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Using Heavy Quark Fragmentation into Heavy Hadrons to Determine QCD Parameters and Test Heavy Quark Symmetry*

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Abstract

We present a detailed analysis of the use of heavy quark fragmentation into heavy hadrons for testing the heavy quark effective theory through comparison of the measured fragmentation parameters of the c and b quarks. Our analysis is entirely model independent. We interpret the known perturbative evolution in a way useful for exploiting heavy quark symmetry at low energy. We first show consistency with perturbative QCD scaling for measurements done solely with c quarks. We then apply the perturbative analysis and the heavy quark expansion to relate measurements from ARGUS and LEP. We place bounds on a nonperturbative quark mass suppressed parameter, and compare the values for the b and c quarks. We find consistency with the heavy quark expansion but fairly sizable QCD uncertainties. We also suggest that one might reduce the systematic uncertainty in the result by not extrapolating to low z.

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1 Introduction

Current e^+e^- colliders have excelled in the precision study of weak interactions. One can also hope to perform detailed tests of the predictions of QCD and heavy quark symmetry. With the copious production of heavy quarks, the study of heavy quark fragmentation functions at e^+e^- colliders might prove one such forum for detailed studies of QCD.

The factorization theorem states that the measured fragmentation function can be written as a convolution of two terms, one which depends on short-distance physics and one which is sensitive to large-distance dynamics. The perturbative contribution has been understood for some time, while heavy quark symmetry provides constraints on the nonperturbative contribution.

The nonperturbative analysis that we follow was derived in ref. [1]. The basic idea is to evaluate the matrix elements of the heavy quark operators at sufficiently low renormalization mass scales, where one can exploit heavy quark symmetry to expand the moments of the fragmentation function in Λ/m, where Λ is some QCD related scale, m is the heavy quark mass, and the coefficients in the expansion are mass independent. With this parameterization of the nonperturbative features of the fragmentation function at low mass scale, one can evolve the moments via perturbative QCD to the high scale Q^2 at which they are measured. One can then use the experimental results to extract the unknown nonperturbative parameters. Since the same parameters enter the c and b quark fragmentation functions and moments, one can test how well the heavy quark expansion converges.

In this paper we apply this approach to the normalized second moments of the c and b quark fragmentation functions, measured at different center of mass energies. We study the second moment because it is best measured experimentally and because it involves a single nonperturbative parameter at order Λ/m.

Our analysis consists of two parts. We first relate the second moments of the c quark fragmentation function measured at different center of mass energies, namely by ARGUS or CLEO and at PEP and LEP. This part of the analysis involves solely perturbative QCD evolution. In the second part of the analysis we combine both perturbative QCD and the leading order heavy quark expansion to relate the moments of the c and b quark fragmentation functions.

We will see that there is no inconsistency with preliminary measurements. We will also see that by comparing the fragmentation function of c quarks at different center of mass energy one might hope to constrain the QCD scale. When using heavy quark symmetry to relate the b and c quark fragmentation functions, the calculation is less predictive, due to the large uncertainty in higher order QCD effects at the heavy quark mass scale. How useful the relations ultimately prove is sensitive to the size of Λ_{QCD}.

In Section 2 we describe in detail the perturbative QCD scaling accurate to subleading logarithmic order, and we summarize the nonperturbative results of ref. [1] relevant for the purposes of this paper. In Section 3 we present our results. Section 4 is a digression from our main analysis in which we suggest that the experimental results be presented differently. Rather than extrapolating the data to low z, we suggest a "cutoff" moment, where one only integrates from some cutoff value z_0. We suggest that factorization, though no longer exact, might work adequately. We conclude in the final section.

2 Theoretical Background

In this section, we give the theoretical framework with which we analyze heavy quark fragmentation. We employ perturbative QCD in conjunction with the heavy quark expansion to relate the heavy quark fragmentation functions measured at different center of mass energies. Because it is best measured and involves only a single nonperturbative parameter (if we only keep the leading quark mass dependence), we will ultimately relate not the fragmentation function itself, but the normalized second moment.

The perturbative analysis uses refs. [2], [3] and the nonperturbative analysis is from ref. [1]. From the latter reference, we use only the results that the moments of the fragmentation function have a heavy quark mass expansion, and that the correct choice of fragmentation variable corresponds to the ratio of energies. The discussion which follows primarily summarizes the perturbative scaling which was not done explicitly in ref. [1]. We frame the analysis entirely in terms of the standard procedure which is used when applying the heavy quark theory, namely scaling, matching to the heavy quark theory, and evaluating matrix elements in the heavy quark theory. With the perturbative scaling, the measured fragmentation function will be used to determine the unknown nonperturbative parameters of the fragmentation function. The tests of the theory come when we relate the fragmentation function measured at different center of mass energies, and when we relate the fragmentation parameters for different flavors of heavy quark.

In this paper, we work to subleading logarithmic order. We do not incorporate explicitly Sudakov logarithms in the nonperturbative analysis, since they are already included in the parameterization of the heavy quark fragmentation function at low renormalization mass scales. Furthermore, the heavy quark expansion constrains us to the study of low moments, because of the large combinatoric factor which multiplies terms in the expansion of higher order moments. Therefore, as argued in ref. [1], Sudakov logs are not important in our approach, since low moments are not dominated by z in the regime where Sudakov logs are large.

We use the following notation: σ is the cross section for the inclusive e^+e^- annihilation into a hadron, H, $\hat{\sigma}_i$ is the short-distance cross section for producing a parton i, and $\hat{f}_i$ is the fragmentation function for producing the hadron H from the parton i. According to the factorization theorem, the cross section for the inclusive annihilation, $e^+e^- \to H + X$, can be written as

$$\frac{d\sigma(z,Q,m)}{dz} = \sum_i \int_z^1 \frac{dy}{y} \frac{d\hat{\sigma}_i(y,Q,\mu)}{dy} \hat{f}_i(z/y,\mu,m) \tag{1}$$

where z is the ratio of the energy of the hadron to the beam energy in the center of mass frame ($0 < z < 1$). The factorization formula (1) states that the physical cross section is the convolution of two terms. The first, $d\hat{\sigma}_i(y,Q,\mu)/dy$, is the short-distance cross section for producing a parton i (where i can be a heavy or light quark or antiquark, or a gluon). It is insensitive to the low-energy features of the process and therefore does not depend upon the mass of the heavy quark.

The inclusive short-distance cross section for producing a parton i in e^+e^- annihilation via γ- and Z-exchange, is given by

$$\frac{d\hat{\sigma}_i(z,\theta,Q)}{dz\,d\cos\theta} = \lambda_i \left\{ \frac{3}{8}(1+\cos^2\theta)\; C_{i,T}(z) + \frac{3}{4}\sin^2\theta\; C_{i,L}(z) + \frac{3}{4}\cos\theta\; C_{i,A}(z) \right\} \tag{2}$$

where $z = 2q\cdot p/q^2$, p_μ is the four-momentum of the parton, q_μ is the four-momentum of the virtual boson, with $\sqrt{q^2} = Q$, and θ is the c.m. angle between the parton and the beam. We have taken $\mu = Q$, so that $d\hat{\sigma}_q(z,\theta,Q)$ is μ–independent. The notation $C_{i,1}(z) = C_{i,T}(z)$ and $C_{i,2}(z) = C_{i,T}(z) - 2C_{i,L}(z)$ is also used in the literature.

In eq. (2), the factor λ_q for quarks is just the total cross section for producing a quark

$$\lambda_q \equiv \hat{\sigma}_q = \frac{4\pi\alpha_{em}^2}{Q^2}\{e_e^2 e_q^2 + 2e_e e_q v_e v_q \mathrm{Re}\chi(Q^2) + (v_e^2 + a_e^2)(v_q^2 + a_q^2)|\chi(Q^2)|^2\} \tag{3}$$

where

$$\chi(Q^2) = \frac{Q^2}{Q^2 - M_Z^2 + iQ^2\Gamma_Z/M_Z} \tag{4}$$

and e_e, v_e, a_e, e_q, v_q, a_q are the charges and Z couplings of the electron and quarks, respectively. For gluons, λ_G is given by

$$\lambda_G = \sum_q (\hat{\sigma}_q + \hat{\sigma}_{\bar{q}}) \tag{5}$$

Integrating (2) over the angular variables we obtain

$$\frac{1}{\lambda_i}\frac{d\hat{\sigma}_i}{dz} \equiv C_i(z) = C_{i,T}(z) + C_{i,L}(z) \tag{6}$$

The coefficient functions C_i are calculable in perturbative QCD as a power series in $\alpha(\mu)$:

$$C_i(z,Q,\mu) = C_i^{(0)}(z) + \frac{\alpha(\mu)}{2\pi}C_i^{(1)}(z,Q,\mu) + \ldots \tag{7}$$

The leading order term in the quark coefficient function $C_{q,L}(z)$ is suppressed by a factor m^2/Q^2 and therefore negligible at high energies, but the $\mathcal{O}(\alpha)$ contribution is not suppressed, so at subleading order we need to include both contributions to C_q. The coefficient function for gluons, C_G, is of order α.

The second factor appearing in equation (1) is the fragmentation function $\hat{f}_i(z,\mu,m)$, which describes how the parton i combines with surrounding partons to produce the observed hadron H, in a much longer lapse of time. It is insensitive to the high-energy part of the process and therefore depends on m but not on Q.

The scale μ is an arbitrary scale which separates the low- from the high-energy dynamics. If we know $\hat{f}_i(z,\mu_0,m)$ at some scale μ_0, we can obtain $\hat{f}_i(z,\mu,m)$ by using the Altarelli-Parisi evolution equations, which organize the large logarithms $\log(\mu/\mu_0)$ to the desired order in perturbation theory. In practice, we will work to subleading order, using the calculated anomalous dimensions from ref. [3]. As we will see, $C_i(z,Q,\mu)$ contains terms of the form $\log Q^2/\mu^2$, so we will choose $\mu^2 \sim Q^2$ to avoid large logarithms. On the other hand, we take μ_0 of order of the heavy quark mass, m, where the results of the heavy quark effective theory apply.

