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# String Fields as Limit of Functions and Surface Terms in String Field Theory

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**Abstract**

We consider the String Field Theory proposed by Witten in the discretized approach, where the string is considered as the limit $N \to \infty$ of a collection of $N$ points. In this picture the string functional is the limit of a succession of functions of an increasing number of variables; an object with some resemblances to distributions. Attention is drawn to the fact that the convergence is not of the uniform kind, and that therefore exchanges of limits, sums and integral signs can cause problems, and be ill defined. In this context we discuss some surface term found by Woodard, which arise in integrations by parts, and argue that they depend crucially on the choice of the successions of function used to define the identity and vertices of the theory.

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## 1 String Fields

Recently a lot of progress has been made towards the construction of a consistent String Field Theory, at least in the case of the bosonic strings. In this paper we will discuss some issue connected with the proposal based on non-commutative geometry made by Witten in [1]. In particular we discuss the issues connected with an operational definition of String Fields and the (unwanted) presence of surface term noted by Woodard [2] in some cases when integrations by parts are performed. The presence of those surface terms would require the modification of the action in the background free version of Witten's theory based on a purely cubic action [3]. Surface terms are discussed in section 2.

In dealing with these problems we will *discretize* the string, i.e. consider it as a collection of $N$ points, $N$ will eventually be sent to infinity, thus recovering string theory as the continuum limit. In this picture string fields are the limit of a succession of functions, much like distributions. In section 3, after a quick reminder of the theory of distributions, we discuss how string fields and vertices can be seen as some sort of distributions. The validity of this approach comes from the successes it has when actual calculations of vertices and amplitudes are performed, as we have shown in [4,5]. In this paper (among other things) we will discuss some of the possible pitfalls caused by the fact that the succession of functions cannot converge to the functional in the uniform sense.

In section 4 we will show how a proper choice of the succession of functions leading to the Identity functional could solve the problem of the unwanted surface terms in the purely cubic action.

Let us begin with a discussion of the differences and similarities between functions and functionals.

The basic objects of a string field theory are the string fields, those are generalizations of the fields of a regular field theory. A field $\psi(x)$ associates a number for each point of space-time, namely $\psi(x)$ is a continuous map from space-time into the complex numbers. A string field $\Psi[x(\sigma)]$ instead associates a complex number for each string configuration, where by string configuration we mean a map from the interval $[0,\pi]$ into space-time$^\dagger$(with the appropriate boundary conditions). We indicate a string configuration (in short a string) by $x^\mu(\sigma)$, or $x(\sigma)$ as we will usually suppress the Lorentz index. The fields in a

$^\dagger$Note that by a string we indicate a *parametrized* image regular field theory are functions, those of a string field theory are functionals.

Consider for example the function

$$\psi(x) = \exp(-x^2).$$

One could generalize it to string functionals in a variety of different ways, one could for example consider:

$$\begin{aligned}
\Psi[x(\sigma)]_1 &= \exp(-\int d\sigma x(\sigma)^2) \\
\Psi[x(\sigma)]_2 &= \exp(-(\int x(\sigma)^2) \\
\Psi[x(\sigma)]_3 &= \int d\sigma \exp(-x(\sigma)^2) \\
\Psi[x(\sigma)]_4 &= \exp(-x(\pi/2)^2
\end{aligned}$$

All of the above choices are legitimate functionals.

On the space of functionals it is possible to define an inner product as follows

$$<\Psi|\Phi> = \int \mathcal{D}x(\sigma)\Psi[x(\sigma)]\Phi[(\bar{x}(\sigma)] \tag{1.1}$$

where

$$\bar{x}(\sigma) = x(\pi - \sigma)$$

The $*$ product has an identity, $I[x(\sigma)]$, defined by the property:

$$\Psi * I = I * \Psi = \Psi$$

for each functional $\Psi[x(\sigma)]$. One way to represent $I[x(\sigma)]$ is the following:

$$I[x_L; x_R] = \delta\left(x(\sigma) - x(\pi - \sigma)\right) \tag{1.2}$$

for $\sigma = [0, \pi/2]$. Here $x_L$ and $x_R$ are the left and right halves of the string respectively. In other words $I[x(\sigma)]$ is a functional which vanishes for all strings, except for strings for which $x(\sigma) = x(\pi - \sigma)$, or $x_L = \bar{x}_R$. In this case then the value of $I[x(\sigma)]$ is infinity (in the sense of a $\delta$ function) for each point of the half string.

