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Isospin Breaking in $K \to \pi\pi$ Decays*

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Abstract

We perform a complete analysis of isospin breaking in $K \to 2\pi$ amplitudes in chiral perturbation theory, including both strong isospin violation ($m_u \neq m_d$) and electromagnetic corrections to next-to-leading order in the low-energy expansion. The unknown chiral couplings are estimated at leading order in the $1/N_c$ expansion. We study the impact of isospin breaking on CP conserving amplitudes and rescattering phases. In particular, we extract the effective couplings g_8 and g_{27} from a fit to $K \to \pi\pi$ branching ratios, finding small deviations from the isospin-limit case. The ratio $\mathrm{Re}A_0/\mathrm{Re}A_2$ measuring the $\Delta I = 1/2$ enhancement is found to decrease from 22.2 ± 0.1 in the isospin limit to 20.3 ± 0.5 in the presence of isospin breaking. We also analyse the effect of isospin violation on the CP violation parameter ϵ', finding a destructive interference between three different sources of isospin violation. Within the uncertainties of large-N_c estimates for the low-energy constants, the isospin violating correction for ϵ' is below 15 %.

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Contents

1 Introduction

A systematic treatment of isospin violation in nonleptonic weak interactions is needed for many phenomenological applications. The generically small effects induced by electromagnetic corrections and by the quark mass difference $m_u - m_d$ are enhanced in subdominant amplitudes with $\Delta I > 1/2$ because of the $\Delta I = 1/2$ rule. For one, a quantitative understanding of the $\Delta I = 1/2$ rule itself is only possible with isospin violating effects included. Another area of application is CP violation in the $K^0 - \overline{K^0}$ system where isospin breaking is crucial for a precision calculation of ϵ'/ϵ.

Isospin violation in $K \to 2\pi$ decays has already been addressed in recent works [1, 2, 3, 4, 5, 6, 7, 8]. In this paper, we reanalyse the $K \to \pi\pi$ decay amplitudes to perform a comprehensive study of all isospin violating effects to next-to-leading order in the low-energy expansion of the standard model. More precisely, we shall work to first order in α and in $m_u - m_d$ throughout, but to next-to-leading order in the chiral expansion. In view of the observed octet dominance of the nonleptonic weak interactions, we therefore calculate to $\mathcal{O}(G_8 p^4, G_8(m_u - m_d)p^2, e^2 G_8 p^2)$ and to $\mathcal{O}(G_{27} p^4)$ for octet and 27-plet amplitudes, respectively.

At this order, many a priori unknown low-energy constants (LECs) appear. With few exceptions to be discussed in Sec. 5, we adopt leading large-N_c estimates for the LECs. The advantage is that we employ a systematic approximation scheme with solid theoretical foundation that can in principle be carried through beyond leading order. On the other hand, the importance of subleading large-N_c effects is at present not known in general. We shall estimate the uncertainties by varying the two scales entering those estimates: the renormalization scale for evaluating Wilson coefficients (short-distance scale) and the chiral scale (long-distance scale) at which the large-N_c results are supposed to apply.

In performing electromagnetic corrections, a careful analysis of radiative events is necessary as emphasized in Ref. [5]. We shall perform such an analysis for the new KLOE measurement [9] of the ratio $\Gamma(K_S \to \pi^+\pi^-[\gamma])/\Gamma(K_S \to \pi^0\pi^0)$ with a fully inclusive

$\pi^+\pi^-[\gamma]$ final state. The KLOE result influences the phase difference $\chi_0 - \chi_2$ of the two isospin amplitudes strongly. Together with this phase difference, the effective weak octet and 27-plet couplings G_8, G_{27} will be the primary output of our analysis. With that output, several phenomenological issues can be addressed such as the relation of the phases χ_0, χ_2 to the s-wave pion-pion scattering phase shifts or the impact of isospin breaking on ϵ'/ϵ.

The content of the paper is as follows. In the subsequent section, we introduce the decay amplitudes and the relevant effective chiral Lagrangians. The amplitudes at leading order in the low-energy expansion are presented in Sec. 3. The amplitudes at next-to-leading order are investigated in Sec. 4, distinguishing between $\pi^0 - \eta$ mixing and all other contributions arising at that order. The amplitudes are divided into various parts depending on the source of isospin violation. The local amplitudes of next-to-leading order are explicitly given here. Sec. 5 analyses the LECs at leading order in $1/N_c$. To determine weak and electroweak LECs for $N_c \to \infty$, one needs input for the strong [up to $\mathcal{O}(p^6)$] and electromagnetic couplings [up to $\mathcal{O}(e^2p^2)$], in addition to the relevant Wilson coefficients. We discuss to which extent the necessary information is available. The numerical calculations of the various amplitudes to next-to-leading order in chiral perturbation theory (CHPT) are presented in Sec. 6. Dispersive and absorptive components of the loop amplitudes are given together with CP-even and CP-odd parts of the local amplitudes. Those amplitudes are then used in Sec. 7 to extract the lowest-order nonleptonic couplings G_8, G_{27} and the phase difference $\chi_0 - \chi_2$ from $K \to \pi\pi$ data. We compare those quantities at lowest and next-to-leading order, the latter with and without isospin violation included. With this information, we then analyse the relation of the phase difference $\chi_0 - \chi_2$ to the corresponding difference of $\pi\pi$ phase shifts. In Sec. 8 we discuss isospin violating contributions to the parameter ϵ' of direct CP violation in $K^0 \to \pi\pi$ decays. Sec. 9 contains our conclusions. Various technical aspects are treated in several appendices: next-to-leading-order effective chiral Lagrangians; explicit loop amplitudes; an alternative convention for LECs of lowest order; details for the analysis of the phase difference.

2 Nonleptonic weak interactions in CHPT

In this section, we define our notation for the $K \to \pi\pi$ amplitudes and we introduce the relevant effective chiral Lagrangians.

2.1 Decay amplitudes

Using the isospin decomposition of two-pion final states, we write the $K \to \pi\pi$ amplitudes in the charge basis in terms of three amplitudes[1] $\mathcal{A}_{\Delta I}$ that are generated by the ΔI component of the electroweak effective Hamiltonian in the limit of isospin conservation:

$$\begin{aligned}
\mathcal{A}_{+-} &= \mathcal{A}_{1/2} + \frac{1}{\sqrt{2}}\left(\mathcal{A}_{3/2} + \mathcal{A}_{5/2}\right) \\
\mathcal{A}_{00} &= \mathcal{A}_{1/2} - \sqrt{2}\left(\mathcal{A}_{3/2} + \mathcal{A}_{5/2}\right) \\
\mathcal{A}_{+0} &= \frac{3}{2}\left(\mathcal{A}_{3/2} - \frac{2}{3}\mathcal{A}_{5/2}\right) \ .
\end{aligned} \tag{2.1}$$

In the standard model, the $\Delta I = 5/2$ piece is absent in the isospin limit, thus reducing the number of independent amplitudes to two. Each amplitude $\mathcal{A}_n$ has a dispersive ($\mathcal{D}isp\,\mathcal{A}_n$) and an absorptive ($\mathcal{A}bs\,\mathcal{A}_n$) component. In order to carry out phenomenological applications and to keep the notation as close as possible to the standard analysis in the isospin limit, we write

$$\begin{aligned}
A_0\, e^{i\chi_0} &= \mathcal{A}_{1/2} \\
A_2\, e^{i\chi_2} &= \mathcal{A}_{3/2} + \mathcal{A}_{5/2} \\
A_2^+\, e^{i\chi_2^+} &= \mathcal{A}_{3/2} - 2/3\, \mathcal{A}_{5/2} \ ,
\end{aligned} \tag{2.2}$$

where we explicitly separate out the phases χ_I. In the limit of CP conservation, the amplitudes A_0, A_2 and A_2^+ are real and positive. In the isospin limit, the A_I are the standard isospin amplitudes and the phases χ_I are identified with the s-wave $\pi\pi$ scattering phase shifts $\delta_I(\sqrt{s} = M_K)$.

For the phenomenological analysis (see Secs. 7 and 8), we therefore adopt the following parametrization of $K \to \pi\pi$ amplitudes:

$$\begin{aligned}
\mathcal{A}_{+-} &= A_0\, e^{i\chi_0} + \frac{1}{\sqrt{2}}\, A_2\, e^{i\chi_2} \\
\mathcal{A}_{00} &= A_0\, e^{i\chi_0} - \sqrt{2}\, A_2\, e^{i\chi_2} \\
\mathcal{A}_{+0} &= \frac{3}{2}\, A_2^+\, e^{i\chi_2^+} \ .
\end{aligned} \tag{2.3}$$

This parametrization holds for the infrared-finite amplitudes where the Coulomb and infrared parts (defined in Sec. 4) have been removed from $\mathcal{A}_{+-}$.

[1] We shall use the invariant amplitudes $\mathcal{A}_n$ defined as follows:

$$\langle (\pi\pi)_n | T\left(e^{\,i \int dx\, \mathcal{L}(x)} \right) | K \rangle \ = \ i\,(2\pi)^4\, \delta^{(4)}(P_f - P_i) \ \times (-i\, \mathcal{A}_n) \ .$$

In the absence of electromagnetic interactions $\mathcal{A}_{5/2} = 0$ and therefore $A_2 = A_2^+$. To set the stage, we extract the isospin amplitudes A_0, A_2 and the phase difference $\chi_0 - \chi_2$ from a fit to the three $K \to \pi\pi$ branching ratios [9, 10]:

$$\begin{aligned} A_0 &= (2.715 \pm 0.005) \cdot 10^{-7} \text{ GeV} \\ A_2 &= (1.225 \pm 0.004) \cdot 10^{-8} \text{ GeV} \\ \chi_0 - \chi_2 &= (48.6 \pm 2.6)^\circ \ . \end{aligned} \tag{2.4}$$

These values hold in the isospin limit except that the physical pion masses have been used for phase space. The substantial reduction in the phase difference $\chi_0 - \chi_2$ (from about 58° during the past 25 years [10]) is entirely due to the new KLOE measurement of the ratio $\Gamma(K_S \to \pi^+\pi^-(\gamma))/\Gamma(K_S \to \pi^0\pi^0)$ [9].

2.2 Effective chiral Lagrangians

In the presence of isospin violation, the physics of $K \to \pi\pi$ decays involves an interplay of the nonleptonic weak, the strong and the electromagnetic interactions. Consequently, a number of effective Lagrangians are needed to describe those transitions. We use the well-known Lagrangian for strong interactions to $\mathcal{O}(p^6)$ [11, 12, 13], the nonleptonic weak Lagrangian to $\mathcal{O}(G_F p^4)$ [14, 15, 16, 17], the electromagnetic Lagrangian to $\mathcal{O}(e^2 p^2)$ [18, 19] and, finally, the electroweak Lagrangian to $\mathcal{O}(e^2 G_8 p^2)$ [20, 21, 22].

Only the leading-order (LO) Lagrangians are written down explicitly here. The relevant parts of the next-to-leading-order (NLO) Lagrangians can be found in App. A along with further details.

Strong Lagrangian:

$$\begin{aligned} \mathcal{L}_{\text{strong}} &= \frac{F^2}{4} \langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \rangle \\ &+ \sum_i L_i \, O_i^{p^4} + \sum_i X_i \, F^{-2} \, O_i^{p^6} \ . \end{aligned} \tag{2.5}$$

F is the pion decay constant in the chiral limit, the $SU(3)$ matrix field U contains the pseudoscalar fields and the scalar field χ accounts for explicit chiral symmetry breaking through the quark masses m_u, m_d, m_s. The relevant operators $O_i^{p^4}$ are listed in App. A. The LECs X_i of $O(p^6)$ will only enter through the large-N_c estimates of the electroweak couplings in Sec. 5. $\langle A \rangle$ denotes the $SU(3)$ flavour trace of A.

Nonleptonic weak Lagrangian:

$$\begin{aligned} \mathcal{L}_{\text{weak}} &= G_8 F^4 \langle \lambda D^\mu U^\dagger D_\mu U \rangle + G_{27} F^4 \left(L_{\mu 23} L_{11}^\mu + \frac{2}{3} L_{\mu 21} L_{13}^\mu \right) \\ &+ \sum_i G_8 N_i F^2 \, O_i^8 + \sum_i G_{27} D_i F^2 \, O_i^{27} \ + \ \text{h.c.} \end{aligned} \tag{2.6}$$

The matrix $L_\mu = iU^\dagger D_\mu U$ represents the octet of $V - A$ currents to lowest order in derivatives; $\lambda = (\lambda_6 - i\lambda_7)/2$ projects onto the $\bar{s} \to \bar{d}$ transition. Instead of G_8, G_{27} we will also use the dimensionless couplings g_8, g_{27} defined as

$$G_{8,27} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \, g_{8,27} \ . \tag{2.7}$$

One of the main tasks of this investigation will be the determination of g_8, g_{27} in the presence of isospin violation to NLO. The LECs N_i, D_i of $\mathcal{O}(G_F p^4)$ are dimensionless. The monomials O_i^8, O_i^{27} relevant for $K \to 2\pi$ transitions can be found in App. A.

Electromagnetic Lagrangian:

$$\begin{aligned} \mathcal{L}_{\rm elm} &= e^2 Z F^4 \langle Q U^\dagger Q U \rangle \\ &+ e^2 \sum_i K_i F^2 \, O_i^{e^2 p^2} \ . \end{aligned} \tag{2.8}$$

The quark charge matrix is given by $Q = \mathrm{diag}(2/3, -1/3, -1/3)$. The lowest-order LEC can be determined from the pion mass difference to be $Z \simeq 0.8$. The NLO LECs K_i are dimensionless and the relevant monomials $O_i^{e^2p^2}$ can again be found in App. A.

Electroweak Lagrangian:

$$\begin{aligned} \mathcal{L}_{\rm EW} &= e^2 G_8 g_{\rm ewk} F^6 \langle \lambda U^\dagger Q U \rangle \\ &+ e^2 \sum_i G_8 Z_i F^4 \, O_i^{EW} \ + \ \text{h.c.} \end{aligned} \tag{2.9}$$

The value of the LO coupling $g_{\rm ewk}$ is discussed in Sec. 5. The LECs Z_i are dimensionless and the associated monomials O_i^{EW} are collected in App. A. We do not include isospin violating corrections for 27-plet amplitudes.

The low-energy couplings L_i, N_i, D_i, K_i, Z_i are in general divergent. They absorb the divergences appearing in the one-loop graphs via the renormalization

$$\begin{aligned} L_i &= L_i^r(\nu_\chi) + \Gamma_i \, \Lambda(\nu_\chi) \\ N_i &= N_i^r(\nu_\chi) + n_i \, \Lambda(\nu_\chi) \\ D_i &= D_i^r(\nu_\chi) + d_i \, \Lambda(\nu_\chi) \\ K_i &= K_i^r(\nu_\chi) + \kappa_i \, \Lambda(\nu_\chi) \\ Z_i &= Z_i^r(\nu_\chi) + z_i \, \Lambda(\nu_\chi) \ , \end{aligned} \tag{2.10}$$

where ν_χ is the chiral renormalization scale and the divergence is included in the factor

$$\Lambda(\nu_\chi) = \frac{\nu_\chi^{d-4}}{(4\pi)^2} \left\{ \frac{1}{d-4} - \frac{1}{2}\Big[\log(4\pi) + \Gamma'(1) + 1 \Big] \right\} \ . \tag{2.11}$$

The divergent parts of the couplings are all known [11, 15, 16, 19, 22] and they allow for a nontrivial check of the loop calculation. On the other hand, many of the renormalized LECs contributing to the decay amplitudes are not known. Our strategy will be to use LO large-N_c estimates. A comprehensive discussion of all relevant LECs will be presented in Sec. 5.

3 Amplitudes at leading order $[\mathcal{O}(G_F p^2, e^2 G_8 p^0)]$

With the most general effective chiral Lagrangian of the previous section, we can now proceed with the construction of physical amplitudes. At LO in the low-energy expansion, the procedure is straightforward: chiral power counting tells us that the amplitudes are obtained by summing all tree-level Feynman diagrams with one insertion from either $\mathcal{L}_{\rm weak}$ of $\mathcal{O}(G_F p^2)$ or $\mathcal{L}_{\rm EW}$ of $\mathcal{O}(e^2 G_8 p^0)$, at most one insertion of $\mathcal{L}_{\rm elm}$ of $\mathcal{O}(e^2 p^0)$ and any number of insertions from the $\mathcal{O}(p^2)$ part of the strong Lagrangian (2.5).

In addition to contributions proportional to the electroweak coupling $g_{\rm ewk}$, isospin breaking occurs also in the pseudoscalar mass matrix, generating in particular non-diagonal terms in the fields (π_3, π_8) ($\pi^0 - \eta$ mixing). Upon diagonalizing the tree-level mass matrix one obtains the relation between the LO mass eigenfields (π^0, η) and the original fields (π_3, π_8) (to first order in $m_u - m_d$):

$$\begin{pmatrix} \pi_3 \\ \pi_8 \end{pmatrix} = \begin{pmatrix} 1 & -\varepsilon^{(2)} \\ \varepsilon^{(2)} & 1 \end{pmatrix} \begin{pmatrix} \pi^0 \\ \eta \end{pmatrix}_{\rm LO} , \qquad (3.1)$$

with the tree-level $\pi^0 - \eta$ mixing angle $\varepsilon^{(2)}$ given by

$$\varepsilon^{(2)} = \frac{\sqrt{3}}{4} \, \frac{m_d - m_u}{m_s - \widehat{m}}, \qquad (3.2)$$

where $\widehat{m}$ stands for the mean value of the light quark masses,

$$\widehat{m} = \frac{1}{2}(m_u + m_d) \ . \qquad (3.3)$$

The physical amplitudes are then obtained by considering the relevant Feynman graphs with insertions from the LO effective Lagrangian expressed in terms of the LO mass eigenfields.

Apart from $\pi^0 - \eta$ mixing, isospin breaking manifests itself also in the mass differences between charged and neutral mesons, due to both the light quark mass difference and electromagnetic contributions. We choose to express all masses in terms of those of the neutral kaon and pion (denoted from now on as M_K and M_π, respectively). In terms of quark masses and LO couplings (B_0 is related to the quark condensate in the chiral limit by $\langle 0|\overline{q}q|0\rangle = -F^2 B_0$), the pseudoscalar meson masses read:

$$\begin{aligned} M_\pi^2 &= 2B_0\, \widehat{m} \\ M_{\pi^\pm}^2 &= M_\pi^2 + 2\, e^2 Z F^2 \\ M_K^2 &= B_0\,(m_s + m_d) \\ M_{K^\pm}^2 &= M_K^2 - \frac{4\,\varepsilon^{(2)}}{\sqrt{3}}\, B_0(m_s - \widehat{m}) + 2\, e^2 Z F^2 \\ M_\eta^2 &= \frac{1}{3}\left(4M_K^2 - M_\pi^2\right) - \frac{8\,\varepsilon^{(2)}}{3\sqrt{3}}\, B_0(m_s - \widehat{m}) \ . \end{aligned} \qquad (3.4)$$

We are now in the position to write down the three independent amplitudes relevant for $K \to \pi\pi$ decays. In the physical "charge" basis the LO amplitudes are

$$
\begin{aligned}
\mathcal{A}_{+-} &= \frac{2}{3}\sqrt{2}\,G_{27}F\left(M_K^2 - M_\pi^2\right) + \sqrt{2}\,G_8F\left[M_K^2 - M_\pi^2 - e^2F^2\left(g_{\rm ewk} + 2Z\right)\right] \\
\mathcal{A}_{00} &= -\sqrt{2}\,G_{27}F\left(M_K^2 - M_\pi^2\right) + \sqrt{2}\,G_8F\left(M_K^2 - M_\pi^2\right)\left(1 - \frac{2}{\sqrt{3}}\,\varepsilon^{(2)}\right) \qquad (3.5) \\
\mathcal{A}_{+0} &= \frac{5}{3}\,G_{27}F\left(M_K^2 - M_\pi^2\right) + G_8F\left[\left(M_K^2 - M_\pi^2\right)\frac{2}{\sqrt{3}}\,\varepsilon^{(2)} - e^2F^2\left(g_{\rm ewk} + 2Z\right)\right] .
\end{aligned}
$$

We recall that we do not include isospin violation for the 27-plet amplitudes. In the isospin basis, more convenient for phenomenological applications, the LO amplitudes are given by (see Eq. (2.1) for the relation between the two bases)

$$
\begin{aligned}
\mathcal{A}_{1/2} &= \frac{\sqrt{2}}{9}\,G_{27}F\left(M_K^2 - M_\pi^2\right) \\
&+ \sqrt{2}\,G_8F\left[\left(M_K^2 - M_\pi^2\right)\left(1 - \frac{2}{3\sqrt{3}}\,\varepsilon^{(2)}\right) - \frac{2}{3}\,e^2F^2\left(g_{\rm ewk} + 2Z\right)\right] \qquad (3.6) \\
\mathcal{A}_{3/2} &= \frac{10}{9}\,G_{27}F\left(M_K^2 - M_\pi^2\right) + G_8F\left[\left(M_K^2 - M_\pi^2\right)\frac{4}{3\sqrt{3}}\,\varepsilon^{(2)} - \frac{2}{3}\,e^2F^2\left(g_{\rm ewk} + 2Z\right)\right] \\
\mathcal{A}_{5/2} &= 0 .
\end{aligned}
$$

The parameter F can be identified with the pion decay constant F_π at this order. The effect of strong isospin breaking (proportional to $\varepsilon^{(2)}$) is entirely due to $\pi^0 - \eta$ mixing at LO. Electromagnetic interactions contribute through mass splitting (terms proportional to Z) and insertions of $g_{\rm ewk}$. As a consequence of imposing CPS symmetry [23] on the effective Lagrangian, electromagnetic corrections to the octet weak Hamiltonian do not generate a $\Delta I = 5/2$ amplitude at LO in the quark mass expansion.

4 Amplitudes at next-to-leading order [$\mathcal{O}(G_F p^4, e^2 G_8 p^2)$]

Let us now outline the construction of NLO amplitudes. As always, chiral power counting is the guiding principle: it tells us that both one-loop and tree-level diagrams now contribute. In the one-loop diagrams, one has to consider one insertion from either $\mathcal{L}_{\rm weak}$ of $\mathcal{O}(G_F p^2)$ or $\mathcal{L}_{\rm EW}$ of $\mathcal{O}(e^2 G_8 p^0)$, at most one insertion of $\mathcal{L}_{\rm elm}$ of $\mathcal{O}(e^2 p^0)$ and any number of insertions from the $\mathcal{O}(p^2)$ part of the strong Lagrangian, with the LO effective Lagrangians expressed in terms of the LO mass eigenfields. For the tree-level diagrams, one has to apply one insertion from the NLO effective Lagrangian and any number of insertions from the LO Lagrangian. The strangeness changing vertex can come from either the LO or NLO effective Lagrangians. This straightforward prescription leads to a large number of explicit diagrams for each mode, due to several topologies and several possibilities to insert isospin breaking vertices from the LO effective Lagrangian (in the weak vertex, in the strong vertex, in the internal propagators, in the external legs). We begin with the well-defined class of NLO corrections to the pseudoscalar meson propagators, focusing afterwards on the other corrections.

4.1 $\pi^0 - \eta$ mixing at NLO

As in the LO case, it is convenient to first analyse isospin breaking in the two-point functions (inverse propagators) and to define renormalized fields in which the propagator has a diagonal form with unit residues at the poles[2]. At NLO two main new features arise:

- Not only the (π_3, π_8) mass matrix acquires off-diagonal matrix elements, but also the purely kinetic part of the propagator does so.
- Electromagnetic interactions contribute to this phenomenon, in addition to the up-down mass splitting.

