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CONFORMAL SYMMETRY AND DIFFERENTIAL REGULARIZATION OF THE THREE-GLUON VERTEX*

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ABSTRACT

The conformal symmetry of the QCD Lagrangian for massless quarks is broken both by renormalization effects and the gauge fixing procedure. Renormalized primitive divergent amplitudes have the property that their form away from the overall coincident point singularity is fully determined by the bare Lagrangian, and scale dependence is restricted to δ-functions at the singularity. If gauge fixing could be ignored, one would expect these amplitudes to be conformal invariant for non-coincident points. We find that the one-loop three-gluon vertex function $\Gamma_{\mu\nu\rho}(x, y, z)$ is conformal invariant in this sense, if calculated in the background field formalism using the Feynman gauge for internal gluons. It is not yet clear why the expected breaking due to gauge fixing is absent. The conformal property implies that the gluon, ghost and quark loop contributions to $\Gamma_{\mu\nu\rho}$ are each purely numerical combinations of two universal conformal tensors $D_{\mu\nu\rho}(x, y, z)$ and $C_{\mu\nu\rho}(x, y, z)$ whose explicit form is given in the text. Only $D_{\mu\nu\rho}$ has an ultraviolet divergence, although $C_{\mu\nu\rho}$ requires a careful definition to resolve the expected ambiguity of a formally linearly divergent quantity. Regularization is straightforward and leads to a renormalized vertex function which satisfies the required Ward identity, and from which the beta-function is easily obtained. Exact conformal invariance is broken in higher-loop orders, but we outline a speculative scenario in which the perturbative structure of the vertex function is determined from a conformal invariant primitive core by interplay of the renormalization group equation and Ward identities.

Other results which are relevant to the conformal property include the following:

1) An analytic calculation shows that the linear deviation from the Feynman gauge is not conformal invariant, and a separate computation using symbolic manipulation confirms that among $D_\mu b_\mu$ background gauges, only the Feynman gauge is conformal invariant.

2) The conventional (*i.e.* non-background) gluon vertex function is not conformal invariant because the Slavnov–Taylor identity it satisfies is more complicated than the simple Ward identity for the background vertex, and a superposition of $D_{\mu\nu\rho}$ and $C_{\mu\nu\rho}$ necessarily satisfies a simple Ward identity. However, the regulated conventional vertex can be expressed as a multiple of the tensor $D_{\mu\nu\rho}$ plus an ultraviolet finite non-conformal remainder. Mixed vertices with both external background and quantum gluons have similar properties.

I. INTRODUCTION

The differential regularization procedure[1] gives a simple, practical method for calculation of the renormalization group functions and the explicit forms of correlation functions in massless φ^4 theory. The supersymmetric Wess–Zumino model appears equally simple to treat by this method,[2] and complete calculations have been done in these theories through three-loop order.

A problematic feature of the procedure emerged in the various one-loop calculations for gauge theories presented in Ref. [1]. Ward identities must be studied explicitly to fix the various mass scales which are the parameters of the regularization scheme. For example in massless quantum electrodynamics the renormalized electron vertex function and self-energy are (in the notation of Sections II.C and II.B of Ref. [1])

$$\begin{aligned} V_\lambda(x,y,z) &= -2\gamma_b\gamma_\lambda\gamma_a V_{ab}(x-z,y-z) \\ V_{ab}(x,y) &= \left(\frac{\partial}{\partial x_a}+\frac{\partial}{\partial y_a}\right)\left[\frac{1}{x^2}\,\frac{\partial}{\partial y_b}\left(\frac{1}{y^2}\right)\frac{1}{(x-y)^2}\right] \\ &\quad -\frac{1}{x^2}\,\frac{1}{(x-y)^2}\left(\frac{\partial}{\partial y_a}\,\frac{\partial}{\partial y_b}-\frac{1}{4}\delta_{ab}\Box_y\right)\frac{1}{y^2}-\frac{\pi^2}{4}\delta_{ab}\delta(y)\Box\frac{\ln M_V^2x^2}{x^2} \\ \Sigma(x) &= \frac{1}{4}\,\partial\!\!\!/\,\Box\frac{\ln M_\Sigma^2x^2}{x^2} \quad , \end{aligned} \tag{1.1}$$

The Ward identity

$$\frac{\partial}{\partial z_\lambda}V_\lambda(x,y,z) = [\delta(z-x)-\delta(z-y)]\,\Sigma(x-y) \tag{1.2}$$

is violated (by terms proportional to $[\delta(z-x)-\delta(z-y)]\,\partial\!\!\!/\,\delta(x-y)$) unless the mass scales are chosen to satisfy $\ln(M_\Sigma/M_V)=1/4$, and it is not difficult to demonstrate this.

The three-gluon vertex is a fundamental correlation function of non-Abelian gauge theories. It is linearly divergent by power counting and provides a test of the compatibility of

differential regularization with Ward identities in a more singular situation than previously explored. In this note we report one-loop results for the three-gluon vertex. These results are quite simple, because of the somewhat surprising property that the bare amplitude in real space is conformal invariant (if calculated in a special gauge). It is not clear whether the conformal property is relevant beyond one-loop and whether simple calculations of higher loop contributions are possible. However, one can outline a certain conformal scenario based on the combined constraints of renormalization group equations and Ward identities on these questions.

The first aspect of the three-gluon vertex we considered were the Ward identities it must satisfy. If the vertex is calculated by standard methods in a covariant gauge, then the analogue of (1.2) is a complicated Slavnov–Taylor identity involving not only the divergence of the vertex and the gluon self-energy, but also the vertex with external ghost lines and ghost self-energy which enter non-linearly (see Section 2.5 of Ref. [3] for the explicit form of this Slavnov–Taylor identity). Fortunately, the structure of the Ward identities is far simpler if one-particle irreducible amplitudes are calculated in the background field formalism developed for gauge theories by DeWitt,[4] 't Hooft,[5] and Abbott,[6] because the generating functional is invariant under gauge transformations of the background. We therefore use the background field method as the basis for our work. Although we work only to one-loop order, the method is quite general, and it is known[7, 8] that the correct S-matrix is obtained when 1PI amplitudes are assembled in tree structures. The background field method is equivalent to ordinary field theory in a special gauge.[6]

Let us now discuss conformal invariance and its effect in our work. Conformal transformations in d-dimensional Euclidean space may be defined as the transformation of points

given by

$$x_\mu \to x'_\mu = \frac{x_\mu + c_\mu x^2}{1 + 2c\cdot x + c^2 x^2} \quad , \tag{1.3}$$

where c_μ is a constant vector. One can show[9] that conformal and Lorentz invariance imply invariance under the discrete inversion $x_\mu \to x'_\mu = x_\mu/x^2$, and that the transformation (1.3) can be described as inversion followed by translation by c_μ followed by a second inversion. The full group containing conformal, scale, and Lorentz transformations plus translations is $\mathcal{O}(d+1,1)$.

If the correlation functions of a quantum field theory were conformal invariant, then the spatial dependence of two- and three-point functions would be almost completely fixed. Consider, for example, conserved currents $J^a_\mu(x)$ of scale dimension three in $d=4$ dimensions which obey the current commutation relations of a Lie algebra with structure constants f^{abc}. Then covariant two- and three-point functions obey the Ward identities

$$\frac{\partial}{\partial z_\rho}\left\langle J^a_\mu(x) J^b_\mu(y) J^c_\rho(z)\right\rangle = -f^{cad}\delta(z-x)\left\langle J^d_\mu(x) J^b_\nu(y)\right\rangle - f^{cbd}\delta(z-y)\left\langle J^a_\mu(x) J^d_\nu(y)\right\rangle \quad . \tag{1.4}$$

With these assumptions, one can follow Schreier[9] who uses the inversion property

$$J'^a_\mu\left(\frac{x_\sigma}{x^2}\right) = x^6\left(\delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}\right) J^a_\nu\left(x_\sigma\right) \tag{1.5}$$

to show that the two-point function, if conformal invariant, must take the gauge invariant form

$$\begin{aligned}\left\langle J^a_\mu(x) J^b_\nu(y)\right\rangle &= -\frac{1}{2}k\,\delta^{ab}\frac{1}{(x-y)^4}\frac{\partial}{\partial x_\mu}\,\frac{\partial}{\partial y_\nu}\ln(x-y)^2 \\ &= k\,\delta^{ab}\left[\frac{\delta_{\mu\nu}}{(x-y)^6} - \frac{2(x-y)_\mu(x-y)_\nu}{(x-y)^8}\right] \\ &= -\frac{1}{12}k\delta^{ab}\left(\frac{\partial}{\partial x_\mu}\,\frac{\partial}{\partial x_\nu} - \delta_{\mu\nu}\Box\right)\frac{1}{(x-y)^4}\end{aligned} \tag{1.6}$$

where k is a constant, while conformal invariant three-point functions must be linear combinations of two possible conformal tensors

$$\left\langle J^a_\mu(x) J^b_\nu(y) J^c_\rho(z) \right\rangle = f^{abc} \left(k_1 D^{\rm sym}_{\mu\nu\rho}(x,y,z) + k_2 C^{\rm sym}_{\mu\nu\rho}(x,y,z) \right) \tag{1.7}$$

where $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$ and $C^{\rm sym}_{\mu\nu\rho}(x,y,z)$ are permutation odd tensor functions, obtained from the specific tensors

$$D_{\mu\nu\rho}(x,y,z) = \frac{1}{(x-y)^2(z-y)^2(x-z)^2} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} \ln(x-y)^2 \frac{\partial}{\partial z_\rho} \ln\left(\frac{(x-z)^2}{(y-z)^2} \right) \tag{1.8}$$

$$C_{\mu\nu\rho}(x,y,z) = \frac{1}{(x-y)^4} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial z_\alpha} \ln(x-z)^2 \frac{\partial}{\partial y_\nu} \frac{\partial}{\partial z_\alpha} \ln(y-z)^2 \frac{\partial}{\partial z_\rho} \ln\left(\frac{(x-z)^2}{(y-z)^2} \right) \tag{1.9}$$

by adding cyclic permutations

$$\begin{aligned} D^{\rm sym}_{\mu\nu\rho}(x,y,z) &= D_{\mu\nu\rho}(x,y,z) + D_{\nu\rho\mu}(y,z,x) + D_{\rho\mu\nu}(z,x,y) \\ C^{\rm sym}_{\mu\nu\rho}(x,y,z) &= C_{\mu\nu\rho}(x,y,z) + C_{\nu\rho\mu}(y,z,x) + C_{\rho\mu\nu}(z,x,y) \quad . \end{aligned} \tag{1.10}$$

(Although not required for a first reading of this paper, we note that only four of the six tensors which appear in (1.10) are linearly independent since the combination $D_{\mu\nu\rho}(x,y,z) + \frac{1}{2} C_{\mu\nu\rho}(x,y,z)$ is cyclically symmetric. A convenient basis, equivalent to that of Schreier, is given by $C^{\rm sym}_{\mu\nu\rho}(x,y,z)$ and $D_{\mu\nu\rho}(x,y,z)$, $D_{\nu\rho\mu}(y,z,x)$, $D_{\rho\mu\nu}(z,x,y)$. There are no permutation even combinations of this basis, so that the d-symbol d^{abc} cannot appear in (1.7). It will be seen later that k_1 and k are related by the Ward identity (1.4), while k_2 is an independent constant.)

Many readers may now think that these considerations are irrelevant to the three-gluon vertex and even suspicious, because it is well-known that correlation functions in massless four-dimensional renormalizable field theories are not conformal invariant. Invariance fails because a scale is introduced in the renormalization procedure in (virtually) all such theories,

while in gauge theories conformal invariance is also broken by the gauge fixing procedure. It turns out that the first difficulty is easier to explain than the second, at least in one-loop order. It is true that conformal invariance fails because of renormalization, but in real space at one-loop order renormalization affects only singular δ-functions $\delta(x-y)\delta(y-z)$, while for non-coincident points the renormalized and bare amplitudes coincide. Thus real space one-loop amplitudes can well be conformal invariant away from short distance singularities.

Indeed Baker and Johnson[10] considered the three-point current correlation function in a theory containing spinor doublet ψ_i with Abelian gauge interactions. The "triangle function" of the $SU(2)$ current operator $J^a_\mu = \frac{1}{2}\bar{\psi}_i \tau^a_{ij} \gamma_\mu \psi_j$ was shown to have the conformal structure (1.7 – 1.10) not only in one-loop, where it is fairly trivial, but also in two-loop order where it vastly simplified the calculation. Conformal invariance held at the two-loop level, because subdivergences cancel due to the Abelian Ward identity $Z_1 = Z_2$, so renormalization again has no effect for non-coincident points.

The idea that amplitudes away from singularities have the conformal symmetry of the bare Lagrangian is not sufficient to explain a conformal structure for the three-gluon vertex function, because the gauge fixing terms in the Lagrangian break conformal invariance. Indeed, the gluon propagator does not transform[10] as expected from the formal inversion property

$$A'_\mu\left(\frac{x_\sigma}{x^2}\right) = x^2\left(\delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}\right) A_\mu\left(x_\sigma\right) \quad . \tag{1.11}$$

However, one can show that the one-loop gluon vertex and self-energy in the background field formalism satisfy Ward identities of the same simple form as (1.4), and further that the quark, Faddeev–Popov ghost, and gluon contributions satisfy the Ward identities separately. The quark triangle function is independent of the background method and the same in one-loop

order as in the Abelian theory above and thus conformal invariant. Our calculations then showed that the ghost triangle function was conformal invariant, while the gluon triangle and seagull graphs combine into an amplitude with the conformal structure (1.7 – 1.10) if the Feynman gauge is used for internal gluons. Separate computations show that the background field vertex is not conformal invariant away from the Feynman gauge, and it is easy to see that the three-gluon vertex in ordinary field theory in the Feynman gauge is not conformal invariant. Thus the conformal property appears to be very specialized, and, apart from the calculations themselves, we do not have any qualitative explanation for it. For example, a functional identity which expresses the conformal variation of the generating functional does not suggest that it vanishes without detailed calculation.

Whether by accident or part of Nature's design, the fact that quark, ghost, and virtual gluon contributions to the vertex are all numerical combinations of the invariant tensors $D^{\rm sym}_{\mu\nu\rho}$ and $C^{\rm sym}_{\mu\nu\rho}$ vastly simplifies the task of regularization. The bare amplitude $D^{\rm sym}_{\mu\nu\rho}$ has an ultraviolet divergent Fourier transform, but it is easily regulated using the ideas of differential regularization. The story of the tensor $C^{\rm sym}_{\mu\nu\rho}$ is slightly more involved. Although formally linearly divergent, its Fourier transform is ultraviolet finite but subject to shift ambiguities similar to those of the fermion triangle anomaly. A shift term changes $C^{\rm sym}_{\mu\nu\rho}$ by a linear polynomial in momenta which is proportional to the bare Yang–Mills vertex (see (1.12)). Regularization is required to specify the ambiguity in $C^{\rm sym}_{\mu\nu\rho}$, and consideration of conformal invariance and differential regularization lead to a simple regulated form which contributes trivially to the Ward identity (1.4). The result of these considerations is that the regulated form of $D^{\rm sym}_{\mu\nu\rho}$ alone determines both the Ward identity and the β-function. The quark, ghost, and gluon contribution to these quantities are easily found from the regulated form. It is also

reasonably straightforward to find the relation between the mass scales in a regularized vertex and self-energy which guarantees that the Ward identity holds at the renormalized level.

The hard-nosed, unimaginative reader will note, as is correct, that conformal symmetry *per se* plays no direct role in our work, and our results can be more prosaically explained by the fact that the bare amplitudes of the background field method in Feynman gauge are linear combinations of the two tensor structure above. However, it can hardly be mere coincidence that these tensors are conformally covariant, although this curiosity is not now understood from a general standpoint.

Further, differential regularization is particularly useful here simply because it is a real space method in which regularized and bare amplitudes of primitively divergent graphs agree for separated points. The conformal structure found for the renormalized amplitudes is independent of the regularization method used. In any method, it would be evident in real space if anyone cared to look. In momentum space, conformal transformations are integral transformations. Both this and scale-dependent renormalization obscure the conformal properties.

To gain some perspective, we discuss further some non-conformal invariant correlators such as the three-gluon vertex in conventional Feynman gauge field theory. Here one can write the bare amplitude as a multiple of $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$ plus a remainder which has a finite Fourier transform. (The coefficient of $D^{\rm sym}_{\mu\nu\rho}$ differs from that in the conformal invariant background field vertex.) Thus the divergent part of the amplitude is described by a conformal tensor, and the reason is very simple. All versions of the three-gluon vertex are linearly divergent and have the full Bose permutation symmetry. The renormalization scale (or ultraviolet cutoff) dependence is uniquely determined by the tensor form and discrete symmetry to be a multiple

of the tree approximation Yang–Mills vertex, namely the linear polynomial

$$V_{\mu\nu\rho}(k_1, k_2, k_3) = \delta_{\mu\nu}\,(k_1 - k_2)_\rho + \delta_{\nu\rho}\,(k_2 - k_3)_\mu + \delta_{\rho\mu}\,(k_3 - k_1)_\nu \tag{1.12}$$

in momentum space. The regularized form of the tensor $D^{\rm sym}_{\mu\nu\rho}(x, y, z)$ also has the same scale-dependence, and there is a unique multiple of this tensor which fully accounts for the ultraviolet divergence.