This point of view considers string functionals as maps from the space of string configurations into complex numbers. In particular one usually considers the wave function of a first quantized string as the functional, in a Fock space approach to string field theory [6,7,8]. This approach, which is equivalent to the discretized approach [5], reproduces correctly the results of the dual model, at least at tree level. In some related works, Chan and Tsou [9] have shown how Witten String Theory can be considered as the gauge theory of a point with a tail, where the gauge degree of freedom is the position of the tail. Bowick and Rajeev [10] have shown how string theory can be considered as the Kähler geometry of loop space. Both those two approaches are based on string functionals seen as maps from a space of strings into the complex numbers.

## 2 Surface Terms in String Field Theory

There are two techniques used to handle Witten String Field Theory. One can either consider conformal techniques, or techniques based on functionals to represent the string fields. We will follow this second alternative. In particular in the bulk of this section we will address the problem of some peculiar surface terms one encounters when integrating by parts, putting restrictions on the space of allowed gauge parameters. Woodard [2] has pointed out that, if one tries to prove the equivalence between Witten's action [1] and the purely cubic action of [3], one runs into troubles and finds infrared obstructions in the form of surface terms. Iwazaki [11] has found that similar surface terms can spoil the axiom $\oint Q\Psi = 0$. Here we will concentrate mainly on the work of Woodard.

The equation considered in [2] is the integration by parts law

$$Q_L\Psi{*}\Phi + (-)^{\Psi}\Psi{*}Q_R\Phi = 0 \tag{2.1}$$

for general string fields $\Psi$ and $\Phi$. An analogous integration by parts involves the momentum operator $P$.

$$P_L\Psi{*}\Phi + (-)^{\Psi}\Psi{*}P_R\Phi = 0 \tag{2.2}$$

where $P_L$ is the momentum $p(\sigma) = -i\delta/\delta x(\sigma)$ integrated over the left half of the string. And analogously $P_R$ is the momentum integrated over the right half of the string. Incidentally it has been shown that this last equation is crucial in showing that the translation operator is an inner derivation of the $*$ algebra [12].

Using conformal arguments it is possible to prove both (2.1) and (2.2), as it is done for example in [13], no surface terms appear and the second equality can be proven considering the vanishing of the functional integration over the two dimensional world sheet of the conserved current $j^{\mu}_{\alpha} = \partial_\alpha x^\mu$ with $\alpha = \sigma, \tau$, around a closed contour sandwiched between the right half of $\Psi$ and the left half of $\Phi$. The first equality can be proved with the use of the BRST current. If one however uses a string functionals formulation of the theory one does find the surface terms. Let us see this in more detail for (2.2), the calculation for (2.1) follows similar lines and can be found in [2].

With the usual definition of $*$ one has

$$(P_R\Psi{*}\Phi)[x_L; x_R] = \int \mathcal{D}y(\sigma) \left[\int_{\pi/2}^{\pi} d\sigma \ \frac{\delta}{\delta y(\sigma)}\Psi[x_L; y]\right] \Phi[\bar{y}; x_R] \tag{2.3}$$

If one considers the integration in a Volume $\mathcal{V}$, that is one constraints the strings to lie inside a box, one readily gets, after integrating by parts:

$$(P_R\Psi*\Phi)[x_L;x_R]+(\Psi*P_L\Phi)[x_L;x_R]=$$

$$=\int_{\pi/2}^{\pi}d\sigma\Psi[x_L;y(\sigma)\ \Phi[\bar{y}(\sigma);x_R]\Big|_{y(\sigma)\in\partial\mathcal{V}}\qquad \sigma\in[\pi/2,\pi] \tag{2.4}$$

Let us analyse in more detail the r.h.s. of eq. (2.4). The surface term receive a contribution for each string whose left half has at least one point on the surface of $\mathcal{V}$. If the volume $\mathcal{V}$ has a boundary, in general the r.h.s. of eq. (2.4) does not vanish for generic $\Psi[x(\sigma)]$ and $\Phi[x(\sigma)]$ and will therefore receive contribution from the value of the functionals on all the strings with points at infinity. Either strings on the boundary, or very long strings. Of course if some dimensions are compactified the surface terms for that directions would not appear, but for the surface term to disappear altogether in this way one would have to compactify all 26 dimensions of space-time!

## 3 String Fields and Vertices as Distributions

The earliest proposal to consider the string as the limit $N \to \infty$ of a discrete set of $N$ points function goes back to Giles and Thorn [14] in 1977, in their work they used both $\sigma$ and $\tau$ to be discrete, effectively considering the motion of strings on a lattice. The light cone gauge was considered and it was found that the continuous limit could only be recovered in the case of 26 dimensions. Nearly ten years afterwards, Thorn [15] completed the project considering a theory of relativistic interacting strings, always in the light cone gauge. Later Srednicki and Woodard [16] applied those ideas to Witten String Field Theory, and among other things they have shown that the discretization is an effective regulator for the theory. In [4,5] we showed that the discretization method is an useful and effective tool in the calculations of amplitudes.