Results on NLO mixing effects induced by quark mass splitting already appear in Refs. [11, 24, 8], while electromagnetically induced effects were considered in [25, 26]. Here we follow the formalism outlined in Ref. [24], treating strong and electromagnetic effects simultaneously. In the LO mass eigenfield basis, the NLO inverse propagator (an eight-by-eight matrix) can be written as follows:

$$\begin{aligned} \widehat{\Delta}(q^2)^{-1} &= q^2\,\mathbf{1} - \widehat{M}^2 - \widehat{\Pi}(q^2) \\ \widehat{\Pi}(q^2) &= \widehat{C}\,q^2 + \widehat{D}\,, \end{aligned} \tag{4.1}$$

where $\widehat{M}^2$ is the diagonal LO mass matrix and $\widehat{C}$, $\widehat{D}$ are symmetric matrices, which are diagonal except for their restriction to the (π^0, η) subspace. The relation between the LO and NLO mass eigenfields (collected in a vector ϕ_a) can be summarized as follows:

$$\phi_{\rm LO} = \left(\mathbf{1} + \frac{\widehat{C}}{2} + \widehat{W}\right)\,\phi_{\rm NLO}\ , \tag{4.2}$$

where $\widehat{W}$ is an antisymmetric matrix, non-vanishing only in the (π^0, η) subspace. Except for the reduction to this subspace, Eq. (4.2) is just the familiar field renormalization, with wave function renormalization given by $Z_i = 1 + \widehat{C}_{ii}$. Focusing on the (π^0, η) sector, we note that $\widehat{W}$ is characterized by a single entry called $\varepsilon^{(4)}$ [24]. This quantity is UV finite and represents a natural generalization to $\mathcal{O}(p^4)$ of the tree-level mixing angle $\varepsilon^{(2)}$. Explicitly, the relation between the (π^0, η) mass eigenfields at LO and NLO is given by

$$\left(\begin{array}{c} \pi^0 \\ \eta \end{array}\right)_{\rm LO} = \left(\begin{array}{cc} 1 + \dfrac{\widehat{C}_{\pi^0\pi^0}}{2} & -\varepsilon^{(4)} + \dfrac{\widehat{C}_{\eta\pi^0}}{2} \\ \varepsilon^{(4)} + \dfrac{\widehat{C}_{\eta\pi^0}}{2} & 1 + \dfrac{\widehat{C}_{\eta\eta}}{2} \end{array}\right) \left(\begin{array}{c} \pi^0 \\ \eta \end{array}\right)_{\rm NLO} . \tag{4.3}$$

[2]This way, no further wave function renormalization effect has to be included.

Eqs. (3.1) and (4.3) give the full relation between the original fields (π_3, π_8) and the NLO mass eigenfields. We do not report here the factors $\widehat{C}_{ab}$, as they are UV divergent and make sense only in combination with other terms in the full amplitudes. We do report, however, the expression for $\varepsilon^{(4)}$ because the replacement $\varepsilon^{(2)} \to \varepsilon^{(2)} + \varepsilon^{(4)}$ in Eqs. (3.5,3.6) gives rise to a well defined (UV finite) subset of the NLO corrections. Breaking up $\varepsilon^{(4)}$ into contributions from strong (S) and electromagnetic (EM) isospin breaking, one gets

$$
\begin{aligned}
\varepsilon^{(4)} &= \varepsilon^{(4)}_{\rm S} + \varepsilon^{(4)}_{\rm EM} \\
\varepsilon^{(4)}_{\rm S} &= -\frac{2\,\varepsilon^{(2)}}{3(4\pi F)^2(M_\eta^2 - M_\pi^2)}\Big\{(4\pi)^2\, 64\,[3L_7 + L_8^r(\nu_\chi)]\,(M_K^2 - M_\pi^2)^2 \\
&\quad - M_\eta^2(M_K^2 - M_\pi^2)\log\frac{M_\eta^2}{\nu_\chi^2} + M_\pi^2(M_K^2 - 3M_\pi^2)\log\frac{M_\pi^2}{\nu_\chi^2} \\
&\quad - 2M_K^2(M_K^2 - 2M_\pi^2)\log\frac{M_K^2}{\nu_\chi^2} - 2M_K^2(M_K^2 - M_\pi^2)\Big\} \qquad (4.4)\\
\varepsilon^{(4)}_{\rm EM} &= \frac{2\sqrt{3}\,\alpha}{108\,\pi\,(M_\eta^2 - M_\pi^2)}\Big\{ - 9M_K^2 Z\left(\log\frac{M_K^2}{\nu_\chi^2} + 1\right) \\
&\quad + 2M_K^2(4\pi)^2\Big[2U_2^r(\nu_\chi) + 3U_3^r(\nu_\chi)\Big] \\
&\quad + M_\pi^2(4\pi)^2\Big[2U_2^r(\nu_\chi) + 3U_3^r(\nu_\chi) - 6U_4^r(\nu_\chi)\Big]\Big\}\ .
\end{aligned}
$$

The electromagnetic LECs U_i are linear combinations of the K_i (defined in Sec. 5.5).

4.2 Remaining NLO contributions: a guided tour

Having dealt with the propagator corrections in the previous section, we now describe the remaining contributions to the $K \to \pi\pi$ amplitudes at NLO, starting with the one-loop terms. There are two main classes of contributions: loops involving only pseudoscalar mesons (Fig. 1) and loops involving virtual photons (Fig. 2). In the isospin limit, contributions to the amplitudes arise from the topologies of Fig. 1, by inserting the LO weak vertices proportional to G_8 or G_{27} in the Lagrangians (2.6) and (2.9). Given the large suppression of G_{27}/G_8, we consider in this work only isospin breaking effects generated through the octet component of the effective Lagrangian. We are therefore interested in the terms proportional to $\varepsilon^{(2)}G_8$ (strong isospin breaking) and e^2G_8 (electromagnetic isospin breaking).

Strong isospin breaking terms ($\varepsilon^{(2)}G_8$) at NLO come from several sources:

- Explicit terms $\sim (m_u - m_d)$ in the strong vertices of Fig. 1, obtained by expressing $\mathcal{L}_{\rm strong}$ in terms of the LO mass eigenfields.

- Mass corrections in the internal propagators, for which we use the LO diagonal form (and the corresponding mass relations of Eq. (3.4)).

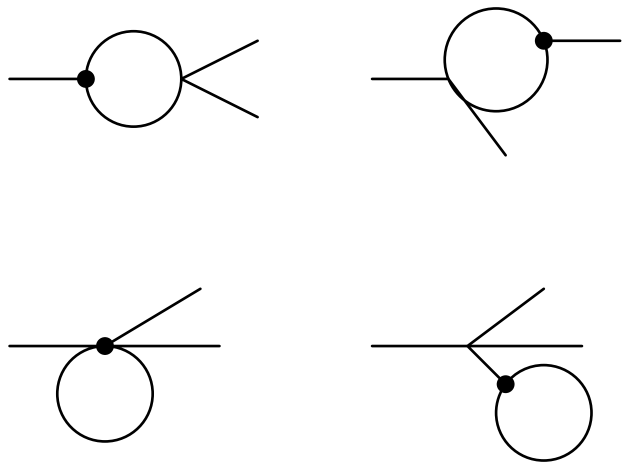

Figure 1: Topologies for purely mesonic loop diagrams contributing to $K \to \pi\pi$: the filled circles indicate $\Delta S = 1$ vertices of lowest order. Wave function renormalization diagrams are not shown.

- Mass corrections arising when external momenta are taken on-shell (using again Eq. (3.4)).

The combination of these effects leads in principle to non-linear contributions in the isospin breaking parameter. We have chosen to expand the final expressions for the amplitudes to first order in $\varepsilon^{(2)}$.

Electromagnetic isospin breaking terms ($e^2 G_8$) at NLO can be naturally divided into three categories:

- $e^2 Z G_8$: these arise exactly in the same way as the strong isospin breaking terms (see discussion above).

- $e^2 G_8 g_{\rm ewk}$: these arise from insertions of the $g_{\rm ewk}$ vertices of $\mathcal{L}_{\rm EW}$ in the topologies of Fig. 1, keeping all other contributions (masses and strong vertices) in the isospin limit.

- $e^2 G_8$: these arise from the photonic diagrams of Fig. 2, using the LO weak vertices of $\mathcal{L}_{\rm weak}$ proportional to G_8. This class of contributions to $\mathcal{A}_{+-}$ is infrared divergent. We regulate the infrared divergence by means of a fictitious photon mass M_γ. The cancellation of infrared divergences only happens when one considers an inclusive sum of $K \to \pi\pi$ and $K \to \pi\pi\gamma$ decay rates and we postpone details on this point to Sec. 7. At this stage, we split the photonic correction to $\mathcal{A}_{+-}$ into an "infrared component" $\mathcal{A}_{+-}^{\rm IR}(M_\gamma)$ (to be treated in combination with real photons) and a structure dependent part $\mathcal{A}_{+-}^{(\gamma)}$, which is infrared finite and has to be used together with the non-photonic amplitudes in Eq. (2.1). Clearly, an arbitrary choice appears here as one can shift

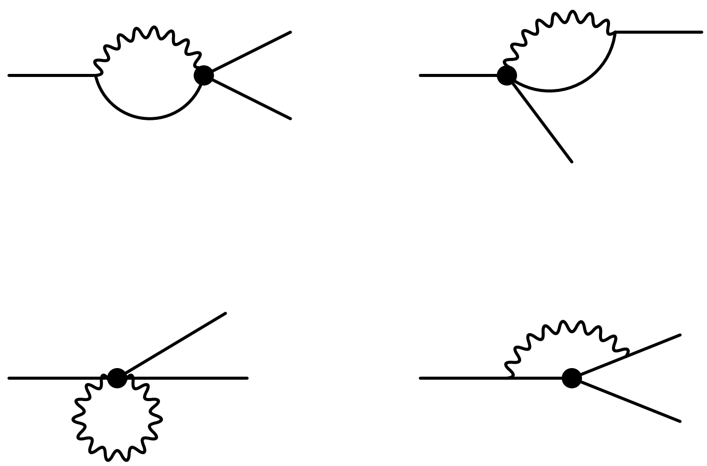

Figure 2: Topologies for meson-photon loop diagrams contributing to $K \to \pi\pi$: the filled circles indicate $\Delta S = 1$ octet vertices of lowest order.

infrared finite terms from $\mathcal{A}_{+-}^{(\gamma)}$ to $\mathcal{A}_{+-}^{\rm IR}(M_\gamma)$. This also implies that the isospin amplitudes all depend on this choice. The observables, however, are only affected by this ambiguity at order α^2. $\mathcal{A}_{+-}^{\rm IR}(M_\gamma)$ has the following structure:

$$\mathcal{A}_{+-}^{\rm IR}(M_\gamma) = \sqrt{2}\, G_8 F \left(M_K^2 - M_\pi^2 \right) \alpha\, B_{+-}(M_\gamma) \ , \tag{4.5}$$

in terms of the function $B_{+-}(M_\gamma)$ reported in App. B.

This concludes our description of one-loop contributions to $K \to \pi\pi$ amplitudes. The NLO local contributions arise from tree-level graphs with insertions of one NLO vertex and any number of LO vertices, according to the topologies depicted in Fig. 3.

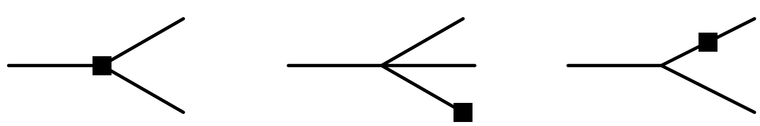

Figure 3: Diagrams for NLO local contributions: the filled square denotes a NLO vertex.

4.3 Structure of the amplitudes

Having identified the various diagrammatic contributions to the physical amplitudes, we now introduce a general parametrization that explicitly separates isospin conserving and isospin breaking parts and allows to keep track of the various sources of isospin breaking. Let n be the label for any amplitude. Then, including the leading isospin breaking corrections (proportional to G_8), one has:

$$\begin{aligned}\mathcal{A}_n &= G_{27}\,F_\pi\left(M_K^2-M_\pi^2\right)\mathcal{A}_n^{(27)} \\ &+ G_8\,F_\pi\left\{\left(M_K^2-M_\pi^2\right)\left[\mathcal{A}_n^{(8)}+\varepsilon^{(2)}\,\mathcal{A}_n^{(\varepsilon)}\right]-e^2\,F_\pi^2\left[\mathcal{A}_n^{(\gamma)}+Z\,\mathcal{A}_n^{(Z)}+g_{\rm ewk}\,\mathcal{A}_n^{(g)}\right]\right\}.\end{aligned} \tag{4.6}$$

The meaning of the amplitudes $\mathcal{A}_n^{(X)}$ can be inferred from the superscript X. $\mathcal{A}_n^{(8)}, \mathcal{A}_n^{(27)}$ represent the octet and 27-plet amplitudes in the isospin limit. $\mathcal{A}_n^{(\varepsilon)}$ represents the effect of strong isospin breaking, while the electromagnetic contribution is split into a part induced by photon loops $\mathcal{A}_n^{(\gamma)}$ and the parts induced by insertions of Z and $g_{\rm ewk}$ vertices ($\mathcal{A}_n^{(Z)}$ and $\mathcal{A}_n^{(g)}$, respectively).

At the order we are working, each of the amplitudes $\mathcal{A}_n^{(X)}$ can be decomposed as follows:

$$\mathcal{A}_n^{(X)} = \begin{cases} a_n^{(X)}\,\left[1+\Delta_L\mathcal{A}_n^{(X)}+\Delta_C\mathcal{A}_n^{(X)}\right] & \text{if} \quad a_n^{(X)}\neq 0 \\ \\ \Delta_L\mathcal{A}_n^{(X)}+\Delta_C\mathcal{A}_n^{(X)} & \text{if} \quad a_n^{(X)}=0 \end{cases} \tag{4.7}$$

with

$$\begin{aligned} a_n^{(X)} &: \quad \text{LO contribution [Eqs. (3.5,3.6)]} \\ \Delta_L\mathcal{A}_n^{(X)} &: \quad \text{NLO loop correction} \\ \Delta_C\mathcal{A}_n^{(X)} &: \quad \text{NLO local correction .}\end{aligned}$$

The amplitudes $a_n^{(X)}$, $\Delta_L\mathcal{A}_n^{(X)}$ and $\Delta_C\mathcal{A}_n^{(X)}$ are dimensionless and we have chosen to normalize the NLO contributions to the LO contributions whenever possible. Moreover, in Eq. (4.6) we have traded the constant F for F_π, the physical pion decay constant at NLO. The relation between the two is given explicitly by [11, 27]

$$\begin{aligned} F &= F_\pi\left\{1-\frac{4}{F^2}\left[L_4^r(\nu_\chi)\left(M_\pi^2+2M_K^2\right)+L_5^r(\nu_\chi)M_\pi^2\right]\right. \\ &+ \frac{1}{2(4\pi)^2F^2}\left[2M_\pi^2\log\frac{M_\pi^2}{\nu_\chi^2}+M_K^2\log\frac{M_K^2}{\nu_\chi^2}\right] \\ &+ \left.\frac{2\varepsilon^{(2)}}{\sqrt{3}}\left(M_K^2-M_\pi^2\right)\left[\frac{8L_4^r(\nu_\chi)}{F^2}-\frac{1}{2(4\pi)^2F^2}\left(1+\log\frac{M_K^2}{\nu_\chi^2}\right)\right]\right\}. \end{aligned} \tag{4.8}$$

Both $\Delta_L\mathcal{A}_n^{(X)}$ and $\Delta_C\mathcal{A}_n^{(X)}$ individually are UV divergent and scale dependent. Only in their sum the UV divergence and the scale dependence cancel, providing a valuable check on the calculation. The explicit form of the various loop contributions is given in App. B while the local amplitudes are reported in the next subsection.

4.4 Local amplitudes

The NLO $K \to \pi\pi$ local amplitudes receive contributions from the NLO couplings L_i, N_i, D_i, K_i, Z_i in the effective Lagrangians of Sec. 2. Following Ref. [28], it is convenient to define the combinations

$$\begin{aligned}
\widetilde{\Delta}_C &= -\frac{M_K^2}{F^2}\,(4\,L_5^r + 32\,L_4^r) - \frac{M_\pi^2}{F^2}\,(12\,L_5^r + 16\,L_4^r) \\
\widetilde{\Delta}_C^{(\mathrm{ew})} &= -\frac{M_K^2}{F^2}\,(4\,L_5^r + 48\,L_4^r) - \frac{M_\pi^2}{F^2}\,(20\,L_5^r + 24\,L_4^r)\ .
\end{aligned} \tag{4.9}$$

In terms of the quantities defined above, the finite parts of the NLO local amplitudes have the form reported below. In this section we use the notation D_i, N_i, Z_i as a shorthand for the ratios of NLO to LO chiral couplings $(g_8 D_i)/g_8, (g_8 N_i)/g_8, (g_8 Z_i)/g_8$.

$\Delta I = 1/2$ amplitudes

$$\begin{aligned}
\Delta_C \mathcal{A}_{1/2}^{(27)} &= \widetilde{\Delta}_C + \frac{M_K^2}{F^2}\,(D_4^r - D_5^r - 9\,D_6^r + 4\,D_7^r) \\
&+ \frac{2\,M_\pi^2}{F^2}\,(-6\,D_1^r - 2\,D_2^r + 2\,D_4^r + 6\,D_6^r + D_7^r) \\
\Delta_C \mathcal{A}_{1/2}^{(8)} &= \widetilde{\Delta}_C - \frac{2\,M_K^2}{F^2}\,(-N_5^r + 2\,N_7^r - 2\,N_8^r - N_9^r) \\
&- \frac{2\,M_\pi^2}{F^2}\,(-2\,N_5^r - 4\,N_7^r - N_8^r + 2\,N_{10}^r + 4\,N_{11}^r + 2\,N_{12}^r) \\
\Delta_C \mathcal{A}_{1/2}^{(\varepsilon)} &= \widetilde{\Delta}_C - \frac{(M_K^2 - M_\pi^2)}{F^2}\,(96 L_4^r + 32\,(3 L_7^r + L_8^r)) \\
&- \frac{2\,M_K^2}{F^2}\,(N_5^r + 6\,N_6^r + 12\,N_7^r - 8\,N_8^r - N_9^r - 4\,N_{10}^r - 8\,N_{12}^r - 12\,N_{13}^r) \\
&+ \frac{2\,M_\pi^2}{F^2}\,(14\,N_5^r + 6\,N_6^r + 24\,N_7^r - 5\,N_8^r - 26\,N_{10}^r - 24\,N_{11}^r - 10\,N_{12}^r - 12\,N_{13}^r) \\
\Delta_C \mathcal{A}_{1/2}^{(Z)} &= \widetilde{\Delta}_C^{(\mathrm{ew})} - \frac{4\,M_K^2}{F^2}\,(2\,N_7^r - N_8^r - N_9^r) + \frac{2\,M_\pi^2}{F^2}\,(2\,N_5^r + 4\,N_7^r + N_8^r) \\
\Delta_C \mathcal{A}_{1/2}^{(g)} &= \widetilde{\Delta}_C^{(\mathrm{ew})} \\
\Delta_C \mathcal{A}_{1/2}^{(\gamma)} &= \frac{2\sqrt{2}}{3}\left[\frac{M_K^2}{F^2}\,(6\,U_1^r + 4\,U_2^r + U_3^r) - \frac{M_\pi^2}{6\,F^2}\,(36\,U_1^r + 22\,U_2^r + 3\,U_3^r + 2\,U_4^r)\right. \\
&+ \frac{(M_K^2 - M_\pi^2)}{6\,F^2}\,(-8\,Z_3^r + 24\,Z_4^r - 9\,Z_5^r - 6\,Z_7^r + 3\,Z_8^r + 3\,Z_9^r + 2\,Z_{10}^r - 2\,Z_{11}^r - 2\,Z_{12}^r) \\
&+ \left.\frac{M_K^2}{F^2}\,(2\,Z_1^r + 4\,Z_2^r) + \frac{M_\pi^2}{F^2}\,(4\,Z_1^r + 2\,Z_2^r - Z_6^r)\right]
\end{aligned} \tag{4.10}$$

$\Delta I = 3/2$ amplitudes

$$
\begin{aligned}
\Delta_C \mathcal{A}_{3/2}^{(27)} &= \widetilde{\Delta}_C + \frac{M_K^2}{F^2}\,(D_4^r - D_5^r + 4\,D_7^r) + \frac{2\,M_\pi^2}{F^2}\,(-2\,D_2^r + 2\,D_4^r + D_7^r) \\
\Delta_C \mathcal{A}_{3/2}^{(\varepsilon)} &= \widetilde{\Delta}_C - \frac{32\,(M_K^2 - M_\pi^2)}{F^2}\,(3L_7^r + L_8^r) \\
&- \frac{2\,M_K^2}{F^2}\,(N_5^r + 6\,N_6^r - 2\,N_8^r - N_9^r - 4\,N_{10}^r - 8\,N_{12}^r - 12\,N_{13}^r) \\
&+ \frac{2\,M_\pi^2}{F^2}\,(2\,N_5^r + 6\,N_6^r + N_8^r - 2\,N_{10}^r - 10\,N_{12}^r - 12\,N_{13}^r) \\
\Delta_C \mathcal{A}_{3/2}^{(Z)} &= \widetilde{\Delta}_C^{(\mathrm{ew})} + \frac{M_K^2}{5\,F^2}\,(12\,N_5^r - 16\,N_7^r + 20\,N_8^r + 8\,N_9^r) \\
&+ \frac{M_\pi^2}{5\,F^2}\,(8\,N_5^r + 16\,N_7^r + 10\,N_8^r + 12\,N_9^r) \\
\Delta_C \mathcal{A}_{3/2}^{(g)} &= \widetilde{\Delta}_C^{(\mathrm{ew})} \\
\Delta_C \mathcal{A}_{3/2}^{(\gamma)} &= \frac{2}{3}\left[-\frac{M_K^2}{F^2}\,\frac{4}{5}\,U_3^r - \frac{M_\pi^2}{F^2}\left(\frac{2}{3}\,U_2^r + \frac{1}{5}\,U_3^r - \frac{2}{3}\,U_4^r\right)\right. \\
&+ \frac{(M_K^2 - M_\pi^2)}{3\,F^2}\left(-4\,Z_3^r + \frac{24}{5}\,Z_4^r - 3\,Z_8^r - 3\,Z_9^r - 2\,Z_{10}^r - \frac{8}{5}\,Z_{11}^r - \frac{8}{5}\,Z_{12}^r\right) \\
&+ \left.\frac{M_K^2}{F^2}\,(2\,Z_1^r + 4\,Z_2^r - Z_6^r) + \frac{M_\pi^2}{F^2}\,(4\,Z_1^r + 2\,Z_2^r)\right]
\end{aligned} \tag{4.11}
$$

$\Delta I = 5/2$ amplitudes

$$
\begin{aligned}
\Delta_C \mathcal{A}_{5/2}^{(Z)} &= \frac{4}{3}\left[\frac{(M_K^2 - M_\pi^2)}{5\,F^2}\,(-12\,N_5^r - 24\,N_7^r + 12\,N_9^r)\right] \\
\Delta_C \mathcal{A}_{5/2}^{(\gamma)} &= \frac{2}{3}\left[\frac{(M_K^2 - M_\pi^2)}{15\,F^2}\,(-18\,U_3^r + 36\,Z_4^r + 18\,Z_{11}^r + 18\,Z_{12}^r)\right]
\end{aligned} \tag{4.12}
$$

5 LECs at leading order in $1/N_c$

Owing to the presence of very different mass scales ($M_\pi < M_K < \Lambda_\chi \ll M_W$), the gluonic corrections to the underlying flavour-changing transition are amplified by large logarithms.

The short-distance logarithmic corrections can be summed up with the use of the operator product expansion [29] and the renormalization group [30], all the way down to scales $\mu_{\rm SD} < m_c$. One gets in this way an effective $\Delta S = 1$ Lagrangian, defined in the three-flavour theory [31, 32, 33, 34],

$$\mathcal{L}_{eff}^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud}\, V_{us}^* \sum_i C_i(\mu_{\rm SD})\, Q_i(\mu_{\rm SD}) \,, \tag{5.1}$$

which is a sum of local four-fermion operators Q_i, constructed with the light degrees of freedom ($m < \mu_{\rm SD}$), modulated by Wilson coefficients $C_i(\mu_{\rm SD})$ which are functions of the heavy masses ($M > \mu_{\rm SD}$) and CKM parameters:

$$\begin{aligned} C_i(\mu_{\rm SD}) &= z_i(\mu_{\rm SD}) \;+\; \tau\, y_i(\mu_{\rm SD}) \\ \tau &= -\frac{V_{td}\, V_{ts}^*}{V_{ud}\, V_{us}^*} \,. \end{aligned} \tag{5.2}$$

The low-energy electroweak chiral Lagrangian arises from the bosonization of the short-distance Lagrangian (5.1) below the chiral symmetry breaking scale Λ_χ. Chiral symmetry fixes the allowed operators, at a given order in momenta, but the calculation of the corresponding CHPT couplings is a difficult non-perturbative dynamical question, which requires to perform the matching between the two effective field theories.

The $1/N_c$ expansion provides a systematic approximation scheme to this problem. At leading order in $1/N_c$ the matching between the three-flavour quark theory and CHPT can be done exactly because the T-product of two colour-singlet quark currents factorizes. Since quark currents have well-known realizations in CHPT the hadronization of the weak operators Q_i can then be done in a straightforward way. As a result, the electroweak chiral couplings depend upon strong and electromagnetic low-energy constants of order p^2, p^4, p^6 and $e^2 p^2$, respectively.