The one-loop vertex function with one background gluon and two quantum gluons is a linearly divergent subgraph of the two-loop background field vertex. Although it does not have a full Bose symmetry, the known renormalization structure[11] of the background field formalism requires that its renormalization scale-dependence is that of (1.12). So it also can be written as a multiple of $D^{\rm sym}_{\mu\nu\rho}(x, y, z)$ plus a remainder which is ultraviolet convergent. This representation may be useful in the study of the two-loop background field vertex.

In Section II, we present the background field formalism employed in our work, and in Section III, we discuss the computations which established the conformal properties of the background field vertex. In Section IV, we discuss the regularization of $D^{\rm sym}_{\mu\nu\rho}$ and $C^{\rm sym}_{\mu\nu\rho}$. The renormalized Ward identity and the mass scale relation for the regularized vertex and self-energy are studied in Section V. In Section VI we show how the β-function is obtained from the regulated vertex. In Section VII we outline a speculative scenario about the role of conformal invariance in higher-loop calculations, and there is a brief statement of the conclusions in Section VIII. Appendix A is devoted to the study of the linear deviation from Feynman gauge, while some results concerning mixed vertices are presented in Appendix B. In Appendix C we summarize our work on the conformal properties in a general background field gauge.

II. THE BACKGROUND FIELD METHOD

We now outline the background field formalism used in our work following Abbott[6] and the treatment in Ref. [1] with some minor but not insignificant changes.

Given the gauge potentials $A^a_\mu(x)$ and the structure constants f^{abc} of a semi-simple Lie algebra, the Yang–Mills field strength and action are

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \\ S[A] &= \frac{1}{4g^2} \int d^4x F^a_{\mu\nu} F^a_{\mu\nu} \quad . \end{aligned} \tag{2.1}$$

One now introduces the background/quantum split

$$\begin{aligned} A^a_\mu(x) &= B^a_\mu(x) + g b^a_\mu(x) \\ F^a_{\mu\nu} &= B^a_{\mu\nu} + g\left(D_\mu b^a_\nu - D_\nu b^a_\mu\right) + g^2 f^{abc} b^b_\mu b^c_\nu \\ D_\mu b^a_\nu &= \partial_\mu b^a_\nu + f^{abc} B^b_\mu b^c_\nu \end{aligned} \tag{2.2}$$

where, unless otherwise specified, D_μ denotes a covariant derivative with background connection. The action $S[B + gb]$ is separately invariant under background gauge transformations

$$\begin{aligned} \delta B^a_\mu &= D_\mu \theta^a \quad , \\ \delta b^a_\mu &= f^{abc} b^b_\mu \theta^c \end{aligned} \tag{2.3}$$

and under quantum gauge transformations

$$\begin{aligned} \delta B^a_\mu &= 0 \\ \delta b^a_\mu &= D_\mu \alpha^a + g f^{abc} b^b_\mu \alpha^c \\ &\equiv D_\mu \left(B + gb\right) \alpha^a \quad . \end{aligned} \tag{2.4}$$

The gauge fixing action

$$S_{gf}[b] = \frac{1}{2a} \int d^4x \left(D_\mu b^a_\mu\right)^2 \tag{2.5}$$

is invariant under background field transformations only, as is the associated Faddeev–Popov operator

$$M[B,b] = D_\mu D_\mu (B + gb) \quad . \tag{2.6}$$

We define the functional

$$e^{-\Omega[B,J]} = \int [db_\mu] \det M \exp - \left\{ S[B+gb] - S[B] + S_{gf}[b] + \int d^4x \, J^a_\mu(x) b^a_\mu \right\} \quad . \tag{2.7}$$

The source $J^a_\mu(x)$ is given by

$$J^a_\mu(x) = \frac{1}{g} D_\alpha B^a_{\alpha\mu}(x) + j^a_\mu(x) \quad . \tag{2.8}$$

The purpose of the first term is to cancel the linear "tadpole" in $S[B+gb]-S[B]$, while $j^a_\mu(x)$ is the source for quantum gluons. For $j^a_\mu(x) \equiv 0$, $\Omega[B,J]_{j=0}$ contains one-particle irreducible graphs with external B fields and internal b lines beginning in one-loop order, plus some non-1PI graphs beginning in two-loop order. The 1PI graphs can be systematically treated by a Legendre transform and they contribute to the gauge invariant effective action of the theory which agrees with the conventional effective action in an unconventional gauge.[5,6] The Legendre transform is not discussed here because it is not required for the one-loop computations which are the major part of the present work. For our schematic discussion of two-loop order in Section VII, it is necessary to note that one-loop subdiagrams with external quantum gluons are required and these are obtained by functional differentiation of $\Omega[B,J]$ with respect to j^a_μ followed by amputation on external b-lines.

The most important property of the background field formalism for our purpose is its invariance under background gauge transformations (2.3). A gauge transformation of B^a_μ is

compensated, except in the source term involving j_μ, by a gauge rotation of the integration variable b_μ, leading to the functional Ward identity

$$D_\mu \frac{\delta\Omega[B,J]}{\delta B^a_\mu(x)} = -f^{abc} j^b_\mu(x) \frac{\delta}{\delta j^c_\mu(x)} \Omega[B,J] \quad . \tag{2.9}$$

When there are no external quantum gluons the right side vanishes. As a special case of (2.9) one finds that the background three-gluon vertex and self-energies are related by

$$\begin{aligned} \frac{\partial}{\partial z_\rho} \frac{\delta^3\Omega[B,J]_{j=0}}{\delta B^a_\mu(x)\delta B^b_\nu(y)\delta B^c_\rho(z)} = &- f^{cad}\delta(z-x)\frac{\delta^2\Omega[B,J]_{j=0}}{\delta B^d_\mu(x)\delta B^b_\nu(y)} \\ &- f^{cbd}\delta(z-y)\frac{\delta^2\Omega[B,J]_{j=0}}{\delta B^a_\mu(x)\delta B^d_\nu(y)} \end{aligned} \tag{2.10}$$

which is exactly of the same form as the identity (1.4) satisfied by current correlation functions.

To implement perturbation theory, one needs the explicit form of the integrand of (2.7)

$$\begin{aligned} &S[B+gb] - S[B] + S_{gf}[B] + \frac{1}{g}\int d^4x\, D_\alpha B^a_{\alpha\mu} b^a_\mu \\ &\quad = \int d^4x \left[\frac{1}{2} D_\mu b^a_\nu D_\mu b^a_\nu + f^{abc} B^a_{\mu\nu} b^b_\mu b^c_\nu + \frac{1}{2}\left(\frac{1}{a}-1\right)(D\cdot b)^2 + \mathcal{L}_q\right] \end{aligned} \tag{2.11}$$

where

$$\mathcal{L}_q = g f^{abc}\left(D_\mu b^a_\nu\right) b^b_\mu b^c_\nu + \frac{g^2}{4} f^{abc} f^{ade} b^b_\mu b^c_\nu b^d_\mu b^e_\nu \tag{2.12}$$

describes quantum gluon vertices which are required in background field calculations beyond one-loop. As in [1] we rewrite the integrand of (2.11) in terms of b_μ kinetic terms and mixed b-B interaction terms

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}\partial_\mu b^a_\nu \partial_\mu b^a_\nu + \frac{1}{2}\left(\frac{1}{a}-1\right)\partial_\mu b^a_\mu \partial_\nu b^a_\nu \\ \mathcal{L}_1 &= f^{abc} B^a_\mu b^b_\nu \partial_\mu b^c_\nu \\ \mathcal{L}_2 &= \frac{1}{2} f^{abc} f^{ade} B^b_\mu B^d_\mu b^c_\nu b^e_\nu \\ \mathcal{L}_3 &= f^{abc} B^a_{\mu\nu} b^b_\mu b^c_\nu \\ \mathcal{L}_4 &= \left(\frac{1}{a}-1\right)\left[f^{abc} B^b_\mu b^c_\mu \partial_\nu b^a_\nu + \frac{1}{2} f^{abc} f^{ade} B^b_\mu b^c_\mu B^d_\nu b^e_\nu\right] \quad . \end{aligned} \tag{2.13}$$

The quantum field propagator is

$$\left\langle b^a_\mu(x) b^b_\nu(y) \right\rangle = \frac{\delta^{ab}}{4\pi^2} \left[\frac{\delta_{\mu\nu}}{(x-y)^2} + \frac{a-1}{4} \partial_\mu \partial_\nu \ln(x-y)^2 \right] \quad . \tag{2.14}$$

One sees that both propagator and interaction terms simplify in the Feynman gauge ($a = 1$) which was the initial motivation for this internal gauge choice.

For computational purposes, one represents the Faddeev–Popov determinant as a functional integral over anti-commuting ghosts $c^a(x)$, $\bar{c}^a(x)$ with action

$$\begin{aligned} S_{\rm gh}[c, \bar{c}] &= \int d^4x \left[D_\mu \bar{c}^a D_\mu c^a + \mathcal{L}'_g \right] \\ &= \int d^4x \left[\mathcal{L}^g_0 + \mathcal{L}^g_i + \mathcal{L}^{'g} \right] \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \mathcal{L}^g_0 &= \partial_\mu \bar{c}^a \partial_\mu c^a \\ \mathcal{L}^g_i &= f^{abc} B^a_\mu \bar{c}^b \overleftrightarrow{\partial}_\mu c^c + f^{abc} f^{ade} B^b_\mu B^d_\mu \bar{c}^c c^e \\ \mathcal{L}^{'g} &= g f^{abc} D_\mu \bar{c}^a b^b_\mu c^c \quad . \end{aligned} \tag{2.16}$$

Note that $\overleftrightarrow{\partial}_\mu$ is an anti-symmetric derivative, and that the ghost propagator is

$$\left\langle c^a(x) \bar{c}^b(y) \right\rangle = \frac{1}{4\pi^2} \, \frac{\delta^{ab}}{(x-y)^2} \quad . \tag{2.17}$$

The only new feature of this treatment, compared with Refs. [1] and [6] is that the tadpole terms were simply neglected previously, because they do not contribute to 1PI diagrams. Now the tadpole is cancelled explicitly, through the form (2.8) of the source. This makes a difference only for some of the functional identities used in Section III and Appendix A.

There are more recent versions of the background field formalism[12] which are more general than that used here. When applied to non-Abelian gauge theories, they agree with the present version to one-loop order, if the Landau gauge, rather than the Feynman gauge, is used for internal gluons. In two-loop order, there are other differences. Since the three-gluon vertex in the Landau gauge is not conformal invariant it does not seem that these formalisms are useful for further investigation of the conformal property.

III. BARE AMPLITUDES AND CONFORMAL TENSORS

One-particle irreducible (1PI) diagrams contributing to the one-loop correction to the three-gluon vertex can be classified in three groups, each of which provides a different conformally invariant contribution to the effective action. Graphs with (1) ghost loops, (2) fermion loops,and (3) gluon loops are separately conformally invariant,namely they are linear combinations of the conformal tensors $C^{\rm sym}_{\mu\nu\rho}(x,y,z)$ and $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$ of (1.8 – 1.10), with different coefficients. The conformal property of the ghost and fermion diagrams are clearly associated with invariant Lagrangians, but that of the gluon graphs presently lacks a simple explanation. In this section we describe in some detail the calculations that led us to recognize the conformal property, which is a regularization-independent result requiring, however, a real space approach.

From (2.15) one sees that the part of the ghost Lagrangian required in one-loop calculations is simply $D_\mu \bar{c}^a D_\mu c^a$ which coincides with that of a minimally coupled scalar field in the adjoint representation. This is conformal invariant, as one can see, for example, by combining (1.11) for $B^a_\mu(x)$ with

$$c^{'a}(x') = x^2 c^a(x) \tag{3.1}$$

for the ghost (and antighost). Thus the ghost contribution to the three-gluon vertex will be conformal invariant at one-loop order. At the computational level, it is the antisymmetric derivative $\overleftrightarrow{\partial}_\mu$ in (2.16) that is crucial for the conformal property. The ghost interaction term $\mathcal{L}'^g$ of (2.15) is not conformal invariant.

To see which linear combination of the conformal tensors $C^{\rm sym}_{\mu\nu\rho}(x,y,z)$ and $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$ describes the ghost contribution we examine the two 1PI diagrams of Fig. 1. Graph (1.a) vanishes because the Wick contractions give an algebraic factor of the type $f^{abc}B^a_\mu(x)B^b_\mu(x) \;=\; 0$.

Graph (1.b) instead gives the following contribution to $-\Omega_3[B]$, *i.e.* the part of the effective action cubic in B,

$$\begin{aligned}(1.\mathrm{b}) = -\frac{1}{3}\,\frac{C}{(4\pi^2)^3} f^{abc} \int d^4x\, d^4y\, d^4z\, B^a_\mu(x) B^b_\nu(y) B^c_\rho(z) \\ \times \left[T_{\mu\rho\nu}(x-z, y-z) - \frac{3}{2} V_{\mu\nu\rho}(x-z, y-z) \right]\end{aligned} \tag{3.2}$$

where we have introduced the tensors

$$T_{\mu\nu\rho}(x-z, y-z) = \partial^x_\mu \frac{1}{(x-z)^2} \partial^y_\nu \frac{1}{(y-z)^2} \partial^x_\rho \frac{1}{(x-y)^2} \quad , \tag{3.3a}$$

$$V_{\mu\nu\rho}(x-z, y-z) = \partial^x_\mu \partial^x_\nu \frac{1}{(x-y)^2}\, \frac{1}{(y-z)^2} \overleftrightarrow{\partial}^z_\rho \frac{1}{(x-z)^2} \quad , \tag{3.3b}$$

C is the Casimir operator in the adjoint representation ($C = N$ for $SU(N)$) and we have used $f^{ade} f^{bef} f^{cfd} = \frac{C}{2} f^{abc}$. For the contribution to the three-point function from (3.2) we get

$$\begin{aligned}\frac{\delta^3(1.\mathrm{b})}{\delta B^a_\mu(x) \delta B^b_\nu(y) \delta B^c_\rho(z)} = -\frac{C\, f^{abc}}{(4\pi^2)^3} \Big[T_{\mu\rho\nu}(x-z, y-z) + T_{\rho\nu\mu}(x-z, y-z) \\ - \frac{3}{2} V^{\mathrm{sym}}_{\mu\nu\rho}(x-z, y-z) \Big] \quad ,\end{aligned} \tag{3.4}$$

where $V^{\mathrm{sym}}_{\mu\nu\rho}$ is constructed as in (1.10) by adding cyclic permutations. By manipulating the derivatives it is possible to show that (3.4) can be rewritten in terms of the conformal tensors of (2.10) according to

$$\frac{\delta^3(1.\mathrm{b})}{\delta B^a_\mu(x) \delta B^b_\nu(y) \delta B^c_\rho(z)} = -\frac{C}{(4\pi^2)^3}\, \frac{f^{abc}}{3} \left[\frac{1}{2} C^{\mathrm{sym}}_{\mu\nu\rho}(x,y,z) + 4 D^{\mathrm{sym}}_{\mu\nu\rho}(x,y,z) \right] \quad . \tag{3.5}$$

It is worth mentioning that if the analogous quantity is computed in the usual quantum field theoretical approach, instead of the background field method, one would get a non-conformally invariant amplitude proportional to the quantity $T_{\mu\rho\nu}(x-z, y-z) + T_{\rho\nu\mu}(x-z, y-z)$. This is to be expected since the ghost action for conventional Lorentz gauge fixing is not conformally invariant.

Let us now analyze the fermion loop contributions. The Euclidean fermion action is (we assume only one flavor)

$$S^F = i \int d^4x \, \bar{\psi} \gamma_\mu \left(\partial_\mu + A_\mu \right) \psi = \int d^4x \left(\mathcal{L}_0^F + \mathcal{L}_i^F \right) \tag{3.6}$$

where $A_\mu = -iA_\mu^a T^a$, T^a are the Hermitian gauge group generators and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ are Euclidean Hermitian Dirac matrices. The action (3.6) is conformally invariant and the conformal properties of the fermion loop triangle were already studied in the Abelian case.[10] In Yang–Mills theories there are two diagrams contributing to the one-loop three-gluon vertex (Fig. 2). The fermion propagator in real space reads

$$\left\langle \psi^i(x) \bar{\psi}^j(y) \right\rangle = -\frac{i}{4\pi^4} \delta^{ij} \gamma^\mu \partial_\mu^x \frac{1}{(x-y)^2} \tag{3.7}$$

where $i, j = 1, \ldots, N$ for $SU(N)$ are the representation indices. Both diagrams are obtained from the Wick contractions in $\left\langle \mathcal{L}_i^F(x) \mathcal{L}_i^F(y) \mathcal{L}_i^F(z) \right\rangle$ which gives the contribution to $\Omega_3[B]$

$$\begin{aligned} (2.\mathrm{a}) + (2.\mathrm{b}) = -\frac{1}{3} \, \frac{i}{(4\pi^2)^3} \int d^4x \, d^4y \, d^4z \, B_\mu^a(x) B_\nu^b(y) B_\rho^c(z) \\ \times \left[\mathrm{Tr} \left\{ T^b T^a T^c \right\} \cdot \mathrm{tr} \left\{ \gamma_\lambda \gamma_\mu \gamma_\sigma \gamma_\rho \gamma_\tau \gamma_\nu \right\} T_{\sigma\tau\lambda}(x-z, y-z) \right] \quad . \end{aligned} \tag{3.8}$$

The trace of three generators is

$$\mathrm{Tr} \left\{ T^a T^b T^c \right\} = \frac{1}{4} \left(d^{abc} + i f^{abc} \right) \tag{3.9}$$

and in (3.8) the terms containing the symmetric part d^{abc} vanish by symmetry properties of the trace on the γ matrices. Using the trace one finds, after a simple calculation, the explicit conformal invariant expression for the fermion loop contribution to the vertex

$$\frac{\delta \left[(2.\mathrm{a}) + (2.\mathrm{b}) \right]}{\delta B_\mu^a(x) \delta B_\nu^b(y) \delta B_\rho^c(z)} = \frac{4}{3} \, \frac{f^{abc}}{(4\pi^2)^3} \left[-\frac{1}{2} C_{\mu\nu\rho}^{\mathrm{sym}}(x,y,z) + 2 D_{\mu\nu\rho}^{\mathrm{sym}}(x,y,z) \right] \quad . \tag{3.10}$$

The conformal invariance of the one-loop diagrams containing gluon loops is much more surprising, since the gauge fixing term (2.5) breaks conformal invariance and, as discussed in the introduction, the gluon propagator does not transform properly. The 1PI gluon diagrams are shown in Fig. 3. It turns out that diagram (3.e) is separately conformal invariant in Feynman gauge ($a = 1$) because the antisymmetric $\overleftrightarrow{\partial}_\mu$ in $\mathcal{L}_1$ of (2.13) does "match" nicely with the effective "scalar-like" gluon propagator of (2.14). Indeed the amplitude of (3.e) is simply -2 times that of the ghost loop (3.5) and thus embodies the expected ghost cancellation of two of the four degrees of freedom of b_μ.

