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Final State Interactions in Kaon Decays

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Abstract

We quantify the important effect of strong final state interactions in the weak $K \to 2\pi$ amplitudes, using the measured π-π phase shifts with $J = 0$ and $I = 0, 2$. The main results of this analysis, with their implications for ε'/ε and the $\Delta I = 1/2$ rule, have been already presented in a previous paper [1]. Here we provide a detailed formal derivation of those results and further discuss the Standard Model prediction of ε'/ε.

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1 Introduction

It is well known that, at centre–of–mass energies around the kaon mass, the strong S–wave π–π scattering generates a large phase-shift difference $(\delta_0^0 - \delta_0^2)(M_K^2) = 45° \pm 6°$ between the $I = 0$ and $I = 2$ partial waves [2]. In the usual description of $K \to 2\pi$ decays, this effect is explicitly taken into account, through the following decomposition of the relevant isospin amplitudes with $I = 0$ and $I = 2$:

$$\mathcal{A}_I \equiv A[K \to (\pi\pi)_I] \equiv A_I \, e^{i\delta_0^I} . \tag{1.1}$$

It is also known [3–12] that final state interactions (FSI) play an important role in the observed enhancement of the $I = 0$ decay amplitude, $A_0/A_2 \approx 22.2$. The presence of such a large phase-shift difference clearly signals a corresponding dispersive FSI effect in the moduli of the isospin amplitudes, because the real and imaginary parts are related by analyticity and unitarity.

The size of the induced FSI correction can be roughly estimated from the available one-loop analyses of $K \to 2\pi$ [12–14] in Chiral Perturbation Theory (χPT). At lowest order in the momentum expansion, $O(p^2)$, the decay amplitudes do not contain any strong phase. Those phases originate in the final rescattering of the two pions and, therefore, are generated by chiral loops which are of higher order in momenta. Since the strong phases are quite large, one should expect large higher–order unitarity corrections. The one-loop calculations [12–14] show in fact that the pion loop diagrams provide an important enhancement of the A_0 amplitude, of about 40%. However, the phase-shift δ_0^0 predicted by the one-loop calculation is still lower than its measured value, which indicates that a further enhancement could be expected at higher orders.

Although the importance of FSI in $K \to 2\pi$ has been known for more than a decade, their impact on the direct CP-violating parameter ε'/ε has been overlooked in the so–called *Standard Model predictions* of this parameter, presented in refs. [15] and [16]. Not surprisingly, those predictions fail to reproduce the experimental measurements [17].

The lattice investigations of kaon decay amplitudes have been only able, up to now, to compute the one-pion $\langle\pi|\mathcal{H}_{\Delta S=1}|K\rangle$ matrix elements. In order to get the physical two-pion decay amplitudes, they rely on the lowest–order χPT relation between $K \to \pi$ and $K \to 2\pi$, which, as mentioned before, does not include any FSI and underestimates the $I = 0$ $K \to 2\pi$ amplitude by at least 40%.

In refs. [15], the large–N_C limit is used to fix the CP–violating $K \to 2\pi$ decay amplitudes. Since the strong phases δ_J^I are zero at leading order in the $1/N_C$ expansion, the FSI enhancement has not been taken into account, either. Other approaches [18, 19] include some one-loop corrections and find larger values for the A_0 amplitude. Although those are model–dependent

estimates, they provide an indication of the importance of higher–order pion–loop contributions.

A proper way to account for the FSI effects has been addressed in ref. [1], where it has been shown that the strong rescattering of the two final pions generates a large enhancement of ε'/ε. The resulting Standard Model prediction [1,20],

$$\mathrm{Re}\left(\varepsilon'/\varepsilon\right)|_{SM} = (17 \pm 6) \times 10^{-4} \,, \qquad (1.2)$$

is in good agreement with the present experimental world average [21]

$$\mathrm{Re}\left(\varepsilon'/\varepsilon\right)|_{exp} = (19.3 \pm 2.4) \times 10^{-4} \,. \qquad (1.3)$$

In the following we provide a detailed discussion of the approach advocated in ref. [1] and further study its implications for ε'/ε and the $\Delta I = 1/2$ rule. The paper is organized as follows. In section 2 we formulate the Omnès problem for a general amplitude with two pions in the final state and derive its solution (for any number of subtractions). It is shown how the Omnès dispersive factor, solution of the Omnès problem, provides an all–order resummation of the infrared chiral logarithms that contribute to FSI. This is a universal process–independent factor, which only depends on the quantum numbers (I and J) of the final two–pion state. There is of course a polynomial (local) ambiguity, which encodes the process–dependent ultraviolet dynamics.

To clarify the physics involved in the Omnès resummation, we present in section 3 explicit results for the scalar pion form factor. This quantity is known to two loops in the chiral expansion and, therefore, provides a simple example where the power of our approach can be easily shown and the uncertainties quantified. The much more involved case of $K \to 2\pi$ transitions is discussed in section 4, while section 5 presents the Standard Model prediction of ε'/ε. We conclude in section 6 with a few summarizing comments. We have relegated to the appendices the details on experimental π–π phase shifts, one-loop χPT results for $K \to 2\pi$ and some remarks on recent literature on the subject.

2 Omnès approach to FSI

Let us consider a generic amplitude (or form factor) $A_J^I(s)$, with two pions in the final state which have total angular momentum and isospin given by J and I, respectively, and invariant mass $s \equiv q^2 \equiv (p_1 + p_2)^2$. As indicated by the name, FSI refer to the final dynamics of the two pions and not to the particular process leading to this final state. Therefore, we will not specify the physical amplitude and will look for a way to resum the strong π–π interactions to all orders in the chiral expansion.

Let us define the amplitude $A_J^I(s)$ being analytic on the complex s plane except for a cut $L \equiv [4M_\pi^2, \infty)$ along the real positive s axis. For real values $s < 4M_\pi^2$ the amplitude is real; this implies that the values of the amplitude above and below the cut are complex conjugate of each other: $A_J^I(s + i\epsilon) = A_J^I(s - i\epsilon)^*$. Above the threshold, $s \geq 4M_\pi^2$, $A_J^I(s)$ has a discontinuity across the cut and develops an absorptive (imaginary) part.

Cauchy's theorem implies that $A_J^I(s)$ can be written as a dispersive integral along the physical cut:

$$A_J^I(s) = \frac{1}{\pi} \int_L dz \, \frac{\operatorname{Im} A_J^I(s)}{z - s - i\epsilon} + \text{subtractions} . \qquad (2.1)$$

The convergence of the dispersive integral is dictated by the specific form of the function $A_J^I(s)$. Depending on the particular asymptotic behaviour of $A_J^I(s)$ at the extremes of the cut L, a number of subtractions has to be performed to make the integral convergent.

Let us further assume that $A_J^I(s)$ corresponds to some weak or electromagnetic transition, in the presence of strong interactions. Thus, above the cut, $A_J^I(s+i\epsilon) = \langle (\pi\pi)_J^I | O | i(q) \rangle$ where O could be some effective (low–energy) electroweak Hamiltonian or a current. Working to first order in the small electroweak coupling, the unitarity condition allows then to write the imaginary part of $A_J^I(s)$ as a sum over the contributions from all possible on–shell intermediate states which couple to the initial and the final state (properly normalized in momentum space):

$$\operatorname{Im} A_J^I(s + i\epsilon) = \frac{1}{2} \sum_n \langle (\pi\pi)_J^I | T^\dagger | n \rangle \, \langle n | O | i(q) \rangle , \qquad (2.2)$$

where T is the scattering T–operator. To derive this result, one makes use of Time–Reversal invariance; we will comment later on the proper way to bypass this assumption when analyzing CP–violating observables.

Note that since $\operatorname{Im} A_J^I$ is real, also the r.h.s. of eq. (2.2) is real. Below the first inelastic threshold, only the elastic channel contributes to the sum; one gets then:

$$\begin{aligned}
\operatorname{Im} A_J^I = \left(\operatorname{Im} A_J^I \right)_{2\pi} &= e^{-i\delta_J^I} \sin \delta_J^I \, A_J^I = e^{i\delta_J^I} \sin \delta_J^I \, A_J^{I*} \\
&= \sin \delta_J^I \, |A_J^I| = \tan \delta_J^I \operatorname{Re} A_J^I ,
\end{aligned} \qquad (2.3)$$

implying that the phase of the decay amplitude $A_J^I(s+i\epsilon) = |A_J^I(s)| \exp\left(i\delta_J^I \right)$ is equal to the phase of T_J^I, the $\pi\pi \to \pi\pi$ partial–wave scattering amplitude (Watson's theorem [22]). Eq. (2.3) expresses the imaginary part of the amplitude $A_J^I(s)$ in terms of the amplitude itself, or its real part, and the $\pi\pi$ phase–shift $\delta_J^I(s)$.

Inserting eq. (2.3) in the dispersion relation (2.1), one obtains an integral equation for $A(s)$ of the Omnès type,

$$A_J^I(s) = \sum_{k=0}^{n-1} \frac{(s-s_0)^k}{k!} \left. \frac{d^k A_J^I}{ds^k}\right|_{s=s_0} + \frac{(s-s_0)^n}{\pi} \int_{4M_\pi^2}^\infty \frac{dz}{(z-s_0)^n} \frac{\tan\delta_J^I(z)\,\mathrm{Re}A_J^I(z)}{z-s-i\epsilon},$$
$$(2.4)$$

which has the well–known Omnès [4, 23–25] solution:

$$A_J^I(s) \;=\; Q_{J,n}^I(s,s_0)\;\exp\left\{I_{J,n}^I(s,s_0)\right\}, \tag{2.5}$$

where

$$I_{J,n}^I(s,s_0) \;\equiv\; \frac{(s-s_0)^n}{\pi} \int_{4M_\pi^2}^\infty \frac{dz}{(z-s_0)^n} \frac{\delta_J^I(z)}{z-s-i\epsilon}, \tag{2.6}$$

$Q_{J,0}^I(s,s_0) \equiv 1$ and

$$Q_{J,n}^I(s,s_0) \;\equiv\; \exp\left\{\sum_{k=0}^{n-1} \frac{(s-s_0)^k}{k!} \frac{d^k}{ds^k} \log\left\{A_J^I(s)\right\}\Big|_{s=s_0}\right\}, \qquad (n\geq 1). \tag{2.7}$$

Strictly speaking, this equation is only valid below the first inelastic threshold ($s \leq 16M_\pi^2$). However, the contributions from higher–mass intermediate states are suppressed by phase space. The production of a larger number of meson pairs is also of higher order in the chiral expansion.

