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A Abdurrahman and J Bordes

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The Relationship Between the Comma Theory and Witten's String Field Theory (I)

A. Abdurrahman

Department of Particle Physics, Rutherford Appleton Laboratory
Chilton DIDCOT, Oxon, OX11 0QX, U.K.

and

J. Bordes[1]

Departament de Física Teórica, Universitat de Valencia
Dr. Moliner 50, E-46100, Burjassot, Spain.

ABSTRACT

The comma representation of interacting string field theory is further elucidated. The proof, that the Witten's vertex solves the comma overlap equations, is established. In this representation associativity of the star algebra is seen to hold. The relationship of the symmetry K in the standard formulation of Witten's string field theory (WSFT) to that in the comma theory is discussed.

[1]Also at IFIC, Centro Mixto Universitat de Valencia-CSIC

1 Introduction

It has been first suggested by Witten [1] that it is may be possible to formulate string field theory as an infinite dimensional local matrix algebra. This suggestion lead to the formulation of string field theory as more or less a generalization of Yang-Mills theory of an extended object, known as the "comma" [2]. In [3,4] a Fock space realization (operator construction) of the comma theory was obtained by writing down the overlap equations that follow from the formulation of the theory in the comma language. In the language of matrices, vertices were written as traces and the explicit form of the Witten's vertex was regained after integrating out the midpoint degrees of freedom. However, the ambiguity related to the midpoint[2] was not settled. One was not sure how to view the midpoint, $\chi(\pi/2)$ ($\varphi(\pi/2)$), since it was common to both formulations of the string field theory (i.e., the Witten's theory and the comma theory). Because of this problem, one was not able to show directly that the Witten's vertex is indeed a solution of the comma overlaps and therefore one was not clear about the precise nature of the relationship between the two theories. Investigation of other problems (such as midpoint ghost insertions) required for the K and the $BRST$ symmetries in the original theory, were made cumbersome by the need to use the full string formulation at some stages of the investigation. To overcome these problems and to give a direct proof that the Witten's vertex is indeed a solution of the comma overlaps we need to modify the comma definition employed in [3,4] (more in the line of reference [1]). The modified comma coordinates are defined

[2]Another approach in which the midpoint plays a central role has been discussed in ref. [5], although in a different context from refs. [3,4].

through the relations

$$\chi^r(\sigma) = \begin{cases} X(\sigma) \ , if\ r = 1\,, \\ X(\pi - \sigma) \ , if\ r = 2\ , \\ \sigma \in [0, \frac{\pi}{2})\,. \end{cases} \tag{1.1}$$

Note that the only difference between this new definition of the comma and that in ref. [3,4] is the exclusion of the midpoint $\chi(\pi/2)$. Theses coordinates are subjected to the constraint,

$$\lim_{\sigma \to \frac{\pi}{2}} \chi^L(\sigma) = \lim_{\sigma \to \frac{\pi}{2}} \chi^R(\sigma)\,. \tag{1.2}$$

This is in the spirit of the WSFT developed in ref. [1]. $X(\sigma)$ are the full string coordinates at fixed time $\tau = 0$ (space-time indices will be suppressed throughout the paper). In this modified approach to the comma, the midpoint is excluded from the degrees of freedom and is used to constrain the emerging system (i.e., the comma degrees of freedom). The Fourier expansion at $\tau = 0$ of the full string coordinate $X(\sigma)$ is given by

$$X(\sigma) = X_0 + \sqrt{2} \sum_{n=1}^{\infty} X_n \cos n\sigma, \quad \sigma \in [0, \pi]\,.$$

If one expands the comma coordinates (1.1) in a Fourier series then they can be related to the full string coordinates. The comma boundary conditions are dictated by the boundary conditions of the full string and the comma definition. Choosing an even extension to the interval $(\pi/2, \pi]$, only the even modes in the Fourier expansions of the comma coordinates survive. Hence,

$$\chi^r(\sigma) = \chi_0 + \sqrt{2} \sum_{n=1}^{\infty} \chi^r_{2n} \cos 2n\sigma, \quad \sigma \in [0, \pi/2)\,, \tag{1.3}$$

where

$$\begin{aligned} \chi^r_0 &= X_0 + (-)^r \frac{2\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{(-)^n}{2n-1} X_{2n-1}, \\ \chi^r_{2n} &= X_{2n} + 2(-)^r \sum_{n=1}^{\infty} B_{2n\,2m-1} X_{2m-1}, \end{aligned} \tag{1.4}$$

and $r = 1,2$ refers to the left (L) and right (R) parts of the string[3] respectively. The change of representation matrix (B) is given by

$$B_{nm} = \frac{(-)^{n+m+1}}{\pi}\left(\frac{1}{n+m} - \frac{1}{n-m}\right) . \tag{1.5}$$

Equation (1.4) can be solved for X_n ($n \geq 0$) with the help of the identities

$$\begin{aligned} \sum_{n=1}^{\infty} B_{2n\,2k-1} B_{2n\,2m-1} &= \frac{1}{4}\delta_{km} , \\ \sum_{k=1}^{\infty} \frac{2m}{2k-1} B_{2n\,2k-1} B_{2k-1\,2m} &= -\frac{1}{4}\delta_{nm} . \end{aligned} \tag{1.6}$$

Hence

$$\begin{aligned} X_{2n} &= \frac{1}{2}\sum_{r=1}^{2} \chi^r_{2n} \quad , n \geq 0, \\ X_{2n-1} &= \sum_{r=1}^{2}(-)^{r+1} \sum_{m=1}^{\infty} \frac{2m}{2n-1} B_{2n-1\,2m} \chi^r_{2m} \quad , n \geq 1 . \end{aligned} \tag{1.7}$$

However, in the second relation of equation (1.7) there are redundant degrees of freedom. Now, the constraint on the comma modes (1.2) can be explicitly solved and what results are the modes with no subsidiary condition. Hence one gets

$$X_{2n-1} = \frac{\sqrt{2}}{\pi}\frac{(-)^n}{2n-1}\sum_{r=1}^{2}(-)^r \chi_0 + \sum_{r=1}^{2}(-)^r \sum_{m=1}^{\infty} B_{2m\,2n-1}\chi^r_{2m} \quad , n \geq 1 , \tag{1.8}$$

for the second equation in (1.7). The comma modes (χ^r_{2n}) have been treated so far as classical objects. There are many ways to quantize a

[3]Throughout the paper we will refer to the left and right parts of the string by 1 and 2 respectively; however to make things more transparent and to avoid confusion, sometimes we may refer to the left and right parts of the string by the letters L and R respectively. When dealing with more than one string the indices may become confusing; therefore indices referring to the parts of the string will always be written as superscripts while those labeling the string will be written as subscripts whenever possible.

system (of course all of them are equivalent). A standard method is to interpret the oscillator modes χ^r_{2n} as $q-$ *operators* and define their conjugate momenta, $\wp^r_{2n} = -i\frac{\partial}{\partial\chi^r_{2n}}$, satisfying

$$[\chi^r_{2n}, \wp^s_{2m}] = i\delta^{rs}\delta_{nm} \,. \tag{1.9}$$

These operators are easily related to the full string operators ones using equations (1.7). Thus

$$\begin{aligned} \wp^r_0 &= \frac{1}{2}p_0 + \sqrt{2}\frac{(-)^r}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{2n-1}p_{2n-1} \,, \\ \wp^r_{2n} &= \frac{1}{2}p_{2n} + (-)^r\sum_{m=1}^{\infty}B_{2n2m-1}p_{2m-1} \,, \quad n \geq 1 \,. \end{aligned} \tag{1.10}$$

They satisfy the desired commutation relations as can be verified using the identities in equation (1.6). The inverse relations are given by

$$\begin{aligned} p_{2n} &= \sum_{r=1}^{2}\wp^r_{2n} \,, \quad n \geq 0 \,, \\ p_{2n-1} &= \frac{2\sqrt{2}}{\pi}\frac{(-)^n}{2n-1}\sum_{r=1}^{2}\wp^r_0 + 2\sum_{r=1}^{2}(-)^r\sum_{m=1}^{\infty}B_{2m2n-1}\wp^r_{2m}, \; n \geq 1 \,. \end{aligned} \tag{1.11}$$

We will focus our attention on the orbital part, the treatment for the ghost part follows the same line. Now the Fock space of the comma theory (where the degrees of freedom for left and right sectors of the string live in different Fock spaces) is easily constructed. One introduces the operators b^r_n and $b^{r\,\dagger}_n (r = 1, 2 \,, n \geq 1)$ satisfying

$$\left[b^r_n, b^{s\,\dagger}_m\right] = \delta^{rs}\delta_{nm} \,. \tag{1.12}$$

It is possible to introduce such operators in the usual way by taking the appropriate combinations of the position and momentum operators, namely,

$$b^r_0 = -i\left(\chi^r_0 + \frac{i}{2}\wp^r_0\right) \,, \quad b^r_n = -i\sqrt{\frac{n}{2}}\left(\chi^r_{2n} + \frac{i}{n}\wp^r_{2n}\right) \,. \tag{1.13}$$