The remaining diagrams of Fig. 3 are not separately conformal invariant but their sum is. We describe the calculations as follows. The graphs (3.a,b,c,f) vanish because symmetric tensors in the group indices are contracted with f^{abc}. For the graph (3.h) we have the interesting property that, after partial integration of all derivatives, the terms in which Dirac δ occur, *i.e.* terms in which two background fields are at the same point cancel exactly with the seagull diagrams (3.d). The same type of local terms, instead cancel among themselves in the graph (3.g). Adding up all these contributions, after a lengthy but straightforward calculation, it is possible to express also the gluon loop contributions to the three point function in terms of the conformal tensors

$$\frac{\delta^3\left((3.\mathrm{d})+(3.\mathrm{e})+(3.\mathrm{g})+(3.\mathrm{h})\right]}{\delta B^a_\mu(x)\delta B^b_\nu(y)\delta B^c_\rho(z)} = \frac{C}{(4\pi^2)^3}\,\frac{f^{abc}}{3}\left[7C^{\mathrm{sym}}_{\mu\nu\rho}(x,y,z) - 40D^{\mathrm{sym}}_{\mu\nu\rho}(x,y,z)\right] \quad . \tag{3.11}$$

In conclusion, the one-loop correction to the three-gluon vertex, computed in the background field framework in Feynman gauge, is conformal invariant. The sum of all contributions of the three groups of graphs for the three point function with N_f fermions is

$$\begin{aligned}\frac{\delta^3\Omega[B,J]}{\delta B^a_\mu(x)\delta B^b_\nu(y)\delta B^c_\rho(z)} = \frac{f^{abc}}{3\,(4\pi^2)^3}\Bigg[&-\left(\frac{13}{2}C - 2N_f\right)C^{\mathrm{sym}}_{\mu\nu\rho}(x,y,z)\\ &+\left(44C - 8N_f\right)D^{\mathrm{sym}}_{\mu\nu\rho}(x,y,z)\Bigg] \quad .\end{aligned} \tag{3.12}$$

The coefficients of the divergent tensor in (3.12) are exactly those necessary to satisfy the Ward identity (2.10) and, consequently they are directly related to the well-known $-\frac{11}{3}C + \frac{2}{3}N_f$ of the Yang–Mills β function. The details of these questions are discussed in Sections V and VI.

The unexpected conformal property of the gluon graphs requires explanation, and we have attempted to explain it via a Slavnov–Taylor-like identity which describes the conformal variation of the generating functional $\Omega[B,J]_{j=0}$. The infinitesimal form of the conformal transformation (1.3) is easily described[13] in terms of conformal Killing vectors defined as

$$x_\mu \longrightarrow x_\mu + v_\mu(x,\epsilon) \equiv x_\mu + \epsilon_\mu x^2 - 2x_\mu \epsilon\cdot x \quad . \tag{3.13}$$

These vectors satisfy

$$\begin{aligned} &\partial_\mu v_\nu + \partial_\nu v_\mu - \frac{1}{2}\delta_{\mu\nu}\partial\cdot v = 0 \\ &\partial\cdot v = -8\epsilon\cdot x \\ &\Box v_\mu = 4\epsilon_\mu \quad . \end{aligned} \tag{3.14}$$

The standard conformal transformations of vector and scalar fields are

$$\begin{aligned} \delta'_\epsilon A_\mu &= v_\nu\partial_\nu A_\mu + A_\nu\partial_\mu v_\nu \\ \delta'_\epsilon \varphi &= v_\nu\partial_\nu\varphi - 2\epsilon\cdot x\varphi \quad . \end{aligned} \tag{3.15}$$

Because of the background gauge invariance of our formalism, it is more convenient to add the field-dependent gauge transformation with parameter $\theta^a = v_\rho B^a_\rho$ and use the gauge covariant conformal variations[14]

$$\begin{aligned} \delta_\epsilon B^a_\mu &= -v_\nu B^a_{\nu\mu} \\ \delta_\epsilon b^a_\mu &= v_\nu D_\nu b^a_\mu + b_\nu\partial_\mu v_\nu \\ \delta_\epsilon c^a &= v_\nu D_\nu c^a - 2\epsilon\cdot x\, c^a \quad . \end{aligned} \tag{3.16}$$

It is not difficult to see that $S[B+bg]$ and $S[B]$ are conformal invariant, and that

$$\delta_\epsilon D\cdot b^a = \left(v\cdot D + \frac{1}{2}\partial\cdot v\right) D\cdot b^a + 4\epsilon\cdot b \tag{3.17}$$

leading to the simple variation of the gauge fixing term (2.5)

$$\delta_\epsilon S_{gf}[b] = \frac{4}{a}\int d^4x\, \epsilon\cdot b^a D\cdot b^a \quad . \tag{3.18}$$

Thus the effect of a conformal transformation of D^a_μ and b^a_μ is to change the gauge fixing $D\cdot b^a$ as follows

$$D\cdot b^a \longrightarrow D\cdot b^a + 4\epsilon\cdot b^a \quad . \tag{3.19}$$

We now describe qualitatively the effect of a conformal transformation on $\Omega[B,J]$ in (2.7). Ignoring ghost variations for the moment, we combine transformations of the background B^a_μ and source

$$\delta_\epsilon J^a_\mu = v_\nu D_\nu J^a_\mu + J^a_\nu \partial_\mu v_\nu + \frac{1}{2}\partial\cdot v J_\mu \tag{3.20}$$

and make the analogous change in the integration variable b_μ. The net effect in (2.7) is the change (3.18) of the gauge fixing term. We then make the further quantum gauge transformation

$$b^a_\mu = b^{'a}_\mu - D_\mu[B+b]\theta^a \quad , \qquad \theta^a = 4M^{-1}\,\epsilon\cdot b^a \tag{3.21}$$

to restore the original gauge and transfer the change to the source term. The process just described leads to the functional identity (for connected graphs)

$$\begin{aligned}
\frac{1}{4}\delta_\epsilon\Omega[B,J] &= \frac{1}{a}\left\langle \int d^4x\, \epsilon\cdot b^a\, D\cdot b^a \right\rangle \\
&= \left\langle \int d^4x\, J^a(x)\cdot D[B+b]\int d^4y\, M^{-1}(x,y)^{ab}\epsilon\cdot b^b(y) \right. \\
&\quad \left. + \int d^4x\, \epsilon\cdot D[B+b]M^{-1}(x,y=x)^{aa'=a} \right\rangle \\
&= \left\langle \int d^4x\, J^a(x)\cdot D[B+b]c^a(x)\int d^4y\, \bar{c}^b(y)\epsilon\cdot b^b(y) \right. \\
&\quad \left. + \int d^4x\, \epsilon\cdot D[B+b]c^a(x)\bar{c}^a(x) \right\rangle \quad .
\end{aligned} \tag{3.22}$$

The first term on the second line is the induced change in the source term, and the next term is the formal contribution of the change of the integration measure due to (3.21).

What about ghosts? We already know that the ghost action is conformal invariant if $b^a_\mu = 0$, and it is easy to confirm this using (3.16). There are no other ghost variation effects at the one-loop level, and the functional identity (3.22) correctly expresses the conformal variation of $\Omega[B, J]$ and the bare one-loop amplitudes it generates. An alternative way to pass from the first to the last line of (3.22) is to apply $\int d^4x\, \epsilon \cdot \big(\delta/\delta J^a_\mu(x)\big)$ to the functional Slavnov–Taylor identity given in (A.2). At higher order there are additional ghost effects which we have not treated. Thus (3.22) may well be a one-loop-only result, but that is all we need.

To see if (3.22) sheds any light on our background field results, we set $j^a_\mu = 0$. Using (2.8) and its covariant conservation property, we rewrite (3.22) as

$$\begin{aligned} \frac{1}{4}\delta_\epsilon \Omega[B, J]_{j=0} = \Big\langle f^{abc} \int d^4x\, D_\nu B^a_{\nu\mu}(x) b^b_\mu(x) c^c(x) \int d^4y\, \bar{c}^d(y) \epsilon \cdot b^d(y) \\ + \int d^4x\, (\epsilon \cdot D[B + b] c^a(x))\, \bar{c}^a(x) \Big\rangle \quad . \end{aligned} \tag{3.23}$$

To account for the experimental results of this paper, the third variational derivative of (3.23) with respect to B^a_μ must vanish in the Feynman gauge but not for general values of a. One sees no direct reason for this in (3.23), other than through a detailed computation of the contributing diagrams which would be a tedious job. As a check against possible error in (3.23), we did study the diagrams which contribute to the second variational derivative. Here it is not difficult to show that contributions to the first and second term in (3.23) vanish separately. This confirms that the background field self-energy is conformal invariant, and is consistent with the fact that the form (1.6) is obtained from direct calculation.

Three-gluon vertices with one or more external quantum gluons are studied in the Appendices. It can be argued very simply, using Ward identities, that they cannot be conformal invariant. It is implicit in the discussions of (1.4) – (1.7) of the Introduction, Section I, and confirmed in Sections IV and V below, that a linear combination of conformal tensors $D^{\rm sym}_{\mu\nu\rho}$ and $C^{\rm sym}_{\mu\nu\rho}$ satisfies a simple Ward identity, specifically that the divergence $\partial/\partial z_\rho$ produces a sum similar to (1.4) of two gauge-invariant self-energies of the form (1.6). However, the divergence of the vertex function for three external quantum gluons involves the more complicated mathematical structure of the Slavnov–Taylor identity (A.3). Thus the three-gluon vertex of conventional field theory cannot be conformal invariant.

Mixed vertices with both background and quantum external gluons, satisfy simple Ward identities when the divergence is taken in the background field. This follows from (2.6) and the fact that mixed and quantum self-energies both take the gauge invariant form (1.6). However, the divergence on a quantum vertex is again more complicated, as one can see from (A.3). This is not quite enough to conclude that mixed vertices are not conformal invariant, because the mixed vertices have the reduced Bose symmetry of a single conformal tensor $D_{\mu\nu\rho}(x,y,z)$ and this tensor also has the curious property that the $\partial/\partial z_\rho$ divergence is that of a simple Ward identity, while the $\partial/\partial x_\mu$ and $\partial/\partial y_\nu$ divergences are more complicated.

The argument that the mixed vertices are not conformal invariant can be completed in several ways, and we choose an argument which gives an additional piece of information. The renormalization properties of the background field formalism have been studied by Kluberg–Stern and Zuber.[11] They find that the counterterm for the overall divergence of a three-gluon vertex with any combination of external background and quantum lines takes the form of the bare Yang–Mills vertex, that is the cubic term of (2.1). This is confirmed by our one-loop

calculations, and it has the implication that the part of the one-loop amplitude which requires regularization can be written as a numerical multiple of $D^{\rm sym}_{\mu\nu\rho}$. The remainder is ultraviolet finite. For mixed vertices, this means that the remainder cannot simply be a non-fully symmetric combination of the linearly independent tensors $D_{\mu\nu\rho}$ and cyclic permutations, because any such combination requires regularization. Mixed vertices are therefore not conformal invariant, although we hope that the fact that their divergent part is a multiple of $D^{\rm sym}_{\mu\nu\rho}$ will facilitate study of the Slavnov–Taylor identities and help with two-loop calculations in the background field method.

It is relevant to ask whether other one-loop vertex functions of gauge theories can have the conformal properties found here. The example (1.1) of the electron vertex function in quantum electrodynamics shows that this is not the case, and we discuss this briefly here. The inversion property of a fermion field is

$$\psi'(x') = x^2 \gamma_5 \gamma \cdot x \, \psi(x) \quad . \tag{3.24}$$

Using this and (1.11), it is not difficult to see that the amplitude

$$\tilde{V}_\lambda(x,y,z) = -2\gamma_a\gamma_\lambda\gamma_b V_{ab}(x-z, y-z) \tag{3.25}$$

transforms properly under conformal transformation, but (1.1) does not. Indeed $\tilde{V}_\lambda$ is the one-loop electromagnetic vertex in a Lagrangian in which a fermion is coupled via $\bar{\psi}\psi\phi$ to a massless scalar field, so conformal invariance is expected! The blame for the non-invariance in the case of quantum electrodynamics rests squarely on the shoulder of the gauge fixing procedure, which affects the virtual photon propagator. One can easily see from (1.1) that the difference $V_\lambda - \tilde{V}_\lambda$ is a total derivative and therefore ultraviolet finite. Thus we find again that the part of a non-conformal vertex that requires regularization is conformal for non-coincident points.

IV. REGULARIZATION OF CONFORMAL COVARIANT TENSORS

In the previous section the bare, primitive divergent Feynman amplitudes for the three-gluon vertex were expressed in terms of the conformal tensors $D^{\rm sym}_{\mu\nu\rho}$ and $C^{\rm sym}_{\mu\nu\rho}$. The regularization problem for the physical amplitudes is therefore solved by regulating these tensors, and regularization is required because the short-distance singularities make the Fourier transforms diverge. Indeed, each tensor $D_{\mu\nu\rho}$ and $C_{\mu\nu\rho}$ of (1.8) and (1.9) corresponds to a formally linear divergent loop integral in momentum space. However, the combination $C^{\rm sym}_{\mu\nu\rho}$ of (1.10) is ultraviolet finite. Thus the various contributions to the three-gluon vertex involve a universal divergent tensor $D^{\rm sym}_{\mu\nu\rho}$. In this section we present two distinct regularized expressions for $D^{\rm sym}_{\mu\nu\rho}$ using the method of differential regularization. We also discuss the properties of $C^{\rm sym}_{\mu\nu\rho}$ which, although finite, requires regularization to make its Fourier transform unambiguous.

In the first approach to regularize the conformal tensor $D_{\mu\nu\rho}(x,y,z)$, it is convenient to write it in terms of the tensors $T_{\mu\nu\rho}(x-z,y-z)$ and $V_{\mu\nu\rho}(x-z,y-z)$ introduced in (3.3). Note that $T_{\mu\nu\rho}$ has been regulated already in Ref. [1] and $V_{\mu\nu\rho}$ may be easily regulated following the same basic ideas. A straightforward calculation shows that

$$D_{\mu\nu\rho}(x,y,z) = \frac{1}{4}\delta_{\mu\nu}\left(T_{\lambda\rho\lambda} + T_{\rho\lambda\lambda} - T_{\lambda\lambda\rho}\right) - \frac{1}{2}V_{\mu\nu\rho}(x-z,y-z) \tag{4.1}$$

where $T_{\alpha\beta\gamma} \equiv T_{\alpha\beta\gamma}(x-z,y-z)$ here and in the following.