It is convenient to define the singlet ($\hat{f}^S$) and non-singlet ($\hat{f}_q^{NS}$) linear combinations of the quark fragmentation functions:

$$\begin{aligned} \hat{f}_q^{(+)} &= \hat{f}_q + \overline{\hat{f}}_q \\ \hat{f}^S &= \sum_{q=1}^{N_F} \hat{f}_q^{(+)} \\ \hat{f}_q^{NS} &= \hat{f}_q^{(+)} - \frac{1}{2N_F}\hat{f}^S \end{aligned} \tag{8}$$

where N_F is the number of flavours and $\overline{\hat{f}}_q$ denotes the fragmentation function of the antiquark $\bar{q}$.

According to the result of ref. [1], we can expand $\hat{f}_Q(z, \mu_0, m)$, the heavy quark fragmentation function, for $\mu_0 \approx m$, [1] as

$$\hat{f}_Q(z, m, m) = \frac{1}{\epsilon}\hat{a}\left(\frac{\frac{1}{z} - \frac{m}{M_H}}{\epsilon}\right) + \hat{b}\left(\frac{\frac{1}{z} - \frac{m}{M_H}}{\epsilon}\right) \tag{9}$$

where M_H is the mass of the heavy hadron and $\epsilon \sim 1 - \frac{m}{M_H}$. The funny dependence on z is because the expansion is simplest in terms of the variable $1/z$. It is important to use $z = E/E_{beam}$ (or its inverse) as the fragmentation parameter. Other fragmentation variables will permit nonvanishing higher twist contributions at leading order in the heavy quark mass expansion and α_{QCD} [1]. With this choice of fragmentation variable, higher twist effects should be negligible (this is of course only significant for relatively low Q^2 (relative to m^2) experiments, like ARGUS and CLEO).

We will not use the full form of this result, as the measured fragmentation functions are too poor to do a detailed fit, which involves many nonperturbative parameters. Instead, as suggested in ref. [1], we will consider only the moments, defined as

$$\hat{\Gamma}_N^{HQ}(\mu, m) = \int_0^1 dz z^{N-1} \hat{f}_Q(z, \mu, m) \tag{10}$$

The low order moments have the advantage of being better measured, and of being describable in terms of a small number of nonperturbative parameters (when the heavy quark expansion is exploited).

According to the assumptions of the heavy quark theory, the fragmentation functions(and their moments) for the light quarks and gluons at the low renormalization scale can be taken to vanish.

By taking the moments of the function of eq. (9), we derive a heavy quark expansion for each of the moments. In general, at order Λ/m, there are two independent parameters in terms of which any moment can be expanded with known N dependence [1]. However, since we will restrict our numerical analysis to the measured second moment, it is convenient to define a single nonperturbative parameter and expand the second moment as

$$\hat{\Gamma}_2^{HQ}(m, m) = 1 - \frac{a}{m} \tag{11}$$

From previous work [4], one can also conclude that there is a heavy quark mass expansion for the moments. The power of the approach in [1] is that one can address the higher twist corrections to determine the best choice of fragmentation variable, that one can relate higher order moments, and the one can in principle include higher order mass suppressed terms. The parameter a is the single nonperturbative parameter which is required to describe the second moment of the fragmentation function if we work at order Λ/m. Notice that because the fragmentation function is not defined in terms of an on shell matrix element, the parameter a is not determined theoretically, but can be fit. (This is in contrast with the distribution function, for which one can derive $a(x) = \delta(x - m/M)$, with corrections of order $(\Lambda/m)^2$.) In ref. [1], a was defined in units of $\overline{\Lambda} = M_H - m$. In this paper, we have absorbed $\overline{\Lambda}$ in the definition of a, which is now a dimensional parameter. It should be kept in mind however, that a is expected to be of order Λ_{QCD}. The parameter a depends on the hadron type. Higher order terms can be incorporated–however if the heavy quark expansion is valid, these terms

[1] This expansion is true in the HQET, so $\hat{f}_Q$ defined here is related by a nontrivial perturbative matching condition to $\hat{f}_Q$ used in eq. (1).

should be suppressed, and we should get an answer accurate at about the 10 % level by just keep the first mass suppressed contribution.

This expansion is valid only for $\mu_0 \approx m$. It is therefore necessary to match onto the full theory, and scale the result to $\mu \approx Q$. We now describe the perturbative scaling and matching procedure.

The fragmentation functions obey the Altarelli-Parisi evolution equations

$$\frac{\partial \hat{f}^{\tau}(z,\mu,m)}{\partial \log \mu^2} = \sum_{\tau'} \int_z^1 P_{\tau\tau'}(z/y,\alpha(\mu)) \hat{f}^{\tau'}(y,\mu,m) \frac{dy}{y} \qquad \tau,\tau' = S,G \tag{12}$$

$$\frac{\partial \hat{f}_q^{NS}(z,\mu,m)}{\partial \log \mu^2} = \int_z^1 P_{NS}(z/y,\alpha(\mu)) \hat{f}_q^{NS}(y,\mu,m) \frac{dy}{y} \tag{13}$$

Only singlet quarks and gluons mix under evolution [2]; the non-singlet equation (13) decouples and can be solved independently.

The evolution functions $P_{\tau\tau'}(z,\alpha(\mu))$ and $P_{NS}(z,\alpha(\mu))$ can be expanded in a power series in $\alpha(\mu)$:

$$P(z,\alpha(\mu)) = \frac{\alpha(\mu)}{2\pi} P^{(0)}(z) + \left(\frac{\alpha(\mu)}{2\pi}\right)^2 P^{(1)}(z) + \mathcal{O}(\alpha^3) \tag{14}$$

The leading log contributions ($P^{(0)}_{\tau\tau'}$, $P^{(0)}_{NS} = P^{(0)}_{SS}$) are the well-known Altarelli-Parisi splitting functions [5].

The definitions of C_i and $\hat{f}_i$, and consequently of $P(z,\alpha(\mu))$, are unique only at leading log order. At the next-to-leading level they depend on the factorization scheme used to separate the collinear singularities. There are complete calculations available in the literature in two different factorization schemes, which we will denote I and II for definiteness. Scheme I is defined in [6] by Curci, Furmanski and Petronzio, where they give the quark coefficient functions and non-singlet evolution function in z-space, as well as their Mellin transforms to N-space. The evolution function matrix in z-space for the singlet sector, including quark and gluon mixing, has been computed in [7] and the gluon coefficient functions can be found in [2]. Matching conditions for quarks and gluons within this scheme are calculated in [3]. Alternatively, in [8] Floratos, Ross and Sachrajda define Scheme II, which is used in [9] to compute coefficient functions for quarks and gluons as well as non-singlet and singlet evolution functions both in z-space and N-space [3].

The results for $P^{(1)}_{\tau\tau'}$ in [7] and [9] do not agree; there is a difference for $P^{(1)}_{GG}(z)$ in the term proportional to the colour factor C_G^2. The timelike evolution functions (14) are obtained from the spacelike ones (adequate to deep inelastic scattering processes) by analytic continuation. The discrepancy mentioned above is in fact a consequence of the discrepancy in the corresponding spacelike evolution functions previously computed [2].

In principle one should evolve the fragmentation functions $\hat{f}_i(z,\mu,m)$ by using eqs. (12) and (13), i.e., including mixing between singlet quarks and gluons. However we have explicitly verified using Scheme I that incorporating mixing never modifies our result by more than about 1% for the b quark and 8% for the c quark. Therefore, for the purpose of an expansion to order Λ/m, we could safely neglect mixing. The reason mixing is small is that the gluon fragmentation function is never very

[2]Note that the anomalous dimension matrix for timelike processes is transposed aside from the dependence on N_f.

[3]However, expressions for the timelike moments of the coefficient functions and anomalous dimensions given in the appendix B of ref. [9] correspond to $C_N^{(1)}$ and $P_N^{(1)}$, despite the $N+1$ index.

big and the gluon coefficient function is $\mathcal{O}(\alpha)$. Our result agrees with the fact that the probability of producing a heavy quark indirectly, via a secondary gluon, is very small at $Q^2 \sim M_Z^2$. The authors of ref. [10] have shown that the so-called Branco et al. terms, which contribute to the heavy quark inclusive cross section, can not be larger than 0.5% for b-quarks and 4% for c-quarks and one would expect that the remaining terms are of the same order of magnitude. This is very important to our analysis, since it means that we do not need to scale the first moment (or equivalently the normalization factor) between different momentum scales. If we are only dealing with energies less than or equal to the Z mass, the normalization is approximately independent of energy scale; that is very few additional heavy quarks are produced through QCD processes. Therefore, we consider only non-singlet evolution in the remainder of this paper.

Beyond leading log level, the non-singlet evolution function has an additional contribution due to $q\bar{q}$ mixing. However, we have found that it is negligible at the energies which are currently of interest. Therefore, we use the non-singlet anomalous dimensions given in [3], which correspond to the factorization scheme I and do not include $q\bar{q}$ mixing.

We are interested in the evolution of the moments of the heavy quark fragmentation function, $\hat{\Gamma}_N(\mu, m)$. The next-to-leading order evolution equation for $\hat{\Gamma}_N$ has a very simple form:

$$\frac{\partial \hat{\Gamma}_N(\mu, m)}{\partial \log \mu^2} = P_N(\alpha(\mu))\hat{\Gamma}_N(\mu, m) \tag{15}$$

where

$$P_N(\alpha(\mu)) = \frac{\alpha(\mu)}{2\pi} P_N^{(0)} + \left(\frac{\alpha(\mu)}{2\pi}\right)^2 P_N^{(1)} \tag{16}$$

is the Mellin transform of the Altarelli-Parisi splitting function in next-to leading order. The formula for $P_N^{(1)}$ is given in the appendix of ref. [3]. The expression for $\alpha(\mu)$ accurate to next-to-leading order is

$$\alpha(\mu) = \frac{1}{b_0 \log(\mu^2/\Lambda^2)} \left(1 - \frac{b_1 \log[\log(\mu^2/\Lambda^2)]}{b_0^2 \log(\mu^2/\Lambda^2)}\right) \tag{17}$$

with

$$b_0 = \frac{33 - 2N_F}{12\pi} \qquad b_1 = \frac{153 - 19N_F}{24\pi^2} \tag{18}$$