5.1 Weak couplings of $\mathcal{O}(G_F p^2), \mathcal{O}(e^2 G_8 p^0)$

At lowest-order [$\mathcal{O}(G_F p^2)$, $\mathcal{O}(e^2 G_8 p^0)$], the chiral couplings of the nonleptonic electroweak Lagrangians (2.6) and (2.9) have the following large-N_c values:

$$\begin{aligned} g_8^\infty &= -\frac{2}{5}\, C_1(\mu_{\rm SD}) + \frac{3}{5}\, C_2(\mu_{\rm SD}) + C_4(\mu_{\rm SD}) - 16\, L_5\, B(\mu_{\rm SD})\, C_6(\mu_{\rm SD}) \\ g_{27}^\infty &= \frac{3}{5}\, [C_1(\mu_{\rm SD}) + C_2(\mu_{\rm SD})] \\ (e^2 g_8\, g_{\rm ewk})^\infty &= -3\, B(\mu_{\rm SD})\, C_8(\mu_{\rm SD}) - \frac{16}{3}\, B(\mu_{\rm SD})\, C_6(\mu_{\rm SD})\, e^2\, (K_9 - 2K_{10}) \,. \end{aligned} \tag{5.3}$$

The operators Q_i ($i \neq 6, 8$) factorize into products of left- and right-handed vector currents, which are renormalization-invariant quantities. Thus, the large-N_c factorization of these operators does not generate any scale dependence. The only anomalous dimensions that survive for $N_c \to \infty$ are the ones corresponding to Q_6 and Q_8 [35]. These operators factorize into colour-singlet scalar and pseudoscalar currents, which are $\mu_{\rm SD}$ dependent. The CHPT evaluation of the scalar and pseudoscalar currents provides, of course, the right $\mu_{\rm SD}$ dependence, since only physical observables can be realized in the low-energy theory. What one actually finds is the chiral realization of the renormalization-invariant products $m_q\, \bar{q}(1,\gamma_5)q$. This generates the factors

$$B(\mu_{\rm SD}) \equiv \left(\frac{B_0^2}{F^2}\right)^{\infty} = \left[\frac{M_K^2}{(m_s+m_d)(\mu_{\rm SD})\, F_\pi}\right]^2 \left[1 - \frac{16 M_K^2}{F_\pi^2}\,(2L_8 - L_5) + \frac{8M_\pi^2}{F_\pi^2}\, L_5 + \frac{8(2M_K^2+M_\pi^2)}{F_\pi^2}\,(3L_4 - 4L_6)\right] \qquad (5.4)$$

in Eq. (5.3), which exactly cancel [35, 36, 37, 38, 39] the $\mu_{\rm SD}$ dependence of $C_{6,8}(\mu_{\rm SD})$ at large N_c. There remains a dependence at next-to-leading order.

Explicitly, the large-N_c expressions imply[3]

$$\begin{aligned} g_8^\infty &= \left(1.10 \pm 0.05_{(\mu_{\rm SD})} \pm 0.08_{(L_5)} \pm 0.05_{(m_s)}\right) \\ &\quad + \tau \left(0.55 \pm 0.15_{(\mu_{\rm SD})} \pm 0.20_{(L_5)}\, {}^{+0.25}_{-0.16}{}_{(m_s)}\right) \\ g_{27}^\infty &= 0.46 \pm 0.01_{(\mu_{\rm SD})} \\ (g_8\, g_{\rm ewk})^\infty &= \left(-1.37 \pm 0.86_{(\mu_{\rm SD})} \pm 0.25_{(K_i)}\, {}^{+0.57}_{-0.35}{}_{(m_s)}\right) \\ &\quad - \tau \left(21.7 \pm 4.5_{(\mu_{\rm SD})} \pm 1.0_{(K_i)}\, {}^{+9.1}_{-5.6}{}_{(m_s)}\right) \;, \end{aligned} \qquad (5.5)$$

where the first uncertainty has been estimated by varying the renormalization scale $\mu_{\rm SD}$ between 0.77 and 1.3 GeV, the second one reflects the error on the strong LECs of order p^4 and e^2p^2, and the third indicates the uncertainty induced by m_s [40] which has been taken in the range [28] $(m_s + m_d)(\mu_{\rm SD} = 1{\rm GeV}) = (156 \pm 25)$ MeV. While the CP-odd component of $g_{\rm ewk}$ is dominated by the electroweak penguin contribution (proportional to $\tau\, y_8(\mu_{\rm SD})$), the CP-even part receives contributions of similar size from both strong (Q_6) and electroweak (Q_8) penguin operators. Its large uncertainty within this approach reflects the GIM mechanism ($z_8(\mu_{\rm SD} > m_c) = 0$). For the CP-even component, there exists an independent estimate, consistent with the one given here within the large uncertainties:

$$\frac{{\rm Re}\,(g_8 g_{\rm ewk})}{{\rm Re}\, g_8} = \begin{cases} -0.99 \pm 0.30 & [4] \\ -1.24 \pm 0.77_{(\mu_{\rm SD})} \pm 0.40_{(L_5, K_i)} & \text{Eq. (5.5)} \end{cases} \; . \qquad (5.6)$$

In this work we shall always use the latter value, in order to perform a consistent analysis at leading order in $1/N_c$.

[3] According to the discussion presented in the following subsections, we use here $L_5^r(M_\rho) = (1.0 \pm 0.3) \cdot 10^{-3}$ and $(K_9^r - 2K_{10}^r)(M_\rho) = -(9.3 \pm 4.6) \cdot 10^{-3}$.

Finally, the large-N_c matching also produces the so-called weak mass term (see Appendix A for notations):

$$\mathcal{L}_{\rm wmt} = G_8' F^4 \,\langle \lambda \,\chi_+^U \rangle + \text{ h.c. } , \tag{5.7}$$

with

$$\begin{aligned} G_8' &= -\frac{G_F}{\sqrt{2}}\, V_{ud} V_{us}^* \, g_8' \\ (g_8')^\infty &= -16 \left(L_8 + \frac{1}{2}\, H_2 \right) \, B(\mu_{\rm SD})\, C_6(\mu_{\rm SD}) \ . \end{aligned} \tag{5.8}$$

We eliminate this term with an appropriate field redefinition [23, 41, 15], of the form

$$U \;\rightarrow\; e^{i\alpha}\, U \, e^{i\beta} \ , \tag{5.9}$$

where the chiral rotation parameters (α and β) are proportional to G_8'. When applied to the strong effective Lagrangians of order p^4 and $e^2 p^2$, the above redefinition generates monomials of the NLO Lagrangians of order $G_8 p^4$ and $e^2 G_8 p^2$. The corresponding contributions to the couplings $g_8 N_i$ (of the form $L_n \times (L_8 \,+\, 1/2\, H_2)$) and $g_8 Z_i$ (of the form $K_n \times (L_8 \,+\, 1/2\, H_2)$) need to be added to the results obtained by direct matching at large-N_c. The complete results (reported in the next section) are independent of the unphysical LEC H_2 of $\mathcal{O}(p^4)$ [11].

5.2 Weak couplings of $\mathcal{O}(G_F p^4), \mathcal{O}(e^2 G_8 p^2)$

The large-N_c matching at the next-to-leading chiral order fixes the couplings $G_8 N_i$, $G_{27} D_i$ and $G_8 Z_i$ of the nonleptonic weak and electroweak Lagrangians (2.6) and (2.9). The operators Q_3 and Q_5 start to contribute at $\mathcal{O}(G_F p^4)$, while the electroweak penguin operators Q_7, Q_9 and Q_{10} make their first contributions at $\mathcal{O}(e^2 G_8 p^2)$. The contributions from the operator Q_6 at $\mathcal{O}(G_8 p^4)$ involve the strong CHPT Lagrangian of $\mathcal{O}(p^6)$ [13] (to avoid confusion with the Wilson coefficients C_i, the corresponding dimensionless couplings [42] are denoted here as X_i). With the definitions

$$\begin{aligned} \widetilde{C}_1(\mu_{\rm SD}) &= -\frac{2}{5} C_1(\mu_{\rm SD}) + \frac{3}{5} C_2(\mu_{\rm SD}) + C_4(\mu_{\rm SD}) \\ \widetilde{C}_2(\mu_{\rm SD}) &= +\frac{3}{5} C_1(\mu_{\rm SD}) - \frac{2}{5} C_2(\mu_{\rm SD}) + C_3(\mu_{\rm SD}) - C_5(\mu_{\rm SD}) \ , \end{aligned} \tag{5.10}$$

the non-vanishing couplings contributing to $K \to \pi\pi$ amplitudes are:

$$\begin{aligned} (g_{27}\, D_4) &= 4 \;\, L_5 \, g_{27}^\infty \\ (g_8\, N_5) &= -2\, L_5\, \widetilde{C}_1(\mu_{\rm SD}) + C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, [-16\, X_{14} + 32\, X_{17} - 24\, X_{38} - 4\, X_{91}] \\ (g_8\, N_6) &= 4\, L_5\, \widetilde{C}_1(\mu_{\rm SD}) + C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, [-32\, X_{17} - 32\, X_{18} + 32\, X_{37} + 16\, X_{38}] \\ (g_8\, N_7) &= 2\, L_5\, \widetilde{C}_1(\mu_{\rm SD}) + C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, [-32\, X_{16} - 16\, X_{17} + 8\, X_{38}] \\ (g_8\, N_8) &= 4\, L_5\, \widetilde{C}_1(\mu_{\rm SD}) + C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, [-16\, X_{15} - 32\, X_{17} + 16\, X_{38}] \end{aligned}$$

$$
\begin{aligned}
(g_8\, N_9) &= C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, [-64\, L_5\, L_8 - 8\, X_{34} + 8\, X_{38} + 4\, X_{91}] \\
(g_8\, N_{10}) &= C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, [-48\, X_{19} - 8\, X_{38} - 2\, X_{91} - 4\, X_{94}] \\
(g_8\, N_{11}) &= C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, [-32\, X_{20} + 4\, X_{94}] \\
(g_8\, N_{12}) &= C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, [128\, L_8\, L_8 + 16\, X_{12} - 16\, X_{31} + 8\, X_{38} - 2\, X_{91} - 4\, X_{94}] \\
(g_8\, N_{13}) &= C_6(\mu_{\rm SD})\, B(\mu_{\rm SD}) \left[256\, L_7\, L_8 - \frac{32}{3}\, X_{12} - 16\, X_{33} + 16\, X_{37} + \frac{4}{3}\, X_{91} + 4\, X_{94}\right] .
\end{aligned}
\tag{5.11}
$$

Bosonization of the four-quark operators Q_i in (5.1) leads to the following expressions[4] for the LECs Z_i:

$$
\begin{aligned}
(g_8\, Z_1) &= \widetilde{C}_1(\mu_{\rm SD}) \left(\frac{1}{3}\, K_{12} - K_{13}\right) + 64\, C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, L_8 \left[-\frac{1}{3}\, K_9 + \frac{5}{3}\, K_{10} + K_{11}\right] \\
&\quad -24\, \frac{C_8(\mu_{\rm SD})\, B(\mu_{\rm SD})}{e^2}\, L_8 \\
(g_8\, Z_2) &= \frac{4}{3}\, \widetilde{C}_1(\mu_{\rm SD})\, K_{13} - \frac{4}{3}\, 64\, C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, (K_{10} + K_{11})\, L_8 \\
(g_8\, Z_3) &= \widetilde{C}_1(\mu_{\rm SD})\, K_{13} - 64\, C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, (K_{10} + K_{11})\, L_8 \\
(g_8\, Z_4) &= -\widetilde{C}_1(\mu_{\rm SD})\, K_{13} + 64\, C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, L_8\, (K_{10} + K_{11}) \\
(g_8\, Z_5) &= \frac{4}{3}\, \widetilde{C}_1(\mu_{\rm SD})\, (4\, K_1 + 3\, K_5 + 3\, K_{12}) - \frac{64}{3}\, C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, (2K_7 + K_9)\, L_5 + \frac{C_{10}(\mu_{\rm SD})}{e^2} \\
(g_8\, Z_6) &= \widetilde{C}_1(\mu_{\rm SD}) \left(-\frac{2}{3}\, (K_5 + K_6) + 2\, (K_{12} + K_{13})\right) \\
&\quad -\frac{32}{3}\, C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, (K_9 + K_{10} + 3K_{11})\, L_5 - 12\, \frac{C_8(\mu_{\rm SD})\, B(\mu_{\rm SD})}{e^2}\, L_5 \\
(g_8\, Z_7) &= \widetilde{C}_1(\mu_{\rm SD})\, (8\, K_2 + 6\, K_6 - 4\, K_{13}) - 32\, C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, (2K_8 + K_{10} + K_{11})\, L_5 \\
(g_8\, Z_8) &= \widetilde{C}_1(\mu_{\rm SD}) \left(\frac{8}{3}\, K_3 + 4\, K_{12}\right) + \frac{4}{3} \left(\widetilde{C}_1(\mu_{\rm SD}) + \widetilde{C}_2(\mu_{\rm SD})\right) K_5 + \frac{3}{2e^2}\, (C_9(\mu_{\rm SD}) + C_{10}(\mu_{\rm SD})) \\
(g_8\, Z_9) &= -\frac{4}{3}\, \widetilde{C}_1(\mu_{\rm SD})\, (K_4 + K_{12} + K_{13}) + \frac{4}{3} \left(\widetilde{C}_1(\mu_{\rm SD}) + \widetilde{C}_2(\mu_{\rm SD})\right) K_5 - \frac{3}{2}\frac{C_7(\mu_{\rm SD})}{e^2} \\
(g_8\, Z_{10}) &= -2\, \widetilde{C}_1(\mu_{\rm SD})\, K_{13} + 4\, \left(\widetilde{C}_1(\mu_{\rm SD}) + \widetilde{C}_2(\mu_{\rm SD})\right) K_6 \\
(g_8\, Z_{11}) &= 2\, \widetilde{C}_1(\mu_{\rm SD})\, (K_4 + K_{13}) \\
(g_8\, Z_{12}) &= -4\, \widetilde{C}_1(\mu_{\rm SD})\, K_3 \\
(g_8\, Z_{13}) &= \frac{2}{3}\, \widetilde{C}_1(\mu_{\rm SD})\, (K_5 - K_{12} - K_{13}) + \frac{64}{3}\, C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, (K_{10} + K_{11})\, L_5 \\
(g_8\, Z_{14}) &= \widetilde{C}_1(\mu_{\rm SD})\, (-2\, K_6 + 4\, K_{13}) - 32\, C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, (K_{10} + K_{11})\, L_5 \\
(g_8\, Z_{15}) &= \widetilde{C}_1(\mu_{\rm SD})\, (-2\, K_6 + 4\, K_{13}) - 32\, C_6(\mu_{\rm SD})\, B(\mu_{\rm SD})\, (K_{10} + K_{11})\, L_5\ .
\end{aligned}
\tag{5.12}
$$

We recall here that a matching ambiguity arises when working to next-to-leading order in the chiral expansion and at leading order in $1/N_c$: we cannot identify at which value of the chiral renormalization scale ν_χ the large-N_c estimates for the LECs apply. This turns out

[4] Z_{13}, Z_{14}, Z_{15} do not contribute to $K \to \pi\pi$ amplitudes.

to be a major uncertainty in this approach. In order to account for this uncertainty, we vary the chiral renormalization scale between 0.6 and 1 GeV. The corresponding changes in the amplitudes are sub-leading effects in $1/N_c$ and we take them as indication of the uncertainty associated with working at leading order in $1/N_c$.

Finally, from the above expressions we see that in order to estimate the weak NLO LECs at leading order in $1/N_c$, one requires as input several combinations of strong LECs of order p^4, p^6 and e^2p^2. Below we summarize our knowledge of the needed parameters.

5.3 Strong couplings of $\mathcal{O}(p^4)$

It is well known that the limit $N_c \to \infty$ provides an excellent description of the $\mathcal{O}(p^4)$ CHPT couplings at $\nu_\chi \sim M_\rho$ [43]. The leading-order contribution of Q_6 involves the LEC L_5. The large-N_c value of this coupling can be estimated from resonance exchange [18]. Within the single-resonance approximation (SRA) [43, 44], taking $F = F_\pi$ and $M_S = 1.48$ GeV [45], one finds $L_5^\infty = F^2/(4M_S^2) = 1.0 \cdot 10^{-3}$. In our analysis we assign a 30% error to this parameter so that the adopted range for L_5 reaches at the upper end the value implied by the p^4 fit and at the lower end the value obtained in the p^6 fit of Ref. [46]. The combination $(2L_8 - L_5)^\infty$ can also be determined through resonance exchange. The only non-zero contribution comes from the exchange of pseudoscalar resonances. Within the SRA one gets [43]:

$$(2L_8 - L_5)^\infty \;=\; -\frac{F^2}{8M_P^2} \;\approx\; -\frac{F^2}{16M_S^2} \;=\; -\frac{1}{4}\,L_5^\infty \;=\; -0.25 \cdot 10^{-3}\,. \tag{5.13}$$

The factor $B(\mu_{\rm SD})$ in (5.4) and the $\mathcal{O}(p^4)$ corrections $\widetilde{\Delta}_C$, $\widetilde{\Delta}_C^{\rm (ew)}$ and $\Delta_C\mathcal{A}_n^{(\epsilon)}$ introduce additional dependences on the strong chiral couplings L_4, L_6 and $(3L_7 + L_8)$. At large N_c, $L_4^\infty = L_6^\infty = 0$ and

$$(3L_7 + L_8)^\infty \;=\; -\frac{(4M_K^2 - 3M_\eta^2 - M_\pi^2)F_\pi^2}{24\,(M_\eta^2 - M_\pi^2)^2} - \frac{1}{4}\,(2L_8 - L_5)^\infty \;=\; -0.15 \cdot 10^{-3}\,. \tag{5.14}$$

The same numerical estimate is obtained within the SRA, taking for L_7^∞ the known contribution from η_1 exchange [18].

5.4 Strong couplings of $\mathcal{O}(p^6)$

A systematic analysis of the LECs of $\mathcal{O}(p^6)$ is still missing. Resonance contributions to some of the X_i have been studied in Refs. [46, 47].

Resonance dominance (that can be justified within large-N_c QCD) implies that the LECs of $\mathcal{O}(p^6)$ occurring in the bosonization of the penguin operator Q_6 are determined by scalar exchange. The mass splitting in the lightest scalar nonet strongly influences those LECs.

We have estimated the relevant X_i with the scalar resonance Lagrangian discussed in Ref. [45] (setting $g_4^S = 0$). The relevant resonance parameters in the nonet limit are c_d, c_m, M_S, and e_m^S, the latter governing the mass splitting within the scalar nonet. We use $c_m = c_d = F_\pi/2$, as determined from short-distance constraints [43], and

$$M_S = 1.48 \text{ GeV}, \qquad e_m^S = 0.2 \tag{5.15}$$

from a phenomenological analysis of mass spectra [45] (these numbers correspond to scenario A of Ref. [45]). Even within resonance saturation, this is not a complete calculation of the relevant LECs of $\mathcal{O}(p^6)$ but we expect it to capture the most significant physics. We refrain from reporting explicit numerics for the individual LECs here. Numerical values for the relevant combinations are reported in the next section.

Finally, one can include nonet breaking effects within the framework of Ref. [45]. In the chiral resonance Lagrangian, these effects are needed in order to understand the scalar mass spectrum (the coupling k_m^S and γ_S of Ref. [45]). Once the resonances are integrated out, nonet breaking effects, sub-leading in $1/N_c$, appear in the X_i and therefore in the weak LECs $g_8 N_i$. Although this is far from being a complete analysis of sub-leading corrections it gives already an indication of their size. For all the quantities of physical interest, inclusion of k_m^S and γ_S produces shifts within our estimate of $1/N_c$ corrections based on varying the chiral renormalization scale (see discussion above).

5.5 Strong couplings of $\mathcal{O}(e^2p^2)$

Four combinations of the K_i appear directly in the local amplitudes $\Delta_C A_n^{(\gamma)}$ of $\mathcal{O}(e^2 G_8 p^2)$:

$$\begin{aligned} U_1 &= K_1 + K_2 \\ U_2 &= K_5 + K_6 \\ U_3 &= K_4 - 2K_3 \\ U_4 &= K_9 + K_{10} \; . \end{aligned} \tag{5.16}$$

Within our large-N_c estimates, also other combinations of K_i appear through the couplings $g_8 g_{\rm ewk}$ and $g_8 Z_i$. The ones relevant for $K \to \pi\pi$ decays are $K_7, K_8, K_9, K_{10}, K_{11}, K_{12}, K_{13}$. It turns out that all the relevant combinations can be obtained from existing estimates [48, 49], which we now briefly review.

The LECs K_i can be expressed as convolutions of a QCD correlation function with the electromagnetic propagator. Therefore, their calculation involves an integration over the internal momenta of the virtual photon, which makes reliable numerical estimates difficult even at large N_c. In contrast to the strong LECs L_i^r, the dependence of the U_i^r on the CHPT renormalization scale ν_χ is already present at leading order in $1/N_c$. In addition, the K_i depend also on the short-distance QCD renormalization scale $\mu_{\rm SD}$ and on the gauge parameter ξ. Whenever numerical estimates are reported in the following, they refer to the Feynman gauge ($\xi = 1$) and $\mu_{\rm SD} = 1$GeV.

A first attempt to estimate the couplings K_i, using the extended Nambu-Jona-Lasinio model at long distances, has found the results [48]:

$$\begin{aligned}
[3\,U_1^r + U_2^r]\,(\nu_\chi = M_\rho) &= (2.85 \pm 2.50)\cdot 10^{-3} \\
[U_1^r + 2\,K_{11}^r]\,(\nu_\chi = M_\rho) &= -(2.5 \pm 1.0)\cdot 10^{-3} \\
U_4^r(\nu_\chi = M_\rho) &= (2.7 \pm 1.0)\cdot 10^{-3} \\
K_{10}^r(\nu_\chi = M_\rho) &= (4.0 \pm 1.5)\cdot 10^{-3}\,.
\end{aligned} \tag{5.17}$$

The last two equations imply (adding the errors linearly):

$$K_9^r(\nu_\chi = M_\rho) = -(1.3 \pm 2.5)\cdot 10^{-3}\,. \tag{5.18}$$

Moreover, in the limit $N_c \to \infty$, one has the relation [48]

$$U_3^r \;=\; 2\,U_1^r \tag{5.19}$$

and the couplings K_7, K_8 are subleading. We therefore take $K_{7,8}^r(M_\rho) = 0$.

The remaining couplings needed were obtained at large N_c in Ref. [49] through the evaluation of the relevant correlation functions in terms of narrow hadronic resonances. Within the SRA, one gets [49]:

$$\begin{aligned}
K_{11}^r &= \frac{1}{8\,(4\pi)^2}\left\{-(\xi+3)\,\ln\left(\frac{\mu_{\rm SD}^2}{M_V^2}\right) + \left(\xi - \frac{3}{2}\right)\,\ln\left(\frac{\nu_\chi^2}{M_V^2}\right) - \xi - \frac{27}{4} + \frac{33}{2}\,\ln 2\right\} \\
K_{12}^r &= \frac{1}{4\,(4\pi)^2}\left\{(\xi - \frac{3}{2})\,\ln\left(\frac{\nu_\chi^2}{M_V^2}\right) - \xi\,\ln\left(\frac{\mu_{\rm SD}^2}{M_V^2}\right) - \xi - \frac{17}{4} + \frac{9}{2}\,\ln 2\right\} \\
K_{13}^r &= \frac{3}{4\,(4\pi)^2}\left\{1 + (1-\xi)\left[\frac{1}{12} + \frac{1}{2}\,\ln\left(\frac{M_V^2}{2\nu_\chi^2}\right)\right]\right\}\,.
\end{aligned} \tag{5.20}$$

Taking $\mu_{\rm SD} = 1$ GeV, $\nu_\chi = M_V$ and $\xi = 1$, this gives $K_{11}^r = 1.26\cdot 10^{-3}$, $K_{12}^r = -4.2\cdot 10^{-3}$ and $K_{13}^r = 4.7\cdot 10^{-3}$. Inserting the SRA prediction from (5.20) into (5.17), we get:

$$U_1^r(\nu_\chi = M_\rho) \;=\; -5.0\cdot 10^{-3}\,, \qquad U_2^r(\nu_\chi = M_\rho) \;=\; 17.9\cdot 10^{-3}\,. \tag{5.21}$$

A direct evaluation of U_1^r and U_2^r is in principle possible within the SRA [49]. However, it requires an analysis of resonance couplings beyond the known results of Ref. [18].

