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Abstract

An approach to Witten String Field Theory based on the discretization of the world sheet is adopted. We use it to calculate tree amplitudes with the formulation of the theory based on string functionals. The results are evaluated numerically and turn out to be very accurate, giving, for a string approximated by 600 points, values within 0.02 % of the prediction of the dual model. The method opens a way to calculate amplitudes in String Field Theory using non-flat backgrounds as well as compactified dimensions.

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The aim of this letter is to discuss, and test, a way to do calculations in the field theory for strings proposed by Witten [1]. The method is based on the discretization of the string, i.e. its substitution by a finite set of points, the number of which is eventually sent to infinity to recover the continuum limit. We will test this method by calculating the ratio of the 3 tachyons amplitude over the tachyon-tachyon-vector amplitude. This quantity has been already calculated by Gross and Jevicki [2], by Cremmer, Schwimmer and Thorn [3] and by Samuel [4] in the context of Witten String Field Theory. The results of [2,3,4] agree with the ones of the dual model. We will discuss this calculation following [2]. In their paper Gross and Jevicki use Fock space vectors and δ function overlaps to construct the string vertices; while their method is very elegant, it is quite involved and cumbersome to use as well, and moreover has little or no flexibility should one wish to make changes such as, for example, a different space-time background (possibly compactified).

We shall see that the discretization method is instead very straightforward, leading basically to gaussian integrals, which can be solved exactly. In the calculation presented here we find the drawback that it is not always possible to do all the calculations analytically, in particular we will encounter one matrix whose inverse we did not succeed to find analytically. It is however quite easy to perform the calculations numerically, for various values of the number of points. The method passes its test, since, already for 600 points, the value of the ratio is within 2 parts in 10^4 of the correct value. The real advantage of the method however lies in its flexibility. Once a calculation has been set up changes such as the background space time may be easily managed, possibly by modifying just a computer code. In the conclusions we will come back to the possible perspectives offered by this method.

This letter is organized as follows: first we give an extremely short review of the relevant aspects of Witten String Field Theory (see for example [5] for a more complete review), and the calculations of [2]. The purpose of this is mainly to set notations. We introduce the discretized approach and discuss its relationship with the Fock space one. We then present the calculation of the ratio of the two amplitudes and discuss its results. We conclude with some final remarks. More details of this calculation, as well as the calculation of other amplitudes and a more complete discussion will be presented in a future paper [6].

In this paper we will ignore ghosts, since we are interested for the moment in calculating only tree amplitudes on shell, and we consider only open strings.

The objects one deals with in a string theory are string configurations, which we denote by $x^\mu(\sigma,\tau)$, where τ is an evolution parameter and, for a fixed τ, $x^\mu(\sigma)$ is an image of the interval $[0,\pi]$ into space-time (hereafter we will usually suppress the index μ). $x(\sigma)$ has the usual boundary conditions for open strings $x'(0) \ = \ x'(\pi) \ = 0$, where the prime denotes differentiation with respect to σ.

For $x(\sigma)$ one has the usual mode expansion

$$x(\sigma) \ = \ \frac{x_0}{\sqrt{2}} + \sqrt{2}\sum_{n=0}^{\infty} x_n \cos(n\sigma) \tag{1}$$

where the x_n are called oscillator modes, and $x_0/\sqrt{2}$ is the position of the center of mass.

The string hamiltonian is (at $\tau = 0$)

$$H \ = \int_0^\pi d\sigma \left\{p(\sigma)^2 \ + \ x(\sigma)'^2\right\}$$

where $p(\sigma) = \dot{x}(\sigma) = \partial_\tau x(\sigma,\tau)$ calculated at $\tau = 0$. In terms of the x_n H is

$$H = p_0^2 + \omega_0^2 x_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(p_k^2 + \omega_k^2 x_k^2) \tag{2}$$

where

$$\omega_n = n, \quad n = 0,\ldots,\infty$$

Even if $\omega_0 = 0$ we have kept its formal dependence in (2) for reasons which will become clear in the following. The hamiltonian (2) consists of an infinity of single harmonic oscillators with frequency ω_n, except for the zero mode, which is instead a free particle, reflecting the translational mode of the whole string. We can therefore easily first quantize the theory defining creation and annihilation operators a_n and $a_n^\dagger$ in the usual way.