Then eq. (15) takes the form

$$\frac{\partial \hat{\Gamma}_N(\mu, m)}{\partial \log \mu^2} = \left[P_N^{(0)} + \frac{\alpha(\mu)}{2\pi}\left(P_N^{(1)} - \frac{2\pi b_1}{b_0} P_N^{(0)}\right)\right] \hat{\Gamma}_N(\mu, m) \tag{19}$$

which can be solved analytically, yielding

$$\hat{\Gamma}_N(\mu, m) = \hat{\Gamma}_N(\mu_0, m) \left[\frac{\alpha(\mu_0)}{\alpha(\mu)}\right]^{\frac{P_N^{(0)}}{2\pi b_0}} exp\left\{\frac{\alpha(\mu_0) - \alpha(\mu)}{4\pi^2 b_0}\left(P_N^{(1)} - \frac{2\pi b_1}{b_0} P_N^{(0)}\right)\right\} \tag{20}$$

The physical fragmentation function (eq. (1)) is obtained by convolution of the fragmentation function with the hard cross-section, evaluated at the scale μ. Therefore, the normalized moments

$$\sigma_N(Q, m) = \frac{1}{\sigma}\int_0^1 dz z^{N-1} \frac{d\sigma(z, Q, m)}{dz} \tag{21}$$

are given by

$$\sigma_N(Q,m) = C_N(Q,\mu)\left[\frac{\alpha(\mu_0)}{\alpha(\mu)}\right]^{\frac{P_N^{(0)}}{2\pi b_0}} exp\left\{\frac{\alpha(\mu_0)-\alpha(\mu)}{4\pi^2 b_0}\left(P_N^{(1)} - \frac{2\pi b_1}{b_0}P_N^{(0)}\right)\right\}\hat{\Gamma}_N(\mu_0,m) \tag{22}$$

where $C_N(Q,\mu)$ is the Mellin transform of the coefficient functions (eq. (7)),

$$C_N(Q,\mu) = 1 + \frac{\alpha(\mu)}{2\pi}C_N^{(1)}(Q,\mu) + \mathcal{O}(\alpha^2) \tag{23}$$

The expression for $C_N^{(1)}(Q,\mu)$ is [3] [4]

$$\begin{aligned} C_N^{(1)}(Q,\mu) \;&=\; C_F\left[\log\frac{Q^2}{\mu^2}\left(\frac{3}{2}+\frac{1}{N(N+1)} - 2S_1(N)\right)\right. \\ &+S_1^2(N) + S_1(N)\left(\frac{3}{2} - \frac{1}{N(N+1)}\right) + 5S_2(N) \\ &\left.+\frac{1}{N} - 2\frac{2N+1}{N^2(N+1)^2} + \frac{1}{(N+1)^2} - \frac{3}{2}\frac{1}{N+1} - \frac{9}{2}\right] \end{aligned} \tag{24}$$

where $C_F = 4/3$ and

$$S_l(N) = \sum_{j=1}^{N}\frac{1}{j^l} \tag{25}$$

We will take $\mu = Q$ from now on, so that $C_N^{(1)}$ does not depend upon μ. We therefore use the notation $C_N^{(1)} = C_N^{(1)}(\mu,\mu)$.

In eq. (22), $\hat{\Gamma}_N(\mu_0,m)$ is the moment of the heavy quark fragmentation function in the full theory at the scale $\mu_0 \sim m$. Note that it is critical to the use of a subleading log calculation that we know the μ–dependence of the matrix element. Even though the matrix element itself is nonperturbative, the scale dependence is not, and is contained in the matching coefficient. We note that in reality, there is still some scaling which goes on below the heavy quark mass scale because of the scale dependence of the spin–dependent mass suppressed operator [5]. However, since the maximum scaling between the b and c mass scales of the gluon magnetic moment operator is 20%, and it is only one of several contributions to the Λ/m term, this effect is never numerically more significant than the higher order mass suppressed corrections. We therefore neglect scaling in the heavy quark theory itself; the μ dependence is then just that from the matching between the full and heavy quark theories. Because this matching is done at a fairly high momentum scale, of order the heavy quark mass, the scale dependence is calculable order by order in perturbation theory. We then have

$$\hat{\Gamma}_N(\mu_0,m) = \left(1 + \frac{\alpha(\mu_0)}{2\pi}d_N^{(1)}(\mu_0,m)\right)\hat{\Gamma}_N^{HQ}(m) \tag{26}$$

We take $\hat{\Gamma}_N^{HQ}(m)$ as independent of μ_0, but because we work at a finite order in perturbative theory we will see that there is a fairly substantial dependence on μ_0 which cannot be neglected. This will be clear in section 3.

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[4] $C_N^{(1)}(Q,\mu) = \hat{a}_N^{(1)}(Q,\mu)$ in the notation of ref. [3].

[5] We thank Eric Braaten for pointing this out.

The matching condition which agrees with the factorization prescription for $C_N^{(1)}(Q,\mu)$ and $P_N^{(1)}$ can be obtained from ref. [3] and reads

$$\begin{aligned} d_N^{(1)}(\mu_0,m) &= \log\frac{\mu_0^2}{m^2}P_N^{(0)} + C_F\left[-2S_1^2(N) + 2S_1(N)\left(1+\frac{1}{N(N+1)}\right)\right.\\ &\left. -2S_2(N) - \frac{2}{(N+1)^2} - \frac{1}{N(N+1)} + 2\right] \end{aligned} \tag{27}$$

where

$$P_N^{(0)} = C_F\left(\frac{3}{2} + \frac{1}{N(N+1)} - 2S_1(N)\right) \tag{28}$$

Using (26) we can write eq. (22) as

$$\sigma_N(Q,m) \equiv \langle z^{N-1}(Q,m)\rangle = \langle z(Q,m)^{N-1}\rangle_{pert}\langle z(m)^{N-1}\rangle_{nonpert} \tag{29}$$

where

$$\begin{aligned} \langle z(Q,m)^{N-1}\rangle_{pert} &= C_N(Q,\mu)\left[\frac{\alpha(\mu_0)}{\alpha(Q)}\right]^{\frac{P_N^{(0)}}{2\pi b_0}}\left(1+\frac{\alpha(\mu_0)}{2\pi}d_N^{(1)}(\mu_0,m)\right)\\ &\times\ exp\left\{\frac{\alpha(\mu_0)-\alpha(Q)}{4\pi^2 b_0}\left(P_N^{(1)} - \frac{2\pi b_1}{b_0}P_N^{(0)}\right)\right\} \end{aligned} \tag{30}$$

$$\langle z(m)^{N-1}\rangle_{nonpert} = \hat{\Gamma}_N^{HQ}(m) \tag{31}$$

It is important to clarify the issue of the factorization scheme dependence. The separate results for the anomalous dimensions and the coefficient functions depend on the factorization scheme, but in the convolution there is a cancellation of the scheme dependence order-by-order in α [2]. Neglecting next-to-next-to-leading terms and using eq. (23), we can write the coefficient of $\alpha(Q)$ in the exponential of eq. (30) as $P_N^{(1)} - 2\pi b_0 C_N^{(1)}$. This combination can be shown to be independent of the renormalization scheme at the subleading level, and we have explicitly checked that by using both schemes I and II. However the coefficient of $\alpha(\mu_0)$ still depends on the factorization scheme and therefore so does the matching condition (27). In order to obtain an expression in terms of renormalization scheme independent quantities, we use the equation

$$C_N^{HQ}(\mu_0,m) = 1 + \frac{\alpha(\mu_0)}{2\pi}d_N^{(1)}(\mu_0,m) + \frac{\alpha(\mu_0)}{2\pi}C_N^{(1)} \tag{32}$$

where $C_N^{HQ}(\mu_0,m)$ is the short-distance cross section for producing a heavy quark of mass m. We then rewrite eq. (30) as

$$\begin{aligned} \langle z(Q,m)^{N-1}\rangle_{pert} &= C_N^{HQ}(\mu_0,m)\left[\frac{\alpha(\mu_0)}{\alpha(Q)}\right]^{\frac{P_N^{(0)}}{2\pi b_0}}\\ &\times\ exp\left\{\frac{\alpha(\mu_0)-\alpha(Q)}{4\pi^2 b_0}\left(P_N^{(1)} - 2\pi b_0 C_N^{(1)} - \frac{2\pi b_1}{b_0}P_N^{(0)}\right)\right\} \end{aligned} \tag{33}$$

Eqs. (33) and (30) are equivalent at the subleading level. Moreover, the evolution factor entering in eq. (33) is now independent of the factorization scheme and thus the matching condition (32), derived in the scheme of [6], is also adequate for the scheme used in [9].

However, since we are using results calculated within a single scheme it is not necessary to use scheme independent expressions as in (33). In fact, this turns out to be the preferred procedure. This is because in order to obtain a scheme independent result, we added terms which are technically higher order, but are in fact quite large, due to the large size of $C_N^{(1)}$. For example, $\frac{\alpha(m_c/2)}{2\pi}C_N^{(1)}$ is as big as 0.8 for $\Lambda_5 = 225$ MeV. So we are probably introducing spurious μ dependence by adding these unduly large terms. We therefore used eq. (30) in our calculations.

To summarize, we have shown that the measured moment of the fragmentation function can be expressed as in eq. (29). The perturbative contribution $\langle z(Q,m)\rangle_{pert}$ is given by (30) or (33), where the former has smaller μ dependence and is actually what one obtains on integrating the renormalization group equation. Once one has calculated $\langle z(Q,m)\rangle_{pert}$, one can extract the nonperturbative parameter a, defined in eq. (11). This parameter is independent of Q, the energy of the experiment, and so long as one always deals with the same kind of hadron, it should be also independent of m.

Unfortunately however there are still fairly large QCD uncertainties in the perturbative scaling given that we are working only to subleading logarithmic order and that the QCD scale is not so well known. The problem is that we need to know $\langle z(Q,m)\rangle_{pert}$ sufficiently well to extract $\langle z(m)\rangle_{nonpert}$ at the level of a/m. For large m, the required accuracy is greater, though the perturbation theory is better. We find that the error in extracting a from the b and c fragmentation functions is in fact comparable. In order to determine the accuracy of our procedure, we will consider different values of μ_0. Higher order perturbative effects may also be estimated from the $\mathcal{O}(\alpha^2)$ terms neglected in eq. (33). We find that these uncertainties are never as large as the μ_0 dependence. So we estimate the uncertainty from higher order effects by considering different values for the renormalization scale μ_0. The particular choices and results are given in the next section.

