Differential Regularization of a Non-relativistic Anyon Model *

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ABSTRACT

Differential regularization is applied to a field theory of a non-relativistic charged boson field ϕ with $\lambda(\phi^*\phi)^2$ self-interaction and coupling to a statistics-changing $U(1)$ Chern-Simons gauge field. Renormalized configuration-space amplitudes for all diagrams contributing to the $\phi^*\phi^*\phi\phi$ 4-point function, which is the only primitively divergent Green's function, are obtained up to 3-loop order. The renormalization group equations are explicitly checked, and the scheme dependence of the β-function is investigated. If the renormalization scheme is fixed to agree with a previous 1-loop calculation, the 2- and 3-loop contributions to $\beta(\lambda, e)$ vanish, and $\beta(\lambda, e)$ itself vanishes when the "self-dual" condition relating λ to the gauge coupling e is imposed.

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I. INTRODUCTION

The differential renormalization procedure [1] is a novel method for perturbative calculations in quantum field theory in which real space correlation functions are simultaneously regularized and renormalized. Neither explicit cutoff nor singular counter terms are required, yet no ultraviolet divergences appear. The method has now been applied to many relativistic field theories, and a systematic formulation [2], obtained with the use of BPHZ topological analysis of Feynman diagrams, has recently been given.

Field theories with non-relativistic kinematics are also of physical interest, and we report in this paper on the application of differential regularization to a theory in two spatial dimensions which has been used to describe the anyon Aharonov-Bohm and fractional quantum Hall effects. The particular theory [3] we study has evolved from earlier work [4,5]. It describes a non-relativistic boson field with quartic self-interaction and coupling to a 2+1 dimensional abelian gauge field with Chern-Simons kinematics.

It was shown in [3] that the classical Lagrangian of the theory is scale invariant, but one expects this symmetry to be broken by renormalization effects at the quantum level. These scaling properties are similar to those of renormalizable massless relativistic theories in 3+1 dimensions, so the theory is a natural place to apply a new regularization method. The theory is also chiral so that dimensional regularization is problematic.

The renormalization properties of the theory have been studied at the 1-loop level [6,7] using a momentum cutoff to handle ultraviolet divergences. Scale symmetry breaking is found for general values of the coupling constants, but scale invariance is restored when the couplings are related by the "self-duality" condition which has special significance in the classical theory [3].

In this work we use the differential regularization method to renormalize the divergent Green's functions of the theory through 3-loop order. This method is based on the observation that bare amplitudes which are primitively divergent in momentum space are well defined for separated points in real $\vec{x}, t$ space, although too singular at short distance to possess a Fourier transform. These amplitudes are then renormalized by the following two-step procedure:

(1) Use differential identities (typically containing a new mass scale parameter M) to express singular bare amplitudes as derivatives of less singular Fourier transformable functions.

(2) Define the renormalized Fourier transform and other integrals of the originally singular amplitudes by formal partial integration of derivatives.

Multi-loop amplitudes are then handled by inserting the differential identities for their primitively divergent subgraphs, manipulating differential operators according to step (2) and using new differential identities as needed. The heuristic reason that this procedure is consistent is that the ultraviolet singular surface terms which are neglected in rule (2) correspond to singular counter terms which occur in traditional regularization methods. The operational test of consistency is that the amplitudes obtained obey renormalization group equations (RGE's) in the scale parameter M, with β-functions and anomalous dimension defined through the equations themselves.

New features occur when these ideas are implemented in a non-relativistic field theory. Because of the lack of symmetry between space and time coordinates, differential identities of rather different structure from those for relativistic theories must be found. Due to the combined non-relativistic and Chern-Simons kinematics the gauge coupling is not renormalized, so that there is only a β-function for the quartic coupling. Nevertheless this β-funtion is scheme-dependent beyond one-loop order, as expected for a theory with two or more couplings. In differential renormalization, scheme dependence appears because of the freedom to introduce different mass scales M, M' in some of the differential identities used to regulate the overall divergence of graphs which are not related by symmetry. If this occurs in a given order of perturbation theory the β-function typically depends on $\log M/M'$ in the next order and beyond. Our calculations show that there are two scale parameters which appear at the one-loop level and one more at the three-loop level. The dependence of the two- and three-loop β-function on the ratio of the one-loop scale parameters is obtained. If the mass ratio is fixed to make our one-loop result agree with [6,7], then the two- and three-loop β-function vanishes. The remaining one-loop contribution then vanishes when the two couplings satisfy the "self-duality" condition of [3]. Thus previous results on the relation of "self-duality" to scale invariance in the quantum theory have now been extended through three-loop order.

Our approach to this problem has been purely field-theoretic; we have simply obtained renormalized real space Green's functions which obey RGE's. We have not compared results with closely related quantum mechanical studies of anyons [8], although it appears that some difficulties of perturbative quantum mechanical treatments can be resolved using field-theoretic methods. Finally, we can only express the hope that our work can be applied to elucidate physical properties of anyon systems.[1]

The Lagrangian of the theory under study is

$$L = \phi^* \left(iD_t + \frac{1}{2m}\mathbf{D}^2 \right) \phi + \alpha \epsilon_{ij} \left(\frac{1}{2}\partial_t A_i A_j - A_0 \partial_i A_j \right) - \lambda(\phi^*\phi)^2 \tag{1.1}$$

with covariant derivatives

$$\begin{aligned} D_t &= \partial_t + ieA_0 \\ \mathbf{D} &= \nabla - ie\mathbf{A} \end{aligned} \tag{1.2}$$

In the Coulomb gauge the only non-vanishing components of the gauge field propagator take the instantaneous form [2]

$$\begin{aligned} \langle A_i(x) A_0(0) \rangle &= iT_i(x) \\ T_i(x) &= \frac{\delta(t)}{2\pi\alpha}\, \epsilon_{ij}\, \frac{\mathbf{x}_j}{\mathbf{x}^2} \end{aligned} \tag{1.3}$$

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[1]During the writing phase of this work, references [9] came to our attention through the bulletin board. Three-loop calculations of the finite V, T partition function of the theory at the self-dual point are presented in these papers.

[2]We use the notation $x = (\mathbf{x}, t)$.

as required by the physical role of A_μ as a statistical gauge field without propagating degrees of freedom. The Schrödinger propagator is

$$\begin{aligned} \langle T\phi(x)\phi^*(0)\rangle &= iG(x) \\ G(x) &= -\frac{m}{2\pi t}\theta(t)e^{\frac{im\mathbf{x}^2}{2t}} \end{aligned} \tag{1.4}$$

It satisfies

$$\left(i\partial_t + \frac{\nabla^2}{2m}\right)G(x) = \delta(t)\delta(\mathbf{x}) \tag{1.5}$$

$$\lim_{t\to O^+} G(x) = -i\delta(\mathbf{x}). \tag{1.6}$$

Perturbation calculations may be performed [7] with the propagators (1.3), (1.4) and the interaction vertices from (1.1). The superficial degree of divergence for the different Green functions can be calculated by taking into account that the scale dimension of time is twice that of the space coordinate.

Many Feynman graphs vanish due to the special kinematics of the theory. There is no particle production so the number of ϕ-field lines is conserved in intermediate states of any graph. Many graphs with internal gauge field lines vanish when one adds the contributions where attachments of A_0 and A_i are exchanged. These features of the theory imply that there are no quantum loop contributions to the propagators (1.3) and (1.4), the $\phi^*\phi A$ vertex and the $\phi^*\phi AA$ "Compton" amplitude. This means that the only superficially divergent Green's function is the $\phi^*\phi^*\phi\phi$ 4-point function, and there are no anomalous dimensions in the theory. From the 4-point amplitude one can construct 6-, 8-, 10-point functions, etc., using a skeleton expansion without encountering further ultraviolet divergences.

In Section II below we confine our attention to the scalar subtheory of (1.1). We discuss the basic differential identity for the 1-loop scalar bubble graph, and its Fourier transform. The convolution theorem can then be used to compute and sum multiple bubbles and obtain the full off-shell scattering amplitude. In Section III, the regularization of the 1-loop graphs of the full theory is obtained. Two- and three-loop contributions are considered in Sections IV and V, respectively. The verification of the RGE's and the β-function are discussed in Section VI.