We have written the most general result, for a given number of subtractions n, performed at a generic subtraction point s_0 outside the physical cut. The dispersive integral $I_{J,n}^I(s,s_0)$ is uniquely determined up to a polynomial ambiguity (that does not produce any imaginary part of the amplitude), which depends on the number of subtractions and the subtraction point. This can be readily seen through the use of the following iterative formula for the real part of $I_{J,n}^I(s,s_0)$:

$$\mathrm{Re}\,I_{J,n}^I(s,s_0) \;=\; \mathrm{Re}\,I_{J,n-1}^I(s,s_0) \;-\; (s-s_0)^{n-1} \lim_{s\to s_0} \frac{\mathrm{Re}\,I_{J,n-1}^I(s,s_0)}{(s-s_0)^{n-1}}, \quad (2.8)$$

where the second term on the r.h.s. depends on s only through the polynomial factor $(s-s_0)^{n-1}$. The non–polynomial part of $I_{J,n}^I(s,s_0)$, containing the infrared chiral logarithms, does not have any dependence on the number of subtractions or the subtraction point. The polynomial ambiguity is of course canceled by the subtraction function $Q_{J,n}^I(s,s_0)$.

Thus, the Omnès solution predicts the chiral logarithmic corrections in a universal way, independently of the number of subtractions or the subtraction point, and provides their exponentiation to all orders in the chiral expansion. The polynomial ambiguity of $I_{J,n}^I(s,s_0)$ and the subtraction function $Q_{J,n}^I(s,s_0)$ can be fixed, at a given order in the chiral expansion, by

matching the Omnès formula (2.5) with the χPT prediction of $A_J^I(s)$. It remains a polynomial ambiguity at higher orders.

A special case, which turns out to be relevant in the treatment of the weak $K \to 2\pi$ amplitudes, is the one where the amplitude $A_J^I(s)$ has a zero of a given order p at some point $s = \zeta$. In this case, once the zero is factorized through the relation $A_J^I(s) = (s - \zeta)^p \overline{A_J^I}(s)$, the dispersive representation of eq. (2.5) is valid for the function $\overline{A_J^I}(s)$, so that

$$A_J^I(s) \;=\; (s - \zeta)^p \, \overline{A_J^I}(s) \;=\; (s - \zeta)^p \, \overline{Q}_{J,n}^I(s, s_0) \, \exp\left\{ I_{J,n}^I(s, s_0) \right\}, \quad (2.9)$$

with $\overline{Q}_{J,n}^I(s, s_0)$ the analogous of the expansion $Q_{J,n}^I(s, s_0)$ of eq. (2.7) for the function $\overline{A_J^I}(s)$ and $I_{J,n}^I(s, s_0)$ as defined in (2.6).

3   The scalar pion form factor

The scalar form factor of the pion is the simplest quantity where the Omnès problem can be solved [26, 27] in order to resum final state interactions of a two-pion state with total angular momentum $J = 0$ and isospin $I = 0$. It is defined by the matrix element of the $SU(2)$ quark scalar density

$$\langle \pi^i(p') | \bar{u}u + \bar{d}d | \pi^k(p) \rangle \;\equiv\; \delta^{ik} \, F_S^\pi(t) \,. \quad (3.1)$$

At low momentum transfer, χPT provides a systematic expansion of $F_S^\pi(t)$ in powers of $t \equiv (p' - p)^2$ and the light quark masses [28, 29]:

$$F_S^\pi(t) = F_S^\pi(0) \left\{ 1 + g(t) + O(p^4) \right\} \,. \quad (3.2)$$

The value at $t = 0$ coincides with the pion sigma term. It can be written as an expansion in powers of the light quark masses as follows:

$$F_S^\pi(0) = \left(\frac{\partial}{\partial m_u} + \frac{\partial}{\partial m_d} \right) M_\pi^2 = 2B_0 + O(m_q) \,, \quad (3.3)$$

where B_0 is a coupling of the lowest–order χPT Lagrangian which is related to the quark–antiquark vacuum condensate. The $O(p^2)$ correction $g(t)$ contains contributions from one-loop diagrams and tree–level terms of the $O(p^4)$ χPT Lagrangian. It is given by [28, 29]:

$$g(t) \;=\; \frac{t}{f^2} \left\{ \left(1 - \frac{M_\pi^2}{2t} \right) \bar{J}_{\pi\pi}(t) + \frac{1}{4} \, \bar{J}_{KK}(t) + \frac{M_\pi^2}{18t} \, \bar{J}_{\eta\eta}(t) + 4(L_5^r + 2L_4^r)(\mu) \right.$$

$$\left. + \frac{5}{4(4\pi)^2} \left(\ln \frac{\mu^2}{M_\pi^2} - 1 \right) - \frac{1}{4(4\pi)^2} \ln \frac{M_K^2}{M_\pi^2} \right\}, \quad (3.4)$$

with $f \approx f_\pi = 92.4$ MeV.

Only two terms of the strong χPT Lagrangian of $O(p^4)$ contribute to $g(t)$. The μ dependence of their corresponding chiral couplings $L_4^r(\mu)$ and $L_5^r(\mu)$ exactly cancels the one from the chiral loops appearing through the logarithm $\ln(\mu^2/M_\pi^2)$. At the standard reference value $\mu = M_\rho$, $[L_5^r + 2L_4^r](M_\rho) = (0.8 \pm 1.1) \times 10^{-3}$ [30–32].

The functions $\bar{J}_{\pi\pi}(t)$, $\bar{J}_{KK}(t)$ and $\bar{J}_{\eta\eta}(t)$ are ultraviolet finite and, together with the logarithms, they are produced by the one-loop exchange of $\pi\pi$, $K\bar{K}$ and $\eta\eta$ intermediate states. They have the form:

$$\bar{J}_{PP}(t) = \frac{1}{(4\pi)^2}\left\{2 - \sigma_P \ln\left(\frac{\sigma_P + 1}{\sigma_P - 1}\right)\right\} \qquad ; \qquad \sigma_P \equiv \sqrt{1 - \frac{4M_P^2}{t}}. \quad (3.5)$$

Below the first inelastic threshold, the absorptive part of the scalar form factor is generated by $\pi\pi$ exchange, through the one-loop function $\bar{J}_{\pi\pi}(t)$. Thus, all non-analytic contributions originate from the final rescattering of the two pions and could be studied within the chiral $SU(2) \otimes SU(2)$ framework [28], where the K and η modes are integrated out. In fact, for values of t such that $t \ll 4M_P^2$,

$$\bar{J}_{PP}(t) = \frac{1}{(4\pi)^2}\left[\frac{1}{6}\frac{t}{M_P^2} + \frac{1}{60}\frac{t^2}{M_P^4} + \dots\right], \quad (3.6)$$

implying that, below the $P\bar{P}$ threshold, $\bar{J}_{PP}(t)$ has a very smooth behaviour and is strongly suppressed. In the case of the pion scalar form factor this means that at values of $t \ll 4M_K^2, 4M_\eta^2$ the one-loop functions $\bar{J}_{KK}(t)$ and $\bar{J}_{\eta\eta}(t)$ only give small analytic corrections, which are numerically negligible with respect to the local $(L_5^r + 2L_4^r)(M_\rho)$ contribution.

The two loop corrections to the pion form factor have been already computed [33]. However, we prefer to keep the discussion at the one-loop level only, in order to make easier the comparison with $K \to \pi\pi$ where two-loop corrections are not yet available.

Let us consider now the Omnès problem for $F_S^\pi(t)$ and fix the subtraction polynomial performing a matching with the one-loop χPT result. We can write the Omnès solution in the form:

$$F_S^\pi(t) = \Omega_0(t, t_0) \cdot F_S^\pi(t_0) \approx \Omega_0(t, t_0) \cdot F_S^\pi(0)\left\{1 + g(t_0)\right\}. \quad (3.7)$$

Thus, knowing the form factor at some low–energy subtraction point t_0, where the momentum expansion can be trusted, the Omnès factor $\Omega_0(t, t_0)$ provides an evolution of the result to higher values of t, through the exponentiation of infrared effects related to FSI. The once–subtracted solution reads:

$$\Omega_0^{(1)}(t, t_0) = \exp\left\{\frac{t - t_0}{\pi}\int_{4M_\pi^2}^{\bar{z}}\frac{dz}{z - t_0}\frac{\delta_0^0(z)}{z - t - i\epsilon}\right\} \equiv \Re_0^{(1)}(t, t_0)\, e^{i\delta_0^0(t)}. \quad (3.8)$$

We have split the integral into its real and imaginary part, making explicit that the phase of the Omnès factor is just the original phase-shift $\delta_0^0(t)$. The Omnès exponential generates its corresponding dispersive factor $\Re_0^{(1)}(t,t_0)$.

The integral has been cut at the upper edge $\bar{z}$, which represents the first inelastic threshold. Above $\bar{z}$ the representation (3.8) is no longer valid and a coupled–channel analysis is required to solve the Omnès problem. From the behaviour of the S–wave $\pi\pi$ phase-shift $\delta_0^0(z)$ (see appendix A), it is immediate to infer that the elastic integral evaluated up to $\bar{z} \sim 1$ GeV^2 will slightly underestimate the exact result obtained with the inclusion of inelastic contributions. We shall discuss this point in more detail later.

The solution for $F_S^\pi(t)$ given in eq. (3.7), being a physical quantity, must be independent of the subtraction point t_0, while the Omnès factor and the amplitude in front of it do depend on t_0. For illustrative purposes, we take $t = M_K^2$ (the scale relevant for $K \to \pi\pi$) and show in Table 1 the resulting value of $|F_S^\pi(M_K^2)/F_S^\pi(0)|$ for different subtraction points $t_0 = 0$, M_π^2, $2M_\pi^2$, $3M_\pi^2$, $4M_\pi^2$ and M_K^2. The upper limit of the integration range has been fixed at $\bar{z} = 1$ GeV^2. Our Omnès integral (3.8) is not defined for $t_0 = M_K^2$, because it lies above the threshold of the non–analyticity cut; however, by its definition in eq. (3.7), $\Omega(M_K^2, t_0 = M_K^2) = 1$ holds.

t_0	$g(t_0)$	$\Re_0^{(1)}(M_K^2, t_0)$	$\Re_0^{(2)}(M_K^2, t_0)$	$|F_S^\pi(M_K^2)/F_S^\pi(0)|$	
				$n=1$	$n=2$
0	0	(1.23) 1.45	(1.55) 1.56	(1.23) 1.45	(1.55) 1.56
M_π^2	0.042	(1.21) 1.40	(1.47) 1.44	(1.26) 1.46	(1.53) 1.50
$2M_\pi^2$	0.091	(1.17) 1.34	(1.38) 1.31	(1.28) 1.46	(1.51) 1.43
$3M_\pi^2$	0.15	(1.12) 1.26	(1.26) 1.13	(1.29) 1.45	(1.45) 1.30
$4M_\pi^2$	0.26	(1.03) 1.11	—	(1.30) 1.40	—
M_K^2	$0.54 - 0.46\,i$	$\equiv 1$	$\equiv 1$	1.61	1.61

Table 1: The one-loop function $g(t_0)$, the Omnès factor $\Re_0^{(n)}(t,t_0)$ and the modulus of $F_S^\pi(t)/F_S^\pi(t_0)$ are shown at $t = M_K^2$ for different values of the subtraction point $t_0 \in [0, M_K^2]$ and for $n = 1, 2$ subtractions. At a given t_0, the first value (within brackets) is obtained with the $O(p^2)$ χPT prediction for δ_0^0, while the fit to the experimental phase-shift data has been used in the second one. The integrals have been cut at $\bar{z} = 1$ GeV^2.