3 Results

In this section, we apply the procedure described in the previous section, where it was shown how perturbative QCD and the heavy quark expansion can be used to relate moments of fragmentation functions of different flavors measured at different Q^2 values. These predictions can be compared with the data from ARGUS, CLEO, PEP, PETRA and LEP. Since current results from LEP are only preliminary, we present our analysis in such a way that it can be readily applied with improved measurements. To do this we center the values on our plots on the preliminary measurement and we extract the nonperturbative parameter over the full range of experimentally allowed numbers (that is, the measured value with 2 sigma errors).

Recall the uncertainties in our predictions arise from two different types of QCD uncertainties. There is uncertainty in the perturbative part of the calculation due to both the poorly known value of α_{QCD} and the fact that we work only to subleading logarithmic order. For quantities involving perturbative scaling only for large Q^2 these uncertainties are expected to be small. But in order to apply heavy quark symmetries, one always needs to scale to the quark mass scales. For such low energy, higher order QCD corrections and the uncertainty in the exact value of α_{QCD} can be important.

It therefore makes sense to divide our analysis into two parts. We first consider simply measuring $\langle z_c\rangle$ at two different energy scales (sufficiently larger than the c quark mass that higher twist contributions should be small). These values should be related by *only* perturbative QCD, at scales for which the subleading calculation should prove reliable. The measurements seem to be consistent with perturbative QCD scaling over the allowed range of Λ_{QCD}. Turned around, it means that with sufficiently accurate data, one can constrain Λ_{QCD} (although probably not as well as with other methods).

Following this analysis, we proceed to incorporate the full formalism, employing both perturbative QCD and a nonperturbative expansion to relate b and c quark fragmentation functions. Here we will find sizable QCD uncertainty. Nonetheless, we will see that the data is substantially self consistent.

Notice that throughout the analysis of section 2 we have assumed fragmentation into a specific final state, although experiments at very high center of mass energy such as LEP do not distinguish hadron species. This should not be a problem when comparing inclusive measurements such as those done at LEP, since the normalized second moment for the inclusive measurement has the same heavy quark expansion for the b and c quark.

This is potentially a problem if different experiments select differently on the final hadronic state. The problem occurs because the nonperturbative parameter a depends on the hadron type. The nonperturbative contribution to the mean $\langle z \rangle$ experimentally measured can be a different linear combination of the different $a's$ at the different experiments. However, measurements from ARGUS show [6] that $\langle z_D \rangle \approx \langle z_{D^*} \rangle \approx \langle z_{\Lambda_c} \rangle$ within errors. Preliminary measurements from CLEO seem to also support this claim. If this is the case, we can to a good approximation neglect the differences between the different hadron types when comparing the measurements at different center of mass energy.

Moreover, for inclusive measurements at sufficiently high energy, the dependence on hadron type can be ignored, since the perturbative QCD scaling is the same for all hadrons so that it is always the same linear combination of $a's$ which is measured at any energy. This is obvious, since μ can be chosen so that only the coefficient function depends on Q.

Finally, for measurements which select on a specific final state (eg D^*), there is no problem.

Before we begin, we outline the differences in our approach from previous studies [11]. First, we make no attempt to fit the entire curve. We look only at the mean $\langle z \rangle$, which is much better measured than the fragmentation function itself. Furthermore, the mean $\langle z \rangle$ depends only on QCD parameters and one single nonperturbative parameter a, where a is defined by

$$\langle z(m) \rangle_{nonpert} = 1 - \frac{a}{m} \tag{34}$$

Our result is therefore independent of any assumed functional form of the fragmentation function. Furthermore, $\langle z \rangle_{nonpert}$ is much more stable against variations in μ and the QCD scale than the parameters used to fit a particular functional form. For example, in ref. [11], where the functional form $z_{nonpert} = z^{\alpha}(1-z)^{\beta}$ was used, the parameters α and β varied enormously in comparison with $\langle z \rangle_{nonpert}$.

Second, we use eq. (30) rather than (33) to calculate the perturbative factor $\langle z(Q,m) \rangle_{pert}$. As explained earlier, this result is better behaved with respect to μ-variation, and it is in fact what is obtained by integrating the renormalization group equations.

Third, we do not incorporate explicitly Sudakov logs in the nonperturbative fragmentation function.

Fourth, we use the experimental results on the mean $\langle z \rangle = \langle E \rangle / E_{beam}$. It is important to use the variable $\langle z \rangle$ because its non-perturbative contribution scale linearly in the mass of the heavy quark, according to eq. (34).

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[6] We thank G. Bonvicini for analyzing the data from the Ph.D. Thesis of P. C-Ho Kim and H.C.J. Seywerd (ARGUS Collaboration).

Finally, we use updated experimental data from LEP which leads to a harder b quark fragmentation function. As a result we find better agreement with the heavy quark effective theory predictions than in ref. [11].

It should be noted that there is some dependence in the measured $\langle z \rangle$ value on the assumed functional form (contributing to the systematic error) because the data is not measured to arbitrarily low z, so functional forms of the fragmentation function are assumed in order to extrapolate to low z. In section 4 we will show that one can use moments with a cutoff greater than zero to reduce this dependence on the functional form.

In both parts of the following analysis, we will allow for different values of Λ_{QCD} within the allowed range. We estimate the importance of the neglected higher order perturbative QCD contributions by testing the stability of our results with respect to a change in renormalization scale. It is an important feature of the analysis that the μ_0 dependence is entirely perturbative for a sufficiently heavy quark. Nonetheless, $\alpha(\mu_0)$ at the scale of the heavy quark mass is so large that it is not clear how well the perturbation theory converges. We estimate this uncertainty by allowing for a sizable variation in μ_0, between $2m$ and $m/2$. This might be too large a range, but without a higher order calculation, it is impossible to determine the accuracy of the subleading calculation.

When we invoke the heavy quark expansion, we work in the approximation of only retaining the leading mass dependent correction, suppressed by a single power of the heavy quark mass. Of course one can readily incorporate higher order terms. But we would like to test whether the heavy quark expansion works sufficiently well to get agreement (at least at the 10 % level) even without incorporating higher order contributions. This is important, as the heavy quark effective theory is still essentially untested at the Λ/m level.

3.1 Perturbative Analysis: Relating $\langle z_c \rangle$ at Different Energy Scales

In this section we focus on what can be learned using solely perturbative QCD evolution, without the heavy quark effective theory. Even if it turns out that QCD perturbation theory at subleading order is not sufficiently accurate at the low scale $\mu_0^2 \sim m^2$, it certainly should give a good description of the evolution between scales above the mass of the Υ. Thus if we compare the moments of the c quark fragmentation function measured by CLEO or ARGUS, and at PEP, PETRA and LEP, the measurements should be related through the perturbative evolution between the scales of the experiments. Again, this is with the caveat that if a different linear combination of hadron states is selected, the relations only need hold if the nonperturbative fragmentation function is the same for each hadron type, as seems to be the case. Furthermore, since D^*'s are the dominant contribution at these experiments, one would expect the perturbative relation to hold fairly well. In fact, with an exclusive measurement on D^*'s at both experiments, the perturbative scaling result would certainly apply.

According to the evolution equations, the $N = 2$ measured moments of the fragmentation function for the c quark are given by

$$\langle z_c(Q) \rangle = C_2(Q,\mu)\hat{\Gamma}_2(\mu, m_c) \tag{35}$$

Therefore the ratio between the mean $\langle z_c(Q) \rangle$ measured at different energy scales is

$$\frac{\langle z_c(Q_2) \rangle}{\langle z_c(Q_1) \rangle} = \frac{C_2(Q_2,\mu)}{C_2(Q_1,\mu)} \tag{36}$$

Λ_5 (MeV)	$\mu = Q_1/2$	$\mu = Q_1$	$\mu = 2Q_1$
75	0.813	0.825	0.832
125	0.794	0.808	0.816
175	0.779	0.795	0.804
225	0.766	0.784	0.794
275	0.755	0.774	0.785

Table 1: $\frac{C_2(Q_2 = 91\text{GeV}, \mu)}{C_2(Q_1 = 10.55\text{GeV}, \mu)}$

We relate the $N = 2$ moments measured at ARGUS ($Q_1 = 10.6$ GeV) or CLEO ($Q_1 = 10.55$ GeV) to those measured at PEP ($Q_2 = 29$ GeV) and LEP ($Q_2 = 91$ GeV). Since the experimental data from ARGUS and LEP have smaller errors, we focus our discussion mainly on the results from those experiments. The difference between the perturbative calculation for ARGUS and CLEO energies is negligible, so we only present the theoretical results for $Q_1 = 10.55$ GeV.

Since for the theoretical computation of the right hand side in eq. (36) we only need to evolve the coefficient functions between Q_1^2 and Q_2^2, which are well above Λ_{QCD}, we expect the perturbative evolution with next-to-leading accuracy to work well. The evolution equations for the coefficient functions are related to that for the fragmentation function by

$$\frac{\mu dC_N(Q, \mu)}{d\mu} = -\frac{\mu d\hat{\Gamma}_N(\mu, m)}{d\mu} \tag{37}$$

We choose the scale $\mu \sim Q_1$, so $C_2(Q_1, \mu)$ is given by eqs. (23) and (24), with $Q = Q_1$. For $C_2(Q_2, \mu)$ we also use the evolution equation (22) and eq. (37) to obtain

$$C_2(Q_2, \mu) = \left(1 + \frac{\alpha(Q_2)}{2\pi} C_2^{(1)}\right) \left[\frac{\alpha(\mu)}{\alpha(Q_2)}\right]^{\frac{P_N^{(0)}}{2\pi b_0}} exp\left\{\frac{\alpha(\mu) - \alpha(Q_2)}{4\pi^2 b_0} \left(P_N^{(1)} - \frac{2\pi b_1}{b_0} P_N^{(0)}\right)\right\} \tag{38}$$

To account for theoretical uncertainties of the prediction, we computed the ratio $C_2(Q_2, \mu)/C_2(Q_1, \mu)$ using the values of Λ in the five flavor theory, $\Lambda_5 = 75$, 125, 175, 225, 275 MeV and, as an estimate of higher order effects, we varied the scale μ between $Q_1/2$ and $2Q_1$.

The QCD perturbative results for the ratio $C_2(Q_2, \mu)/C_2(Q_1, \mu)$, with $Q_1 = 10.55$ GeV and $Q_2 = 91$ GeV are shown in Table 1. Notice that the results do not depend strongly on the scale μ, as one would expect since subsubleading effects at the energy scales we are considering should be rather small.

