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The Space of String Configurations in String Field Theory

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Abstract

In this paper we consider the set of maps from the interval $[0, \pi]$ which constitute the argument of the functionals of a String Field Theory. We show that in order to correctly reproduce results of the dual model one has to include all square integrable functions in the functional integral, or Ω_0 in terms of Sobolev spaces.

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One way to construct a String Field Theory[†] is to generalize the concept of the point x, considered as the argument of a field $\psi(x)$, to the string $x(\sigma)$, now seen as the argument of a *functional* $\Psi[x(\sigma)]$. The $x(\sigma)$ are maps from the interval $[0,\pi]$ with proper boundary conditions, in the case of open strings, or the circle in the case of closed ones, into some manifold. This leaves open the question: Which maps?, Should they be continuous, with some of their derivatives?, Square integrable? ... In this paper we will discuss some of the features of this infinite dimensional space, which in this paper we will call String Configuration Space.

Horowitz and Witt have argued in [4] that this space is independent of the dimensions of space time, and depends at most weakly on its other properties, like its topology etc. They found Sobolev spaces to be a useful tool in this sort of investigation, and argued that one has to include at least discontinuous maps as arguments of the functionals. Here we work out in detail the contributions of various spaces to an actual numerical computation of an amplitude in Witten's [5] approach to Open String Field Theory. We will work with a flat background and ignore ghosts in what follows, we choose an amplitude for which the ghost variable does not contribute. We find that in order to reproduce the correct amplitude the configuration space has to include all square integrable functions (Ω_0 in terms of Sobolev spaces).

Given an open string, that is a map $x(\sigma)$, with boundary conditions $x'(0) = x'(\pi) = 0$, consider the Fourier expansion

$$x(\sigma) \;=\; \frac{x_0}{\sqrt{2}} + \sqrt{2}\sum_{n=1}^{\infty} x_n \cos(n\sigma).$$

We can then define, for each real number s, the Sobolev spaces as the set of maps for which

$$\sum_{0}^{\infty} x_n^2 (1+n^2)^s < \infty.$$

Among the properties of Sobolev spaces (see [4] and references therein) there is the inclusion relation

$$\Omega_p \subset \Omega_q \quad (p > q). \tag{1}$$

Moreover Ω_0 is the set of square integrable maps. For s negative Ω_s contains distributions. Furthermore:

$$\Omega_s \subset C^n \;\;,\;\; s > \frac{1}{2} + n,$$

[†]For reviews see [1,2,3] and references therein.

where C^n is the set of continuous functions with their first n derivatives.

In order to determine which are the important strings contributing to the functional integrals appearing in String Field Theory, we will follow the following strategy. We will consider the tree level amplitude for the scattering of a vector with two tachyon states, normalized to the tachyon–tachyon–tachyon scattering, in Witten's [5] approach to string Field Theory. These amplitudes involve functional integrals in string space. We will consider these functional integrals restricted to some appropriate subsets of this space, and will show that the whole of Ω_0 is necessary to reproduce the dual model result.

The functional integrals are difficult to treat, and necessitate a regulator. We will work in the discretized version of the theory introduced in [6,7,8,9]. Instead of continuous strings $x(\sigma)$ we will consider N points on it, collected in a vector $\vec{x}$, with components $x(\sigma_i) \equiv x(\frac{i\pi}{N})$. The string functionals are now functions $\Psi^N(\vec{x})$, and the product and integral of the theory can be defined as

$$\begin{aligned}
(\Psi{*}\Phi)(x_L;x_R) &= \lim_{N\to\infty}(\Psi{*}\Phi)^N(x_L;x_R) = \\
&= \lim_{N\to\infty}\sqrt{\kappa_N}\int dy(\sigma_M),\ldots,dy(\sigma_N)\ \Psi^N(x_L;y(\sigma_M),\ldots,y(\sigma_N)) \\
&\quad \Phi^N(y(\sigma_N),\ldots,y(\sigma_M);x_R) \\
\oint\Psi &= \lim_{N\to\infty}\kappa_N\int dx(\sigma_0),\ldots,dx(\sigma_M) \\
&\quad \Psi^N(x(\sigma_0),\ldots,x(\sigma_M),x(\sigma_M),\ldots,x(\sigma_0))
\end{aligned} \tag{2}$$

where $M = N/2(N \text{ even})$ or $M = (N-1)/2(N \text{ odd})$ and κ_N is a normalization constant, $x_L(x_R)$ corresponds to the left (right) halves of the string $x(\sigma)$ and, in both cases the integrations are performed over half string.