The dominant contributions to $g(t_0)$ come from the logarithms and the $\pi\pi$ one-loop function $\bar{J}_{\pi\pi}(t)$. The corrections from $\bar{J}_{KK}(t_0)$ and $\bar{J}_{\eta\eta}(t_0)$ stay within 1% of the total one–loop correction at all non-zero subtraction points, while the local $[L_5^r + 2L_4^r](M_\rho)$ term gives a contribution smaller than 10%.

The χPT calculation of $F_S^\pi(t_0)$ is obviously better at lower values of t_0 where the one-loop correction $g(t_0)$ is smaller. At $t_0 = 4M_\pi^2$ a sizeable 26% effect is already found, while at $t_0 = M_K^2$ the correction is so large than one should worry about higher–order contributions. The Omnès exponential allows us to predict $F_S^\pi(M_K^2)$ in a much more reliable way, through the evolution of safer results at lower t_0 values.

The resulting values for the scalar form factor remain very stable within the whole range of subtraction points. Increasing the value of t_0 one just moves higher–order χPT corrections from the Omnès factor to the amplitude $F_S^\pi(t_0)$. At $t_0 = M_\pi^2$ the one-loop corrections are still almost zero, while the Omnès factor contains all the higher–order effects. At the highest possible subtraction point $t_0 = 4M_\pi^2$, i.e. the threshold of the non–analyticity cut, the bulk of the higher–order corrections has been moved to $F_S^\pi(t_0)$, while the Omnès factor approaches one, the value that it assumes at $t_0 = M_K^2$, by construction.

Together with the more accurate results obtained with the experimental phase-shifts, we have shown in Table 1 (within brackets) the corresponding numerical values using the lowest–order χPT prediction for $\delta_0^0(z)$ in eq. (A.4). As expected, the $O(p^2)$ χPT approximation to $\delta_0^0(z)$ underestimates the dispersive integral and, therefore, the true results, which are obtained with the experimental phase-shifts.

As shown in Table 2, the dispersive correction factors $\Re_0^{(1)}(M_K^2, t_0)$ increase with increasing values of the upper integration limit $\bar{z}$. This clearly indicates that we are underestimating the FSI effect. However, one cannot trust the numerical results obtained for $\bar{z} > 1$ GeV^2 because a coupled–channel analysis is required above the inelastic threshold.

	$\Re_0^{(1)}(M_K^2, t_0)$			$\Re_0^{(2)}(M_K^2, t_0)$		
	$\bar{z} = 1$	$\bar{z} = 2$	$\bar{z} = 3$	$\bar{z} = 1$	$\bar{z} = 2$	$\bar{z} = 3$
$t_0 = 0$	1.45	1.58	1.62	1.56	1.58	1.59
$t_0 = M_\pi^2$	1.40	1.51	1.55	1.44	1.46	1.47
$t_0 = 2M_\pi^2$	1.34	1.44	1.47	1.31	1.32	1.33
$t_0 = 3M_\pi^2$	1.26	1.35	1.38	1.13	1.14	1.14
$t_0 = 4M_\pi^2$	1.11	1.19	1.21	—	—	—

Table 2: Dependence of $\Re_0^{(n)}(M_K^2, t_0)$ on the upper edge $\bar{z}$ (in GeV^2 units) of the dispersive integral, for various choices of t_0 and $n = 1, 2$. The fit to the experimental data for δ_0^0 has been used.

We can suppress the sensitivity to the high integration range by using

a twice–subtracted dispersion relation. The corresponding Omnès factor is given by:

$$
\begin{aligned}
\Omega_0^{(2)}(t, t_0) &= \exp\left\{(t - t_0)\frac{g'(t_0)}{1 + g(t_0)} + \frac{(t - t_0)^2}{\pi} \int_{4M_\pi^2}^{\bar{z}} \frac{dz}{(z - t_0)^2} \frac{\delta_0^0(z)}{z - t - i\epsilon}\right\} \\
&\equiv \Re_0^{(2)}(t, t_0)\, e^{i\delta_0^0(t)}\,.
\end{aligned}
\tag{3.9}
$$

Table 2 shows that with $n = 2$ the numerical results remain indeed stable under variations of $\bar{z}$. Moreover, as shown in Table 1, the $O(p^2)$ approximation to $\delta_0^0(z)$ works now much better, giving results in good agreement with the ones obtained from the experimental phase-shifts.

Notice, that $\Omega_0^{(2)}(t, t_0)$ is not defined at $t_0 = 4M_\pi^2$, because the derivative $g'(t_0)$ has a discontinuity at the threshold of the physical cut. Since $F_S^\pi(t)$ is an analytic function in the cut s–plane, its Taylor expansion around the subtraction point t_0 ($\leq 4M_\pi^2$) has a convergence radius $|t_0 - 4M_\pi^2|$, which becomes zero at $t_0 = 4M_\pi^2$. Thus, subtraction points close to this threshold singularity [34] should be avoided [35].

Since the derivative $g'(t_0)$ has been fixed at the one-loop level only (i.e. has been estimated at the lowest non-trivial order), there is a corresponding uncertainty which gets somehow increased by its exponentiation. This explains why the predicted values of $|F_S^\pi(M_K^2)/F_S^\pi(0)|$ in Table 1 are less stable for $n = 2$ than for $n = 1$ under changes of the subtraction point t_0. A twice–subtracted Omnès solution requires a more precise knowledge of the subtraction function. With $n = 2$, the scalar form factor slightly decreases for increasing values of t_0, in the same way as $\Re_0^{(1)}(t, t_0)$, the tree–level once–subtracted solution, does. This could be easily improved by using the available two-loop χPT results [33]. Nevertheless, since chiral corrections are smaller at lower values of t_0, we can safely conclude that the true value of $|F_S^\pi(M_K^2)/F_S^\pi(0)|$ is between 1.5 and 1.6. Taking the experimental phase-shift uncertainties into account, we finally get:

$$
|F_S^\pi(M_K^2)/F_S^\pi(0)| = 1.55 \pm 0.10\,.
\tag{3.10}
$$

The naive one-loop χPT prediction, $|F_S^\pi(M_K^2)/F_S^\pi(0)| = 1.61$, turns out to be within the 1σ range of our final result (3.10). The advantage of using the dispersive Omnès resummation is that one can pin down the true value with an acceptable accuracy (7%), in spite of having a 60% one-loop correction.

4 $K \to \pi\pi$ amplitudes

The analysis of $K \to \pi\pi$ transition amplitudes is technically more complicated. Since the electroweak scale M_W, where the short–distance quark transition takes place, is much larger than the long–distance hadronic scale, there

are large short–distance logarithmic contributions which can be summed up using the Operator Product Expansion [36] and the renormalization group. One gets an effective $\Delta S = 1$ Lagrangian, defined in the three–flavour theory [37, 38],

$$\mathcal{L}_{\text{eff}}^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_i C_i(\nu) \, Q_i(\nu) \, , \qquad (4.1)$$

which is a sum of local four–fermion operators Q_i, constructed with the light degrees of freedom, modulated by Wilson coefficients $C_i(\nu)$ which are functions of the heavy masses M_W, M_Z, m_t, m_b and m_c that have been integrated out. The overall renormalization scale ν separates the short– $(M > \nu)$ and long– $(m < \nu)$ distance contributions, which are contained in $C_i(\nu)$ and Q_i, respectively. The physical amplitudes are of course independent of ν; thus, the explicit scale (and scheme) dependence of the Wilson coefficients should cancel exactly with the corresponding dependence of the Q_i matrix elements between on-shell states. We have explicitly factored out the Fermi coupling G_F and the Cabibbo–Kobayashi–Maskawa (CKM) matrix elements V_{ij} containing the usual Cabibbo suppression of K decays.

Our knowledge of $\Delta S = 1$ transitions has improved qualitatively in recent years, thanks to the completion of the next-to-leading logarithmic order calculation of the Wilson coefficients [39, 40]. All gluonic corrections of $O(\alpha_s^n t^n)$ and $O(\alpha_s^{n+1} t^n)$ are already known, where $t \equiv \ln(M_1/M_2)$ refers to the logarithm of any ratio of heavy mass scales $M_1, M_2 \geq \nu$. Moreover, the full m_t/M_W dependence (at lowest order in α_s) has been taken into account. In order to predict physical amplitudes, however, one is still confronted with the calculation of hadronic matrix elements of the four–quark operators. This is a very difficult problem, which so far remains unsolved.

The chiral symmetry properties of the effective Lagrangian (4.1) determine its corresponding χPT realization, in terms of the QCD Goldstone bosons

$$\Phi = \begin{pmatrix} \sqrt{\frac{1}{2}}\pi^0 + \sqrt{\frac{1}{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\sqrt{\frac{1}{2}}\pi^0 + \sqrt{\frac{1}{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix}, \qquad (4.2)$$

parametrized through the exponential $U = \exp(\sqrt{2}i\Phi/f)$. At a given order in the momentum expansion, chiral symmetry fixes the allowed chiral operators and, therefore, the structure of the physical weak amplitudes. The only remaining problem is the calculation of the chiral couplings from the effective short–distance Lagrangian.