II. SCALAR SECTOR

In this section we will study the scalar sector of the theory defined by the Lagrangian (1.1). The tree level contribution to the amputated 4-point function is

$$\Gamma^a(x_i) = -4i\lambda\delta(x_1 - x_2)\delta(x_1 - x_3)\delta(x_1 - x_4) \tag{2.1}$$

where we have used the simplified notation $\delta(x) \equiv \delta(t)\delta(\mathbf{x})$.

In a non-relativistic theory, the only non-vanishing one-loop diagram is the direct channel bubble graph depicted in Fig. 1c. The bare amplitude in configuration space is given by

$$\Gamma^c(x_i) = 8\lambda^2\delta(x_1 - x_2)\delta(x_3 - x_4)G^2(x_3 - x_1) \tag{2.2}$$

where $G(x_3 - x_1)$ is the scalar propagator given by eq. (1.4). This amplitude is well-defined for separated points, but the factor $1/t^2$ is too singular at short distances, yielding a logarithmically divergent Fourier transform.

The bare amplitude can be regulated by the following differential identity

$$\begin{aligned} G^2_{reg}(x) &= -\frac{im^2}{4\pi^2}\left(i\partial_t + \frac{\nabla^2}{4m}\right)F(x) \\ F(x) &= \log iM^2 t \frac{\theta(t)}{t} e^{\frac{im\mathbf{x}^2}{t}} \end{aligned} \tag{2.3}$$

which is an exact relation away from the origin constructed to incorporate the ideas discussed in the introduction.

i) The differential operator acts on the function $F(x)$, which is less singular than $G^2(x)$ and has a well-defined Fourier transform. Using Gaussian integration it is straightforward to obtain

$$\hat{F}(\omega, \mathbf{k}) = -\frac{\pi}{m}\frac{\log(\mathbf{k}^2/4m - \omega - i\epsilon)\gamma/M^2]}{\mathbf{k}^2/4m - \omega - i\epsilon}, \tag{2.4}$$

where $\gamma = 1.781\ldots$ is Euler's constant.

ii) The coincident point singularity of (2.2) is defined as in distribution theory using formal partial integration of derivatives in any integral. For the Fourier transform, this prescription gives

$$\hat{\Gamma}^c(\omega, \mathbf{k}) = -\frac{2im\lambda^2}{\pi}\log\left[\left(\frac{\mathbf{k}^2}{4m} - \omega - i\epsilon\right)\frac{\gamma}{M^2}\right] \tag{2.5}$$

iii) Most important is the fact that the parameter M, which is required for dimensional reasons in (2.3), plays the role of the renormalization group scale parameter. To see this we can compute the effect of a variation of M

$$\begin{aligned} M\frac{\partial}{\partial M}G^2_{reg}(x) &= -\frac{im^2}{2\pi^2}\left(i\partial_t + \frac{\nabla^2}{4m}\right)\frac{\theta(t)}{t}e^{\frac{im\mathbf{x}^2}{t}} \\ &= \frac{im}{2\pi}\delta(x) \end{aligned} \tag{2.6}$$

where we have used (1.6) and the fact that the $\log iM^2t$ factor in (2.3) multiplies a Schrödinger propagator with mass $2m$. Comparing (2.1, 2.2, 2.6), we see that the effect of a change in M can be compensated by a change in the coupling constant. Indeed from the renormalization group equation

$$M\frac{\partial}{\partial M}\Gamma_{(s)}(x_i) = -\beta(\lambda)\frac{\partial}{\partial\lambda}\Gamma_{(s)}(x_i) \tag{2.7}$$

with $\Gamma_{(s)}$ taken as the sum of the tree and renormalized bubble amplitudes one finds the one-loop β-function

$$\beta(\lambda) = \frac{m\lambda^2}{\pi} \tag{2.8}$$

iv) The factor of i in $\log iM^2t$ is required to obtain a real analytic amplitude in momentum space with non-zero imaginary part for $\omega > \mathbf{k}^2/4m$ as required by unitarity. Alternatively, as in the relativistic case [1], one could start in the imaginary time formalism, $t = -i\tau$, in which all bare amplitudes and the differential identities used to regulate them are real functions of x and τ. Then $\log iM^2t$ is automatically obtained for real Schrödinger time.

In the scalar theory it is easy to sum multi-bubble graphs and obtain the full scattering amplitude. These graphs are convolutions in real-space and the renormalized amplitudes in p-space can be defined as products of the renormalized 1-bubble amplitude (2.5). The scattering amplitude is then

$$A(\omega, \mathbf{k}) = -4i\lambda \sum_{n=0}^{\infty} \left(\frac{i}{4\lambda} \hat{\Gamma}^c(\omega, \mathbf{k}) \right)^n . \tag{2.9}$$

In the c.m. frame, $\mathbf{k} = 0$, one obtains

$$A(\omega, 0) = -\frac{4i\lambda}{1 - \frac{m\lambda}{2\pi} \log \frac{-(\omega + i\epsilon)\gamma}{M^2}} . \tag{2.10}$$

The above remarks are intended to show how the differential renormalization procedure reproduces previous results on the scalar sector [10,11], obtained in most cases with a momentum cutoff and renormalization at a momentum scale $\mathbf{p}^2 = \mu^2$ (here $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$, where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the momenta of the incident particles). Since the on-shell c.m. total energy is given by $\omega = \mathbf{p}^2/m$, our renormalized amplitude can be brought into full agreement with previous work if the scale parameters are related by $\mu^2 = mM^2/\gamma$.

III. ONE LOOP ORDER

In this section we shall illustrate how differential regularization works for the full theory to one-loop order. The tree graph of Fig. 1b does not require regularization, but for completeness we give its x-space amplitude,

$$\Gamma^b(x_i) = \frac{e^2}{m} \delta(x_1 - x_3) T_i(x_1 - x_2) \frac{\partial}{\partial \mathbf{x}_4^i} \delta(x_4 - x_2) \tag{3.1}$$

The one-loop graphs are shown in Fig. 1c-1f. In addition to the diagrams explicitly drawn one must add, as appropriate in each case, i)the exchange of A_0 and A_i vertices, ii)the opposite orientation of each triangle with A_0, A_0, A_i^2 vertices, iii)the time-reflected graph with initial and final states interchanged, iv)exchange of x_3 and x_4 as required by Bose symmetry. It would be tedious to discuss these permutations explicitly, but it is important to include them when the RGE's are discussed in section VI.

Power counting leads us to expect that all one-loop graphs for the 4-point function are ultraviolet divergent. However, as we shall show, the only 1-loop graph which actually requires renormalization (apart from the bubble already discussed in section II) is the seagull graph shown in Fig. 1d. We therefore begin with this graph whose bare amplitude is given by the following expression,

$$\Gamma^d(x_i) = \frac{e^4}{m}\delta(x_2 - x_4)G(x_3 - x_1)T_i(x_2 - x_1)T_i(x_2 - x_3). \tag{3.2}$$

Note that in this expression there is a factor

$$\theta(t_3 - t_1)\delta(t_3 - t_1) \tag{3.3}$$

where the θ and δ function come from the boson and gauge field propagator respectively. We then use that,

$$\theta(t)\delta(t) = \frac{1}{2}\delta(t) \tag{3.4}$$

a property that can be deduced from the convolution theorem. Using this together with the value at $t = 0$ of the boson propagator (1.6), we obtain,

$$\Gamma^d(x_i) = -\frac{ie^4}{8m\pi^2\alpha^2}\delta(x_2 - x_4)\delta(x_1 - x_3)\delta(t_3 - t_2)\frac{1}{(\mathbf{x}_2 - \mathbf{x}_1)^2} \tag{3.5}$$

Notice that as the boson propagator at $t = 0$ is proportional to a delta function, the triangle "collapses" to an effective gauge field bubble. The function $\frac{1}{(\mathbf{x}_1-\mathbf{x}_2)^2}$ is too singular at short distances and as a result, its Fourier transform is logarithmically divergent. In the spirit of differential regularization, we use the following identity, valid at all points except $\mathbf{x} = 0$,

$$\frac{1}{\mathbf{x}^2} = \frac{1}{8}\nabla^2 \log^2 M_s^2\mathbf{x}^2 \tag{3.6}$$

Note that we have introduced a different mass scale from that in the identity (2.3) associated with the bubble graph. A priori, one can use independent scales for different graphs which are not related by a Ward identity. Different choices of M's correspond to different renormalization schemes and in section VI we shall see how this affects the β-function.