One can then build a Fock space acting with the $a^\dagger$'s on the vacuum $|0>$, which is the vector annihilated by all the a's. Given a generic vector $|\Psi>$, the associated functional in a Schrödinger-like representation is

$$<X|\Psi>= \Psi[x(\sigma)]$$

where

$$|X>= \prod_{n=0}^{\infty} |x_n> \ .$$

In particular the tachyon of momentum p and the vector with distribution of momenta $A(p)$ will be represented in the Fock space by

$$
\begin{aligned}
|T> &= e^{ip\cdot x_0/\sqrt{2}}|0> \\
|V> &= e^{ip\cdot x_0/\sqrt{2}} A_\mu(p) a_1^{\mu\dagger}|0>
\end{aligned}
$$

Thus the study of first quantized strings is equivalent to the study of functionals $\Psi[x(\sigma)]$, that is objects which associate a number for each string, which in this particular approach is seen as an infinite collection of oscillator modes. On the other hand those functionals, which we will call string fields, are the objects of interest in a string field theory. As it is known Witten [1] proposed an action

$$A = \oint \Psi * Q\Psi + \frac{2}{3}\Psi * \Psi * \Psi$$

Where Q is the BRST operator (which acts as some sort of differentiation for the string fields). The $*$ and $\oint$ operators are defined as follows

$$
\begin{aligned}
(\Psi * \Phi)[x_L, x_R] &= \int \mathcal{D}y(\sigma)\ \Psi[x_L; y]\ \Phi[\bar{y}; x_R] \\
\oint \Psi &= \int \mathcal{D}x_L(\sigma)\ \mathcal{D}x_R(\sigma) \Psi[x_L; x_R]
\end{aligned}
$$

where x_L, x_R and y are 'half strings', that is maps from the interval $[0, \pi/2]$ into space time. x_L represents the left half of the string, $x_L(\sigma) = x(\sigma)$, while $x_R(\sigma)$ is the right half, $x_R(\sigma) = x(\pi/2 + \sigma)$. $\bar{y}(\sigma) = y(\pi - \sigma)$ is the half string y (which acts as dummy variable) parametrized from right to left.

An amplitude (vertex) between three states Ψ_1 Ψ_2 and Ψ_3 is simply the value of $\oint \Psi_1 * \Psi_2 * \Psi_3$.

To calculate $*$ and $\oint$ among vectors one has to define vertices $|V_N>$, vectors of the N-string Fock space, and then evaluate

$$\oint \Psi_1 * \Psi_2 * \Psi_3 = < V_3 | \Psi_1 > | \Psi_2 > | \Psi_3 > \tag{3}$$

This calculation has been done [2,3,4] and the result reproduces the correct value as given in the dual model, thus proving the validity of Witten String Field Theory, at least at this level. Unfortunately the evaluation of V_3 is rather complicated and, as we already mentioned, the techniques used depend on the background.

In this letter we adopt a different approach, first pioneered by Giles and Thorn [7] and Thorn [8] and later applied to Witten String Field Theory by

Srednicki and Woodard[9], some related work has appeared in [10]. The idea is to interpret the string functionals as limits (in a sense which will be specified below) of N points functions. As N goes to infinity one should then recover the usual results of string field theory.

In other words, one discretizes the string world sheet in the σ direction in such a way that the $x(\sigma)$ variable is substituted by the set of variables $x(\sigma_0), x(\sigma_1), \ldots, x(\sigma_N)$, where $\sigma_i = i\pi/N$.

The functional $\Psi[x(\sigma)]$ will be seen as the limit for N going to infinity of a succession of N-variable functions $\Psi^N(x(\sigma_0), \ldots, x(\sigma_N))$, and the functional integral measure

$$\mathcal{D}x(\sigma) = \lim_N \kappa \prod_{i=0}^{N} dx(\sigma_i),$$

where κ is a normalization constant we need not calculate for our present purposes.

$\Psi[x(\sigma)]$ is an object which associates a number for each string configuration $x(\sigma)$, that is a number for each image of the interval $[0, \pi]$ into space time (with the appropriate boundary conditions). A N-variable function Ψ^N instead associates a number for each set of N points. With a prescription to extract N points from each string configuration we can compare the values of $\Psi[x(\sigma)]$ and Ψ^N for each string. If we now consider Ψ^N to be an element of a succession of functions as N changes, we can have that, on a given string $\Psi^N \to \Psi[x(\sigma)]$ as $N \to \infty$.

We will use the following definition for the convergence of a succession of functions to a string functional:

$$\lim_N \Psi^N = \Psi[x(\sigma)] \;\Rightarrow\; \forall \varepsilon \; \forall x(\sigma) \; \exists N_{min} \text{ such that } \left|\Psi[x(\sigma)] - \Psi^N\right| < \varepsilon \quad \forall N > N_{min} \tag{4}$$

One important thing to notice is that the convergence criterion defined in (4) is not of the uniform kind. This is not a disaster, it means that there might be problems in exchanging limits, integrals and infinite sums among themselves. In the calculations presented here we have taken care not to interchange limits and integrals once we have defined the $*$ and $\oint$ operations.