In the context of String Field Theory the idea is to identify string functionals with functions of $2N$ variables, and then send $N$ to $\infty$. More precisely, given a string functional $\Psi[x(\sigma)]$, consider a sequence of functions, of a progressively increasing number of variables, $\Psi^N(\vec{x})$. Where $\vec{x}$ is an $2N$ component vector ‡

If one consider, as components of $\vec{x}$ $2N$ equidistant points on the string:

$$\vec{x} = (x(\sigma_0), x(\sigma_1), \dots, x(\sigma_{2N-1})),$$
$$x(\sigma_i) = x(\pi i/(2N-1)) \quad i = 0, \dots, 2N-1,$$

at each level $\Psi^N$ gives a value for each string, we will say that $\Psi^N$ converges to $\Psi[x(\sigma)]$ if

$$\forall \varepsilon \;\; \forall x(\sigma) \;\; \exists N_{min} \text{ such that } \left|\Psi[x(\sigma)] - \Psi^N\right| < \varepsilon \quad \forall N > N_{min} \tag{3.1}$$

One important thing to notice is that the convergence criterion defined in (3.1) is not of the uniform kind. In other words, given a string $x(\sigma)$ it is always possible to find an $N$ large enough that the difference between $\Psi[x(\sigma)]$ and $\Psi^N(\vec{x})$ is smaller than a given $\epsilon$. What is impossible is to find an $N$ such that the difference between functional and function is smaller than $\epsilon$ *for any* $x$. Consider for example the simple functional

$$L[x(\sigma)] = \int_o^\pi d\sigma x(\sigma)^2.$$

‡ We choose the number of points of the string to be even in order to avoid midpoint complications. The content of this paper would not change in the presence of a midpoint. The midpoint does however play an important role in Witten String Field Theory, and can be ignored only in a first treatment.

$L[x(\sigma)]$ associates to each string the square of its length in space time. The succession of functions can be easily seen to be:

$$L^N(\vec{x}) = \frac{\pi}{2N-1} \sum_{i=0}^{2N-1} x(\sigma_i)^2.$$

Obviously, given a string, no matter how twisted or long, for $N$ large enough $L^N$ will approximate $L$ arbitrarily well, but it will be impossible to find an $N$ which is large enough for any string, it will in fact be always possible to find a string with a large spike between two successive $x(\sigma_i)$, $L[x(\sigma)]$ will be sensitive to it, while $L^N$ will miss completely this feature. We will return on the uniform versus non uniform convergence in the next subsection, and see that its effect is that it is in general impossible to exchange limits, integrals and infinite sums among themselves.

With the above definition we can define the scalar product (1.1) as a limiting process:

$$<\Psi|\Phi>= \lim_N \kappa_N \int \prod_{i=0}^{2N-1} dx(\sigma_i)\Psi^N(\vec{x})\Phi^N(\vec{x}) \qquad (3.2)$$

where $\kappa_N$ is a normalization factor and

$$\vec{x} = (x(\sigma_{2N-1}), \ldots, x(\sigma_0))$$

In the previous definition the limit came before the integral, there is no obvious reason for this, except that we would not otherwise know how to perform the calculations. In [4,5] it is shown that the definition (3.2) does enable one to obtain the correct value for tree amplitudes.

Note however that $I[x(\sigma)]$ is not a 'good' functional, in the sense that it does not associate a number for each string. We defined it in the first section in an heuristic way as some sort of $\delta$ function. As it is known the Dirac $\delta$ is not a function, but a different object called a distribution. Distributions can sometimes be defined as the limit of a succession of functions, and this is similar to our definition (3.1) of string functionals as limit of functions. To highlight the similarities in the next subsection we present a very brief review, mainly based on examples, of some elementary aspects of distribution theory. We will make no attempt to be rigorous nor complete, and we refer to the mathematical literature [17,18,19] for a complete treatment. In particular what we here call distributions are what in the mathematical literature are called tempered distributions.

### 3.1 Distributions

The archetype of all distributions is the Dirac $\delta$. As it is known it is impossible to properly define it as a function, and it can only be defined either by its properties under integration, or as a limit of a succession of functions. That is, given a generic function $f(x)$ we can defined $\delta$ by

$$\int_a^b \delta(x-y)f(x)dx = f(y) \text{ if } y \in [a,b], \qquad 0 \text{ otherwise} \tag{3.3}$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$

Alternatively it is possible to define

$$\delta(x) = \lim_n \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$$

or as the limit of other successions.