After using equation (3.6), we obtain the regularized form

$$\Gamma^d_{reg}(x_i) = -\frac{ie^4}{64m\pi^2\alpha^2}\delta(x_2 - x_4)\delta(x_1 - x_3)\delta(t_3 - t_2)\nabla_1^2 \log^2 M_s^2(\mathbf{x}_1 - \mathbf{x}_2)^2 \tag{3.7}$$

which has now a well defined Fourier transform, since it can be shown using the same technique as in the Appendix of [1] that

$$\int d^2xe^{i\mathbf{px}} \log^2 M^2\mathbf{x}^2 = -\frac{8\pi}{\mathbf{p}^2} \log \frac{4M^2}{\gamma^2\mathbf{p}^2} \tag{3.8}$$

From this expression we can calculate the contribution of this graph to the 4-point function in momentum space. After taking into account the graphs obtained by permutation, we have in the center of mass frame,

$$\hat{\Gamma}^D(\mathbf{p}) = -\frac{ie^4}{2\pi\alpha^2}\log\left(\frac{\tilde{M}_s^2}{2\mathbf{p}^2|\sin\theta|}\right) \tag{3.9}$$

which coincides with the result in ref. [7] if the identification $\mu = \tilde{M}_s = \frac{2M_s}{\gamma}$ is made.

Comparing (3.9) and the momentum space amplitude of the bubble graph (2.5) in the c.m. frame we also see that the renormalization scheme adopted in [7] corresponds to the following relation between scales

$$M_s^2 = \rho_1 M^2 = \frac{m\gamma}{4}M^2 \tag{3.10}$$

We end this section by showing that the remaining one-loop graphs do not require renormalization. It can be easily shown that contribution of the triangle graph shown in Fig. 1e cancels against the one coming from the graph with the $< A_0 A_i >$ propagator running in the opposite direction. Of course, any higher-loop diagram that contains this one as subdiagram will automatically vanish, therefore they are not included in Fig. 1 and we shall omit them from our discussion.

We shall show next that contributions of the box graphs are finite. Let us consider as an example the diagram shown in Fig. 1f, which is given by

$$\Gamma^f(x_i) = -\frac{e^4}{m^2}G(x_3-x_1)\left(\frac{\partial}{\partial\mathbf{x}_2^i}\frac{\partial}{\partial\mathbf{x}_4^j}G(x_4-x_2)\right)T_i(x_2-x_1)T_j(x_4-x_3) \tag{3.11}$$

Because the gauge field propagator is transverse,

$$\nabla_{\mathbf{x}}\cdot\mathbf{T}(x-y) = 0 \tag{3.12}$$

one can immediately extract the derivatives as total derivatives and rewrite the amplitude as,

$$\Gamma^f(x_i) = -\frac{e^4}{m^2}\frac{\partial}{\partial\mathbf{x}_2^i}\frac{\partial}{\partial\mathbf{x}_4^j}[G(x_3-x_1)G(x_4-x_2)T_i(x_2-x_1)T_j(x_4-x_3)] \tag{3.13}$$

The Fourier transform of this amplitude can be expressed now as a finite loop momentum integral and the total derivatives are simply factors of external momenta. The other box graphs with $A_o \leftrightarrow A_i$ permutations can be treated similarly.

IV. TWO LOOP ORDER

At two-loop order there are two divergent graphs: the double bubble shown in Fig. 1g and the "ice cream" graph shown in Fig. 1h.

A regularized expression for the double bubble is obtained immediately using the regulated expression (2.3) for each bubble subgraph,

$$\Gamma^g_{reg}(x_i) = 16i\lambda^3\delta(x_1-x_2)\delta(x_3-x_4)\int d^3u G^2_{reg}(x_3-u)G^2_{reg}(u-x_1) \tag{4.1}$$

This integral can be explicitly computed and we obtain

$$\Gamma^g_{reg}(x_i) = i\left(\frac{m\lambda}{\pi}\right)^3 \delta(x_1-x_2)\delta(x_3-x_4)\left(i\partial_{t_3}+\frac{\nabla_3^2}{4m}\right)S(x_3-x_1) \tag{4.2}$$

where

$$S(x) = \frac{\theta(t)}{t}e^{\frac{im\mathbf{x}^2}{t}}\left(\log^2 iM^2t - \frac{\pi^2}{6}\right) \tag{4.3}$$

Now, let us turn to the regularization of the ice cream graph. The bare amplitude is given by

$$\Gamma^h(x_i) = \frac{e^4\lambda}{2m\pi^2\alpha^2}\delta(x_1-x_2)\delta(t_3-t_4)G(x_3-x_2)G(x_4-x_1)\frac{1}{(\mathbf{x}_3-\mathbf{x}_4)^2} \tag{4.4}$$

For simplicity, let us set $x_1 = 0$ and introduce coordinates $\mathbf{x}$, $\mathbf{y}$ and t such that $x_3 = (\mathbf{x}, t)$ and $x_4 = (\mathbf{y}, t)$. The singular part of the graph is then contained in

$$f^h(\mathbf{x},\mathbf{y},t) = G(\mathbf{x},t)G(\mathbf{y},t)\frac{1}{(\mathbf{x}-\mathbf{y})^2}. \tag{4.5}$$

We begin by regularizing the divergent seagull subgraph (see Eq. (3.6))

$$f^h(\mathbf{x},\mathbf{y},t) \to f^h(\mathbf{x},\mathbf{y},t) = \frac{1}{32}G(\mathbf{x},t)G(\mathbf{y},t)(\nabla_x-\nabla_y)^2\log^2 M_s^2(\mathbf{x}-\mathbf{y})^2 \tag{4.6}$$

After manipulating derivatives, this expression can be rewritten as

$$\begin{aligned} f^h(\mathbf{x},\mathbf{y},t) = \frac{1}{32}\Big\{ & -(\vec{\nabla}_{\mathbf{x}}-\vec{\nabla}_{\mathbf{y}})\left[\log^2(M_s^2(\mathbf{x}-\mathbf{y})^2)(\overleftrightarrow{\nabla}_{\mathbf{x}}-\overleftrightarrow{\nabla}_{\mathbf{y}})G(\mathbf{x},t)G(\mathbf{y},t)\right] \\ & +\log^2 M_s^2(\mathbf{x}-\mathbf{y})^2(\vec{\nabla}_x-\vec{\nabla}_y)^2G(\mathbf{x},t)G(\mathbf{y},t)\Big\} \end{aligned} \tag{4.7}$$

The first term in this expression, being a total divergence, is finite by power counting. In order to improve the second term, note that the product of two non-relativistic propagators satisfies the identity

$$(\vec{\nabla}_x-\vec{\nabla}_y)^2G(\mathbf{x},t)G(\mathbf{y},t) = -\left[(\vec{\nabla}_x+\vec{\nabla}_y)^2+4im\partial_t\right]G(\mathbf{x},t)G(\mathbf{y},t) - 4im\delta(\mathbf{x})\delta(\mathbf{y})\delta(t) \tag{4.8}$$

To regularize (4.7) we drop the last term in this identity, because it simply gives a singular surface term in the Fourier transform of f^h. In this way, we obtain the regularized expression

$$\begin{aligned} f^h_{reg}(\mathbf{x},\mathbf{y},t) = -\frac{1}{32}\Big\{ & (\vec{\nabla}_{\mathbf{x}}-\vec{\nabla}_{\mathbf{y}})\left[\log^2 M_s^2(\mathbf{x}-\mathbf{y})^2(\overleftrightarrow{\nabla}_{\mathbf{x}}-\overleftrightarrow{\nabla}_{\mathbf{y}})G(\mathbf{x},t)G(\mathbf{y},t)\right] \\ & +\left[(\vec{\nabla}_{\mathbf{x}}+\vec{\nabla}_{\mathbf{y}})^2+4im\partial_t\right]\log^2 M_s^2(\mathbf{x}-\mathbf{y})^2G(\mathbf{x},t)G(\mathbf{y},t)\Big\} \end{aligned} \tag{4.9}$$

This can be inserted in (4.4), restoring the original space-time points, to obtain a regularized form for the amplitude $\Gamma^h(x_i)$.