The effect of strangeness–changing non-leptonic weak interactions with $\Delta S = 1$ is incorporated [41] in the low–energy chiral theory as a perturbation to the strong effective Lagrangian. At lowest order, the most general effective bosonic Lagrangian, with the same $SU(3)_L \otimes SU(3)_R$ transformation

properties as the short–distance Lagrangian (4.1), contains three terms:

$$\mathcal{L}_2^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left\{ g_8 \ f^4 \ \langle \lambda L_\mu L^\mu \rangle + g_{27} \ f^4 \left(L_{\mu 23} L_{11}^\mu + \frac{2}{3} L_{\mu 21} L_{13}^\mu \right) \right.$$

$$\left. + e^2 f^6 g_{EW} \ \langle \lambda U^\dagger \mathcal{Q} U \rangle \right\}. \tag{4.3}$$

The flavour–matrix operator $L_\mu = -i U^\dagger D_\mu U$ represents the octet of $V - A$ currents at lowest order in derivatives, $\mathcal{Q} = \text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ is the quark charge matrix, $\lambda \equiv (\lambda^6 - i\lambda^7)/2$ projects onto the $\bar{s} \to \bar{d}$ transition $[\lambda_{ij} = \delta_{i3}\delta_{j2}]$ and $\langle A \rangle$ denotes the flavour trace of A.

The chiral couplings g_8 and g_{27} measure the strength of the two parts of the effective Lagrangian (4.1) transforming as $(8_L, 1_R)$ and $(27_L, 1_R)$, respectively, under chiral rotations. Chiral symmetry forces the lowest–order Lagrangian to contain at least two derivatives (Goldstone bosons are free particles at zero momenta). In the presence of electroweak interactions, however, the explicit breaking of chiral symmetry generated by the quark charge matrix $\mathcal{Q}$ induces the $O(p^0)$ operator $\langle \lambda U^\dagger \mathcal{Q} U \rangle$ [42, 43], transforming as $(8_L, 8_R)$ under the chiral group. In the usual chiral counting $e^2 \sim O(p^2)$ and, therefore, the g_{EW} term is also of order p^2. One additional term [44] proportional to the quark mass matrix, which transforms as $(8_L, 1_R)$, has not been written since it does not contribute to the physical $K \to \pi\pi$ matrix elements [12, 13, 45]. At next–to–leading order in the chiral expansion, i.e. $O(p^4)$, a set of additional weak counterterms will contribute [12, 13, 46] together with the strong chiral operators L_i introduced in [29].

The Lagrangian (4.3) gives the lowest–order contribution to the $K \to 2\pi$ matrix elements. At generic values of the squared centre–of–mass energy $s = (p_{\pi 1} + p_{\pi 2})^2$, the $I = 0, 2$ amplitudes are given by

$$A_0(s) = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sqrt{2} f \left\{ \left(g_8 + \frac{1}{9} g_{27} \right) (s - M_\pi^2) - \frac{2}{3} f^2 e^2 g_{EW} \right\},$$

$$A_2(s) = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{2}{9} f \left\{ 5 g_{27} (s - M_\pi^2) - 3 f^2 e^2 g_{EW} \right\}. \tag{4.4}$$

We have made the usual isospin decomposition:

$$A[K^0 \to \pi^+ \pi^-] \equiv \mathcal{A}_0 + \frac{1}{\sqrt{2}} \mathcal{A}_2,$$

$$A[K^0 \to \pi^0 \pi^0] \equiv \mathcal{A}_0 - \sqrt{2} \mathcal{A}_2, \tag{4.5}$$

$$A[K^+ \to \pi^+ \pi^0] \equiv \frac{3}{2} \mathcal{A}_2,$$

where the amplitudes $\mathcal{A}_I \equiv A_I \exp\{i\delta_0^I\}$ contain the strong phase-shifts, which are zero at tree level.

For the discussion of the CP–conserving amplitudes we will neglect[3] the tiny electroweak correction proportional to $e^2 g_{EW}$. Taking the measured phase-shifts into account, eqs. (4.4) allow us to extract the lowest–order weak couplings from the experimental information on $K \to 2\pi$ decays [50]:

$$\left| g_8 + \frac{1}{9} g_{27} \right| \simeq 5.1 \,, \qquad |g_{27}/g_8| \simeq 1/18 \,. \qquad (4.6)$$

The huge difference between these two couplings shows the well–known enhancement of the octet $|\Delta I| = 1/2$ transitions.

Let us now apply the Omnès procedure to the $\Delta S = 1$ decay amplitudes. This is more subtle than for the scalar pion form factor, because we need to consider an off-shell kaon of mass squared $s = (p_{\pi 1} + p_{\pi 2})^2$, instead of a physical momentum transfer s. Since we are just studying the corrections induced by FSI between the two pions, the kaon can be formally considered as an external source, provided all SU(3) symmetry constraints are satisfied. As we saw explicitly for the scalar form factor, the FSI corrections that are summed up through the Omnès exponential are actually an SU(2) effect, generated by pion loops. Intuitively, we are just correcting a local weak $K \to \pi\pi$ transition with a chain of pion–loop bubbles, incorporating the strong $\pi\pi \to \pi\pi$ rescattering to all orders in the chiral expansion.

In the absence of $e^2 g_{EW}$ corrections, the tree–level isospin amplitudes have a zero at $s = M_\pi^2$, because the on-shell amplitudes should vanish in the SU(3) limit [44, 47, 51, 52]. We must take this important constraint into account, when making the Omnès summation of FSI effects, factorizing the zero explicitly as indicated in eq. (2.9). This is what was done in ref. [1], using a once–subtracted Omnès factor, to evolve the tree-level χPT results from $s_0 = M_\pi^2$ to the physical point $s = M_K^2$.

At higher orders in χPT there are small corrections proportional to $(M_K^2 - M_\pi^2)$ instead of $(s - M_\pi^2)$, which originate in the explicit breaking of chiral symmetry provided by the quark mass matrix. According to the general one-loop analysis presented in appendix B, those tiny effects can be neglected to a very good approximation. However, there is no need to do it. In full generality, the isospin amplitudes can be decomposed as

$$\mathcal{A}_I(s) = \tilde{a}_I(s) \left(s - M_\pi^2 \right) + \delta\tilde{a}_I(s) \left(M_K^2 - M_\pi^2 \right) \,, \qquad (4.7)$$

where $\delta\tilde{a}_I(s)$ is zero at lowest order[4]. Since there is a single strong phase, for a given isospin, the unitarity relation (2.3) is also valid for the individual functions $\tilde{a}_I(s)$ and $\delta\tilde{a}_I(s)$. Therefore, the Omnès problem can be solved

---

[3] A general analysis of isospin breaking and electromagnetic corrections to $K \to 2\pi$ transitions is presented elsewhere [47–49].

[4] To make the decomposition (4.7) unique, we require the function $\delta\tilde{a}_I(s)$ to depend on s only logarithmically.

separately for the two pieces. Combining them, we can write our result for the physical on-shell amplitude in the simpler form:

$$
\begin{aligned}
\mathcal{A}_I \equiv \mathcal{A}_I(M_K^2) &= \left(M_K^2 - M_\pi^2\right) a_I(M_K^2) \\
&= \left(M_K^2 - M_\pi^2\right) \Omega_I(M_K^2, s_0) \, a_I(s_0) \qquad (4.8) \\
&= \left(M_K^2 - M_\pi^2\right) \Re_I(M_K^2, s_0) \, a_I(s_0) \, e^{i\delta_0^I(M_K^2)} \, ,
\end{aligned}
$$

where $a_I(s) \equiv \tilde{a}_I(s) + \delta\tilde{a}_I(s)$.

The once–subtracted Omnès factor $\Omega_I^{(1)}(M_K^2, s_0)$ is universal, i.e. the same for $f(s) = F_S^\pi(s)$, $\tilde{a}_I(s)$ or $\delta\tilde{a}_I(s)$, because it only depends on the strong phase-shift $\delta_0^I(s)$ [see eq. (3.8)]. This is no longer true with two sub-tractions [eq. (3.9)], because $\Omega_I^{(2)}(s, s_0)$ contains an explicit dependence on $f'(s_0)/f(s_0)$. Nevertheless, given the smallness of the non-leading $\delta\tilde{a}_I(s)$ contribution, it is a very good numerical approximation to take also a global Omnès exponential for $a_I(s)$ in the twice-subtracted case.

Let us define $a_0(s) \equiv a_0^{(8)}(s) + a_0^{(27)}(s)$, thus separating the $(8_L, 1_R)$ and $(27_L, 1_R)$ contributions to the isoscalar amplitude. The complete one-loop χPT results for the different decay amplitudes are given in appendix B. Their s dependences can be written in a rather transparent way:

$$
\begin{aligned}
a_0^{(8)}(s) &= a_0^{(8)}(0) \left\{1 + g_0^{(8)}(s) + O(p^4)\right\} , \\
a_0^{(27)}(s) &= a_0^{(27)}(0) \left\{1 + g_0^{(27)}(s) + O(p^4)\right\} , \qquad (4.9) \\
a_2(s) &= a_2(0) \left\{1 + g_2(s) + O(p^4)\right\} ,
\end{aligned}
$$

where

$$
\begin{aligned}
g_0^{(8)}(s) &= \frac{s}{f^2} \left\{ \left(1 - \frac{M_\pi^2}{2s}\right) \bar{J}_{\pi\pi}(s) - \frac{1}{4}\left(1 - \frac{M_K^2}{s}\right) \bar{J}_{KK}(s) + \frac{M_\pi^2}{18s} \bar{J}_{\eta\eta}(s) \right. \\
&\quad \left. + C_5^8(\mu) + \frac{3}{4(4\pi)^2}\left(\ln\frac{\mu^2}{M_\pi^2} - 1\right) + \frac{1}{4(4\pi)^2}\ln\frac{M_K^2}{M_\pi^2} \right\} , \quad (4.10)
\end{aligned}
$$

$$
\begin{aligned}
g_0^{(27)}(s) &= \frac{s}{f^2} \left\{ \left(1 - \frac{M_\pi^2}{2s}\right) \bar{J}_{\pi\pi}(s) - \frac{3}{2}\left(1 - \frac{M_K^2}{s}\right) \bar{J}_{KK}(s) - \frac{M_\pi^2}{2s} \bar{J}_{\eta\eta}(s) \right. \\
&\quad \left. + C_5^{27}(\mu) - \frac{1}{2(4\pi)^2}\left(\ln\frac{\mu^2}{M_\pi^2} - 1\right) + \frac{3}{2(4\pi)^2}\ln\frac{M_K^2}{M_\pi^2} \right\} , \quad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
g_2(s) &= \frac{s}{f^2} \left\{ -\frac{1}{2}\left(1 - \frac{2M_\pi^2}{s}\right) \bar{J}_{\pi\pi}(s) \right. \\
&\quad \left. + \bar{C}_5^{27}(\mu) - \frac{1}{2(4\pi)^2}\left(\ln\frac{\mu^2}{M_\pi^2} - 1\right) \right\} . \qquad (4.12)
\end{aligned}
$$

The corrections coming from the $\delta\tilde{a}_I$ terms, included in these results, are very small. Denoting by $\tilde{g}_I(s)$ the corresponding functions for the un-corrected $\tilde{a}_I$ amplitudes, the differences $\Delta g_I(s) \equiv g_I(s) - \tilde{g}_I(s)$ only get

contributions from the $K\bar{K}$ and $\eta\eta$ loop functions which, as shown in section 3, are numerically suppressed at low values of s. Moreover, they get an additional suppression factor M_π^2 (see appendix B). Since the $K\bar{K}$ and $\eta\eta$ intermediate states cannot give rise to $I = 2$, one gets $\Delta g_2(s) = 0$ ($\delta\tilde{a}_2$ does not have any s dependence at this order), while the isoscalar differences are given by $\Delta g_0^{(8)}(s) = M_\pi^2 [9\bar{J}_{KK}(s) + 8\bar{J}_{\eta\eta}(s)]/(36f^2)$ and $\Delta g_0^{(27)}(s) = M_\pi^2 [3\bar{J}_{KK}(s) - 4\bar{J}_{\eta\eta}(s)]/(2f^2)$. Even at $s = M_K^2$, these differences are completely negligible: $\Delta g_0^{(8)}(M_K^2) \sim 1 \times 10^{-3}$ and $\Delta g_0^{(27)}(M_K^2) \sim -2 \times 10^{-4}$.

