

Towards a determination of the chiral couplings at NLO in $1/N_C$: $L_8^r(\mu)$ and $C_{38}^r(\mu)$

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Abstract

We present a dispersive method which allows to investigate the low-energy couplings of chiral perturbation theory at the next-to-leading order (NLO) in the $1/N_C$ expansion, keeping full control of their renormalization scale dependence. Using the resonance chiral theory Lagrangian, we perform a NLO calculation of the scalar and pseudoscalar two-point functions, within the single-resonance approximation. Imposing the correct QCD short-distance constraints, one determines their difference $\Pi(t) \equiv \Pi_S(t) - \Pi_P(t)$ in terms of the pion decay constant and resonance masses. Its low momentum expansion fixes then the low-energy chiral couplings L_8 and C_{38} . At $\mu_0 = 0.77$ GeV, we obtain $L_8^r(\mu_0)^{SU(3)} = (0.6 \pm 0.4) \cdot 10^{-3}$ and $C_{38}^r(\mu_0)^{SU(3)} = (2 \pm 6) \cdot 10^{-6}$.

1 The large- N_C limit

In recent years we have witnessed a spectacular progress in our understanding of low-energy effective field theories [1, 2, 3]. In particular, chiral perturbation theory (χ PT) has been established as a very powerful tool to incorporate the chiral symmetry constraints when analysing the strong interactions in the non-perturbative regime [4, 5, 6]. The precision required in present phenomenological applications makes necessary to include corrections of $\mathcal{O}(p^6)$. While many two-loop χ PT calculations have been already performed [7, 8, 9, 10, 11, 12, 13, 14, 15], the large number of unknown low-energy couplings (LECs) appearing at this order puts a clear limit to the achievable accuracy [16].

The limit of an infinite number of quark colours has proved very useful to bridge the gap between χ PT and the underlying QCD dynamics [17, 18]. Assuming confinement, the strong dynamics at $N_C \rightarrow \infty$ is given by tree diagrams with infinite sums of hadron exchanges, which correspond to the tree approximation of some local effective Lagrangian [19, 20]. Resonance chiral theory (R χ T) provides a correct framework to incorporate the massive mesonic states within an effective Lagrangian formalism [21]. Integrating out the heavy fields one recovers the χ PT Lagrangian with explicit values of the chiral LECs in terms of resonance parameters. Moreover, the short-distance properties of QCD impose stringent constraints on the low-energy parameters [22].

Truncating the infinite tower of meson resonances to the lowest states with 0^{-+} , 0^{++} , 1^{--} and 1^{++} quantum numbers (single-resonance approximation, SRA), one gets a very successful prediction of the $\mathcal{O}(p^4 N_C)$ χ PT couplings in terms of only three parameters: M_V , M_S and the pion decay constant F [17]. Some $\mathcal{O}(p^6)$ LECs have been already predicted in this way, by studying an appropriate set of three-point functions [23]. More recently, the program to determine all $\mathcal{O}(p^6)$ LECs at leading order in $1/N_C$ has been put on very solid grounds, with a complete classification of the needed terms in the R χ T Lagrangian [24].

Since chiral loop corrections are of next-to-leading order (NLO) in the $1/N_C$ expansion, the large- N_C determination of the LECs is unable to control their renormalization-scale dependence. For couplings related with the scalar sector this introduces large uncertainties, because their μ dependence is very sizable. A first analysis of resonance loop contributions to the running of $L_{10}^r(\mu)$ was attempted in Ref. [25]. More recently, a NLO determination of the χ PT coupling $L_9^r(\mu)$ has been achieved, through a one-loop calculation of the vector form factor in R χ T [26]. In spite of all the complexity associated with the still not so well understood renormalization of R χ T [26, 27], this calculation has shown the potential predictability at the NLO in $1/N_C$.

In this article we present a NLO determination of the couplings $L_8^r(\mu)$ and $C_{38}^r(\mu)$. Using analyticity and unitarity we can avoid all technicalities associated with the renormalization procedure, reducing the calculation to tree-level diagrams plus dispersion relations. This allows to understand the underlying physics in a much more transparent way. In particular, the subtle cancellations among many unknown renormalized couplings found in Ref. [26] and the relative simplicity of the final result can be better understood in terms of the imposed short-distance constraints.