We shall show next that the rest of the two-loop graphs in Fig. 1 are indeed finite. The bare amplitude for diagram i is

$$\Gamma^i(x_i) = -\frac{ie^6}{24\pi^2\alpha^2}\left[\frac{\partial}{\partial \mathbf{x}_2^i}G(x_4-x_2)\right]G(x_3-x_1)T_i(x_2-x_1)\frac{1}{(\mathbf{x}_4-\mathbf{x}_3)^2} \tag{4.10}$$

We first regularize the subdivergence $\mathbf{x}_3 \to \mathbf{x}_4$ by using the identity (3.6)

$$\Gamma^i(x_i) = -\frac{ie^6}{24\pi^2\alpha^2}\left[\frac{\partial}{\partial \mathbf{x}_2^i}G(x_4-x_2)\right]G(x_3-x_1)T_i(x_2-x_1)\frac{1}{8}\nabla^2\log^2 M_s^2(\mathbf{x}_4-\mathbf{x}_3)^2 \tag{4.11}$$

Using now the property (3.12), the overall divergence can be regularized by writing the diagram as total derivative,

$$\Gamma^i(x_i) = -\frac{ie^6}{24\pi^2\alpha^2}\frac{\partial}{\partial \mathbf{x}_2^i}\left[G(x_4-x_2)G(x_3-x_1)T_i(x_2-x_1)\frac{1}{8}\nabla^2\log^2 M_s^2(\mathbf{x}_4-\mathbf{x}_3)^2\right] \tag{4.12}$$

Diagram i is now regularized. Nevertheless, let us point the following fact. Though this diagram depends on M_s, the sum of this diagram plus the one where the $< A_0 A_i >$ line runs in the opposite direction *does not*. This can be understood in the following way: as easily seen from (4.12), the scale derivative of Γ^i is proportional to Γ^e, and we have already shown that this diagram is zero once the corresponding $A_0 \leftrightarrow A_i$ exchange is taken into account. We then see that although a scale dependence is introduced in intermediate steps to properly handle subdivergences, this dependence can disappear in the final amplitude. As we shall see, a similar situation occurs in several three-loop diagrams.

The last two-loop diagram, Fig. 1j, can be shown to be finite. After some manipulation of derivatives, the amplitude can be written as

$$\Gamma^j(x_i) = -\frac{e^6}{m^3}\frac{\partial}{\partial \mathbf{x}_2^i}\frac{\partial}{\partial \mathbf{x}_4^k}T_i(x_2-x_1)T_k(x_4-x_3)\frac{\partial}{\partial \mathbf{x}_4^j}\int_{u,v}G(x_3-v)G(x_4-u)G(u-x_2)G(v-x_1)T_j(u-v) \tag{4.13}$$

The Fourier transform of this amplitude is a two-loop energy-momentum integral which is convergent by power counting because it contains explicit factors of external momenta.

V. THREE LOOP ORDER

There are eight diagrams contributing to the 4-point function at three loops (see Fig. 1) although we shall show that only four of them are divergent, namely diagrams k-n. To regularize three-loop bare amplitudes we first identify the divergent subgraphs and treat them in the same way as we have done at lower orders. The regularization of overall divergences (when all the external points come close) may require new DR identities. Our main concern is to compute the contribution of every diagram to the RGE's (see section VI) and thus in some cases we won't give the explicit closed form of the diagram, since it is not needed, but only an integral representation that has an ultraviolet finite Fourier transform and from which we can obtain the relevant contributions of the diagram to the Callan-Symanzik equations.

Let us start with the triple bubble diagram. We regulate each bubble subgraph using eq. (2.3) and we obtain

$$\Gamma^k_{reg}(x_i) = -32\lambda^4\delta(x_1-x_2)\delta(x_3-x_4)\int_{u,v} G^2_{reg}(u-x_1)G^2_{reg}(v-u)G^2_{reg}(x_3-v) \tag{5.1}$$

This expression is the convolution of three factors which have a finite Fourier transform, therefore so does the full result, and we shall find the contribution to the RGE's explicitly from (5.1).

Diagram l has also the structure of a convolution of two subdiagrams we have already regulated, namely the one bubble graph (2.3) and the "ice cream cone" (4.9). Using these we obtain

$$\Gamma^l_{reg}(x_i) = \frac{i\lambda^2 e^4}{m(\pi\alpha)^2}\delta(x_1-x_2)\delta(t_3-t_4) \times \int dt d^2\mathbf{u} G^2_{reg}(u-x_1;M) f^h_{reg}(\mathbf{x}_3-\mathbf{u},\mathbf{x}_4-\mathbf{u},t_3-t;M_s) \tag{5.2}$$

where, as we have explicitly indicated, the mass scales used in regulating the subdivergences are different. Again, (5.2) is an adequate regulated expression, since each factor in the convolution has a well defined Fourier transform, and thus it is all we need to find the contribution of the diagram to the Callan-Symanzik equations.

The regularization of diagram m is more involved. The bare amplitude is

$$\Gamma^m(x_i) = \frac{1}{64m^2}\left(\frac{e^2}{\pi\alpha}\right)^4 \delta(t_2-t_1)\delta(t_4-t_3) f^m(x_i) \tag{5.3}$$

where

$$f^m(x_i) = \frac{1}{(\mathbf{x}_2-\mathbf{x}_1)^2(\mathbf{x}_4-\mathbf{x}_3)^2} G(x_3-x_1)G(x_4-x_2) \tag{5.4}$$

After relabelling points $\mathbf{x}_1 \to \mathbf{x}$, $\mathbf{x}_2 \to \mathbf{y}$, $\mathbf{x}_3 \to \mathbf{z}$, $\mathbf{x}_4 \to 0$, $t_1 = t_2 \to 0$, $t_3 = t_4 \to t$ and regulating the two divergent seagull subgraphs by using (3.6), we obtain the partially regulated expression:

$$f^m(\mathbf{x},\mathbf{y},\mathbf{z},t) = \frac{1}{16^2} G(\mathbf{z}-\mathbf{x},t)G(-\mathbf{y},t)(\nabla_{\mathbf{x}}-\nabla_{\mathbf{y}})^2 \log^2 M_s^2(\mathbf{x}-\mathbf{y})^2 \nabla^2_{\mathbf{z}} \log^2 M_s^2\mathbf{z}^2 \tag{5.5}$$

Now we integrate by parts the differential operator $(\nabla_{\mathbf{x}}-\nabla_{\mathbf{y}})^2$ and we use the relation (4.8) satisfied by the product of two scalar propagators. Note that when we insert the exact identity (4.8) in (5.5) we have to keep the term that contains the δ-functions because it is not a purely local contribution (as it was in the "ice cream cone" amplitude (4.8), where we dropped it), but it multiplies a factor which requires further regularization.