Let us now introduce a set of test functions, namely a set of functions continuous with all their derivatives which vanish exponentially as $|x| \to \infty$. A distribution on a class of test functions is then a functional from the space of test functions into the real numbers. That is an object which gives a number for each function. We will use bra and ket notation with round brackets as delimiter to indicate the action of distributions on functions. So if $(a|$ is a distribution and $|f)$ a function, $(a|f)$ will be a number. With this notation $(\delta(x-y)|f(y)) = f(x)$.

Any function $|g)$ defines a distribution $(g|$ with the definition

$$(g|f) = \int_{-\infty}^{\infty} dx\ g(x)f(x) \tag{3.4}$$

Note that (3.4) is a way to define a function as well. Thus to identify $g$ one can either give its value at each point $x$, or the value of the integral (3.4) for each test function $f$. The function $g$ does not necessarily have to vanish at infinity.

In general a distribution $(a|$ can be defined as the limit of a succession of functions $a_n$ as follows:

$$(a|f) = (a_n|f) = \lim_n \int_{-\infty}^{\infty} a_n(x)f(x)dx \tag{3.5}$$

With a slight notational abuse we will indicate

$$\lim_n a_n = a$$

For example in the case of the Dirac $\delta$, if we define

$$\delta_n = n/\sqrt{\pi}\ e^{-nx^2} \tag{3.6}$$

it is easy to see that

$$\lim_n \delta_n = \delta$$

The $\delta_n$'s are useful to recover a function $g(x)$ (if it exists) from a distribution $(g|$ by defining

$$g(x) = \lim_n g_n(x) \equiv \lim_n (g|\delta_n) \quad . \tag{3.7}$$

One can also define a derivative $(a'|$ of a distribution $(a|$ by the rule

$$(a'|f) = (a|f')$$

where $f$ is any test function. Note that if $f = \lim_n f_n$, then $f' = \lim_n f'_n$ only if we can integrate by parts without problems, that is if there are no surface terms and we can exchange limits and integrals. This may not always be possible.

A continuous distribution will have to satisfy some continuity requirements, the usual one is the following: given a succession of functions $\phi_i,\ i = 1, 2, \ldots$ which converges to a function $\phi$ as $i \to \infty$

$$\lim_i \phi_i = \phi \tag{3.8}$$

then we must have

$$\lim_i (a|\phi_i) = (a|\phi) \tag{3.9}$$

The convergence of (3.8) is usually intended in the uniform sense:

$$\lim_i \phi_i = \phi \ \Rightarrow\ \forall\epsilon\ \exists i_{min} \text{ such that} |\phi_i(x) - \phi(x)| < \epsilon\ \forall i > i_{min}\ \forall x\ . \tag{3.10}$$

It is possible however to define a weaker form of convergence as

$$\lim_i \phi_i = \phi \ \Rightarrow\ \forall\epsilon\ \forall x\ \exists i_{min} \text{ such that} |\phi_i(x) - \phi(x)| < \epsilon\ \forall i > i_{min}\ . \tag{3.11}$$

Note that the only difference between the two definitions of convergence is the position of the $\forall x$.

An example of succession of functions which is weakly but non uniformly convergent is:

$$\phi_i = e^{-(x-i)^2}$$

As $i$ grows the exponent becomes infinitively negative and one could conclude that $\lim \phi_i = 0$. Let us examine this claim in the light of the two convergence criteria we just presented. Given an $x$ and an $\epsilon$ it is possible to find $i_{min} = \sqrt{-\log \epsilon + x^2}$ such that $|\phi_i| < \epsilon$ for $i > i_{min}$. We can conclude therefore that $\lim \phi_i = 0$ in the sense of (3.11). But it is impossible to find an $i_{min}$ for which $\phi_i < \epsilon \; \forall x$, in fact it will be enough to take $x = i$ to have $\phi_i(x) = 1$. The succession therefore does not converge to 0 in the uniform sense of (3.10). A consequence is that it is not possible to exchange limits and integral signs, in this case

$$\lim_i \int_{-\infty}^{\infty} dx e^{-(x-i)^2} = \lim_i \sqrt{\pi} = \sqrt{\pi}$$

while

$$\int_{-\infty}^{\infty} dx \lim_i e^{-(x-i)^2} = 0$$

All the concepts expressed so far can be straightforwardly extended to the d-dimensional case without any trouble.