Now, we briefly summarize the regularization of $T_{\mu\nu\rho}$ given in Ref. [1]. The procedure maintains explicitly the $x \leftrightarrow y$, $\mu \leftrightarrow \nu$ antisymmetry of the tensor and consists of moving the derivatives to the left in order to have a piece with two total derivatives, which has a finite Fourier transform, and some remaining terms whose singular parts lie only in the trace. Thus, $T_{\mu\nu\rho}$ may be written as

$$T_{\mu\nu\rho}(x,y) = F_{\mu\nu\rho}(x,y) + S_{\mu\nu\rho}(x,y) \quad , \tag{4.2}$$

where we have set $z = 0$ for simplicity. $F_{\mu\nu\rho}(x,y)$, which has a well-defined Fourier transform, is

$$\begin{aligned} F_{\mu\nu\rho}(x,y) = {} & \partial^x_\mu \partial^y_\nu \left[\frac{1}{x^2 y^2} \partial^x_\rho \frac{1}{(x-y)^2} \right] + \partial^x_\mu \left[\frac{1}{x^2 y^2} \left(\partial_\nu \partial_\rho - \frac{1}{4} \delta_{\nu\rho} \Box \right) \frac{1}{(x-y)^2} \right] \\ & - \partial^y_\nu \left[\frac{1}{x^2 y^2} \left(\partial_\mu \partial_\rho - \frac{1}{4} \delta_{\mu\rho} \Box \right) \frac{1}{(x-y)^2} \right] \\ & - \frac{1}{x^2 y^2} \left[\partial^x_\mu \partial^x_\nu \partial^x_\rho - \frac{1}{6} \left(\delta_{\mu\nu} \partial^x_\rho + \delta_{\mu\rho} \partial^x_\nu + \delta_{\nu\rho} \partial^x_\mu \right) \Box \right] \frac{1}{(x-y)^2} \quad , \end{aligned} \tag{4.3}$$

while $S_{\mu\nu\rho}(x,y)$ contains the trace terms and thus derivatives of $\delta(x-y)$ times the factor $1/x^4$, which can be regularized in the usual way, *i.e.*, by using the identity

$$\frac{1}{x^4} = -\frac{1}{4} \Box \frac{\ln M^2 x^2}{x^2} \quad , \tag{4.4}$$

yielding

$$S_{\mu\nu\rho}(x,y) = -\frac{\pi^2}{12} \left[\delta_{\mu\nu} \left(\partial^x_\rho - \partial^y_\rho \right) + \delta_{\mu\rho} \left(\partial^x_\nu + 2\partial^y_\nu \right) - \delta_{\nu\rho} \left(2\partial^x_\mu + \partial^y_\mu \right) \right] \delta(x-y) \Box \frac{\ln M^2 x^2}{x^2} \quad . \tag{4.5}$$

With the same procedure, it is easy to see that also the singular part of $V_{\mu\nu\rho}$ lies only in the trace, and, when it is regularized by means of (4.4), we get

$$\begin{aligned} V_{\mu\nu\rho}(x,y) = {} & \left(\partial^y_\rho - \partial^x_\rho \right) \left[\frac{1}{x^2 y^2} \left(\partial_\mu \partial_\nu - \frac{1}{4} \delta_{\mu\nu} \Box \right) \frac{1}{(x-y)^2} \right] \\ & + \frac{2}{x^2 y^2} \left[\partial^x_\mu \partial^x_\nu \partial^x_\rho - \frac{1}{6} \left(\delta_{\mu\nu} \partial^x_\rho + \delta_{\mu\rho} \partial^x_\nu + \delta_{\nu\rho} \partial^x_\mu \right) \Box \right] \frac{1}{(x-y)^2} \\ & + \frac{\pi^2}{12} \left[-\delta_{\mu\nu} \left(\partial^x_\rho - \partial^y_\rho \right) + 2\delta_{\mu\rho} \left(\partial^x_\nu - \partial^y_\nu \right) + 2\delta_{\nu\rho} \left(\partial^x_\mu - \partial^y_\mu \right) \right] \delta(x-y) \Box \frac{\ln M^2 x^2}{x^2} \quad . \end{aligned} \tag{4.6}$$

Notice that the second and third rank tensor traces in (4.3) and (4.6) are independent and a regularization with several independent mass scale parameters[1] M_i is consistent with the $x \leftrightarrow y$, $\mu \leftrightarrow \nu$ antisymmetry, and could have been used. However, in the cyclically symmetric combinations $C^{\rm sym}_{\mu\nu\rho}$ and $D^{\rm sym}_{\mu\nu\rho}$, which are the final objects of interest for the three-gluon vertex,

the mass ratios $\ln M_i/M_j$ appear as coefficients of the bare Yang–Mills vertex ((1.12) in p-space). Thus the mass ratio ambiguity simply corresponds to a finite choice of regularization scheme in differential regularization, and we have chosen the simplest scheme in which only a single scale M appears from the beginning.

Substituting the regulated forms of $T_{\mu\nu\rho}$ and $V_{\mu\nu\rho}$ in (4.1), one finds the following regulated expression for the tensor $D_{\mu\nu\rho}(x,y,z)$:

$$\begin{aligned} D_{\mu\nu\rho}(x,y,z) &= \frac{1}{2}\left(\partial^x_\rho - \partial^y_\rho\right)\left[\frac{1}{(x-z)^2(y-z)^2}\left(\partial_\mu\partial_\nu - \frac{1}{4}\delta_{\mu\nu}\Box\right)\frac{1}{(x-y)^2}\right] \\ &- \frac{1}{(x-z)^2(y-z)^2}\left[\partial^x_\mu\partial^x_\nu\partial^x_\rho - \frac{1}{6}\left(\delta_{\mu\nu}\partial^x_\rho + \delta_{\mu\rho}\partial^x_\nu + \delta_{\nu\rho}\partial^x_\mu\right)\Box\right]\frac{1}{(x-y)^2} \\ &+ \frac{1}{4}\delta_{\mu\nu}\left(\partial^x_\lambda\partial^y_\rho + \partial^x_\rho\partial^y_\lambda - \delta_{\lambda\rho}\partial^x_\sigma\partial^y_\sigma\right)\left[\frac{1}{(x-z)^2(y-z)^2}\partial^x_\lambda\frac{1}{(x-y)^2}\right] \\ &- \frac{\pi^2}{12}\left\{-2\delta_{\mu\nu}\left(\partial^x_\rho - \partial^y_\rho\right) + \delta_{\mu\rho}\left(\partial^x_\nu - \partial^y_\nu\right) + \delta_{\nu\rho}\left(\partial^x_\mu - \partial^y_\mu\right)\right\}\left[\delta(x-y)\Box\frac{\ln M^2(x-z)^2}{(x-z)^2}\right] \quad . \end{aligned} \tag{4.7}$$

From this equation, the regulated form of $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$ may be easily obtained by adding the cyclic permutations $D_{\nu\rho\mu}(y,z,x)$ and $D_{\rho\mu\nu}(z,x,y)$.

The Fourier transform of (4.7), computed using formal integration by parts of the total derivatives, involves a sum of essentially conventional Feynman loop integrals which are convergent because they contain a combination of external momentum factors and traceless tensors. Standard methods, for example, combining denominators using Feynman parameters, can be used to evaluate the loop integrals.

We now discuss an alternative differential regularization of the singular vertex amplitude $D_{\mu\nu\rho}(x,y,z)$ of (1.8). We first put $D_{\mu\nu\rho}$ into a form which suggests a simple regularization by writing out (1.8) as

$$D_{\mu\nu\rho}(x,y,z) = \frac{1}{3}\left[\left(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\frac{1}{(x-y)^4}\right]\left(\frac{(x-z)_\rho}{(x-z)^2} - \frac{(y-z)_\rho}{(y-z)^2}\right)\frac{(x-y)^2}{(y-z)^2(x-z)^2} \quad . \tag{4.8}$$

This form of the bare amplitude contains the product of a factor more singular than in the original form times a vanishing factor, and it is analogous to the form used to regulate the primitive non-planar three-loop graph for the four-point function of ϕ^4-theory.[1] We then regulate the singular factor using (4.4) and obtain

$$\begin{aligned} D_{\mu\nu\rho}(x,y,z) = \frac{1}{24}\left[(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\,\Box\left(\frac{\ln M^2(x-y)^2}{(x-y)^2}\right)\right] \\ \times (x-y)^2\left(\frac{\partial}{\partial x_\rho} - \frac{\partial}{\partial y_\rho}\right)\frac{1}{(x-z)^2(y-z)^2} \quad . \end{aligned} \tag{4.9}$$

We will demonstrate that this expression gives a satisfactory regularization by showing that it has a well-defined Fourier transform when the formal partial integration rule of differential regularization is used. This regularized form is considerably simpler than the previous (4.7), although it does involve the somewhat peculiar technique of artificially raising the degree of singularity of part of the bare amplitude. In this case we have a test of the compatibility of this technique with Ward identities. The regulated Ward identities for the analogous form of $D^{\rm sym}_{\mu\nu\rho}$ are discussed in the next section, but we note here that the Ward-like identity

$$\begin{aligned} &\frac{\partial}{\partial z_\rho}D_{\mu\nu\rho}(x,y,z) = -\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)_\rho D_{\mu\nu\rho}(x,y,z) \\ &= -\frac{1}{24}\left[(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\,\Box\frac{\ln M^2(x-y)^2}{(x-y)^2}\right](x-y)^2\,(\Box_x - \Box_y)\,\frac{1}{(x-z)^2(y-z)^2} \\ &= \frac{\pi^2}{6}\,(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\,\Box\frac{\ln M^2(x-y)^2}{(x-y)^2}\cdot(\delta(x-z) - \delta(y-z)) \quad , \end{aligned} \tag{4.10}$$

relates $D_{\mu\nu\rho}$ to a regulated, gauge invariant self-energy. This suggests that the divergences of $D_{\mu\nu\rho}$ are indeed controlled by our procedure.

A more complete proof that the regularization (4.9) is correct requires the Fourier transform

$$D_{\mu\nu\rho}(p_1,p_2) = \int d^4x\, d^4y\, e^{i(p_1\cdot x + p_2\cdot y)} D_{\mu\nu\rho}(x,y,0) \quad . \tag{4.11}$$

To carry out the x and y integrations we insert

$$(x-y)^2\left(\frac{\partial}{\partial x_\rho}-\frac{\partial}{\partial y_\rho}\right)\frac{1}{x^2y^2}=i\int\frac{d^4k_1\,d^4k_2}{(2\pi)^4}e^{-i(k_1\cdot x+k_2\cdot y)}\left(\frac{\partial}{\partial k_{1\sigma}}-\frac{\partial}{\partial k_{2\sigma}}\right)^2\left[\frac{(k_1-k_2)_\rho}{k_1^2\,k_2^2}\right] \tag{4.12}$$

and

$$(\delta_{\mu\nu}\Box-\partial_\mu\partial_\nu)\,\Box\frac{\ln M^2(x-y)^2}{(x-y)^2}=-\int\frac{d^4k_3}{(2\pi)^2}e^{-ik_3\cdot(x-y)}\left(\delta_{\mu\nu}k_3^2-k_{3\mu}k_{3\nu}\right)\ln\frac{k_3^2}{\bar{M}^2}$$

where the Fourier transform table of Ref. [1] has been used. We then obtain

$$\begin{aligned}D_{\mu\nu\rho}(p_1,p_2)=&\frac{\pi^2 i}{6}\int d^4k\ln\left(\frac{(k+p_1)^2}{\bar{M}^2}\right)\\&\times\left[(k+p_1)^2\,\delta_{\mu\nu}-(k+p_1)_\mu\,(k+p_1)_\nu\right]\Box_k\left[\frac{(2k+p_1+p_2)_\rho}{k^2\,(k+p_1+p_2)^2}\right]\quad.\end{aligned} \tag{4.13}$$

To test that this somewhat unconventional loop integral is finite, it is sufficient to examine the leading terms as $k\to\infty$ which are formally of order $1/k^3$ and $1/k^4$. These leading terms are

$$\left((k+p_1)^2\,\delta_{\mu\nu}-(k+p_1)_\mu\,(k+p_1)_\nu\right)\Box_k\left(-\frac{\partial}{\partial k_\rho}\left(\frac{1}{k^2}\right)-\frac{1}{2}\,(p_1+p_2)_\lambda\,\frac{\partial}{\partial k_\lambda}\,\frac{\partial}{\partial k_\rho}\left(\frac{1}{k^2}\right)\right)\quad. \tag{4.14}$$

Since $\Box 1/k^2=0$ for $k\neq 0$, these vanish identically, so the loop integral in (4.13) is ultraviolet finite. Further, shifts in the loop momenta k are permitted, since the $1/k^3$ term is absent. If one develops the asymptotic series in k further, one sees that the first term which contributes to (4.13) has three powers of the external momenta. Notice that since the leading term in the integral is proportional to at least the momenta p_i, this is equivalent in coordinate space to an amplitude which has at least three derivatives of a singular function with a well-defined transform.

We now study the conformal tensor $C^{\rm sym}_{\mu\nu\rho}(x,y,z)$. The individual terms of (1.9) have an ultraviolet divergent Fourier transform. The divergence cancels in the cyclic sum of (1.10) but there remains a shift ambiguity proportional to the bare gluon vertex (1.12). At first sight it was surprising to find that $C^{\rm sym}_{\mu\nu\rho}(x,y,z)$ was finite, but it is actually a direct consequence of the fact that the cutoff-dependent part of any permutation odd tensor $A^{\rm sym}_{\mu\nu\rho}(x,y,z)$ is proportional to (1.12) in p-space. Suppose we had picked any pair of permutation odd conformal tensors, say $A^{\rm sym}_{\mu\nu\rho}(x,y,z)$ and $B^{\rm sym}_{\mu\nu\rho}(x,y,z)$, rather than $C^{\rm sym}_{\mu\nu\rho}$ and $D^{\rm sym}_{\mu\nu\rho}$ of (1.10). Then by examination of the cutoff-dependent part of $A^{\rm sym}_{\mu\nu\rho}(x,y,z)$ and $B^{\rm sym}_{\mu\nu\rho}(x,y,z)$, we could select a linear combination with finite Fourier transform.

The simplest way to confirm that $C^{\rm sym}_{\mu\nu\rho}$ has the properties stated above is to relate this tensor to the Feynman amplitudes for the ghost and quark loop contributions to the three-gluon vertex. From (3.5) and (3.10), one obtains

$$f^{abc}C^{\rm sym}_{\mu\nu\rho}(x,y,z) = -\left(4\pi^2\right)^3 \left\{ \frac{2}{C}\,\frac{\delta^3(1.{\rm b})}{\delta B^a_\mu(x)\delta B^b_\nu(y)\delta B^c_\rho(z)} + \frac{\delta^3\left[(2.{\rm a})+(2.{\rm b})\right]}{\delta B^a_\mu(x)\delta B^b_\nu(y)\delta B^c_\rho(z)} \right\} \quad . \tag{4.15}$$

We then use standard p-space Feynman rules for these loop graphs. The result is a loop integral with a formal linear divergence $\int d^4k\, k_\mu k_\nu k_\rho/k^6$, but no log divergent terms with numerator quadratic in k.

The finiteness of $C^{\rm sym}_{\mu\nu\rho}(x,y,z)$ can also be shown in real space, without calculation of the Fourier transform. It is not difficult to show that each contribution to the cyclic sum (1.10) for $C^{\rm sym}_{\mu\nu\rho}$ can be expressed in terms of $T_{\mu\nu\rho}$ and $V_{\mu\nu\rho}$, for example

$$\begin{aligned} C_{\mu\nu\rho}(x,y,z) &= -\delta_{\mu\nu}\left(T_{\lambda\rho\lambda}+T_{\rho\lambda\lambda}-T_{\lambda\lambda\rho}\right)+\frac{1}{2}\delta_{\mu\rho}\left(T_{\nu\lambda\lambda}-T_{\lambda\nu\lambda}+T_{\lambda\lambda\nu}\right) \\ &-\frac{1}{2}\delta_{\rho\nu}\left(T_{\mu\lambda\lambda}-T_{\lambda\mu\lambda}+T_{\lambda\lambda\mu}\right)+2\left(T_{\mu\rho\nu}+T_{\rho\nu\mu}\right)-V_{\nu\rho\mu}(y-x,z-x)-V_{\rho\mu\nu}(z-y,x-y) \quad . \end{aligned} \tag{4.16}$$

We then regulate these quantities as in (4.2 – 4.6), but include different scale masses M_c for independent traces. We then find that the overall scale-dependence cancels in $C^{\rm sym}_{\mu\nu\rho}$ but $\ln M_i/M_j$ terms multiply the real space form of (1.12) remain. This is the signal in differential regularization of a quantity with finite but ambiguous Fourier transform, and it is very similar to the axial fermion triangle anomaly in Section II.D of Ref. [1].

The regularized form of $C^{\rm sym}_{\mu\nu\rho}(x,y,z)$ found by the procedure above could be used as the regularized contribution of this tensor to the three-gluon vertex, but it is a very complicated form and we have found a much simpler form by combining ideas of conformal invariance, differential regularization and Ward identities.

One suspects that the tensor $C^{\rm sym}_{\mu\nu\rho}$ satisfies a trivial bare Ward identity

$$\frac{\partial}{\partial z_\rho} C^{\rm sym}_{\mu\nu\rho}(x,y,z) = 0 \tag{4.17}$$

because the ultraviolet divergence, which would normally be present on both sides of a non-trivial identity of the form (1.4), has cancelled. One can confirm this by using conformal invariance to take the limit[10] as one of points goes to ∞ (for example, $y_\mu \to \infty$). It is easy to compute the $\partial/\partial z_\rho$ derivatives of $C_{\mu\nu\rho}(x,y,z)$ and its cyclic permutations directly from (1.9) in this limit. The non-local contribution ($z \neq x$) vanishes trivially in the cyclic sum, and one also shows that there is no quasi-local $\delta(z-x)\sum_{\mu\nu}(y)$ term, thus verifying (4.17). We call (4.17) a bare Ward identity because the Ward identity of a regularized form of $C^{\rm sym}_{\mu\mu\rho}$ in general contains ultra-local terms $\delta(z-x)\,(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)\,\delta(x-y)$. Such terms correspond to the intrinsic ambiguity of $C^{\rm sym}_{\mu\nu\rho}$, and they cannot be detected at large y.