We then rewrite (5.5) as

$$\begin{aligned} f^m(\mathbf{x},\mathbf{y},\mathbf{z},t) = -\frac{1}{16^2}\Big\{ &(\vec{\nabla}_{\mathbf{x}}-\vec{\nabla}_{\mathbf{y}})\left[\log^2 M_s^2(\mathbf{x}-\mathbf{y})^2(\overleftrightarrow{\nabla}_{\mathbf{x}}-\overleftrightarrow{\nabla}_{\mathbf{y}})G(\mathbf{z}-\mathbf{x},t)G(-\mathbf{y},t)\nabla^2_{\mathbf{z}}\log^2 M_s^2\mathbf{z}^2\right] \\ &+\left[(\vec{\nabla}_{\mathbf{x}}+\vec{\nabla}_{\mathbf{y}})^2+4im\partial_t\right]\log^2 M_s^2(\mathbf{x}-\mathbf{y})^2 G(\mathbf{z}-\mathbf{x},t)G(-\mathbf{y},t)\nabla^2_{\mathbf{z}}\log^2 M_s^2\mathbf{z}^2 \\ &+32im\delta(t)\delta(\mathbf{z}-\mathbf{x})\delta(\mathbf{y})\frac{\log^2 M_s^2\mathbf{x}^2}{\mathbf{x}^2}\Big\} \end{aligned} \tag{5.6}$$

Note that once we have regulated the subdivergence $\mathbf{z} \sim 0$ by using (3.6), we do not need to integrate by parts the operator $\nabla^2_{\mathbf{z}}$ as we did with $(\nabla_{\mathbf{x}} - \nabla_{\mathbf{y}})^2$. The reason is that the first two terms are already total derivatives with respect to external points and, since the diagram is only logarithmically divergent, these terms have a finite Fourier transform, defined by partial integration neglecting surface terms. The remaining term is regulated using the new DR identity

$$\frac{\log^2 M_s^2 \mathbf{x}^2}{\mathbf{x}^2} = \frac{1}{48} \nabla^2 \log^4 M_s^2 \mathbf{x}^2 \tag{5.7}$$

Thus the final regulated amplitude is

$$\begin{aligned}
\Gamma^m_{reg}(x_i) = &- \left(\frac{e^4}{2m}\right)^2 \frac{1}{(8\pi\alpha)^4} \delta(t_2 - t_1)\delta(t_4 - t_3) \\
&\times \left\{ (\vec{\nabla}_1 - \vec{\nabla}_2) \left[\log^2 M_s^2 (\mathbf{x}_1 - \mathbf{x}_2)^2 (\overleftrightarrow{\nabla}_1 - \overleftrightarrow{\nabla}_2) G(x_3 - x_1) G(x_4 - x_2) \nabla^2 \log^2 M_s^2 (\mathbf{x}_3 - \mathbf{x}_4)^2 \right] \right. \\
&+ \left[(\vec{\nabla}_1 + \vec{\nabla}_2)^2 + 4im\partial_{t_3} \right] \log^2 M_s^2 (\mathbf{x}_1 - \mathbf{x}_2)^2 G(x_3 - x_1) G(x_4 - x_2) \nabla^2 \log^2 M_s^2 (x_3 - x_4)^2 \\
&\left. + \frac{2im}{3} \delta(t_3 - t_1) \delta(\mathbf{x}_1 - \mathbf{x}_3) \delta(\mathbf{x}_2 - \mathbf{x}_4)) \nabla^2 \log^4 M_s^2 (\mathbf{x}_1 - \mathbf{x}_2)^2 \right\}
\end{aligned} \tag{5.8}$$

The bare amplitude of diagram n is given by

$$\Gamma^n(x_i) = \frac{2i}{m} \left(\frac{\lambda e^2}{\pi\alpha}\right)^2 \delta(x_1 - x_2)\delta(x_3 - x_4) I(x_3 - x_1) \tag{5.9}$$

where

$$I(x_3 - x_1) = \int_{u,v} G(u - x_1) G(v - x_1) G(x_3 - u) G(x_3 - v) \delta(t_u - t_v) \frac{1}{(\mathbf{u} - \mathbf{v})^2} \tag{5.10}$$

This is the first diagram which involves an internal integration which is not a convolution. Indeed this integral contains a logarithmic subdivergence for $\mathbf{u} \sim \mathbf{v}$. Our strategy is to regulate the singular factor $1/(\mathbf{u} - \mathbf{v})^2$ using (3.6) and then to perform the integral over internal points using formal partial integration. After this is done we will cure the overall divergence when $x_3 \sim x_1$.

We relabel points as $x_1 \to 0$, $x_3 \to (\mathbf{x}, t)$, $t_u = t_v \to t'$. After regulating the divergent seagull subgraph by using (3.6) we integrate by parts the laplacian operators. Total derivatives with respect to internal points can be dropped and we obtain the partially regulated form

$$\begin{aligned}
I(\mathbf{x}, t) = &\frac{1}{32} \int dt' d^2\mathbf{u} d^2\mathbf{v} \log^2 M_s^2 (\mathbf{u} - \mathbf{v})^2 \\
&\times \left\{ \left[(\vec{\nabla}_{\mathbf{u}} - \vec{\nabla}_{\mathbf{v}})^2 G(\mathbf{u}, t') G(\mathbf{v}, t') \right] G(\mathbf{x} - \mathbf{u}, t - t') G(\mathbf{x} - \mathbf{v}, t - t') \right. \\
&+ G(\mathbf{u}, t') G(\mathbf{v}, t') \left[(\vec{\nabla}_{\mathbf{u}} - \vec{\nabla}_{\mathbf{v}})^2 G(\mathbf{x} - \mathbf{u}, t - t') G(\mathbf{x} - \mathbf{v}, t - t') \right] \\
&\left. + 2 \left[(\vec{\nabla}_{\mathbf{u}} - \vec{\nabla}_{\mathbf{v}}) G(\mathbf{u}, t') G(\mathbf{v}, t') \right] \cdot \left[(\vec{\nabla}_{\mathbf{u}} - \vec{\nabla}_{\mathbf{v}}) G(\mathbf{x} - \mathbf{u}, t - t') G(\mathbf{x} - \mathbf{v}, t - t') \right] \right\}
\end{aligned} \tag{5.11}$$

The first term, which we shall denote $I_1(\mathbf{x}, t)$, can be easily regulated by means of the DR identity (4.8). The delta term can be dropped because its effect is cancelled by the counterterm which cancels the corresponding singular surface term in the ice cream cone subgraph. After partially integrating the differential operator and replacing derivatives with respect to internal points by derivatives with respect to $\mathbf{x}$, t, we can rewrite $I_1(\mathbf{x},t)$ as

$$\begin{aligned} I_1(\mathbf{x},t) = &-\frac{1}{32}(4im\partial_t + \nabla^2_{\mathbf{x}}) \\ &\times \int dt' d^2\mathbf{u} d^2\mathbf{v} \log^2 M_s^2(\mathbf{u}-\mathbf{v})^2 G(\mathbf{u},t')G(\mathbf{v},t')G(\mathbf{x}-\mathbf{u},t-t')G(\mathbf{x}-\mathbf{v},t-t') \end{aligned} \tag{5.12}$$

The integral in (5.12) is now finite by power counting and it can be done using Gaussian integration, leading to the regulated result:

$$\begin{aligned} I_1(\mathbf{x},t) = &\frac{m^2}{128\pi^2}(4im\partial_t + \nabla^2_{\mathbf{x}}) \left\{ \frac{\theta(t)}{t} e^{\frac{im\mathbf{x}^2}{t}} \right. \\ &\left. \times \left[\log^2 iM_s^2 t - 2(\log\rho_1 + 2)\log iM_s^2 t + k \right] \right\} \end{aligned} \tag{5.13}$$

with $k = \log^2\rho_1 + 4\log\rho_1 + 8 - \pi^2/6$ and $\rho_1 = m\gamma/4$.

Note that (5.13) has a finite Fourier transform, since $I_1(\mathbf{x},t)$ was logarithmically divergent and it contains total derivatives with respect to external points. It can be shown that the second integral in (5.11) is equal to $I_1(\mathbf{x},t)$, so we now focus on the last and most involved one

$$I_3(\mathbf{x},t) = -\frac{m^2}{16}\int dt' d^2\mathbf{u} d^2\mathbf{v} \frac{(\mathbf{u}-\mathbf{v})^2}{t'(t-t')} \log^2 M_s^2(\mathbf{u}-\mathbf{v})^2 G(\mathbf{u},t')G(\mathbf{v},t')G(\mathbf{x}-\mathbf{u},t-t')G(\mathbf{x}-\mathbf{v},t-t') \tag{5.14}$$