Notice the strong similarity with the scalar form factor result in eq. (3.4). The isoscalar functions $g_0^{(8)}(s)$ and $g_0^{(27)}(s)$ get exactly the same $\bar{J}_{\pi\pi}(s)$ contribution than the scalar form factor function $g(s)$, while the corresponding contribution to $g_2(s)$ has opposite sign. The polynomial factors in front of the $\pi\pi$ loop function, $(s - M_\pi^2/2)$ and $(M_\pi^2 - s/2)$ for $I = 0$ and 2 respectively, clearly identify the corresponding lowest–order χPT phase-shifts, given in eq. (A.4) [$\text{Im}\,\bar{J}_{\pi\pi}(s) = \theta(s - 4M_\pi^2)\,\sigma_\pi(s)/(16\pi)$].

In addition, at $\mu^2 = M_K^2$, one recognizes the same local infrared logarithmic enhancement of all isoscalar g functions, $\ln(M_K^2/M_\pi^2)/(4\pi)^2$; the corresponding factor in g_2 decreases the $I = 2$ amplitude. For arbitrary values of the chiral scale μ, this logarithmic correction is split in $\ln(\mu^2/M_\pi^2)$ and $\ln(M_K^2/M_\pi^2)$ terms, which are slightly different for the different g functions. However, all isoscalar g functions contain exactly the same infrared $\ln M_\pi^2$ contribution.

Thus, the s dependence of the weak decay amplitudes is indeed dominated by infrared effects related to the FSI of the two final pions. Moreover, in the isoscalar case, by comparison with the scalar form factor, we see explicitly that this is a universal effect related to the quantum numbers of the $\pi\pi$ state.

The particular dynamics leading to this final state gives rise also to local contributions, which are different in each case. We saw in section 3 that at the usual reference scale $\mu = M_\rho$ the contribution from the local term $[L_5^r + 2L_4^r](M_\rho)$ is small. For the weak amplitudes this needs to be further investigated; the usual factorization models [46] amount to $C_5^8(M_\rho) = C_5^{27}(M_\rho) = \bar{C}_5^{27}(M_\rho) = 0$.

The main effects of the short–distance dynamics, not related to FSI, are contained in the particular values of the different amplitudes $a_I(s)$ at $s = 0$. This physics needs to be analyzed independently, because it cancels out from the Omnès relation. The Omnès factor only allows us to relate the amplitudes at two different values of s, but does not give any information on their global normalization.

Taking a low subtraction point where higher–order corrections are expected to be small, we can just multiply the tree–level formulae (4.4) with the experimentally determined Omnès exponentials, as done in ref. [1]. For

$I = 0$ we already have the result obtained in the previous section,

$$\Re_0(M_K^2, 0) = 1.55 \pm 0.10\,, \tag{4.13}$$

which improves the lowest-order estimates made in refs. [1,4] at $s_0 = M_\pi^2$.

In the $I = 2$ channel the inelasticity effect is absent at least up to 1.6 GeV. Evaluating the once-subtracted dispersive integral over the measured phase-shifts up to $\bar{z} = (1.6 \text{ GeV})^2$, we get

$$\Re_2^{(1)}(M_K^2, 0) = 0.92 \pm 0.03\,, \tag{4.14}$$

to be compared with the earlier estimates $\Re_2^{(1)}(M_K^2, M_\pi^2) = 0.96$ ($\bar{z} = 1 \text{ GeV}^2$) [53] and $\Re_2^{(1)}(M_K^2, M_\pi^2) = 0.89 \pm 0.03$ [8]. The error bar in (4.14) takes into account uncertainties in the fits to the phase–shift data and higher energy contributions.

The corrections induced by FSI in the moduli of the decay amplitudes $\mathcal{A}_I$ generate an additional enhancement of the $\Delta I = 1/2$ to $\Delta I = 3/2$ ratio,

$$\Re_0(M_K^2, 0)/\Re_2(M_K^2, 0) = 1.68 \pm 0.12\,. \tag{4.15}$$

This factor multiplies the enhancement already found at short distances. This is a quite large correction, which improves previous calculations of $A_I(M_K^2)$. Taking the $\Re_I$ correction into account, the experimental A_I amplitudes imply the following corrected values for the lowest–order $\Delta S = 1$ chiral couplings:

$$\left| g_8 + \frac{1}{9} g_{27} \right| \approx 3.3 \qquad , \qquad |g_{27}| \approx 0.31\,. \tag{4.16}$$

These "experimental" numbers are not very far from the short–distance estimates obtained in the first of refs. [54].

5   Standard Model prediction of ε'/ε

One further subtlety has to be taken into account in the discussion of CP–violating isospin amplitudes. The derivation of eq. (2.2) for the absorptive parts makes use of Time–Reversal invariance, so that the procedure can be strictly applied only to CP–conserving amplitudes. This is not a problem, however, because we are working to first order in the weak Fermi coupling.

The CP–odd phase is hidden in the Wilson coefficients of the short–distance $\Delta S = 1$ Lagrangian (4.1), which can be decomposed as

$$C_i(\nu) = z_i(\nu) + \tau\, y_i(\nu) \qquad ; \qquad \tau = -\frac{V_{td} V_{ts}^*}{V_{ud} V_{us}^*}\,. \tag{5.1}$$

Since CP violation is only originated by the short–distance ratio of CKM matrix elements τ, we can always write

$$\mathcal{A}_I = \mathcal{A}_I^{CP} + \tau\, \mathcal{A}_I^{C\!\!\!/P} \tag{5.2}$$

and apply the Omnès procedure to the amplitudes $\mathcal{A}_I^{CP}$ and $\mathcal{A}_I^{C\!\!\!/P}$, which respect Time–Reversal invariance.

The CP-conserving piece of $\tau\, \mathcal{A}_I^{C\!\!\!/P}$ is negligible in comparison with $\mathcal{A}_I^{CP}$. Therefore, in a more standard notation, $\mathrm{Re}A_I \approx A_I^{CP}$ and $\mathrm{Im}A_I = \mathrm{Im}(\tau)\, A_I^{C\!\!\!/P}$, where "real" and "imaginary" refer to CP–even and CP–odd since the absorptive phases have been already factored out through $\mathcal{A}_I = A_I\, e^{i\delta_0^I}$.

The most striking consequence of the correction factors $\Re_{0,2}$ is a sizeable modification of the numerical short–distance estimates[5] for the direct CP–violation parameter ε'/ε. A handy way of writing this quantity, used in all theoretical short–distance calculations up to date, can be as follows [15]

$$\frac{\varepsilon'}{\varepsilon} = \mathrm{Im}\left(V_{ts}^* V_{td}\right)\ e^{i\Phi}\ \left[P^{(1/2)} - P^{(3/2)} \right], \tag{5.3}$$

where the phase $\Phi = \Phi_{\varepsilon'} - \Phi_\varepsilon \simeq 0$ and the quantities $P^{(1/2)}$ and $P^{(3/2)}$ contain the contributions from the hadronic matrix elements of four–quark operators with $\Delta I = 1/2$ and $3/2$ respectively:

$$\begin{aligned}
P^{(1/2)} &= r \sum_i y_i(\nu)\, \langle Q_i(\nu)\rangle_0 \left(1 - \Omega_{IB}\right), \\
P^{(3/2)} &= \frac{r}{\omega} \sum_i y_i(\nu)\, \langle Q_i(\nu)\rangle_2 .
\end{aligned} \tag{5.4}$$

Here, $\langle Q_i\rangle_I \equiv \langle(\pi\pi)_I|Q_i|K\rangle$, r and ω are given by

$$r = \frac{G_F}{2|\varepsilon|} \frac{\omega}{\mathrm{Re}A_0} \qquad , \qquad \omega = \frac{\mathrm{Re}A_2}{\mathrm{Re}A_0}, \tag{5.5}$$

and the parameter

$$\Omega_{IB} = \frac{1}{\omega} \frac{(\mathrm{Im}A_2)_{IB}}{\mathrm{Im}A_0} \tag{5.6}$$

parametrizes isospin breaking corrections.

A detailed analysis of ϵ'/ϵ, within the Standard Model, will be given in ref. [20]. Here we just want to illustrate the important role of FSI and how their proper inclusion modifies the ϵ'/ϵ prediction in a very important way.

Since the hadronic matrix elements are quite uncertain theoretically, the CP–conserving amplitudes $\mathrm{Re}A_I$, and thus the factors r and ω, are set to

_____

[5] The correction factors $\Re_I^{(1)}(M_K^2, M_\pi^2)$ were already considered in ref. [55] to estimate ϵ'/ϵ within the $SU(2)_L \otimes SU(2)_R \otimes U(1)$ model of CP violation.

their experimentally determined values; this automatically includes the FSI effect. All the rest in the numerator is *theoretically* predicted via short–distance calculations, because the leading contributions come [18] from the operators Q_6 and Q_8 whose matrix elements cannot be directly measured from $K \to 2\pi$ decay rates.