Consider now the Tachyon–Tachyon–Vector (TTV) tree level amplitude normalized to the Tachyon–Tachyon–Tachyon (TTT) amplitude. In this paper we will ignore ghosts since they do not contribute to the ratio of amplitudes we are considering here. In a sense here we are only concerned with the bosonic part of the string configuration space relevant to string theory.

$$(\mathrm{TTT})^N = \kappa'\int\prod_{i=0}^{M-1} dx(\sigma_i)\ dy(\sigma_i)\ dz(\sigma_i)\ \mathcal{Z} \tag{3}$$

$$(\mathrm{TTV})^N = \kappa' p_3 2\omega_1\int\prod_{i=0}^{M-1} dx(\sigma_i)dy(\sigma_i)dz(\sigma_i)\frac{i}{N}\vec{\mathbf{B}}^{\,T}_{1R}(1,0,-1)\begin{pmatrix}\vec{y}\\ \vec{x}\\ \vec{z}\end{pmatrix}\mathcal{Z} \tag{4}$$

with

$$\mathcal{Z} = \exp\left\{\frac{i}{N}\vec{\mathbf{B}}_{0R}{}^{T}(1,1)\left[p_1\begin{pmatrix}\vec{y}\\ \vec{x}\end{pmatrix} + p_2\begin{pmatrix}\vec{x}\\ \vec{z}\end{pmatrix} + p_3\begin{pmatrix}\vec{z}\\ \vec{y}\end{pmatrix}\right]\right\}$$

$$\exp\left\{\frac{-1}{(N)^2}\left[(\vec{y}^T,\vec{x}^T)\mathbf{A}\begin{pmatrix}\vec{y}\\ \vec{x}\end{pmatrix} + (\vec{x}^T,\vec{z}^T)\mathbf{A}\begin{pmatrix}\vec{x}\\ \vec{z}\end{pmatrix} + (\vec{z}^T,\vec{y}^T)\mathbf{A}\begin{pmatrix}\vec{z}\\ \vec{y}\end{pmatrix}\right]\right\}$$

in the above equation, $\vec{\mathbf{B}}\,{}^{T}_{iR} = (B_{iM}, B_{iM+1}, \ldots, B_{iN})$ where $B_{nm} = c_{nm}cos(n\sigma_m)$ ($c_{nm} = 1/2$ for $m = 0, N$ and $c_{nm} = 1$ otherwise), $\omega_k = \frac{sin(k\pi/2N)}{\pi/2N}$ and the vectors $\vec{x}, \vec{y}$ and $\vec{z}$ refer to half strings in an obvious notation. The matrix $\mathbf{A}$ is given by:

$$\begin{aligned} A_{ij} &= A_{i+M,j+M} = \sum_{k=0}^{N}\omega_k B_{k,i+M}B_{k,j+M} \\ A_{i+M,j} &= A_{i,j+M} = \sum_{k=0}^{N}\omega_k B_{k,M-i-1}B_{k,j+M} \quad i,j = 0,\ldots,M-1 \end{aligned}$$

p_3 is the momentum of the vector and κ' is a normalization constant related to κ_N and the normalization of the functionals. It will drop out in the ratio.

We have that [7,9,10], defining $\mathcal{R} = \dfrac{(\mathrm{TTV})}{(\mathrm{TTT})}$

$$\mathcal{R} = -p_3^{\mu}\omega_1(p_{1\mu} - p_{2\mu})C_{10}\exp\left[-\frac{1}{4}(m_V^2 - m_T^2)C_{00}\right] \tag{5}$$

where the relevant factors C_{ij} are given by:

$$C_{ij} = \vec{\mathbf{B}}^{T}_{(iR)}\left[\frac{1}{\mathbf{M}}\right]\vec{\mathbf{B}}_{(jR)} \tag{6}$$

being $M_{ij} = 2A_{i+M,j+M} - A_{M-i-1,j+M}$.

Numerical calculations show that, for $N = 600$

$$C_{00} = 0.52336 \quad C_{10} = -0.38495 \tag{7}$$

which gives

$$\mathcal{R} = 0.5\; p_3(p_1 - p_2)$$

in agreement with the dual model result.

We will now repeat the above calculations reducing the functional integral of $\oint$ to a particular domain.