Let us consider the two-point correlation functions of two scalar or pseudoscalar currents, in the chiral limit. Of particular interest is their difference $\Pi(t) \equiv \Pi_S(t) - \Pi_P(t)$, which is

identically zero in QCD perturbation theory. When $t \rightarrow \infty$, this correlator vanishes as $1/t^2$, with a coefficient proportional to $\alpha_s \langle \bar{q}\Gamma q \bar{q}\Gamma q \rangle$ [28, 29]. The low-momentum expansion of $\Pi(t)$ is determined by χ PT to have the form [4, 7]

$$\begin{aligned} \Pi(t) = B_0^2 & \left\{ \frac{2F^2}{t} + 32L_8^r(\mu) + \frac{\Gamma_8}{\pi^2} \left(1 - \ln \frac{-t}{\mu^2} \right) \right. \\ & \left. + \frac{t}{F^2} \left[32C_{38}^r(\mu) - \frac{\Gamma_{38}^{(L)}}{\pi^2} \left(1 - \ln \frac{-t}{\mu^2} \right) + \mathcal{O}(N_C^0) \right] + \mathcal{O}(t^2) \right\}, \end{aligned} \quad (1)$$

with $\Gamma_8 = 5/48$ [3/16] and $\Gamma_{38}^{(L)} = -5L_5^r/6 [-3L_5^r/2]$ in the $SU(3)_L \otimes SU(3)_R [U(3)_L \otimes U(3)_R]$ effective theory [4, 7]. The correlator is proportional to $B_0^2 \equiv \langle \bar{q}q \rangle^2 / F^4$, which guarantees the right dependence with the QCD renormalization scale. The couplings F^2 , L_8 and C_{38}/F^2 are $\mathcal{O}(N_C)$, while Γ_8 and $\Gamma_{38}^{(L)}/F^2$ are $\mathcal{O}(1)$ and represent a NLO effect.¹

In the large- N_C limit, $\Pi(t)$ has the general form

$$\Pi(t) = 2B_0^2 \left\{ \sum_i \frac{8c_{m_i}^2}{M_{S_i}^2 - t} - \sum_i \frac{8d_{m_i}^2}{M_{P_i}^2 - t} + \frac{F^2}{t} \right\}, \quad (2)$$

which involves an infinite number of scalar and pseudoscalar meson exchanges. This expression can be easily obtained within $R\chi T$, with c_{m_i} and d_{m_i} being the relevant meson couplings. For a finite number of resonances,² one finds that imposing the right high-energy behaviour ($\sim 1/t^2$) constrains the resonance parameters to obey the relations [33]:

$$\sum_i (c_{m_i}^2 - d_{m_i}^2) = \frac{F^2}{8}, \quad \sum_i (c_{m_i}^2 M_{S_i}^2 - d_{m_i}^2 M_{P_i}^2) = \tilde{\delta}, \quad (3)$$

where $\tilde{\delta} \equiv 3\pi\alpha_s F^4/4 \approx 0.08\alpha_s F^2 \times (1 \text{ GeV})^2$. Truncating the infinite sums to their first contributing states and neglecting $\tilde{\delta}$, these relations fix the corresponding scalar and pseudoscalar couplings in terms of the resonance masses:

$$c_m^2 = \frac{F^2}{8} \frac{M_P^2}{M_P^2 - M_S^2}, \quad d_m^2 = \frac{F^2}{8} \frac{M_S^2}{M_P^2 - M_S^2}. \quad (4)$$

Note that Eqs. (4) imposes $M_P \geq M_S$. On the other hand, the low-energy expansion of (2) determines [22, 24]

$$\begin{aligned} L_8 &= \sum_i \left\{ \frac{c_{m_i}^2}{2M_{S_i}^2} - \frac{d_{m_i}^2}{2M_{P_i}^2} \right\} \approx \frac{F^2}{16M_S^2} + \frac{F^2}{16M_P^2}. \\ C_{38} &= \sum_i \left\{ \frac{c_m^2 F^2}{2M_S^4} - \frac{d_m^2 F^2}{2M_P^4} \right\} \approx \frac{F^4}{16M_P^2 M_S^2} \left(1 + \frac{M_P^2}{M_S^2} + \frac{M_S^2}{M_P^2} \right). \end{aligned} \quad (5)$$

Using the approximate constraint $M_P/\sqrt{2} \approx M_S \sim 1 \text{ GeV}$ [34], this gives $L_8 \approx 3F^2/(32M_S^2) \approx 0.7 \cdot 10^{-3}$ and $C_{38} \approx 7F^4/(64M_S^4) \approx 7 \cdot 10^{-6}$. However, one does not know at which scale μ these predictions apply.

¹Note that we have defined a dimensionless C_{38} and C_{38}^r ; in the notation of Ref. [7] that corresponds to $F^2 C_{38}$ and C_{38}^r respectively.

²Some issues related to the truncation of the spectrum to a finite number of resonances are discussed in Refs. [30, 31, 32].

