# Updates to the Dualized Standard Model on Fermion Masses and Mixings

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**Abstract**

The Dualized Standard Model has scored a number of successes in explaining the fermion mass hierarchy and mixing pattern. This note contains updates to those results including (a) an improved treatment of neutrino oscillation free from previous assumptions on neutrino masses, and hence admitting now the preferred LMA solution to solar neutrinos, (b) an understanding of the limitation of the 1-loop calculation so far performed, thus explaining the two previous discrepancies with data, and (c) an analytic derivation and confirmation of the numerical results previously obtained.

# 1 Introduction

The Dualized Standard Model (DSM) suggested a few years ago [1, 2, 3] has scored some, to us, notable successes in explaining the intricate and otherwise mysterious mass and mixing patterns of quarks and leptons. With only 3 adjustable parameters, the model is able to reproduce already at 1-loop level the following mass and mixing parameters all to within present experimental errors [4, 5]: all 9 CKM matrix elements $|V_{rs}|$ for quarks [6], the 2 MNS lepton mixing matrix elements $|U_{\mu 3}|$ and $|U_{e3}|$ bounded by neutrino oscillation experiments with respectively atmospheric [7] and reactor neutrinos [8], plus the 3 mass ratios $m_c/m_t, m_s/m_b$ and $m_\mu/m_\tau$ [6], together accounting for 8 independent parameters of the Standard Model. However, the treatment so far published, culminating in [5] to which we refer the reader for details, contains some shortcomings which were not apparent at the beginning but have since, with time, been recognized and remedied. In particular, (a) some assumptions were made in $\nu$ oscillations which are later found to be unnecessary, (b) the limitations of the 1-loop approximation made which were not recognized before are now seen to explain why the other masses and mixing angles: $m_u, m_d, m_e, U_{e2}$, were not well reproduced, while (c) the results obtained only numerically before are now seen to follow also from simple analytic considerations. The purpose of this note is to remove these old shortcomings, resulting in both a tighter and a more transparent scheme, while affording at the same time an analytic check on the previous numerical results.

We begin with a very brief summary of the basic features of the DSM scheme necessary for our presentation. For explanations and details, the reader is referred to the literature cited. The DSM scheme is based on a nonabelian generalization of electric-magnetic duality [9] which offers an explanation for the existence of exactly 3 fermion generations and a suggestion of how the generation symmetry is broken. This leads in turn to the construction of a Higgs potential and Yukawa coupling which give the tree-level fermion mass matrix a factorized form [1] and allows its radiative corrections to be calculated. Further, the fermion mass matrix is found to retain a factorized form even after 1-loop radiative corrections:

$$m = m_T \begin{pmatrix} x \\ y \\ z \end{pmatrix} (x, y, z), \tag{1}$$

where only the normalization $m_T$ depends on the fermion species, i. e. $U$ or

$D$ type quarks, or charged leptons $L$ or neutrinos $N$. The vector $(x, y, z)$, however, now rotates in generation space with changing scales $\mu$, and satisfies the RG equation:

$$\frac{d}{d(\ln \mu^2)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{3}{64\pi^2} \rho^2 \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{z}_1 \end{pmatrix}, \tag{2}$$

with

$$\tilde{x}_1 = \frac{x(x^2 - y^2)}{x^2 + y^2} + \frac{x(x^2 - z^2)}{x^2 + z^2}, \quad \text{cyclic}, \tag{3}$$

and $\rho^2$ the Yukawa coupling strength [4][1]. In (2), one sees that the vector $(x, y, z)$, taken to be normalized from now on, is stationary in orientation with respect to change in $\mu$ at $(x, y, z) = (1, 0, 0)$ and $\frac{1}{\sqrt{3}}(1, 1, 1)$, which means that these are to be interpreted as fixed points of rotation. Further, the sign of the derivative is such that as $\mu$ changes, the vector $\mathbf{r} = (x, y, z)$ traces out a trajectory on the unit sphere in 3-D generation space starting from the "high-energy" fixed point $(1, 0, 0)$ at infinite scale and ending at the "low-energy" fixed point $\frac{1}{\sqrt{3}}(1, 1, 1)$ at zero scale.