Motivated by the Ward identity (4.17), we sought a mathematical representation of a permutation odd tensor with the conformal inversion property (1.5) of a three-point function

of currents in which the trivial gauge property was carried by explicit derivatives in x, y, and z. This led us to the *ansatz*

$$Q_{\mu\nu\rho}(x,y,z) = \left(\delta_{\mu a}\partial^x_b - \delta_{\mu b}\partial^x_a\right)\left(\delta_{\nu c}\partial^y_d - \delta_{\nu d}\partial^y_c\right)\left(\delta_{\rho e}\partial^z_f - \delta_{\rho f}\partial^z_e\right)$$
$$\times\, \partial^x_a\partial^y_c \ln(x-y)^2 \partial^x_b\partial^z_e \ln(x-z)^2 \partial^y_d\partial^z_f \ln(z-y)^2 \tag{4.18a}$$

$$= -\left(\delta_{\mu a}\Box - \partial_\mu\partial_a\right)^x \left(\delta_{\nu b}\Box - \partial_\nu\partial_b\right)^y \left(\delta_{\rho c}\Box - \partial_\rho\partial_c\right)^z$$
$$\times\, \partial^x_a \ln(x-y)^2 \partial^y_b \ln(y-z)^2 \partial^z_c \ln(z-x)^2 \quad . \tag{4.18b}$$

To see that (4.18a) has the correct inversion property, we note that for fixed y, z

$$\left(\delta_{\mu a}\partial^x_b - \delta_{\mu b}\partial^x_a\right)\left[\partial^x_a \ln(x-y)^2 \partial^x_b \ln(x-z)^2\right] = \partial^x_b\left[\partial^x_\mu \ln(x-y)^2 \partial^x_b \ln(x-z)^2 - (b \leftrightarrow \mu)\right] \quad . \tag{4.19}$$

This equation has the structure $j_\mu = \partial_b F_{\mu b}$ where $F_{\mu b}$ is an anti-symmetric tensor with the same conformal properties as a field strength. Thus (4.19) is conformal invariant for the same reason that Maxwell's equations are conformal invariant, namely $\partial_b F_{\mu b}$ has the inversion property of a dimension three current. In (4.18a), this same property is symmetrically incorporated in all three variables. The form (4.18b) is obtained by "partial integration" of one derivative in each log factor, using the gauge property.

The conformal tensor $Q_{\mu\nu\rho}(x,y,z)$ must be a linear combination of $C^{\rm sym}_{\mu\nu\rho}$ and $D^{\rm sym}_{\mu\nu\rho}$, and the trivial gauge property suggests that it must be proportional to $C^{\rm sym}_{\mu\nu\rho}$. The derivatives in (4.18) are very tedious to compute, so we employed a symbolic manipulation program to calculate $Q_{\mu\nu\rho}$ explicitly, and compare with $C^{\rm sym}_{\mu\nu\rho}$ in (1.9 – 1.10). The result is the equality

$$C^{\rm sym}_{\mu\nu\rho}(x,y,z) = \frac{1}{16} Q_{\mu\nu\rho}(x,y,z) \quad . \tag{4.20}$$

One remaining subtlety arises because the symbolic manipulation program ignores $\delta(x-y)$ terms which could arise from expressions such as $\Box\left(1/(x-y)^2\right)$. Therefore, we checked by hand calculation that there are no hidden quasi-local terms, so that the relation (4.20) is correct.

Because of the many external derivatives in (4.18b), $Q_{\mu\nu\rho}$ can be regarded as a regularization of $C^{\rm sym}_{\mu\nu\rho}$ because it assigns a unique Fourier transform, when the partial integration rule of differential regularization is used. We find it astonishing that the tensor $C^{\rm sym}_{\mu\nu\rho}$ whose Fourier transform is linearly divergent by power counting can be presented in the form (4.18b) whose Fourier transform contains six powers of external momentum and is therefore represented by highly convergent loop integrals.

To see that this regularized version of the finite tensor $C^{\rm sym}_{\mu\nu\rho}$ makes no finite contribution to the Ward identity we shall compute the Fourier transform of it. The Fourier transform of $C^{\rm sym}_{\mu\nu\rho}$ is

$$\begin{aligned} C^{\rm sym}_{\mu\nu\rho} &= 2\pi^2 i\left(\delta_{\mu a}p_1^2 - p_{1\mu}p_{1a}\right)\left(\delta_{\nu b}p_2^2 - p_{2\nu}p_{2b}\right)\left(\delta_{\rho c}p_3^2 - p_{3\rho}p_{3c}\right) \\ &\cdot \frac{\partial}{\partial p_{1a}}\,\frac{\partial}{\partial p_{2b}}\left(\frac{\partial}{\partial p_{1c}} - \frac{\partial}{\partial p_{2c}}\right)\int \frac{d^4k}{k^2\left(k-p_1\right)^2\left(k+p_2\right)^2} \end{aligned} \tag{4.21}$$

where $p_1+p_2+p_3=0$. We can see from this representation that as any momentum component vanishes, $C^{\mu\nu\rho}$ goes to zero so that there is no contribution to the Ward identity. It might seem that as $p_2\to 0$ there would be an infrared singularity of the loop integration at zero but a careful study of (5.21) for small p_2 indicates that it goes smoothly to zero as $p_2\to 0$. Therefore $C^{\rm sym}_{\rm reg}$ gives no contribution to the Ward identity.

V. WARD IDENTITY AND THE MASS SCALE SHIFT

In this section we will consider the Ward identity (2.10) which relates the three-gluon vertex and the self-energy. The vertex is a linear combination of the tensors $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$

and $C^{\rm sym}_{\mu\nu\rho}(x,y,z)$. Since $C^{\rm sym}_{\mu\nu\rho}$ satisfies a trivial Ward identity, our main task is to study the $\partial/\partial z_\rho$ divergence of $D^{\rm sym}_{\mu\nu\rho}$. We will first obtain the bare form of the Ward identity, and then study the regulated version associated with each of the regulated forms of $D^{\rm sym}_{\mu\nu\rho}$ discussed in Section IV. The purpose is to show that the proper relation between renormalized vertex $V_{\mu\nu\rho}$ and self-energy $\Sigma_{\mu\nu}$ can be achieved, as in (1.1) – (1.2) by specific choice of the mass scale parameter of the differential regularization procedure.

The bare Ward identity satisfied by $D^{\rm sym}_{\mu\nu\rho}$ is easily obtained by following the same approach outlined for $C^{\rm sym}_{\mu\nu\rho}$. Exploiting conformal invariance, we take advantage of the algebraic simplifications that occur in the limit as one of the points (we choose y_μ) goes to ∞. Evaluating derivatives, we find that $(\partial/\partial z_\rho)D_{\mu\nu\rho}(x,y,z)$ gives only a local contribution. Each of the other cyclic permutations in (1.10) contains non-local terms ($x \neq y$) which cancel in the sum leaving only $\delta(x-z)$. The net result is

$$\frac{\partial}{\partial z_\rho} D^{\rm sym}_{\mu\nu\rho}(x,y,z) \underset{y\to\infty}{\longrightarrow} -12\pi^2\delta(x-z)\left(\delta_{\mu\nu} - \frac{2y_\mu y_\nu}{y^2}\right)\frac{1}{y^6} \quad . \tag{5.1}$$

From this we infer, using $x,\mu \leftrightarrow y,\nu$ permutation symmetry of $D^{\rm sym}_{\mu\nu\rho}$ and translation invariance, the bare Ward identity

$$\begin{aligned} \frac{\partial}{\partial z_\rho} D^{\rm sym}_{\mu\nu\rho}(x,y,z) &= \pi^2\left[\delta(x-z)-\delta(y-z)\right]\left(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box\right)\frac{1}{(x-y)^4} \\ &\equiv \left[\delta(x-z)-\delta(y-z)\right]\Sigma_{\mu\nu}(x-y) \quad . \end{aligned} \tag{5.2}$$

Comparing this result with the Ward identity (1.4) obeyed by current correlation functions, we see that the coefficients k and k_1 in (1.6) and (1.7) are related by

$$12\pi^2 k_1 = k \tag{5.3}$$

while k_2 in (1.7) is an independent constant, since $C^{\rm sym}_{\mu\nu\rho}(x,y,z)$ does not contribute to the Ward identity.

The regulated form of the Ward identity is a more delicate matter because it tests the treatment of the overall singularity at $x \sim y \sim z$ of $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$. In any regularization procedure, there is an unresolved ambiguity which for a linearly divergent quantity with the discrete symmetries of $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$ is simply a finite multiple of the bare Yang–Mills vertex. Similarly the ambiguity in a gauge invariant self-energy function is a multiple of $(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)\,\delta(x-y)$. In differential regularization, these ambiguities are reflected in the dependence of regulated amplitudes on the mass scales M which are chosen in (4.4). Thus we study the regulated form of (5.2) with self-energy scale M_Σ

$$\frac{\partial}{\partial z_\rho} D^{\rm sym}_{\mu\nu\rho}(x,y,z) = \frac{\pi^2}{4}\left(\delta(x-z)-\delta(y-z)\right)\left(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu\right)\Box\frac{\ln M_\Sigma^2(x-y)^2}{(x-y)^2} \quad , \tag{5.4}$$

and we require that this be satisfied for both regularizations of $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$ given in Section IV. For the regularized form (4.7), we use vertex mass scale M_{V_1}, and for the form (4.9), the scale M_{V_2}. In each case we will find that the Ward identity is satisfied, if relations of the form

$$\ln\left(\frac{M_{V_1}}{M_\Sigma}\right) = a_1 \quad , \qquad \ln\left(\frac{M_{V_2}}{M_\Sigma}\right) = a_2 \tag{5.5}$$

hold. Since these relations fix the ambiguity in the vertex up to an overall scale, the two forms (4.7) and (4.9) will then coincide as renormalized amplitudes if

$$\ln\left(\frac{M_{V_1}}{M_{V_2}}\right) = a_1 - a_2 \tag{5.6}$$

Since the bare $D^{\rm sym}_{\mu\nu\rho}$ and bare self-energy are properly related away from coincident points $x = y = z$, it is sufficient to study a restricted form of (5.4) in order to fix the mass scale ratios of (5.5). For the first regulated version (4.7) it is convenient to use the integrated form

of (5.4)

$$\frac{\partial}{\partial z_\rho}\int d^4y\, D^{\rm sym}_{\mu\nu\rho}(x,y,z) = -\frac{\pi^2}{4}\left(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu\right)\Box\frac{\ln M_\Sigma^2(x-z)^2}{(x-z)^2} \quad . \tag{5.7}$$

For the regularization (4.9) a Fourier transform of (5.4) is more convenient, as we discuss below.

Thus our first task is to insert the regulated form (4.7) and its cyclic permutations in the left-hand side of (5.7), do the integral d^4y and compute the $\partial/\partial z_\rho$ divergence. All integrals can be done using the intermediate results

$$\begin{aligned}
&\int^R d^4y\frac{1}{(x-y)^2}\,\frac{1}{y^2} = -\pi^2\left[\ln\frac{x^2}{R^2}-1\right]\\
&\int^R d^4y\frac{1}{y^4} = -\frac{1}{4}\int^R d^4y\,\Box\frac{\ln M^2y^2}{y^2} = \pi^2\left[\ln M^2R^2-1\right]\\
&\left(\partial_\mu\partial_\nu-\frac{1}{4}\delta_{\mu\nu}\Box\right)\int d^4y\frac{1}{(x-y)^2y^2} = -\pi^2\left(\delta_{\mu\nu}-\frac{4x_\mu x_\nu}{x^2}\right)\frac{1}{x^2}\\
&\left[\partial_\mu\partial_\nu\partial_\rho-\frac{1}{6}\left(\delta_{\mu\nu}\partial_\rho+\delta_{\mu\rho}\partial_\nu+\delta_{\nu\rho}\right)\Box\right]\int d^4y\frac{1}{(x-y)^2y^2} =\\
&= -16\pi^2\left[x_\mu x_\nu x_\rho-\frac{1}{6}x^2\left(\delta_{\mu\nu}x_\rho+\delta_{\nu\rho}x_\mu+\delta_{\rho\mu}x_\nu\right)\right]\frac{1}{x^6}
\end{aligned} \tag{5.8}$$

An infrared cutoff R at some large value of y is required because individual terms in the contribution of the permutation $D_{\rho\mu\nu}(z,x,y)$ to (5.7) are infrared divergent, although this divergence cancels in the full contribution. The last two integrals are obtained by explicit differentiation of the first result. The second integral is evaluated using the differential regularization recipe in which the singular contribution for small y is ignored. The large y contribution is obtained from the divergence theorem. One uses translational invariance to replace $x\to x-z$ in applying (5.8) to (5.7).

We record the results of the three cyclic permutations:

$$
\begin{aligned}
\frac{\partial}{\partial z_\rho}\int d^4y D_{\mu\nu\rho}(x,y,z) &= \frac{\pi^2}{6}\left(\partial_\mu\partial_\nu-\delta_{\mu\nu}\Box\right)\Box\frac{\ln M_{V_1}^2(x-y)^2}{(x-z)^2} \\
&\quad+\frac{\pi^4}{9}\left(\partial_\mu\partial_\nu-\frac{1}{4}\delta_{\mu\nu}\Box\right)\delta(x-z) \\
\frac{\partial}{\partial z_\rho}\int d^4y\, D_{\nu\rho\mu}(y,z,x) &= -\frac{\pi^2}{12}\left(\partial_\mu\partial_\nu-\delta_{\mu\nu}\Box\right)\Box\frac{\ln M_{V_1}^2(x-y)^2}{(x-z)^2} \\
&\quad+\frac{\pi^4}{9}\left(-\frac{5}{4}\partial_\mu\partial_\nu+2\delta_{\mu\nu}\Box\right)\delta(x-z) \\
\frac{\partial}{\partial z_\rho}\int d^4y\, D_{\rho\mu\nu}(z,x,y) &= \frac{\pi^2}{6}\left(\partial_\mu\partial_\nu-\delta_{\mu\nu}\Box\right)\Box\frac{\ln M_{V_1}^2(x-z)^2}{(x-z)^2} \\
&\quad-\frac{\pi^4}{9}\left(\frac{1}{2}\partial_\mu\partial_\nu+\delta_{\mu\nu}\Box\right)\delta(x-z) \quad .
\end{aligned}
\tag{5.9}
$$

We add these contributions to obtain the final gauge invariant result

$$
\frac{\partial}{\partial z_\rho}\int d^4y\, D^{\rm sym}_{\mu\nu\rho}(x,y,z) = -\frac{\pi^2}{4}\left(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu\right)\Box\left[\frac{\ln M_{V_1}^2(x-z)^2}{(x-z)^2}+\frac{1}{12}\,\frac{1}{(x-z)^2}\right] \quad , \tag{5.10}
$$

which, when compared with (5.7) yields the mass scale relation

$$
\ln\frac{M_{V_1}^2}{M_\Sigma^2} = -\frac{1}{12} \quad . \tag{5.11}
$$

We next consider the second regularized form (4.9) of $D^{\rm sym}_{\mu\nu\rho}$. Since the Fourier transform (4.13) is quite simple, we choose to work in momentum space. The Fourier transform of (5.4) is

$$
i\left(p_1+p_2\right)_\rho D^{\rm sym}_{\mu\nu\rho}\left(p_1,p_2\right) = \Sigma_{\mu\nu}\left(p_2\right)-\Sigma_{\mu\nu}\left(p_1\right) \quad . \tag{5.12}
$$

The restriction we use to fix mass scales is the analogue of the original Ward identity of quantum electrodynamics and is obtained by applying $\partial/\partial p_{2\lambda}$ to (5.12), and then setting $p_2=-p_1=-p$, which is equivalent to $p_3=0$. The result is

$$
\begin{aligned}
iD^{\rm sym}_{\mu\nu\lambda}(p,-p) &= -\frac{\partial}{\partial p_\lambda}\Sigma_{\mu\nu}(p) \\
&= \pi^4\frac{\partial}{\partial p_\lambda}\left[\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)\ln\frac{p^2}{\bar{M}_\Sigma^2}\right]
\end{aligned}
\tag{5.13}
$$

where we have used (5.4) and the Fourier transform result of Appendix A of Ref. [1] to obtain the last line. Note that $\bar{M}_\Sigma = 2M_\Sigma/\gamma$ where $\gamma = 1.781$ is Euler's constant.

From (4.11) and permutations, we find that

$$D^{\rm sym}_{\mu\nu\lambda}(p,-p) = D_{\mu\nu\lambda}(p,-p) + D_{\nu\lambda\mu}(-p,0) + D_{\lambda\mu\nu}(0,p) \quad . \tag{5.14}$$

To evaluate $D_{\mu\nu\lambda}(p,-p)$ is simple; we have already shown that this term obeys a kind of Ward identity by itself. One finds directly from (4.13) using

$$\Box_p \frac{\partial}{\partial p_\lambda}\left(\frac{1}{p^2}\right) = -4\pi^2 \frac{\partial}{\partial p_\lambda}\delta(p) \quad ,$$

$$D_{\mu\nu\lambda}(p,-p) = -\frac{2\pi^4 i}{3}\,\frac{\partial}{\partial p_\lambda}\left[\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\ln\frac{p^2}{\bar{M}^2_{V_2}}\right] \quad . \tag{5.15}$$

Next we compute the remaining terms in (5.14) where an elementary integral must be evaluated, namely

$$\begin{aligned}&\int d^4k\,\ln\left(\frac{k^2}{\bar{M}^2_{V_2}}\right)\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right)\Box_k\left(\frac{(2k-p)_\lambda}{k^2(k-p)^2}\right)\\ &= -2\pi^2\frac{p_\lambda p_\mu p_\nu}{p^2} + p_\lambda\delta_{\mu\nu}\left(-4\pi^2\ln\frac{p^2}{\bar{M}^2_{V_2}}\right) - \left(p_\mu\delta_{\lambda\nu} + p_\nu\delta_{\lambda\mu}\right)\left(2\pi^2\ln\frac{p^2}{\bar{M}^2_{V_2}} + \pi^2\right)\end{aligned} \tag{5.16}$$

Thus

$$D_{\nu\lambda\mu}(-p,0) + D_{\lambda\mu\nu}(0,p) = \left(-\frac{\pi^4 i}{3}\right)\frac{\partial}{\partial p_\lambda}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\left(\ln\frac{p^2}{\bar{M}^2_{V_2}} - \frac{1}{2}\right) \tag{5.17}$$

and so we obtain

$$D^{\rm sym}_{\mu\nu\lambda}(p,-p) = \left(-\pi^4 i\right)\frac{\partial}{\partial p_\lambda}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\left(\ln\frac{p^2}{\bar{M}^2_{V_2}} - \frac{1}{6}\right) \tag{5.18}$$

We see that the only difference between (5.18) and (5.13) is a linear term in p which is the expected local violation of the Ward identity due to regularization ambiguities. The Ward identity is satisfied exactly if we fix the mass scale ratio

$$\ln\left(\frac{M^2_{V_2}}{M^2_\Sigma}\right) = -\frac{1}{6} \quad . \tag{5.19}$$

Using (5.11) we see that

$$\ln\left(\frac{M_{V_1}^2}{M_{V_2}^2}\right) = -\frac{1}{12} + \frac{1}{6} = \frac{1}{12} \quad . \tag{5.20}$$

This calculation illustrates the "robustness" of the differential regularization method. Two rather different regularizations of the same amplitude are simply related by a proper choice of scale parameters.