2 NLO corrections

At the NLO in $1/N_C$, $\Pi(t)$ has a contribution from one-particle exchanges, with the structure in Eq. (2), plus one-loop corrections $\Delta\Pi(t)$ generating absorptive contributions from two-particle exchanges. The corresponding spectral functions of the scalar and pseudoscalar correlators take the form:

$$\begin{aligned}\frac{1}{\pi}\text{Im}\Pi_S(t) &= 2B_0^2 \left\{ 8c_m^2 \delta(t - M_S^2) + \frac{3\rho_S(t)}{16\pi^2} \right\}, \\ \frac{1}{\pi}\text{Im}\Pi_P(t) &= 2B_0^2 \left\{ F^2 \delta(t) + 8d_m^2 \delta(t - M_P^2) + \frac{3\rho_P(t)}{16\pi^2} \right\},\end{aligned}\quad (6)$$

with

$$\begin{aligned}\rho_S(t) &= \frac{\theta(t)}{2} |F_S^{\pi\pi}(t)|^2 + \theta(t - M_P^2) \left(1 - \frac{M_P^2}{t}\right) |F_S^{P\pi}(t)|^2 \\ &+ \theta(t - M_A^2) \frac{t^2}{4M_A^4} \left(1 - \frac{M_A^2}{t}\right)^3 |F_S^{A\pi}(t)|^2 + \dots\end{aligned}\quad (7)$$

$$\begin{aligned}\rho_P(t) &= \theta(t - M_V^2) \frac{t^2}{4M_V^4} \left(1 - \frac{M_V^2}{t}\right)^3 |F_P^{V\pi}(t)|^2 \\ &+ \theta(t - M_S^2) \left(1 - \frac{M_S^2}{t}\right) |F_P^{S\pi}(t)|^2 + \dots\end{aligned}\quad (8)$$

We have adopted the single-resonance approximation and, moreover, we have only taken explicitly into account the lowest-mass two-particle exchanges: two Goldstone bosons or one Goldstone and one heavy resonance. In the energy region we are interested in, exchanges of two heavy resonances or higher multiplicity states are kinematically suppressed. Our normalization takes into account the different flavour-exchange possibilities. The relevant two-particle cuts are governed by the following scalar,

$$\begin{aligned}F_S^{\pi\pi}(t) &= \frac{M_S^2}{M_S^2 - t}, \\ F_S^{P\pi}(t) &= \sqrt{1 - \frac{M_S^2}{M_P^2} \frac{M_S M_P}{M_S^2 - t}}, \\ F_S^{A\pi}(t) &= 0,\end{aligned}\quad (9)$$

and pseudoscalar,

$$\begin{aligned}F_P^{V\pi}(t) &= -2\sqrt{1 - \frac{M_V^2}{M_A^2} \frac{M_V^2 M_P^2}{(M_P^2 - t)t}}, \\ F_P^{S\pi}(t) &= \sqrt{1 - \frac{M_S^2}{M_P^2} \frac{M_S^2 M_P^2}{(M_P^2 - t)t}},\end{aligned}\quad (10)$$

form factors [35].

The $R\chi T$ couplings [21, 24] generating these form factors have been determined imposing a good high-energy behaviour of the corresponding spectral functions, i.e. that the individual form factor contributions to $\rho_S(t)$ and $\rho_P(t)$ should vanish at infinite momentum transfer. Moreover, we have used the constraints (4) and the analogous relations (Weinberg sum rules and good high-energy behaviour of the vector form factor) emerging in the vector/axial-vector sector. It is quite remarkable that these short-distance constraints completely determine the form factors in terms of the resonance masses [35]. The form factor $F_S^{A\pi}(t)$ turns out to be identically zero, within the SRA.³

Using its known analyticity properties, $\Delta\Pi(t)$ can be obtained from the spectral functions through a dispersion relation, up to a subtraction term which has the same structure as the tree-level scalar and pseudoscalar resonance exchanges. Therefore, the unknown subtraction constants can be absorbed by a redefinition of c_m , d_m , M_S and M_P at NLO in $1/N_C$:

$$\Pi(t) = 2 B_0^2 \left\{ \frac{8 c_m^{r2}}{M_S^{r2} - t} - \frac{8 d_m^{r2}}{M_P^{r2} - t} + \frac{F^2}{t} + \Delta\Pi(t)|_\rho \right\}. \quad (11)$$