Let us consider the strings $x(\sigma)$ belonging to the space $\Omega'_p(R)$, identified by (a limit $N \to \infty$ is understood)

$$\Omega'_p(R) = \left\{x(i),\ i = 0\ldots N\ such\ that\ \ N^{p-1}\sum_{i=0}^{N}x(i)^2\ <\ R^2\right\} \tag{8}$$

where p is any real number. An inclusion relation, analogous to (1) is readily obtained:

$$\Omega'_p(R) \subset \Omega'_q(R) \quad (p > q). \tag{9}$$

Those spaces can be related to the Sobolev spaces mentioned in the introduction in the limit $R \to \infty$. Let us indicate $\Omega'_p = \lim_R \Omega'_p$. Considering $\sum_{n=0}^N n^p x_n^2 < N^p \sum_{n=0}^N x_n^2 = N^{p-1} \sum_{n=0}^N x(\sigma_n)$ for $p > 0$ (and conversely for $p < 0$) we have the following inclusion relations:

$$\begin{aligned} \Omega_p &\subset \Omega'_p \ (p < 0) \\ \Omega_0 &\equiv \Omega'_0 \\ \Omega'_p &\subset \Omega_p \ (p > 0). \end{aligned}$$

To see what happens when we restrict the domain of integration, let us start from equation (3) written in the following form:

$$(\mathrm{TTT})_p^N = \kappa \int D\vec{X} \exp\left[-\vec{X}\mathbf{A}'\vec{X}\right] \exp\left[i\vec{K}\vec{X}\right] \Theta_p(R) \tag{10}$$

Where $\vec{X} = (\vec{x}, \vec{y}, \vec{z})$ is a vector whose components are the three half strings, $\vec{K} = \vec{B}_R(p_1 + p_2, p_2 + p_3, p_3 + p_1)$, $\mathbf{A}'$ is a block rearrangement of $\mathbf{A}$ and $\Theta_p(R)$ is a function which restricts the integration to the proper domain $\Omega'_p(R)$. For instance, if we were interested in restricting the integration range into a box of side $2R$, the form of $\Theta_p(R)$ would be just the product of step functions of the form $\theta(x(\sigma_i) - R)\theta(x(\sigma_i) + R)$. In the general case of interest for us (8) however, the actual form of $\Theta_p(R)$ is not easy to work out and we will rely on some analytical approximations (see below).

Before going on let us comment on possible infinities coming from translational invariance. The matrix $\mathbf{A}'$ appearing in (10) is singular in the limit $\omega_0 \to 0$ that one has to take at the end of the calculation. We can however understand the meaning of this infinity by isolating the singularity. Precisely, if we diagonalize the matrix $\mathbf{A}'$ and single out the terms containing ω_0, that gives the singular piece, one gets the factor:

$$\frac{R}{[1 + 4R^2\omega_0]^{1/2}} \exp\left[-\omega_0^{-1}(\sum_{i=1}^{3} p_i)^2\right] \tag{11}$$

that, as we send ω_0 to zero gives $R\delta(\sum p_i)$, here δ is a Kronecker function corresponding to the conservation of momentum in a finite space. Taking the limit $R \to \infty$ one recovers the usual Dirac delta function.

Now, to proceed further, we can simulate the behaviour of Θ_p *for large R* using the product of Gaussian–like functions:

$$\prod_{i=0}^{N} \exp\left(-N^{p-1}\frac{x(i)^2}{R^2}\right). \tag{12}$$

which stresses the contribution of the strings in the space Ω'_p.

To better understand the meaning of this approximation, it is interesting to work out in detail an example. Let us take p=1 in (8), that implies considering Ω'_1. The product of exponentials now stresses the contribution from the strings such that $|x(\sigma_i)| \leq R$ for $i = 0, \ldots, N$, i.e. the ones contained in a box of side $2R$. We can connect this result with the product of step functions mentioned above. In fact, taking the Fourier transform of $\Theta_1(R)$ one has the expression:

$$\Theta_1(R) = \left(\frac{1}{2\pi}\right)^{N/2} \int_{-\infty}^{\infty} D\vec{Q}\ \ f(\vec{Q}, R)\ \ e^{i\vec{Q}\vec{x}}$$

where the function $f(\vec{Q}, R)$ is given by:

$$f(\vec{Q}, R) = \prod_{i=0}^{N}\ \ 2\frac{sinQ_iR}{Q_i}$$

which, as $R \to \infty$, gives $\delta(\vec{Q})$. Hence, what we have effectively done in (12) is to approximate this Dirac function by its expression in terms of product of Gaussian functions which gives the same behaviour for $f(\vec{Q}, R)$ in the limit of large R. Integrating back in $D\vec{Q}$ one gets the approximation of $\Theta_1(R)$ given in (12).