The mean $\langle z_{D^{*+}} \rangle$ for the fragmentation function of a c quark into D^{*+} mesons has been measured both at ARGUS [12] and LEP [13]. The results are

$$\begin{aligned} \langle z_{D^{*+}}(Q_1 = 10.6\text{GeV}) \rangle &= 0.647 \pm 0.006 \\ \langle z_{D^{*+}}(Q_2 = 91\text{GeV}) \rangle &= 0.495 \pm 0.013 \end{aligned} \tag{39}$$

In order to relate the inclusive measurements from PEP and LEP with the ARGUS measurements of $\langle z_D \rangle$, $\langle z_{D^*} \rangle$ and $\langle z_{\Lambda_c} \rangle$ we have to average over all the final states. [7] If we assume heavy quark

[7] In fact, there are ARGUS measurements only for D^{*+} mesons, but we assume $\langle z_{D^{*0}} \rangle = \langle z_{D^{*+}} \rangle$.

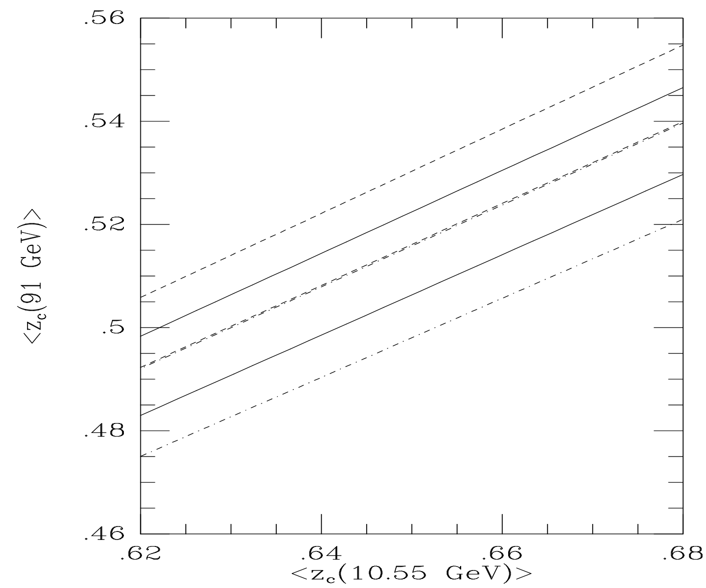

Figure 1: $\langle z_c(Q_2 = 91\text{GeV})\rangle$ as a function of $\langle z_c(Q_1 = 10.55\text{GeV})\rangle$, for $\Lambda_5 = 125$ MeV (dashed line), 175 MeV (solid), and 225 MeV (dashed-dotted). For each value of Λ_5 the lower line corresponds to $\mu = Q_1/2$ and the upper line to $\mu = 2Q_1$.

symmetry and that a c quark fragments into baryons about 12% of the time, the mean value of z_c is

$$\begin{aligned} \langle z_c(Q_1 = 10.6\text{GeV})\rangle &= 0.12\langle z_{\Lambda_c}\rangle + 0.88\left[\frac{1}{4}\langle z_D\rangle + \frac{3}{4}\langle z_{D^*}\rangle\right] \\ &= 0.640 \pm 0.009 \end{aligned} \quad (40)$$

It would be very useful to test the assumed ratio of 3:1 for D^* relative to D production is in fact correct. Preliminary results from CLEO seem to give a bigger value, so we allow for a wide range $0.62 \le \langle z_c(Q_1)\rangle \le 0.68$. The mean $\langle z_c\rangle$ measured inclusively at PEP [14] is

$$\langle z_c(Q_2 = 29\text{GeV})\rangle = 0.526 \pm 0.03 \quad (41)$$

while at LEP

$$\langle z_c(Q_2 = 91\text{GeV})\rangle = 0.487 \pm 0.011 \quad (42)$$

The numbers shown in Table 1 are all that is needed to compare the measurements from ARGUS or CLEO and LEP. In Figure 1 we present this result graphically for a slightly narrower range of Λ_5, namely $\Lambda_5 = (175 \pm 50)$ MeV, according to ref. [15]. For each value of $\langle z_c(Q_1 = 10.55\text{GeV})\rangle$ given on the horizontal axis, we plot the value of $\langle z_c(Q_2 = 91\text{GeV})\rangle$ which should be measured according to eq. (36). There are two lines for each value of Λ_5, corresponding to $\mu = Q_1/2$ and $\mu = 2Q_1$. So for a given value of α_{QCD} the prediction for $\langle z_c(Q_2 = 91\text{GeV})\rangle$ lies between these lines.

We see that from the ARGUS measurement $\langle z_c(Q_1 = 10.6\text{GeV})\rangle$ =0.640, $\langle z_c(Q_2 = 91\text{GeV})\rangle$ is predicted to be in the range $0.490 \le \langle z_c(Q_2 = 91\text{GeV})\rangle \le 0.522$. If we allow $\langle z_c(Q_1 = 10.6\text{GeV})\rangle$ to vary within 2σ around the measured value we obtain $0.477 \le \langle z_c(Q_2 = 91\text{GeV})\rangle \le 0.537$. Therefore, the results are consistent at the 2σ level but they seem to favour large values of Λ_{QCD}.

If we restrict the analysis to D^{*+} mesons, from $\langle z_{D^{*+}}(Q_1 = 10.6\text{GeV})\rangle = 0.647$ we predict $0.496 \le \langle z_{D^{*+}}(Q_2 = 91\text{GeV})\rangle \le 0.528$ ($0.487 \le \langle z_{D^{*+}}(Q_2 = 91\text{GeV})\rangle \le 0.538$ within 2σ). It is very important to relate the exclusive measurements on D^{*+} mesons at both experiments, because in this case the nonperturbative fragmentation function is exactly the same (as it does not depend on Q^2) and we are sure to be testing perturbative QCD, without invoking heavy quark symmetry at all. With sufficient statistics, this could prove to be a useful measurement of Λ_{QCD}.

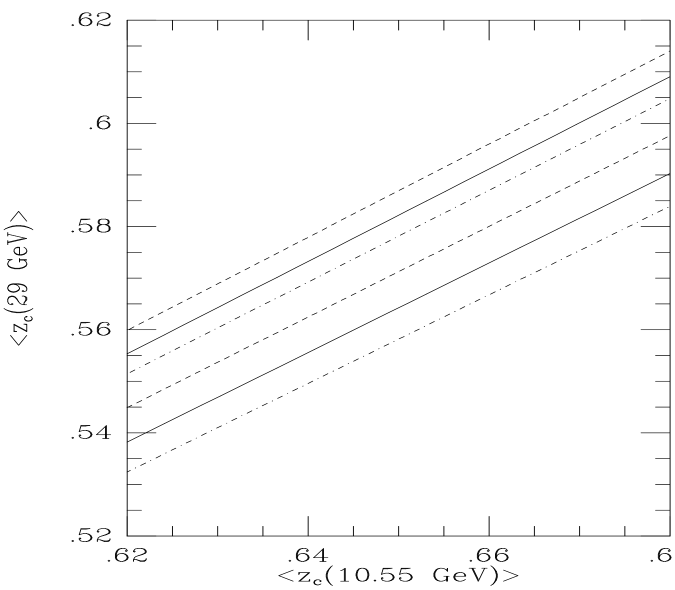

Figure 2: $\langle z_c(Q_2 = 29\text{GeV})\rangle$ as a function of $\langle z_c(Q_1 = 10.55\text{GeV})\rangle$, for the same values of Λ_5 and μ as in Figure 1.

We have also related $\langle z_c(Q_1 = 10.6\text{GeV})\rangle$ and $\langle z_c(Q_2 = 29\text{GeV})\rangle$. The result is shown in Figure 2, for the same values of Λ_5 and μ as in Figure 1. In this case we obtain $0.549 \le \langle z_c(Q_2 = 29\text{GeV})\rangle \le 0.578$ from the ARGUS measurement $\langle z_c(Q_1 = 10.6\text{GeV})\rangle$ =0.640 ($0.534 \le \langle z_c(Q_2 = 29\text{GeV})\rangle \le 0.594$ within 2σ). We also see that although the QCD perturbative calculation tends to give a too large prediction for $\langle z_c(Q_2 = 29\text{GeV})\rangle$, since the experimental error of this measurement is bigger than at LEP, the theoretical prediction lies always within 2σ of the measured value. Therefore, it is not possible to constrain further Λ_{QCD} by relating $\langle z_c(Q_1 = 10.55\text{GeV})\rangle$ and $\langle z_c(Q_2 = 29\text{GeV})\rangle$. Obviously, the same applies when we compare $\langle z_c(Q_1 = 29\text{GeV})\rangle$ and $\langle z_c(Q_2 = 91\text{GeV})\rangle$.

One can do a similar analysis for the b quark, relating $\langle z_b(Q_1 = 29\text{GeV})\rangle$ measured at PEP and $\langle z_b(Q_2 = 91\text{GeV})\rangle$ from LEP. However, the QCD perturbative evolution is in fact inconsistent with the experimental measurements [14], [13], namely

$$\langle z_b(29\text{GeV})\rangle = 0.715 \pm 0.03 \quad (43)$$

$$\langle z_b(91\text{GeV})\rangle = 0.714 \pm 0.012 \quad (44)$$

Presumably, the b measurement at 29 GeV is not reliable.

We conclude that perturbative QCD predictions are consistent with current data on c quark fragmentation function within experimental errors. However the data seems to favor large Λ_{QCD}. Figure 1 shows that a measurement of $\langle z_c\rangle$ from CLEO would be very useful in constraining further Λ_{QCD}.

The measured second moment of the b quark fragmentation function at $Q = 29\text{GeV}$ seem to be inconsistent with perturbative QCD and the LEP measurement at $Q = 91\text{GeV}$. For the c quark, the

Λ_5 (MeV)	$\mu_0 = m_c/2$	$\mu_0 = m_c$	$\mu_0 = 2m_c$
75	0.842	0.865	0.888
125	0.794	0.832	0.865
175	0.743	0.800	0.845
225	0.685	0.769	0.826
275	0.618	0.736	0.809

Table 2: $\langle z(Q,m)\rangle_{pert}$ for $Q = 10.55$ GeV, $m = 1.5$ GeV.

second moment measured at $Q = 29$GeV does not provide further bounds, since the experimental error is larger than in later experiments. Therefore, in the following section we only consider the more accurate measurements of the heavy quark fragmentation functions at center of mass energies $Q = 10.55$ GeV and $Q = 91$ GeV.