The creation operators ($b_n^{r\dagger}, n \geq 0$) are given by the same expressions with ($i \rightleftharpoons -i$). It can be checked that these operators satisfy the required commutation relations. Introducing the comma vacua $|0>^r$ ($r = 1, 2$), satisfying

$$b_n^r|0>^r = 0 \ , r = 1,2 \ \text{ and } n \geq 1 \, ,$$

one obtains the Fock space[4] corresponding to each comma (half string) by repeated application of the comma creation operators on the comma vacua. The annihilation and creation operators in the comma theory ($b_n^r, b_n^{r\dagger}$) when related to the conventional annihilation and creation operators ($a_n, a_n^\dagger$) give

$$\begin{aligned} b_0^r &= \frac{1}{4}(3a_0 - a_0^\dagger) + \frac{(-)^r}{2\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{(2n-1)^{3/2}}\left[(4 + (2n-1))a_{2n-1}\right. \\ &\quad \left. -(4-(2n-1))a_{2n-1}^\dagger\right] \, , \\ b_n^r &= \frac{1}{\sqrt{2}}a_{2n} + \frac{(-)^{r+1}}{\sqrt{2}}\sum_{m=1}^{\infty}\frac{2m-1}{2n}^{1/2}\left[A_{2m-1\,2n}a_{2m-1} - S_{2m-1\,2n}a_{2m-1}^\dagger\right] \, , \end{aligned} \tag{1.14}$$

where

$$\begin{pmatrix} A_{2m-1\,2n} \\ S_{2m-1\,2n} \end{pmatrix} = B_{2m-1\,2n} \mp B_{2n\,2m-1} \, ,$$

and $b_0^{r\dagger}$, $b_n^{r\dagger}$ are given by the same expressions with $a_n \rightleftharpoons a_n^\dagger$. Now it is important to make sure that these relations are consistent with the constraint in the definition of the commas. Rewriting the constraint, eq. (1.2), in the comma annihilation-creation basis we have

$$\sum_{r=1}^{2}(-)^{r+1}(b_0^r - b_0^{r\dagger}) = -2\sum_{n=1}^{\infty}\frac{(-)^n}{\sqrt{n}}\sum_{r=1}^{2}(-)^{r+1}(b_n^r - b_n^{r\dagger}) \, , \tag{1.15}$$

which can be easily verified to hold in the annihilation-creation basis of the full string using the following identity

$$\psi(x) - \psi(y) = \sum_{n=0}^{\infty}\left(\frac{1}{y+n} - \frac{1}{x+n}\right) \, ,$$

[4]The complete space on which the comma states reside is given by the tensor product (completion), $\overline{\mathcal{H}^1 \bigotimes \mathcal{H}^2}$.

and the fact that $\psi(\frac{1}{2}-n)=\psi(\frac{1}{2}+n)$ for n integer. The inverse relations of (1.14) are given by

$$
\begin{aligned}
a_0 &= \sum_{r=1}^{2}\left(\frac{3}{4}b_0^r+\frac{1}{4}b_0^{r\dagger}\right), \quad a_{2n}=\frac{1}{\sqrt{2}}\sum_{r=1}^{2}b_n^r \quad ,n\geq 1\,, \\
a_{2n-1} &= \frac{1}{\sqrt{2}}\sum_{r=1}^{2}(-)^r\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^n}{(2n-1)^{3/2}}\left[((2n-1)+4)b_0^r-((2n-1)-4)b_0^{r\dagger}\right]\right. \\
&\quad \left.-\sum_{m=1}^{\infty}\left(\frac{2n-1}{2m}\right)^{1/2}\left[A_{2n-12m}b_m^r+S_{2n-12m}b_m^{r\dagger}\right]\right)\,, \qquad (1.16)
\end{aligned}
$$

with $a_n^\dagger$ given by the same expressions with $b_n^r \rightleftharpoons b_n^{r\dagger}$. It is not hard to see, using the properties of the matrix B and the commutation relations for the comma operators defined earlier, that the standard commutation relations for the annihilation-creation operators (a and $a^\dagger$) are indeed satisfied.

2 Comma Vertices

In the comma formulation of string field theory the elements of the theory are defined by δ-function type overlaps. The N interaction vertex is given by

$$V[\chi_1^r,\chi_2^r,...,\chi_N^r,\varphi^r]=e^{iQ^\varphi(\phi/2)}\prod_{i=1}^{N}\prod_{\sigma=0}^{\pi/2}\delta(\chi_i^1(\sigma)-\chi_{i-1}^2(\sigma))\delta(\varphi_i^1(\sigma)-\varphi_{i-1}^2(\sigma))\,.$$

The index i refers to the ith string (it is understood that $i=0$ and N are identified). The ghost δ-function have the same structure as the coordinates ones and Q^φ is the ghost number insertion. In the oscillator Hilbert space of the comma theory, the δ functions, for the coordinates[5], translate into operator overlap equations, namely

$$\left[\chi_i^L(\sigma)-\chi_{i-1}^R(\sigma)\right]|V_N>=0\,,\sigma\in[0,\pi/2)\,, \qquad (2.1)$$

and $i=1,2,..,N$. In addition, conservation of momentum requires

[5]The ghost degrees of freedom, in the bosonized representation, have the same structure apart from some mid-point insertions which will be addressed later.

$$\left[\wp_j^L(\sigma) + \wp_{j-1}^R(\sigma)\right] |V_N >= 0\,. \tag{2.2}$$

These are now the overlaps defining equations for the comma vertices. Now we are in position to construct the explicit form of the comma vertices. We will start with the case $|V_1>$, since it is the identity vertex $|I>$ with respect to the $*$ in Witten's string field theory. For $N=1$, we have

$$\begin{aligned}\left[\chi^1(\sigma) - \chi^2(\sigma)\right] |I> &= 0\,,\\ \left[\wp^1(\sigma) + \wp^2(\sigma)\right] |I> &= 0\,,\end{aligned}$$

where the superscripts 1 and 2 refer to the left and right parts of the string respectively. In terms of the oscillator modes the above overlaps result in

$$\begin{aligned}\sum_{r=1}^{2}(-)^{r+1}(b_n^r - b^{\dagger}{}_n^r)|I> &= 0\,,\\ \sum_{r=1}^{2}(b_n^r + b^{\dagger}{}_n^r)|I> &= 0.\end{aligned}$$

It is trivial to solve the above equations, assuming that $|I>$ has the form

$$|I>= e^{-\frac{1}{2}(b^{\dagger}|\mathcal{I}|b^{\dagger})}|0>^1\,|0>^2\,,$$

then it is clear that

$$\mathcal{I}_{nm}^{rs} = (\delta^{r+1s} + \delta^{s+1r})\delta_{nm}$$

solves the above overlaps. The cases $N \geq 2$ are simplified if one rewrites the overlaps in terms of complex coordinates. Following Gross and Jevicki [6] we define

$$\mathcal{Q}_k^r(\sigma) = \frac{1}{\sqrt{N}}\sum_{l=1}^{N}\chi_l^r(\sigma)e^{\frac{2\pi ilk}{N}}\,,\ r=1,2,$$

and similar ones for the momenta. The corresponding creation and annihilation operators are defined similarly and (for $N \geq 3$) satisfy the commutation relations

$$\left[\mathcal{B}_n^r, \overline{\mathcal{B}^s}_{-m}\right] = \delta^{rs}\delta_{nm}, \qquad \left[\mathcal{B}_n^r, \mathcal{B}_{-m}^s\right] = 0\,.$$

The advantage of this new set of variables is that it leads to the separation of degrees of freedom in the overlap equations. For the case $N = 2$ the overlaps now are simply

$$\begin{aligned} \mathcal{Q}_1^L(\sigma) &= -\mathcal{Q}_1^R(\sigma) \ , \sigma \in [0, \pi/2), \\ \mathcal{Q}_2^L(\sigma) &= \mathcal{Q}_2^R(\sigma) \ , \sigma \in [0, \pi/2) \end{aligned} \tag{2.3}$$

These two equations are the same as the overlaps for the identity vertex (apart from a " $-$ " sign in the first one). Hence,the form of the vertex follows immediately from the form of the identity vertex. It is simply

$$|V_2> = e^{-\frac{1}{2}(\mathcal{B}_2^\dagger|\mathcal{I}|\mathcal{B}_2^\dagger)+\frac{1}{2}(\mathcal{B}_1^\dagger|\mathcal{I}||\mathcal{B}_1^\dagger)} \prod_{i=1}^{2} |0>_i^1 \, |0>_i^2$$

or in the original creation operators $({b^r}_n^\dagger)$

$$|V_2> = e^{-\sum_{i=1}^{2} {b^L}_{i\,n}^\dagger {b^R}_{i-1}^\dagger} \prod_{i=1}^{2} |0>_i^L \, |0>_i^R \ .$$