6 Numerical results

We are now in the position to quantify the size of NLO contributions to the relevant isospin amplitudes, due to both chiral loops and local couplings in the effective theory. The master formulas for the amplitudes at NLO are given in Eq. (4.6) and (4.7). They are organized in such a way as to easily identify the distinct sources of IB and to separate the LO from the NLO contributions in the chiral expansion. In Tables 1, 2 and 3 we report explicit results for the isospin amplitudes $\mathcal{A}_n$, $n = 1/2, 3/2, 5/2$, quoting for each component the following quantities:

- The LO contributions $a_n^{(X)}$.
- The NLO loop corrections $\Delta_L \mathcal{A}_n^{(X)}$, consisting of absorptive and dispersive components. The dispersive component depends on the chiral renormalization scale ν_χ (fixed at 0.77 GeV).
- The NLO local contributions to the CP-even and CP-odd amplitudes, denoted respectively by $[\Delta_C \mathcal{A}_n^{(X)}]^+$ and $[\Delta_C \mathcal{A}_n^{(X)}]^-$. Our estimates of $[\Delta_C \mathcal{A}_n^{(X)}]^\pm$ at the scale $\nu_\chi = 0.77$ GeV are based on the leading $1/N_c$ approximation. We discuss below the uncertainty associated with this method.

The definition of $[\Delta_C \mathcal{A}_n^{(X)}]^\pm$ is:

$$[\Delta_C \mathcal{A}_n^{(X)}]^+ \; = \; \begin{cases} \dfrac{\mathrm{Re}\left(G_{27}\ \Delta_C \mathcal{A}_n^{(27)}\right)}{\mathrm{Re}(G_{27})} & X = 27 \\ \dfrac{\mathrm{Re}\left(G_8 g_{\rm ewk}\ \Delta_C \mathcal{A}_n^{(g)}\right)}{\mathrm{Re}(G_8 g_{\rm ewk})} & X = g \\ \dfrac{\mathrm{Re}\left(G_8\ \Delta_C \mathcal{A}_n^{(X)}\right)}{\mathrm{Re}(G_8)} & X = 8, Z, \epsilon, \gamma \end{cases} \tag{6.1}$$

$$[\Delta_C \mathcal{A}_n^{(X)}]^- \; = \; \begin{cases} \dfrac{\mathrm{Im}\left(G_{27}\ \Delta_C \mathcal{A}_n^{(27)}\right)}{\mathrm{Im}(G_{27})} & X = 27 \\ \dfrac{\mathrm{Im}\left(G_8 g_{\rm ewk}\ \Delta_C \mathcal{A}_n^{(g)}\right)}{\mathrm{Im}(G_8 g_{\rm ewk})} & X = g \\ \dfrac{\mathrm{Im}\left(G_8\ \Delta_C \mathcal{A}_n^{(X)}\right)}{\mathrm{Im}(G_8)} & X = 8, Z, \epsilon, \gamma\ . \end{cases} \tag{6.2}$$

The uncertainty in $[\Delta_C \mathcal{A}_n^{(X)}]^\pm$ has two sources, related to the procedure used to estimate the NLO local couplings (see Sec. 5), and we quote them separately in the tables. The first one corresponds to the short-distance input, essentially the renormalization scale used to evaluate the Wilson coefficients. We estimate this uncertainty by varying the scale $\mu_{\rm SD}$ between 0.77 GeV and 1.3 GeV. The second one derives from working at leading order in the large-N_c expansion. At this order, there is a matching ambiguity because we do not know at which value of the chiral scale the estimates apply. Therefore, we vary the chiral renormalization scale ν_χ between 0.6 and 1 GeV. The results show that the second uncertainty (long-distance) dominates over the first one (short-distance) in most cases. Moreover, one should keep in mind that the errors quoted for the $[\Delta_C \mathcal{A}_n^{(X)}]^\pm$ are strongly correlated. In phenomenological applications we shall take such correlations into account.

Some comments on the numerical results are now in order. From chiral power counting, the expected size of NLO corrections is at the level of $\sim 0.2-0.3$, reflecting $M_K^2/(4\pi F_\pi)^2 \simeq 0.2$. This estimate sets the reference scale in the following discussion.

Table 1: Numerics for $\mathcal{A}_{1/2}$: $a_{1/2}^{(X)}$, $\Delta_L\mathcal{A}_{1/2}^{(X)}$, $\Delta_C\mathcal{A}_{1/2}^{(X)}$

(X)	$a_{1/2}^{(X)}$	$\Delta_L\mathcal{A}_{1/2}^{(X)}$	$[\Delta_C\mathcal{A}_{1/2}^{(X)}]^+$	$[\Delta_C\mathcal{A}_{1/2}^{(X)}]^-$
(27)	$\frac{\sqrt{2}}{9}$	1.02 + 0.47 i	0.01 ± 0 ± 0.60	0.01 ± 0 ± 0.60
(8)	$\sqrt{2}$	0.27 + 0.47 i	0.03 ± 0.01 ± 0.05	0.17 ± 0.01 ± 0.05
(ε)	$-\frac{2\sqrt{2}}{3\sqrt{3}}$	0.26 + 0.47 i	− 0.17 ± 0.03 ± 0.05	1.56 ± 0.06 ± 0.05
(γ)	–	− 1.38	− 0.30 ± 0.05 ± 0.30	− 12.6 ± 2.5 ± 0.30
(Z)	$\frac{4\sqrt{2}}{3}$	− 1.06 + 0.79 i	− 0.08 ± 0.01 ± 0.18	0.17 ± 0.01 ± 0.18
(g)	$\frac{2\sqrt{2}}{3}$	0.27 + 0.47 i	− 0.15 ± 0 ± 0.05	− 0.15 ± 0 ± 0.05

Table 2: Numerics for $\mathcal{A}_{3/2}$: $a_{3/2}^{(X)}$, $\Delta_L\mathcal{A}_{3/2}^{(X)}$, $\Delta_C\mathcal{A}_{3/2}^{(X)}$

(X)	$a_{3/2}^{(X)}$	$\Delta_L\mathcal{A}_{3/2}^{(X)}$	$[\Delta_C\mathcal{A}_{3/2}^{(X)}]^+$	$[\Delta_C\mathcal{A}_{3/2}^{(X)}]^-$
(27)	$\frac{10}{9}$	− 0.04 − 0.21 i	0.01 ± 0 ± 0.05	0.01 ± 0 ± 0.05
(ε)	$\frac{4}{3\sqrt{3}}$	− 0.69 − 0.21 i	− 0.15 ± 0.02 ± 0.50	1.74 ± 0.06 ± 0.50
(γ)	–	− 0.47	0.59 ± 0.02 ± 0.10	1.70 ± 0.35 ± 0.10
(Z)	$\frac{4}{3}$	− 0.86 − 0.78 i	0.02 ± 0.01 ± 0.30	0.16 ± 0.01 ± 0.30
(g)	$\frac{2}{3}$	− 0.50 − 0.21 i	− 0.15 ± 0 ± 0.20	− 0.15 ± 0 ± 0.20

Table 3: Numerics for $\mathcal{A}_{5/2}$: $a_{5/2}^{(X)}$, $\Delta_L\mathcal{A}_{5/2}^{(X)}$, $\Delta_C\mathcal{A}_{5/2}^{(X)}$

(X)	$a_{5/2}^{(X)}$	$\Delta_L\mathcal{A}_{5/2}^{(X)}$	$[\Delta_C\mathcal{A}_{5/2}^{(X)}]^+$	$[\Delta_C\mathcal{A}_{5/2}^{(X)}]^-$
(γ)	–	− 0.51	− 0.20 ± 0 ± 0.10	− 0.11 ± 0.01 ± 0.10
(Z)	–	− 0.93 − 1.15 i	− 0.14 ± 0.01 ± 0.40	0.01 ± 0.01 ± 0.40

The following pattern seems to emerge from our results. On one hand, whenever the absorptive loop correction is small, the dispersive correction is dominated by the local contribution. Therefore, it is rather sensitive to the chiral renormalization scale and to the values of LECs. In these cases, the size of NLO corrections is rather uncertain, at least within the approach we follow here in evaluating the LECs. Extreme examples of this behaviour are provided by $\Delta_C \mathcal{A}_{1/2}^{(27)}$ (of little phenomenological impact) and $\Delta_C \mathcal{A}_{3/2}^{(\epsilon)}$ (which is instead quite relevant phenomenologically).

On the other hand, whenever the absorptive loop correction is large, the dispersive component is dominated by the non-polynomial part of the loops and it is relatively insensitive to the chiral renormalization scale and to the values of LECs. In all relevant cases we have checked that the absorptive component of $\Delta_L \mathcal{A}_n^{(X)}$ is consistent with perturbative unitarity. Therefore we conclude that in these cases the size of NLO corrections is rather well understood, being determined by the physics of final state interactions. Typical examples of this behaviour are given by $\Delta_L \mathcal{A}_{1/2,3/2}^{(Z)}$, which have an important phenomenological impact.

We conclude this section with some remarks on apparently anomalous results.

- $\Delta_L \mathcal{A}_{1/2,3/2}^{(Z)}$ is $O(1)$. As discussed above, the physics underlying this result is well understood, being related to the absorptive cut in the amplitude. The key point is that this feature is absent at LO in the chiral expansion. It first shows up at NLO, setting the natural size of the loop corrections. NNLO terms in the chiral expansion are then expected to scale as NLO $\times(0.2 - 0.3)$, since corrections to the absorptive cut behave this way. Therefore $\Delta_L \mathcal{A}_{1/2,3/2}^{(Z)} \sim O(1)$ does not imply a breakdown of the chiral expansion.

- $[\Delta_C \mathcal{A}_{1/2,3/2}^{(\epsilon)}]^-$ is $O(1)$. This result is determined by the large numerical coefficients multiplying the couplings $N_{6,7,8,13}^r$, which turn out to have natural size within the leading $1/N_c$ approximation. We observe, however, that in the case of the $\Delta I = 3/2$ amplitude (phenomenologically relevant) the leading $1/N_c$ approximation is afflicted by a large uncertainty due to high sensitivity to ν_χ. This uncertainty mitigates the apparent breakdown of chiral power counting.

- Yet another surprising result is the one for $[\Delta_C \mathcal{A}_{1/2}^{(\gamma)}]^-$. The underlying reason is in the large size of the CP violating component of the Wilson coefficients $C_{9,10}$. Again, the operators $Q_{9,10}$ only make their first contribution at NLO in the chiral expansion.

7 Phenomenology I: CP conserving amplitudes

This section is devoted to a phenomenological analysis of $K \to \pi\pi$ decays including all sources of isospin breaking. The theoretical parametrization of the amplitudes is based on the NLO CHPT analysis discussed in the previous sections. Our goal is to extract information on the pure weak amplitudes (or equivalently on the couplings g_8 and g_{27}) and to clarify the role of isospin breaking in the observed $K \to \pi\pi$ rescattering phases. All along we keep track of both experimental errors and the theoretical uncertainties related to our estimates of the NLO couplings at leading order in the $1/N_c$ expansion.

7.1 Including the radiative modes

When considering electromagnetic effects at first order in α, only an inclusive sum of $K \to \pi\pi$ and $K \to \pi\pi\gamma$ widths is theoretically meaningful (free of IR divergences) and experimentally observable.

We denote the appropriate observable widths by $\Gamma_{n[\gamma]}(\omega)$ for $n = +-, 00, +0$. These widths depend in general on the amount of radiative events included in the data sample, according to specific experimental cuts on the radiative mode. This dependence is compactly represented by the parameter ω. Denoting by $\mathcal{A}_n$ the IR finite amplitudes as defined in Eq. (2.3), the relevant decay rates can be written as

$$\Gamma_{n[\gamma]}(\omega) = \frac{1}{2\sqrt{s_n}} \, |\mathcal{A}_n|^2 \, \Phi_n \, G_n(\omega) \ . \tag{7.1}$$

Here $\sqrt{s_n}$ is the total cms energy (the appropriate kaon mass) and Φ_n is the appropriate two-body phase space. The infrared factors $G_n(\omega)$ are defined as

$$G_n(\omega) = 1 + \frac{\alpha}{\pi} \left[2\pi \, \mathrm{Re} B_n(M_\gamma) + I_n(M_\gamma;\omega) \right] \ . \tag{7.2}$$

Note that $G_n(\omega)$ is different from 1 only in the $K^0 \to \pi^+\pi^-$ and $K^+ \to \pi^+\pi^0$ modes. The factor $B_n(M_\gamma)$ arises from the IR divergent loop amplitude (its definition for $n = +-$ is given in Eq. (4.5)), while $\alpha/\pi \, I_n(M_\gamma;\omega)$ is the $K \to \pi\pi\gamma$ decay rate normalized to the non-radiative rate. The latter term depends on the treatment of real photons (hence on ω) and is infrared divergent. The combination of IR divergences induced by virtual and real photons cancels in the sum, leaving the ω dependent factor $G_n(\omega)$.

We discuss here in some detail the expression for $G_{+-}(\omega)$, which plays an important phenomenological role. On the other hand, the inclusion of $G_{+0}(\omega)$ only produces an effect of order $\alpha \mathcal{A}_{3/2}$ (or αG_{27}) and therefore represents a sub-leading correction. Its numerical effect will be taken into account, following the analysis of Ref. [5]. The explicit form of $B_{+-}(M_\gamma)$, the virtual photon contribution to $G_{+-}(\omega)$, can be found in Eq. (B.1). The real photon contribution $\alpha/\pi \, I_n(M_\gamma;\omega)$ arises from the decay $K^0(P) \to \pi^+(p_+)\pi^-(p_-)\gamma(k)$ and it has the form

$$I_{+-}(M_\gamma;\omega) = \frac{2}{M_K^2 \sqrt{1 - \frac{4M_\pi^2}{M_K^2}}} \int_{s_-(\omega)}^{s_{\max}} ds \, f_{+-}(s;M_\gamma) \tag{7.3}$$

where

$$\begin{aligned} s &= (p_+ + p_-)^2 \ , \qquad s_{\min} = 4M_\pi^2 \ , \qquad s_{\max} = (M_K - M_\gamma)^2 && (7.4) \\ f_{+-}(s;M_\gamma) &= M_\pi^2 \left(\frac{1}{X_+} - \frac{1}{X_-} \right) + \frac{s - 2M_\pi^2}{M_K^2 - s - M_\gamma^2} \log\left(\frac{X_+}{X_-} \right) && (7.5) \\ X_\pm &= \frac{1}{2}\left(M_K^2 - s - M_\gamma^2 \right) \pm \frac{1}{2} \sqrt{1 - \frac{4M_\pi^2}{s}} \lambda^{1/2}(M_K^2, s, M_\gamma^2) && (7.6) \\ \lambda(x,y,z) &= x^2 + y^2 + z^2 - 2(xy + xz + yz) \ . && (7.7) \end{aligned}$$

The infrared divergence comes from the upper end of the integration in the dipion invariant mass ($s \sim s_{\rm max}$). We have verified by analytic integration in the range $M_K(M_K - 2\omega) < s < s_{\rm max}$ that $I_{+-}(M_\gamma;\omega)$ has the correct M_γ dependence to cancel the infrared singularity generated by virtual photons. For $\omega/M_K \ll 1$, the analytic expression of $I_{+-}(M_\gamma;\omega)$ can be found in Eq. (21) of Ref. [5]. The corresponding function $G_{+-}(\omega)$ is plotted in Fig. 2 of Ref. [5].

Recently, the KLOE collaboration has reported a high-precision measurement of the ratio Γ_{+-}/Γ_{00} [9], where the result refers to the fully inclusive treatment of radiative events. In order to use the KLOE measurement in our analysis, we need to calculate the fully inclusive rate (no cuts on the $\pi\pi\gamma$ final state). We have done this by numerical integration of Eq. (7.3) and we find

$$G_{+-}\Big|_{\rm inclusive} = 1 \ + \ 0.67 \cdot 10^{-2} \ . \tag{7.8}$$

7.2 Constraints from measured branching ratios

CP conserving $K \to \pi\pi$ phenomenology is based on the following input from Eq. (7.1) for $n = +-, 00, +0$:

$$|\mathcal{A}_n| = \left(\frac{2\sqrt{s_n}\,\Gamma_n}{G_n\,\Phi_n} \right)^{1/2} \equiv C_n \ . \tag{7.9}$$

It is convenient to express these equations in terms of the isospin amplitudes A_0, A_2, A_2^+ and the phase shift $\chi_0 - \chi_2$ (as defined in Eq. (2.2)). With $r = (C_{+-}/C_{00})^2$ one obtains[5]

$$\begin{aligned} A_2^+ &= \frac{2}{3}\,C_{+0} \\ (A_0)^2 + (A_2)^2 &= \frac{2}{3}\,C_{+-}^2 + \frac{1}{3}\,C_{00}^2 \\ \frac{A_2}{A_0}\cos(\chi_0 - \chi_2) &= \frac{r - 1 + (\frac{A_2}{A_0})^2(2r - \frac{1}{2})}{\sqrt{2}(1+2r)} \ . \end{aligned} \tag{7.10}$$

In general, in the presence of isospin breaking, these three independent experimental constraints are not sufficient to fix the three isospin amplitudes (A_0, A_2, A_2^+ or $\mathcal{A}_{1/2}, \mathcal{A}_{3/2}, \mathcal{A}_{5/2}$) plus the phase difference ($\chi_0 - \chi_2$). In the previous sections, we have shown how CHPT relates the amplitude $\mathcal{A}_{5/2}$ to $\mathcal{A}_{1/2}$, thus effectively reducing the number of independent amplitudes. Including also all other isospin breaking effects, we can extract the couplings g_8 and g_{27} from Eqs. (7.10).

7.3 CHPT fit to $K \to \pi\pi$ data

Using Eqs. (7.10) as starting point, we perform a fit to g_8, g_{27} and the phase difference $\chi_0 - \chi_2$. In order to do so, we employ the CHPT parametrization for the A_I. The detailed

[5]Note that in the last equation one has to use A_2 and not A_2^+ (as done in the isospin conserving analyses). For this reason the extraction of the phase shift is related to the $\Delta I = 5/2$ amplitude.

relations between $\mathcal{A}_{1/2,3/2,5/2}$ (presented in Secs. 3, 4) and A_0, A_2, A_2^+, to first order in isospin breaking, are reported in Section 8.1. We leave the phase difference $\chi_0 - \chi_2$ as a free parameter because one-loop CHPT fails in reproducing the strong s-wave phase shifts.

Apart from g_8 and g_{27}, the amplitudes A_n depend on the LO couplings $g_8 g_{\rm ewk}, Z$ and on a large set of NLO couplings. Given our large-N_c estimates for $g_8 g_{\rm ewk}$ and for the ratios of NLO over LO couplings $(g_8 N_i)/g_8$, $\cdots$ (see Sec. 5), we study the constraints imposed on g_8, g_{27} by the experimental branching ratios. In this process we keep track of the theoretical uncertainty induced by the use of a specific approximation in estimating the relevant LECs (leading order in the large-N_c expansion). In practice, this reduces to studying the dependence of the amplitude (and of the output values of g_8 and g_{27}) on two parameters: the short distance scale ($\mu_{\rm SD}$) and the chiral renormalization scale (ν_χ).

In summary, the experimental input to the fit is given by the three partial widths $\Gamma_{+-,00,+0}$ (kaon lifetimes and branching ratios) [10] and by the new KLOE measurement for Γ_{+-}/Γ_{00} [9]. The theoretical input is given by the NLO CHPT amplitudes as well as the estimates for $g_{\rm ewk}$ and the NLO couplings. As primary output we report ${\rm Re}\, g_8, {\rm Re}\, g_{27}$ and $\chi_0 - \chi_2$. Derived quantities of interest for phenomenological applications will be reported subsequently.

(1) Using the **NLO isospin conserving** amplitudes (IC-fit), we find

$$\begin{aligned} {\rm Re}\, g_8 &= 3.665 \quad \pm 0.007\,({\rm exp}) \quad \pm 0.001\,(\mu_{\rm SD}) \quad \pm 0.137\,(\nu_\chi) \\ {\rm Re}\, g_{27} &= 0.297 \quad \pm 0.001\,({\rm exp}) \quad \pm 0.014\,(\nu_\chi) \\ \chi_0 - \chi_2 &= 48.6 \quad \pm 2.6\,({\rm exp})\ . \end{aligned} \tag{7.11}$$

Using instead the tree-level (LO) amplitudes in the isospin limit would lead to ${\rm Re}\, g_8 = 5.09 \pm 0.01$ and ${\rm Re}\, g_{27} = 0.294 \pm 0.001$. This result is in qualitative agreement with Ref. [50]: NLO chiral corrections enhance the $I = 1/2$ amplitude by roughly 30%.

(2) Using the full **NLO isospin breaking** amplitudes (IB-fit), we find[6]

$$\begin{aligned} {\rm Re}\, g_8 &= 3.650 \quad \pm 0.007\,({\rm exp}) \quad \pm 0.001\,(\mu_{\rm SD}) \quad \pm 0.143\,(\nu_\chi) \\ {\rm Re}\, g_{27} &= 0.303 \quad \pm 0.001\,({\rm exp}) \quad \pm 0.001\,(\mu_{\rm SD}) \quad \pm 0.014\,(\nu_\chi) \\ \chi_0 - \chi_2 &= 54.6 \quad \pm 2.2\,({\rm exp}) \quad \pm 0.9\,(\nu_\chi)\ . \end{aligned} \tag{7.12}$$

In the IB case, a tree-level (LO) fit leads to ${\rm Re}\, g_8 = 5.11 \pm 0.01$ and ${\rm Re}\, g_{27} = 0.270 \pm 0.001$.

A few remarks are in order:

- Using NLO amplitudes, both g_8 and g_{27} receive small shifts after inclusion of IB corrections. While this could be expected for g_8, it results from a cancellation of different effects in the case of g_{27} (at tree level the inclusion of isospin breaking

[6]The uncertainty in $\varepsilon^{(2)} = (1.061 \pm 0.083) \cdot 10^{-2}$ [51] produces errors one order of magnitude smaller than the smallest uncertainty quoted above.

reduces g_{27} by roughly 10%). Note also that competing loop effects reduce the ν_χ-dependence of $\mathrm{Re}\, g_{27}$ (IB-fit) to only ± 0.002. As a more realistic estimate of the long-distance error we have chosen to quote the ν_χ-dependence induced by each one of the competing effects (for example the isospin-conserving loops).

- The output for g_8 and g_{27} is sensitive to the input used for the strong LECs L_i of $\mathcal{O}(p^4)$. The results of Eqs. (7.11) and (7.12) correspond to the central values quoted in Sec. 5. We have repeated the fit with non-central input and have found the variations in g_8 and g_{27} to be below 5%.

- In obtaining the results in Eqs. (7.11) and (7.12) we have used the large-N_c predictions for the ratios $(g_8 N_i)/g_8$. We have also employed the alternative procedure of using large N_c directly for the couplings $g_8 N_i$. In this case we find $g_8 = 3.99$ and $g_{27} = 0.289$, reflecting the change in size of the p^4 local amplitudes. All other quantities of phenomenological interest are stable under this change in the fitting procedure.

- Some derived quantities of phenomenological interest are the ratios of isospin amplitudes: $\mathrm{Re}A_0/\mathrm{Re}A_2$, $\mathrm{Re}A_0/\mathrm{Re}A_2^+$, $f_{5/2} \equiv \mathrm{Re}A_2/\mathrm{Re}A_2^+ - 1$. From our fit we find

$$
\begin{aligned}
\left[\frac{\mathrm{Re}A_0}{\mathrm{Re}A_2}\right]_{\mathrm{IB-fit}} &= 20.33 \quad \pm 0.07\,(\mathrm{exp}) \quad \pm 0.01\,(\mu_{\mathrm{SD}}) \quad \pm 0.47\,(\nu_\chi) \\
\left[\frac{\mathrm{Re}A_0}{\mathrm{Re}A_2^+}\right]_{\mathrm{IB-fit}} &= 22.09 \quad \pm 0.09\,(\mathrm{exp}) \quad \pm 0.01\,(\mu_{\mathrm{SD}}) \quad \pm 0.05\,(\nu_\chi) \\
\left[f_{5/2}\right]_{\mathrm{IB-fit}} &= \Big(8.6 \quad \pm 0.03\,(\mathrm{exp}) \quad \pm 0.01\,(\mu_{\mathrm{SD}}) \quad \pm 2.5\,(\nu_\chi)\Big) \cdot 10^{-2} \, .
\end{aligned}
\tag{7.13}
$$

In the absence of isospin breaking, one finds instead $f_{5/2} = 0$ and $\mathrm{Re}A_0/\mathrm{Re}A_2 = 22.16 \pm 0.09$.