The explicit expression of $\Delta\Pi(t)|_\rho$ is relegated to appendix A. At large values of t , the one-loop contribution has the behaviour

$$\Delta\Pi(t)|_\rho = \frac{F^2}{t} \delta_{\text{NLO}}^{(1)} + \frac{F^2 M_S^2}{t^2} \left(\delta_{\text{NLO}}^{(2)} + \tilde{\delta}_{\text{NLO}}^{(2)} \ln \frac{-t}{M_S^2} \right) + \mathcal{O}\left(\frac{1}{t^3}\right). \quad (12)$$

Since the logarithmic term $\ln(-t)/t^2$ should vanish, one obtains the constraint

$$\tilde{\delta}_{\text{NLO}}^{(2)} = \frac{M_S^2}{32\pi^2 F^2} \left\{ 6 \left(1 - \frac{M_V^2}{M_A^2} \right) \frac{M_P^4}{M_S^4} + 3 \left(1 - 2 \frac{M_P^2}{M_S^2} \right) \right\} = 0, \quad (13)$$

leading to

$$\left(1 - \frac{M_V^2}{M_A^2} \right) = \frac{M_S^2}{M_P^2} \left(1 - \frac{M_S^2}{2 M_P^2} \right), \quad (14)$$

which requires $M_A \leq \sqrt{2} M_V$. Imposing the right short-distance behaviour ($\sim 1/t^2$) in $\Pi(t)$, one gets

$$\begin{aligned} F^2 (1 + \delta_{\text{NLO}}^{(1)}) - 8 c_m^{r2} + 8 d_m^{r2} &= 0, \\ F^2 M_S^2 \delta_{\text{NLO}}^{(2)} - 8 c_m^{r2} M_S^{r2} + 8 d_m^{r2} M_P^{r2} &= -8 \tilde{\delta}, \end{aligned} \quad (15)$$

where the corrections

$$\delta_{\text{NLO}}^{(m)} = \frac{3 M_S^2}{32\pi^2 F^2} \left\{ 1 + \left(1 - \frac{M_S^2}{M_P^2} \right) \xi_{S\pi}^{(m)} + 2 \left(\frac{M_P^2}{M_S^2} - 1 \right) \xi_{P\pi}^{(m)} - \frac{2 M_P^2}{M_S^2} \left(1 - \frac{M_V^2}{M_A^2} \right) \xi_{V\pi}^{(m)} \right\} \quad (16)$$

³We are using the SRA and only operators constructed with up to $\mathcal{O}(p^2)$ chiral tensors are allowed in the $R\chi T$ Lagrangian. While the inclusion of extra resonances would certainly generate a non-zero value for $F_S^{A\pi}(t)$, its numerical impact in our final results would be small, because the short-distance constraints would force a compensating effect with the other contributions. The same comment applies to possible higher-derivative chiral structures, which generate a bad high-energy behaviour and, therefore, would need to be severely tuned.

are known functions of the resonance masses:

$$\begin{aligned}
\xi_{S\pi}^{(1)} &= 1 - \frac{6M_S^2}{M_P^2} + \left(\frac{4M_S^2}{M_P^2} - \frac{6M_S^4}{M_P^4} \right) \ln \left(\frac{M_P^2}{M_S^2} - 1 \right), \\
\xi_{P\pi}^{(1)} &= 1 + \frac{M_P^2}{M_S^2} \ln \left(1 - \frac{M_S^2}{M_P^2} \right), \\
\xi_{V\pi}^{(1)} &= 1 + \frac{3M_V^2}{M_P^2} \left[\frac{M_V^2}{M_P^2} - \frac{3}{2} + \left(1 - \frac{M_V^2}{M_P^2} \right)^2 \ln \left(\frac{M_P^2}{M_V^2} - 1 \right) \right], \\
\xi_{S\pi}^{(2)} &= -4 + \left(2 - \frac{4M_S^2}{M_P^2} \right) \ln \left(\frac{M_P^2}{M_S^2} - 1 \right), \\
\xi_{P\pi}^{(2)} &= 1 + \ln \left(\frac{M_P^2}{M_S^2} - 1 \right), \\
\xi_{V\pi}^{(2)} &= \frac{M_P^2}{M_S^2} \left(1 - \ln \frac{M_S^2}{M_V^2} \right) - \frac{2M_V^2}{M_S^2} \left(1 - \frac{M_V^2}{M_P^2} \right) \\
&\quad + \left(\frac{M_P^2}{M_S^2} + \frac{2M_V^2}{M_S^2} \right) \left(1 - \frac{M_V^2}{M_P^2} \right)^2 \ln \left(\frac{M_P^2}{M_V^2} - 1 \right).
\end{aligned} \tag{17}$$