Next we consider the case N=3. In the complex coordinates the overlap conditions $\chi_j^L(\sigma) = \chi_{j-1}^R(\sigma)$ for the three string vertex read

$$\begin{aligned} \mathcal{Q}^L(\sigma) &= e^{2\pi i/3}\mathcal{Q}^R(\sigma) \ , \sigma \in [0, \pi/2) \, , \\ \mathcal{Q}_3^L(\sigma) &= \mathcal{Q}_3^R(\sigma) \ , \sigma \in [0, \pi/2) \, , \end{aligned}$$

where $\mathcal{Q}^r(\sigma) \equiv \mathcal{Q}_1^r(\sigma) = (\overline{\mathcal{Q}_3^r(\sigma)})$. For the complex momenta the overlap conditions $\wp_j^L(\sigma) = -\wp_{j-1}^R(\sigma)$ translate into

$$\begin{aligned} \mathcal{P}^L(\sigma) &= -e^{2\pi i/3}\mathcal{P}^R(\sigma) \ , \sigma \in [0, \pi/2) \, , \\ \mathcal{P}_3^L(\sigma) &= -\mathcal{P}_3^R(\sigma) \ , \sigma \in [0, \pi/2) \, , \end{aligned}$$

where $\mathcal{P}^r(\sigma) \equiv \mathcal{P}_1^r(\sigma) = (\overline{\mathcal{P}_3^r(\sigma)})$. The vertex $V_3({b_1^r}^\dagger, {b_2^r}^\dagger, {b_3^r}^\dagger)$, therefore, separates into a product of two pieces depending on ${\mathcal{B}^r}_3^\dagger$ and on ${\mathcal{B}^r}^\dagger = {\mathcal{B}_1^r}^\dagger$, $\overline{{\mathcal{B}^r}^\dagger} = {\mathcal{B}^r}_2^\dagger$ respectively. The first factor is identical to that in $|I>$. Thus one has

$$|V_3> = exp\left(-\frac{1}{2}(\mathcal{B}_3^\dagger|\mathcal{I}|\mathcal{B}_3^\dagger) - (\mathcal{B}^\dagger|\mathcal{H}|\overline{\mathcal{B}^\dagger})\right) \prod_{i=1}^{3} |0>_i^1 \, |0>_i^2 \, ,$$

where $\mathcal{I}$ is the same as that for $|I>$ and $\mathcal{H}$ is an infinite dimensional matrix to be determined. In order to determine $\mathcal{H}$ we first note that the overlap conditions on $\mathcal{Q}^r(\sigma)$ and $\mathcal{P}^r(\sigma)$ imply that their Fourier components satisfy

$$\begin{aligned} \left[\mathcal{Q}^L_{2n} - e^{2\pi i/3}\mathcal{Q}^R_{2n}\right]|V_3> &= 0\ , n \geq 0 \\ \left[\mathcal{P}^L_{2n} + e^{2\pi i/3}\mathcal{P}^R_{2n}\right]|V_3> &= 0\ , n \geq 0 \end{aligned} \qquad (2.4)$$

As well as their complex conjugates. Applying these equations to the three vertex yields

$$\begin{aligned} (\mathcal{H}^{rL}_{kn} - e^{2\pi i/3}\mathcal{H}^{rR}_{kn}) + (\delta^{rL} - e^{2\pi i/3}\delta^{rR})\delta_{kn} &= 0\ , n \geq 0\ , r = 1,2,\ , \\ (\mathcal{H}^{rL}_{kn} + e^{2\pi i/3}\mathcal{H}^{rR}_{kn}) - (\delta^{rL} + e^{2\pi i/3}\delta^{rR})\delta_{kn} &= 0\ , n \geq 0\ , r = 1,2\ , \end{aligned} \qquad (2.5)$$

and their complex conjugates. These equations are easily solved for the matrix elements of $\mathcal{H}$. Thus one has,

$$\mathcal{H}^{rs}_{nm} = e^{\frac{2\pi i}{3}(r-s)}(\delta^{r+1s} + \delta^{s+1r})\delta_{nm}$$

(with $\mathcal{H}^T = \overline{\mathcal{H}}$). We have therefore the explicit form of the 3-interaction vertex which is of central importance in the theory, expressed in the complex creation-annihilation comma operator basis. Combining the two pieces of the vertex (i.e., $\mathcal{I}$ and $\mathcal{H}$) and rewriting everything in the original creation operators (i.e., ${b^r_n}^\dagger$) we obtain

$$|V_3> = e^{-\sum_{i=1}^{3} {b^L}^\dagger_{in} {b^R}^\dagger_{i-1\,n}} \prod_{i=1}^{3} |0>^L_i\ |0>^R_i\ .$$

At this point it is worth comparing this expression to that of the full string given in [6,7,8,9,10]. Our vertex is extremely simple, it basically says that one must sew the left half of the string i to the right half of the string $i-1$. In the case of the full string operator formulation given in the above mentioned references this is not obvious from the form of the vertex. Also our expression can easily be generalized to higher vertices while in the case of the full string operator formulation, the calculation is quite cumbersome for a general value of N (see [11] for the construction of the general N-vertex), in fact the general expression in the comma formulation is simply,

$$|V_N>= e^{-\sum_{i=1}^{N} b^L{}^\dagger_{i\,n} b^R{}^\dagger_{i-1\,n}} \prod_{i=1}^{N} |0>_i^L\,|0>_i^R\,, \qquad (2.6)$$

which can be put in a more formal form as

$$|V_N>= e^{-\frac{1}{2}(b^\dagger|H|b^\dagger)} \prod_{i=1}^{N} |0>_i^L\,|0>_i^R$$

where

$$H^{rs}_{in\,jm} = (\delta^{r+1\,s}\delta_{i-1\,j} + \delta^{s+1\,r}\delta_{j-1\,i})\delta_{nm}\,.$$

Now the interesting question to ask is "are these two theories equivalent?". In other words do the vertices in the full string creation-annihilation operator basis solve the overlap equations for the comma theory? And do the two theories have the same symmetries? We would like to address these two questions in the next section.

3 Full String Vertices and the Comma Overlaps

The fact that the Witten's vertex solves the comma overlaps only prove that the Witten's vertex is a solution of the comma overlaps and not necessarily the only solution[6]. If this turn to be the case it will be interesting to see what other solutions are admitted by the comma formulation; certainly one of them will be the comma vertex itself if one can show that it is different from the Witten's vertex (i.e., possesses different properties from the Witten's vertex.), these questions will be addressed later in the paper.

The proof that the operator form of the Witten's vertex solves the comma overlaps is not a trivial one, since it involves double infinite sums (the second coming from integrating σ over the range $[0, \pi/2)$ in formulating the comma theory). Here the double infinite sums may not converge

[6]This statement is true, since no one has yet proven that Witten's interaction fixes the form of the vertex uniquely.

absolutely and the convergence may depend on the order of the sums (so the expressions may be ambiguous!) The case of the full string [6,7,8,9,10] is different, the expression for the vertices involve absolutely convergent sums. This ambiguity is not an accident, we have seen in [3,4] that Witten's theory can be viewed as an infinite dimensional local matrix algebra where the star product " $*$ " becomes matrix multiplication over infinite dimensional matrices that does not conserve associativity. The proof of the cases $N = 1, 2$ is trivial due to the simple form of the overlap matrices in the full string formulation. However for the cases $N = 3$ and higher, the form of the overlap matrices are quite complicated. Although we can show that the Witten's 3-Vertex satisfies the comma overlaps directly [12] it is simpler to use the Witten's 4-vertex, $|V_4^W>$, since

$$|V_3^W> = <I_4^W|V_4^W>, \tag{3.1}$$

where $|I_4^W>$ is just the identity vertex corresponding to the $4th$ string. To establish that the Witten's $4 - Vertex$ solves the comma δ-function overlaps we recall that the comma interaction requires that $\chi_j^L(\sigma) = \chi_{j-1}^R(\sigma)$ for $\sigma \in [0, \pi/2)$ and $j = 1, .., 4$ (with $j - 1 = 0 \equiv 4$). In complex coordinates the overlap equations take the form

$$\begin{aligned} \mathcal{Q}^L(\sigma) &= i\mathcal{Q}^R(\sigma) \;, \sigma \in [0, \pi/2)\,, \\ \mathcal{Q}_j^L(\sigma) &= (-)^{j/2}\mathcal{Q}_j^R(\sigma)\,, \quad j = 2, 4; \ \sigma \in [0, \pi/2), \end{aligned}$$

where $\mathcal{Q}^r(\sigma) \equiv \mathcal{Q}_1^r(\sigma) = (\overline{\mathcal{Q}_3^r(\sigma)})$ and similarly another set for the complex momenta $\mathcal{P}_I^r(\sigma)$. In the Fourier space of the comma, the overlaps for $\mathcal{Q}(\sigma)$ read

$$\begin{aligned} \mathcal{Q}_{2n}^L &= i\mathcal{Q}_{2n}^R\,, \quad n \geq 0, \\ \mathcal{Q}_{j2n}^L &= (-)^{j/2}\mathcal{Q}_{j2n}^R\,, \quad j = 2, 4\;; n \geq 0, \end{aligned} \tag{3.2}$$

and similar ones for $\mathcal{P}(\sigma)$[7]. Now recall that the form of the full string