7.4 Isospin breaking in the phases

This section is devoted to understanding isospin breaking in the rescattering phases of $K \to \pi\pi$. If isospin is conserved Watson's theorem predicts $\chi_0 - \chi_2 = \delta_0 - \delta_2 \sim 45°$. For a long time this prediction has not been fulfilled by the data, as one typically encountered $\chi_0 - \chi_2 \sim 60°$.

The situation has recently improved. Using the KLOE data [9] and working in the isospin limit, our fit (7.11) gives $\chi_0 - \chi_2 \sim 49°$ and so there seems to be no more phase problem. However, the inclusion of isospin breaking appears to reintroduce the issue. In order to understand what is going on, we analyse in detail the various factors determining $\chi_0 - \chi_2$.

The last of Eqs. (7.10) can be rewritten as

$$
\frac{A_2^+}{A_0} \cos(\chi_0 - \chi_2) = \frac{r - 1 + (\frac{A_2}{A_0})^2 (2r - \frac{1}{2})}{\sqrt{2}\,(1 + 2r)(1 + f_{5/2})} \, . \tag{7.14}
$$

Let us first calculate the right-hand side without an $I = 5/2$ amplitude ($f_{5/2} = 0$). With the old value $r = 1.1085$ as input (PDG2000 [52]), one obtains

$$\frac{A_2^+}{A_0} \cos(\chi_0 - \chi_2) = 0.02461 \ . \tag{7.15}$$

With $A_2^+/A_0 = 0.045$ (based on the IC-fit), this leads to the standard puzzle that $\chi_0 - \chi_2$ is much bigger than 45°:

$$\chi_0 - \chi_2 = 57^\circ \ . \tag{7.16}$$

What could be the reasons for this discrepancy of about 30 % ($\cos 45^\circ / \cos 57^\circ = 1.30$)? Let us consider several effects:

- First of all, the right-hand side of (7.14) has changed with the recent KLOE result [9] $r = 1.1345$ to give

$$\frac{A_2^+}{A_0} \cos(\chi_0 - \chi_2) = 0.02987 \ . \tag{7.17}$$

  This is a sizable correction of about 18 % and it goes more than half-way in the right direction to decrease the phase difference.

- Taking into account isospin breaking introduces an $I = 5/2$ amplitude via the ratio $f_{5/2}$ in Eq. (7.14). According to our results (7.13),

$$f_{5/2} = \Big(6.5\,(\text{loops}) \ + \ 2.1\,(\text{local}) \ \pm \ 2.5(\nu_\chi)\Big) \cdot 10^{-2} \ , \tag{7.18}$$

  increasing again the discrepancy. This value is dominated by loop contributions. Even if one changes the sign of the relevant combination of LECs, $f_{5/2}$ would still be positive. Note that (7.18) amounts to a correction of ~ 8 % (in the "wrong" direction). As already noted by Wolfe and Maltman [7], it seems impossible to solve the phase problem with a reasonable choice of counterterms.

- Finally, the "infrared factor" for the $+-$ mode must be taken into account. This is straightforward with the inclusive measurement of KLOE. We find $r = 1.127$, which increases again the discrepancy. Including also the effect of $f_{5/2}$, we obtain

$$\frac{A_2^+}{A_0} \cos(\chi_0 - \chi_2) = 0.02611 \ , \tag{7.19}$$

  leading to $\chi_0 - \chi_2 = (54.6 \pm 2.4)^\circ$.

Before addressing the question whether this result is in disagreement with the $\pi\pi$ phase shift prediction [53] $\delta_0 - \delta_2 = (47.7 \pm 1.5)^\circ$, it is mandatory to study the effect of isospin breaking on the phases themselves. We use the general decomposition

$$\chi_I \ = \ \delta_I + \gamma_I \qquad (I = 0, 2) \ , \tag{7.20}$$

where γ_I represents an isospin breaking correction. The γ_I are related to isospin breaking dynamics in $\pi\pi$ rescattering as well as to the presence of radiative channels [5, 54]. Since the analysis of Ref. [5], new information on radiative corrections in $\pi\pi$ scattering has become available, allowing for a reevaluation of $\gamma_0 - \gamma_2$.

7.5 Optical theorem and $\gamma_0 - \gamma_2$

The $K \to \pi\pi$ amplitudes at NLO in CHPT allow for a perturbative evaluation of $\gamma_{0,2}$. We find[7]

$$\begin{aligned}\gamma_0 &= (-0.18 \pm 0.02)^\circ \\ \gamma_2 &= (3.0 \pm 0.4)^\circ \ ,\end{aligned} \tag{7.21}$$

where the error is obtained by varying the chiral renormalization scale ν_χ as in the main fit. Setting the NLO local terms to zero would lead to results within the range quoted in Eq. (7.21). This evaluation incorporates the constraints of the optical theorem at leading order in perturbation theory. In practice, this only reflects the $\mathcal{O}(e^2p^0)$ mixing between the $I = 0$ and $I = 2$ $\pi\pi$ channels, completely missing both higher-order corrections and the new physical effect due to the radiative channel $\pi\pi\gamma$. In order to improve upon these perturbative results, a more general analysis of the optical theorem for $K^0 \to \pi\pi$ amplitudes is required. We shall follow here the approach of Ref. [5], except for a few details. The main novelty lies in the final stage, in which one needs an explicit calculation of isospin breaking effects in $\pi\pi$ scattering: we use the results obtained at $\mathcal{O}(e^2p^2)$ in CHPT in Refs. [55, 56].

We now summarize the steps involved in the optical theorem analysis of Ref. [5], relegating some technical details to App. D. For this section, CP is assumed to be conserved.

1. The first step is to work out the consequences of the optical theorem for $K^0 \to \pi\pi$ amplitudes, considering the following intermediate states: $\pi^+\pi^-, \pi^0\pi^0$ and $\pi^+\pi^-\gamma$. For the radiative amplitudes describing $K^0 \to \pi^+\pi^-\gamma$ and $\pi^+\pi^-\gamma \to \pi\pi$ we use the leading Low parametrization, thus neglecting possible structure dependent terms[8]. In this approximation the radiative amplitudes are known in terms of the non-radiative ones. Under the assumptions listed above, and collecting the $K^0 \to \pi\pi$ amplitudes in a two-component vector $\mathcal{A}$, the optical theorem has the following form:

$$\mathcal{A}bs\, \mathcal{A} = \beta \left(\mathcal{T}^\dagger + \mathcal{R}\right) \mathcal{A} \tag{7.22}$$

   where $\beta = \sqrt{1 - 4M_{\pi^0}^2/M_K^2}$, while $\mathcal{T}$ and $\mathcal{R}$ are two-by-two matrices: $\mathcal{T}$ is related to the s-wave projection of the $\pi\pi$ T-matrix, while $\mathcal{R}$ encodes the effect of both radiative modes and phase space corrections induced by mass splitting.
   The explicit form of Eq. (7.22) is best derived by working with $\pi\pi$ states in the charge basis $(\pi^+\pi^-, \pi^0\pi^0)$ where it is more transparent to deal properly with IR singularities and phase space corrections. Special care is needed in removing the IR and Coulomb singularities from the amplitudes $\mathcal{A}_{+-}$, $\mathcal{T}_{+-,00}$ and $\mathcal{T}_{+-,+-}$. This step involves an arbitrary choice, which only affects the intermediate states of the analysis but not

[7] The results depend on the ratio g_8/g_{27}, for which we use the IB-fit output.
[8] This is known to be an excellent approximation for $K^0 \to \pi^+\pi^-\gamma$.

the final results. We adopt the following prescription[9]:

$$
\begin{aligned}
A(K^0 \to \pi^+\pi^-) &= \mathcal{A}_{+-} \quad \exp\{\alpha\, B_{\pi\pi}\} \\
A(\pi^0\pi^0 \to \pi^+\pi^-) &= T_{+-,00} \;\; \exp\{\alpha\, B_{\pi\pi}\} \\
A(\pi^+\pi^- \to \pi^+\pi^-) &= T_{+-,+-} \; \exp\{\alpha\,(2\, B_{\pi\pi} + C_{\pi\pi})\}\ ,
\end{aligned}
\tag{7.23}
$$

where the infrared singularity is separated in the factors $B_{\pi\pi}$ and $C_{\pi\pi}$, whose form is reported in App. D. These factors depend only on the charges and kinematical configuration of the external particles.

2. In order to make contact with standard treatments, it is convenient to represent Eq. (7.22) in the "isospin" basis for $\pi\pi$ amplitudes. Explicit relations between charge and isospin amplitudes are reported in Ref. [5]. In the isospin basis the matrices have the form

$$
\mathcal{T} = \begin{pmatrix} \mathcal{T}_{00} & \mathcal{T}_{02} \\ \\ \mathcal{T}_{20} & \mathcal{T}_{22} \end{pmatrix} \ , \quad \mathcal{R} = \begin{pmatrix} \frac{2}{3}\,(\Delta_{+-}\mathcal{T}^*_{00} + \delta_{+-}) & \frac{\sqrt{2}}{3}\,(\Delta_{+-}\mathcal{T}^*_{00} + \delta_{+-}) \\ \\ \frac{\sqrt{2}}{3}\,(\Delta_{+-}\mathcal{T}^*_{22} + \delta_{+-}) & \frac{1}{3}\,(\Delta_{+-}\mathcal{T}^*_{22} + \delta_{+-}) \end{pmatrix} \ ,
\tag{7.24}
$$

where, using the notation

$$
\langle f \rangle \;\equiv\; \int_{-1}^{+1} d(\cos\theta)\ f(\cos\theta)\ ,
\tag{7.25}
$$

the various quantities have the following structure:

$$
\begin{aligned}
\mathcal{T}_{\rm ab} &= \frac{1}{64\pi}\,\langle T_{\rm ab}\rangle \\
\Delta_{+-} &= -\frac{2(M^2_{\pi^\pm} - M^2_{\pi^0})}{\beta^2 M_K^2} + 2\alpha\,{\rm Re}(B_{\pi\pi}) + \frac{e^2}{\Phi_{+-}}\int d\Phi_{+-\gamma}\ f_1^{\rm rad} \\
\delta_{+-} &= \frac{\alpha}{32\pi}\,\langle T_{+-,+-}\cdot C_{\pi\pi}\rangle + \frac{\alpha}{4\Phi_{+-}}\int d\Phi_{+-\gamma}\ T_{+-,+-}\cdot f_2^{\rm rad}\ .
\end{aligned}
\tag{7.26}
$$

$\mathcal{T}_{\rm ab}$ is the s-wave projection of the b $\to$ a $\pi\pi$ scattering amplitude. The factor Δ_{+-} receives a contribution from phase space corrections (pion mass splitting), one from virtual photons ($B_{\pi\pi}$) and one from real photons ($f_1^{\rm rad}$). Likewise, δ_{+-} reflects both virtual corrections ($C_{\pi\pi}$) and real photon effects ($f_2^{\rm rad}$). The definition of the phase space factors $d\Phi_{+-}$ and $d\Phi_{+-\gamma}$, as well as of $f_{1,2}^{\rm rad}$ is reported in App. D. We remark here that Δ_{+-} and δ_{+-} are free of infrared singularities, as these cancel in the sum of real and virtual photon contributions.

[9]The $\pi\pi$ amplitudes are functions of two of the three Mandelstam variables s, t, u. In the following we set $s = M_K^2$ and trade the other independent variable for the cms scattering angle θ. Moreover, the explicit dependence on $\cos\theta$ is suppressed in order to keep the expressions compact.

3. At this point one needs a general parametrization of the matrix $\mathcal{T}_{IJ}$, the T-matrix restricted to the dimension-two subspace of $\pi\pi$ channels. Assuming T-invariance (but not unitarity of the S-matrix restricted to this subspace), an explicit form is given by

$$\mathcal{T} = \frac{1}{\beta} \begin{pmatrix} \dfrac{(\eta_0\, e^{2i\delta_0} - 1)}{2i} & a\, e^{i(\delta_0+\delta_2)} \\ & \\ a\, e^{i(\delta_0+\delta_2)} & \dfrac{(\eta_2\, e^{2i\delta_2} - 1)}{2i} \end{pmatrix} \tag{7.27}$$

in terms of five parameters (two phase shifts, two inelasticities and one off-diagonal amplitude). If one assumes that only one extra state couples to the ones considered here (namely the $\pi^+\pi^-\gamma$ state), then the inelasticities are correlated, as noted in Ref. [54]. Since our subsequent discussion does not depend on the inelasticities, we do not elaborate further on this point.

4. The next step is to assume an ansatz for the $K \to \pi\pi$ amplitudes of the type

$$\mathcal{A}_I = A_I\, e^{i(\delta_I+\gamma_I)} \qquad (I = 0, 2) \tag{7.28}$$

and work out the constraints imposed upon γ_I by Eq. (7.22). Solving for $\sin\gamma_0$ and $\sin\gamma_2$ to first order in α and taking into account $A_2/A_0 \ll 1$, one finds [5]

$$\begin{aligned} \sin\gamma_0 &= \beta\,(\mathrm{Re}(\mathcal{R}_{00}) - \tan\delta_0\,\mathrm{Im}(\mathcal{R}_{00})) \simeq \mathcal{O}(\alpha\sin\delta_0) \\ \sin\gamma_2 &= \beta\,\frac{A_0}{A_2}\,\left[|\mathcal{T}_{20}| + \frac{1}{\cos\delta_2}\,(\mathrm{Re}(\mathcal{R}_{20})\cos\delta_0 - \mathrm{Im}(\mathcal{R}_{20})\sin\delta_0)\right]\ . \end{aligned} \tag{7.29}$$

The key feature of Eq. (7.29) is that in the expression for γ_2 the isospin breaking effects get once again enhanced by the factor $A_0/A_2 \sim 22$.

5. The final step consists in evaluating $\mathcal{T}_{02}$, Δ_{+-} and δ_{+-}, for which we need an explicit expression for the $\pi\pi$ amplitudes with isospin breaking [55, 56], as well as explicit expressions of $B_{\pi\pi}$ and $C_{\pi\pi}$. The details of the calculations cannot be given in a concise way and we report here only the results.
For $\mathcal{T}_{02}$ we find

$$\mathcal{T}_{02} = \frac{\sqrt{2}\,(M^2_{\pi^\pm} - M^2_{\pi^0})}{24\pi F_\pi^2}\left(1 + \Delta_{02}^{e^2p^2}\right)\ . \tag{7.30}$$

Using the results of Refs. [55, 56] and their estimate of the relevant LECs K_i, we obtain

$$\Delta_{02}^{e^2p^2} = (0.78 \pm 0.83) + i\,0.54 \tag{7.31}$$

where we have added the various errors in quadrature.
For the radiative factors the calculation cannot be done in a fully analytic form and

we employ Monte Carlo integration to deal with the real-photon contribution to δ_{+-}. We find

$$\begin{aligned} \Delta_{+-} &= -0.81 \cdot 10^{-2} \\ \delta_{+-} &= 0.09 \cdot 10^{-2} \ . \end{aligned} \tag{7.32}$$

Using these input values in Eq. (7.29), we arrive at

$$\begin{aligned} \gamma_0 &= -0.2^\circ \\ \gamma_2 &= (6 \pm 3)^\circ \ . \end{aligned} \tag{7.33}$$

The conclusion is that the optical theorem estimate of $\gamma_0 - \gamma_2$ agrees roughly with the perturbative estimate (7.21) and that it tends to worsen the discrepancy between the theoretical prediction of $\delta_0 - \delta_2$ [53] and its phenomenological extraction from $K \to \pi\pi$ decays. Explicitly one has

$$\begin{aligned} (\delta_0 - \delta_2)_{\pi\pi\to\pi\pi} &= (47.7 \ \pm \ 1.5)^\circ \\ (\delta_0 - \delta_2)_{K\to\pi\pi} &= (60.8 \ \pm \ 2.2\,(\mathrm{exp}) \ \pm \ 0.9\,(\nu_\chi) \ \pm \ 3.0\,(\gamma_2))^\circ \ . \end{aligned} \tag{7.34}$$

Although the precise KLOE measurement [9] of the ratio of $K_S \to \pi\pi$ rates has considerably improved the situation we still obtain a difference of about 13° for the phase shift difference in the isospin limit between the two determinations in Eq. (7.34). The theoretical error is much bigger in the present case due to uncertainties in the NLO LECs. However, we observe that more than half of this difference is due to the $\Delta I = 5/2$ loop amplitude that depends only on the well-established lowest-order electromagnetic LEC Z in the Lagrangian (2.8). In order to obtain a phase shift difference in the isospin limit below 50°, the local amplitude with $\Delta I = 5/2$ would have to be more than twice as big and of opposite sign. While such an explanation cannot be totally excluded at this time, the discrepancy in the two entries of Eq. (7.34) certainly warrants further study.

The $\Delta I = 5/2$ amplitude induced by isospin violation in the octet amplitude is small because it only arises at NLO and it is of purely electromagnetic origin. One may wonder whether isospin violation in the 27-plet amplitude, which occurs already at leading order, could compete. Whereas isospin violating contributions to the $\Delta I = 1/2, 3/2$ amplitudes proportional to G_{27} are certainly negligible, the effect on the $\Delta I = 5/2$ amplitude is worth investigating.

It is straightforward to calculate isospin violation in the LO 27-plet amplitudes in (3.6) due to mass differences and $\pi^0 - \eta$ mixing. We are only interested in the resulting $\Delta I = 5/2$ amplitude, entirely due to the quark mass difference:

$$\mathcal{A}_{5/2}^{(27)} = \frac{2\sqrt{3}}{9}\, G_{27} F_\pi (M_K^2 - M_\pi^2)\, \varepsilon^{(2)} \ . \tag{7.35}$$

This amplitude may now be compared to the corresponding 5/2 amplitude in (4.6):

$$\mathcal{A}_{5/2} = -e^2 G_8 F_\pi^3 \, (\mathcal{A}_{5/2}^{(\gamma)} + Z \mathcal{A}_{5/2}^{(Z)}) \ . \tag{7.36}$$

With the numerical information of Table 3 and Eq. (7.12), we obtain for the ratio

$$\frac{\mathcal{A}_{5/2}^{(27)}}{\mathcal{D}isp\ \mathcal{A}_{5/2}} \simeq 6 \cdot 10^{-2}\ . \tag{7.37}$$

The conclusion is that the 27-plet contribution to the $\Delta I = 5/2$ amplitude is of the same sign and only about 6 % of the octet contribution. The impact on the $\Delta I = 1/2, 3/2$ amplitudes is of course much smaller still. Isospin violation in the 27-plet amplitude can safely be neglected.

8 Phenomenology II: CP violation

The main contents of this section have already been published in Ref. [57]. They are included here for completeness.

8.1 Isospin violation and ϵ'

The direct CP violation parameter ϵ' is given by

$$\epsilon' = -\frac{i}{\sqrt{2}}\, e^{i(\chi_2-\chi_0)}\, \frac{\mathrm{Re}A_2}{\mathrm{Re}A_0} \left[\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} - \frac{\mathrm{Im}A_2}{\mathrm{Re}A_2}\right]\ . \tag{8.1}$$

The expression (8.1) is valid to first order in CP violation. Since $\mathrm{Im}A_I$ is CP odd the quantities $\mathrm{Re}A_I$ and χ_I are only needed in the CP limit ($I = 0, 2$).

To isolate the isospin breaking corrections in ϵ', we write the amplitudes A_0, A_2 more explicitly as

$$\begin{aligned} A_0\, e^{i\chi_0} &= \mathcal{A}_{1/2}^{(0)} + \delta\mathcal{A}_{1/2} \\ A_2\, e^{i\chi_2} &= \mathcal{A}_{3/2}^{(0)} + \delta\mathcal{A}_{3/2} + \mathcal{A}_{5/2}\ , \end{aligned} \tag{8.2}$$

where the superscript (0) denotes the isospin limit and $\delta\mathcal{A}_{1/2,3/2}$, $\mathcal{A}_{5/2}$ are first order in isospin violation. In the limit of isospin conservation, the amplitudes $\mathcal{A}_{\Delta I}$ would be generated by the ΔI component of the electroweak effective Hamiltonian.

To the order we are considering, the amplitudes $\mathcal{A}_{\Delta I}$ have both absorptive and dispersive parts. To disentangle the (CP conserving) phases generated by the loop amplitudes from the CP violating phases of the various LECs, we express our results explicitly in terms of $\mathcal{D}isp\ \mathcal{A}_{\Delta I}$ and $\mathcal{A}bs\ \mathcal{A}_{\Delta I}$. Writing Eq. (8.2) in the generic form

$$A_I\, e^{i\chi_I} = \mathcal{A}_n \equiv \mathcal{A}_n^{(0)} + \delta\mathcal{A}_n\ , \tag{8.3}$$

we obtain to first order in CP violation:

$$\begin{aligned} \mathrm{Re}A_I &= \sqrt{(\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_n])^2 + (\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}_n])^2} \\ \mathrm{Im}A_I &= (\mathrm{Re}A_I)^{-1}\ (\mathrm{Im}[\mathcal{D}isp\,\mathcal{A}_n]\,\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_n] + \mathrm{Im}[\mathcal{A}bs\,\mathcal{A}_n]\,\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}_n]) \\ e^{i\chi_I} &= (\mathrm{Re}A_I)^{-1}\ (\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_n] + i\,\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}_n])\ . \end{aligned} \tag{8.4}$$

Using the second equality in Eq. (8.3), one can now expand $\mathrm{Re}A_I$ and $\mathrm{Im}A_I$ to first order in isospin breaking. With the notation $|\mathcal{A}_n^{(0)}| = \sqrt{(\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_n^{(0)}])^2 + (\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}_n^{(0)}])^2}$, we find

$$\begin{aligned}
\mathrm{Re}A_I &= \left|\mathcal{A}_n^{(0)}\right| + \left|\mathcal{A}_n^{(0)}\right|^{-1} \Big(\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_n^{(0)}]\ \mathrm{Re}[\mathcal{D}isp\,\delta\mathcal{A}_n] + \mathrm{Re}[\mathcal{A}bs\,\mathcal{A}_n^{(0)}]\ \mathrm{Re}[\mathcal{A}bs\,\delta\mathcal{A}_n]\Big) \quad (8.5)\\
\mathrm{Im}A_I &= \left|\mathcal{A}_n^{(0)}\right|^{-1} \Big\{\mathrm{Im}[\mathcal{D}isp\,\mathcal{A}_n^{(0)}]\,\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_n^{(0)}] + \mathrm{Im}[\mathcal{A}bs\,\mathcal{A}_n^{(0)}]\,\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}_n^{(0)}]\Big\}\\
&+ \left|\mathcal{A}_n^{(0)}\right|^{-1} \Big\{\mathrm{Im}[\mathcal{D}isp\,\delta\mathcal{A}_n]\,\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_n^{(0)}] + \mathrm{Im}[\mathcal{D}isp\,\mathcal{A}_n^{(0)}]\,\mathrm{Re}[\mathcal{D}isp\,\delta\mathcal{A}_n]\\
&\qquad + \mathrm{Im}[\mathcal{A}bs\,\delta\mathcal{A}_n]\,\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}_n^{(0)}] + \mathrm{Im}[\mathcal{A}bs\,\mathcal{A}_n^{(0)}]\,\mathrm{Re}[\mathcal{A}bs\,\delta\mathcal{A}_n]\Big\}\\
&- \left|\mathcal{A}_n^{(0)}\right|^{-3} \Big\{\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_n^{(0)}]\,\mathrm{Re}[\mathcal{D}isp\,\delta\mathcal{A}_n] + \mathrm{Re}[\mathcal{A}bs\,\mathcal{A}_n^{(0)}]\,\mathrm{Re}[\mathcal{A}bs\,\delta\mathcal{A}_n]\Big\} \times\\
&\quad \Big\{\mathrm{Im}[\mathcal{D}isp\,\mathcal{A}_n^{(0)}]\,\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_n^{(0)}] + \mathrm{Im}[\mathcal{A}bs\,\mathcal{A}_n^{(0)}]\,\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}_n^{(0)}]\Big\}\ , \qquad (8.6)
\end{aligned}$$

where the first term in each equation above represents $\mathrm{Re}A_I^{(0)}$ and $\mathrm{Im}A_I^{(0)}$, respectively.

