x)/u\right), \end{aligned} \tag{128}$$

in terms of the Higgs field H and the Goldstones $\vec{\phi}$, and taking the limit $\lambda >> 1$ (heavy Higgs), we can rewrite $\mathcal{L}(\Phi)$ in the standard chiral form:

$$\mathcal{L}(\Phi) = \frac{u^2}{4} \langle D_\mu U^\dagger D^\mu U \rangle + O\left(H\right). \tag{129}$$

In the unitary gauge $U = 1$, this $O(p^2)$ Lagrangian reduces to the usual bilinear gauge-mass term.

As we know already, (129) is the universal model-independent interaction of the Goldstone bosons induced by the assumed pattern of SCSB, $SU(2)_L \otimes SU(2)_C \longrightarrow SU(2)_{L+C}$. The scattering of electroweak Goldstone bosons (or equivalently longitudinal gauge bosons) is then described by the same formulae as the scattering of pions, changing f by u [94]. To the extent that the present data are still not sensitive to the virtual Higgs effects, we have only tested up to now the symmetry properties of the scalar sector encoded in Eq. (129).

In order to really prove the particular scalar dynamics of the Standard Model, we need to test the model-dependent part involving the Higgs field H. If the Higgs turns out to be too heavy to be directly produced (or if it does not exist at all!), one could still investigate the higher-order effects [95–103] by applying the standard chiral-expansion techniques in a completely straightforward way. The Standard Model gives definite predictions for the corresponding chiral couplings of the $O(p^4)$ Lagrangian, which could be tested in future high-precision experiments. It remains to be seen if the experimental determination of the higher-order electroweak chiral couplings will confirm the renormalizable Standard Model Lagrangian, or will constitute an evidence of new physics

14 Summary

ChPT is a powerful tool to study the low-energy interactions of the pseudoscalar-meson octet. This effective Lagrangian framework incorporates all the constraints implied by the chiral symmetry of the underlying Lagrangian at the quark–gluon level, allowing for a clear distinction between genuine aspects of the Standard Model and additional assumptions of variable credibility, usually related to the problem of long-distance dynamics. The low-energy amplitudes of the Standard Model are calculable in ChPT, except for some coupling constants which are not restricted by chiral symmetry. These constants reflect our lack of understanding of the QCD confinement mechanism and must be determined experimentally for the time being. Further progress in QCD can only improve our knowledge of these chiral constants, but it cannot modify the low-energy structure of the amplitudes.

ChPT provides a convenient language to improve our understanding of the long-distance dynamics. Once the chiral couplings are experimentally known, one can test different dynamical models, by comparing the predictions that they give for those couplings with their phenomenologically determined values. The final goal would be, of course, to derive the low-energy chiral constants from the Standard Model Lagrangian itself. Although this is a very difficult problem, the recent attempts done in this direction look quite promising.

It is important to emphasize that:

1. ChPT is not a model. The effective Lagrangian generates the more general S-matrix elements consistent with analyticity, perturbative unitarity and the assumed symmetries. Therefore, ChPT is the effective theory of the Standard Model at low energies.

2. The experimental verification of the ChPT predictions does not provide a test of the detailed dynamics of the Standard Model; only the implications of the underlying symmetries are being proved. Any other model with identical chiral-symmetry properties would give rise to the same low-energy structure.

3. The dynamical information on the underlying fundamental Lagrangian is encoded in the chiral couplings. Different short-distance models with identical symmetry properties will result in the same effective Lagrangian, but with different values for the low-energy couplings. In order to actually test the non-trivial low-energy dynamics of the Standard Model, one needs first to know the Standard Model predictions for the chiral couplings.

In these lectures I have presented the basic formalism of ChPT and some selected phenomenological applications. There are many more applications of the chiral framework. Any system which contains Goldstone bosons can be studied in a similar way. A discussion of further topics in ChPT can be found in refs. [3–14].

References

[1] E. Euler, Ann. Phys. (Leipzig) 26 (1936) 398;
E. Euler and W. Heisenberg, Z. Phys. 98 (1936) 714.

[2] S. Weinberg, Physica 96A (1979) 327.

[3] J. Bijnens, Int. J. Mod. Phys. A8 (1993) 3045.

[4] J.F. Donoghue, "Chiral symmetry as an experimental science", in "Medium energy antiprotons and the quark–gluon structure of hadrons", eds. R. Landua, J.M. Richard and R. Klapisch (Plenum Press, New York, 1991) p. 39.

[5] G. Ecker, "Chiral Perturbation Theory", in "Quantitative Particle Physics", eds. M. Levy et al. (Plenum Publ. Co.,New York, 1993);
"The Standard Model at low energies", lectures given at the 6$^{\mathrm{th}}$ Summer School on Intermediate Energy Physics, Prague, 1993, Wien preprint UWThPh-1993-31 (to appear in Czech J. Phys.);
"Chiral realization of the non-leptonic weak interactions", in Proc. 24th Int. Symposium on the theory of elementary particles, ed. G. Weigt (Zeuthen, 1991).