In general, the result for the ratio of vertices (5), restricted to the domain Ω'_p, is now given by:

$$\mathcal{R}_p = -p_3^{\mu}\omega_1(p_{1\mu} - p_{2\mu})C_{10}^{(p)}(R)\exp\left[-\frac{1}{4}(m_V^2 - m_T^2)C_{00}^{(p)}(R)\right] \tag{13}$$

where the coefficients $C_{ij}^{(p)}$ are:

$$C_{ij}^{(p)} = \vec{\mathbf{B}}_{iR}^T\left[\frac{1}{\frac{N^{p+1}}{R^2} + \mathbf{M}}\right]\vec{\mathbf{B}}_{jR}. \tag{14}$$

$\mathcal{R}_p$ is the contribution to $\mathcal{R}$ of the strings belonging to Ω'_p, if $\mathcal{R}_p$ vanishes this means that the contribution of those strings is 0. We are interested in finding the smallest Ω'_p for which $\mathcal{R}_p = \mathcal{R}$.

	$N^n\vec{\mathbf{B}}_{0R}^T\mathbf{M}^{-(n+1)}\vec{\mathbf{B}}_{0R}$			$N^n\vec{\mathbf{B}}_{1R}^T\mathbf{M}^{-(n+1)}\vec{\mathbf{B}}_{0R}$		
$N\backslash n$	0	1	2	0	1	2
10	0.554	0.768	1.129	−0.398	−0.580	−0.864
20	0.535	0.675	0.885	−0.390	−0.512	−0.677
40	0.528	0.638	0.791	−0.387	−0.483	−0.605
80	0.525	0.621	0.751	−0.385	−0.471	−0.574
160	0.524	0.613	0.733	−0.385	−0.465	−0.560
320	0.524	0.610	0.724	−0.385	−0.462	−0.553
640	0.523	0.608	0.720	−0.385	−0.461	−0.550

Table 1: *Numerical result of the calculation of the parameter* $N^n\vec{\mathbf{B}}_{iR}^T\mathbf{M}^{-(n+1)}\vec{\mathbf{B}}_{jR}$ *for different values of the number of points* N.

As always, in this approach, we find quantities that cannot be handled analytically, however, as it was shown in [7,9] this sort of calculations are very well behaved and an accurate result is obtained from a numerical calculation already for a small number of points on the string.

We can see that for $p > 0$, and in particular for $p = 1$, the limits $N \to \infty$ and $R \to \infty$ cannot be safely interchanged in (13), in particular taking the limit over N, that corresponds to the continuous string in our approach, gives a zero result for any finite radius R. Problems in the exchange of limits in string field theories are discussed in [8]. This means that the strings in Ω'_p, $p > 0$ give a zero contribution to the functional integral.

On the other hand, if $p \leq 0$ we get a finite result for any radius R. In particular, in the case $p = 0$ we can expand the coefficients (14): in powers of $\frac{1}{R^2}$ to obtain:

$$C_{ij}^{(1)} = \sum_{n=0} \frac{(-)^n}{R^{2n}} N^n\vec{\mathbf{B}}_{iR}^T\mathbf{M}^{-(n+1)}\vec{\mathbf{B}}_{jR}$$

now the terms appearing in the series are finite, as opposite to the case $p > 0$ where the series is divergent as N goes to ∞. In this case, the first few terms of the sequence are given in Table 1. We have taken up to $N = 640$ for which an accurate result is obtained [7,9].

One can see that the first term corresponds to the values of (7). Whereas the remaining represent the corrections for finite R which vanish as $R \to \infty$.

$N\backslash\mathcal{R}_1^{-1}$	$R=0.1$	$R=1$	$R=10$	$R=100$
10	1.7 10^2	2.1 10^2	1.9 10^2	1.8 10^2
20	3.9 10^2	4.3 10^2	3.5 10^2	3.6 10^2
40	7.2 10^2	7.1 10^2	6.8 10^2	6.3 10^2
80	1.4 10^3	1.4 10^3	1.4 10^3	1.4 10^3
160	2.9 10^3	2.7 10^3	2.9 10^3	2.7 10^3
320	6.2 10^3	6.5 10^3	5.4 10^3	5.7 10^3
640	1.1 10^4	1.2 10^4	9.6 10^3	9.7 10^3

Table 2: *The results of the Montecarlo calculation for $\mathcal{R}_1^{-1}$ for various values of R and various number of points on the string. The error on the values are less than 5% for all values of N except for $N = 640$ for which there is an error of 7%*

As we mentioned before, the vanishing result in the ratio of amplitudes is true for Ω'_p, for any real $p > 0$ hence, using the inclusion relations between the spaces Ω'_p, we can conclude that the minimum space necessary to achieve the correct result is the corresponding to the square integrable function Ω_0.