Note that Eqs. (15) determine the NLO couplings c_m^r and d_m^r :

$$\begin{aligned}
c_m^{r^2} &= \frac{F^2}{8} \frac{M_P^{r^2}}{M_P^{r^2} - M_S^{r^2}} \left(1 + \delta_{\text{NLO}}^{(1)} - \frac{M_S^2}{M_P^2} \delta_{\text{NLO}}^{(2)} - \frac{8}{M_P^2 F^2} \tilde{\delta} \right), \\
d_m^{r^2} &= \frac{F^2}{8} \frac{M_S^{r^2}}{M_P^{r^2} - M_S^{r^2}} \left(1 + \delta_{\text{NLO}}^{(1)} - \delta_{\text{NLO}}^{(2)} - \frac{8}{M_S^2 F^2} \tilde{\delta} \right).
\end{aligned} \tag{18}$$

3 $L_8^r(\mu)$ at NLO

The low-momentum expansion of the $R\chi T$ correlator in Eq. (11) reproduces the $U(3)_L \otimes U(3)_R$ χ PT result (1), with a definite prediction for the LEC $L_8^r(\mu)$:

$$\begin{aligned}
\bar{L}_8^{U(3)} &\equiv \left[L_8^r(\mu) + \frac{\Gamma_8}{32\pi^2} \ln \frac{\mu^2}{M_S^2} \right]_{U(3)} \\
&= \frac{F^2}{16} \left(\frac{1}{M_S^{r^2}} + \frac{1}{M_P^{r^2}} \right) \left\{ 1 + \delta_{\text{NLO}}^{(1)} - \frac{M_S^{r^2} \delta_{\text{NLO}}^{(2)} + 8 \tilde{\delta} / F^2}{M_S^{r^2} + M_P^{r^2}} \right\} - \frac{3\Delta}{256\pi^2},
\end{aligned} \tag{19}$$

with

$$\begin{aligned}
\Delta &= 1 - \left(1 - \frac{M_V^2}{M_A^2} \right) \left[\frac{17}{6} - \frac{7M_V^2}{M_P^2} + \frac{4M_V^4}{M_P^4} + \left(1 - \frac{4M_V^2}{M_P^2} \right) \left(1 - \frac{M_V^2}{M_P^2} \right)^2 \ln \left(\frac{M_P^2}{M_V^2} - 1 \right) \right] \\
&\quad + \left(\frac{M_P^2}{M_S^2} - 1 \right) \left[2 + \left(\frac{2M_P^2}{M_S^2} - 1 \right) \ln \left(1 - \frac{M_S^2}{M_P^2} \right) + \frac{M_S^2}{6M_P^2} + \frac{M_S^4}{M_P^4} - \frac{4M_S^6}{M_P^6} \right] \\
&\quad + \frac{M_S^4}{M_P^4} \left(1 - \frac{M_S^2}{M_P^2} \right) \left(3 - \frac{4M_S^2}{M_P^2} \right) \ln \left(\frac{M_P^2}{M_S^2} - 1 \right).
\end{aligned} \tag{20}$$

We have used the relations in Eqs. (18) to eliminate the explicit dependence on the effective couplings c_m^r and d_m^r .

Eq. (19) modifies the large- N_C result in (5) with NLO corrections $\delta_{\text{NLO}}^{(1)}$, $\delta_{\text{NLO}}^{(2)}$ and Δ , which are fully known in terms of resonance masses. We have also taken into account the tiny correction $\tilde{\delta}$. Moreover, our calculation has generated the right renormalization-scale dependence, giving rise to an absolute prediction for the scale-independent parameter $\bar{L}_8^{U(3)}$. Since we are working within the large- N_C framework, the Goldstone-nonet loops reproduce the non-analytic $\ln(-t)$ structure that arises in $U(3)_L \otimes U(3)_R$ χ PT. To make contact with the usual $SU(3)_L \otimes SU(3)_R$ theory, we still need to integrate out the singlet η_1 field. Computing the massive one-loop η_1 contribution to $\Pi(t)$, one easily gets the known relation [36] between the corresponding L_8 couplings in the two chiral effective theories. At $\mathcal{O}(p^4)$ in the $U(3)_L \otimes U(3)_R$ theory, one finds:

$$\bar{L}_8^{SU(3)} = \bar{L}_8^{U(3)} + \frac{\Gamma_8^{SU(3)} - \Gamma_8^{U(3)}}{32\pi^2} \ln \frac{M_{\eta_1}^2}{M_S^2} = \bar{L}_8^{U(3)} - \frac{1}{384\pi^2} \ln \frac{M_{\eta_1}^2}{M_S^2}. \quad (21)$$