[7] The comma 4-vertex described by the above overlaps is $|V_4 \quad >= exp \left(-\frac{1}{2}\mathcal{B}_4^\dagger\mathcal{I}\mathcal{B}_4^\dagger + \frac{1}{2}\mathcal{B}_2^\dagger\mathcal{I}\mathcal{B}_2^\dagger - \mathcal{B}^\dagger\mathcal{K}\overline{\mathcal{B}^\dagger}\right) \prod_{i=1}^4 |0>_i^1 |0>_i^2$ where $\mathcal{K}_{nm}^{rs} = e^{-i\pi(r-s)/6}\mathcal{H}_{nm}^{rs}$.

vertex (Witten's 4-vertex) in oscillator basis (see ref. [6]) is given by

$$|V_4^W> = exp\left(-\frac{1}{2}(A_4^\dagger|C|A_4^\dagger) + \frac{1}{2}(A_2^\dagger|C|A_2^\dagger) - (A^\dagger|V|\overline{A^\dagger})\right)\prod_{i=1}^{4}|0>_i\,, \tag{3.3}$$

where $C_{nm} = (-)^n\delta_{nm}$ and V_{nm} is an infinite dimensional matrix [6] constructed from the binomial coefficients $\begin{pmatrix} -1/2 \\ n \end{pmatrix}$. To verify that the Witten's $4-vertex$ solves the Comma overlaps, (3.2); we first note that the second equation is the same as the overlap equation for the identity vertex and therefore the proof follows from the form of the vertex. Hence we are only left with the first one to verify. With the help of the change of representation formulas, equation (1.14), we are able to show that the comma overlaps are satisfied by the full string vertices. To do this we need to express the comma overlaps, (3.2), in terms of the full string Fourier coefficients and show that they are satisfied by the Witten's $4-Vertex$, i.e., we have to show that

$$\begin{aligned}\left[(1-i)\mathcal{Q}_0 - \frac{2\sqrt{2}}{\pi}(1+i)\sum_{n=1}^{\infty}\frac{(-)^n}{2n-1}\mathcal{Q}_{2n-1}\right]|V_4^W> &= 0\,,\\ \left[(1-i)\mathcal{Q}_{2n} - 2(1+i)\sum_{m=1}^{\infty}B_{2n\,2m-1}\mathcal{Q}_{2m-1}\right]|V_4^W> &= 0\end{aligned} \tag{3.4}$$

hold. Similar equations for the complex momenta $\mathcal{P}_{2n}$ are easily established. The proof of the two equations is very similar; so we only need to consider one of them[8]; we do the harder one (the second equation). Commuting the annihilation operators in (3.4) through the creation operators in $|V_4^W>$ yields a sum of creation operators acting on $|V_4^W>$, hence

$$-\frac{i}{2\sqrt{2}}\sum_{m=0}^{\infty}[........]\,A_m^\dagger|V_4^W>\,,$$

where the expression in the squared bracket is given by

$$\frac{1}{\sqrt{2n}}(1-i)V_{m\,2n}+\delta_{m\,2n}) - 2(1+i)\sum_{k=1}^{\infty}\frac{1}{\sqrt{2k-1}}B_{2n\,2k-1}(V_{m\,2k-1}+\delta_{m\,2k-1})\,. \tag{3.5}$$

[8]We have checked that indeed both of them are satisfied for the Witten's Vertex.

Since the states $A_m^\dagger|V_4^W>$ are linearly independent, the expression in (3.5) must vanish for all values of m. Now there are three cases to consider $m = 0,\ 2l,\ 2l-1\ (l \geq 1)$. For $m = 0$, equation (3.5) reduces to

$$\left[\frac{1}{\sqrt{2n}}(1-i)V_{0\,2n} - 2(1+i)\sum_{k=1}^{\infty}\frac{1}{\sqrt{2k-1}}B_{2n\,2k-1}V_{0\,2k-1}\right]. \tag{3.6}$$

Using the expression for V_{nm} in ref. [6], eq. (3.6) becomes

$$\left[(1-i)(V_{00}-1)\frac{u_{2n}}{2n} + 2i(1+i)(V_{00}-1)\sum_{k=1}^{\infty}B_{2n\,2k-1}\frac{\nu_{2k-1}}{2k-1}\right]$$

where

$$u_{n=2k} = \begin{pmatrix} -1/2 \\ n/2 \end{pmatrix} \quad \text{and} \quad \nu_{n=2k-1} = \begin{pmatrix} -1/2 \\ (n-1)/2 \end{pmatrix} \tag{3.7}$$

are the Fourier coefficients in the expansion of

$$\left(\frac{1+ie^{\xi}}{1-ie^{\xi}}\right)^{1/2}, \tag{3.8}$$

(for more details see ref. [6]). The sum in the above equation when carried out gives

$$\sum_{k=1}^{\infty}B_{2n\,2k-1}\frac{\nu_{2k-1}}{2k-1} = \sum_{k=1}^{\infty}B_{2n\,2k-1}\frac{\begin{pmatrix} -1/2 \\ k-1 \end{pmatrix}}{2k-1} = \frac{1}{2}\frac{u_{2n}}{2n}, \tag{3.9}$$

where we have used the fact that (see the Appendix A)

$$\sum_{k=0}^{\infty}\frac{(-)^k}{k+a}\begin{pmatrix} -1/2 \\ n \end{pmatrix} = \frac{\Gamma(1/2)\Gamma(a)}{\Gamma(a+1/2)}.$$

This proves that equation (3.5) is identically zero for $m = 0$. The next case to consider is $m = 2l$. Now eq. (3.5) becomes

$$\left[\frac{1}{\sqrt{2n}}(1-i)(V_{2l\,2n} + \delta_{l\,n}) - 2(1+i)\sum_{k=1}^{\infty}\frac{1}{\sqrt{2k-1}}B_{2n\,2k-1}(V_{2l\,2k-1} + \delta_{2l\,2k-1})\right] \tag{3.10}$$

where

$$\begin{aligned} V_{2l\,2n} &= -\frac{(2l)^{1/2}(2n)^{1/2}}{2l+2n}u_{2l}u_{2n} - (1-V_{00})\frac{u_{2l}u_{2n}}{(2l)^{1/2}(2n)^{1/2}}, \\ V_{2l\,2k-1} &= -\frac{(2l)^{1/2}(2k-1)^{1/2}}{2l-2k+1}u_{2l}\nu_{2k-1} + i(1-V_{00})\frac{u_{2l}\nu_{2k-1}}{(2l)^{1/2}(2k-1)^{1/2}} \end{aligned}$$

To evaluate the sums we make use of the identity[9]

$$\sum_{k=0}^{\infty}\left(\frac{1}{2n+2k+1}-\frac{1}{2n-2k-1}\right)\frac{\nu_{2k+1}}{2l-2k-1}=\frac{\pi}{2}\frac{(-)^n}{2n+2l}u_{2n}+\frac{(-)^n}{2n}\frac{2}{u_{2n}}\delta_{nl}\,.$$

Thus

$$\begin{aligned}\sum_{k=1}^{\infty}B_{2n\,2k-1}\frac{V_{2l\,2k-1}}{(2k-1)^{1/2}} &= \\ &= \frac{i}{2}\left(\frac{(2l)^{1/2}}{2n+2l}u_{2l}u_{2n}-\frac{1}{(2n)^{1/2}}\delta_{nl}+(1-V_{00})\frac{u_{2l}u_{2n}}{(2l)^{1/2}(2n)}\right).\end{aligned}$$

Substituting in eq. (3.10) we get zero. The last case to consider is $m=2l-1\geq 1$. In this case eq. (3.5) reduces to

$$\left[\frac{1}{\sqrt{2n}}(1-i)V_{2l-1\,2n}-2(1+i)\sum_{k=1}^{\infty}\frac{1}{\sqrt{2k-1}}B_{2n\,2k-1}(V_{2l-1\,2k-1}+\delta_{l\,k})\right],$$

where

$$V_{n\,m}=(-)^{n+m}V_{m\,n}$$

and

$$V_{2l-1\,2k-1}=-\frac{(2l-1)^{1/2}(2k-1)^{1/2}}{(2l-1)+(2k-1)}\nu_{2l-1}\nu_{2k-1}-(1-V_{00})\frac{\nu_{2l-1}\nu_{2k-1}}{(2l-1)^{1/2}(2k-1)^{1/2}}\,.$$

Now it is clear that the expression in the square bracket vanish if one uses the following identity,

$$\begin{aligned}\sum_{k=1}^{\infty}B_{2n\,2k-1}\frac{V_{2l-1\,2k-1}}{(2k-1)^{1/2}} &= -(2l-1)^{1/2}\nu_{2l-1}\left[\frac{1}{2}\frac{u_{2n}}{2l-1-2n}+\frac{B_{2n\,2l-1}}{(2l-1)\nu_{2l-1}}\right] \\ &- \frac{1}{2}(1-V_{00})\frac{u_{2n}\nu_{2l-1}}{(2l-1)^{1/2}(2n)},\end{aligned}$$

which is derived in Appendix A.