It is convenient to rewrite

$$\frac{1}{t'(t-t')} = \frac{1}{t}\left(\frac{1}{t'} + \frac{1}{t-t'}\right) \tag{5.15}$$

so $I_3(\mathbf{x},t)$ becomes the sum of two terms, which can be shown to be equal using a change of the integration variables. Then we use the identity

$$m^2(\mathbf{u}-\mathbf{v})^2 \frac{\theta(t-t')}{(t-t')^3} e^{\frac{im}{2(t-t')}[(\mathbf{u}-\mathbf{x})^2+(\mathbf{v}-\mathbf{x})^2]} = (4im\partial_t + \nabla^2_{\mathbf{x}}) \frac{\theta(t-t')}{t-t'} e^{\frac{im}{2(t-t')}[(\mathbf{u}-\mathbf{x})^2+(\mathbf{v}-\mathbf{x})^2]} \tag{5.16}$$

to obtain

$$\begin{aligned} I_3(\mathbf{x},t) = &-\left(\frac{m}{2\pi}\right)^4 \frac{1}{8t}(4im\partial_t + \nabla^2_{\mathbf{x}})\theta(t) \\ &\times \int_0^t dt' \int d^2\mathbf{u} d^2\mathbf{v} \log^2 M_s^2(\mathbf{u}-\mathbf{v})^2 \frac{e^{\frac{im}{2t'}(\mathbf{u}^2+\mathbf{v}^2)}}{t'^2} \frac{e^{\frac{im}{2(t-t')}[(\mathbf{u}-\mathbf{x})^2+(\mathbf{v}-\mathbf{x})^2]}}{t-t'} \end{aligned} \tag{5.17}$$

We can easily perform the integral over internal points and we find the partially regulated form

$$\begin{aligned} I_3(\mathbf{x},t) = &\frac{m^2}{32\pi^2 t}(4im\partial_t + \nabla^2_{\mathbf{x}}) \left\{ \theta(t) e^{\frac{im\mathbf{x}^2}{t}} \right. \\ &\times \left[\frac{1}{2}\log^2 iM_s^2 t - (\log\rho_1 + 2)\log iM_s^2 t \right. \\ &\left.\left. + \left(\frac{1}{2}\log^2\rho_1 + 2\log\rho_1 + 4 - \frac{\pi^2}{12}\right)\right]\right\} \end{aligned} \tag{5.18}$$

which still contains an overall divergence when $\mathbf{x} \sim 0$, $t \sim 0$. To regulate it, we integrate by parts the time derivative so that we are left with a term which is a total external derivative and therefore has a finite Fourier transform, plus three remaining singular terms that can be easily regulated by using (2.3) and two new DR identities, namely

$$\frac{\theta(t)}{t^2} e^{\frac{im\mathbf{x}^2}{t}} \log^n iM_s^2 t = -\frac{i}{n+1}\left(i\partial_t + \frac{\nabla_{\mathbf{x}}^2}{4m}\right)\frac{\theta(t)}{t} e^{\frac{im\mathbf{x}^2}{t}} \log^{n+1} iM_s^2 t \qquad n = 1, 2 \tag{5.19}$$

The final regulated form of $I_3(\mathbf{x}, t)$ is then

$$\begin{aligned} I_3(\mathbf{x}, t) = & \frac{m^2}{32\pi^2}(4im\partial_t + \nabla_{\mathbf{x}}^2)\left\{\frac{\theta(t)}{t} e^{\frac{im\mathbf{x}^2}{t}}\right. \\ & \times \left[\frac{1}{6}\log^3 iM_s^2 t - \frac{1}{2}(\log\rho_1 + 1)\log^2 iM_s^2 t\right. \\ & \left.\left. + \left(\frac{1}{2}\log^2\rho_1 + \log\rho_1 + 2 - \frac{\pi^2}{12}\right)\log iM_c^2 t\right]\right\} \end{aligned} \tag{5.20}$$

Note the presence of a new mass scale M_c in (5.20), which appears when regularizing the last term in (5.18). It is easily seen that the choice of this mass parameter does not modify the β-function at the three-loop level, so we choose its value such that it cancels the local term (last one) in (5.13) and it simplifies the expression of the renormalized amplitude. Restoring the original space-time points we obtain

$$\begin{aligned} \Gamma^n_{reg}(x_i) = & \frac{i}{8}\left(\frac{m\lambda e^2}{\pi^2\alpha}\right)^2 \delta(x_1 - x_2)\delta(x_3 - x_4)\left(i\partial_{t_3} + \frac{\nabla_3^2}{4m}\right)\left\{\frac{\theta(t_3 - t_1)}{t_3 - t_1} e^{\frac{im(\mathbf{x}_3 - \mathbf{x}_1)^2}{t_3 - t_1}}\right. \\ & \times \left[\frac{1}{3}\log^3 iM_s^2(t_3 - t_1) - \log\rho_1 \log^2 iM_s^2(t_3 - t_1)\right. \\ & \left.\left. + \left(\log^2\rho_1 - \frac{\pi^2}{6}\right)\log iM_s^2(t_3 - t_1)\right]\right\} \end{aligned} \tag{5.21}$$

We will show now that the remaining diagrams of Fig. 1 although superficially log divergent are instead finite. The contributions of diagrams o, p, q are given by,

$$\begin{aligned} \Gamma^o_{reg}(x_i) = & \frac{i}{32}\left(\frac{e^4}{m\pi\alpha}\right)^2 \frac{\partial}{\partial\mathbf{x}_4^j}\left\{T_j(x_4 - x_3)\delta(t_1 - t_2)\nabla_1^2 \log^2 M_s^2(\mathbf{x}_1 - \mathbf{x}_2)^2\right. \\ & \left. \times \frac{\partial}{\partial\mathbf{x}_2^i}\int_{u,v} G(u - x_1)G(v - x_2)G(x_3 - u)G(x_4 - v)T_i(v - u)\right\} \end{aligned} \tag{5.22}$$

$$\begin{aligned} \Gamma^p_{reg}(x_i) = & \frac{i}{32}\left(\frac{e^4}{m\pi\alpha}\right)^2 \frac{\partial}{\partial\mathbf{x}_2^i}\frac{\partial}{\partial\mathbf{x}_4^j}\left\{T_i(x_2 - x_1)T_j(x_4 - x_3)(\nabla_1 + \nabla_3)^2\right. \\ & \left. \times \int dt d^2\mathbf{u} d^2\mathbf{v} \log^2 M_s^2(\mathbf{u} - \mathbf{v})^2 G(u - x_1)G(v - x_2)G(x_3 - u)G(x_4 - v)\right\} \end{aligned} \tag{5.23}$$

$$\begin{aligned} \Gamma^q_{reg}(x_i) = & -\frac{\lambda}{16}\left(\frac{e^3}{m\pi\alpha}\right)^2 \delta(x_1 - x_2)\frac{\partial}{\partial\mathbf{x}_4^j}\left\{T_j(x_4 - x_3)(\nabla_1 + \nabla_3)^2\right. \\ & \left. \times \int dt d^2\mathbf{u} d^2\mathbf{v} \log^2 M_s^2(\mathbf{u} - \mathbf{v})^2 G(u - x_1)G(v - x_1)G(x_3 - u)G(x_4 - v)\right\} \end{aligned} \tag{5.24}$$

As in previous cases, the subdivergences have been regularized and derivatives manipulated in order to make the integral over internal variables as well as the Fourier transform respect to external points well defined. As in the case of diagram i, although each of these diagrams depends on M_s, this dependence disappears when we take into account the graphs obtained by the permutation $A_0 \leftrightarrow A_i$. It is easy to prove that the scale derivative of each of these graphs is proportional to a graph that contains graph e as a subgraph and as a consequence the scale derivative of these graphs is zero.

We conclude with the triple box (diagram r). The contribution of this diagram to the amputated 4-point function is given by

$$\begin{aligned}\Gamma^r(x_i) = & \frac{1}{4}\left(\frac{e^2}{m}\right)^4 \frac{\partial}{\partial \mathbf{x}_2^i}\frac{\partial}{\partial \mathbf{x}_4^j} T_i(x_2-x_1)T_j(x_4-x_3) \\ & \times \int_{u,v,y,z} G(u-x_1)G(y-u)G(x_3-y)T_k(v-u)T_l(z-y) \\ & \left\{G(x_4-z)\ \frac{\overleftrightarrow{\partial}}{\partial \mathbf{z}^l}\ G(z-v)\ \frac{\overleftrightarrow{\partial}}{\partial \mathbf{v}^k}\ G(v-x_2)\right\} \end{aligned} \tag{5.25}$$

Because of the "transversality" of the gauge field propagator, all derivatives inside the integral can be rewritten as derivatives with respect to external points and brought to the left to obtain the regulated form

$$\begin{aligned}\Gamma^r_{reg}(x_i) = & -\left(\frac{e^2}{m}\right)^4 \frac{\partial}{\partial \mathbf{x}_2^i}\frac{\partial}{\partial \mathbf{x}_4^j}\left\{T_i(x_2-x_1)T_j(x_4-x_3)\frac{\partial}{\partial \mathbf{x}_2^k}\frac{\partial}{\partial \mathbf{x}_4^l}\right. \\ & \times \int_{u,v,y,z} G(u-x_1)G(y-u)G(x_3-y)T_k(v-u) \\ & \left. T_l(z-y)G(x_4-z)G(z-v)G(v-x_2)\right\} \end{aligned} \tag{5.26}$$

Away from coincident external points, one can see that the integrals over internal points are now finite by power counting. Furthermore, the amplitude (5.26) has a well defined Fourier transform since it contains two total derivatives with respect to external points.