Let us now consider a two dimensional example, the distribution $\tilde{\delta}(x, y)$ defined as the limit

$$\tilde{\delta}(x, y) = \lim_n \tilde{\delta}_n = \lim \frac{n}{\sqrt{\pi}} e^{-n^2(x-y)^2}$$

The tilde above the $\delta$ signifies that this distribution is not the two dimensional Dirac $\delta$ which instead is the limit of

$$\frac{n^2}{\pi} e^{-n^2x^2 - n^2y^2}$$

Under integration of a two variables function, $\tilde{\delta}$ behaves as follows:

$$\int_{-\infty}^{\infty} dy \; f(x, y)\tilde{\delta}(y, z) = f(x, z)$$

and

$$\int_{-\infty}^{\infty} dy \; dx \; f(x, y)\tilde{\delta}(y, x) = \int_{-\infty}^{\infty} dx \; f(x, x)$$

Note that the last two equations are reminiscent of Witten's star product of a string field with the Identity $I[x(\sigma)]$ and integral respectively.

Consider now the distribution

$$\tilde{\delta}'(x, y) = \lim_n \frac{\partial}{\partial x} \tilde{\delta}_n(x, y) = \lim_n \frac{-2n^3(x-y)}{\sqrt{\pi}} e^{-n^2(x-y)^2} \qquad (3.12)$$

For most purposes $\tilde{\delta}'$ acts as a derivative of $\tilde{\delta}$, for example:

$$\int_{-\infty}^{\infty} dy\ f(x,y)\tilde{\delta}'(y,x) = -\frac{d}{dx}f(x,x)$$

However some subtleties have to be taken into account when one attempts to integrate by parts. Consider the integration by parts in a box of size $l$

$$-l \leq x \leq l \qquad -l \leq y \leq l$$

we will send $l \to \infty$ after the integration. We then consider

$$\lim_{l\to\infty}\lim_{n}\int_{-l}^{l} dy\ f(x,y)\tilde{\delta}_n'(x,y) = -\frac{d}{dx}f(x,x) + \lim_{l\to\infty}\lim_{n}\left[f(x,y)\frac{n}{\sqrt{\pi}}e^{-n^2(x-y)^2}\right]_{y=-l}^{y=l} \tag{3.13}$$

The surface term of equation (3.13) does not vanish for $x = y$ when $l \to \infty$, moreover it is not clear that one gets the same value when interchanging the two limits. The reason for this is that we are in the presence of non uniform convergence.

These problems can however be cured by considering the distribution

$$\hat{\delta}(x,y) = \lim_{n}\frac{n}{\sqrt{\pi}}e^{-(x-y)^2n^2-\frac{1}{n^p}(x+y)^2} \qquad p > 0 \tag{3.14}$$

It is easy to see that, as a distribution $\hat{\delta} = \tilde{\delta}$, in fact

$$\int_{-\infty}^{\infty} dy\ f(x,y)\hat{\delta}(y,z) = f(x,z) \tag{3.15}$$

Things are different if one considers the distribution $\hat{\delta}'$ defined by

$$\hat{\delta}'(x,y) = \lim_{n}\frac{\partial}{\partial x}\hat{\delta}_n(x,y) = \lim_{n}\frac{-2n^3(x-y)-2n^{1-p}(x+y)}{\sqrt{\pi}}e^{-n^2(x-y)^2-\frac{1}{n^p}(x+y)^2} \tag{3.16}$$

Again $\hat{\delta}'$ behaves as the derivative of $\hat{\delta}$, but now we have the second factor in the exponentials, which acts as a convergence term, there are no more problems regarding to surface terms:

$$\lim_{l\to\infty}\lim_{n}\int_{-l}^{l} dy\ f(x,y)\hat{\delta}_n'(x,y) =$$

$$= -\frac{d}{dx}f(x,x) + \lim_{l\to\infty}\lim_{n}\left[f(x,y)\frac{n}{\sqrt{\pi}}e^{-n^2(x-y)^2-\frac{1}{n^p}(x+y)^2}\right]_{y=-l}^{y=l} \tag{3.17}$$

Now the surface term vanishes for either $l \to \infty$ or $n \to \infty$, the convergence term in the definition of the distribution has solved the surface terms problems without affecting the properties of the distribution. We will perform a similar trick in the next subsection.

### 3.2 String Fields

Let us first of all notice that the definition of a string field, as given in (3.1) is quite like the one of a distribution. In both cases one has the limit of a succession of functions, which is not a function itself. The main difference being that in the convergence defined in (3.1) the functions are from a different space at each step. Moreover objects such as $I[x(\sigma)]$, and the vertices as we will see, are not even functionals in the usual sense of a map from the space of strings into the complex numbers, they are the functional equivalent of a distribution, they are defined only with respect to their properties under integration. We will call those objects distributionals.

So far we have defined functionals as maps, in analogy with the case of functions we can define a distributional from a functional with a formula similar to (3.4).