This shows that the comma overlaps (3.2) are satisfied by the Witten's $4-vertex$. Exactly the same procedure is followed to prove the $\mathcal{P}-$overlaps. They are seen to hold too. All the sums needed to carry out the proofs can be obtained using the results in Appendix A. To complete the proof we have to see if the ghost part of the Witten's $4-vertex$

[9]This identity can be easily derived using the results in Appendix A.

satisfies (violates) the comma overlaps in exactly the same way as in the standard formulation. The ghost part of the Witten's 4-vertex is given by

$$|V_4^\phi >= e^{3i\phi(\pi/2)}|V_4^{\phi,0}>,$$

where $|V_4^{\phi,0}>$ has the same form as the orbital part of the vertex. The ghost factor $e^{3i\phi(\pi/2)}$ corresponding to ghost number 3 is the right ghost number, since one must require $|V_3^\phi >=< I_4^\phi|V_4^\phi >$. Expanding the phase factor and commuting the annihilation operator through the creation part of the vertex results in doubling the creation part of the insertion. Thus one has

$$|V_4^\phi >= e^{3\sum_{n=1}^\infty \frac{(-)^n}{\sqrt{2n}} A_{4\ 2n}^\dagger}|V_4^{\phi,0}>.$$

The quadratic part of the vertex, $|V_4^{\phi,0}>$, satisfies the comma overlaps since it has the same structure as the orbital part which solves the comma overlaps as we have seen. However, when one includes the ghost insertion this is no longer the case. To see this one first observes that the comma overlaps for V_4 are blind to the phase factor[10] (insertion) apart from

$$\begin{aligned} \mathcal{Q}_{4\ 2n}^L &= \mathcal{Q}_{4\ 2n}^R \quad , n \geq 0 , \\ \mathcal{P}_{4\ 2n}^L &= -\mathcal{P}_{4\ 2n}^R \quad , n \geq 0 . \end{aligned}$$

In fact the first of these equations is also blind to the insertion factor, since it contains only odd modes in the annihilation-creation operators A_4 which clearly commute with the even modes in the phase factor. On the other hand, the second equation contains even modes of the operator A_4 and therefore is not satisfied by the vertex due to the insertion. To see this notice that

$$\mathcal{P}_{4\ 2n}^r exp\left(3\sum_{n=1}^\infty \frac{(-)^n}{\sqrt{2n}} A_{4\ 2n}^\dagger\right) = exp\left(3\sum_{n=1}^\infty \frac{(-)^n}{\sqrt{2n}} A_{4\ 2n}^\dagger\right)\left[-\frac{3}{2}\frac{(-)^n}{\sqrt{2n}} + \mathcal{P}_{4\ 2n}^r\right],$$

where $r = 1, 2$ refers to the left and right parts of the string respectively. Thus commuting the overlaps through the insertion factor and collecting terms we obtain

$$exp\left(3\sum_{n=1}^\infty \frac{(-)^n}{\sqrt{2n}} A_{4\ 2n}^\dagger\right)\left[-3\frac{(-)^n}{\sqrt{2n}} + \mathcal{P}_{4\ 2n}^L = -\mathcal{P}_{4\ 2n}^R\right].$$

[10]The reason for this is that the other overlaps describe different strings in the complex coordinates as we have seen before.

Now it is clear that the overlaps in the square bracket are not satisfied by the quadratic part of the ghost vertex because of the presence of a $c - number$. This is the same violation seen in the operator formulation of Witten's string field theory (see ref. [6]). Therefore the comma overlaps are satisfied (violated) by the Witten's $4 - vertex$ in exactly the same way as in the case of the standard formulation [7,8,6,9,10]. It follows from (3.1) that the Witten's $3 - vertex$ also solves the comma overlaps, since the $|I^W>$ and the $|V_4^W>$ *vertices* solve the comma overlaps. This completes the demonstration that the Witten's vertex is a solution to the comma overlaps. In the next section we will address the remaining questions raised earlier. At this stage one can not help to look at the relationship between the dual model vertex of Caneschi, Schwimmer and Veneziano (CSV) and the comma vertex. The CSV vertex [13,14] is given by

$$< V^{CSV}| =< 0,0,0|e^{\frac{1}{2}\alpha_n^i M_{nm}^{ij} \alpha_m^j} ,$$

where $n \geq 1$ and $m \geq 0$ refer to the modes while the $i(j)$ index refers to the ith (jth) string and take the values $1, 2, 3$. The CSV coefficients are given by

$$M_{nm}^{12} = M_{nm}^{23} = M_{nm}^{31} = (-)^m \frac{1}{n} \begin{pmatrix} n \\ m \end{pmatrix}$$

and all other $M's$ vanish. It is not hard to see, by direct substitution in (2.4), that the CSV vertex does not satisfy the comma overlaps and therefore is not a solution to the comma theory. To see this in another way note that the CSV vertex is related to the Witten's vertex by $< V^{CSV}| =< V_3^W|\mathbf{O}^{-1}$, where $\mathbf{O}$ is the conformal operator derived in [15,16]. Now it is not hard to see from the explicit form of the conformal operator $\mathbf{O}$ that it fails to commute with the comma overlaps. It follows that the CSV vertex is not a solution to the comma theory, since the Witten's vertex is. However, it is worth noticing that there is no self coupling in either of the comma $3 - Vertex$ or the CSV vertex; therefore it is reasonable to investigate a theory in which the string is made up of two pieces coupled together at two end points and strings are allowed to interact whenever their endpoints touch or they overlap as in Witten's theory. Then one should study the theory for different values of the coupling and see if one can get a consistent string theory. It is possible that Witten's theory and the dual model emerge as special cases of this theory. However, it remains to give a meaning to these formal statements.

4 Symmetries and Other Problems

The role of ghosts becomes significant when one considers the properties of the theory. The ghost vertices in the bosonized version are of the same form as the coordinate ones apart from some midpoint ghost insertions. First we would like to consider invariance under reparametrizations generated by

$$K_n = L_n - (-)^n L_{-n} .$$

It was established in [1] in the bosonic representation and proved more rigorously using the fermionic operator representation of the ghosts in [18] that K symmetry requires specific ghost insertions at the midpoint. Now we have established that the full string vertices are in fact solutions to the comma overlaps. Therefore it is important to see if the K symmetry in the comma representation requires the same insertions in the comma vertices and if in fact K continue to be a symmetry of the comma theory. For the identity vertex, $|I>$, the K_n invariance of the integration requires that

$$0 = \int K_n A =< A|K_n|I> ,$$

where $K_n = K_n^{\chi+\varphi}$, $|I>=|I^{\chi+\varphi}>$ and $|I^\varphi>= e^{-i\frac{3}{2}\phi(\pi/2)}|I_0^\varphi>$ (with $|I_0^\varphi>$ having the same form as $|I^\chi>$). In fact the action K^χ on $|I^\chi>$ gives $-\frac{D}{2}\frac{n}{2}(-)^{n/2}\delta_{n\in 2Z}$. The effect of the ghost will be to cancel this anomaly when considering $K^{\chi+\varphi}$. In the comma representation the phase factor reads

$$exp\left(\frac{-3i}{2}\phi(\pi/2)\right) = exp\left(\sum_{n=0}^{\infty}\sum_{r=1}^{2}\lambda_n^r(b_n^r - b_n^{r\dagger})\right) ,$$

where

$$\lambda_n^r = \lambda_n = \frac{3}{4}\frac{(-)^n}{\sqrt{4\delta_{n0}+n}} , r = 1, 2 ; n \geq 0 .$$

Commuting the annihilation operator through the quadratic form in $|I_0^\varphi>$ we have

$$exp\left(-2\sum_{n=0}^{\infty}\sum_{r=1}^{2}\lambda_n^r b_n^{r\dagger}\right) . \tag{4.1}$$

Now we are ready to compute the effect of commuting K_n^φ through the phase factor in (4.1). It is not hard to see that for $n = odd$, K_n commutes with the phase factor. Only $K_{n=even}^\varphi$ contributes to the anomaly. The Virasoro generators L_n^φ for the ghost in the comma theory are given in Appendix B; here we only recall the pieces that contribute to the anomaly. Thus (using Appendix B) for the phase factor we have:

$$\frac{1}{2}\sum_{N=1}^{M-1}\sqrt{(2M-2N)(2N)}(\frac{1}{\sqrt{2}})^2\sum_{r,s=1}^{2}b_{M-N}^r b_N^s$$

$$\longrightarrow \sum_{r,s=1}^{2}\sum_{N=1}^{M-1}\sqrt{(2M-2N)(2N)}\lambda_N^s\lambda_{M-N}^r = \frac{9}{2}(M-1)(-)^M .$$

The linear term in L^φ is:

$$\sqrt{2M}(\sum_{r=1}^{2}\wp_0^r - 3M)\frac{1}{\sqrt{2}}\sum_{s=1}^{2}b_M^s \longrightarrow \frac{9}{2}(1+2M)(-)^M .$$