Within this interpretation we define the $*$ and $\oint$ as follows

$$
\begin{aligned}
(\Psi * \Phi)(x_L; x_R) &= \lim_{N\to\infty} (\Psi * \Phi)^N (x_L; x_R) = \\
&= \lim_{N\to\infty} \sqrt{\kappa_N} \int dy(\sigma_M), \ldots, dy(\sigma_N)\ \Psi^N(x_L; y(\sigma_M), \ldots, y(\sigma_N)) \\
&\quad \Phi^N(y(\sigma_N), \ldots, y(\sigma_M); x_R) \\
\oint \Psi &= \lim_{N\to\infty} \kappa_N \int dx(\sigma_0), \ldots, dx(\sigma_N) \Psi^N(x(\sigma_0), \ldots, x(\sigma_N))
\end{aligned} \tag{5}
$$

where M=N/2 (N odd) or M=(N-1)/2 (N even).

Note that we have defined $*$ and $\oint$ with the limits before the integrals, there is no a priori reason to do that, except that only in this case we know how to calculate the right hand side of the equations, which is a regular integral among functions.

Another observation to make is that possibly not all string configurations are reproduced in this picture, and some of those can be relevant for the functional integrals. The same situation appears in the Fock space representation, where for example it is impossible to obtain closed strings [11]. We will now discuss the relation between the discrete and the oscillator pictures of the theory in some more detail. We plan to come back to this problem in a future publication [6].

Let us now quantize the discretized string, that we represent by the vector $\vec{x}$, a vector whose components are the $x(\sigma_i)$'s. In analogy with eq. (1) one can make a finite Fourier expansion of $\vec{x}$ finding an identical formula to (1), but now the sum goes up to N. In matrix notation:

$$\vec{x} = \mathbf{B}\vec{X}^N \tag{6}$$

where

$$B_{nm} = \begin{pmatrix} 1/2 & 1 & \cdots & 1 & \cdots & 1/2 \\ \vdots & & & & & \vdots \\ 1/2 & cos(n\sigma_1) & \cdots & cos(n\sigma_m) & \cdots & (-)^n/2 \\ \vdots & & & & & \vdots \\ 1/2 & -1 & \cdots & (-)^m & \cdots & (-)^N/2 \end{pmatrix} \tag{7}$$

and $\vec{X}^N$ is a vector whose components are the x_n^N, the Fourier components of the finite expansion of $x(\sigma_i)$. Since these components converge to the x_n of (1) to recover the continuous limit, hereafter we will skip the superscript N on the x_n's.

The hamiltonian (2) becomes in the discretized case

$$H = p_0^2 + \frac{1}{2}\sum_{k=1}^{N-1} p_k^2 + p_N^2 + \frac{1}{2}\sum_{k=0}^{N} \omega_k^2 x_k^2 \tag{8}$$

corresponding to a set of N coupled harmonic oscillators with frequency

$$\omega_k = \frac{sin\frac{k\pi}{2N}}{\pi/(2N)}, \quad k=0\ldots N$$

this is basically the result obtained by Thorn in [8].

The equation (6) gives the finite Fourier decomposition for the N discretized string. Since the matrix **B** is non singular, there is an equivalence between the N points $x(\sigma_i)$ and the N oscillators x_n. Taking the limit $N \to \infty$ one easily gets the expansion (1), thus showing the equivalence between the two approaches.

We can now proceed to the construction of the states in the discretized approach. Although one can deduce those states from the hamiltonian (8) directly in the discretized picture, we will deduce them from the states in the Fock space. Since the hamiltonian (8) corresponds to a set of harmonic oscillator, again one can define in the usual way creators and annihilators. We still keep a formal dependence on ω_0, when at the end we will send ω_0 to zero this will give the δ function of conservation of momentum.