VI. THE BETA-FUNCTION

Although the one-loop $\beta(g)$ for non-Abelian gauge theories was calculated in Ref. [1] using differential regularization, we shall recompute it here from the present viewpoint which emphasizes our principal result that the gluon vertex function is the linear combination of conformal tensors given in (3.12).

We use the notation (7.1) and the renormalization group equation (7.4), which requires that the classical and regulated one-loop contributions to the vertex function, denoted respectively by $\Gamma^0_{\mu\nu\rho}$ and $\Gamma^1_{\mu\nu\rho}$ are related by

$$\begin{aligned} M\frac{\partial}{\partial M}\Gamma^1_{\mu\nu\rho}(x,y,z) &= -\beta(g)\frac{\partial}{\partial g}\Gamma^0_{\mu\nu\rho}(x,y,z) \\ &= 2\frac{\beta(g)}{g^3}\left\{\delta_{\mu\nu}\left(\partial^x_\rho - \partial^y_\rho\right) + \delta_{\nu\rho}\left(\partial^y_\mu - \partial^z_\mu\right) + \delta_{\rho\mu}\left(\partial^z_\nu - \partial^x_\nu\right)\right\}\delta(x-y)\delta(y-z) \end{aligned} \tag{6.1}$$

where the classical term is easily computed directly from $S[B]$ in (2.1). Using (3.12) and the fact that only the regulated tensor $D^{\rm sym}_{\mu\nu\rho}$ is scale-dependent, we see that the left-hand side of (6.1) is simply

$$M\frac{\partial}{\partial M}\Gamma^1_{\mu\nu\rho}(x,y,z) = \frac{1}{48\pi^6}\left(11C - 2N_f\right) M\frac{\partial}{\partial M}D^{\rm sym}_{\mu\nu\rho}(x,y,z) \quad . \tag{6.2}$$

It is rather trivial to compute the scale derivative of the regularized form (4.7) of $D_{\mu\nu\rho}$ and add permutations to obtain

$$M\frac{\partial}{\partial M}D^{\rm sym}_{\mu\nu\rho}(x,y,z) = -2\pi^4\left\{\delta_{\mu\nu}\left(\partial^x_\rho - \partial^y_\rho\right) + \delta_{\nu\rho}\left(\partial^y_\mu - \partial^z_\mu\right) + \delta_{\rho\mu}\left(\partial^z_\nu - \partial^x_\nu\right)\right\}\delta(x-y)\delta(y-z). \tag{6.3}$$

From (6.1) – (6.3) one immediately finds the well-known result

$$\beta(g) = -\frac{g^3}{48\pi^2}\left(11C - 2N_f\right) + \mathcal{O}(g^5) \quad . \tag{6.4}$$

It is a useful test of the ideas underlying the second regularization of $D_{\mu\nu\rho}$ in (4.9) to see that the same result can be obtained from this form. The scale derivative of (4.9) gives an expression which is difficult to interpret as a product of delta functions unless an integral with a smooth function is performed, so it is natural to study the momentum form (4.13), where the scale derivative is

$$M\frac{\partial}{\partial M}D_{\mu\nu\rho}(p_1,p_2) = -\frac{i\pi^2}{3}\int d^4k\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right)\Box_k\frac{\left(2k-p_1+p_2\right)_\rho}{\left(k-p_1\right)^2\left(k+p_2\right)^2} \tag{6.5}$$

where we have made the (permitted) shift $k \to k - p_1$ of the loop momentum in (4.13). Dimensional and symmetry arguments tell us that the integral must have the form

$$M\frac{\partial}{\partial M}D_{\mu\nu\rho}(p_1,p_2) = A\left(p_1-p_2\right)_\rho\delta_{\mu\nu} + B\left[\left(p_1-p_2\right)_\mu\delta_{\nu\rho} + \left(p_1-p_2\right)_\nu\delta_{\mu\rho}\right] \tag{6.6}$$

where A and B are purely numerical constants. To compute A and B it is sufficient to evaluate (6.5) with $p_2 = -p_1$. This leads to

$$Ap_\rho\delta_{\mu\nu} + B\left[p_\mu\delta_{\nu\rho} + p_\nu\delta_{\mu\rho}\right] = -\frac{i\pi^2}{3}\int d^4k\left(k_2\delta_{\mu\nu} - k_\mu k_\nu\right)\Box_k\frac{(k-p)_\rho}{(k-p)^4} \quad . \tag{6.7}$$

In this form the integral is elementary, since

$$\Box_k\frac{(k-p)_\rho}{(k-p)^4} = \frac{1}{2}\,\frac{\partial}{\partial p_\rho}\Box_k\frac{1}{(k-p)^2} = -2\pi^2\frac{\partial}{\partial p_\rho}\delta(k-p) \tag{6.8}$$

so

$$Ap_\rho\delta_{\mu\nu} + B\left[p_\mu\delta_{\nu\rho} + p_\nu\delta_{\mu\rho}\right] = \frac{i2\pi^4}{3}\,\frac{\partial}{\partial p_\rho}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right) \tag{6.9}$$

which gives

$$A = \frac{i4\pi^4}{3} \quad , \qquad B = -\frac{i2\pi^4}{3} \quad . \tag{6.10}$$

We insert this into (6.6) and add cyclic permutations to obtain

$$\begin{aligned} M\frac{\partial}{\partial M}D^{\mathrm{sym}}_{\mu\nu\rho}\left(p_1,p_2,p_3\right) &= M\frac{\partial}{\partial M}\left[D_{\mu\nu\rho}\left(p_1,p_2\right)\right. \\ &\left. +D_{\nu\rho\mu}\left(p_2,p_3=-(p_1+p_2)\right)+D_{\rho\mu\nu}\left(p_3=-(p_1+p_2),p_1\right)\right] \\ &= i2\pi^4\left\{\delta_{\mu\nu}\left(p_1-p_2\right)_\rho+\delta_{\nu\rho}\left(p_2-p_3\right)_\mu+\delta_{\rho\mu}\left(p_1-p_2\right)_\nu\right\} \quad . \end{aligned} \tag{6.11}$$

If this result is inserted in the Fourier transform of (6.2) and (6.1), we again find the beta-function (6.4).

VII. OUTLOOK BEYOND ONE-LOOP

Our approach to the three-gluon vertex has been largely "experimental," and we do not yet have a theoretical explanation of the gauge-specific conformal property found at one-loop order. Nevertheless it is of some interest to consider the possible role of conformal symmetry beyond one-loop. We discuss here an admittedly speculative scenario based on the interplay of renormalization group equations and Ward identities. We will see that exact conformal invariance cannot hold in higher order because of the twin problems of subdivergences and gauge-dependence. However, one might have a situation in which, in a given order of perturbation theory, the three-gluon vertex or the three-point current correlation function has a certain conformal invariant primitive core which is a linear combination of $D^{\mathrm{sym}}_{\mu\nu\rho}$ and $C^{\mathrm{sym}}_{\mu\nu\rho}$ plus conformal breaking terms which are determined by the renormalization group in terms of lower-order amplitudes.

To simplify notation let us define 1PI two- and three-point functions in the background field formalism as

$$\begin{aligned} \frac{\delta^2}{\delta B^a_\mu(x)\delta B^b_\nu(y)}\left(S[B]+\Omega[B,J]_{j=0}\right) &\equiv \delta^{ab}\Gamma_{\mu\nu}(x-y) \\ \frac{\delta^3}{\delta B^a_\mu(x)\delta B^b_\nu(y)\delta B^c_\rho(z)}\left(S[B]+\Omega[B,J]_{j=0}\right) &\equiv f^{abc}\Gamma_{\mu\nu\rho}(x,y,z) \quad . \end{aligned} \tag{7.1}$$

The Ward identity (2.10) then becomes

$$\frac{\partial}{\partial z_\rho}\Gamma_{\mu\nu\rho}(x,y,z) = \left[\delta(y-z)-\delta(x-z)\right]\Gamma_{\mu\nu}(x-y) \tag{7.2}$$

We use the results of Kluberg–Stern and Zuber[11] and Abbott[6] on the renormalization properties of the background method to determine the renormalization group equations satisfied by $\Gamma_{\mu\nu}$ and $\Gamma_{\mu\nu\rho}$. First we note that our background field $B^a_\mu(x)$ has no anomalous dimension to all orders in perturbation theory because $B^a_\mu(x)$ is related to $A^a_\mu(x)$ of Refs. [6] and [11] by

$$\begin{aligned} B^a_\mu(x) &= g_{\rm bare}\, A^a_\mu(x)_{\rm bare} \\ &= g\, Z_g\sqrt{Z_A}\, A^a_\mu(x)_{\rm ren} \\ &= g\, A^a_\mu(x)_{\rm ren} \end{aligned} \tag{7.3}$$

since $Z_g\sqrt{Z_A}=1$. It is also known that renormalization of the gauge parameter a is required to make the two-point function of the quantum field b^a_μ multiplicatively renormalizable in any gauge except Landau gauge. These facts suggest that the renormalization group equations take the form

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}+\delta(g)a\frac{\partial}{\partial a}\right]\Gamma_{\mu\nu}(x-y)=0 \tag{7.4}$$

$$\left[M\frac{\partial}{\partial M}+\beta(g)+\delta(g)a\frac{\partial}{\partial a}\right]\Gamma_{\mu\nu\rho}(x,y,z)=0 \tag{7.5}$$

where $\beta(g) = g^3\beta_1 + g^5\beta_2 + \ldots$ and $\delta(g) = g^2\delta_1 + g^4\delta_2 + \ldots$ and the subscripts 1 and 2 are the loop-order of the expansion coefficients of $\beta(g)$ and $\delta(g)$. We remind readers that in our conventions the ℓ-loop contributions to $\Gamma_{\mu\nu}$ and $\Gamma_{\mu\nu\rho}$ carry the power $g^{2\ell-2}$.

Although (7.4) – (7.5) are generally valid we use them here only for separated points. Therefore the classical contributions can be dropped, and the differential regularization of the overall singularity in a given order of perturbation theory is irrelevant. Thus the result of Section II.G of Ref. [1] for $\Gamma_{\mu\nu}$ in one-loop order can be written as

$$\Gamma_{\mu\nu}(x) = -\frac{\beta(g)}{\pi^2 g^3}\left(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\frac{1}{x^4} \quad , \qquad x \neq 0 \quad . \tag{7.6}$$

Let us bring in some information about higher-order terms in $\Gamma_{\mu\nu}$. Using the structure of (7.4) and the fact that β_1 and β_2 are gauge-independent, one sees that $\Gamma_{\mu\nu}$ is described by

$$\Gamma_{\mu\nu}(x) = -\frac{1}{\pi^2}\left(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\left[\frac{1}{x^4}\left(\frac{\beta(g)}{g^3} - g^4\beta_1\beta_2 \ln M^2x^2\right)\right] \tag{7.7}$$

through three-loop order. The last term is the result of uncancelled subdivergences in three-loop order, and the only allowed gauge dependence in (7.7) is the coefficient β_3.

What can be said about $\Gamma_{\mu\nu\rho}$? First let us refer to the calculation of the linear deviation from the Feynman gauge in Appendix A, and denote by $R_{\mu\nu\rho}(x,y,z)$ the variational derivative of the final result (A.7) which is a one-loop contribution to $\frac{\partial}{\partial a}\Gamma^1_{\mu\nu\rho}$ at $a = 1$. Note also that $R_{\mu\nu\rho}$ satisfies a trivial Ward identity, *e.g.* $\partial^z_\rho R_{\mu\nu\rho}(x,y,z) = 0$. It then follows from (7.5) that the two-loop contribution to the vertex function $\Gamma^2_{\mu\nu\rho}$ in Feynman gauge satisfies

$$M\frac{\partial}{\partial M}\Gamma^2_{\mu\nu\rho} = -g^2\delta_1 R_{\mu\nu\rho} \quad . \tag{7.8}$$

We shall write down the following solution of (7.8)

$$\Gamma^2_{\mu\nu\rho}(x,y,z) = \frac{1}{6}g^2\left\{\delta_1 \ln\left[-\left(\Box_x\Box_y\Box_x\right)/M^6\right]R_{\mu\nu\rho}(x,y,z) + N_{\mu\nu\rho}(x,y,z)\right\} \tag{7.9}$$

where $N_{\mu\nu\rho}$ is a permutation odd tensor which is independent of scale M. This solution may not be unique, but it does illustrate one way in which our conformal scenario can work. Note that a small change in the scale parameter corresponds to a perturbative correction to the Feynman gauge condition, $a = 1 + \mathcal{O}(g^2)$, and especially that the scale-dependent term satisfies a trivial Ward identity. Thus the only *a priori* constraint on $N_{\mu\nu\rho}$ is that it satisfies the Ward identity (7.2) with the two-loop contribution $\Gamma^2_{\mu\nu}$ on the right-hand side. One solution of this is the conformal tensor

$$N_{\mu\nu\rho} = -\frac{1}{\pi^4}\beta_2 D^{\rm sym}_{\mu\nu\rho} + \gamma_2 C^{\rm sym}_{\mu\nu\rho} \tag{7.10}$$

where we have used (5.3), and the constant γ_2 is undetermined because $C^{\rm sym}_{\mu\nu\rho}$ satisfies a trivial Ward identity. The analysis presented here cannot substitute for the very difficult job of a complete two-loop calculation, yet it incorporates all the general properties which the true amplitudes must satisfy. The tensor $N_{\mu\nu\rho}$ could be the conformal invariant core of the two-loop vertex function.

Note that it is the fact that $\Gamma^1_{\mu\nu\rho}$ is gauge-dependent that forces the scale-dependence in (7.9) and indicates that subdivergences do not cancel in $\Gamma^2_{\mu\nu\rho}$. The situation would be the same even if the linear deviation from Feynman gauge was conformal invariant, so that $R_{\mu\nu\rho} \sim C^{\rm sym}_{\mu\nu\rho}$. In the Landau gauge, $a = 0$, the problematic $\partial/\partial a$ term in (7.5) disappears, and there can be no subdivergences in $\Gamma^2_{\mu\nu\rho}$. However the calculations described in Appendix C indicate that $\Gamma^1_{\mu\nu\rho}$ is not conformal invariant in Landau gauge. It is still possible that Γ^2 takes the conformal form (7.10) in this gauge, but this does not seem to be interesting.

As a separate question, one can also study the conformal properties of gauge invariant operators such as the color singlet currents $J_{\mu AB} = \bar{q}_A \gamma_\mu q_B$ in a colored quark theory where

the quark fields $q_A(x)$ are labelled by explicit flavor indices A (with color and spin labels suppressed). The three-point function of such currents should have similar properties to that of the $SU(2)$ currents $J^a_\mu = \frac{1}{2}\bar{\psi}_i \tau^a_{ij} \gamma_\mu \psi_j$ in the model discussed by Baker and Johnson in which each field $\psi_i(x)$, with $i = 1, 2$, is coupled to an Abelian gauge field. In both cases, current correlation functions are independent of the gauge condition chosen for the internal gauge field. Since, as we saw in Section III, a conformal transformation can be compensated by a gauge transformation, the correlators of the J^a_μ have the property that conformal invariance may be broken by the renormalization procedure, but gauge-fixing is not a problem.

Let us factor out fermion flavor indices as in (7.1) and use a notation in which the space-time part of the two- and three-point current correlators are denoted by $\hat{\Gamma}_{\mu\nu}(x-y)$ and $\hat{\Gamma}_{\mu\nu\rho}(x, y, z)$. For non-coincident points, these gauge-independent amplitudes obey renormalization group equations of the simple form

$$\left[M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} \right] \hat{\Gamma}_{\mu\nu}(x-y) = 0 \tag{7.11}$$

$$\left[M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} \right] \hat{\Gamma}_{\mu\nu\rho}(x, y, z) = 0 \tag{7.12}$$

and the Ward identity (7.2) holds. One can see that $\beta(g)\frac{\partial}{\partial g}$ acts on $\hat{\Gamma}^2_{\mu\nu}$ as $\hat{\Gamma}^2_{\mu\nu\rho}$ to produce terms of order g^4 in these equations, thus showing that $\hat{\Gamma}^3_{\mu\nu}$ and $\hat{\Gamma}^3_{\mu\nu\rho}$ are scale-dependent because of subdivergences. However these same equations show that subdivergences cancel in two-loop contributions.