3.2 Relating $\langle z_b\rangle$ and $\langle z_c\rangle$

In this section, we incorporate QCD directions and the leading order heavy quark expansion in order to relate the measured second moment of the b and c quark fragmentation functions, the first measured at LEP and the latter measured at ARGUS, CLEO and LEP.

From Section 2, we know that for each value measured at a particular value of Q^2,

$$\langle z(Q,m)\rangle = \langle z(Q,m)\rangle_{pert}\langle z(m)\rangle_{nonpert}. \tag{45}$$

We find the first term via the Altarelli–Parisi evolution we described in Section 2, and use the measured value (actually a range of possible values) to determine the nonperturbative factor.

In Tables 2–4, we present the factor $\langle z(Q,m)\rangle_{pert}$ for $Q = 10.55$ GeV, $m = 1.5$ GeV; $Q = 91$ GeV, $m = 1.5$ GeV; and $Q = 91$ GeV, $m = 4.5$ GeV, where we have taken $\mu_0 = 2m, m, m/2$ and the values of Λ_5 shown in the first column. With these perturbative factors, we can extract $\langle z(m)\rangle_{nonpert}$ from measurements of the fragmentation function at CLEO and ARGUS and LEP (providing the subleading calculation is adequate).

The measured values at ARGUS and LEP are the following [12], [13]

$$\begin{aligned}
\langle z_c(10.6\text{GeV})\rangle &= 0.640 \pm 0.009 \\
\langle z_c(91\text{GeV})\rangle &= 0.487 \pm 0.011 \\
\langle z_b(91\text{GeV})\rangle &= 0.714 \pm 0.012
\end{aligned} \tag{46}$$

The numbers from LEP are still preliminary. Because the exact numbers are not yet known, and in order that our analysis can be applied when more exact numbers are measured, in each case we extract the parameter a over a range of values, namely

$$\begin{aligned}
0.62 \leq \quad &\langle z_c(10.55\text{GeV})\rangle \quad \leq 0.68 \\
0.46 \leq \quad &\langle z_c(91\text{GeV})\rangle \quad \leq 0.51 \\
0.69 \leq \quad &\langle z_b(91\text{GeV})\rangle \quad \leq 0.74
\end{aligned} \tag{47}$$

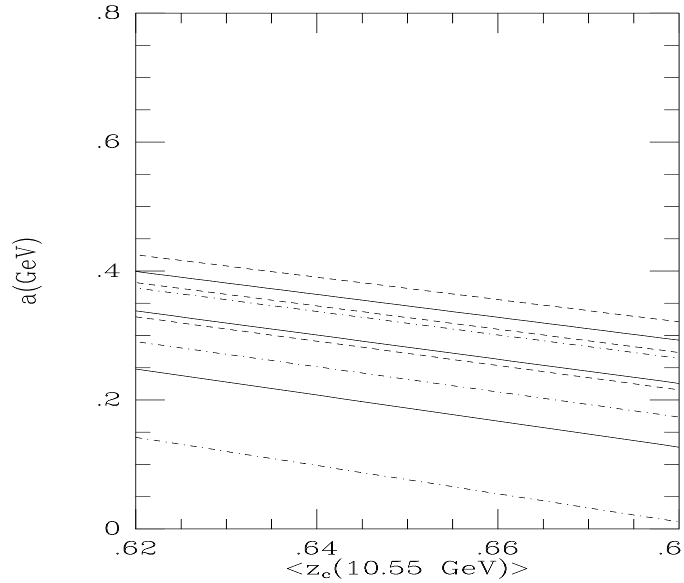

Figure 3: Non-perturbative parameter a (in GeV) as a function of $\langle z(Q,m)\rangle$ for $Q = 10.55$GeV and $m = 1.5$GeV, with $\Lambda_5 = 125$ MeV (dashed line), 175 MeV (solid), and 225 MeV (dashed-dotted). For each value of Λ_5 we have taken $\mu_0 = 2m, m, m/2$, the higher line corresponding to the larger μ_0.

Λ_5 (MeV)	$\mu_0 = m_c/2$	$\mu_0 = m_c$	$\mu_0 = 2m_c$
75	0.695	0.713	0.733
125	0.641	0.672	0.699
175	0.590	0.636	0.671
225	0.537	0.603	0.647
275	0.478	0.570	0.625

Table 3: $\langle z(Q,m)\rangle_{pert}$ for $Q = 91$ GeV, $m = 1.5$ GeV.

Using the perturbative factors from Tables 2-4, in Figures 3-5, we present (on the vertical axis) the resulting values of a (in units of GeV) which would be extracted if the value on the horizontal axis is measured. We have presented the data graphically, rather than in tabular form to illustrate the sensitivity to the exact value which is measured. The values of Q^2 and m are as in the tables.

In each figure we give the extracted parameters for $\Lambda_5 = 125$ MeV (dashed line), 175 MeV (solid line), and 225 MeV (dotted line) (the current range in the particle data book [15]). For each value of Λ_5 we present three lines, corresponding to $\mu_0 = 2m, m, m/2$. The largest values of a in each case correspond to the largest value of μ_0. This is readily understood, since there is more QCD scaling for smaller μ_0, so a smaller a is required to get a particular measured value.

By way of illustration, we give the predictions for a corresponding to $\Lambda_5 = 175$ MeV. First consider the mean value of the measurement. In Figure 3, one can see that from the ARGUS measurement $\langle z_c(10.6\text{GeV})\rangle = 0.640$ we predict 208MeV $\leq a \leq$ 365MeV . Similarly, from the LEP measurements $\langle z_c(91\text{GeV})\rangle = 0.487$ and $\langle z_b(91\text{GeV})\rangle = 0.714$ in Figures 4-5 we obtain 264MeV $\leq a \leq$ 413MeV and 425MeV $\leq a \leq$ 552MeV , respectively. Therefore we find no overlap between the a parameter

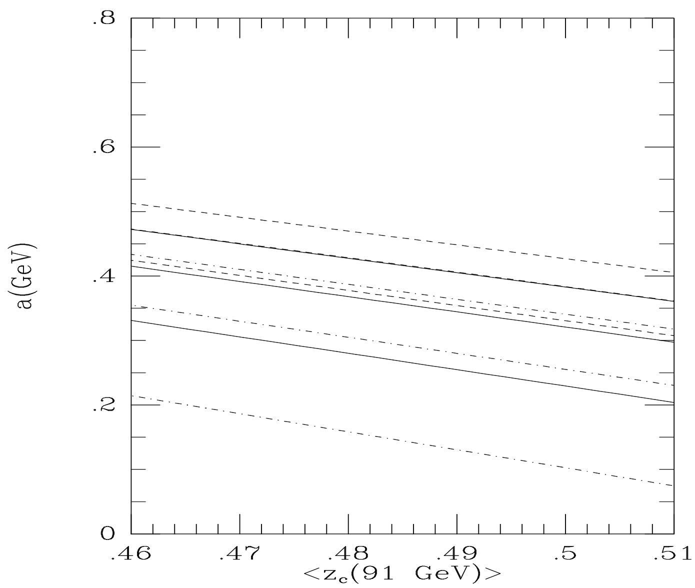

Figure 4: a (in GeV) as a function of $\langle z(Q,m)\rangle$ for $Q = 91\text{GeV}$ and $m = 1.5\text{GeV}$. The values of Λ_5 and μ_0 are as in Figure 3.

Λ_5 (MeV)	$\mu_0 = m_b/2$	$\mu_0 = m_b$	$\mu_0 = 2m_b$
75	0.829	0.836	0.846
125	0.807	0.816	0.828
175	0.789	0.800	0.814
225	0.772	0.786	0.802
275	0.757	0.773	0.791

Table 4: $\langle z(Q,m)\rangle_{pert}$ for $Q = 91$ GeV, $m = 4.5$ GeV.

determined from the measured mean values $\langle z_c\rangle$ and $\langle z_b\rangle$. For $\Lambda_5 = 125$ MeV there is no overlap either, and for $\Lambda_5 = 225$ MeV there is overlap only between the LEP measurements. However, if we allow for a 2σ variation around the $\langle z\rangle$ measured in each experiment there is substantial overlap: from the ARGUS measurement of $\langle z_c\rangle$ we predict 175MeV $\leq a \leq$ 396MeV , and from $\langle z_c\rangle$ at LEP 208MeV $\leq a \leq$ 461MeV , while from $\langle z_b\rangle$ we obtain 287MeV $\leq a \leq$ 686MeV .

There are certain qualitative features of agreement which are good to note. First, the parameter a is never larger than 800 MeV, and is very likely smaller (especially if the value is indeed in the overlap region of the c and b quark results). This is reassuring, as one could not presume to do a heavy quark expansion for the c quark for a parameter a much larger than this. Furthermore, we see there is indeed substantial overlap for each possible value of Λ_{QCD} between the allowed range of a for each experiment. Remember, according to the heavy quark effective theory, one would predict the parameter a is the same in each case up to higher order corrections, of order $(\Lambda/m)^2$, which one expects to be of order 10%. It is also clear that the allowed values of a as determined from the two $\langle z_c\rangle$ measurements are in very good agreement. Both LEP measurements have greater overlap than the ARGUS measurement has with $\langle z_b\rangle$, therefore a measurement from CLEO would be very useful.

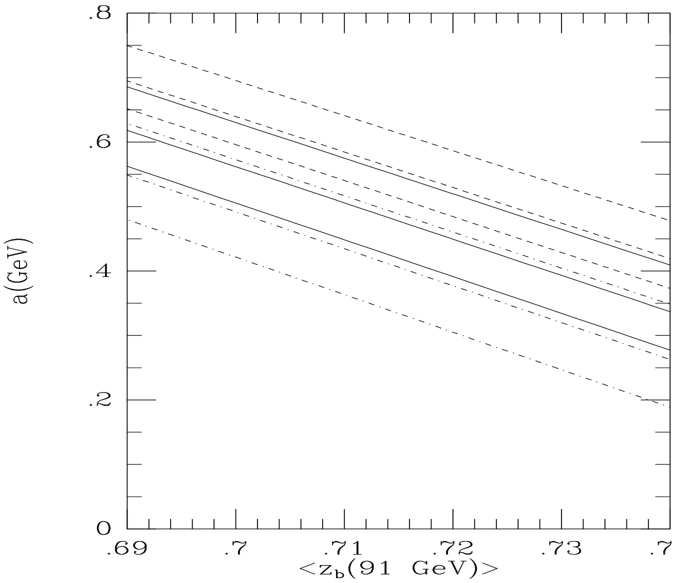

Figure 5: a (in GeV) as a function of $\langle z(Q,m)\rangle$ for $Q = 91\text{GeV}$ and $m = 4.5\text{GeV}$. The values of Λ_5 and μ_0 are as in Figure 3.