The vacuum state for the hamiltonian (8) is just given by the gaussian of the oscillator modes:

$$<\vec{X}|0>= exp\left\{-\frac{1}{2}\sum_{n=0}^{N}\omega_n x_n^2\right\}$$

Where now

$$|\vec{X}>=\prod_{n=0}^{N}|x_n> \quad .$$

Any other state can be obtained from this expression by repeated application of the creator operators on the vacuum $|0>$. Using the relation (7) one can then obtain the functional corresponding to any state in the Fock space. For our purposes it will be sufficient to explicitly show the functionals for tachyon and vector.

$$\begin{aligned} T[x(\sigma)] &= exp\left\{\frac{i}{N}\,p\,B_{0i}\,x(\sigma_i) - \frac{1}{N^2}\,\vec{x}^T\mathbf{A}\vec{x}\right\} \\ V[x(\sigma)] &= 2\omega_1^{1/2}\,\frac{i}{N}\,B_{1i}\,x(\sigma_i)^\mu A(p)_\mu\, exp\left\{\frac{i}{N}\,p\,B_{0i}\,x(\sigma_i) - \frac{1}{N^2}\,\vec{x}^T\mathbf{A}\vec{x}\right\} \end{aligned}$$

and the matrix **A** is defined by

$$A_{ij} = \sum_{k=0}^{N}\omega_k B_{ki}B_{kj} \tag{9}$$

These results will in general allow to calculate the $*$ and $\oint$ between different oscillation modes of the string (one particle states) in terms of string functionals. The recipe is straightforward, once the state is specified one uses (6) to get the corresponding N-variable functions. Those functions, introduced in (5) will allow the numerical or analytical calculation of the multiple integral. The final result is then obtained taking the limit $N \to \infty$. In particular for the three string vertex (3) one considers

$$V_3(\Psi_1, \Psi_2, \Psi_3) =$$
$$= \lim_{N\to\infty} \int \prod_{i=0}^{M-1} dx(\sigma_i)dy(\sigma_i)dz(\sigma_i)\Psi_1^N(x(\sigma_{M-1}),\ldots,x(\sigma_0);y(\sigma_0),\ldots,y(\sigma_{M-1}))$$
$$\Psi_2^N(y(\sigma_{M-1}),\ldots,y(\sigma_0);z(\sigma_0),\ldots,z(\sigma_{M-1}))$$
$$\Psi_3^N(z(\sigma_{M-1}),\ldots,z(\sigma_0);x(\sigma_0),\ldots,x(\sigma_{M-1}))$$

Since middle point subtleties do not to play any role in this calculation we will use an even number of points, $N = 2M - 1$.

The calculations for the two amplitudes in question are straightforward, but long and tedious, heavily relaying on the symmetries of the matrix $\mathbf{B}$. They can be done exactly up to a point however, since all the integrals are gaussian. Details will be presented elsewhere [6]. The final result for the two amplitudes is

$$A(TTT) = \kappa' \int \prod_{i=0}^{M-1} dx(\sigma_i)\ dy(\sigma_i)\ dz(\sigma_i)$$
$$exp\left\{\frac{i}{2M-1}\ \vec{B}^{\,t}_{0R}(1,1)\left[p_1\begin{pmatrix}\vec{y}\\ \vec{x}\end{pmatrix}+p_2\begin{pmatrix}\vec{x}\\ \vec{z}\end{pmatrix}+p_3\begin{pmatrix}\vec{z}\\ \vec{y}\end{pmatrix}\right]\right\}$$
$$exp\left\{\frac{-1}{(2M-1)^2}\left[(\vec{y}^{\,t},\vec{x}^{\,t})\mathbf{A}'\begin{pmatrix}\vec{y}\\ \vec{x}\end{pmatrix}+(\vec{x}^{\,t},\vec{z}^{\,t})\mathbf{A}'\begin{pmatrix}\vec{x}\\ \vec{z}\end{pmatrix}+(\vec{z}^{\,t},\vec{y}^{\,t})\mathbf{A}'\begin{pmatrix}\vec{z}\\ \vec{y}\end{pmatrix}\right]\right\} \tag{10}$$

$$A(TTV) = \kappa' A(p_3)2\omega_1^{1/2}\int \prod_{i=0}^{M-1} dx(\sigma_i)dy(\sigma_i)dz(\sigma_i)\frac{i}{2M-1}\vec{B}^{\,t}_{1R}(1,0,-1)\begin{pmatrix}\vec{y}\\ \vec{x}\\ \vec{z}\end{pmatrix}$$
$$exp\left\{\frac{i}{2M-1}\vec{B}_{0R}{}^{t}(1,1)\left[p_1\begin{pmatrix}\vec{y}\\ \vec{x}\end{pmatrix}+p_2\begin{pmatrix}\vec{x}\\ \vec{z}\end{pmatrix}+p_3\begin{pmatrix}\vec{z}\\ \vec{y}\end{pmatrix}\right]\right\}$$
$$exp\left\{\frac{-1}{(2M-1)^2}\left[(\vec{y}^{\,t},\vec{x}^{\,t})\mathbf{A}'\begin{pmatrix}\vec{y}\\ \vec{x}\end{pmatrix}+(\vec{x}^{\,t},\vec{z}^{\,t})\mathbf{A}'\begin{pmatrix}\vec{x}\\ \vec{z}\end{pmatrix}+(\vec{z}^{\,t},\vec{y}^{\,t})\mathbf{A}'\begin{pmatrix}\vec{z}\\ \vec{y}\end{pmatrix}\right]\right\} \tag{11}$$