Just as one defines functions of more than one variable one can similarly define functionals of many variables, the simplest example of which would be the product of two string functionals: $\Psi[x(\sigma)]\Phi[y(\sigma)]$. Let us now define a class of distributionals which we will generically call $<\!V_n|$. The $<\!V_n|$'s act on functionals of $n$ variables to give a complex number. In particular let us define

$$<\!V_2|\Psi\!>|\Phi\!> = <\!\Psi|\Phi\!> \tag{3.18}$$

$V_2$ can be used to obtain distributionals from functionals by defining

$$<\!\Psi| = <\!V_2|\Psi\!>|I\!>|\cdot\!> \tag{3.19}$$

$<\!\Psi|$ is obviously a distributional (when saturated with a functional we get a number). In general, for $p > q$,

$$<\!V_p|\Psi_1\!>\cdots|\Psi_q\!>|\cdot\!>\cdots|\cdot\!>$$

is a $p-q$ distributional.

Let us now define the $*$ product as

$$<\!\Psi*\Phi| = <\!V_3|\Psi\!>|\Phi\!>|\cdot\!> \tag{3.20}$$

The only problem is that $<\!\Psi*\Phi|$ is a distributional, we want to obtain a functional from it. To do this we will follow a procedure similar to the one we outlined in the previous subsection. Let us consider the $V_n$ and in particular $V_2$ as the limit of functions, just as we did for functionals, i.e.

$$V_2[x(\sigma), y(\sigma)] = \lim_N V_2^N(\vec{x}, \vec{y}) \tag{3.21}$$

Then we can define

$$\Psi[x(\sigma)] = \lim_N <\Psi[y]|V_2^N(\vec{x}, \vec{y})> \tag{3.22}$$

where now the integration implicit in the r.h.s. runs only over $y(\sigma)$.

Obviously in some cases the limit in (3.22) might not converge to a functional at all, just as in (3.6) the $\delta_n$'s did not converge to a function.

Another property we will require is the following:

$$\begin{aligned} <V_n|\Psi_1>|\Psi_2>\cdots|\Psi_n> &= <V_{n-1}|\Psi_1*\Psi_2>\Psi_3>\cdots|\Psi_n> \\ &= <V_{n-1}|\Psi_1>|\Psi_2*\Psi_3>\cdots|\Psi_n> \quad \text{etc.} \end{aligned} \tag{3.23}$$

and its generalizations.

The integral is defined by

$$\oint \Psi = <V_1|\Psi> \tag{3.24}$$

Let us examine in a little more detail the properties of $V_1$. From (3.23) follows that

$$<V_n|\Psi_1>\cdots|\Psi_n> = \oint \Psi_1 * \cdots \Psi_n \tag{3.25}$$

Another important thing to notice is that $V_1$ acts as the identity in the $*$ algebra:

$$<V_1*\Psi| = <V_3|V_1>|\Psi>|\cdot> = <V_2|\Psi|\cdot> = <\Psi| \tag{3.26}$$

Notice that $V_1$ appears as a ket in this last equation, this has to be understood in the limiting sense we will specify below. Since $V_1$ acts as the identity we will identify it with $I$, using the two symbols in an interchangeable way.

Although one could envisage more general cases, in Witten theory one has:

$$<V_n|\Psi_1>|\cdots|\Psi_n> = \int \mathcal{D}x_1\cdots\mathcal{D}x_n\delta(x_{1_L} - \bar{x}_{2_R})\cdots\delta(x_{n_L} - \bar{x}_{1_R})$$

$$\Psi_1[x_{1_L}, x_{1_R}]\cdots\Psi_n[x_{n_L}, x_{n_R}] \tag{3.27}$$

Or

$$V_n[x_1, \ldots, x_n] = \delta(x_{1_L} - \bar{x}_{2_R})\cdots\delta(x_{n_L} - \bar{x}_{1_R}) \tag{3.28}$$

To write $V_n$ as a functional is of course as ill defined as considering Dirac's $\delta$ as a function, and it must be understood in exactly the same sense. As in the case of distributions, in order to make sense of distributionals we have to define them as limits of well defined functionals, which in turn are defined as limits of $N$ variables functions. One could for example define for $I$

$$I[x(\sigma)] = \lim_{N\to\infty} \prod_{j=0}^{N} \frac{N}{\sqrt{\pi}} e^{-N^2(x(\sigma_{2N-j})-x(\sigma_j))^2} \tag{3.29}$$

The definition of $I$ given in (3.29) is quite useful, it says that, considered as a function, $I^N[\vec{x}]$ is exponentially vanishing, unless the string has

$$x(\sigma_i) = x(\sigma_{2N-i}) \qquad \forall i$$

and in this case the value goes as $N^N$, therefore, unless the string has $x(\sigma) = x(\pi - \sigma)\ \forall\sigma$, or $x_L = \bar{x}_R$, all the $I^N$ will be exponentially small for $N$ large enough.