VI. RENORMALIZATION GROUP EQUATIONS

In this section we shall show that the renormalized 4-point function satisfies the RGE equation and we compute the β-function to three loops. We recover the one loop result of [7], namely the zero of the β-function for the self-dual case, and we find that the relation between the couplings for which the β-function vanishes is scheme dependent beyond one loop. Scheme dependence arises from the two scale parameters M and M_s introduced initially at one-loop order, and renormalized amplitudes depend on the parameter $\rho = M_s^2/M^2$. As explained below (5.20), an additional scale parameter in the regularization of graph n was fixed to simplify its renormalized amplitude. This scale parameter affects the RGE's beginning at four loops.

The RGE equation for the 4-point correlation function is

$$\left[M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}\right]\Gamma_{reg}=0 \tag{6.1}$$

Recall the anomalous dimension term is absent because there is no field renormalization in this theory. In order to check that (6.1) is satisfied, we explicitly compute the scale derivatives of the renormalized amplitudes obtained in the previous sections and compare order by order in perturbation theory with $\frac{\partial}{\partial\lambda}\Gamma_{reg}$. The contribution to the β-function in each order can be uniquely identified from the purely local term, in which $\frac{\partial}{\partial\lambda}$ acts on the tree contribution (2.1). Non-local contributions then provide an independent consistency check, since they must appear with the correct coefficients to test that subdivergences have been correctly handled by our method.

The β-function can be written as a series expansion in the couplings λ, e

$$\beta(\lambda,e)=a_1\lambda^2+b_1e^4+a_2\lambda^3+c_2\lambda e^4+\dots\ , \tag{6.2}$$

where the subscript in the coefficients indicates the loop order. In a theory with more than one coupling constant the β-function is renormalization scheme independent through one loop order but not beyond. We will use capital letter superscripts to denote that the amplitudes include now not only the contribution of the particular diagram drawn in Fig. 1 but also the contributions of graphs related by Bose symmetry as well as time reflection. All permutations of A_0, A_i and A_i^2 vertices are also added.

We have already computed the one loop β-function in the purely scalar subtheory (see (2.8)), while from (3.7) one can immediately obtain the scale derivative of the seagull graph

$$\begin{aligned}M\frac{\partial}{\partial M}\Gamma^D_{reg}&=-\frac{ie^4}{\pi m\alpha^2}\delta(x_1-x_2)\delta(x_1-x_3)\delta(x_1-x_4)\\&=\frac{e^4}{4\pi m\alpha^2\lambda}\Gamma^A\end{aligned} \tag{6.3}$$

Thus

$$a_1=\frac{m}{\pi} \tag{6.4}$$

$$b_1=-\frac{1}{4\pi m\alpha^2} \tag{6.5}$$

and we see that

$$M\frac{\partial}{\partial M}(\Gamma^C_{reg}+\Gamma^D_{reg})=0 \tag{6.6}$$

when

$$\lambda^2=\frac{e^4}{4m^2\alpha^2} \tag{6.7}$$

showing that scale invariance is recovered for the self-dual point [3].

At two-loop order the β-function is renormalization scheme dependent, as we shall see. The scale derivative of the double bubble (4.1) is trivially found to be

$$M\frac{\partial}{\partial M}\Gamma^G_{reg} = -\frac{2m\lambda}{\pi}\Gamma^C_{reg} \tag{6.8}$$

The case of the "ice cream cone" diagram is considerably more complicated. Although it is true that the regularized amplitude has a well defined scale derivative, we must be able to recognize in the result the amplitude of graphs that have already appeared at one loop order. However, if we calculate the scale derivative of f^h_{reg} from (4.9) we find

$$M\frac{\partial}{\partial M}f^h_{reg}(\mathbf{x},\mathbf{y},t) = -\frac{1}{8}(\vec{\nabla}_{\mathbf{x}} - \vec{\nabla}_{\mathbf{y}})\left[\log M_s^2(\mathbf{x}-\mathbf{y})^2(\overleftrightarrow{\nabla}_{\mathbf{x}} - \overleftrightarrow{\nabla}_{\mathbf{y}})G(\mathbf{x},t)G(\mathbf{y},t)\right] \\ -\frac{1}{8}\left[(\vec{\nabla}_{\mathbf{x}} + \vec{\nabla}_{\mathbf{y}})^2 + 4im\partial_t\right]G(\mathbf{x},t)G(\mathbf{y},t)\log M_s^2(\mathbf{x}-\mathbf{y})^2 \tag{6.9}$$

which is difficult to interpret immediately. One can show that (6.9) and the regulated bubble (2.3) differ at most by a finite local term independent of M. In order to determine this possible β-function contribution, we compute the Fourier transform of (6.9) (see appendix) and compare it with the regularized expression for the bubble. The result is

$$M\frac{\partial}{\partial M}f^h_{reg}(\mathbf{x},\mathbf{y},t;M_s) = 2\pi\delta(\mathbf{x}-\mathbf{y})\left[G^2_{reg}(\mathbf{x},t;M) + \frac{im}{4\pi}\log\frac{\rho}{\rho_1}\delta(t)\delta(\mathbf{x})\right] \tag{6.10}$$

where $\rho = M_s^2/M^2$ and $\rho_1 = m\gamma/4$. Note that we used M_s to regulate the subdivergence of the graph, but the scale derivative involves the one loop bubble. The RGE requires that the renormalized bubble amplitude appear with the assigned scale parameter, *i.e.*, M.

Combining (6.10) with the "ice cream cone" amplitude given by Eqs. (4.4), (4.5), and adding the contribution of the diagrams related by Bose symmetry and time inversion, we obtain

$$M\frac{\partial}{\partial M}\Gamma^H_{reg}(M_s) = \frac{e^4}{2\pi m\lambda\alpha^2}\Gamma^C_{reg}(M) + \frac{e^4}{(2\pi\alpha)^2}\log\frac{\rho_1}{\rho}\Gamma^A \tag{6.11}$$

It is easy to check that (6.8) and (6.11) are consistent with the one loop result and yield the two loop contributions:

$$a_2 = 0 \tag{6.12}$$

$$c_2 = \frac{1}{(2\pi\alpha)^2}\log\frac{\rho}{\rho_1} \tag{6.13}$$

One can see explicitly that the coefficient c_2 depends on the mass ratio we have chosen, $\rho = M_s^2/M^2$. This means that the point for which the β-function vanishes is also scheme dependent, and in general there will be higher order corrections to the one-loop relation (6.7). However, in the renormalization scheme defined by

$$M_s^2 = \rho_1 M^2 = \frac{m\gamma}{4}M^2 \tag{6.14}$$

we have $\rho = \rho_1$ and thus there is no contribution to the two-loop β-function. As a consequence, in this scheme the relation (6.7) between the couplings which gives a finite theory is valid through two loops.