[6] J. Gasser, "The QCD vacuum and chiral symmetry", in "Hadrons and Hadronic Matter", eds. D. Vautherin et al. (Plenum Press, New York, 1990) p. 87;
"Chiral dynamics", in Proc. Workshop on Physics and Detectors for DAΦNE, ed. G. Pancheri (Frascati, 1991) p. 291.

[7] H. Leutwyler, "Chiral Effective Lagrangians", in "Recent Aspects of Quantum Fields", eds. H. Mitter and M. Gausterer, Lecture Notes in Physics, vol. 396 (Springer Verlag, Berlin, 1991).

[8] U.G. Meißner, "Recent Developments in Chiral Perturbation Theory", Bern preprint BUTP-93/01.

[9] A. Pich, Nucl. Phys. B (Proc. Suppl.) 23A (1991) 399;
"η decays and chiral Lagrangians", in Proc. Workshop on Rare Decays of Light Mesons", ed. B. Mayer (Gif-sur-Yvette, 1990), p. 43.

[10] E. de Rafael, Nucl. Phys. B (Proc. Suppl.) 10A (1989) 37.

[11] A.J. Buras, J.-M. Gerard and W. Huber (eds.), Proc. of the Ringberg Workshop on Hadronic Matrix Elements and Weak Decays, Ringberg Castle, Germany, 1988, Nucl. Phys. B (Proc. Suppl.) 7A (1989).

[12] U.-G. Meißner (ed.), Proc. of the Workshop on Effective Field Theories of the Standard Model, Dobogókö, Hungary, 1991 (World Sientific, Singapore, 1992).

[13] H. Georgi, "Weak Interactions and Modern Particle Theory" (Benjamin / Cummings, Menlo Park, 1984).

[14] J.F. Donoghue, E. Golowich and B.R. Holstein, "Dynamics of the Standard Model" (Cambridge Univ. Press, Cambridge, 1992).

[15] J. Gasser and H. Leutwyler, Nucl. Phys. B250 (1985) 465; 517; 539.

[16] A. Pich and E. de Rafael, Nucl. Phys. B367 (1991) 313.

[17] S. Adler and R.F. Dashen, "Current Algebras", (Benjamin, New York, 1968); V. de Alfaro, S. Fubini, G. Furlan and C. Rossetti, "Currents in Hadron Physics" (North-Holland, Amsterdam, 1973).

[18] J. Goldstone, Nuovo Cim. 19 (1961) 154.

[19] J. Schwinger, Ann. Phys. 2 (1957) 407;
M. Gell-Mann and M. Lévy, Nuovo Cim. 16 (1960) 705.

[20] J. Gasser and H. Leutwyler, Ann. Phys. 158 (1984) 142.

[21] S. Weinberg, Phys. Rev. Lett. 17 (1966) 616.

[22] M. Gell-Mann, R.J. Oakes and B. Renner, Phys. Rev. 175 (1968) 2195.

[23] S. Weinberg, in "A Festschrift for I.I. Rabi", ed. L. Motz (Acad. of Sciences, New York, 1977) p. 185.

[24] M. Gell-Mann, Phys. Rev. 106 (1957) 1296;
S. Okubo, Prog. Theor. Phys. 27 (1962) 949.

[25] R. Dashen, Phys. Rev. 183 (1969) 1245.

[26] J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95;
E. Witten, Nucl. Phys. B223 (1983) 422.

[27] S.L. Adler, Phys. Rev. 177 (1969) 2426;
J.S. Bell and R. Jackiw, Nuovo Cim. 60A (1969) 47.

[28] W.A. Bardeen, Phys. Rev. 184 (1969) 1848.

[29] A. Manohar and H. Georgi, Nucl. Phys. B234 (1984) 189.

[30] J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95.

[31] E. Witten, Nucl. Phys. B223 (1983) 422.

[32] J. Bijnens, G. Ecker and J. Gasser, "Introduction to Chiral Symmetry", in "The DAΦNE Physics Handbook", eds. L. Maiani, G. Pancheri and N. Paver (Frascati 1992) Vol. I, p. 107.

[33] J. Bijnens and F. Cornet, Nucl. Phys. B296 (1988) 557;
J. Bijnens, Nucl. Phys. B337 (1990) 635;
C. Riggenbach, J. Gasser, J.F. Donoghue and B.R. Holstein, Phys. Rev. D43 (1991) 127.

[34] H. Leutwyler and M. Roos, Z. Phys. C25 (1984) 91.

[35] W.R. Molzon et al., Phys. Rev. Lett. 41 (1978) 1213.