The different input parameters are defined in the chiral limit. We take the ranges [4, 17, 37, 38, 39] $M_V = (770 \pm 5)$ MeV, $M_S^r = (1.14 \pm 0.16)$ GeV, $M_P^r = (1.3 \pm 0.1)$ GeV, $M_{\eta_1} = (0.85 \pm 0.05)$ GeV and $F = (89 \pm 2)$ MeV, and use Eq. (14) to fix M_A , keeping the constraint $M_P \geq M_S$ from Eqs. (4) and imposing $M_A \geq 1$ GeV. The correction $\tilde{\delta}$ turns out to be negligible. One obtains the numerical prediction

$$\bar{L}_8^{SU(3)} = (0.4 \pm 0.4) \cdot 10^{-3}. \quad (22)$$

The largest uncertainties originate in the badly known values of M_S^r and M_P^r , which already appear in the leading order prediction (5). To account for the higher-mass intermediate states which have been neglected in (6), we have added an additional truncation error equal to $0.12 \cdot 10^{-3}$, the contribution of the heaviest included channel ($P\pi$). All errors have been added in quadrature. At the usual χ PT renormalization scale $\mu_0 = 0.77$ GeV, Eq. (22) implies

$$L_8^r(\mu_0)^{SU(3)} = (0.6 \pm 0.4) \cdot 10^{-3}, \quad (23)$$

to be compared with the value $L_8^r(\mu_0)^{SU(3)} = (0.9 \pm 0.3) \cdot 10^{-3}$, usually adopted in $\mathcal{O}(p^4)$ phenomenological analyses, or $L_8^r(\mu_0)^{SU(3)} = (0.62 \pm 0.20) \cdot 10^{-3}$, obtained from the $\mathcal{O}(p^6)$ fit of Ref. [11].

The sizable numerical difference between $\bar{L}_8^{SU(3)}$ and $L_8^r(\mu_0)^{SU(3)}$ shows the large sensitivity of this coupling to the χ PT renormalization scale. This is a general trend for those LECs which are dominated by scalar or pseudoscalar resonance exchanges. Therefore, to perform accurate phenomenological applications one needs to control the renormalization scale dependence, which requires a determination of the χ PT couplings at NLO in $1/N_C$, like the one presented here for L_8 .

4 $C_{38}^r(\mu)$ at NLO

Following the same procedure used in Section 3, one is able to find a NLO prediction of the LEC $C_{38}^r(\mu)$:

$$\begin{aligned} \bar{C}_{38}^{U(3)} &\equiv \left[C_{38}^r(\mu) - \frac{\Gamma_{38}^{(L)}}{32\pi^2} \ln \frac{\mu^2}{M_S^2} \right]_{U(3)} = \frac{F^4}{16M_P^{r2}M_S^{r2}} \left(1 + \frac{M_P^{r2}}{M_S^{r2}} + \frac{M_S^{r2}}{M_P^{r2}} \right) \times \\ &\times \left\{ 1 + \delta_{\text{NLO}}^{(1)} - \frac{(M_S^{r2} \delta_{\text{NLO}}^{(2)} + 8\tilde{\delta}/F^2) (M_S^{r2} + M_P^{r2})}{M_P^{r4} + M_S^4 + M_P^{r2}M_S^2} \right\} - \frac{9F^2\Delta'}{512\pi^2M_S^2}, \end{aligned} \quad (24)$$

with

$$\begin{aligned} \Delta' &= 1 + \frac{1}{3} \left(\frac{M_P^2}{M_S^2} - 1 \right) \left[6 - \frac{M_S^2}{M_P^2} + 2 \left(\frac{3M_P^2}{M_S^2} - 2 \right) \ln \left(1 - \frac{M_S^2}{M_P^2} \right) \right] \\ &+ \frac{1}{3} \left(1 - \frac{M_S^2}{M_P^2} \right) \left[\frac{1}{6} + \frac{2M_S^2}{3M_P^2} + \frac{3M_S^4}{M_P^4} - \frac{10M_S^6}{M_P^6} + \left(\frac{8M_S^6}{M_P^6} - \frac{10M_S^8}{M_P^8} \right) \ln \left(\frac{M_P^2}{M_S^2} - 1 \right) \right] \\ &+ \frac{1}{3} \left(1 - \frac{M_V^2}{M_A^2} \right) \left[\frac{M_S^2}{2M_V^2} - \frac{28M_S^2}{3M_P^2} + \frac{19M_V^2M_S^2}{M_P^4} - \frac{10M_V^4M_S^2}{M_P^6} \right. \\ &\quad \left. + \left(\frac{4M_S^2}{M_P^2} - \frac{10M_V^2M_S^2}{M_P^4} \right) \left(1 - \frac{M_V^2}{M_P^2} \right)^2 \ln \left(\frac{M_P^2}{M_V^2} - 1 \right) \right]. \end{aligned} \quad (25)$$