It remains to compute the action of L^φ on $|I_0^\varphi>$. This is the same as the action of L^χ on $|I^\chi>$. But since we have not done the orbital part here let us do this one. Now the linear term does not contribute to this anomaly since $|I_0^\varphi>$ is quadratic in the creation operators. Therefore only

$$\frac{1}{2}\sum_{N=1}^{M-1}\sqrt{(2M-2N)(2N)}\left(\frac{1}{\sqrt{2}}\right)^2\sum_{r,s=1}^{2}b_{M-N}^r b_N^s$$

and

$$\frac{1}{2}\quad\sum_{N=1}^{M-1}\sqrt{(2M-(2N-1))(2N-1)}\;\frac{1}{\sqrt{2}}\sum_{r,s=1}^{2}(-)^{r+s}\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^{M-N+1}}{(2M-2N+1)^{3/2}}\times\right.$$
$$\left[((2M-2N+1)+4)b_0^r-((2M-2N+1)-4)b_0^{r\dagger}\right]+$$
$$\left.\sum_{k=1}^{\infty}\left(\frac{2M-(2N-1)}{2k}\right)^{1/2}\left[A_{2M-(2N-1)\;2k}b_k^r+S_{2M-(2N-1)\;2k}b_k^{r\dagger}\right]\right)$$
$$\times\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^N}{(2N-1)^{3/2}}\left[((2N-1)+4)b_0^s-((2N-1)-4)b_0^{s\dagger}\right]\right.$$

$$+\sum_{l=1}^{\infty}\left(\frac{2N-1}{2l}\right)^{1/2}\left[A_{2N-1\ 2l}b_l^r+S_{2N-1\ 2l}b^{r\dagger}_l\right]\Bigg)$$

contribute to the anomaly. The first equation when acting on $|I_0^\varphi>$ gives

$$-\frac{M}{2}(-)^M\delta_{M\in 2Z}\,.$$

The action of the second equation gives a finite piece plus a divergent piece, however the divergent piece cancels against another divergent piece coming from the same term in $L^\varphi_{2M}\dagger$, which is given by the second equation with $b^r \rightleftharpoons b^{r\dagger}$, making the difference of the two terms finite. Thus one gets

$$-\frac{M}{2}(-)^M\delta_{M\in 2Z-1}\,,$$

where we have used the identity (see appendix A)

$$\sum_{k=1}^{\infty}\frac{1}{2k}\left(B_{2n-1\ 2k}B_{2k\ 2m-(2n-1)}+B_{2m-(2n-1)\ 2k}B_{2k\ 2n-1}\right)$$
$$=\begin{cases}\frac{1}{2m}\left(\frac{8}{\pi^2m^2}-1\right)\delta_{m2n-1} & \\ \left(\frac{2}{\pi}\right)^2\frac{m(-)^{m+1}}{(2n-1)^2(2m-(2n-1))^2} & ,\texttt{if}\ m\neq 2n-1\,.\end{cases}\tag{4.2}$$

Putting everything together

$$K^\varphi_{2M}\longrightarrow\frac{9}{2}(M-1)(-)^M+\frac{9}{2}(2M-1)(-)^M-\frac{M}{2}(-)^M=\frac{26}{2}M(-)^M\,.$$

Clearly, this cancels the orbital anomaly thus proving the symmetry K. This shows that the midpoint insertions are the same in both theories for the identity. Therefore integration means the same thing in both theories at this level of rigor[11]. The 2-vertex is anomaly free in both theories. Both the orbital and ghost parts of the string are invariant under the symmetry K separately. For the $2-vertex$ in the comma formulation this is true for the following reason. The Virasoro generators contain terms which are either linear or quadratic in the creation-annihilation operators, the linear term does not give rise to a $c-number$ when it is commuted through the quadratic piece. The quadratic term in the Virasoro generators is of the form $b_i^r b_i^s$; the fact that the same string index "i" appears in both "$b's$"

[11]One still needs to check the $BRST$ invariance as well.

means that, the action of the $i-th$ operator on the exponential will bring down a creation operator with index different from i (since, in the comma theory there is no self coupling in the 2-vertex creation operators) which commute with the second operator giving rise to no $c-number$ anomaly. This is precisely the reason why one does not need any midpoint insertion in the case of the two vertex and therefore the $K-symmetry$ is present not only in $D=26$ but in any space time dimension (the same remark is true in the case of the full string $2-vertex$). In fact in the comma theory the K symmetry does not require any midpoint ghost insertions for all vertices apart from the the identity vertex (i.e., $|I>$) as can be easily seen from the form of the $N-vertex$ in the comma theory. However, the case for higher vertices is different in the full string theory [6]. At this point it is worth looking at how the midpoint insertions are seen by the comma theory for higher vertices? To see this we recall that, in the comma theory, the ghost $N-Vertex$ is given by

$$|V_N^{\varphi}>=e^{iQ_N^{\varphi}\phi(\pi/2)}|V_N^{\varphi,0}> \ .$$

For $|V_3^{\varphi}>$ the phase factor (insertion) is

$$exp\Big(\frac{3i}{2}\phi(\pi/2)\Big)=\sum_{j=1}^{3}\Big(\frac{i}{2}\phi(\pi/2)\Big)=\sum_{j=1}^{3}\sum_{r=1}^{2}\sum_{n=0}^{\infty}\lambda_{j\ n}^{r}(b_{j\ n}^{r}-b_{j\ n}^{r}{}^{\dagger}), \quad (4.3)$$

where $\lambda_{j\ n}^{r}=-\frac{1}{3}\lambda_n$ for all r, j and n. Commuting this phase factor through the creation operators in $|V_3^{\varphi,0}>$ doubles the factor of the creation operator in the phase factor. Hence

$$|V_3^{\varphi}>=\mathcal{F}_3|V_3^{\varphi,0}>,$$

where

$$\mathcal{F}_3=-exp\left(\sum_{j=1}^{3}\sum_{r=1}^{2}\sum_{n=0}^{\infty}\lambda_{j\ n}^{r}b_{j\ n}^{r}{}^{\dagger}\right) .$$

Thus for the phase factor we have

$$\frac{1}{2}\sum_{N=1}^{M-1}\sqrt{(2M-2N)(2N)}(\frac{1}{\sqrt{2}})^2\sum_{j=1}^{3}\sum_{r,s=1}^{2}b_{j\ M-N}^{r}b_{j\ N}^{s}$$

$$\longrightarrow \sum_{j=1}^{3}\sum_{r,s=1}^{2}\sum_{N=1}^{M-1}\sqrt{(2M-2N)(2N)}\lambda_{j\ N}^{s}\lambda_{j\ M-N}^{r}=\frac{3}{2}(M-1)(-)^{M} .$$

For the linear term in L^{φ} we have

$$\sqrt{2M}\sum_{j=1}^{3}(\sum_{r=1}^{2}\wp^{r}_{j\,0}-3M)\frac{1}{\sqrt{2}}\sum_{s=1}^{2}b^{s}_{j\,M}\longrightarrow$$

$$\frac{1}{\sqrt{2}}\sqrt{2M}\quad\sum_{rsj}4\lambda^{s}_{j\,M}\lambda^{r}_{j\,0}+\frac{3}{\sqrt{2}}\sqrt{2M}(2M)\sum_{sj}\lambda^{s}_{j\,M}=\frac{3}{2}(-)^{M}-9M(-)^{M}.$$

Hence

$$\sum_{j=1}^{3}K^{\varphi}_{j\,2M}\,\mathcal{F}_3\longrightarrow-\frac{15}{2}M(-)^{M}=-\frac{13.5}{3^2}M(-)^{M}-\frac{5}{2.3^2}M(-)^{M},$$

which is precisely what one gets in the standard formulation [6,18] of WSFT. The same procedure can be repeated for the comma 4-vertex giving

$$\sum_{j=1}^{4}K^{\varphi}_{j\,2M}\,\mathcal{F}_4\longrightarrow-\frac{27}{2}M(-)^{M}=\frac{1}{2}M(-)^{M}(-26-1)\,.$$

Which is again the standard result. Although the comma theory treats the midpoint insertions in the same way as the standard theory it does not require them for consistency (for $N\geq 2$)! The same thing happens when considering the $BRST$ symmetry. The $BRST$ symmetry requires the same midpoint insertions in the comma theory for the identity vertex as in the case of the full string identity vertex [18]. For higher vertices ($N\geq 3$) the $BRST$ symmetry in the comma theory does not require any midpoint insertions for consistency (for details see ref. [17]) unlike the case of the standard theory [18]. This seems to suggest that both theories are different in some way not yet obvious to us in spite of the fact that all vertices are solution to the comma overlaps. Now it will be interesting to see what sort of operator (if exists) interpolate between these solutions [17]. Before concluding this paper; it will be useful to explore some other features of the full string theory within the frame work of the comma theory. Let us recall that in ref. [19] translations were shown to be inner derivations. To prove that the authors of [19] had to show that the properties

$$P^{L}_{0}+P^{R}_{0}=p_0,\tag{4.4}$$

(where P_0^L and P_0^R are the integrals of the space-time momentum density over the intervals $\sigma = 0 \to \pi/2$ and $\pi/2 \to \pi$ respectively and p is the total centre of mass momentum) and

$$(P_0^R A_1) * A_2 + A_1 * (P_0^L A_2) = 0, \tag{4.5}$$

(where A is a string field) hold. The star product of two states is given by Gross and Jevicki,

$$|A_1 * A_2> = < A_1| < A_2||V_3> .$$

Thus to prove (4.5) one only has to show that the integration by parts law [19] is satisfied

$$< A_1| < A_2|(P_{1\,0}^R + P_{2\,0}^L)|V_3> = 0, \tag{4.6}$$

where the indices 1 and 2 refers to string one and string two respectively. Eq. (4.6) is a consequence of

$$(P_{1\,0}^R + P_{2\,0}^L)|V_3> = 0. \tag{4.7}$$

It is a straight forward to see that these properties continue to hold in the comma theory. Eq. (4.4) is just eq. (1.11) for $n = 0$ which was established earlier in the comma formalism. Eq. (4.7) is a consequence of integrating[12] (from $\sigma = 0 \to \pi/2$) the overlap equation (2.2) defining the comma vertex for the particular values $j = 2$ and $N = 3$. This can be easily seen by integrating (2.2) from $\sigma = 0 \to \pi/2$ using the Fourier expansion of the comma conjugate momenta, $\wp_j^r(\sigma)$,

$$\wp_i^r(\sigma) = \frac{2}{\pi}\wp_{i0}^r + \frac{2\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\wp_{i2n}^r cos 2n\sigma .$$