However, the value of the nonperturbative parameter a which is extracted is clearly strongly dependent both on the precise value of Λ_{QCD} and the renormalization scale μ_0, indicating the importance of subsubleading logarithmic corrections. This is because the difference between the measured value and the perturbative contribution is much more strongly μ_0–dependent (defined as the fraction of the value of the quantity) than the perturbative contribution itself. So for example, if $\langle z_b\rangle$ measured at LEP is 0.71, the parameter a for $\Lambda_5 = 125$ MeV, can vary between about 537 MeV and 637 MeV, even though the perturbative factor varies only between 0.81 and 0.83. In order to determine an effect of size Λ_{QCD}/m, the perturbative contribution needs to be known at accuracy more precise than this. Clearly, the subleading calculation is not adequate for a precise extraction of a. This problem is not endemic to the particular test of the heavy quark theory we have done. It seems likely that in order to determine Λ/m corrections, one either must take ratios in which the QCD corrections cancel, or do an even higher order calculation.

However, the parameter $\langle z\rangle_{nonpert}$ is more stable against QCD uncertainties. This is actually the quantity upon which the measured value depends. When we relate two different measured values of $\langle z\rangle$, the variation with μ_0 and Λ_{QCD} might not be so large. That is, we have not at all correlated the choice of μ_0/m for the b and c quarks.

In Figures 6-7, for the assumed values of $\langle z_c(Q)\rangle$ which are given on the horizontal axis, we determine the nonperturbative parameter a. We then use this to predict the value of $\langle z_b(91\text{GeV})\rangle$ which should be measured for the b quark (this was the procedure followed in ref. [11] but for the fragmentation function itself). There are two graphs, corresponding to $Q = 10.55$ GeV and $Q = 91$ GeV. In each case, we have allowed for $\Lambda_5 = 125, 175, 225$ MeV. For each of these, we have allowed for $\mu_0 = 2m, m, m/2$.

Notice that the current measurement looks fairly borderline in that $\langle z_b\rangle$ is lower than the favored range. However, this procedure, which is essentially that of [11], probably gives too narrow a range of predictions, since it is probably not justified to always choose exactly the same value of μ_0/m. This

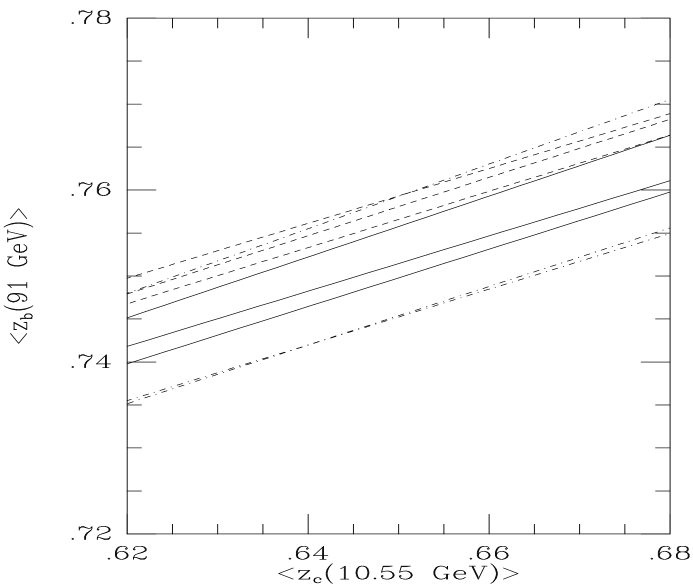

Figure 6: Prediction for $\langle z_b(91\text{GeV})\rangle$ as a function of the measured $\langle z_c(10.55\text{GeV})\rangle$, with $\mu_b/m_b = \mu_c/m_c$. $\Lambda_5 = 125$ MeV (dashed line), 175 MeV (solid), and 225 MeV (dashed-dotted).

procedure underestimates the μ variation, because the values of a one extracts are highly correlated with respect to the choice of μ_0/m. So for example, we get a similar prediction if we take $\mu_0 = m_c$ to extract a and use $\mu_0 = m_b$ to determine $\langle z_b\rangle$ as to if we had used $\mu_0 = 2m_c$ and $2m_b$ respectively.

The optimal choice of μ_0 would correspond to that for which higher order terms are negligible, so the subleading order answer, evaluated at the renormalization point μ_*, is the correct answer. One can then ask the question whether $r = \mu_*/m$ depends on the mass, m. It is straightforward to see that it is roughly independent of m if one is performing perturbation theory in α. However, for a leading log calculation, r will be formally mass independent only so long as the ratio Q/m_1 is much greater than m_1/m_2, where Q is the center of mass energy, m_1 will be the b quark mass, and m_2 will be that of the c quark (assuming common Q). In reality, one probably requires that the scaling between Q and m_1 is much greater than that between m_1 and m_2.

We therefore conservatively allow for independent μ_0 variation in the b and c quark calculations. In the case where there are two values of Q involved, we have allowed the full μ_0 variation of each. In the case where $Q = M_Z$ for both the b and c quarks, we have always taken both μ_0 greater than m or both less than m, given that the value of μ_* is somewhat correlated. This is probably a fairly conservative range. The range might be less.

In Figures 8 and 9, we give the allowed range of possible predictions for $\langle z_b\rangle$ when $\Lambda_5 = 175$ MeV, given a measurement of $\langle z_c\rangle$ at $Q = 10.55$GeV and $Q = 91$GeV respectively. The corresponding curves for $\Lambda_5 = 125$ MeV are slightly more predictive and for $\Lambda_5 = 225$ MeV are somewhat less predictive. The allowed range of predictions is greater than in Figure 6 and 7, but still somewhat predictive. For example, if we vary $\langle z_c\rangle$ within 2σ around the measured value $\langle z_c(10.55\text{GeV})\rangle = 0.640$ we obtain $0.719 \leq \langle z_b(91\text{GeV})\rangle \leq 0.782$, and similarly for $\langle z_c(91\text{GeV})\rangle = 0.487$ we get $0.718 \leq \langle z_b(91\text{GeV})\rangle \leq 0.763$ within 2σ.

Figures 8 and 9 are our main results. They tell us which measurements would be consistent with the heavy quark expansion, in light of the perturbative QCD uncertainties. We see that the range is

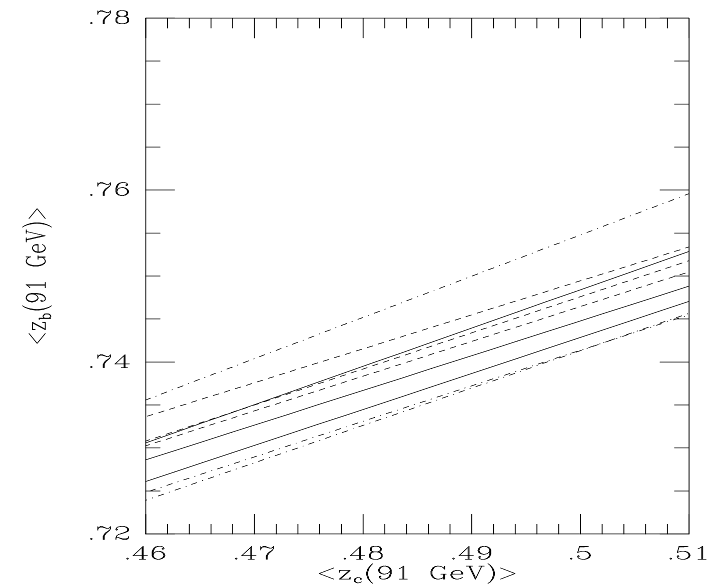

Figure 7: Prediction for $\langle z_b(91\text{GeV})\rangle$ as a function of the measured $\langle z_c(91\text{GeV})\rangle$, with $\mu_b/m_b = \mu_c/m_c$ and the same values of Λ_5 as in Figure 6.

sufficiently narrow that tests of the heavy quark expansion are possible. A measurement outside this range would be an indication of a $(\Lambda/m)^2$ contribution to $\langle z\rangle_{nonpert}$ (or less likely, an indication of nonstandard model physics in low energy QCD scaling). We see that the preliminary numbers do not yet require such higher order terms. Reducing the error may or may not leave us inside the range of predictions given in Figures 8-9.

4 Cutoff Moments

In general, when extracting the moments experimentally, the data does not go down to arbitrarily low z. It is therefore necessary to extrapolate the data into this region. This however introduces model dependence. The model to which one fits will have some effect on the number which is extracted. An uncertainty due to this modeling error can be assigned by attempting to fit to a couple of different functions; however this does not necessarily represent the true error.

In this section, we argue that it might be better to work with truncated moments. By this we mean one can define a moment as the integral from z_0 to 1, rather than from 0 to 1. Ideally, one can choose z_0 as the experimental cut, so long as z_0 is not too large.