We have used again Eqs. (18) to fix c_m^r and d_m^r . As in the case of $L_8^r(\mu)$, one has to make contact with the usual $SU(3)_L \otimes SU(3)_R$ theory by computing the massive one-loop η_1 contribution to $\Pi(t)$. It is straightforward to get the expression that relates \bar{C}_{38} in both theories:

$$\begin{aligned} \bar{C}_{38}^{SU(3)} &= \bar{C}_{38}^{U(3)} - \frac{\Gamma_{38}^{(L)SU(3)} - \Gamma_{38}^{(L)U(3)}}{32\pi^2} \left(\ln \frac{M_{\eta_1}^2}{M_S^2} + \frac{1}{2} \right) - \frac{\Gamma_8^{SU(3)} - \Gamma_8^{U(3)}}{32\pi^2} \frac{F^2}{2M_{\eta_1}^2} \\ &= \bar{C}_{38}^{U(3)} - \frac{F^2}{192\pi^2} \left(\frac{1}{M_S^2} \ln \frac{M_{\eta_1}^2}{M_S^2} + \frac{1}{2M_S^2} - \frac{1}{4M_{\eta_1}^2} \right), \end{aligned} \quad (26)$$

where we have used the LO prediction of the $\mathcal{O}(p^4)$ chiral coupling $L_5 = c_d c_m / M_S^2 = F^2 / (4M_S^2)$ [21].

Taking the same input parameters than in the previous section, one gets the numerical prediction

$$\bar{C}_{38}^{SU(3)} = (-1 \pm 6) \cdot 10^{-6}. \quad (27)$$

At the usual χPT renormalization scale $\mu_0 = 0.77$ GeV, Eq. (27) gives

$$C_{38}^r(\mu_0)^{SU(3)} = (2 \pm 6) \cdot 10^{-6}, \quad (28)$$

showing again the large numerical sensitivity to the choice of scale.

A sizable difference between the phenomenological value of some $\mathcal{O}(p^6)$ chiral couplings and their large- N_C estimates was pointed out in Ref. [40]. Our result shows how the NLO corrections in $1/N_C$ become relevant in some cases. These NLO contributions must be considered for a proper determination of the LECs. Our calculation reproduces the correct scale-dependence of $C_{38}^r(\mu)$ at the NLO in $1/N_C$. However, the $\mathcal{O}(p^6)$ LECs contain additional dependences on μ , which are suppressed by two powers of $1/N_C$ [7]. In order to remove the corresponding numerical uncertainty, it would be necessary to perform the calculation at the next-to-next-to-leading order in the $1/N_C$ expansion.

5 Conclusions

The large- N_C limit provides a solid theoretical framework to understand the success of resonance saturation in low-energy phenomenology [17]. However, this limit is unable to pin down the scale dependence of the χ PT couplings. Although this is a NLO effect in the $1/N_C$ expansion, its numerical impact is very sizable in couplings which are dominated by scalar or pseudoscalar exchanges.

In this paper we have presented a NLO prediction of the $\mathcal{O}(p^4)$ coupling $L_8^r(\mu)$, which exactly reproduces its right renormalization-scale dependence. Moreover, we have also determined the $\mathcal{O}(p^6)$ coupling $C_{38}^r(\mu)$ at the NLO, controlling its μ dependence up to small NNLO effects.

The determination of this two LECs has been made possible through a NLO calculation in $1/N_C$ of the $\Pi(t) \equiv \Pi_S(t) - \Pi_P(t)$ correlator, in the chiral limit. We have used the R χ T Lagrangian, within the SRA, to compute the one and two-particle exchange contributions to the absorptive part of the correlator. It is remarkable that, imposing a good short-distance behaviour for the corresponding scalar and pseudoscalar spectral functions, one fully determines the relevant contributing form factors. Using a dispersion relation, we have reconstructed the correlator, up to a subtraction term which has the same structure as the tree-level one-particle contributions.

The stringent short-distance QCD constraints on $\Pi(t)$ have allowed us to fix the subtraction constants in terms of resonance masses, with the results shown in Eqs. (18). Therefore, we have obtained a complete NLO determination of $\Pi(t)$, which only depends on the pion decay constant F and the meson masses M_V , M_A , M_S and M_P . Its low momentum expansion reproduces the right χ PT expression, with explicit values for the LECs $L_8^r(\mu)$ and $C_{38}^r(\mu)$.