[36] S. R. Amendolia et al., Nucl. Phys. B277 (1986) 168.

[37] H. Leutwyler, Nucl. Phys. B (Proc. Suppl.) 7A (1989) 42.

[38] E.B. Dally et al., Phys. Rev. Lett. 45 (1980) 232; 48 (1982) 375.

[39] Particle Data Group, "Review of Particle Properties", Phys. Rev. D45 (1992) Part II.

[40] G. Donaldson et al., Phys. Rev. D9 (1974) 2960.

[41] Y. Cho et al., Phys. Rev. D22 (1980) 2688.

[42] D.B. Kaplan and A.V. Manohar, Phys. Rev. Lett. 56 (1986) 2004.

[43] J. Gasser and H. Leutwyler, Phys. Rep. 87C (1982) 77.

[44] H. Leutwyler, Nucl. Phys. B337 (1990) 108.

[45] J.F. Donoghue and D. Wyler, Phys. Rev. D45 (1992) 892.

[46] G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B321 (1989) 311.

[47] J.F. Donoghue, C. Ramirez and G. Valencia, Phys. Rev. D39 (1989) 1947.

[48] S. Weinberg, Phys. Rev. Lett. 18 (1967) 507.

[49] U.-G. Meißner, Phys. Rep. 161 (1988) 213.

[50] M. Bando, T. Kugo and K. Yamawaki, Phys. Rep. 164 (1988) 115.

[51] G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, Phys. Lett. B223 (1989) 425.

[52] D. Espriu, E. de Rafael and J. Taron, Nucl. Phys. B345 (1990) 22.

[53] J. Bijnens, C. Bruno and E. de Rafael, Nucl. Phys. B390 (1993) 501.

[54] F.J. Gilman and M.B. Wise, Phys. Rev. D20 (1979) 2392.

[55] A. Pich, B. Guberina and E. de Rafael, Nucl. Phys. B277 (1986) 197.

[56] A. Pich and E. de Rafael, Nucl. Phys. B358 (1991) 311.

[57] G. Ecker, A. Pich and E. de Rafael, Nucl. Phys. B291 (1987) 692.

[58] G. Ecker, A. Pich and E. de Rafael, Phys. Lett. B189 (1987) 363.

[59] G. Ecker, A. Pich and E. de Rafael, Nucl. Phys. B303 (1988) 665.

[60] J. Kambor, J. Missimer and D. Wyler, Nucl. Phys. B346 (1990) 17.

[61] G. Ecker, "Geometrical aspects of the non-leptonic weak interactions of mesons", in Proc. of the IX Int. Conference on the Problems of Quantum Field Theory, Dubna, 1990, ed. M.K. Volkov (JINR, Dubna, 1990);
G. Esposito-Farèse, Z. Phys. C50 (1991) 255.

[62] G. Ecker, J. Kambor and D. Wyler, Nucl. Phys. B394 (1993) 101.

[63] G. Ecker, H. Neufeld and A. Pich, "Non-Leptonic Kaon Decays and the Chiral Anomaly", CERN-TH.6920/93; Phys. Lett. B278 (1992) 337.

[64] J. Bijnens, G. Ecker and A. Pich, Phys. Lett. B286 (1992) 341.

[65] S. Fajfer and J.-M. Gérard, Z. Phys. C42 (1989) 425;
H.-Y. Cheng, Phys. Rev. D42 (1990) 72.

[66] G. Ecker, A. Pich and E. de Rafael, Phys. Lett. B237 (1990) 481.

[67] C. Bruno and J. Prades, Z. Phys. C57 (1993) 585.

[68] G. Isidori and A. Pugliese, Nucl. Phys. B385 (1992) 437.

[69] T.J. Devlin and J.O. Dickey, Rev. Mod. Phys. 51 (1979) 237.

[70] J. Kambor, J. Missimer and D. Wyler, Phys. Lett. B261 (1991) 496.

[71] J. Kambor, J.F. Donoghue, B.R. Holstein, J. Missimer and D. Wyler, Phys. Rev. Lett. 68 (1992) 1818.

[72] G. D'Ambrosio and D. Espriu, Phys. Lett. B175 (1986) 237;
J.L. Goity, Z. Phys. C34 (1987) 341.

[73] H. Burkhardt et al. (CERN NA31), Phys. Lett. B199 (1987) 139.

[74] G. Ecker and A. Pich, Nucl. Phys. B366 (1991) 189.

[75] F.J. Botella and C.S. Lim, Phys. Rev. Lett. 56 (1986) 1651.

[76] T. Mori, "Searches for the Standard Model Higgs Boson at LEP", in Proc. XXVI Int. Conf. on High Energy Physics, Dallas, 1992, ed. J.R. Sanford, AIP Conf. Proceedings No. 272 (AIP, New York, 1993) Vol. II, p. 1321.