To proceed further, we discuss the Abelian model,[10] although we expect that the current correlators of the colored quark theory are similar. From the work of DeRafael and Rosner,[15] one can see that $\hat{\Gamma}_{\mu\nu}(x)$ has the form (7.7) through three–loop order, where $\beta(g)$ is the beta-function of quantum electrodynamics. The results of Baker and Johnson show that the vertex

$\hat{\Gamma}_{\mu\nu\rho}$ is conformal invariant through two-loop order and can be expressed as

$$\hat{\Gamma}_{\mu\nu\rho}(x,y,z) = -\frac{1}{\pi^4}\left[\left(b_1 + b_2 g^2\right) D^{\rm sym}_{\mu\nu\rho}(x,y,z) + \left(a_1 + a_2 g^2\right) C^{\rm sym}_{\mu\nu\rho}(x,y,z)\right] + \mathcal{O}(g^4) \quad (7.13)$$

where a_1 and a_2 are numbers whose exact values are not relevant here.

We represent the unknown three-loop contribution as the sum of a scale-dependent part and a primitive core

$$\hat{\Gamma}^3_{\mu\nu\rho}(x,y,z) = g^4\left[S^3_{\mu\nu\rho}(x,y,z,M) + N^3_{\mu\nu\rho}(x,y,z)\right] \quad . \quad (7.14)$$

Then $S^3_{\mu\nu\rho}$ must satisfy the twin constraints

$$M\frac{\partial}{\partial M} S^3_{\mu\nu\rho} = \frac{2}{\pi^4} b_1\left[b_2 D^{\rm sym}_{\mu\nu\rho} + a_2 C^{\rm sym}_{\mu\nu\rho}\right] \quad (7.15)$$

$$\frac{\partial}{\partial z_\rho} S^3_{\mu\nu\rho}(x,y,z,M) = -\frac{b_1 b_2}{\pi^2}\left[\delta(z-x) - \delta(z-y)\right]\left(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\frac{\ln M^2(x-y)^2}{(x-y)^4} \quad (7.16)$$

It is plausible that a combined solution of (7.15) – (7.16) can be obtained. Then the tensor $N^3_{\mu\nu\rho}$ is constrained only by the simple Ward identity for which one solution is the conformal tensor

$$N^3_{\mu\nu\rho} = -\frac{1}{\pi^4} b_3 D^{\rm sym}_{\mu\nu\rho} + \gamma_3 C^{\rm sym}_{\mu\nu\rho} \quad , \quad (7.16)$$

which would represent the primitive conformal core of the three-loop vertex function.

The last topic to be discussed here is an idea for an unconventional gauge fixing and renormalization procedure for gauge theories, which appears to lead to renormalization group equations without the troublesome $a\frac{\partial}{\partial a}$ term of (7.4) – (7.5). Further study is certainly required to see if this procedure is consistent. If so, then exact conformal invariance can hold at two-loop order for the background gluon vertex function, and the situation in higher-order becomes similar to that of the current correlation function discussed above.

The need to renormalize the gauge parameter a appears first in the one-loop quantum two-point function $\Gamma^b_{\mu\nu}(x)$. In momentum space the conventional renormalized amplitude is[16]

$$\Gamma^b_{\mu\nu}(p) = p^2\delta_{\mu\nu} + \left(\frac{1}{a} - 1\right) p_\mu p_\nu + g^2\left[c_1 + c_2(a-1)\right]\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\ln\frac{p^2}{\bar{M}^2} \tag{7.17}$$

where c_1 and c_2 are known numerical constants. This amplitude was obtained via dimensional regularization, but differential regularization would give the same result without the appearance of explicit divergences and counter terms. Because the one-loop (order g^2) contribution is transverse, this amplitude satisfies the renormalization group equation (to one-loop order)

$$\left[M\frac{\partial}{\partial M} + \delta(g)a\frac{\partial}{\partial a} - 2\gamma(g)\right]\Gamma^b_{\mu\nu}(p) = 0 \tag{7.18}$$

with

$$\gamma(g) = -\frac{1}{2}\delta(g) = -g^2\left[c_1 + c_2(a-1)\right] \tag{7.19}$$

and this is the result expected from the renormalized Lagrangian of Kluberg–Stern and Zuber (see Appendix B).

We now consider the combination of a conventional and a new non-local gauge fixing term

$$\widetilde{S}_{gf} = \frac{1}{2a}\int d^4x (D\cdot b)^2 + \frac{\alpha g^2}{8\pi^2}\int d^4x\, d^4y\, D\cdot b(x)\,\Box\,\frac{\ln M^2(x-y)^2}{(x-y)^2} D\cdot b(y) \quad . \tag{7.20}$$

The last integral is finite at $x \approx y$ if interpreted with the partial integration rule of differential regularization. The last term gives an additional contribution to the momentum space two-point function proportional to $p_\mu p_\nu \ln p^2/\bar{M}^2$ as we can see from the Fourier transform rule of Appendix A of Ref. [1]. The new renormalized two-point function is

$$\widetilde{\Gamma}^b_{\mu\nu}(p) = \Gamma^b_{\mu\nu}(p) + \alpha g^2 p_\mu p_\nu \ln\frac{p^2}{\bar{M}^2} \quad . \tag{7.21}$$

We now choose $\alpha = [c_1 + c_2(a-1)]/a$ and obtain

$$\widetilde{\Gamma}^b_{\mu\nu}(p) = \left\{1 + g^2\left[c_1 + c_2(a-1)\right]\ln\frac{p^2}{\bar{M}^2}\right\}\left[p^2\delta_{\mu\nu} + \left(\frac{1}{a} - 1\right)p_\mu p_\nu\right] \quad . \tag{7.22}$$

This amplitude satisfies the simpler renormalization group equation (to order g^2):

$$\left[M\frac{\partial}{\partial M} - 2\gamma(g)\right]\widetilde{\Gamma}^b_{\mu\nu}(p) = 0 \tag{7.23}$$

with $\gamma(g)$ as given in (7.19).

As in Ref. [1], this equation is obtained essentially by inspection. The standard method of deriving renormalization group equations from the cutoff dependence of renormalization constants can also be implemented in differential regularization[17] by introducing a short-distance cutoff and showing that the surface terms usually neglected in the Fourier transform are actually cancelled by counterterms. In the present situation there is a transverse counterterm $\delta Z\frac{1}{2}\int d^4x\, b_\mu\left[\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu\right]b_\nu$ associated with the one-loop contribution to (7.17), and it seems clear that the surface term associated with the non-local part of (7.20) has the local form $\delta Z'\int d^4x\frac{1}{2}(\partial\cdot b)^2$ with $\delta Z = \delta Z'$ if the relation $\alpha\left[c_1 + c_2(a-1)\right]/a$ is enforced. Thus one would find a net wavefunction renormalization of the local kinetic terms of b_μ in agreement with (7.23).

We now discuss the application of (7.20) in the background formalism. As written, however, it is not applicable because the non-local term is not background gauge invariant. But it can be covariantized if we make the replacement

$$\frac{1}{4\pi^2}\Box\frac{\ln(x-y)^2M^2}{(x-y)^2} \to \ln\left(-D_\mu D_\mu/\bar{M}^2\right) \tag{7.24}$$

where D_μ is the background covariant derivative in the adjoint representation. When $B_\mu = 0$, the left- and right-hand sides of (7.24) coincide, as can be seen by Fourier transformation. It

appears that the usual argument[7,8] that the background field method gives the correct S-matrix can be extended to cover a non-local B_μ-dependent gauge fixing term, and we proceed to analyze the implications for the three-gluon vertex.

Let us denote by $\widetilde{\Gamma}_{\mu\nu\rho}$ the background field vertex function calculated with the new gauge fixing action (7.20). We will discuss only the two-loop contribution $\widetilde{\Gamma}^2_{\mu\nu\rho}$ which is of order g^2, and the lowest order in which the non-local part of (7.20) has an effect. We assume that the appropriate renormalization group equation is the same as (7.12). The question of an anomalous dimension for $B_\mu(x)$ in the new procedure may need reexamination, but it is certain that the one-loop anomalous dimension vanishes which is sufficient to allow us to examine the consequences of (7.20) at two-loop order away from the coincident point singularity. The $\beta(g)\frac{\partial}{\partial g}$ term makes no contribution to to order g^2, and we find only the simple condition

$$M\frac{\partial}{\partial M}\widetilde{\Gamma}^2_{\mu\nu\rho}(x,y,z) = 0 \tag{7.25}$$

which has the direct interpretation that subdivergences cancel among all the graphical contributions to $\widetilde{\Gamma}^2_{\mu\nu\rho}(x,y,z)$. A complete two-loop calculation of $\widetilde{\Gamma}^2_{\mu\nu\rho}$ is very difficult, but the cancellation of subdivergences is a computationally simpler question, and it is a useful test of the idea under discussion. A positive result in no way guarantees that $\widetilde{\Gamma}^2_{\mu\nu\rho}(x,y,z)$ is conformal invariant in the Feynman gauge $a = 1$ or a perturbative modification of this gauge, but there is no reason why this cannot be the case.

The non-local gauge fixing term discussed above removed the $a\frac{\partial}{\partial a}$ term from the renormalization group equation only to one-loop order for $\widetilde{\Gamma}_{\mu\nu}$ and to two-loop order for $\widetilde{\Gamma}_{\mu\nu\rho}$. It seems plausible that the procedure works quite generally if the non-local term in (7.20) is chosen appropriately as a series expansion in g^2, and this is an interesting subject for further investigation.

The conformal scenario we have outlined in this section is very far from proven, but it does indicate a way in which the conformal property can combine with the renormalization group and Ward identities to determine the full structure of vertex functions.

VIII. CONCLUSIONS

In this section we present a partial review of the results of this work, in which the logical relation of the ideas underlying the conformal property is emphasized. Conformal invariance holds for renormalized amplitudes only when the twin difficulties of renormalization scale effects and gauge fixing can be circumvented. The first requires that we study real space amplitudes away from coincident points and choose amplitudes in which subdivergences cancel. The second problem is avoided for gauge-independent correlation functions. However, the standard renormalization program in gauge theories requires consideration of gauge-dependent amplitudes. Here the situation is somewhat different for two-, three- and four-point correlation functions.

In gauge field theory, self-energies and two-point current correlators are constrained by gauge invariance and canonical dimension to have the structure

$$(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)\,\frac{1}{(x-y)^4}\left[c_0 + \sum_{n=1} c_n \left(\ln M^2(x-y)^2\right)^n\right] \quad . \tag{8.1}$$

If subdivergences are absent, $c_n = 0$ for $n \geq 1$, then one obtains the conformal invariant form (1.6). In other words, conformal invariance gives no information beyond gauge invariance and cancellation of subdivergences. We expect that this last condition holds through two-loop order (for currents or external gluons), and that logarithmic corrections beyond that are determined by the renormalization group equations.

For vertex functions of vector fields or conserved currents, conformal symmetry gives restrictions well-beyond gauge and scale invariance, requiring that amplitudes are combinations of the tensors $C^{\rm sym}_{\mu\nu\rho}$ and $D^{\rm sym}_{\mu\nu\rho}$ of (1.10). Any such combination obeys a simple Ward identity of the form (1.4) or (2.10), and this suggests that the background field formalism is relevant. However, a simple Ward identity is not sufficient for conformal invariance, and we have found explicit examples of gauge and scale invariant, but not conformal, tensors in our study of the background field vertex in a general gauge.

Our most striking result is that the one-loop background field vertex in Feynman gauge is conformal invariant, so that the quark, ghost, and gluon loop contributions are each linear combinations of $C^{\rm sym}_{\mu\nu\rho}$ and $D^{\rm sym}_{\mu\nu\rho}$. The practical consequence of the conformal property is the relative ease of regularization, essentially because only $D^{\rm sym}_{\mu\nu\rho}$ has an ultraviolet divergent Fourier transform. It is not difficult to show that the renormalized Ward identity can be satisfied by adjustment of mass scale parameters, M_V in the regularized form of $D^{\rm sym}_{\mu\nu\rho}$ and M_Σ in the self-energy. Calculation of the β-function is also a simple matter.

One may also study three-gluon vertex functions with one or more external quantum gluons, of which some are required for higher-loop computations in the background field formalism. Such vertex functions satisfy Slavnov–Taylor identities which are more complicated than the simple Ward identity (1.4), and essentially for this reason one can rule out conformal invariance. Nevertheless, one can show easily that at one-loop order the regulated form of these vertex functions can be expressed as a multiple of the regulated $D^{\rm sym}_{\mu\nu\rho}$ plus a remainder which is ultraviolet convergent. A complete one-loop study of the differential regularization of these vertices, which includes the vertex function of conventional (non-background) gauge field theory, is an open problem whose solution should be facilitated by the observation above.

Another open problem is the possible conformal property of the background field four-gluon correlator, which is related to the three-gluon vertex by a Ward identity. Conformal symmetry is not a very restrictive property for a four-point function. Nevertheless, it would be useful to know whether the basic primitive divergent one-loop amplitudes with two, three, or four external background gluons share the property of conformal invariance for non-coincident points in Feynman gauge. One can predict that the quark and ghost loop contributions to the four-point function are conformal invariant.

At present we do not have a real explanation of the gauge specific conformal property we found, nor do we know that it has any significance beyond the technical virtue of ease of regularization. Further exploration of the conformal scenario outlined in Section VII may illuminate such questions.

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APPENDIX A

DEVIATION FROM THE FEYNMAN GAUGE

It is natural to ask whether the conformal properties found for the background self-energy and vertex are valid for general values of the gauge parameter a. A complete calculation, using the propagator (2.14) and the interaction vertices $\mathcal{L}_1, \ldots, \mathcal{L}_4$ of (2.13) is feasible for the self-energy and it gives exactly the same result as the Feynman gauge calculation of (7.6) and is thus independent of a. This is of course to be expected, since the one-loop beta-function does not depend on the gauge parameter.

An analytic calculation of the three-gluon vertex in a general gauge is very tedious and we chose to study whether conformal invariance is preserved for a small deviation from Feynman gauge. To this purpose one can expand the functional $\Omega[B, J]_{j=0}$ in a series in a

$$\Omega[B, J] = \Omega[B, J]\Big|_{a=1} + \frac{\partial \Omega[B, J]}{\partial a}\Big|_{a=1} (a-1) + \ldots \tag{A.1}$$

and then analyze the conformal properties of the first coefficient of the expansion $\frac{\partial \Omega}{\partial a}\Big|_{a=1} = \Omega^{(1)}$. Such a coefficient coincides with the vacuum expectation value of the gauge fixing term and by means of a standard Ward-identity [Abers–Lee] of Yang–Mills theories it is possible to relate it to other Green's functions which can be more easily computed.

One can easily adapt to the background field formalism, the derivation of the functional Slavnov–Taylor identity given by Abers and Lee.[18] The result is

$$\begin{aligned} &\int db \det M \exp - \left\{ S[B+b] - S[B] + S_{gf}[b] + \int d^4x \, J^a_\mu b^a_\mu \right\} \\ &\quad \cdot \left\{ \frac{1}{a} G^a[B, b](x) + \int d^4y \, J^b_\mu(y) D^y_\mu[B+b] \left[M^{-1}(y, x) \right]^{ba} \right\} = 0 \end{aligned} \tag{A.2}$$

where $G^a[B,b]$ is the gauge fixing functional, and we must choose $J^a_\mu = D_\nu B^a_{\nu\mu}$ as in Section II, in order to eliminate the linear "tadpole" terms in $S[B+b]-S[B]$. Equation (A.2) can be rewritten in terms of ghost fields as the functional "expectation value"

$$\left\langle \frac{1}{a} D_\mu b^a_\mu(x) + f^{bcd} \int d^4y\, J^b_\mu(y) b^c_\mu(y) c^d(y) \bar{c}^a(x) \right\rangle = 0 \quad , \tag{A.3}$$

where we have used the covariant conservation of the current. From (A.3) it is now easy to obtain a Ward identity for $\Omega^{(1)}[B,J]$ by taking the D^x_ν derivative, then a variation with respect to $J^a_\nu(x)$ and finally integrating in d^4x, we arrive at

$$\begin{aligned} \Omega^{(1)}\left[B, D_\nu B_{\nu\mu}\right] &= -\frac{1}{2} \int d^4x \left\langle D_\mu b^a_\mu D_\nu b^a_\nu \right\rangle \\ &= -\frac{1}{2} f^{bcd} \int d^4x\, d^4y\, D_\rho B^b_{\rho\mu}(y) \left\langle b^c_\mu(y) c^d(y) D^x_\nu \bar{c}^a(x) b^a_\nu(x) \right\rangle \quad . \end{aligned} \tag{A.4}$$

in which it is understood that only connected graphs are included on the right-hand side. The linear deviation of the gluon vertex can now be obtained from all graphs contributing to the third variational derivative of (A.4) with respect to B. (The second variational derivative vanishes because the bilinear part of the effective action is the same in any gauge).