The fact that the mass matrix rotates with changing scales means that the usual definition of the 3 generation states as mass eigenstates has to be refined since it will have to be specified at what scales they are to be so defined. An analysis of the situation, defining each state at its own mass scale, leads to the following conclusion [10]. For each fermion species $T$ (i.e. whether $U, D, L$ or $N$) the state vectors for the 3 generations form an orthonormal triad in generation space which is given in terms of the rotating vector $\mathbf{r}(\mu)$ as follows:

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{r}_1, \\ \mathbf{v}_2 &= -\frac{\mathbf{r}_1 \wedge (\mathbf{r}_1 \wedge \mathbf{r}_2)}{|\mathbf{r}_1 \wedge (\mathbf{r}_1 \wedge \mathbf{r}_2)|}, \\ \mathbf{v}_3 &= \frac{\mathbf{r}_1 \wedge \mathbf{r}_2}{|\mathbf{r}_1 \wedge \mathbf{r}_2|} \end{aligned} \tag{4}$$

where $\mathbf{r}_i = \mathbf{r}(m_i)$ is the rotating vector $\mathbf{r}(\mu)$ taken at the scale $\mu = m_i$, with $m_i$ being the physical mass of the $i$th generation labelled from the heaviest

[1] There was an error in the coefficient of the right-hand side of eq. (2) as given in [4], which means that the numerical value of $\rho$ given in [4] and [5] should be multiplied by a factor of $\sqrt{5/3}$. Other results in these references are not affected.

(1) to the lightest (3). The elements of the mixing matrix (whether CKM [11] for quarks or MNS [12] for leptons) are then given as the scalar products between the up-states $\mathbf{v}_i$ and the down states $\mathbf{v}'_j$, thus:

$$V_{ij} = \mathbf{v}_i . \mathbf{v}'_j. \tag{5}$$

Furthermore, for the fermion species $U, D$ and $L$, the mass of the $i$th generation state is given as the solution for $m_i$ to the equation:

$$m_i = m_T |\mathbf{r}_i . \mathbf{v}_i|^2. \tag{6}$$

This criterion for determining the masses of the 3 generations, however, does not apply to neutrinos which, because of a likely see-saw mechanism [13], may have physical masses different from the Dirac masses appearing in the above equation and hence require a special treatment to be explained later. With the above formulae, once given the trajectory for $\mathbf{r}$, mixing matrix elements and mass ratios between generations can be evaluated, excepting for the moment where neutrinos are involved.

The trajectory for $\mathbf{r}(\mu)$ depends on only 3 parameters, two of them corresponding to the vacuum expectation values of the Higgs fields which specify the rotation trajectory, and the third being the Yakawa coupling strength $\rho$ in (2) which governs the rotation speed. The first 2 parameters are independent of the fermion species $T$, while the third $\rho$ can in principle depend on $T$ but, for consistency with the above prescription for defining masses of the generations, it was found that $\rho$ has also to be $T$-independent (see [4] and later). In [5], these 3 parameters were fitted to $m_c/m_t, m_\mu/m_\tau$ and the Cabibbo angle, which then allows one to calculate the whole trajectory via (2). The result is shown in Figure 1 where from the location of the various states each marked at its own mass scale, the rotation speed with respect to $\mu$ can be gauged. From this, the CKM matrix for quarks was calculated using (5), and gave, as already mentioned, all elements within present experimental limits.

## 2 $\nu$ Oscillations

Neutrinos require a special treatment because of the see-saw mechanism which is likely to give them physical masses different from the Dirac mass appearing in the mass matrix (1). On the one hand, the state vectors for neutrinos, in parallel to other fermions, ought to be defined at the scales of