We now turn to the different sources of isospin violation in the expression (8.1) for ϵ'. We disregard the phase which can be obtained from the $K \to \pi\pi$ branching ratios. The same branching ratios are usually employed to extract the ratio $\omega_S = \mathrm{Re}A_2/\mathrm{Re}A_0$ assuming isospin conservation. Accounting for isospin violation via the general parametrization (2.3), one is then really evaluating $\omega_+ = \mathrm{Re}A_2^+/\mathrm{Re}A_0$ rather than ω_S. The two differ by a pure $\Delta I = 5/2$ effect:

$$\omega_S = \omega_+ \left(1 + f_{5/2}\right) \qquad (8.7)$$

$$f_{5/2} = \frac{\mathrm{Re}A_2}{\mathrm{Re}A_2^+} - 1\ . \qquad (8.8)$$

Because ω_+ is directly related to branching ratios it proves useful to keep ω_+ in the normalization of ϵ', introducing the $\Delta I = 5/2$ correction $f_{5/2}$ [5].

Since $\mathrm{Im}A_2$ is already first order in isospin breaking the formula for ϵ' takes the following form, with all first-order isospin violating corrections made explicit:

$$\epsilon' = -\frac{i}{\sqrt{2}}\, e^{i(\chi_2-\chi_0)}\, \omega_+ \left[\frac{\mathrm{Im}A_0^{(0)}}{\mathrm{Re}A_0^{(0)}}\,(1 + \Delta_0 + f_{5/2}) - \frac{\mathrm{Im}A_2}{\mathrm{Re}A_2^{(0)}}\right]\ , \qquad (8.9)$$

where

$$\Delta_0 = \frac{\mathrm{Im}A_0}{\mathrm{Im}A_0^{(0)}}\frac{\mathrm{Re}A_0^{(0)}}{\mathrm{Re}A_0} - 1\ . \qquad (8.10)$$

With the help of Eqs. (8.5) and (8.6), one obtains

$$\begin{aligned}
\mathrm{Im}A_0^{(0)} &= \left|\mathcal{A}_{1/2}^{(0)}\right|^{-1} \Big\{\mathrm{Im}[\mathcal{D}isp\,\mathcal{A}_{1/2}^{(0)}]\,\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_{1/2}^{(0)}] + \mathrm{Im}[\mathcal{A}bs\,\mathcal{A}_{1/2}^{(0)}]\,\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}_{1/2}^{(0)}]\Big\} \qquad (8.11)\\
\mathrm{Im}A_2 &= \left|\mathcal{A}_{3/2}^{(0)}\right|^{-1} \Big\{\mathrm{Im}[\mathcal{D}isp\,\left(\delta\mathcal{A}_{3/2} + \mathcal{A}_{5/2}\right)]\,\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_{3/2}^{(0)}]
\end{aligned}$$

$$+\,\mathrm{Im}[\mathcal{A}bs\left(\delta\mathcal{A}_{3/2}+\mathcal{A}_{5/2}\right)]\,\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}^{(0)}_{3/2}]\Big\} \tag{8.12}$$

$$\begin{aligned}
\Delta_0 &= -2\left|\mathcal{A}^{(0)}_{1/2}\right|^{-2}\Big(\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}^{(0)}_{1/2}]\;\mathrm{Re}[\mathcal{D}isp\,\delta\mathcal{A}_{1/2}]+\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}^{(0)}_{1/2}]\;\mathrm{Re}[\mathcal{A}bs\,\delta\mathcal{A}_{1/2}]\Big)\\
&+\;\Big[\mathrm{Im}[\mathcal{D}isp\,\mathcal{A}^{(0)}_{1/2}]\,\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}^{(0)}_{1/2}]+\mathrm{Im}[\mathcal{A}bs\,\mathcal{A}^{(0)}_{1/2}]\,\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}^{(0)}_{1/2}]\Big]^{-1}\;\times\\
&\quad\Big\{\mathrm{Im}[\mathcal{D}isp\,\delta\mathcal{A}_{1/2}]\,\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}^{(0)}_{1/2}]+\mathrm{Im}[\mathcal{D}isp\,\mathcal{A}^{(0)}_{1/2}]\,\mathrm{Re}[\mathcal{D}isp\,\delta\mathcal{A}_{1/2}]\\
&\quad+\;\mathrm{Im}[\mathcal{A}bs\,\delta\mathcal{A}_{1/2}]\,\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}^{(0)}_{1/2}]+\mathrm{Im}[\mathcal{A}bs\,\mathcal{A}^{(0)}_{1/2}]\,\mathrm{Re}[\mathcal{A}bs\,\delta\mathcal{A}_{1/2}]\Big\}
\end{aligned} \tag{8.13}$$

$$f_{5/2} = \frac{5}{3}\left|\mathcal{A}^{(0)}_{3/2}\right|^{-2}\Big\{\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}^{(0)}_{3/2}]\,\mathrm{Re}[\mathcal{D}isp\,\mathcal{A}_{5/2}]+\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}^{(0)}_{3/2}]\,\mathrm{Re}[\mathcal{A}bs\,\mathcal{A}_{5/2}]\Big\}\,. \tag{8.14}$$

These expressions are general results to first order in CP and isospin violation but they are independent of the chiral expansion. Working strictly to a specific chiral order, these formulas simplify considerably. We prefer to keep them in their general form but we will discuss later the numerical differences between the complete and the systematic chiral expressions. The differences are one indication for the importance of higher-order chiral corrections.

Although $\mathrm{Im}A_2$ is itself first order in isospin breaking we now make the usual (but scheme dependent) separation of the electroweak penguin contribution to $\mathrm{Im}A_2$ from the isospin breaking effects generated by other four-quark operators:

$$\mathrm{Im}A_2 = \mathrm{Im}A_2^{\mathrm{emp}} \;+\; \mathrm{Im}A_2^{\mathrm{non-emp}}\,. \tag{8.15}$$

In order to perform the above separation within the CHPT approach, we need to identify the electroweak penguin contribution to a given low-energy coupling. In other words, we need a matching procedure between CHPT and the underlying theory of electroweak and strong interactions. Such a matching procedure is given here by working at leading order in $1/N_c$. Then, the electroweak LECs of $\mathcal{O}(G_8 e^2 p^n)$ $(n=0,2)$ in $\mathrm{Im}A_2^{\mathrm{non-emp}}$ must be calculated by setting to zero the Wilson coefficients C_7, C_8, C_9, C_{10} of electroweak penguin operators.

Splitting off the electromagnetic penguin contribution to $\mathrm{Im}A_2$ in this way, we can now write ϵ' in a more familiar way as

$$\epsilon' = -\frac{i}{\sqrt{2}}\,e^{i(\chi_2-\chi_0)}\,\omega_+\,\left[\frac{\mathrm{Im}A_0^{(0)}}{\mathrm{Re}A_0^{(0)}}\,(1-\Omega_{\mathrm{eff}})-\frac{\mathrm{Im}A_2^{\mathrm{emp}}}{\mathrm{Re}A_2^{(0)}}\right] \tag{8.16}$$

where

$$\Omega_{\mathrm{eff}} = \Omega_{\mathrm{IB}}-\Delta_0-f_{5/2} \tag{8.17}$$

$$\Omega_{\mathrm{IB}} = \frac{\mathrm{Re}A_0^{(0)}}{\mathrm{Re}A_2^{(0)}}\cdot\frac{\mathrm{Im}A_2^{\mathrm{non-emp}}}{\mathrm{Im}A_0^{(0)}}\,. \tag{8.18}$$

The quantity Ω_{eff} includes all effects to leading order in isospin breaking and it generalizes the more traditional parameter Ω_{IB}. Although Ω_{IB} is in principle enhanced by the large ratio $\mathrm{Re}A_0^{(0)}/\mathrm{Re}A_2^{(0)}$ the actual numerical analysis shows all three terms in (8.17) to be relevant when both strong and electromagnetic isospin violation are included.

8.2 Numerical results

We present numerical results for the following two cases:

i. We calculate $\Omega_{\rm eff}$ and its components for $\alpha = 0$, i.e., we keep only terms proportional to the quark mass difference (strong isospin violation). In this case, there is a clean separation of isospin violating effects in $\mathrm{Im}A_2$. We compare the lowest-order result of $\mathcal{O}(m_u - m_d)$ with the full result of $\mathcal{O}[(m_u - m_d)p^2]$.

ii. Here we include electromagnetic corrections, comparing again $\mathcal{O}(m_u - m_d, e^2 p^0)$ with $\mathcal{O}[(m_u - m_d)p^2, e^2 p^2]$. In this case, the splitting between $\mathrm{Im}A_2^{\rm emp}$ and $\mathrm{Im}A_2^{\rm non-emp}$ is performed at leading order in $1/N_c$.

The LO entries depend on $\mathrm{Re}g_8/\mathrm{Re}g_{27}$ as well as on $\mathrm{Im}(g_8 g_{\rm ewk})/\mathrm{Im}g_8$. Subleading effects in $1/N_c$ are known to be sizable for the LECs of leading chiral order. We will therefore not use the large-N_c values for $\mathrm{Re}g_8$, $\mathrm{Re}g_{27}$ in the numerical analysis but instead determine these couplings from our fit to the $K \to \pi\pi$ branching ratios.

The other combination of interest is the ratio $\mathrm{Im}(g_8 g_{\rm ewk})/\mathrm{Im}g_8$. In this case, existing calculations beyond factorization [58] suggest that the size of $1/N_c$ effects is moderate, roughly $-(30 \pm 15)\%$. As a consequence, it turns out that the dominant uncertainty comes from the input parameters in the factorized expressions. Finally, one also needs the ratio $\mathrm{Im}(g_8 g_{\rm ewk})^{\rm non-emp}/\mathrm{Im}g_8$: in this case, leading large-N_c implies -3.1 ± 1.8 (error due to input parameters), while the calculation of Ref. [1] gives -1.0 ± 0.5. Given the overlap between the two ranges and the large error in the large-N_c result, we use in the numerics the range implied by leading large-N_c.

At NLO the quantities we need to evaluate depend on the ratio of next-to-leading to leading-order LECs. In Table 4, we use the leading $1/N_c$ estimates for the ratios $G_8 N_i/G_8$, ... The final error for each of the quantities $\Omega_{\rm IB}$, Δ_0, $f_{5/2}$ and $\Omega_{\rm eff}$ is obtained by adding in quadrature the LO error and the one associated to weak LECs at NLO. Moreover, only $f_{5/2}$ and $\mathrm{Re}A_0^{(0)}/\mathrm{Re}A_2^{(0)}$ depend on the ratio g_8/g_{27}. In these cases we rely on the phenomenological value implied by our fit. Some of the errors in Table 4 are manifestly correlated, e.g., in the LO column for $\alpha \neq 0$.

The NLO results are obtained with the full expressions (8.11), ..., (8.14). Using instead the simplified expressions corresponding to a fixed chiral order, the modified results are found to be well within the quoted error bars. We therefore expect our errors to account also for higher-order effects in the chiral expansion.

We have also employed an alternative procedure for estimating the non-leading weak LECs. In contrast to the previous analysis, we now apply large N_c directly to the LECs $G_8 N_i$, ... This amounts to assuming that the failure of large N_c for G_8 is specific to the leading chiral order and that the non-leading LECs are not significantly enhanced compared to the large-N_c predictions. Of course, this implies that the local amplitudes of $\mathcal{O}(G_8 p^4)$ are less important than in the previous case. Consequently, the fitted value for g_8 comes out quite a bit bigger than in Eq. (7.12), whereas g_{27} gets smaller (see Sec. 7.3). However,

	$\alpha = 0$		$\alpha \neq 0$	
	LO	LO+NLO	LO	LO+NLO
$\Omega_{\rm IB}$	11.7	15.9 ± 4.5	18.0 ± 6.5	22.7 ± 7.6
Δ_0	$-\ 0.004$	$-\ 0.41 \pm 0.05$	8.7 ± 3.0	8.4 ± 3.6
$f_{5/2}$	0	0	0	8.3 ± 2.4
$\Omega_{\rm eff}$	11.7	16.3 ± 4.5	9.3 ± 5.8	6.0 ± 7.7

Table 4: Isospin violating corrections for ϵ' in units of 10^{-2}. The first two columns refer to strong isospin violation only ($m_u \neq m_d$), the last two contain the complete results including electromagnetic corrections. LO and NLO denote leading and next-to-leading orders in CHPT. The small difference between the value of $f_{5/2}$ reported here and the one in Eq. (7.13) is due to higher-order effects in isospin breaking (absent in this table according to Eq. (8.14)).

the isospin violating ratios in Table 4 are very insensitive to those changes. Not only are the numerical values in this case well within the errors displayed in Table 4 but they are in fact very close to the central values given there.

Finally, we have investigated the impact of some subleading effects in $1/N_c$ [45]. Although this is not meant to be a systematic expansion in $1/N_c$, the nonet breaking terms considered in [45] may furnish another indication for the intrinsic uncertainties of some of the LECs. The size of those terms depends on the assignment of isosinglet scalar resonances. Since nonet breaking effects are large in the scalar sector they affect most of the entries in Table 4 in a non-negligible way, although always within the quoted uncertainties. Employing scenario A for the lightest scalar nonet [45], $\Omega_{\rm eff}$ in (8.16) decreases from $6.0 \cdot 10^{-2}$ to $-1.4 \cdot 10^{-2}$.

The lessons to be drawn from our analysis of isospin violating corrections for ϵ' are straightforward. Separate parts of those corrections turn out to be sizable. A well-known example are the contributions of strong isospin violation to $\pi^0 - \eta$ mixing where the sum of η and η' exchange generates an $\Omega_{\rm IB}$ of the order of 25 % [59]. However, already at the level of $\pi^0 - \eta$ mixing alone, a complete calculation at next-to-leading order [24] produces a destructive interference in $\Omega_{\rm IB}$. Additional contributions to the $K \to \pi\pi$ amplitudes from strong isospin violation at next-to-leading order essentially cancel out. The final result $\Omega_{\rm IB} = (15.9 \pm 4.5) \cdot 10^{-2}$ is consistent within errors with the findings of Ref. [8]. Inclusion of electromagnetic effects slightly increases $\Omega_{\rm IB}$ and generates sizable Δ_0 and $f_{5/2}$, which interfere destructively with $\Omega_{\rm IB}$ to produce the final result $\Omega_{\rm eff} = (6.0 \pm 7.7) \cdot 10^{-2}$.

It turns out that Δ_0 is largely dominated by electromagnetic penguin contributions. In those theoretical calculations of ϵ' where electromagnetic penguin contributions are explicitly included one may therefore drop Δ_0 to a very good approximation. Finally, if all electromagnetic corrections are included in theoretical calculations of ${\rm Im}A_0/{\rm Re}A_0$, ${\rm Im}A_2/{\rm Re}A_2$ and ${\rm Re}A_2/{\rm Re}A_0$, $\Omega_{\rm eff}$ is essentially given by $\Omega_{\rm IB}$. In this case, $\Omega_{\rm eff} = (16.3 \pm 4.5) \cdot 10^{-2}$ is practically identical to the result based on $\pi^0 - \eta$ mixing only [24].

9 Conclusions

In most processes isospin violation induces a small effect on physical amplitudes. In $K \to \pi\pi$ decays, however, it is amplified by the $\Delta I = 1/2$ rule: isospin breaking admixtures of the dominant $\Delta I = 1/2$ amplitudes can generate sizable corrections to $\Delta I > 1/2$ amplitudes. Understanding isospin violation is crucial for a quantitative analysis of the $\Delta I = 1/2$ rule itself and for a theoretical estimate of ϵ'.

The theoretical description of K decays involves a delicate interplay between electroweak and strong interactions in the confinement regime. Chiral perturbation theory provides a convenient framework for a systematic low-energy expansion of the relevant amplitudes. In this paper we have performed the first complete analysis of isospin violation in $K \to \pi\pi$ decays induced by the dominant octet operators to NLO in CHPT. We have reported explicit expressions for loop and counterterm amplitudes, verifying cancellation of ultraviolet divergences at NLO.

On the phenomenological side, the main features/results of this work are:

1. We have included for the first time both strong and electromagnetic isospin violation in a joint analysis.

2. Nonleptonic weak amplitudes in CHPT depend on a number of low-energy constants: we have used leading large-N_c estimates for those constants which cannot be obtained by a fit to $K \to \pi\pi$ branching ratios (i.e., all NLO couplings and the electroweak coupling of order $e^2 G_8 p^0$). Uncertainties within this approach arise from (i) input parameters in the leading $1/N_c$ expressions as well as from (ii) potentially large subleading effects in $1/N_c$. We have discussed the impact of both (i) and (ii) on the relevant quantities.

3. Using this large-N_c input, we have performed a fit to the CP-even component of the couplings g_8 and g_{27}, both without and with inclusion of isospin breaking. We find that in general the inclusion of NLO effects (loops and counterterms) has a significant impact on the output. The main outcome of the NLO fit is that both g_8 and g_{27} are only mildly affected by isospin breaking (e.g., g_{27} gets shifted upwards by only 2%). While this result is fully expected for g_8, in the case of g_{27} it arises from non-trivial cancellations between LO and NLO corrections. For the ratio measuring the

$\Delta I = 1/2$ enhancement in K^0 decays we find $\mathrm{Re}A_0/\mathrm{Re}A_2 = 20.3 \pm 0.5$, compared to 22.2 ± 0.1 in the isospin limit.

4. Using as input a NLO calculation of electromagnetic corrections to $\pi\pi$ scattering [55, 56], we have used the optical theorem to study the effect of isospin breaking on the final-state-interaction phases [5]. According to our analysis, isospin breaking leads to a discrepancy between the theoretical prediction [53] of $\delta_0 - \delta_2$ from pion-pion scattering and its phenomenological extraction from $K \to \pi\pi$ (see also Refs. [5, 6]). Before drawing a definite conclusion about the possible presence of an additional $\Delta I = 5/2$ amplitude, more work is necessary to understand this discrepancy.

5. We have studied the effect of isospin violation on the direct CP violation observable ϵ'. In this case isospin breaking affects the destructive interference between the two main contributions to ϵ' from normal and electromagnetic penguin operators. Apart from the traditional term $\Omega_{\rm IB}$, we have identified and studied the effect of isospin violation in the ratio $\mathrm{Im}A_0/\mathrm{Re}A_0$, parametrized by the quantity Δ_0 and the purely electromagnetic $\Delta I = 5/2$ amplitude. Both Δ_0 and the $\Delta I = 5/2$ contribution $f_{5/2}$ interfere destructively with $\Omega_{\rm IB}$ to yield a final value $\Omega_{\rm eff} = (6.0 \pm 7.7) \cdot 10^{-2}$ for the overall measure of isospin violation in ϵ'. If electromagnetic penguin contributions are included in theoretical calculations of $\mathrm{Im}A_0/\mathrm{Re}A_0$, Δ_0 can be dropped in $\Omega_{\rm eff}$ to a very good approximation. Finally, if all electromagnetic corrections are included in $\mathrm{Im}A_0/\mathrm{Re}A_0$, $\mathrm{Im}A_2/\mathrm{Re}A_2$ and $\mathrm{Re}A_2/\mathrm{Re}A_0$, $\Omega_{\rm eff}$ is essentially determined by $\Omega_{\rm IB}$ and is practically identical to the result based on $\pi^0 - \eta$ mixing only.

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A NLO effective Lagrangians

In this appendix we collect the relevant parts of the NLO Lagrangians.

First we recall our notation. The gauge-covariant derivative of the matrix field U is denoted $D_\mu U$, the external scalar field χ accounts for explicit symmetry breaking through the quark masses, the matrix $\lambda = (\lambda_6 - i\lambda_7)/2$ projects onto the $\bar{s} \to \bar{d}$ transition and $Q = \mathrm{diag}(2/3, -1/3, -1/3)$ is the quark charge matrix. For compactness of notation, we introduce the definitions

$$\chi_+^U = U^\dagger \chi + \chi^\dagger U \ , \qquad \chi_-^U = U^\dagger \chi - \chi^\dagger U$$
$$Q_U = U^\dagger Q\, U \ . \tag{A.1}$$

Starting with the strong Lagrangian (2.5), we have the familiar terms [11]

$$\begin{aligned}\sum_i L_i\, O_i^{p^4} &= L_4\, \langle D_\mu U^\dagger D^\mu U\rangle\langle\chi_+^U\rangle + L_5\, \langle D_\mu U^\dagger D^\mu U \chi_+^U\rangle \\ &+ L_7\, \langle\chi_-^U\rangle^2 + L_8\, \langle\chi^\dagger U\chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle + \ldots\end{aligned} \tag{A.2}$$

For the explicit form of the strong Lagrangian of $\mathcal{O}(p^6)$ we refer to Ref. [13].

The electromagnetic Lagrangian (2.8) is explicitly given by [19]

$$\begin{aligned}\sum_i K_i O_i^{e^2p^2} &= K_1\, \langle D_\mu U^\dagger D^\mu U\rangle\langle Q^2\rangle + K_2\, \langle D_\mu U^\dagger D^\mu U\rangle\langle Q Q_U\rangle \\ &+ K_3\, (\langle D_\mu U^\dagger Q U\rangle^2 + \langle D_\mu U Q U^\dagger\rangle^2) + K_4\, \langle D_\mu U^\dagger Q U\rangle\langle D_\mu U Q U^\dagger\rangle \\ &+ K_5\, \langle\{D_\mu U^\dagger, D^\mu U\} Q^2\rangle + K_6\, \langle D_\mu U^\dagger D^\mu U Q U^\dagger Q U + D_\mu U D^\mu U^\dagger Q U Q U^\dagger\rangle \\ &+ K_7\, \langle\chi_+^U\rangle\langle Q^2\rangle + K_8\, \langle\chi_+^U\rangle\langle Q Q_U\rangle \\ &+ K_9\, \langle(\chi^\dagger U + U^\dagger\chi) Q^2 + (\chi U^\dagger + U\chi^\dagger) Q^2\rangle \\ &+ K_{10}\, \langle(\chi^\dagger U + U^\dagger\chi) Q U^\dagger Q U + (\chi U^\dagger + U\chi^\dagger) Q U Q U^\dagger\rangle \\ &+ K_{11}\, \langle(\chi^\dagger U - U^\dagger\chi) Q U^\dagger Q U + (\chi U^\dagger - U\chi^\dagger) Q U Q U^\dagger\rangle \\ &+ K_{12}\, \langle D_\mu U^\dagger [D^\mu Q_R, Q] U + D_\mu U [D^\mu Q_L, Q] U^\dagger\rangle \\ &+ K_{13}\, \langle D^\mu Q_R U D_\mu Q_L U^\dagger\rangle \ ,\end{aligned} \tag{A.3}$$

with (l_μ and r_μ denote external spin-1 fields)

$$D_\mu Q_L = \partial_\mu Q - i\,[l_\mu, Q] \ , \qquad D_\mu Q_R = \partial_\mu Q - i\,[r_\mu, Q] \ . \tag{A.4}$$

Turning to the nonleptonic weak Lagrangian, we first display the octet couplings in the notation of Ref. [16]:

$$\begin{aligned}\sum_i N_i O_i^8 &= N_5\, \langle\lambda\{\chi_+^U, D_\mu U^\dagger D^\mu U\}\rangle + N_6\, \langle\lambda D_\mu U^\dagger U\rangle\langle U^\dagger D^\mu U \chi_+^U\rangle \\ &+ N_7\, \langle\lambda\chi_+^U\rangle\langle D_\mu U^\dagger D^\mu U\rangle + N_8\, \langle\lambda D_\mu U^\dagger D^\mu U\rangle\langle\chi_+^U\rangle \\ &+ N_9\, \langle\lambda[\chi_-^U, D_\mu U^\dagger D^\mu U]\rangle + N_{10}\, \langle\lambda(\chi_+^U)^2\rangle + N_{11}\, \langle\lambda\chi_+^U\rangle\langle\chi_+^U\rangle \\ &+ N_{12}\, \langle\lambda(\chi_-^U)^2\rangle + N_{13}\, \langle\lambda\chi_-^U\rangle\langle\chi_-^U\rangle + \ldots\end{aligned} \tag{A.5}$$

The corresponding 27-plet couplings [15] are

$$
\begin{aligned}
\sum_i D_i O_i^{27} &= D_1\ t_{ij,kl}\,\langle\lambda_{ij}\chi_+^U\rangle\langle\lambda_{kl}\chi_+^U\rangle + D_2\ t_{ij,kl}\,\langle\lambda_{ij}\chi_-^U\rangle\langle\lambda_{kl}\chi_-^U\rangle \\
&- D_4\ t_{ij,kl}\,\langle\lambda_{ij}U^\dagger D_\mu U\rangle\langle\lambda_{kl}\{\chi_+^U, U^\dagger D^\mu U\}\rangle \\
&+ D_5\ t_{ij,kl}\,\langle\lambda_{ij}U^\dagger D_\mu U\rangle\langle\lambda_{kl}[\chi_-^U, U^\dagger D^\mu U]\rangle \\
&+ D_6\ t_{ij,kl}\,\langle\lambda_{ij}\chi_+^U\rangle\langle\lambda_{kl}D_\mu U^\dagger D^\mu U\rangle \\
&- D_7\ t_{ij,kl}\,\langle\lambda_{ij}U^\dagger D_\mu U\rangle\langle\lambda_{kl}U^\dagger D^\mu U\rangle\langle\chi_+^U\rangle \qquad \text{(A.6)}
\end{aligned}
$$

with $(\lambda_{ij})_{ab} = \delta_{ia}\delta_{jb}$. The non-zero components of $t_{ij,kl}$ are given by $(i,j,k,l = 1,2,3)$

$$
\begin{aligned}
t_{21,13} &= t_{13,21} = t_{23,11} = t_{11,23} = \frac{1}{3} \\
t_{22,23} &= t_{23,22} = t_{23,33} = t_{33,23} = -\frac{1}{6}\ . \qquad \text{(A.7)}
\end{aligned}
$$

Finally, the relevant part of the electroweak Lagrangian of $O(e^2 G_8 p^2)$ [22] is

$$
\begin{aligned}
\sum_i Z_i O_i^{EW} &= Z_1\langle\lambda\{Q_U, \chi_+^U\}\rangle + Z_2\langle\lambda Q_U\rangle\langle\chi_+^U\rangle + Z_3\langle\lambda Q_U\rangle\langle\chi_+^U Q_U\rangle \\
&+ Z_4\langle\lambda\chi_+^U\rangle\langle Q Q_U\rangle + Z_5\langle\lambda D_\mu U^\dagger D^\mu U\rangle + Z_6\langle\lambda\{Q_U, D_\mu U^\dagger D^\mu U\}\rangle \\
&+ Z_7\langle\lambda D_\mu U^\dagger D^\mu U\rangle\langle Q Q_U\rangle + Z_8\langle\lambda D_\mu U^\dagger U\rangle\langle Q U^\dagger D^\mu U\rangle \qquad \text{(A.8)} \\
&+ Z_9\langle\lambda D_\mu U^\dagger U\rangle\langle Q_U U^\dagger D^\mu U\rangle + Z_{10}\langle\lambda D_\mu U^\dagger U\rangle\langle\{Q, Q_U\}U^\dagger D^\mu U\rangle \\
&+ Z_{11}\langle\lambda\{Q_U, U^\dagger D_\mu U\}\rangle\langle Q D^\mu U^\dagger U\rangle + Z_{12}\langle\lambda\{Q_U, U^\dagger D_\mu U\}\rangle\langle Q_U D^\mu U^\dagger U\rangle + \ldots
\end{aligned}
$$

B Explicit form of NLO loop amplitudes

In this appendix we report explicit expressions for the NLO loop corrections $\Delta_L \mathcal{A}_n^{(X)}$ appearing in the master formulas of Eqs. (4.6) and (4.7).