As a consequence, since the relevant matrix elements $\langle Q_{6,8} \rangle_I$ are usually taken from large–N_C estimates [15] or lattice calculations [16], which do not include FSI corrections, this procedure produces a mismatch with the FSI included phenomenologically in the values of r and ω. This can be easily corrected, introducing in the numerator the dispersion factors $\Re_I$ that we have estimated. This implies [1] a large enhancement of the predicted value of ε'/ε by roughly a factor of 2.

To a very good approximation, the Standard Model prediction for ε'/ε can be written, up to global factors, as [15]

$$\frac{\varepsilon'}{\varepsilon} \sim \left[B_6^{(1/2)}(1 - \Omega_{IB}) - 0.4\, B_8^{(3/2)} \right] , \qquad (5.7)$$

where $B_6^{(1/2)}$ and $B_8^{(3/2)}$ parametrize the matrix elements of the QCD penguin operator Q_6 and the electroweak penguin operator Q_8, respectively, in units of their vacuum insertion approximation values. These parameters are usually taken to be (from $1/N_C$ considerations [15] and Lattice calculations [16]) $B_6^{(1/2)} = 1.0 \pm 0.3$ and $B_8^{(3/2)} = 0.8 \pm 0.2$, while the isospin–breaking factor is set to $\Omega_{IB} \approx 0.25$ [56] with large uncertainties [57, 58]. With those inputs, there is a rather large numerical cancellation between the two terms in eq. (5.7), which results in a predicted central value [15, 16] $\varepsilon'/\varepsilon \approx 7.0 \times 10^{-4}$.

Since those estimates do not include FSI effects, their values should be multiplied by the appropriate factors $\Re_I$. Notice, that the Omnès procedure can be also applied to the individual matrix elements $\langle Q_i \rangle_I$. In order to avoid any possible double counting, we will take as the starting point of our analysis the large–N_C estimate for the relevant matrix elements [38]:

$$\left. B_6^{(1/2)} \right|_{N_C \to \infty} = 1 \; ; \qquad \left. B_8^{(3/2)} \right|_{N_C \to \infty} \approx 1.0 \; . \qquad (5.8)$$

FSI only appear at next-to-leading order in the $1/N_C$ expansion and, therefore, correct the leading values (5.8).

The corrected ε'/ε prediction can be easily obtained, taking into account the following points:

1. The penguin operator Q_6 transforms as $(8_L, 1_R)$ under chiral transformations. At lowest order in the chiral expansion, it corresponds to the first operator in eq. (4.3) (from eq. (5.8) one actually gets the Q_6 contribution to the chiral coupling g_8, in the large–N_C limit). The FSI corrections induced by pion chiral loops modify $B_6^{(1/2)}$ as follows

$$B_6^{(1/2)} = \left. B_6^{(1/2)} \right|_{N_C \to \infty} \times \Re_0(M_K^2, 0) = 1.55 \; .$$

2. The electroweak penguin operator Q_8 corresponds to the chiral operator proportional to g_{EW} in eq. (4.3). As shown in (4.4), it contributes to the two isospin amplitudes, although we only need here the $I = 2$ piece $\langle Q_8 \rangle_2$. The $K \to 2\pi$ matrix element is not proportional to $M_K^2 - M_\pi^2$ because the needed $SU(3)$ breaking is provided by the quark charge matrix (the chiral operator is identically zero for $\mathcal{Q} = I$). The presence or not of this factor does not change the Omnès summation (the corresponding zero just factors out whenever is present). One gets then

$$B_8^{(3/2)} \;=\; B_8^{(3/2)}\Big|_{N_C \to \infty} \;\times\; \Re_2(M_K^2, 0) \;=\; 0.92 \;.$$

3. The isospin–breaking correction coming from π^0–η mixing has been recently calculated at $O(p^4)$ in the chiral expansion, with the result $\Omega_{IB} = 0.16 \pm 0.03$ [59]. This value is smaller than the previous lowest–order estimate $\Omega_{IB} \approx 0.25$ [56]. The term $B_6^{(1/2)} \Omega_{IB}$ in eq. (5.7) should be multiplied by $\Re_2$ and not by $\Re_0$, because it corresponds to two final pions with $I = 2$. Thus,

$$B_6^{(1/2)} \, \Omega_{IB} \;=\; B_6^{(1/2)}\Big|_{N_C \to \infty} \; \Omega_{IB} \times \; \Re_2(M_K^2, 0) \;=\; 0.15 \;.$$

The large FSI correction to the $I = 0$ amplitude gets reinforced by the mild suppression of the $I = 2$ contributions. The net effect is a large enhancement of eq. (5.7), by a factor 2.4, pushing the predicted central value from 7.0×10^{-4} [15, 16] to

$$\varepsilon'/\varepsilon = 17 \times 10^{-4} \;, \tag{5.9}$$

which compares well with the present experimental world average [21] in eq. (1.3).

A more careful analysis, taking into account all hadronic and quark–mixing inputs [20] gives the result quoted in eq. (1.2) for the Standard Model prediction of ε'/ε.

6 Discussion

Many attempts have been made to compute the isospin amplitudes A_I from first principles [3,15,16,18,19,50,54,60–66]. Although those calculations have provided encouraging results, we are still far from getting accurate predictions. Nevertheless, a qualitative understanding of the $K \to \pi\pi$ transition amplitudes is now emerging.

The strong rescattering of the two final pions generates important corrections to the kaon decay amplitudes, enhancing the $I = 0$ piece by about 50% and originating a mild suppression of the $I = 2$ one. FSI alone cannot

explain the measured ratio of $\Delta I = 1/2$ to $\Delta I = 3/2$ transition amplitudes, but they constitute a very important ingredient which reinforces the enhancement already found at short distances. Combined with the $1/N_C$ and χPT expansions, the calculation of the Omnès factors $\Omega_{0,2}(M_K^2, 0)$ allows for a reliable estimate of ε'/ε [1, 20].

The lowest–order approximation in the $1/N_C$ expansion does not provide a good starting point to analyze the CP–conserving $K \to 2\pi$ amplitudes, because the anomalous dimensions of the most important operators Q_i are zero at this order [54]. Thus, at lowest order in $1/N_C$ one misses the dominant physics leading to the well–known short–distance enhancement. That makes difficult to perform precise predictions for the $K \to 2\pi$ decay rates.

The situation is different for the CP–violating amplitudes, which are completely dominated by Q_6 and Q_8. These are precisely the only operators which have a non-zero anomalous dimension at leading order in the $1/N_C$ expansion. The large–N_C approximation works rather well for those operators [54, 60] and their matrix elements can be safely estimated within a 30% accuracy, once the large infrared logarithms related to FSI are properly taken into account.

In the large–N_C limit the four–quark operators factorize into currents which have a known chiral realization at very low energies. The factorization of the operators Q_6 and Q_8 leads to scalar (pseudo-scalar) currents which are not directly measurable; their matrix elements are determined with χPT techniques at leading (next-to-leading for Q_8) non-trivial order in the momentum expansion. This fixes the Q_6 and Q_8 contribution to the $\Delta S = 1$ χPT couplings in the large–N_C limit [1, 20]. A reliable determination of the corresponding $K \to 2\pi$ transition amplitudes can then be performed at low s values where chiral loop corrections are smaller. Once this is accomplished, the Omnès dispersive factors allow us to evolve this result to the physical $s = M_K^2$ point, resumming the large chiral corrections associated with FSI.

The usual vacuum insertion estimate of $\langle Q_6 \rangle_0$, adopted in some *Standard Model* calculations of ε'/ε [15, 16], corresponds to the lowest non-trivial order in both the $1/N_C$ and χPT expansions. This naive estimate misses the large enhancement generated by one-loop χPT corrections [12–14], which originates mainly in the strong rescattering of the two final pions with $I = J = 0$. The FSI correction destroys the accidental numerical cancellation between the Q_6 and Q_8 contributions in eq. (5.7), producing a large increase in the resulting prediction of ε'/ε. The size of the FSI effect can be already determined with the one-loop χPT calculation. The Omnès resummation is only needed to perform a reliable estimate of higher–order corrections and pin down their associated uncertainties.

More work is still needed in order to get a precise quantitative description of kaon decays. In the meanwhile, our analysis demonstrates that it is at least possible to pin down the value of ε'/ε with an accuracy of about 30%. Within

the present uncertainties, the resulting Standard Model theoretical prediction is in good agreement with the measured experimental value, without any need to invoke a new physics source of CP violation.

Acknowledgments

A Experimental π–π phase shifts

For the experimental phase shifts of π–π scattering with $J = 0$ and total isospin $I = 0$ or 2 we used a simple parametrization by A. Schenk [67] that works in the elastic region. A more involved coupled–channel analysis is needed above the first inelastic threshold [68]. For generic I and J, the parametrization provided in [67] is given by:

$$
\begin{aligned}
\tan \delta_J^I(s) \;=\; & \sqrt{\frac{s - 4M_\pi^2}{s}} \left(\frac{s - 4M_\pi^2}{4M_\pi^2}\right)^J \left(\frac{4M_\pi^2 - s_J^I}{s - s_J^I}\right) \\
\times \; & \left\{ a_J^I + \tilde{b}_J^I \left(\frac{s - 4M_\pi^2}{4M_\pi^2}\right) + c_J^I \left(\frac{s - 4M_\pi^2}{4M_\pi^2}\right)^2 \right\} .
\end{aligned} \tag{A.1}
$$

The threshold expansion of the scattering amplitude is reproduced by setting

$$
\tilde{b}_J^I = b_J^I - a_J^I \frac{4M_\pi^2}{s_J^I - 4M_\pi^2} + (a_J^I)^3 \delta_{J0} . \tag{A.2}
$$

For given I, J there are four parameters: a_J^I, b_J^I, c_J^I and s_J^I. The numerical values of a_J^I, b_J^I (which parametrize the threshold behaviour of the scattering amplitude) have been determined by means of χPT, while the remaining two parameters have been extracted from the experimental data. The details of the analysis can be found in the original work [67]. Here we only compile the numerical values of the parameters. For isospin $I = 0$ and $I = 2$, in S–wave, the values of the threshold parameters are:

$$
a_0^0 = 0.20 \quad ; \quad b_0^0 = 0.24 \quad ; \quad a_0^2 = -0.042 \quad ; \quad b_0^2 = -0.075 . \tag{A.3}
$$