We focus our attention on the triangle diagrams with three external fields and neglect seagull diagrams, because the triangles will give us a sufficient condition to disprove conformal invariance for this linear deviation from the Feynman gauge. These triangle diagrams are obtained from the Wick contractions of the quantum fields in the double-integral term in (A.4) with the vertices $\mathcal{L}_1$, $\mathcal{L}_3$ and $\mathcal{L}^g_i$ of the interaction Lagrangian. In several diagrams there is an effective two-point vertex from the fields $\partial_\nu \bar{c}^a(x) b^a_\nu(x)$ in (A.4), and this leads to the integral

$$\int d^4x \frac{1}{(x-y)^2} \partial^x_\mu \frac{1}{(x-z)^2} = 2\pi^2 \frac{(z-y)_\mu}{(z-y)^2} \quad . \tag{A.5}$$

One can show that the sum of all graphs gives the following contribution to $\Omega_3^{(1)}$:

$$-\frac{1}{2\,(4\pi^2)^3}Cf^{abc}\int d^4x\,d^4y\,d^4z\Bigg[B^a_\mu(x)\partial^y_\rho B'^b_{\rho\nu}(y)B'^c_{\nu\sigma}(z)$$
$$\cdot\frac{(y-z)_\sigma}{(y-z)^2}\,\frac{1}{(y-x)^2}\overleftrightarrow{\partial}^x_\mu\frac{1}{(x-z)^2}-B'^a_{\mu\nu}(x)\partial^y_\rho B'^b_{\rho\sigma}(y)B'^c_{\sigma\mu}(z)\frac{1}{(y-z)^2(z-x)^2}\,\frac{(y-x)_\nu}{(y-x)^2}\Bigg] \tag{A.6}$$

where $B'^a_{\mu\nu}(x)=\partial_\nu B^a_\nu-\partial_\nu B^a_\mu$ is the linearized field strength. This quantity is ultraviolet finite due to the presence of three external derivatives. One might then suspect that if (A.6) is conformal invariant, its variational derivative with respect to $B^a_\mu(x)$, $B^b_\nu(y)$, $B^c_\rho(z)$ would be proportional to the ultraviolet convergent tensor in $C^{\rm sym}_{\mu\nu\rho}$. However, partial integration of the three external derivatives produces both genuine "triangular terms" depending on $(x-y)^2$, $(y-z)^2$ and $(z-y)^2$ and "semi-local terms" containing $\delta(z-y)$ *etc.* (which we drop here). The triangular part could then be a combination of $C^{\rm sym}_{\mu\nu\rho}$ and $D^{\rm sym}_{\mu\nu\rho}$ with the divergent part of $D^{\rm sym}_{\mu\nu\rho}$ cancelled by the neglected semi-local terms and seagull graphs.

After partial integration of the external derivatives and tedious algebra to simplify the result, we find that (A.6) can be rewritten as

$$\begin{aligned}
&-\frac{4C}{(4\pi^2)^3}f^{abc}\int d^4x\,d^4y\,d^4z\,B^a_\mu(x)B^b_\nu(y)B^c_\rho(z)\\
&\times\Bigg\{\delta_{\mu\nu}(x-z)_\rho\Bigg[\frac{(x-z)^2+(y-z)^2+(x-y)^2}{(x-y)^4(z-y)^4(x-z)^4}\\
&-\frac{6}{(z-x)^4(x-y)^6}+\frac{2}{(z-y)^4(x-y)^6}\\
&-\frac{4}{(x-y)^4(z-x)^6}+\frac{4(z-y)^2}{(z-x)^6(x-y)^6}\Bigg]\\
&-2\frac{(x-z)_\mu}{(x-z)^4}\,\frac{(x-y)_\nu}{(x-y)^4}\,\frac{(y-z)_\rho}{(y-z)^4}-8\frac{(x-y)_\mu(z-y)_\nu(x-z)_\rho}{(x-y)^6(y-z)^6}\\
&-4\frac{(x-y)_\mu(x-z)_\nu(x-z)_\rho}{(x-y)^6(x-z)^2(z-y)^2}\left[\frac{1}{(z-y)^2}+\frac{1}{(x-z)^2}\right]\\
&+4\frac{(x-z)_\mu(z-y)_\nu(x-z)_\rho}{(z-y)^6(x-y)^2(x-z)^2}\left[\frac{1}{(x-y)^2}+\frac{1}{(x-z)^2}\right]\Bigg\}\ .
\end{aligned} \tag{A.7}$$

As a check of the computations one can show that (A.7) is gauge invariant. Nevertheless, close inspection shows that its (symmetrized) variational derivative cannot be expressed in terms of the conformal tensors $C^{\rm sym}_{\mu\nu\rho}$ and $D^{\rm sym}_{\mu\nu\rho}$ and consequently is not conformal invariant. Both analytic calculation and symbolic manipulation confirm this, so we can conclude that for small deviations from the Feynman gauge the conformal properties of the three-gluon vertex are lost.

APPENDIX B

MIXED THREE GLUON VERTICES

In this Appendix we discuss further the structure of one-loop vertex functions with both background and quantum external gluons. An argument was given in Section III that these vertices are not conformal invariant, but they can be expressed as a multiple of the conformal tensor $D^{\rm sym}_{\mu\nu\rho}$ plus an ultraviolet finite remainder. Our explicit real space computations support this picture, but we do not give full details here, since our main concern is to show that the coefficient of the $D^{\rm sym}_{\mu\nu\rho}$ leads, after regularization of this tensor, to a renormalization scale dependence in agreement with the work of Kluberg–Stern and Zuber.[11] In our notation, the result of these authors for the renormalized action involving three-gluon vertices in the background field formalism is:

$$S^R = \int d^4x \left\{ \frac{1}{g^2 Z_g^2} \mathcal{L}_{\rm YM} \left(B + g Z_g Z_3^{1/2} b \right) + \frac{1}{2a} \left(D_\mu b^a_\mu \right)^2 \right\} \tag{B.1}$$

where $g_0 = g Z_g$ is the bare coupling constant and Z_3 is the wavefunction renormalization factor of the quantum gauge field, $b^a_\mu(x)_{\rm bare} = Z_3^{1/2} b^a_\mu(x)$. In the second term, one sees that renormalization of the gauge fixing parameter a is also needed, with $a_0 = Z_a a$ and $Z_a = Z_3$.

From Eq. (B.1) we see that the counterterm for the overall divergence of any three-gluon vertex has the form of the bare Yang–Mills vertex. In differential regularization this implies that the part of the one-loop amplitudes which requires regularization is a numerical coefficient times the singular part of the tensor $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$. Indeed, this is the only possibility for vertices with three background or three quantum gluons, because of Bose symmetry, but the result is not obvious for mixed vertices.

The Feynman rules for the quantum gluon vertices are given by $\mathcal{L}_q$ and $\mathcal{L}'^g$ in Eqs. (2.12) and (2.16). It is straightforward to apply differential regularization to the 1PI diagrams which contribute to the one-loop amplitude for the mixed vertices, and identify the coefficient of the scale dependent term which is proportional to the semi-local scale dependent part of the regulated tensor $D^{\rm sym}_{\mu\nu\rho}$ in (4.7).

In the mixed vertex with two background and one external quantum gluon, both ghost seagull graphs vanish separately, due to the antisymmetry of the group structure constants f^{abc}. The contribution of the two gluon seagull diagrams is just the divergent part of $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$ times a numerical coefficient, while the triangle graphs yield a multiple of $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$ plus an ultraviolet finite piece. We give here only the mass scale dependence of the final result

$$M\frac{\partial}{\partial M}\ \frac{\delta^3\Omega[B,J]_{j=0}}{\delta j^a_\mu(x)\delta B^b_\nu(y)\delta B^c_\rho(z)} = g\, f^{abc}\frac{C}{(4\pi^2)^3}\ \frac{32}{3}M\frac{\partial}{\partial M}D^{\rm sym}_{\mu\nu\rho}(x,y,z)\ \ . \tag{B.2}$$

In the mixed vertex with one background and two quantum gluons the contributions from all diagrams combine to yield a multiple of the divergent part of $D^{\rm sym}_{\mu\nu\rho}(x,y,z)$, plus an ultraviolet finite remainder from the triangle graphs. The formal scale dependence of the one-loop vertex is

$$M\frac{\partial}{\partial M}\ \frac{\delta^3\Omega[B,J]_{j=0}}{\delta j^a_\mu(x)\delta j^b_\nu(y)\delta B^c_\rho(z)} = g^2\, f^{abc}\frac{C}{(4\pi^2)^3}\ \frac{20}{3}M\frac{\partial}{\partial M}D^{\rm sym}_{\mu\nu\rho}(x,y,z)\ \ . \tag{B.3}$$

The renormalized action given by Eq. (B.1) suggests that the renormalization group equation for a three-gluon vertex function with n_b external quantum gluons, $\Gamma^{(n_b)}_{\mu\nu\rho}$, takes the form

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}+\delta(g)a\frac{\partial}{\partial a}-n_b\gamma(g)\right]\Gamma^{(n_b)}_{\mu\nu\rho}=0 \tag{B.4}$$

where $\gamma(g) = g^2\gamma_1 + g^4\gamma_2 + \ldots$ in the notation of Section VII. We have already discussed the three background gluon vertex in Section VI, where we computed the one-loop coefficient of $\beta(g)$ by means of the renormalization group equation (7.4).

Then we can apply Eq. (B.4) to the vertex with two background and one external quantum gluons. We combine the classical term of order $1/g$ with the one-loop result (B.2) and use the scale derivative of the regulated tensor $D^{\rm sym}_{\mu\nu\rho}$, given by Eq. (6.3). Since the gauge fixing action does not contain BBb terms, the $\delta(g)a\frac{\partial}{\partial a}$ term does not contribute to lowest order. In the final result to order g, $M\frac{\partial}{\partial M}$ acting on the one-loop amplitude is balanced by $g^3\beta_1\frac{\partial}{\partial g} - g^2\gamma_1$ applied to the classical term. Using the value of β_1 in (6.4) we obtain the anomalous dimension in the Feynman gauge

$$\gamma_1 = -\frac{5}{48\pi^2}C \tag{B.5}$$

which agrees with the known result.[3]

Finally we apply a similar test of the renormalization group Eq. (B.4) to the combined classical and one-loop Bbb vertex. Here the $\beta(g)\frac{\partial}{\partial g}$ term does not contribute to order g^2, and the scale derivative is balanced by $g^2\delta_1 a\frac{\partial}{\partial a} - 2g^2\gamma_1$ applied to the classical term. The result confirms that $\delta_1 = -2\gamma_1$ which agrees with the general argument that $\delta(g) = -2\gamma(g)$ (which follows from $Z_a = Z_3$), and also with the relation found from the study of bb two-point function in Section VII.

Using these results for β_1 and γ_1, the renormalization group equation for the three quantum gluon vertex function to one-loop implies that its mass scale dependence in the Feynman gauge must be

$$M\frac{\partial}{\partial M}\ \frac{\delta^3\Omega[B,J]_{j=0}}{\delta j^a_\mu(x)\delta j^b_\nu(y)\delta j^c_\rho(z)} = g^3\, f^{abc}\frac{C}{(4\pi^2)^3}\ \frac{8}{3}M\frac{\partial}{\partial M}D^{\rm sym}_{\mu\nu\rho}(x,y,z)\ \ . \tag{B.7}$$

In this case we have not performed the explicit graphical computation to test this result.

APPENDIX C

THREE-GLUON VERTEX IN A GENERAL GAUGE

It is clearly of interest to see if conformal invariance holds for any value of the gauge parameter a different from the Feynman gauge value $a = 1$. The case of the Landau gauge ($a = 0$) is especially interesting, because the simplified renormalization group equations in this gauge permit exact conformal invariance at the two-loop level as discussed in Section VII.

The analytic calculation of the three-gluon vertex in a general gauge is very tedious so we have used symbolic manipulation based on a core REDUCE program written by J. I. Latorre. It uses symbolic logic to calculate partial derivatives of any translation invariant tensor function of three points x_μ, y_ν, z_ρ.

We compute analytically the Wick contractions for all triangle graphs involving the trilinear interaction terms $\mathcal{L}_1$, $\mathcal{L}_3$, $\mathcal{L}_4$ of Eq. (2.13), using the general gauge propagator (2.14). The contribution to the cubic part of the effective action can be expressed as

$$\Omega^{ijk}[B] = \frac{C}{(4\pi^2)^3} f^{abc} \int d^4x\, d^4y\, d^4z\, B^a_\mu(x) B^b_\nu(y) B^c_\rho(z) A^{ijk}_{\mu\nu\rho}(x,y,z) \tag{C.1}$$

where i, j, k indicates which vertices contribute to a given triangle. The program computes the derivatives in $A^{ijk}_{\mu\nu\rho}(x,y,z)$ and then adds the permutations that are necessary to produce the fully permutation odd contributions $S^{ijk}_{\mu\nu\rho}(z,y,z)$ to the three-gluon vertex function. Each $S^{ijk}_{\mu\nu\rho}(x,y,z)$ is a cubic function of the gauge parameter a. It is easily seen that possible $1/a$ poles from the interaction vertex $\mathcal{L}_4$ cancel, because $\mathcal{L}_4$ always generates the divergence of the propagator

$$\partial^x_\mu \langle b_\mu(x) b_\nu(y) \rangle = a \frac{1}{4\pi^2} \partial^x_\mu \frac{1}{(x-y)^2} \tag{C.2}$$

which cancels the $1/a$ singularity. As explained in Appendix A, we neglect both seagull graphs and δ-functions in the triangle graphs because they are not needed to test the conformal property for $x \neq y \neq z$.

We express the full vertex amplitude as a series in $(a-1)$:

$$J^0_{\mu\nu\rho} + (a-1)J^1_{\mu\nu\rho} + (a-1)^2 J^2_{\mu\nu\rho} + (a-1)^3 J^3_{\mu\nu\rho} \tag{C.3}$$

and we perform the following consistency checks of the computation.

1) Each term $J^i_{\mu\nu\rho}(x,y,z)$ satisfies the divergenceless property

$$\frac{\partial}{\partial z_\rho} J^i_{\mu\nu\rho}(x,y,z) = 0 \quad , \qquad i = 0, \ldots 3 \quad , \tag{C.4}$$

which is required since the Ward identity (2.10) vanishes for non-coincident points $x \neq y \neq z$.

2) The $J^0_{\mu\nu\rho}$ piece agrees with the Feynman gauge result we obtained analytically in Section III.

3) The $J^1_{\mu\nu\rho}$ term agrees with the analytic calculations of linear deviation from Feynman gauge (see Appendix A).

These highly non-trivial checks give confidence in the computer result, so we go on to analyze the conformal properties of the tensors $J^i_{\mu\nu\rho}(x,y,z)$ in Eq. (C.3). $J^0_{\mu\nu\rho}(x,y,z)$ is conformal invariant, as we already knew since it is the only contribution when $a = 1$. For the coefficient of the $(a-1)^3$ term in (C.3) we find the conformal tensor

$$J^3_{\mu\nu\rho}(z,y,z) = \frac{1}{8} C^{\rm sym}_{\mu\nu\rho}(x,y,z) \quad . \tag{C.5}$$

This result may be easily explained, since the piece cubic in the gauge parameter a involves only the term $a\partial_\mu\partial_\nu \ln(x-y)^2$ in the quantum gluon propagator, which has the correct conformal inversion property, and propagators are connected for large a only by vertices from the Yang–Mills action $S[B+gb]$, which is conformal invariant. This implies that $J^3_{\mu\nu\rho}$ must be a linear combination of $C^{\rm sym}_{\mu\nu\rho}$ and $D^{\rm sym}_{\mu\nu\rho}$.

Finally, we analyze the linear and quadratic terms in Eq. (C.3). $J^1_{\mu\nu\rho}(x,y,z)$ has the non-conformal structure given in Eq. (A.7) and $J^2_{\mu\nu\rho}(x,y,z)$ is a far more complicated expression, so we chose to study the structure of these tensors in the limit as one of the points (y_ν) goes to infinity. In this limit, both conformal tensors $C^{\rm sym}_{\mu\nu\rho}$ and $D^{\rm sym}_{\mu\nu\rho}$ have the form

$$\frac{c_1}{y^6}\left(\delta_{\nu\sigma}-\frac{2y_\nu y_\sigma}{y^2}\right)\frac{1}{x^4}\left\{\delta_{\mu\rho}x_\sigma-\delta_{\mu\sigma}x_\rho-\delta_{\rho\sigma}x_\mu+c_2\frac{x_\mu x_\rho x_\sigma}{x^2}\right\} \tag{C.6}$$

where we have set $z=0$ for simplicity and the coefficients c_1 and c_2 depend on the conformal tensor ($c_1=8$, $c_2=4$ for $C^{\rm sym}_{\mu\nu\rho}$; $c_1=-4$, $c_2=-2$ for $D^{\rm sym}_{\mu\nu\rho}$).

We study only the terms containing $\delta_{\mu\rho}$ in $J^1_{\mu\nu\rho}$ and $J^2_{\mu\nu\rho}$, because they give us a sufficient condition to disprove conformal invariance away from the Feynman gauge. The leading term as y_ν goes to infinity is of order $1/y^4$, which is absent in the conformal tensor structure, but it has the same form in both tensors $J^1_{\mu\nu\rho}$ and $J^2_{\mu\nu\rho}$ and therefore it can be eliminated in a suitable combination of them, namely $J^1_{\mu\nu\rho}-8J^2_{\mu\nu\rho}$. However, when we study the next-to-leading order, which is $1/y^6$, we see that this linear combination of $J^1_{\mu\nu\rho}$ and $J^2_{\mu\nu\rho}$ does not have the correct structure of a conformal tensor given by (C.6). We can then conclude that no linear combination of $J^1_{\mu\nu\rho}$ and $J^2_{\mu\nu\rho}$ can be a combination of conformal tensors and therefore among $D_\mu b^a_\mu$ background gauges only in the Feynman gauge the one-loop three-gluon vertex function is conformal invariant.

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FIGURE CAPTIONS

Fig. l: 1PI diagrams involving ghosts which contribute to the one-loop three-gluon vertex.

Fig. 2: 1PI diagrams involving fermion loops which contribute to the three-gluon vertex.

Fig. 3: 1PI diagrams involving gluon loops which contribute to the three-gluon vertex.