B.1 Photonic amplitudes

Let us start with the terms arising from exchange of virtual photons. The amplitude $\mathcal{A}_{+-}$ is infrared divergent and is regulated by introducing a fictitious photon mass M_γ. Moreover, it is convenient to work with a subtracted amplitude, after removing the infrared component $\mathcal{A}_{+-}^{\rm IR}$ (see discussion in Sec. 4). The function $B_{+-}(M_\gamma)$ appearing in our definition of the infrared-divergent amplitude $\mathcal{A}_{+-}^{\rm IR}$ (see Eq. (4.5)) is given by

$$
\begin{aligned}
B_{+-}(M_\gamma) &= \frac{1}{4\pi}\Big[2a(\beta)\log\frac{M_\pi^2}{M_\gamma^2} + \frac{1+\beta^2}{2\beta}h(\beta) + 2 + \beta\log\frac{1+\beta}{1-\beta} \\
&\qquad + i\pi\left(\frac{1+\beta^2}{\beta}\log\frac{M_K^2\beta^2}{M_\gamma^2} - \beta\right)\Big]\ , \qquad \text{(B.1)}
\end{aligned}
$$

where

$$\begin{aligned}
\beta &= (1-4M_\pi^2/M_K^2)^{1/2} \\
a(\beta) &= 1+\frac{1+\beta^2}{2\beta}\log\frac{1-\beta}{1+\beta} \\
h(\beta) &= \pi^2+\log\frac{1+\beta}{1-\beta}\log\frac{1-\beta^2}{4\beta^2}+2f\left(\frac{1+\beta}{2\beta}\right)-2f\left(\frac{\beta-1}{2\beta}\right) \\
f(x) &= -\int_0^x dt\ \frac{1}{t}\log|1-t|\ . \qquad \text{(B.2)}
\end{aligned}$$

The amplitudes $\Delta_L\mathcal{A}_n^{(\gamma)}$ are given by

$$\Delta_L\mathcal{A}_{1/2}^{(\gamma)} = \frac{\sqrt{2}}{(4\pi F_\pi)^2}\left[-\frac{14}{3}M_\pi^2+2M_K^2\left(1+\log\frac{M_\pi^2}{\nu_\chi^2}\right)\right]+\frac{4\sqrt{2}M_K^2}{F_\pi^2}\Lambda(\nu_\chi) \qquad \text{(B.3)}$$

$$\begin{aligned}
\Delta_L\mathcal{A}_{3/2}^{(\gamma)} &= \frac{1}{(4\pi F_\pi)^2}\left[-\frac{14}{3}M_\pi^2+\frac{4}{5}(M_K^2+\frac{3}{2}M_\pi^2)\left(1+\log\frac{M_\pi^2}{\nu_\chi^2}\right)\right] \\
&+ \frac{8}{5F_\pi^2}(M_K^2+\frac{3}{2}M_\pi^2)\Lambda(\nu_\chi) \qquad \text{(B.4)}
\end{aligned}$$

$$\Delta_L\mathcal{A}_{5/2}^{(\gamma)} = \frac{6}{5}\,\frac{M_K^2-M_\pi^2}{(4\pi F_\pi)^2}\left(1+\log\frac{M_\pi^2}{\nu_\chi^2}\right)+\frac{12(M_K^2-M_\pi^2)}{5F_\pi^2}\Lambda(\nu_\chi)\ . \qquad \text{(B.5)}$$

The divergent factor $\Lambda(\nu_\chi)$ is defined in Eq. (2.11).

B.2 Non-photonic amplitudes

The mesonic loop corrections can be expressed in terms of the following basic function (and its derivatives):

$$\begin{aligned}
J(p^2,M_1^2,M_2^2) &= \frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{[k^2-M_1^2]\ [(k-p)^2-M_2^2]} \\
&= \bar{J}(p^2,M_1^2,M_2^2)+J(0,M_1^2,M_2^2)\ . \qquad \text{(B.6)}
\end{aligned}$$

The subtraction term is given by

$$\begin{aligned}
J(0,M_1^2,M_2^2) &= \frac{M_1^2\,T(M_1^2)-M_2^2\,T(M_2^2)}{M_1^2-M_2^2}\ -\ 2\,\Lambda(\nu_\chi) \qquad \text{(B.7)} \\
T(M^2) &= -\frac{1}{(4\pi)^2}\,\log\frac{M^2}{\nu_\chi^2}\ . \qquad \text{(B.8)}
\end{aligned}$$

Expansion around the neutral meson masses generates terms involving derivatives of the function $\bar{J}(p^2,M_1^2,M_2^2)$. In order to deal with such terms we use the notation

$$\bar{J}^{(1,0,0)}(p^2,M_1^2,M_2^2) = \frac{\partial}{\partial p^2}\,\bar{J}(p^2,M_1^2,M_2^2)$$

$$
\begin{aligned}
\bar{J}^{(0,1,0)}(p^2, M_1^2, M_2^2) &= \frac{\partial}{\partial M_1^2}\,\bar{J}(p^2, M_1^2, M_2^2) \\
\bar{J}^{(0,0,1)}(p^2, M_1^2, M_2^2) &= \frac{\partial}{\partial M_2^2}\,\bar{J}(p^2, M_1^2, M_2^2)\ .
\end{aligned} \tag{B.9}
$$

We report below the explicit form of the relevant functions. For this purpose we define

$$
\lambda(t,x,y) = \left[t - \left(\sqrt{x}+\sqrt{y}\right)^2\right]\,\left[t - \left(\sqrt{x}-\sqrt{y}\right)^2\right] \tag{B.10}
$$

and

$$
\Sigma_{12} = M_1^2 + M_2^2\ , \qquad \Delta_{12} = M_1^2 - M_2^2\ . \tag{B.11}
$$

Then

$$
\begin{aligned}
\bar{J}(p^2, M_1^2, M_2^2) &= \frac{1}{32\pi^2}\left[2 + \frac{\Delta_{12}}{p^2}\log\frac{M_2^2}{M_1^2} - \frac{\Sigma_{12}}{\Delta_{12}}\log\frac{M_2^2}{M_1^2}\right. \\
&\left. - \frac{\lambda^{1/2}(p^2, M_1^2, M_2^2)}{p^2}\times\log\left(\frac{[p^2+\lambda^{1/2}(p^2, M_1^2, M_2^2)]^2 - \Delta_{12}^2}{[p^2-\lambda^{1/2}(p^2, M_1^2, M_2^2)]^2 - \Delta_{12}^2}\right)\right] \\
\bar{J}(p^2, M^2, M^2) &= \frac{1}{16\pi^2}\left[2 - \sigma\log\left(\frac{\sigma+1}{\sigma-1}\right)\right], \quad \sigma \equiv \sqrt{\lambda\left(1, M^2/p^2, M^2/p^2\right)}\ .
\end{aligned} \tag{B.12}
$$

The relevant derivative functions are reported below (recalling the symmetry property[10] $\bar{J}^{(0,1,0)}(p^2, M_1^2, M_2^2) = \bar{J}^{(0,0,1)}(p^2, M_2^2, M_1^2)$):

$$
\begin{aligned}
\bar{J}^{(1,0,0)}(p^2, M_1^2, M_2^2) &= \frac{1}{32\pi^2}\left\{-\frac{2}{p^2} - \frac{\Delta_{12}}{(p^2)^2}\log\frac{M_2^2}{M_1^2}\right. \\
&\left. - \frac{(p^2\,\Sigma_{12} - \Delta_{12}^2)}{(p^2)^2\,\lambda^{1/2}(p^2, M_1^2, M_2^2)}\,\log\left(\frac{\Sigma_{12} - p^2 - \lambda^{1/2}(p^2, M_1^2, M_2^2)}{\Sigma_{12} - p^2 + \lambda^{1/2}(p^2, M_1^2, M_2^2)}\right)\right\} \\
\bar{J}^{(0,0,1)}(p^2, M_1^2, M_2^2) &= \frac{1}{32\pi^2}\left\{-\frac{2}{\Delta_{12}} - \frac{\Delta_{12}^2 + 2M_1^2 p^2}{p^2\,\Delta_{12}^2}\log\frac{M_2^2}{M_1^2}\right. \\
&\left. + \frac{p^2 + \Delta_{12}}{p^2\,\lambda^{1/2}(p^2, M_1^2, M_2^2)}\,\log\left(\frac{\Sigma_{12} - p^2 - \lambda^{1/2}(p^2, M_1^2, M_2^2)}{\Sigma_{12} - p^2 + \lambda^{1/2}(p^2, M_1^2, M_2^2)}\right)\right\} \\
\bar{J}^{(0,0,1)}(p^2, M^2, M^2) &= \frac{1}{32\pi^2}\left\{\frac{1}{M^2} + \frac{2}{p^2\,\sigma}\,\log\left(\frac{\sigma+1}{\sigma-1}\right)\right\}\ .
\end{aligned} \tag{B.13}
$$

We recall here that we expand all our amplitudes around the neutral pion and kaon masses M_π and M_K to define the isospin limit (see Ref. [60] for a more general discussion of the splitting between strong and electromagnetic contributions). This applies also to the η mass given in Eq. (3.4). Therefore, in all (loop) amplitudes where M_η^2 appears explicitly it actually stands for $(4M_K^2 - M_\pi^2)/3$ instead of the physical value in (3.4). This concerns all loop functions in Apps. B and C.

[10] In the equal mass case we adopt the definition

$$
\bar{J}^{(0,0,1)}(p^2, M^2, M^2) \equiv \lim_{M_2\to M}\,\bar{J}^{(0,0,1)}(p^2, M_2^2, M^2) = \frac{1}{2}\frac{\partial}{\partial M^2}\bar{J}(p^2, M^2, M^2)\ .
$$

B.2.1 $\Delta I = 1/2$ amplitudes

In this section we list the one-loop corrections to the $\Delta I = 1/2$ amplitude.

$$\begin{aligned}
\Delta_L \mathcal{A}_{1/2}^{(27)} &= -\frac{M_\pi^2}{2\,F_\pi^2}\,\bar{J}(M_K^2, M_\eta^2, M_\eta^2) + \frac{(2\,M_K^2 - M_\pi^2)}{2\,F_\pi^2}\,\bar{J}(M_K^2, M_\pi^2, M_\pi^2) \\
&+ \frac{M_K^4}{3\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2, M_K^2, M_\eta^2) - \frac{M_K^2\,(M_K^2 - 4\,M_\pi^2)}{4\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2, M_K^2, M_\pi^2) \\
&- \frac{3\,(12\,M_K^4 - 11\,M_K^2\,M_\pi^2 + 3\,M_\pi^4)}{8\,F_\pi^2\,(-M_K^2 + M_\pi^2)}\,T(M_\eta^2) + \frac{(6\,M_K^4 - 11\,M_K^2\,M_\pi^2)}{4\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_K^2) \\
&+ \frac{(8\,M_K^4 - 35\,M_K^2\,M_\pi^2 + 25\,M_\pi^4)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\pi^2) + \frac{-M_K^2 + M_\pi^2}{16\,F_\pi^2\,\pi^2} \qquad \text{(B.14)}
\end{aligned}$$

$$\begin{aligned}
\Delta_L \mathcal{A}_{1/2}^{(8)} &= \frac{M_\pi^2}{18\,F_\pi^2}\,\bar{J}(M_K^2, M_\eta^2, M_\eta^2) + \frac{(2\,M_K^2 - M_\pi^2)}{2\,F_\pi^2}\,\bar{J}(M_K^2, M_\pi^2, M_\pi^2) \\
&- \frac{M_K^4}{12\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2, M_K^2, M_\eta^2) - \frac{(M_K^4 - 4\,M_K^2\,M_\pi^2)}{4\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2, M_K^2, M_\pi^2) \\
&- \frac{(36\,M_K^4 - 73\,M_K^2\,M_\pi^2 + 19\,M_\pi^4)}{72\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\eta^2) + \frac{M_K^2}{4\,F_\pi^2}\,T(M_K^2) \\
&+ \frac{(8\,M_K^4 - 35\,M_K^2\,M_\pi^2 + 25\,M_\pi^4)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\pi^2) + \frac{-9\,M_K^2 + 4\,M_\pi^2}{144\,F_\pi^2\,\pi^2} \qquad \text{(B.15)}
\end{aligned}$$

$$\begin{aligned}
\Delta_L \mathcal{A}_{1/2}^{(\epsilon)} &= +\frac{5\,M_\pi^2}{6\,F_\pi^2}\,\bar{J}(M_K^2, M_\eta^2, M_\eta^2) + \frac{3\,M_K^2}{2\,F_\pi^2}\,\bar{J}(M_K^2, M_K^2, M_K^2) \\
&+ \frac{(5\,M_K^2 - 6\,M_\pi^2)}{3\,F_\pi^2}\,\bar{J}(M_K^2, M_\pi^2, M_\eta^2) + \frac{(2\,M_K^2 - M_\pi^2)}{2\,F_\pi^2}\,\bar{J}(M_K^2, M_\pi^2, M_\pi^2) \\
&- \frac{(3\,M_K^4 - 4\,M_K^2\,M_\pi^2)}{12\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2, M_K^2, M_\eta^2) - \frac{(M_K^4 - 2\,M_K^2\,M_\pi^2)}{F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2, M_\pi^2, M_K^2) \\
&- \frac{(22\,M_K^4 - 71\,M_K^2\,M_\pi^2 + 43\,M_\pi^4)}{12\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\eta^2) + \frac{5\,M_K^2\,M_\pi^2}{4\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_K^2) \\
&+ \frac{(4\,M_K^4 - 11\,M_K^2\,M_\pi^2)}{4\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\pi^2) + \frac{-8\,M_K^4 - 22\,M_K^2\,M_\pi^2 + 7\,M_\pi^4}{32\,F_\pi^2\,(4\,M_K^2 - M_\pi^2)\,\pi^2} \\
&+ \frac{4\,M_\pi^2\,(M_K^2 - M_\pi^2)}{9\,F_\pi^2}\,\bar{J}^{(0,0,1)}(M_K^2, M_\eta^2, M_\eta^2) \\
&- \frac{M_K^4\,(M_K^2 - M_\pi^2)}{3\,F_\pi^2\,M_\pi^2}\left[\bar{J}^{(0,0,1)}(M_\pi^2, M_K^2, M_\eta^2) + \bar{J}^{(0,1,0)}(M_\pi^2, M_K^2, M_\eta^2)\right] \qquad \text{(B.16)}
\end{aligned}$$

$$\begin{aligned}
\Delta_L \mathcal{A}_{1/2}^{(Z)} &= -\frac{3\,M_K^2}{8\,F_\pi^2}\,\bar{J}(M_K^2, M_K^2, M_K^2) - \frac{(2\,M_K^2 - 3\,M_\pi^2)}{2\,F_\pi^2}\,\bar{J}(M_K^2, M_\pi^2, M_\pi^2) \\
&- \frac{(2\,M_K^6 + 5\,M_K^4\,M_\pi^2 - 4\,M_K^2\,M_\pi^4)}{24\,F_\pi^2\,M_\pi^4}\,\bar{J}(M_\pi^2, M_K^2, M_\eta^2)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(M_K^6 - 4\,M_K^2\,M_\pi^4)}{4\,F_\pi^2\,M_\pi^4}\,\bar{J}(M_\pi^2,M_K^2,M_\pi^2) \\
&+ \frac{M_\pi^2\,(4\,M_K^2 - M_\pi^2)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\eta^2) - \frac{3\,(7\,M_K^4 - 7\,M_K^2\,M_\pi^2 + M_\pi^4)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_K^2) \\
&+ \frac{(7\,M_K^2 - 18\,M_\pi^2)}{8\,F_\pi^2}\,T(M_\pi^2) + \frac{8\,M_K^4 - 13\,M_K^2\,M_\pi^2 + 2\,M_\pi^4}{128\,F_\pi^2\,M_\pi^2\,\pi^2} \\
&- \frac{(2\,M_K^4 - 3\,M_K^2\,M_\pi^2 + M_\pi^4)}{F_\pi^2}\,\bar{J}^{(0,0,1)}(M_K^2,M_\pi^2,M_\pi^2) \\
&+ \frac{(M_K^6 - 5M_K^4\,M_\pi^2 + 4M_K^2\,M_\pi^4)}{4\,F_\pi^2\,M_\pi^2}\left[\bar{J}^{(0,0,1)}(M_\pi^2,M_K^2,M_\pi^2) + \bar{J}^{(1,0,0)}(M_\pi^2,M_K^2,M_\pi^2)\right] \\
&+ \frac{(M_K^6 - M_K^4\,M_\pi^2)}{12\,F_\pi^2\,M_\pi^2}\left[\bar{J}^{(0,1,0)}(M_\pi^2,M_K^2,M_\eta^2) + \bar{J}^{(1,0,0)}(M_\pi^2,M_K^2,M_\eta^2)\right] \qquad \text{(B.17)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_L\mathcal{A}_{1/2}^{(g)} &= -\frac{3\,M_K^2}{8\,F_\pi^2}\,\bar{J}(M_K^2,M_K^2,M_K^2) + \frac{(2\,M_K^2 - M_\pi^2)}{2\,F_\pi^2}\,\bar{J}(M_K^2,M_\pi^2,M_\pi^2) \\
&- \frac{M_K^4}{8\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2,M_K^2,M_\eta^2) - \frac{(M_K^4 - 4\,M_K^2\,M_\pi^2)}{4\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2,M_K^2,M_\pi^2) \\
&- \frac{(8\,M_K^4 - 6\,M_K^2\,M_\pi^2 + M_\pi^4)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\eta^2) + \frac{(2\,M_K^4 + 7\,M_K^2\,M_\pi^2)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_K^2) \\
&+ \frac{(8\,M_K^4 - 35\,M_K^2\,M_\pi^2 + 21\,M_\pi^4)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\pi^2) + \frac{-5\,M_K^2 + 4\,M_\pi^2}{128\,F_\pi^2\,\pi^2}\ . \qquad \text{(B.18)}
\end{aligned}
$$

B.2.2 $\Delta I = 3/2$ amplitudes

In this section we list the one-loop corrections to the $\Delta I = 3/2$ amplitude.

$$
\begin{aligned}
\Delta_L\mathcal{A}_{3/2}^{(27)} &= -\frac{(M_K^2 - 2\,M_\pi^2)}{2\,F_\pi^2}\,\bar{J}(M_K^2,M_\pi^2,M_\pi^2) - \frac{M_K^4}{24\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2,M_K^2,M_\eta^2) \\
&- \frac{M_K^2\,(5\,M_K^2 - 8\,M_\pi^2)}{8\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2,M_K^2,M_\pi^2) \\
&+ \frac{M_\pi^2\,(4\,M_K^2 - M_\pi^2)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\eta^2) + \frac{(3\,M_K^4 + M_K^2\,M_\pi^2)}{4\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_K^2) \\
&- \frac{(4\,M_K^4 - 22\,M_K^2\,M_\pi^2 + 29\,M_\pi^4)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\pi^2) + \frac{M_K^2 - 2\,M_\pi^2}{32\,F_\pi^2\,\pi^2} \qquad \text{(B.19)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_L\mathcal{A}_{3/2}^{(\epsilon)} &= +\frac{M_K^2}{6\,F_\pi^2}\,\bar{J}(M_K^2,M_\pi^2,M_\eta^2) - \frac{(M_K^2 - 2\,M_\pi^2)}{2\,F_\pi^2}\,\bar{J}(M_K^2,M_\pi^2,M_\pi^2) \\
&- \frac{(21M_K^4 - 8M_K^2\,M_\pi^2)}{24\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2,M_K^2,M_\eta^2) - \frac{(11M_K^4 - 16M_K^2\,M_\pi^2)}{8\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2,M_\pi^2,M_K^2)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{(44\,M_K^4 - 47\,M_K^2\,M_\pi^2 + 9\,M_\pi^4)}{24\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\eta^2) - \frac{(12\,M_K^4 - 17\,M_K^2\,M_\pi^2)}{4\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_K^2) \\
&- \frac{(4\,M_K^4 - 11\,M_K^2\,M_\pi^2 + 15\,M_\pi^4)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\pi^2) + \frac{2\,M_K^2 - M_\pi^2}{16\,F_\pi^2\,\pi^2} \\
&+ \frac{M_K^4\,(M_K^2 - M_\pi^2)}{6\,F_\pi^2\,M_\pi^2}\,\bar{J}^{(0,1,0)}(M_\pi^2, M_K^2, M_\eta^2) \qquad \text{(B.20)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_L \mathcal{A}_{3/2}^{(Z)} = &- \frac{(13\,M_K^2 - 18\,M_\pi^2)}{10\,F_\pi^2}\,\bar{J}(M_K^2, M_\pi^2, M_\pi^2) \\
&- \frac{(10\,M_K^6 + 13\,M_K^4\,M_\pi^2 - 32\,M_K^2\,M_\pi^4 + 24\,M_\pi^6)}{120\,F_\pi^2\,M_\pi^4}\,\bar{J}(M_\pi^2, M_K^2, M_\eta^2) \\
&- \frac{(10\,M_K^6 + 3\,M_K^4\,M_\pi^2 - 28\,M_K^2\,M_\pi^4)}{40\,F_\pi^2\,M_\pi^4}\,\bar{J}(M_\pi^2, M_K^2, M_\pi^2) \\
&+ \frac{(48\,M_K^4 - 40\,M_K^2\,M_\pi^2 + 7\,M_\pi^4)}{40\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\eta^2) - \frac{3\,(21\,M_K^4 - 20\,M_K^2\,M_\pi^2)}{20\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_K^2) \\
&- \frac{(58\,M_K^4 - 22\,M_K^2\,M_\pi^2 - 27\,M_\pi^4)}{40\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\pi^2) + \frac{-M_K^4 + 14\,M_K^2\,M_\pi^2 - 10\,M_\pi^4}{80\,F_\pi^2\,M_\pi^2\,\pi^2} \\
&+ \frac{2\,(M_K^4 - 3\,M_K^2\,M_\pi^2 + 2\,M_\pi^4)}{5\,F_\pi^2}\,\bar{J}^{(0,0,1)}(M_K^2, M_\pi^2, M_\pi^2) \\
&+ \frac{(5M_K^6 - 13M_K^4\,M_\pi^2 + 8M_K^2\,M_\pi^4)}{20\,F_\pi^2\,M_\pi^2}\left[\bar{J}^{(1,0,0)}(M_\pi^2, M_K^2, M_\pi^2) + \bar{J}^{(0,0,1)}(M_\pi^2, M_K^2, M_\pi^2)\right] \\
&+ \frac{(M_K^6 - M_K^4\,M_\pi^2)}{12\,F_\pi^2\,M_\pi^2}\left[\bar{J}^{(0,1,0)}(M_\pi^2, M_K^2, M_\eta^2) + \bar{J}^{(1,0,0)}(M_\pi^2, M_K^2, M_\eta^2)\right] \qquad \text{(B.21)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_L \mathcal{A}_{3/2}^{(g)} = &- \frac{(M_K^2 - 2\,M_\pi^2)}{2\,F_\pi^2}\,\bar{J}(M_K^2, M_\pi^2, M_\pi^2) - \frac{M_K^4}{8\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2, M_K^2, M_\eta^2) \\
&- \frac{(5\,M_K^4 - 8\,M_K^2\,M_\pi^2)}{8\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2, M_K^2, M_\pi^2) \\
&- \frac{(8\,M_K^4 - 6\,M_K^2\,M_\pi^2 + M_\pi^4)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\eta^2) - \frac{(2\,M_K^4 - 5\,M_K^2\,M_\pi^2)}{4\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_K^2) \\
&- \frac{(4\,M_K^4 + 2\,M_K^2\,M_\pi^2 - 3\,M_\pi^4)}{8\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\pi^2) + \frac{M_K^2 - 2\,M_\pi^2}{32\,F_\pi^2\,\pi^2}\;. \qquad \text{(B.22)}
\end{aligned}
$$