where $\vec{B}^{\,t}_{iR} = (B_{iM}, B_{iM+1}, \ldots, B_{i2M-1})$ and the vectors $\vec{x}, \vec{y}$ and $\vec{z}$ refer to half strings in an obvious notation. The matrix $\mathbf{A}'$ (obtained from $\mathbf{A}$ in equation (9)) is given by :

$$\begin{aligned} A'_{ij} &= A'_{i+M,j+M} = A_{i+M,j+M} \\ A'_{i+M,j} &= A'_{i,j+M} = A_{M-i-1,j+M}\ , \quad i,j = 0,\ldots,M-1 \end{aligned}$$

p_3 is the momentum of the vector and κ' is a normalization constant related to κ and the normalization of the functionals. It will drop out in the final result.

It is worth however to work out in a bit of detail the dependence on the frequency ω_0. Eq. (9) shows that in the limit $\omega_0 \to 0$ the matrix $\mathbf{A}$ (and $\mathbf{A}'$) are singular. However, with a bit of work, it is possible to isolate the singular part which turns out to be

$$\frac{1}{\omega_0^{1/2}}\, exp\left\{-\left(\frac{\sum_{i=1}^{3} p_i}{2\sqrt{3}}\right)^2 \frac{1}{\omega_0}\right\}$$

Thus sending ω_0 to 0 one recovers the δ function of the conservation of momentum. This is not surprising because p_0 is the translational mode of the string, and the fact that its associated frequency vanishes is a statement about translational invariance.

Finally, to avoid unnecessary problems with the normalization we divide (11) by (10) to obtain

$$\frac{A(TTV)}{A(TTT)} = [A(p_1+p_2)]^{\mu}(p_2-p_1)_{\mu}\,\omega_1^{1/2}\,\vec{B}^{\,t}_{1R}\mathbf{M}^{-1}\vec{B}_{0R}\,exp\left\{-\frac{1}{2}\,\vec{B}^{\,t}_{0R}\mathbf{M}^{-1}\vec{B}_{0R}\right\} \quad (12)$$

where $M_{ij} = 2A_{i+M,j+M} - A_{M-i-1,j+M}$.

Since the result of the dual model (and of Witten String Field Theory as calculated in [2,3,4]) is

$$\frac{1}{2}\,[A(p_1+p_2)]^{\mu}\,(p_1-p_2)_{\mu}$$

for the method to work we must have that

$$\lim_{N\to\infty}\,\omega_1^{1/2}\,\vec{B}^{\,t}_{1R}\mathbf{M}^{-1}\vec{B}_{0R}\,exp\left\{-\frac{1}{2}\,\vec{B}^{\,t}_{0R}\mathbf{M}^{-1}\vec{B}_{0R}\right\} = 0.5 \qquad (13)$$

Unfortunately we are not able to invert the matrix in (13) analytically, it is however easy to perform the calculation numerically for various values of N.

N	result	error
4	0.629083	25.81 %
6	0.558865	11.77 %
20	0.508975	1.79 %
100	0.500964	0.10 %
200	0.500378	0.07 %
300	0.500219	0.04 %
400	0.500149	0.03 %
600	0.500087	0.02 %

Table 1: The results of the calculation of the ratio of the two amplitudes. In column one the number of points used for the discretization, in column 2 the calculated value, in column 3 the percent error from the correct value of 0.5.

The results are showed in table [1]. From the table it is evident that not only the convergence is quite fast, but that already with a small number of points (20 for example), it is possible to obtain results which are within a few percent of the correct value. This is very encouraging should one attempt more involved calculations, which might require much more computer time.

To conclude, we have shown that the discretization of the world surface provides a viable method to compute amplitudes in Witten String Field Theory, despite the fact that some calculations have to be performed numerically, and some possible initial worries about convergence. In this letter we presented only the calculation relating tachyons and vectors at tree level on shell. But similar amplitudes involving any other excited state can be obtained easily, with a slight modification of the equation (12), the changes are not expected to significantly modify the rate of convergence. For the moment the calculation done was just a test of feasibility. Work is in progress to use this method in problems which other approaches find either difficult or impossible to cope with.

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