It is however not standard to do this, because unless one integrates to zero, the equation for the evolution of the moments does not factorize. In particular, when one defines

$$D_{z_0}^N = \int_{z_0}^1 dz z^{N-1} \hat{f}(z), \tag{48}$$

and uses

$$\frac{d\hat{f}(z)}{d\log\mu^2} = \int_z^1 P(z/y)\hat{f}(y)\frac{dy}{y} \ , \tag{49}$$

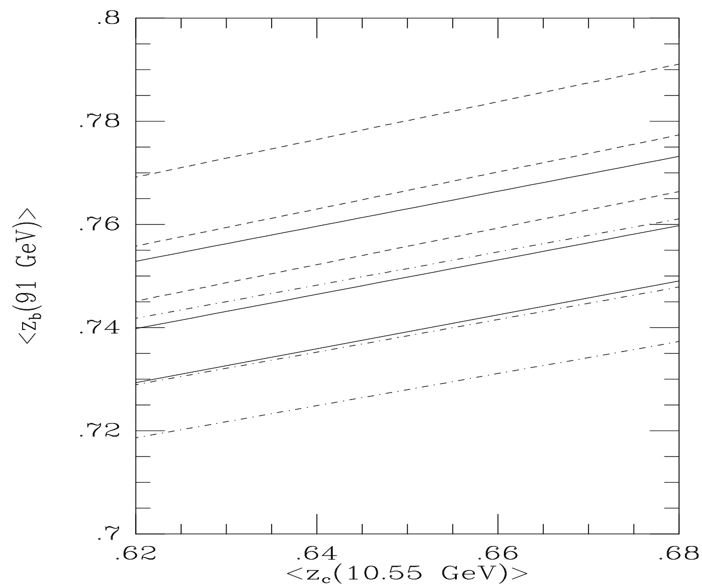

Figure 8: Prediction for $\langle z_b(91\mathrm{GeV})\rangle$ as a function of the measured $\langle z_c(10.55\mathrm{GeV})\rangle$, for $\Lambda_5 = 175$ MeV. For each value of the matching scale $\mu_c = m_c/2$ (dashed line), $\mu_c = m_c$ (solid line) and $\mu_c = 2m_c$ (dashed-dotted line) we plot the results for $\mu_b = m_b/2, m_b, 2m_b$, the higher line corresponding to the larger μ_b.

one finds

$$\frac{dD^N_{z_0}}{d\log\mu^2} = \int_{z_0}^1 dy y^{N-1}\hat{f}(y)\int_{z_0/y}^1 dz z^{N-1}P(z) \tag{50}$$

In the case $z_0 = 0$ we see this equation factorizes into the product of moments. However, because of the y dependence of the endpoint on the z integration, this is not the case for general z_0. The problem here is to evaluate the right hand side, you would actually need to know $\hat{f}(y)$, which is precisely what we wish to avoid. What we would like is an expression solely in terms of moments, so we can follow the same procedure described in section 2.

In this section we will focus again on the second moment and we will show that if we make the approximation that the evolution equation *does* factorize, by taking the lower limit on the z integration to be z_0, the error we make is actually only of order z_0^2 which is generally rather small.

So we consider the difference

$$\int_{z_0}^1 dz z P(z) - \int_{z_0/y}^1 dz z P(z) = \int_{z_0}^{z_0/y} dz z P(z) \tag{51}$$

between the function which truly appears in the evolution equation and the approximation one obtains by taking the lower endpoint of the integral to be z_0. We consider only the leading log anomalous dimension, as the error from the subleading piece is suppressed by α_{QCD} and should be a small correction compared to the error from the leading term.

As discussed in section 2, it is sufficient to consider only nonsinglet evolution which involves the function

$$P(z) \propto \left(\frac{1+z^2}{1-z}\right)_+ \tag{52}$$

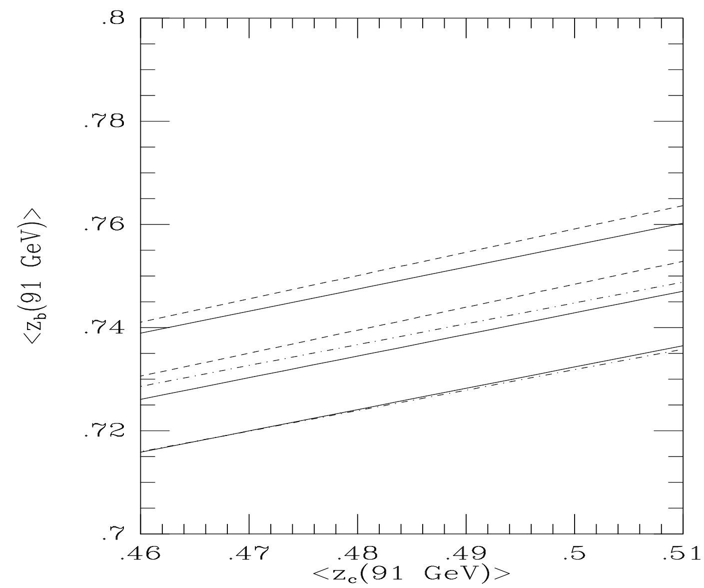

Figure 9: Prediction for $\langle z_b(91\mathrm{GeV})\rangle$ as a function of the measured $\langle z_c(91\mathrm{GeV})\rangle$, for $\Lambda_5 = 175$ MeV. The values of μ_c and μ_b are as in Figure 8, although given that they are somewhat correlated we omit in this figure the combinations $\mu_c = m_c/2, \mu_b = 2m_b$ and $\mu_c = 2m_c, \mu_b = m_b/2$.

It should be kept in mind that the "+" keeps the function properly normalized. However this should be true independent of the arbitrary choice of z_0. We therefore take the coefficient of the δ function which needs to be subtracted to be determined from integrating over all z from 0 to 1. (The fact that this assumption leads to a smaller error is good confirmation of the expectation that this will better approximate the true function). So the error in our approximation is determined from the following difference

$$\int_{z_0}^{z_0/y} dz z \frac{1+z^2}{1-z} \tag{53}$$

which is

$$\frac{3z_0^2}{2}((\frac{1}{y})^2 - 1) + \ldots \tag{54}$$

where the ... represents higher order terms in z_0 and we have expanded the logarithm which is obtained upon integration. The error is only of order z_0^2. Really, it is of order $(z_0/y)^2$. However, so long as the distribution is weighted towards unity, as it is for a the fragmentation function of a heavy quark evaluated at a sufficiently low momentum scale, our statement is approximately correct.

To check that we do not make too large an error when applying this approximation to a heavy quark fragmentation function, we explicitly evaluated the error for different assumed functional forms which were claimed to fit data reasonably well in the past. Notice that we are using the function only to estimate the error, but not to fit the data. Furthermore, the error estimate only involves those values of z which are measured–one never needs the values of the function below z_0. So ultimately, it is possible to estimate the error based on a function which fits the data, without extrapolation.

The test functions we used were a Peterson function with a large range of ϵ and also a function of the form from ref. [11], namely of the form $z^\alpha(1-z)^\beta$. The resulting differences (in percent) between the exact and approximate evolution of the second moment are given in Tables 5 (c quark) and 6 (b

z_0	0.05	0.1	0.15	0.2
$\alpha = 0.35, \ \beta = 3.05$	0.18	0.9	2.0	3.6
$\epsilon_c = 0.14$	0.4	1.7	3.7	6.6
$\epsilon_c = 0.44$	0.7	2.8	6.0	10.

Table 5: Relative difference (in %) between the exact and approximate evolution of the second moment for the c quark.

z_0	0.05	0.1	0.15	0.2
$\alpha = 0.595, \ \beta = 18.67$		0.15	0.3	0.6
$\epsilon_b = 0.016$	0.05	0.18	0.7	1.3
$\epsilon_b = 0.049$	0.1	0.4	1.0	2.0

Table 6: Relative difference (in %) between the exact and approximate evolution of the second moment for the b quark.

quark) for a range of cutoff values, z_0. We see that in no case does the error exceed 10% for the c quark and 2% for the b quark, and in most of the cases it is considerably smaller. These particular numbers correspond to $Q = M_Z$, and we have taken $\Lambda_5 = 200\text{MeV}$ and $\mu_0 = m$, but varying them in the ranges we have allowed for in section 2 does not change our conclusions. It is important that the error is calculated entirely based on the region above z_0, so no assumption is made about extrapolating to the unmeasured region. It can evaluated in a fairly model independent manner.

It is clear that the error is always within the regime of accuracy of our predictions (ie smaller than the uncertainty from higher order QCD effects). Therefore, it is as useful to determine the cutoff moments as it is to determine the true ones, and probably has a smaller systematic error. We suggest the data be presented in the future in this way when there is a cut on z_0 (or as a function of cut values for comparison with other experiments). As we have seen, the error when using our procedure to extract meaningful numbers should be quite small, of order a few percent.

5 Conclusions

In this paper, we have used the measured values of the heavy quark fragmentation second moments to test the heavy quark effective theory. By relating c and b quark measurements we have extracted a mass suppressed heavy quark parameter. We have also related measurements of c quark fragmentation using purely perturbative QCD. The graphs of Figures 1, 8, and 9 are our main results.

From the purely perturbative analysis, we have seen that the measurements of c quark fragmentation at different center of mass energies are consistent with QCD predictions, and tend to favor large values of Λ_{QCD}. This part of the analysis does not involve heavy quark symmetry and could prove an alternative way of determining the strong coupling.

We have found the value of the nonperturbative parameter a as measured with the c quark lies between 79MeV and 422MeV (assuming it is in the overlap of the two measurements) and in the range

199MeV to 750MeV for the b quark. A better determination of Λ_{QCD} would narrow the allowed range of a sizably. For instance, we have obtained that if $\Lambda_5 = 175$ MeV a lies between 208MeV and 396MeV for the c quark and in the range 287MeV to 686MeV for the b quark. Although the range is substantial, it is of interest to have any measurement at the level of Λ/m. Furthermore, it is encouraging that these values are reasonably small and overlap. With better measurement, one might hope to search for deviations at the level of $(\Lambda/m)^2$. However, given the large perturbative QCD uncertainty, it is not yet clear whether this will be possible.

Finally, we have suggested to use "cutoff" moments, which do not involve extrapolation of the experimental data to low z, and therefore will probably have smaller systematic error. We have shown that although factorization is not exact, it is a good approximation within the regime of accuracy of our calculation.

There are several measurements which it would be nice to see done or improved. One would want CLEO to measure the ratio of D to D^* production to test for the importance of higher twist effects. One would also want to take advantage of the many c quark states at CLEO to get a better low energy measurement of the fragmentation function. A measurement of the production of $cc\overline{c}\overline{c}$ at LEP would be a useful confirmation of the claim that multi c quark production will not distort our predictions. A good exclusive measurement on D^*'s at LEP would be very useful when using fragmentation functions to extract Λ_{QCD}.

It would certainly be advantageous to improve the measurements of heavy quark fragmentation functions. Heavy quark fragmentation could prove to be an interesting test of the heavy quark mass expansion and a supplement to existing measurements of α_{QCD}.

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