[77] C.Q. Geng and J.N. Ng, Phys. Rev. D39 (1989) 3330.

[78] R.N. Mohapatra, Prog. Part. Nucl. Phys. 31 (1993) 39;
C.Q. Geng and J.N. Ng, Phys. Rev. D42 (1990) 1509.

[79] P. Herczeg, Phys. Rev. D27 (1983) 1512.

[80] L. Cappiello and G. D'Ambrosio, Nuovo Cim. 99A (1988) 153.

[81] G.D. Barr et al. (CERN NA31), Phys. Lett. B284 (1992) 440; B242 (1990) 523.

[82] V. Papadimitriou et al. (FNAL E731), Phys. Rev. D44 (1991) 573.

[83] L.M. Sehgal, Phys. Rev. D38 (1988) 808; D41 (1990) 161;
T. Morozumi and H. Iwasaki, Progr. Theor. Phys. 82 (1989) 371;
J. Flynn and L. Randall, Phys. Lett. B216 (1989) 221.

[84] A. Cohen, G. Ecker and A. Pich, Phys. Lett. B304 (1993) 347.

[85] L. Cappiello, G. D'Ambrosio and M. Miragliuolo, Phys. Lett. B298 (1993) 423.

[86] C. Alliegro et al. (BNL E777), Phys. Rev. Lett. 68 (1992) 278.

[87] A.J. Buras and M.K. Harlander, "A Top Quark Story: Quark Mixing, CP Violation and Rare Decays in the Standard Model", in "Heavy Flavours", eds. A.J. Buras and M. Lindner, Advanced Series on Directions in High Energy Physics (World Scientific, Singapore 1992);
G. Buchalla, A.J. Buras and M.K. Harlander, Nucl. Phys. B349 (1991) 1;
C. Dib, I. Dunietz and F.J. Gilman, Phys. Lett. B218 (1989) 487; Phys. Rev. D39 (1989) 2639;
J. Flynn and L. Randall, Nucl. Phys. B326 (1989) 31.

[88] K.E. Ohl et al. (BNL E845), Phys. Rev. Lett. 64 (1990) 2755;
A. Barker et al. (FNAL E731), Phys. Rev. D41 (1990) 3546.

[89] S.L. Adler and W.A. Bardeen, Phys. Rev. 182 (1969) 1517.

[90] J.F. Gunion, H.E. Haber, G.L. Kane and S. Dawson, "The Higgs hunter's guide", Frontiers in Physics Lecture Note Series (Addison-Wesley, New York, 1990).

[91] R.S. Chivukula, A. Cohen, H. Georgi, B. Grinstein and A.V. Manohar, Ann. Phys. (N.Y.) 192 (1989) 93;
R.S. Chivukula, A. Cohen, H. Georgi and A.V. Manohar, Phys. Lett. B222 (1989) 258;
S. Dawson, Phys. Lett. B222 (1989) 143;
H. Leutwyler and M.A. Shifman, Phys. Lett. B221 (1989) 384; Nucl. Phys. B343 (1990) 369.

[92] J. Prades and A. Pich, Phys. Lett. B245 (1990) 117.

[93] A. Pich, J. Prades and P. Yepes, Nucl. Phys. B388 (1992) 31.

[94] M.S. Chanowitz and M.K. Gaillard, Nucl. Phys. B261 (1985) 379;
J.M. Cornwall, D.N. Levin and G. Tiktopoulos, Phys. Rev. D10 (1974) 1145.

[95] A. Longhitano, Nucl. Phys. B188 (1981) 118; Phys. Rev. D22 (1980) 1166;
T. Appelquist and C. Bernard, Phys. Rev. D22 (1980) 200.

[96] A. Dobado, D. Espriu and M.J. Herrero, Phys. Lett. B255 (1991) 405;
D. Espriu and M.J. Herrero, Nucl. Phys. B373 (1992) 117;
A. Dobado and M.J. Herrero, Phys. Lett. B228 (1989) 495; B233 (1989) 505.

[97] J.F. Donoghue and C. Ramirez, Phys. Lett. B234 (1990) 361.

[98] B. Holdon and J. Terning, Phys. Lett. B247 (1990) 88;
B. Holdon, Phys. Lett. B259 (1991) 329; B258 (1991) 156.

[99] M. Golden and L. Randall, Nucl. Phys. B361 (1991) 3.

[100] H. Georgi, Nucl. Phys. B363 (1991) 301.

[101] A. Falk, M. Luke and E. Simmons, Nucl. Phys. B365 (1991) 523.

[102] S. Dawson and G. Valencia, Nucl. Phys. B348 (1991) 23; B352 (1991) 27.

[103] A. De Rújula, M.B. Gavela, P. Hernández and E. Massó, Nucl. Phys. B384 (1992) 3.