For a generic vertex this expression readily generalizes to

$$V_n[x_1(\sigma), \ldots, x_n(\sigma)] = \lim_{N\to\infty} \prod_{i=1}^{n} \prod_{j_i=0}^{N} \frac{N}{\sqrt{\pi}} e^{-N^2(x_i(\sigma_{2N-j_{i+1}})-x_{i+1}(\sigma_{j_{i+1}}))^2} \tag{3.30}$$

Where $x_{n+1} = x_1$.

$V_n^N$ is exponentially small unless the right half of the first string overlaps with the left half of the second, the right half of the second with the left of the third and so on.

One has to check that, for example, $<V_2|\Psi>|\Phi> = <\Psi|\Phi>$. In fact

$$<V_2|\Psi>|\Phi> =$$

$$= \lim_N \int d\vec{x}\ d\vec{y}\ \Psi^N(\vec{x})\Phi^N(\vec{y}) \frac{N^2}{\pi} \prod_{i=0}^{N} e^{-N^2(x(\sigma_i)-y(\sigma_{2N-i}))} e^{-N^2(y(\sigma_i)-x(\sigma_{2N-i}))}$$

$$= \lim_N \int d\vec{x}\ d\vec{y}\ \Psi^N(x(\sigma_i), y(\sigma_{2N-i})))\Phi^N(y(\sigma_i), x(\sigma_{2N-i}))) + O\left(\frac{1}{N^2}\right) \tag{3.31}$$

Where

$$d\vec{x} = \prod_{i=0}^{2N} dx(\sigma_i)$$

Finally, with a change of variables, we obtain

$$\lim_N \int d\vec{x}\ \Psi^N(\vec{x})\Phi^N(\bar{\vec{x}})\ + \text{higher orders in } \frac{1}{N}$$

Similar checks proceed in the same way.

With this definition for the vertex we have the usual (and useful) way to represent the $*$ product

$$(\Psi*\Phi)[x_L, x_R] = \lim_N \int d\vec{y}\ \Psi^N(\vec{x}_L, \vec{y})\Phi^N(\vec{y}, x_L) + \text{higher orders in } \frac{1}{N} \tag{3.32}$$

The choice of $I^N$ in (3.29) however is not unique, just as in the case of distributions. One could for example consider the following succession

$$\hat{I}[x(\sigma)] = \lim_{N\to\infty} \prod_{j=0}^{N} \frac{N}{\sqrt{\pi}} e^{-N^2(x(\sigma_{2N-j})-x(\sigma_j))^2 - \frac{1}{N^p}(x(\sigma_{2N-j})+x(\sigma_j))^2} \tag{3.33}$$

where $p > 0$.

Equation (3.33) generalizes in an obvious way to $\hat{V}_n^N$:

$$\hat{V}_n[x_1(\sigma), \ldots, x_n(\sigma)] =$$

$$= \lim_{N\to\infty} \prod_{i=1}^{n} \prod_{j_i=0}^{N} \frac{N}{\sqrt{\pi}} e^{-N^2(x_i(\sigma_{2N-j_{i+1}})-x_{i+1}(\sigma_{j_{i+1}}))^2 - \frac{1}{N^p}(x_i(\sigma_{2N-j_i})+x_{i+1}(\sigma_{j_i+1}))^2} \tag{3.34}$$

As $N \to \infty$, $\hat{I}^N \to I^N$ as well, thus showing that, as distributionals, $\hat{I}$ and $I$ are the same. But again, as in the case of distributions, there are some subtleties in the case of integrations by parts. We will discuss them in the next section.

## 4 Surface Terms in the Discretized Approach

In this section we will consider the surface terms (2.1,2.2) in the discretized approach,using the various successions we introduced in the last section. We will only calculate the surface terms of (2.2) for $P_L I$, the discussion for $Q_L I$ of (2.1) is similar and will be omitted.

In the discretized approach we define $P_L\Psi$ as

$$P_L\Psi = \lim_{N\to\infty} \frac{\pi}{2N-1}\sum_{i=0}^{N} \frac{\partial}{\partial x(\sigma_i)}\Psi^N(\vec{x}) \tag{4.1}$$

If now for the identity we use the succession defined in (3.29) we have that

$$P_L I = \lim_{N\to\infty} \frac{\pi}{2N-1}\sum_{i=0}^{N-1} \frac{\partial}{\partial x(\sigma_i)} I^N =$$

$$= \lim_{N\to\infty} \frac{\pi}{2N-1}\sum_{i=0}^{N-1} \frac{\partial}{\partial x(\sigma_i)} \prod_{j=0}^{N-1} \frac{N}{\sqrt{\pi}} e^{-N^2((x(\sigma_j)-x(\sigma_{2N-j}))^2} =$$

$$= \lim_{N\to\infty} -\frac{\pi}{2N-1}\sum_{i=0}^{N-1} 2N^2((x(\sigma_i)-x(\sigma_{2N-i}))^2 \prod_{j=0}^{N-1} \frac{N}{\sqrt{\pi}} e^{-N^2((x(\sigma_j)-x(\sigma_{2N-j}))^2} \tag{4.2}$$