For the other parameters c_J^I and s_J^I we have taken the range of values determined in [67] in order to take into account the experimental uncertainties. For $I = 0$, we used $c_0^0 = 0.008, 0.0, -0.015$ and $\sqrt{s_0^0} = 840, 865, 890$ MeV. For $I = 2$, we used $c_0^2 = 0$ and $s_0^2 = -920^2, -685^2, -555^2$ MeV^2. The three values for each single parameter correspond to the three solid curves shown in Figure 1 for each isospin. The central line corresponds to the best fit in [67], while the other two extremes enclose the region covered by the experimental data considered in [67].
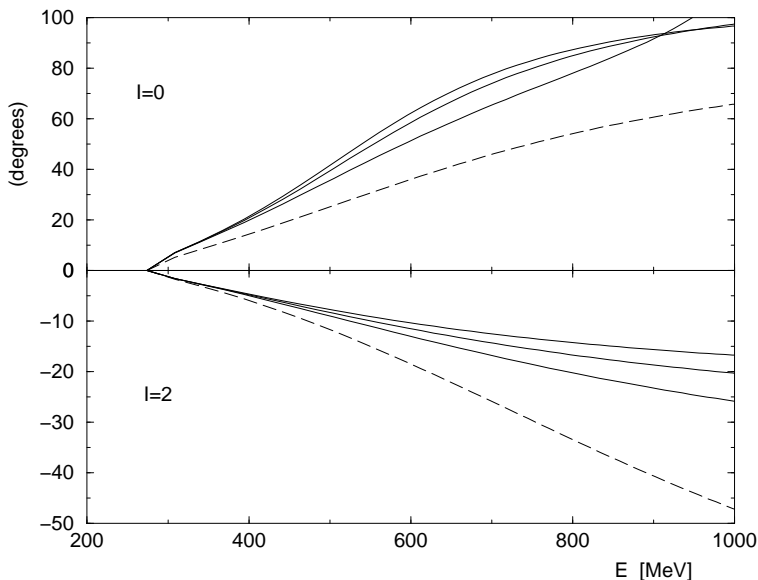

Figure 1: Phase shifts $\delta_0^{0;2}(s)$ used in our numerical analyses. The solid lines enclose the range covered by the experimental data, while the dashed lines show the unitarized lowest–order χPT prediction.

The lowest–order χPT prediction of the phase shifts is also shown with a dashed line in Figure 1. This corresponds to the expression

$$\tan \delta_0^{0;2}(s) = \frac{1}{32\pi f^2}\, \sigma_\pi(s)\, \left(2s - M_\pi^2;\, 2M_\pi^2 - s\right);\qquad \sigma_\pi(s) \equiv \sqrt{1 - \frac{4M_\pi^2}{s}}, \tag{A.4}$$

which is a unitarization of the usual χPT prediction, valid at low values of s where $\tan \delta \sim \delta$. The lowest–order χPT prediction fails already at relatively low energies ~ 500 MeV, specially for $I = 0$. In the $I = 0$ case it underestimates the experimental phase shift, while in the $I = 2$ case it gives a too large, in absolute value, phase shift.

21

B $K \to \pi\pi$ matrix elements at one loop

The one-loop contribution to the physical CP–conserving $K \to \pi\pi$ isospin amplitudes has been computed in Refs. [12, 13]. In this appendix the same amplitudes are calculated at a generic value of the squared invariant mass $s = (p_{\pi 1} + p_{\pi 2})^2$. The complete next–to–leading correction is of order p^4 in the chiral expansion and includes one-loop contributions generated by the lowest–order p^2 Lagrangian (4.3) and tree–level contributions coming from order p^4 counterterms [12, 13, 46].

In the analysis of the CP–conserving amplitudes we have neglected the tiny electroweak corrections which are proportional to $e^2 g_{EW}$ at leading order in the chiral expansion. We then decompose the isospin amplitudes as follows

$$\mathcal{A}_I(s) = \tilde{a}_I(s)\left(s - M_\pi^2\right) + \delta\tilde{a}_I(s)\left(M_K^2 - M_\pi^2\right), \qquad (\text{B.1})$$

where $\delta\tilde{a}_I(s)$ is zero at lowest order. In addition, we define $\tilde{a}_0 \equiv \tilde{a}_0^{(8)} + \tilde{a}_0^{(27)}$ and $\delta\tilde{a}_0 \equiv \delta\tilde{a}_0^{(8)} + \delta\tilde{a}_0^{(27)}$, thus explicitly separating the octet and 27–plet contributions to the $I = 0$ amplitude.

At $O(p^4)$ the octet $I = 0$ function $\tilde{a}_0^{(8)}$ takes the form

$$
\begin{aligned}
\tilde{a}_0^{(8)} = &-\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sqrt{2} f_\pi\, g_8 \left\{ 1 - \frac{1}{2}\left(2s - M_\pi^2\right) B(M_\pi^2, M_\pi^2, s)\right. \\
&+ \frac{1}{4}\left(s - M_K^2 + M_\pi^2\right) B(M_K^2, M_K^2, s) + \frac{1}{6} M_\pi^2\, B(M_\eta^2, M_\eta^2, s) \\
&+ \frac{(M_K^2 - M_\pi^2)}{12 M_\pi^2}\left[(3M_K^2 - 4M_\pi^2)\, B(M_K^2, M_\pi^2, M_\pi^2) + M_K^2\, B(M_K^2, M_\eta^2, M_\pi^2)\right] \\
&+ \frac{67}{12}\mu_\pi + \frac{1}{6}\mu_K - \frac{7}{36}\mu_\eta + \frac{(M_K^2 - M_\pi^2)}{2M_\pi^2}(\mu_\pi - \mu_\eta) - \frac{M_\pi^2}{2M_K^2}\left(\mu_\pi + \frac{\mu_\eta}{9}\right) \\
&+ \frac{1}{f^2}\left[-16(2M_K^2 + M_\pi^2)\, L_4^r(\mu) - 4(M_K^2 + 3M_\pi^2)\, L_5^r(\mu) + M_K^2\, C_1^8(\mu)\right. \\
&\left.\left. + M_\pi^2\, C_4^8(\mu) - \frac{M_\pi^4}{M_K^2} C_6^8(\mu) + s\, C_5^8(\mu)\right]\right\}, \qquad (\text{B.2})
\end{aligned}
$$

while $\delta\tilde{a}_0^{(8)}$ is given by

$$
\begin{aligned}
\delta\tilde{a}_0^{(8)} = &-\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sqrt{2} f_\pi\, g_8 \left\{ -M_\pi^2\left[\frac{1}{4}B(M_K^2, M_K^2, s) + \frac{2}{9}B(M_\eta^2, M_\eta^2, s)\right]\right. \\
&- \frac{1}{12}(5M_K^2 + 4M_\pi^2)\, B(M_K^2, M_\pi^2, M_\pi^2) + \frac{1}{12}M_K^2\, B(M_K^2, M_\eta^2, M_\pi^2) \\
&+ \frac{1}{f^2}\left[M_\pi^2\, C_2^8(\mu) + M_K^2\, C_3^8(\mu) + \frac{M_\pi^4}{M_K^2}C_6^8(\mu)\right] - \frac{5}{6}\mu_\pi - \frac{\mu_\eta}{18} + \frac{4}{3}\mu_K
\end{aligned}
$$

$$-\frac{M_\pi^2}{M_K^2 - M_\pi^2}\left(\frac{\mu_\pi}{2} + \frac{17}{18}\mu_\eta - 2\mu_K\right) - \frac{M_\pi^4}{2M_K^2(M_K^2 - M_\pi^2)}\left(\mu_\pi + \frac{\mu_\eta}{9}\right)\Bigg\}. \quad \text{(B.3)}$$

At the one–loop level, the 27–plet $I = 0$ function $\tilde{a}_0^{(27)}$ is

$$\tilde{a}_0^{(27)} = -\frac{G_F}{\sqrt{2}}V_{ud}V_{us}^*\sqrt{2}f_\pi\frac{1}{9}\,g_{27}\left\{1 - \frac{1}{2}\left(2s - M_\pi^2\right)B(M_\pi^2, M_\pi^2, s)\right.$$

$$+\frac{3}{2}\left(s - M_K^2 + M_\pi^2\right)B(M_K^2, M_K^2, s) - \frac{3}{2}M_\pi^2\,B(M_\eta^2, M_\eta^2, s)$$

$$+\frac{(M_K^2 - M_\pi^2)}{12M_\pi^2}\left[(3M_K^2 - 4M_\pi^2)\,B(M_K^2, M_\pi^2, M_\pi^2) - 4M_K^2\,B(M_K^2, M_\eta^2, M_\pi^2)\right]$$

$$+\frac{67}{12}\,\mu_\pi - \frac{49}{6}\,\mu_K - \frac{3}{4}\,\mu_\eta + \frac{(M_K^2 - M_\pi^2)}{2M_\pi^2}\,(\mu_\pi + 4\mu_\eta - 5\mu_K) + \frac{M_\pi^2}{2M_K^2}\,(\mu_\eta - \mu_\pi)$$

$$+\frac{1}{f^2}\Big[-16(2M_K^2 + M_\pi^2)\,L_4^r(\mu) - 4(M_K^2 + 3M_\pi^2)\,L_5^r(\mu) + M_K^2\,C_1^{27}(\mu)$$

$$\left.+M_\pi^2\,C_4^{27}(\mu) - \frac{M_\pi^4}{M_K^2}\,C_6^{27}(\mu) + s\,C_5^{27}(\mu)\Big]\right\}, \quad \text{(B.4)}$$

and

$$\delta\tilde{a}_0^{(27)} = -\frac{G_F}{\sqrt{2}}V_{ud}V_{us}^*\sqrt{2}f_\pi\frac{1}{9}\,g_{27}\left\{\frac{1}{2}M_\pi^2\left[4B(M_\eta^2, M_\eta^2, s) - 3B(M_K^2, M_K^2, s)\right]\right.$$

$$-\frac{1}{12}(5M_K^2 + 4M_\pi^2)\,B(M_K^2, M_\pi^2, M_\pi^2) - \frac{1}{3}M_K^2\,B(M_K^2, M_\eta^2, M_\pi^2)$$

$$+\frac{1}{f^2}\left[M_\pi^2\,C_2^{27}(\mu) + M_K^2\,C_3^{27}(\mu) + \frac{M_\pi^4}{M_K^2}\,C_6^{27}(\mu)\right] - \frac{5}{6}\,\mu_\pi - 2\,\mu_\eta + \frac{13}{6}\,\mu_K$$

$$\left.-\frac{M_\pi^2}{M_K^2 - M_\pi^2}\left(\frac{\mu_\pi}{2} + \frac{3}{2}\mu_\eta - 2\mu_K\right) + \frac{M_\pi^4}{2M_K^2(M_K^2 - M_\pi^2)}\,(\mu_\eta - \mu_\pi)\right\}. \quad \text{(B.5)}$$