In order to test the validity of this result we have done an extensive numerical test for the Ω'_1 case, ($p = 1$ in 8), in this case we can prove that the contribution for any finite R vanishes. This numerical work does not rely at all on analytical approximations, but it has of course the problem coming from a Montecarlo, that is the number of points and the values of R are quite limited in range.

As we have commented before, by setting $p = 1$ in (12), the corresponding function stresses the contribution of the strings contained entirely inside a box of side $2R$.

We have calculated the value of $(\mathcal{R}_1)^{-1}$ numerically with a Montecarlo for a variety of values of the side R of the box, for an increasing number of points on the string, and for various values of the momenta of tachyons and vector. Table (2) shows the results for the choice of momenta in which the first tachyon has momentum 1 in unities in which the α constant of the string has value 1/2, the second tachyon has momentum 0 and the third tachyon or the vector have momentum -1. The results for different values of the momenta do not differ appreciably.

From the table it is quite evident that the ratio goes to infinity independently

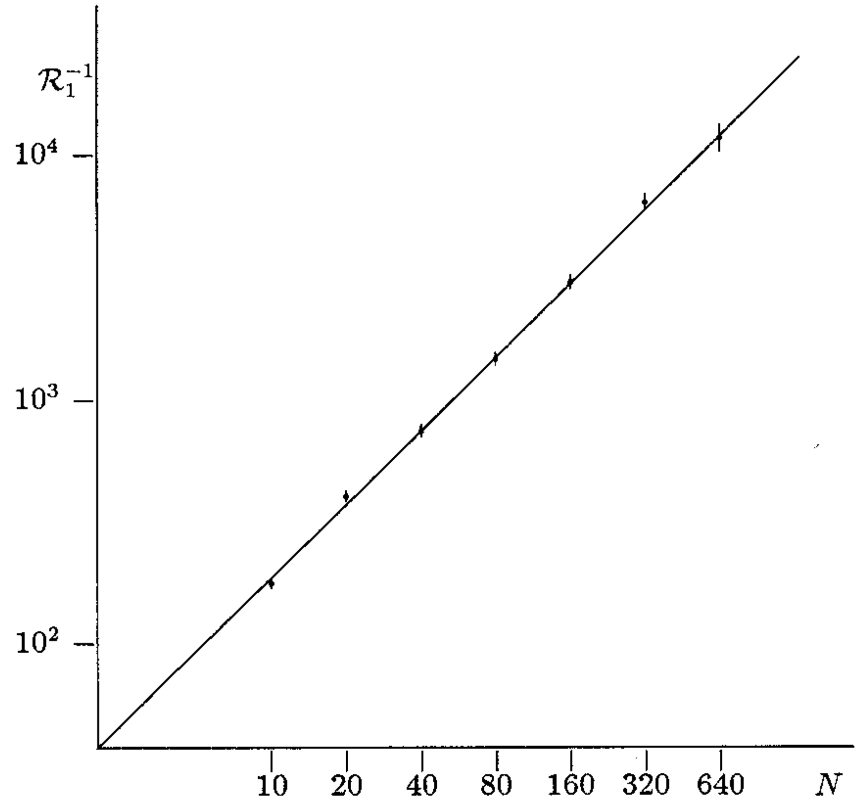

Figure 1: *The ratio* $(\mathcal{R}_1)^{-1}$ *versus the number of points* N

on the value of R. The same can be seen from Fig. (1), where the $R = 0.1$ case is shown. The line shows a behaviour $\sim N$, which clearly fits the data very well. Other values of R would give a very similar picture. It is worth noticing that for $N = 640$, the error due to the discretization is in the hundredths of a percent [7,9].

In conclusion, using numerical methods we have investigated the configuration space of a string field theory. We found that, in order to reproduce the results of the dual model, the measure of the functional integrals must include all square integrable maps, the Sobolev space Ω_0. This space is not a subset of the set of continuous maps, which leaves open the possibility that non–continuous strings may play an important role in a string field theory.

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