If instead we use the succession defined in (3.33) we have

$$P_L I = \lim_{N\to\infty} \frac{\pi}{2N-1}\sum_{i=0}^{N-1} \frac{\partial}{\partial x(\sigma_i)} \hat{I}^N =$$

$$= \lim_{N\to\infty} -\frac{\pi}{2N-1}\sum_{i=0}^{N-1} 2N\left\{n^2((x(\sigma_i)-x(\sigma_{2N-i}))^2 + \frac{1}{N^p}((x(\sigma_i)+x(\sigma_{2N-i}))^2\right\}$$

$$\prod_{j=0}^{N-1} \frac{N}{\sqrt{\pi}} e^{-N^2(x(\sigma_j)-x(\sigma_{2N-j}))^2-\frac{1}{N^p}(x(\sigma_j)+x(\sigma_{2N-j}))^2} \tag{4.3}$$

Those two successions converge to the same distributionals, however, as in the case of (3.17), their behaviour in the presence of an integration by parts is quite different. Let us again consider, as in [2] the string in a box of length $2l$:

$$-l \leq x(\sigma_i) \leq l$$

We now have

$$(P_R\Psi{*}\Phi)[x(\sigma)] = \lim_{N\to\infty} \frac{\pi}{2N-1}\int d\vec{y}\sum_{j=0}^{N} \frac{\partial}{\partial y(\sigma_j)}\Psi^N(\vec{x}_L;\vec{y})\Phi(\vec{y};\vec{x}_R) \tag{4.4}$$

Where now the vectors refer to half strings.

When we perform the integration by parts (4.4) becomes:

$$((P_R\Psi)*\Phi)[x_L,x_R] =$$

$$\lim_{N\to\infty}\frac{\pi}{2N-1}\sum_{i=0}^{N-1}\Psi^N(\vec{x}_L,\vec{y})\Phi^N(\vec{y},\vec{x}_R)|_{y(\sigma)=-l}^{y(\sigma)=l} - (\Psi*P_L\Phi)[x_L,x_R] \qquad (4.5)$$

Where a change of variables $y(\sigma_i) \to y(\sigma_{2N-i})$ has been performed for the second term of the r.h.s.

The first term in (4.5) is the unwanted surface term[§] This term receives a contribution every time one of the $x(\sigma_i)$ lies on the boundary of the box. It can easily be seen that in this case the surface term does not vanish as $l \to \infty$.

The surface terms in (4.5) are however strongly dependent on which succession one chooses to define $I$. In fact if one defines $I$ as the succession of the $\hat{I}^N$ defined in (3.33) one has, in place of(4.5)

$$((P_R\Psi)*\Phi)[x_L,x_R] =$$

$$\lim_{N\to\infty}\frac{\pi}{2N-1}\sum_{i=0}^{N-1}\left(\prod_{j_i=0}^{N-1}\frac{N}{\sqrt{\pi}}e^{-N^2(x(\sigma_j)-x(\sigma_{2N-j}))^2-\frac{1}{N^p}(x(\sigma_i)+x(\sigma_{2N-i}))^2}\right)$$

$$\Psi(\vec{x}_L,\vec{y})\Phi(\vec{y},\vec{x}_R)|_{y(\sigma)=-l}^{y(\sigma)=l} - (\Psi*P_L\Phi)[x_L,x_R] = \qquad (4.6)$$

The surface term in (4.6) vanishes as either $l$ and $N \to \infty$ in whichever order.

Which definition of vertices and identities should one use? Comparison with conformal techniques would suggest the use of the ones defined with the convergence factor. However even such a definition could still have some problems, for example with respect to midpoint singularities. It does not seem possible at this point to give a conclusive answer. One should probably keep a pragmatic point of view in choosing whichever definition is convenient for the calculation at hand. And an open mind to try to avoid the problems and pitfall caused by the not yet precise knowledge of the mathematical objects string fields are, and above all, in case one uses the discretized approach, extreme care should be taken because of the weak convergence of the succession of functions to the integral.

In any case, so far the discretized approach to String Field Theory seems to be a viable computational tool, and while some more work is needed on the formal

[§]note that in (4.4) we were allowed to exchange sum and product because their range is finite side, its successes in reproducing the dual model results make it definitively a reliable method.

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