Finally, the $I = 2$ $\tilde{a}_2$ function is

$$\tilde{a}_2 = -\frac{G_F}{\sqrt{2}}V_{ud}V_{us}^*\frac{10}{9}f_\pi\,g_{27}\left\{1 + \frac{1}{2}\left(s - 2M_\pi^2\right)B(M_\pi^2, M_\pi^2, s)\right.$$

$$+\frac{(M_K^2 - M_\pi^2)}{24M_\pi^2}\left[(15M_K^2 - 8M_\pi^2)\,B(M_K^2, M_\pi^2, M_\pi^2) + M_K^2\,B(M_K^2, M_\eta^2, M_\pi^2)\right]$$

$$-\frac{23}{12}\,\mu_\pi - \frac{1}{4}\,\mu_\eta - \frac{7}{6}\,\mu_K + \frac{(M_K^2 - M_\pi^2)}{4M_\pi^2}\,(5\mu_\pi - \mu_\eta - 4\mu_K)$$

$$+\frac{1}{f^2}\Big[-16(2M_K^2 + M_\pi^2)L_4^r(\mu) - 4(M_K^2 + 3M_\pi^2)L_5^r(\mu) + M_K^2\,\bar{C}_1^{27}(\mu)$$

$$\left.+M_\pi^2\,\bar{C}_4^{27}(\mu) + s\,\bar{C}_5^{27}(\mu)\Big]\right\}, \quad \text{(B.6)}$$

while $\delta\tilde{a}_2$ is given by

$$
\begin{aligned}
\delta\tilde{a}_2 = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{10}{9} f_\pi\, g_{27} & \left\{ \frac{1}{f^2} \left[M_\pi^2\, \bar{C}_2^{27}(\mu) + M_K^2\, \bar{C}_3^{27}(\mu) \right] \right. \\
& + \frac{1}{24} \left[M_K^2\, B(M_K^2, M_\eta^2, M_\pi^2) - (M_K^2 + 8M_\pi^2)\, B(M_K^2, M_\pi^2, M_\pi^2) \right] \\
& \left. - \frac{1}{12}(\mu_\pi + 3\mu_\eta + 4\mu_K) + \frac{M_\pi^2}{M_K^2 - M_\pi^2}\, (2\mu_\pi - \mu_\eta - \mu_K) \right\}.
\end{aligned}
\tag{B.7}
$$

The one-loop function $B(M_1^2, M_2^2, p^2)$ is defined as follows

$$
\begin{aligned}
B(M_1^2, M_2^2, p^2) = \frac{1}{16\pi^2 f^2} & \left[-1 + \ln\left(\frac{M_2^2}{\mu^2}\right) + \frac{1}{2}\ln\left(\frac{M_1^2}{M_2^2}\right)\left(1 + \frac{M_1^2}{p^2} - \frac{M_2^2}{p^2}\right) \right. \\
& \left. + \frac{1}{2}\lambda^{1/2}\left(1, \frac{M_1^2}{p^2}, \frac{M_2^2}{p^2}\right) \ln\left[\frac{p^2 - M_1^2 - M_2^2 + \lambda^{1/2}\left(p^2, M_1^2, M_2^2\right)}{p^2 - M_1^2 - M_2^2 - \lambda^{1/2}\left(p^2, M_1^2, M_2^2\right)}\right] \right],
\end{aligned}
\tag{B.8}
$$

for $p^2 > (M_1 + M_2)^2$ and $p^2 \le (M_1 - M_2)^2$, while

$$
\begin{aligned}
B(M_1^2, M_2^2, p^2) = \frac{1}{16\pi^2 f^2} & \left[-1 + \ln\left(\frac{M_2^2}{\mu^2}\right) + \frac{1}{2}\ln\left(\frac{M_1^2}{M_2^2}\right)\left(1 + \frac{M_1^2}{p^2} - \frac{M_2^2}{p^2}\right) \right. \\
& \left. - \frac{1}{2}\sqrt{-\lambda\left(1, \frac{M_1^2}{p^2}, \frac{M_2^2}{p^2}\right)}\, \arctan\left[\frac{(p^2 - M_1^2 - M_2^2)\sqrt{-\lambda(p^2, M_1^2, M_2^2)}}{(p^2 - M_1^2 - M_2^2)^2 - 2M_1^2 M_2^2}\right] \right]
\end{aligned}
\tag{B.9}
$$

for $(M_1 - M_2)^2 < p^2 \le (M_1 + M_2)^2$ and $-\pi/2 < \arctan(x) < \pi/2$. The function $\lambda(x, y, z)$ is given by

$$
\lambda(x, y, z) = (x + y - z)^2 - 4xy.
\tag{B.10}
$$

The function $B(M_1^2, M_2^2, p^2)$ is related to the function $\bar{J}(s)$ introduced in ref. [29] in the following way

$$
f^2\, B(M_1^2, M_2^2, p^2) = -\bar{J}(s) + \frac{1}{16\pi^2}\left(\ln\frac{M_2^2}{\mu^2} + \frac{M_1^2}{M_1^2 - M_2^2}\ln\frac{M_1^2}{M_2^2}\right).
\tag{B.11}
$$

For $M_1 = M_2 \equiv m$ one gets

$$
f^2\, B(M^2, M^2, p^2) = -\bar{J}(s) + \frac{1}{16\pi^2}\left(\ln\frac{M^2}{\mu^2} + 1\right),
\tag{B.12}
$$

where $\bar{J}(s)$ for $M_1 = M_2$ has been given in eq. (3.5).

The parameters μ_P $(P = \pi, K, \eta)$ contain a logarithmic dependence on the chiral renormalization scale μ generated by one-loop corrections. They are defined as

$$\mu_P = \frac{M_P^2}{32\pi^2 f^2} \ln \frac{M_P^2}{\mu^2}. \tag{B.13}$$

The explicit μ–dependence of μ_P and the functions $B(M_1^2, M_2^2, p^2)$ is canceled by the local contributions L_i^r $(i = 4, 5)$, C_i^8 $(i = 1, \ldots 6)$, C_i^{27} $(i = 1, \ldots 6)$ and $\bar{C}_i^{27}$ $(i = 1, \ldots 5)$.

C   Comments on recent literature

The ideas put forward in our first letter [1] have been further discussed in several recent papers by other authors. We would like to make here some brief comments on these works.

C.1

It has been pointed out in ref. [34] that the Omnès exponential depends on the chosen subtraction point. In that reference the FSI enhancement is minimized, by taking the highest subtraction point below the physical cut, $s_0 = 4M_\pi^2$. This trivial fact is then used to argue that our evaluation of FSI corrections is unreliable. Our detailed analysis of the subtraction point dependence in section 3 shows that this claim is unfounded. Taking a higher value of s_0 one is just shifting FSI corrections from the Omnès exponential to the amplitude in front, but the physical result is of course the same. At $s_0 = 4M_\pi^2$ there is a large one-loop correction to the amplitude, which has been overlooked in ref. [34]. Moreover, $s_0 = 4M_\pi^2$ is a bad choice of subtraction point, because the corresponding Taylor expansion has zero convergence radius [35].

Ref. [34] states that it is not precisely known at which value of s existing lattice estimates correspond to. It is suggested that future lattice calculations could obtain the $K \to \pi\pi$ weak matrix elements at threshold $(s_0 = 4M_\pi^2)$ and the $\Re_I(M_K^2, s_0)$ correction factors could then be used to get the physical amplitudes.

The s ambiguity mentioned in ref. [34] is not present in the low-energy chiral expansion. Our ε'/ε calculation is based on a large–N_C evaluation of the couplings of the $\Delta S = 1$ χPT Lagrangian. Once these chiral couplings are determined the $K \to \pi\pi$ amplitudes can in principle be computed at any value of s. Higher–order chiral corrections are of course smaller at lower values of momenta, which makes advisable to use the chiral expansion al low s values.

C.2

At lowest order in χPT, the four–quark operator Q_8 induces the $O(p^0)$ chiral term proportional to the coupling g_{EW}. The corresponding one–loop correction has been analyzed in ref. [49], where a small positive contribution is obtained. This result agrees with a recent dispersive calculation of $B_8^{(3/2)}$ [69], which finds $B_8^{(3/2)} = 1.11 \pm 0.16 \pm 0.23$.

FSI generate instead a small suppression of the $I = 2$ amplitude. However, there are other chiral corrections not related to FSI which appear at the one-loop level; they are included in the value of $a_2(0)$. Since for $I = 2$ the FSI effect is small, other correction could be equally important and even reverse the sign of the correction for the physical amplitude $a_2(M_K^2)$. A detailed one-loop analysis will be presented elsewhere [47].

C.3

A simplified (the dispersive integral over the phase-shift is not exponentiated) version of our $\Re_I$ factors has been used in ref. [70], which advocates a different and conceptually incorrect interpretation of the chiral corrections related to FSI.

In this reference a non-subtracted dispersion relation is used. The resulting divergence in $I_{0,0}^I(s)$ is regulated cutting the dispersive integral at the upper edge $\bar{z}$, and making the ad-hoc identification $\bar{z} = \nu^2$, with ν the short–distance scale governing the Wilson coefficients of the effective four–quark Hamiltonian (4.1). This generates a ν dependence in the dispersive integral which is claimed to cancel the renormalization–scale dependence of the Wilson coefficients.

The choice $\bar{z} = \nu^2$ and the associated identification of infrared and ultraviolet logarithms is arbitrary and cannot be correct. The FSI logarithms have nothing to do with the underlying short–distance physics. The Omnès factor $\Omega_I(s, s_0)$ relates the isospin amplitudes at two different points s_0 and s, but is unable to fix the global normalization. The short–distance information is hidden in $a_I(0)$ which, moreover, is independent of the scale ν. Thus, the cancellation of ν dependences must be accomplished even in the absence of FSI.

The argument can be better seen analyzing the scalar form factor, which has the same FSI logarithms but a different short–distance contribution. In fact, instead of working with hadronic matrix elements of the scalar current, we can take the corresponding matrix elements of the divergence of the associated vector current. Both quantities are trivially related by a quark–mass factor, through a Ward identity. The FSI phenomena and the associated χPT logarithms are of course identical; but now, there is no anomalous dimension. Since there is no short–distance renormalization scale ν, the identification

$\bar{z} = \nu^2$ is then meaningless.

The same argumentation can be repeated with the Omnès summation of FSI effects in the pion vector form factor [25]. Again, this is a renormalization–group invariant quantity (no anomalous dimension) and does not make any sense to identify FSI phenomena with non-existing short–distance logarithms.

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