Integrating out the η_1 field, one can further connect the $U(3)_L \otimes U(3)_R$ and $SU(3)_L \otimes SU(3)_R$ effective theories. This introduces an additional dependence on M_{η_1} . At the usual scale $\mu_0 = 0.77$ GeV, we finally obtain the numerical predictions:

$$L_8^r(\mu_0)^{SU(3)} = (0.6 \pm 0.4) \cdot 10^{-3}, \quad C_{38}^r(\mu_0)^{SU(3)} = (2 \pm 6) \cdot 10^{-6}. \quad (29)$$

The ideas discussed in this article can be applied to generic Green functions, which opens a way to investigate other chiral LECs at NLO in the large- N_C expansion. Further work along these lines is in progress [35].

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A The one-loop correction $\Delta\Pi(t)|_\rho$

In this appendix we show the explicit expression of the one-loop correction $\Delta\Pi(t)|_\rho$, generated by the considered two-particles exchanges, which has been calculated by using the dispersive method discussed in Section 2.

$$\Delta\Pi_{S-P}(t)|_{\eta\pi} = \frac{n_f}{2} \frac{1}{16\pi^2} \left(\frac{M_S^2}{M_S^2 - t} \right)^2 \left[-1 + \frac{t}{M_S^2} - \ln \left(\frac{-t}{M_S^2} \right) \right], \quad (\text{A.1})$$

$$\begin{aligned} \Delta\Pi_{S-P}(t)|_{V\pi} &= \frac{n_f}{2} \frac{1}{16\pi^2} \left(1 - \frac{M_V^2}{M_A^2} \right) \left(\frac{M_P^2}{M_P^2 - t} \right)^2 \left[\left(1 - \frac{t}{M_P^2} \right) \left(-\frac{2M_V^4}{t^2} - \frac{2M_V^4}{tM_P^2} \right. \right. \\ &\quad \left. \left. + \frac{5M_V^2}{t} + 2 - \frac{9M_V^2}{M_P^2} + \frac{6M_V^4}{M_P^4} \right) + 2 \left(1 - \frac{M_V^2}{t} \right)^3 \ln \left(1 - \frac{t}{M_V^2} \right) \right. \\ &\quad \left. - 2 \left(1 - \frac{4M_V^2}{M_P^2} + \frac{3M_V^2 t}{M_P^4} \right) \left(1 - \frac{M_V^2}{M_P^2} \right)^2 \ln \frac{M_P^2 - M_V^2}{M_V^2} \right], \quad (\text{A.2}) \end{aligned}$$

$$\Delta\Pi_{S-P}(t)|_{A\pi} = 0, \quad (\text{A.3})$$

$$\begin{aligned} \Delta\Pi_{S-P}(t)|_{S\pi} &= \frac{n_f}{2} \frac{1}{16\pi^2} \left(1 - \frac{M_S^2}{M_P^2} \right) \left(\frac{M_P^2}{M_P^2 - t} \right)^2 \left\{ -\frac{2M_S^4}{t^2} + \frac{M_S^2}{t} + \frac{8M_S^4}{M_P^4} - \frac{2M_S^2}{M_P^2} \right. \\ &\quad \left. + \frac{M_S^2 t}{M_P^4} - \frac{6M_S^4 t}{M_P^6} + \frac{2M_S^4}{t^2} \left(1 - \frac{M_S^2}{t} \right) \ln \left(1 - \frac{t}{M_S^2} \right) \right. \\ &\quad \left. + \frac{2M_S^4}{M_P^4} \left(3 - \frac{4M_S^2}{M_P^2} - \frac{2t}{M_P^2} + \frac{3M_S^2 t}{M_P^4} \right) \ln \frac{M_S^2}{M_P^2 - M_S^2} \right\}, \quad (\text{A.4}) \end{aligned}$$

$$\begin{aligned} \Delta\Pi_{S-P}(t)|_{P\pi} &= \frac{n_f}{2} \frac{1}{16\pi^2} \frac{M_S^2(M_P^2 - M_S^2)}{(M_S^2 - t)^2} \left[-2 + \frac{2t}{M_S^2} - 2 \left(1 - \frac{M_P^2}{t} \right) \ln \left(1 - \frac{t}{M_P^2} \right) \right. \\ &\quad \left. + 2 \left(1 - \frac{2M_P^2}{M_S^2} + \frac{M_P^2 t}{M_S^4} \right) \ln \frac{M_P^2 - M_S^2}{M_P^2} \right]. \quad (\text{A.5}) \end{aligned}$$

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