B.2.3 $\Delta I = 5/2$ amplitudes

In this section we report the one-loop $\Delta I = 5/2$ amplitude generated by insertions of $e^2 p^0$ vertices from $\mathcal{L}_{\rm elm}$.

$$
\Delta_L \mathcal{A}_{5/2}^{(Z)} = -\frac{8\,(M_K^2 - M_\pi^2)}{5\,F_\pi^2}\,\bar{J}(M_K^2, M_\pi^2, M_\pi^2) - \frac{2\,(M_K^4 - M_K^2\,M_\pi^2)}{5\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2, M_K^2, M_\pi^2)
$$

$$
\begin{aligned}
&- \frac{2\,(M_K^4 + M_K^2\,M_\pi^2 - 2\,M_\pi^4)}{15\,F_\pi^2\,M_\pi^2}\,\bar{J}(M_\pi^2, M_K^2, M_\eta^2) \\
&- \frac{2\,(4\,M_K^2 - M_\pi^2)}{5\,F_\pi^2}\,T(M_\eta^2) - \frac{4\,M_K^4}{5\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_K^2) \\
&- \frac{2\,(6\,M_K^4 - 19\,M_K^2\,M_\pi^2 + 11\,M_\pi^4)}{5\,F_\pi^2\,(M_K^2 - M_\pi^2)}\,T(M_\pi^2) + \frac{-M_K^4 + 9\,M_K^2\,M_\pi^2 - 10\,M_\pi^4}{40\,F_\pi^2\,M_\pi^2\,\pi^2} \\
&+ \frac{4\,(M_K^4 - 3\,M_K^2\,M_\pi^2 + 2\,M_\pi^4)}{5\,F_\pi^2}\,\bar{J}^{(0,0,1)}(M_K^2, M_\pi^2, M_\pi^2) \\
&- \frac{4\,(M_K^4 - M_K^2\,M_\pi^2)}{5\,F_\pi^2}\,\Big[\bar{J}^{(0,0,1)}(M_\pi^2, M_K^2, M_\pi^2) + \bar{J}^{(1,0,0)}(M_\pi^2, M_K^2, M_\pi^2)\Big]\,. \qquad \text{(B.23)}
\end{aligned}
$$

B.2.4 Divergent parts

For completeness, we list here the divergent parts of the mesonic loop amplitudes. We have checked explicitly that they get absorbed by the independently known renormalization of NLO chiral couplings.

$$
\begin{aligned}
\Big[\Delta_L \mathcal{A}_{1/2}^{(27)}\Big]_{\rm div} &= \frac{-28\,M_K^2 + 17\,M_\pi^2}{2\,F_\pi^2}\,\Lambda(\nu_\chi) \\
\Big[\Delta_L \mathcal{A}_{1/2}^{(8)}\Big]_{\rm div} &= \frac{-27\,M_K^2 + 103\,M_\pi^2}{18\,F_\pi^2}\,\Lambda(\nu_\chi) \\
\Big[\Delta_L \mathcal{A}_{1/2}^{(\epsilon)}\Big]_{\rm div} &= \frac{10\,M_K^2 - 43\,M_\pi^2}{6\,F_\pi^2}\,\Lambda(\nu_\chi) \\
\Big[\Delta_L \mathcal{A}_{1/2}^{(Z)}\Big]_{\rm div} &= \frac{7\,(M_K^2 + M_\pi^2)}{2\,F_\pi^2}\,\Lambda(\nu_\chi) \\
\Big[\Delta_L \mathcal{A}_{1/2}^{(g)}\Big]_{\rm div} &= \frac{-M_K^2 + 10\,M_\pi^2}{2\,F_\pi^2}\,\Lambda(\nu_\chi) \\
\Big[\Delta_L \mathcal{A}_{3/2}^{(27)}\Big]_{\rm div} &= \frac{-M_K^2 - 15\,M_\pi^2}{2\,F_\pi^2}\,\Lambda(\nu_\chi) \\
\Big[\Delta_L \mathcal{A}_{3/2}^{(\epsilon)}\Big]_{\rm div} &= \frac{64\,M_K^2 - 27\,M_\pi^2}{6\,F_\pi^2}\,\Lambda(\nu_\chi) \\
\Big[\Delta_L \mathcal{A}_{3/2}^{(Z)}\Big]_{\rm div} &= \frac{17\,(4\,M_K^2 + M_\pi^2)}{10\,F_\pi^2}\,\Lambda(\nu_\chi) \\
\Big[\Delta_L \mathcal{A}_{3/2}^{(g)}\Big]_{\rm div} &= \frac{8\,M_K^2 + M_\pi^2}{2\,F_\pi^2}\,\Lambda(\nu_\chi) \\
\Big[\Delta_L \mathcal{A}_{5/2}^{(Z)}\Big]_{\rm div} &= \frac{48\,(M_K^2 - M_\pi^2)}{5\,F_\pi^2}\,\Lambda(\nu_\chi)\ . \qquad \text{(B.24)}
\end{aligned}
$$

C Alternative convention for LO LECs

In the effective chiral Lagrangians of Sec. 2.2, the meson decay constant in the chiral limit F is the only dimensionful parameter in addition to the Fermi coupling constant G_F. This is the original convention of Cronin [14] for the nonleptonic weak Lagrangian of lowest order and it is used throughout this work. It has definite advantages, e.g., for the renormalization of the various Lagrangians.

However, this convention has a certain aesthetic drawback in that the $K \to 2\pi$ amplitudes (the $K \to 3\pi$ amplitudes as well, for that matter) depend at NLO on the strong LECs L_4 and L_5 even in the isospin limit. These LECs account for the renormalization of F to F_π and F_K at NLO. The associated uncertainties propagate into the uncertainties of the LO LECs $G_8, \ldots$ Since F_π and F_K are much better known than F, it may be useful for phenomenological purposes to redefine the LO LECs so that they are then free of the uncertainties in L_4^r, L_5^r.

A first step consists in generalizing the convention first used in Ref. [15], albeit with a different notation:

$$\begin{aligned} \bar{G}_8 &= G_8 F^4/F_\pi^4 \,, & \bar{G}_{27} &= G_{27} F^4/F_\pi^4 \\ \bar{g}_{\rm ewk} &= g_{\rm ewk} F^2/F_\pi^2 \,, & \bar{Z} &= Z F^2/F_\pi^2 \,. \end{aligned} \tag{C.1}$$

At lowest order, the barred quantities are identical to the original unbarred ones because we always set $F = F_\pi$ at lowest order. Writing the NLO amplitudes (4.6) in terms of the barred LECs of lowest order, the strong LEC L_4^r disappears completely from all $K \to 2\pi$ amplitudes. To get rid of L_5^r as well (at least in the isospin limit), one can introduce a scale factor F_π/F_K [50]. The amplitudes of Eq. (4.6) then take the following form:

$$\begin{aligned} \mathcal{A}_n &= \bar{G}_{27}\, F_\pi \left(M_K^2 - M_\pi^2\right) \bar{\mathcal{A}}_n^{(27)} \qquad\qquad (\text{C.2}) \\ &+ \bar{G}_8\, F_\pi \left\{ \left(M_K^2 - M_\pi^2\right) \left[\bar{\mathcal{A}}_n^{(8)} + \varepsilon^{(2)}\, \bar{\mathcal{A}}_n^{(\varepsilon)}\right] - e^2\, F_\pi^2 \left[\mathcal{A}_n^{(\gamma)} + \bar{Z}\, \bar{\mathcal{A}}_n^{(Z)} + \bar{g}_{\rm ewk}\, \bar{\mathcal{A}}_n^{(g)}\right]\right\} , \end{aligned}$$

where

$$\bar{\mathcal{A}}_n^{(X)} = \begin{cases} a_n^{(X)}\, \dfrac{F_\pi}{F_K} \left[1 + \Delta_L \bar{\mathcal{A}}_n^{(X)} + \Delta_C \bar{\mathcal{A}}_n^{(X)}\right] & \text{if} \quad a_n^{(X)} \neq 0 \,, \\ \Delta_L \mathcal{A}_n^{(X)} + \Delta_C \mathcal{A}_n^{(X)} & \text{if} \quad a_n^{(X)} = 0 \,. \end{cases} \tag{C.3}$$

The change in notation only affects those amplitudes that are non-zero at lowest order.

The amplitudes $\bar{\mathcal{A}}_n^{(X)}$ are related to the original $\mathcal{A}_n^{(X)}$ as follows (only amplitudes for $n = 1/2$ or $3/2$ are affected):

for $X = 27, 8, \varepsilon$:

$$\Delta_C \bar{\mathcal{A}}_n^{(X)} = \Delta_C \mathcal{A}_n^{(X)}|_{\widetilde{\Delta}_C = L_4^r = 0} + \frac{24(M_K^2 - M_\pi^2)}{F_\pi^2} L_5^r(\nu_\chi) \delta_{n,1/2} \delta_{X,\varepsilon}$$

$$\Delta_L \bar{\mathcal{A}}_n^{(X)} = \Delta_L \mathcal{A}_n^{(X)} + \Delta_K + 3\Delta_\pi - \frac{3}{2}(E_K + 3E_\pi)\delta_{n,1/2}\delta_{X,\varepsilon} \; ; \qquad \text{(C.4)}$$

for $X = Z, g$:

$$\Delta_C \bar{\mathcal{A}}_n^{(X)} = \Delta_C \mathcal{A}_n^{(X)}|_{\widetilde{\Delta}_C^{(\mathrm{ew})}=0}$$

$$\Delta_L \bar{\mathcal{A}}_n^{(X)} = \Delta_L \mathcal{A}_n^{(X)} + \Delta_K + 5\Delta_\pi \; . \qquad \text{(C.5)}$$

From the definitions of F_π and $F_{K^\pm}$ in Ref. [27] one obtains [$T(M^2)$ is defined in (B.8) and M_η^2 stands for $(4M_K^2 - M_\pi^2)/3$ as in all loop amplitudes]

$$\begin{aligned}
\Delta_\pi &= \frac{M_\pi^2}{F_\pi^2} T(M_\pi^2) + \frac{M_K^2}{2F_\pi^2} T(M_K^2) \\
\Delta_K &= \frac{3M_\pi^2}{8F_\pi^2} T(M_\pi^2) + \frac{3M_K^2}{4F_\pi^2} T(M_K^2) + \frac{(4M_K^2 - M_\pi^2)}{8F_\pi^2} T(M_\eta^2) \\
E_\pi &= -\frac{(M_K^2 - M_\pi^2)}{F_\pi^2} T(M_K^2) + \frac{(M_K^2 - M_\pi^2)}{(4\pi)^2 F_\pi^2} \\
E_K &= \frac{3M_\pi^2}{4F_\pi^2} T(M_\pi^2) - \frac{2(M_K^2 - M_\pi^2)}{F_\pi^2} T(M_K^2) - \frac{(8M_K^2 - 5M_\pi^2)}{4F_\pi^2} T(M_\eta^2) \\
&\quad + \frac{3(M_K^2 - M_\pi^2)}{(4\pi)^2 F_\pi^2} \; . \qquad \text{(C.6)}
\end{aligned}$$

As can be seen from Eqs. (C.2,. . . ,C.6), L_4^r has disappeared completely from the amplitudes $\mathcal{A}_n$ whereas L_5^r occurs only in the isospin violating amplitude $\Delta_C \bar{\mathcal{A}}_{1/2}^{(\varepsilon)}$. In spite of its conceptual advantages, we have not used this alternative convention in this paper because L_4, L_5 reappear anyway through the large-N_c relations for the NLO LECs N_i, D_i, Z_i. Moreover, the large-N_c relations for g_8, g_{27} and g_{ewk} would also be affected. Finally, consistent with the expansion to leading order in $1/N_c$, L_4^r and L_5^r are set equal to their large-N_c limits as discussed in Sec. 5.3.

D Details on the optical theorem analysis

In this appendix we report the explicit form of functions needed when studying the unitarity condition in the presence of isospin breaking. Let us start with the IR divergent factors:

$$B_{\pi\pi} = B_{+-}(M_\gamma) \qquad \text{(see Eq.(B.1))} \qquad \text{(D.1)}$$

$$C_{\pi\pi} = 16\pi \Big[(u - 2M_\pi^2)\, G_{+-\gamma}(u) - (t - 2M_\pi^2)\, G_{+-\gamma}(t) \Big] \; . \qquad \text{(D.2)}$$

The definition of the variables t, u and the function $G_{+-\gamma}(x)$ can be found in Refs. [55, 56].

In order to define the remaining ingredients, we need to fix the notation. The four-momenta are denoted as follows:

$$K(P) \longrightarrow \pi^+(q_+)\, \pi^-(q_-)\, \gamma(k) \longrightarrow \pi^+(p_+)\, \pi^-(p_-) \; .$$

The differential phase space is then given by

$$d\Phi_{+-\gamma} = \frac{d^3\, q_+}{(2\pi)^3\, 2q_+^0}\, \frac{d^3\, q_-}{(2\pi)^3\, 2q_-^0}\, \frac{d^3\, k}{(2\pi)^3\, 2k^0}\, (2\pi)^4\, \delta^{(4)}(q_+ + q_- + k - p_+ - p_-)\ . \tag{D.3}$$

Then, after performing the sum over photon polarizations, the radiative amplitudes in leading Low approximation generate the following factors:

$$f_1^{\rm rad} = -\frac{q_+^2}{(q_+\cdot k + \frac{M_\gamma^2}{2})^2} - \frac{q_-^2}{(q_-\cdot k + \frac{M_\gamma^2}{2})^2} + \frac{2q_+\cdot q_-}{(q_+\cdot k + \frac{M_\gamma^2}{2})(q_-\cdot k + \frac{M_\gamma^2}{2})} \tag{D.4}$$

$$\begin{aligned} f_2^{\rm rad} = & \frac{p_+\cdot q_+}{(p_+\cdot k - \frac{M_\gamma^2}{2})(q_+\cdot k + \frac{M_\gamma^2}{2})} + \frac{p_-\cdot q_-}{(p_-\cdot k - \frac{M_\gamma^2}{2})(q_-\cdot k + \frac{M_\gamma^2}{2})} \\ & - \frac{p_+\cdot q_-}{(p_+\cdot k - \frac{M_\gamma^2}{2})(q_-\cdot k + \frac{M_\gamma^2}{2})} - \frac{p_-\cdot q_+}{(p_-\cdot k - \frac{M_\gamma^2}{2})(q_+\cdot k + \frac{M_\gamma^2}{2})}\ . \end{aligned} \tag{D.5}$$

References

[1] V. Cirigliano, J.F. Donoghue and E. Golowich, Phys. Lett. **B 450** (1999) 241.

[2] S. Gardner, G. Valencia, Phys. Lett. **B 466** (1999) 355.

[3] V. Cirigliano, J.F. Donoghue and E. Golowich, Phys. Rev. **D 61** (2000) 093001; Err. ibid. **D 63** (2001) 059903.

[4] V. Cirigliano, J.F. Donoghue and E. Golowich, Phys. Rev. **D 61** (2000) 093002.

[5] V. Cirigliano, J.F. Donoghue and E. Golowich, Eur. Phys. J. **C 18** (2000) 83.

[6] S. Gardner, G. Valencia, Phys. Rev. **D 62** (2000) 094024

[7] C.E. Wolfe and K. Maltman, Phys. Lett. **B 482** (2000) 77.

[8] C.E. Wolfe and K. Maltman, Phys. Rev. **D 63** (2001) 014008.

[9] A. Aloisio et al. (KLOE Collaboration), Phys. Lett. **B 538** (2002) 21.

[10] K. Hagiwara et al., Particle Data Group, Review of Particle Physics, Phys. Rev. **D 66** (2002) 010001.

[11] J. Gasser and H. Leutwyler, Nucl. Phys. **B 250** (1985) 465.

[12] H.W. Fearing and S. Scherer, Phys. Rev. **D 53** (1996) 315.

[13] J. Bijnens, G. Colangelo and G. Ecker, JHEP **9902** (1999) 020.

[14] J.A. Cronin, Phys. Rev. **161** (1967) 1483.

[15] J. Kambor, J. Missimer and D. Wyler, Nucl. Phys. **B 346** (1990) 17.

[16] G. Ecker, J. Kambor and D. Wyler, Nucl. Phys. **B 394** (1993) 101.

[17] J. Bijnens, E. Pallante and J. Prades, Nucl. Phys. **B 521** (1998) 305.

[18] G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. **B 321** (1989) 311.

[19] R. Urech, Nucl. Phys. **B 433** (1995) 234.

[20] J. Bijnens and M.B. Wise, Phys. Lett. **B 137** (1984) 245.

[21] B. Grinstein, S.-J. Rey and M.B. Wise, Phys. Rev. **D 33** (1986) 1495.

[22] G. Ecker et al., Nucl. Phys. **B 591** (2000) 419.

[23] C. Bernard et al., Phys. Rev. **D 32** (1985) 2343.

[24] G. Ecker, G. Müller, H. Neufeld and A. Pich, Phys. Lett. **B 477** (2000) 88.

[25] H. Neufeld and H. Rupertsberger, Z. Phys. **C 68** (1995) 91; Z. Phys. **C 71** (1996) 131.

[26] V. Cirigliano et al., Eur. Phys. J. **C 23** (2002) 121.

[27] M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, Eur. Phys. J. **C 12** (2000) 469.

[28] E. Pallante, A. Pich and I. Scimemi, Nucl. Phys. **B 617** (2001) 441.

[29] K.G. Wilson, Phys. Rev. **179** (1969) 1499;
W. Zimmermann, *Lectures on Elementary Particles and Quantum Field Theory*, Brandeis Summer Institute (1970), Vol. 1 (MIT Press, Cambridge, MA, 1970).

[30] E.C.G. Stueckelberg and A. Petermann, Helv. Phys. Acta **26** (1953) 499;
M. Gell-Mann and F.E. Low, Phys. Rev. **95** (1954) 1300;
C.G. Callan, Jr., Phys. Rev. **D 2** (1970) 1541;
K. Symanzik, Comm. Math. Phys. **18** (1970) 227; **23** (1971) 49.

[31] M.K. Gaillard and B.W. Lee, Phys. Rev. Lett. **33** (1974) 108;
G. Altarelli and L. Maiani, Phys. Lett. **B 52** (1974) 351.

[32] A.I. Vainshtein, V.I. Zakharov and M.A. Shifman, JETP Lett. **22** (1975) 55; Nucl. Phys. **B 120** (1977) 316.

[33] F.J. Gilman and M.B. Wise, Phys. Rev. **D 20** (1979) 2392; Phys. Rev. **D 21** (1980) 3150.

[34] A.J. Buras, M. Jamin, M.E. Lautenbacher and P.H. Weisz, Nucl. Phys. **B 400** (1993) 37 ;
M. Ciuchini, E. Franco, G. Martinelli and L. Reina, Nucl. Phys. **B 415** (1994) 403.

[35] W.A. Bardeen, A.J. Buras and J.-M. Gérard, Nucl. Phys. **B 293** (1987) 787; Phys. Lett. **B 211** (1988) 343, **B 192** (1987) 138, **B 180** (1986) 133;
A.J. Buras and J.-M. Gérard, Nucl. Phys. **B 264** (1986) 371; Phys. Lett. **B 192** (1987) 156.

[36] E. de Rafael, Nucl. Phys. B (Proc. Suppl.) **7A** (1989) 1.

[37] A. Pich, Nucl. Phys. B (Proc. Suppl.) **7A** (1989) 194.

[38] A. Pich and E. de Rafael, Nucl. Phys. **B 358** (1991) 311; Phys. Lett. **B 374** (1996) 186.

[39] M. Jamin and A. Pich, Nucl. Phys. **B 425** (1994) 15.

[40] E. Gámiz et al., JHEP **0301** (2003) 060;
M. Jamin, J.A. Oller and A. Pich, Eur. Phys. J. **C 24** (2002) 237;
K. Maltman and J. Kambor, Phys. Rev. **D 65** (2002) 074013;
S.M. Chen et al., Eur. Phys. J. **C 22** (2001) 31;
H. Wittig, Nucl. Phys. Proc. Suppl. **119** (2003) 59.

[41] R.J. Crewther, Nucl. Phys. **B 264** (1986) 277;
M. Leurer, Phys. Lett. **B 201** (1988) 128.

[42] J. Bijnens, G. Colangelo and G. Ecker, Ann. Phys. **280** (2000) 100.

[43] A. Pich, in Proc. of "The Phenomenology of Large-N_c QCD", Tempe, Arizona, Jan. 2002, Ed. R. Lebed, World Scientific (Singapore, 2002); `hep-ph/0205030`.

[44] S. Peris, M. Perrottet and E. de Rafael, JHEP **9805** (1998) 011.

[45] V. Cirigliano, G. Ecker, H. Neufeld and A. Pich, JHEP **0306** (2003) 012.

[46] G. Amorós, J. Bijnens and P. Talavera, Nucl. Phys. **B 602** (2001) 87.

[47] M. Knecht and A. Nyffeler, Eur. Phys. J. **C 21** (2001) 659.

[48] J. Bijnens and J. Prades, Nucl. Phys. **B 490** (1997) 239.

[49] B. Moussallam, Nucl. Phys. **B 504** (1997) 381.

[50] J. Kambor, J. Missimer and D. Wyler, Phys. Lett. **B 261** (1991) 496.

[51] H. Leutwyler, Phys. Lett. **B 378** (1996) 313.

[52] D.E. Groom et al., Particle Data Group, Review of Particle Physics, Eur. Phys. J. **C 15** (2000) 1.

[53] G. Colangelo, J. Gasser and H. Leutwyler, Nucl. Phys. B **603** (2001) 125.

[54] S. Gardner, U.-G. Meißner and G. Valencia, Phys. Lett. **B 508** (2001) 44.

[55] M. Knecht and R. Urech, Nucl. Phys. **B 519** (1998) 329.

[56] M. Knecht and A. Nehme, Phys. Lett. **B 532** (2002) 55.

[57] V. Cirigliano, G. Ecker, H. Neufeld and A. Pich, Phys. Rev. Lett. **91** (2003) 162001.

[58] J. Bijnens and J. Prades, JHEP **0006** (2000) 035;
V. Cirigliano, J.F. Donoghue, E. Golowich and K. Maltman, Phys. Lett. **B 555** (2003) 71;
T. Hambye, S. Peris and E. de Rafael, JHEP **0305** (2003) 027.

[59] J.F. Donoghue, E. Golowich, B.R. Holstein and J. Trampetic, Phys. Lett. **B 179** (1986) 361;
A.J. Buras and J.-M. Gérard, Phys. Lett. **B 192** (1987) 156;
H.-Y. Cheng, Phys. Lett. **B 201** (1988) 155;
M. Lusignoli, Nucl. Phys. **B 325** (1989) 33.

[60] J. Gasser, A. Rusetsky and I. Scimemi, hep-ph/0305260.