Thus (4.7) is satisfied by construction in the comma theory. One can check the above statement directly by explicitly substituting the oscillator form of the comma 3-vertex (derived earlier) into (4.7) and then integrating over σ. Doing so one gets

$$\left[-H_{10\,im}^{Rs} + \delta^{Rs}\delta_{1i}\delta_{0m} - H_{20\,im}^{Ls} + \delta^{Ls}\delta_{2i}\delta_{0m}\right] b^{r\dagger}_{im}|V_3> .$$

[12] One must check the convergence of the implicit sums over oscillators since integrating over σ corresponds to a second sum.

The expression in the square bracket is zero as can be easily seen using the explicit values of the matrix H obtained before.

It has been shown in [19] that there is an anomaly in the operator associativity of WSFT which in turn implies an associativity anomaly in the star algebra of Witten's string field theory. This can be seen using the fact

$$\left[P^R_{10} + P^L_{20}\,,\, X_1(\pi/2) - X_3(\pi/2)\right] = -\frac{i}{2}|V_4>\,,$$

(since the zero modes do not commute). However, we have seen in ref. [19] that

$$(P^R_{10} + P^L_{20})|V_4>= 0$$

and[13]

$$[X_1(\pi/2) - X_3(\pi/2)]|V_4>= 0$$

which is a clear violation of the uncertainty principle. These anomalies have been discussed before [20] and are characterized by the failure of the Jacobi identity. This anomaly arises because of the coupling between the first and the third strings in the vertex. Now it is not hard to see that the Witten's 4-vertex suffers from the same problem when viewed by the comma theory. To see this one only needs to notice that the above two equations are in fact comma equations. For the first equation this is obvious. For the second equation, recall that from the definition of the comma coordinates; $\lim_{\sigma\to\frac{\pi}{2}}\chi^L(\sigma) = \lim_{\sigma\to\frac{\pi}{2}}\chi^R(\sigma) = X(\pi/2)$ (where L and R refer to the left and right parts of the string respectively). However, when one is working fully in the comma representation, the comma vertices do not seem to suffer from this particular problem. This is due to the fact that in the comma theory, there is no coupling between the first and third strings or the second and the forth strings in the vertex. This can is easily be proven to be equivalent to the following statement; it is not possible to construct an operator that fails to commute with $(P^R_{10} + P^L_{20})$ and at the same time kills the comma $4 - vertex$. In fact, in the comma theory, this is true in general for higher vertices ($N \geq 3$).

[13]Remember that this equation is the statement that the midpoint is not moved in the oscillators representation of WSFT which is not the case in the comma theory.

Hence the above mentioned anomaly disappears in the comma theory and the associativity of the star algebra is retained !

5 Conclusion

We have shown that the operator form of the Witten's vertex given in [7,8,6,9,10] is indeed a solution to the comma theory. The question about the equivalence of the two theories is discussed. On the level of the one and the two vertices (i.e., $|I>$ and $|V_2>$) we have seen that both forms of the vertex (i.e., the comma vertex and the Witten's vertex) possess the same symmetries and in fact can be shown to be equal using the change of representation formulae derived in the introduction[14]. However for higher vertices ($N \geq 3$), while in the full string formulation, the K and the $BRST$ invariance require some specific ghost insertions at the midpoint of the string for consistency, it does not seem to be the case in the comma formulation. In the comma theory both the orbital and ghost parts of the vertices (for $N \geq 2$) are invariant under the K and the $BRST$ symmetries separately. The associativity anomaly in the star algebra of the standard formulation disappears in the comma theory. Now it seems to us that the comma formulation of string field theory is somehow more general than the standard formulation of string field, since beside the comma vertices all the Witten's vertices are solutions to the comma theory. This seems to suggest that Witten's interaction does not lead to a unique solution but to more than one solution. A challenging task is to understand the relationships between theses solutions. Work in this direction is in progress and the result will be reported in the future.

Appendix A

In this appendix we give details of the summation formulae. Many

[14]In fact to do this properly, one also needs to derive the relationship of the full string vacuum to the comma vacua. See ref. [3,4].

other useful formulae can be found in refs. [6,?,11] First consider

$$\sum_{n=0}^{\infty} \frac{(-)^n}{n+a} \begin{pmatrix} -1/2 \\ n \end{pmatrix} = \int_0^1 dt\, t^{a-1} \sum_{n=0}^{\infty} (-)^n t^n \begin{pmatrix} -1/2 \\ n \end{pmatrix}$$
$$= \int_0^1 dt\, t^{a-1} (1-t)^{\frac{1}{2}-1} .$$

where we have used the binomial formula $(1+t)^a = \sum_{n=0}^{\infty} t^n \begin{pmatrix} a \\ n \end{pmatrix}$ to sum the series in the integrand. However this equation defines the $B(a, 1/2)$. Hence

$$\sum_{n=0}^{\infty} \frac{(-)^n}{n+a} \begin{pmatrix} -1/2 \\ n \end{pmatrix} = B(a, 1/2) = \frac{\Gamma(1/2)\Gamma(a)}{\Gamma(a+1/2)} . \tag{A.1}$$

Next we consider

$$\sum_{n=0}^{\infty} \frac{(-)^n}{(n-k-1/2)^2} \begin{pmatrix} -1/2 \\ n \end{pmatrix} = \frac{\partial}{\partial \xi} \sum_{n=0}^{\infty} \frac{(-)^n}{n-\xi-1/2} \begin{pmatrix} -1/2 \\ n \end{pmatrix} |_{\xi=k} =$$
$$= \frac{\partial}{\partial \xi} \left(\frac{\Gamma(1/2)\Gamma(1/2-\xi)}{\Gamma(1-\xi)} \right) |_{\xi=k} = 2\pi \frac{(-)^k}{2k-1} \begin{pmatrix} -1/2 \\ k-1 \end{pmatrix}^{-1} .$$

A special case of the above formula is

$$\sum_{n=0}^{\infty} \frac{(-)^n}{(2n+1)^2} \begin{pmatrix} -1/2 \\ n \end{pmatrix} = \frac{\pi}{4}(\psi(1) - \psi(1/2))$$
$$= \frac{\pi}{4} \left(\frac{1+V_{00}}{1-V_{00}} \right) .$$

Using (A.1) and the explicit expression for the matrix elements B_{nm} it is straightforward to see that

$$\sum_{n=0}^{\infty} B_{2k\,2n+1} \nu_{2n+1} = -\frac{1}{2} \frac{(-)^k \Gamma(k+1/2)}{\Gamma(1/2)\Gamma(1+k)} = -\frac{u_{2k}}{2} . \tag{A.2}$$

It is also easy to see that

$$\sum_{n=0}^{\infty} B_{2k\,2n+1} \frac{\nu_{2n+1}}{2n+1} = \frac{1}{2} \frac{u_{2k}}{2k} . \tag{A.3}$$

The above formulae can be utilized to show that

$$\sum_{m=0}^{\infty} B_{2k\,2m+1} \frac{2m+1}{(2n)-(2m+1)} \nu_{2n+1} = \left[\frac{u_{2k}}{2} + \frac{1}{2\pi} \frac{\delta_{nk}}{u_{2n}} - \frac{2n}{2n+2k} \frac{u_{2k}}{2} \right] . \tag{A.4}$$

Another useful sum to consider is

$$\sum_{k=0}^{\infty} B_{2n\,2k+1}\frac{\nu_{2n+1}}{(2k+1)+(2\ell+1)} = \left[-\frac{1}{(2\ell-1)-(2n)}\frac{u_{2n}}{2} - \frac{1}{2\ell-1}\frac{B_{2n\,2\ell-1}}{\nu_{2\ell-1}}\right],$$

To arrive at the above result we only need to use the fact $\Gamma(1-n)=\infty$ for $n \geq 1$, equation (A.1) and

$$u_{2n} \equiv \begin{pmatrix} -1/2 \\ n \end{pmatrix} = \frac{(-)^n\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}.$$

Another usefull sum to perform is

$$\sum_{k=1}^{\infty} \frac{1}{2k}\left(B_{2x\;2k}B_{2k\;2y} + B_{2y\;2k}B_{2k\;2x}\right) \tag{A.5}$$

Using the following identities:

$$\psi(x)-\psi(y) = \sum_{k=0}^{\infty}\left(\frac{1}{y+x} - \frac{1}{x+y}\right),$$

$$\psi(1-z) = \psi(z) + \pi ctg\pi z\,,$$

$$\psi(x+1) = \psi(x) + \frac{1}{x}$$

the sum in (A.5) reduces to

$$\frac{(-)^{2x+1}}{(2\pi)^2}\left(\frac{2}{x^3} - \frac{\pi}{x^2}ctg\pi(x+1) - \frac{\pi^2}{x}(1-ctg^2\pi(x+1))\right)$$

for $y=x$ and

$$\frac{1}{(2\pi)^2}\frac{(-)^{x+y}}{y-x}\left(\frac{\pi ctg\pi(x+1)}{x} - \frac{\pi ctg\pi(y+1)}{y} + \frac{x^2-y^2}{x^2y^2}\right)$$

for $x \neq y$. Setting $2x = 2n-1$ and $2y = 2m-(2n-1)$, where n and m are integers greter than zero, we recover (4.2).