The scale derivatives of the three loop amplitudes are found to be

$$M\frac{\partial}{\partial M}\Gamma^K_{reg} = -\frac{3m\lambda}{\pi}\Gamma^G_{reg} \tag{6.15}$$

$$M\frac{\partial}{\partial M}\Gamma^L_{reg}(M_s, M) = -\frac{\lambda m}{\pi}\Gamma^H_{reg}(M_s) + \frac{e^4}{2\pi m\lambda\alpha^2}\Gamma^G_{reg}(M) + \frac{e^4}{(2\pi\alpha)^2}\log\frac{\rho_1}{\rho}\Gamma^C_{reg}(M) \tag{6.16}$$

$$M\frac{\partial}{\partial M}\Gamma^M_{reg} = \frac{e^4}{4\pi m\lambda\alpha^2}\Gamma^H_{reg} \tag{6.17}$$

$$M\frac{\partial}{\partial M}\Gamma^N_{reg}(M_s) = \frac{e^4}{4\pi m\lambda\alpha^2}\Gamma^G_{reg}(M) + \frac{e^4}{(2\pi\alpha)^2}\log\frac{\rho_1}{\rho}\Gamma^C_{reg}(M) + \frac{\lambda e^4 m}{4\pi(2\pi\alpha)^2}\log^2\frac{\rho_1}{\rho}\Gamma^A \tag{6.18}$$

where we have specified the mass parameter only when the mass scale used in the regularization of the three loop diagram is not the same as the mass scale of the graph which appears in its scale derivative. In general, it is immediate to obtain these results from the regulated amplitudes discussed in section V and only the scale derivative of diagram m requires some further study. As in the case of the "ice cream cone" (diagram h), one can show by direct computation that the difference between $M\frac{\partial}{\partial M}\Gamma^m_{reg}$ (see (5.8)) and the regulated amplitude of diagram h is purely local, finite and independent of M. Fortunately, to determine this possible local term we only need to compute the Fourier transform of a semi-local piece that does not appear in the regulated form of the "ice cream cone", and we easily find that there is no such local term.

When we substitute the scale derivatives in the RGE (6.1), we find the three-loop contributions to the β-function

$$a_3 = 0 \tag{6.19}$$

$$b_3 = 0 \tag{6.20}$$

$$c_3 = -\frac{m}{4\pi(2\pi\alpha)^2}\log^2\frac{\rho}{\rho_1} \tag{6.21}$$

Note that if we choose $M_s^2 = \rho_1 M^2$ the three loop contribution to the β-function vanishes and thus the theory is finite at the self-dual point defined by (6.7).

To conclude we summarize the result for the β-function

$$\beta(\lambda, e) = \frac{m}{\pi}\lambda^2 - \frac{1}{4\pi m\alpha^2}e^4 + \frac{1}{(2\pi\alpha)^2}\log\frac{\rho}{\rho_1}\lambda e^4 - \frac{m}{16\pi^3\alpha^2}\log^2\frac{\rho}{\rho_1}\lambda^2 e^4 \tag{6.22}$$

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APPENDIX:

In this appendix we shall calculate the scale derivative of the ice cream graph h. In order to do so, we shall calculate its Fourier transform and show how it is related to the bubble graph c. Let us start by recalling that

$$R(\mathbf{x},\mathbf{y},t) = M\frac{\partial}{\partial M} f^h_{reg} = -\frac{1}{8}(\vec{\nabla}_{\mathbf{x}} - \vec{\nabla}_{\mathbf{y}})\left[\log M_s^2(\mathbf{x}-\mathbf{y})^2(\overleftrightarrow{\nabla}_{\mathbf{x}} - \overleftrightarrow{\nabla}_{\mathbf{y}})G(\mathbf{x},t)G(\mathbf{y},t)\right]$$
$$-\frac{1}{8}\left[(\vec{\nabla}_{\mathbf{x}} + \vec{\nabla}_{\mathbf{y}})^2 + 4im\partial_t\right] G(\mathbf{x},t)G(\mathbf{y},t)\log M_s^2(\mathbf{x}-\mathbf{y})^2 \tag{A1}$$

It is convenient to define variables $\mathbf{u} = (\mathbf{x}+\mathbf{y})/2$ and $\mathbf{v} = (\mathbf{x}-\mathbf{y})/2$. After manipulations of derivatives and use of (1.4), the Fourier transform of R can be shown to be

$$\hat{R}(\mathbf{p}_1,\mathbf{p}_2,\omega) = 4\{i\mathbf{p}_2\cdot\mathbf{f}_3 + [-\mathbf{p}_1^2 - \mathbf{p}_2^2 + 4m\omega]f_2\} \tag{A2}$$

where $\mathbf{p}_1 = \mathbf{p}_x + \mathbf{p}_y$, $\mathbf{p}_2 = \mathbf{p}_x - \mathbf{p}_y$, and

$$f_2 = -\frac{1}{8}\int d^2\mathbf{u}d^2\mathbf{v}dt\log 4M_s^2\mathbf{v}^2 G(\sqrt{2}\mathbf{u},t)G(\sqrt{2}\mathbf{v},t)e^{-i\mathbf{p}_1\cdot\mathbf{u}-i\mathbf{p}_2\cdot\mathbf{v}+i\omega t} \tag{A3}$$
$$\mathbf{f}_3 = -\frac{1}{4}\int d^2\mathbf{u}d^2\mathbf{v}dt(\nabla_v\log 4M_s^2\mathbf{v}^2)G(\sqrt{2}\mathbf{u},t)G(\sqrt{2}\mathbf{v},t)e^{-i\mathbf{p}_1\cdot\mathbf{u}-i\mathbf{p}_2\cdot\mathbf{v}+i\omega t} \tag{A4}$$

The calculation of $\mathbf{f}_3$ is straightforward. A conventional non-relativistic loop integral gives the result:

$$\mathbf{p}_2\cdot\mathbf{f}_3 = -\frac{m}{4}\log\left(\frac{-4m\omega+\mathbf{p}_1^2}{-4m\omega+\mathbf{p}_1^2+\mathbf{p}_2^2}\right) \tag{A5}$$

The loop integral representation of f_2 is infrared divergent because it contains the factor $1/q^2$ which is the Fourier transform of the logarithmic term. Instead we calculate f_2 by the following 3-step technique:

i)a trivial calculation of the transform of $G(\sqrt{2}\mathbf{u},t)$ with respect to $\mathbf{u}$ and t,

ii)a not-so-trivial calculation of the transform of $\log 4M_s^2\mathbf{v}^2 G(\sqrt{2}\mathbf{v},t)$ with respect to $\mathbf{v}$ and t,

iii)a final convolution integral over ω to obtain the net result.

Step ii) is done by considering the function $(\mathbf{v}^2)^\alpha G(\sqrt{2}\mathbf{v},t)$, calculating its Fourier transform and taking the derivative with respect to α at $\alpha = 0$ to obtain the logarithmic factor. Several special function formulae [12] are required for this task. The final result is

$$f_2 = \frac{im}{8[-\mathbf{p}_1^2 - \mathbf{p}_2^2 + 4m\omega]}\log\left[\left(\frac{16M_s^2}{\gamma^2}\right)\frac{-4m\omega+\mathbf{p}_1^2}{(-4m\omega+\mathbf{p}_1^2+\mathbf{p}_2^2)^2}\right] \tag{A6}$$

Then,

$$\hat{R}(\mathbf{p}_1,\mathbf{p}_2,t) = -\frac{im}{2}\log\left[\left(-\omega+\frac{\mathbf{p}_1^2}{4m}\right)\frac{m\gamma^2}{4M_s^2}\right] \tag{A7}$$

Now, using (2.2) and comparing with the regulated amplitude of the bubble in momentum space (2.5) we get the relation

$$\hat{R}(\mathbf{p}_x, \mathbf{p}_y, t) = 2\pi \hat{G}^2 \left(\mathbf{p}_x + \mathbf{p}_y, \omega; M\sqrt{\frac{\rho}{\rho_1}} \right) \tag{A8}$$

where $\rho_1 = m\gamma/4$ and $\rho = M_s^2/M^2$, which is the Fourier transform of (6.10).

One should note that the regularized amplitude of graph h, see (4.9), differs from its scale derivative only by the exponent of the logarithmic factor. The loop integral representation of its Fourier transform is also infrared divergent. However the transform is well defined, and a technique similar to that described above is required to calculate it.

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FIG. 1. Figure 1. Diagrams which contribute to the 4-point function $\Gamma(x_1, x_2, x_3, x_4)$ through three-loop order. Dots represent the A_i vertex with derivative coupling. Diagrams related by permutations must be added as explained in section III.