Appendix B

Here we state the Virasoro Generators for the ghost sector in the comma annihilation-creation basis. For the even modes, we have:

$$L^{\varphi}_{2M} = \frac{1}{2}\sum_{N=1}^{\infty}\sqrt{(2N+2M)(2N)}\sum_{r,s=1}^{2} b^{r}_{N}{}^{\dagger} b^{s}_{N+M} + \frac{1}{2}\sum_{N=1}^{\infty}\sqrt{(2N-1)(2N+2M-1)}$$

$$\sum_{r,s=1}^{2}(-)^{r+s}\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^{N}}{(2N-1)^{3/2}}\left[((2N-1)+4)b^{r}_{0}{}^{\dagger}\right.\right.$$

$$\left.\left.-((2N-1)-4)b^{r}_{0}\right] - \sum_{l=1}^{\infty}\left(\frac{2N-1}{2l}\right)^{1/2}\left[A_{2N-12l}b^{r}_{l}{}^{\dagger} + S_{2N-12l}b^{r}_{l}\right]\right)$$

$$\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^{N+M}}{(2N+2M-1)^{3/2}}\left[((2N+2M-1)+4)b^{s}_{0} - ((2N+2M-1)-4)b^{s}{}^{\dagger}_{0}\right]\right.$$

$$\left.-\sum_{k=1}^{\infty}\left(\frac{2N+2M-1}{2k}\right)^{1/2}\left[A_{2N+2M-12k}b^{s}_{k} + S_{2N+2M-12k}b^{s}{}^{\dagger}_{k}\right]\right) +$$

$$\frac{1}{4}\sum_{N=1}^{M-1}\sqrt{(2M-2N)(2N)}\sum_{r,s=1}^{2} b^{r}_{M-N}b^{s}_{N} + \frac{1}{4}\sum_{N=1}^{M}\sqrt{(2M-2N+1)(2N-1)}$$

$$\sum_{r,s=1}^{2}(-)^{r+s}\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^{M-N+1}}{(2M-2N+1)^{3/2}}\left[((2M-2N+1)+4)b^{r}_{0} - ((2M-2N+1)-4)b^{r}{}^{\dagger}_{0}\right]\right.$$

$$\left.-\sum_{l=1}^{\infty}\left(\frac{2M-2N+1}{2l}\right)^{1/2}\left[A_{2M-2N+12l}b^{r}_{l} + S_{2M-2N+12l}b^{r}{}^{\dagger}_{l}\right]\right)$$

$$\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^{N}}{(2N-1)^{3/2}}\left[((2N-1)+4)b^{s}_{0} - ((2N-1)-4)b^{s}{}^{\dagger}_{0}\right]\right) - \sum_{k=1}^{\infty}\left(\frac{2N-1}{2k}\right)^{1/2}$$

$$\left.\left[A_{2N-12k}b^{s}_{k} + S_{2N-12k}b^{s}{}^{\dagger}_{k}\right]\right) + \frac{1}{\sqrt{2}}\sqrt{2M}\left(\sum_{r=1}^{2}(b^{r}_{0} + b^{r}_{0}{}^{\dagger}) - 3M\right)\sum_{s=1}^{2} b^{s}_{M}\,. \tag{B.1}$$

Whereas the odd modes of the Virasoro generators are given by:

$$L^{\varphi}_{2M-1} = \frac{1}{2}\sum_{N=1}^{\infty}\sqrt{(2N)(2N+2M-1)}\sum_{r,s=1}^{2}(-)^{s}b^{r}_{N}{}^{\dagger}\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^{N+M}}{(2N+2M-1)^{3/2}}\right.$$

$$\left[((2N+2M-1)+4)b^{s}_{0} - ((2N+2M-1)-4)b^{r}_{0}{}^{\dagger}\right] - \sum_{k=1}^{\infty}\left(\frac{2N+2M-1}{2k}\right)^{1/2}$$

$$\left[A_{2N+2M-12k}b_k^s + S_{2N+2M-12l}b_k^{s\dagger}\right]\Big) + \frac{1}{2}\sum_{N=1}^{\infty}\sqrt{(2N-1)(2N+2M-2)}$$

$$\sum_{r,s=1}^{2}(-)^r\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^N}{(2N-1)^{3/2}}\left[((2N-1)+4)b_0^{r\dagger} - ((2N-1)-4)b_0^r\right] - \right.$$

$$\left.\sum_{k=1}^{\infty}\left(\frac{2N-1}{2k}\right)^{1/2}\left[A_{2N-12k}b_k^{r\dagger} + S_{2N-12k}b_k^r\right]\right) b_{N+M-1}^s + \frac{1}{4}\sum_{N=1}^{M-1}\sqrt{(2M-2N-1)(2N)}$$

$$\sum_{r,s=1}^{2}(-)^r\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^{M-N}}{(2M-2N-1)^{3/2}}\left[((2M-2N-1)+4)b_0^r - ((2M-2N-1)-4)b_0^{r\dagger}\right]\right.$$

$$\left.-\sum_{k=1}^{\infty}\left(\frac{2M-2N-1}{2k}\right)^{1/2}\left[A_{2M-2N-12k}b_k^r + S_{2M-2N-12k}b_k^{r\dagger}\right]\right) b_N^s$$

$$+\frac{1}{4}\sum_{N=1}^{M-1}\sqrt{(2M-2N)(2N-1)}\sum_{r,s=1}^{2}(-)^s b_{M-N}^r\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^N}{(2N-1)^{3/2}}\right.$$

$$\left[((2N-1)+4)b_0^s - ((2N-1)-4)b_0^{s\dagger}\right] - \sum_{k=1}^{\infty}\left(\frac{2N-1}{2k}\right)^{1/2}$$

$$\left.\left[A_{2N-12k}b_k^s + S_{2N-12k}b_k^{s\dagger}\right]\right) + \sqrt{\frac{2M-1}{2}}\left(\sum_{r=1}^{2}(b_0^r + b_0^{r\dagger}) - 3M + \frac{3}{2}\right)$$

$$\sum_{s=1}^{2}(-)^s\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^M}{(2M-1)^{3/2}}\left[((2M-1)+4)b_0^s - ((2M-1)-4)b_0^{s\dagger}\right]\right.$$

$$\left.-\sum_{k=1}^{\infty}\left(\frac{2M-1}{2k}\right)^{1/2}\left[A_{2M-12k}b_k^s + S_{2M-12k}b_k^{s\dagger}\right]\right). \qquad \text{(B.2)}$$

Finally, the zero mode of the Virasoro Generators (i.e., the Hamiltonian operator) has the form:

$$L_0^{\varphi} = \frac{1}{2}\left(\sum_{r=1}^{2}(b_0^r + b_0^{r\dagger})\right)^2 - \frac{1}{8} + \frac{1}{2}\sum_{N=1}^{\infty}\sum_{r,s=1}^{2}2N\, b_N^{r\dagger}b_N^s + \frac{1}{2}\sum_{N=1}^{\infty}\sum_{r,s=1}^{2}$$

$$(2N-1)(-)^{r+s}\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^N}{(2N-1)^{3/2}}\left[((2N-1)+4)b_0^{r\dagger} - ((2N-1)-4)b_0^r\right]\right.$$

$$\left.-\sum_{l=1}^{\infty}\left(\frac{2N-1}{2l}\right)^{1/2}\left[A_{2N-12l}b_l^{r\dagger} + S_{2N-12l}b_l^r\right]\right)$$

$$\left(\frac{1}{\sqrt{2}\pi}\frac{(-)^N}{(2N-1)^{3/2}}\left[((2N-1)+4)b_0^s - ((2N-1)-4)b_0^{s\dagger}\right]\right.$$

$$
-\sum_{k=1}^{\infty}\left(\frac{2N-1}{2k}\right)^{1/2}\left[A_{2N-12k}b_k^s+S_{2N-12k}b_k^{s\dagger}\right]\bigg)\ . \qquad (B.3)
$$

It is tedious, otherwise straightforward, to show that the desired commutation relation for the comma Virasoro generators are indeed satisfied.

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