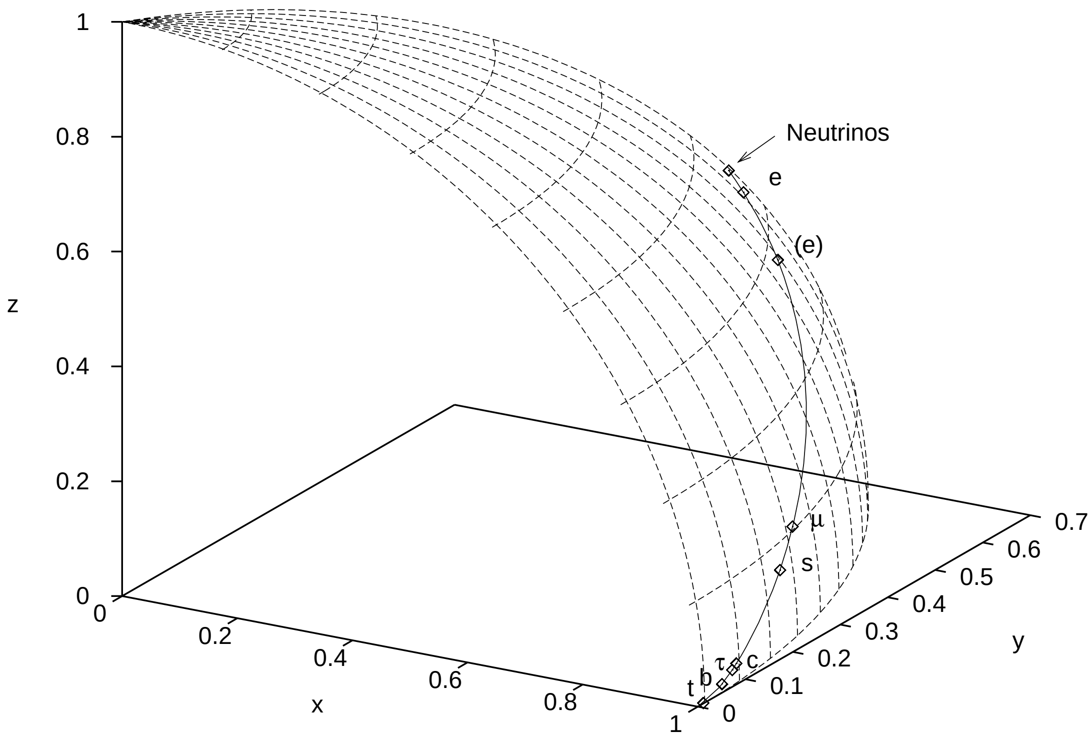


Figure 1: Trajectory traced out by the vector $\mathbf{r}(\mu)$ on the unit sphere as calculated in [5] with the various fermion states marked each at its own mass scale.

their physical masses, while on the other the equation (6) applies only to states whose physical and Dirac masses coincide, and hence not directly to neutrinos. It would appear therefore that some additional assumption will have to be made on the relationship between the physical and Dirac masses of neutrinos before the above scheme can be applied, and this was the approach we took in [14].

It turns out, however, that as far as reproducing the MNS mixing matrix elements is concerned, which is at present our main conern, then no such new assumption is needed. This arises as follows. The electron neutrino mass is now restricted by e.g. tritium decay experiments to below 3 eV [6], while oscillation experiments on solar [7, 15] and atmospheric [7] neutrinos limit the mass differences between generations to less than 0.05 eV, implying that all 3 generations have masses of at most order eV. According to the calculation cited in Figure 1, however, at scales of order eV, the vector $\mathbf{r}$ is already so close to the low energy fixed point $\frac{1}{\sqrt{3}}(1,1,1)$ as to be indistinguishable in the figure. Referring next to the definition of the state vectors in (4), one sees that in this situation when all 3 generations are close together, the 3 vectors in the triad become respectively just the position vector, the tangent vector to the trajectory, and the vector normal to both the above, all taken at the same point. In other words, one has in particular for the state vector of the heaviest neutrino $\nu_3$ just:

$$\mathbf{v}_1 = \mathbf{v}_{\nu_3} = \mathbf{r}_0 = \frac{1}{\sqrt{3}}(1,1,1). \tag{7}$$

(Notice that neutrinos are conventionally labelled in the opposite order to that adopted in (4) above, i.e. from the lightest ($\nu_1$) to the heaviest ($\nu_3$)).

Charged leptons being ordinary Dirac particles, their state vectors can be calculated by the method detailed in the preamble above in the same manner as for quarks. The result of the calculation performed in [5] and cited in Figure 1 gave:

$$\begin{aligned} |\tau\rangle &= (0.996732, 0.076223, 0.026756), \\ |\mu\rangle &= (-0.075925, 0.774100, 0.628494), \\ |e\rangle &= (0.027068, -0.628482, 0.777354). \end{aligned} \tag{8}$$

From these and (7) above, using (5), one easily obtains the MNS mixing matrix elements:

$$\begin{aligned} U_{\mu 3} = \langle\mu|\nu_3\rangle &= 0.7660, \\ U_{e3} = \langle e|\nu_3\rangle &= 0.1016, \end{aligned} \tag{9}$$

which are seen to be within the experimental limits obtained for these quantities by respectively the atmospheric [7] and reactor [8] neutrino experiments:

$$\begin{aligned} |U_{\mu 3}| &\sim 0.56 - 0.83, \\ |U_{e3}| &\sim 0.00 - 0.15. \end{aligned} \tag{10}$$

One notices that the above result is independent of any assumptions about the Dirac masses of neutrinos, in contrast to the previous treatment of [14]. In particular, because of the assumptions made then on neutrino masses, the treatment in [14] is valid only for the range of masses permitted by the "vacuum oscillation" solution to the solar neutrino problem. The present result in (9), on the other hand, having been derived independently of assumptions on neutrino masses, is valid for any solution to the solar neutrino problem, in particular for the large mixing angle MSW [16] solution. This is of great practical significance, given the increasing preference of recent data for the LMA solution to the exclusion of the others [17, 18].

In principle, one can use the trajectory calculated in [5] and displayed in Figure 1 to evaluate also the tangent and normal vectors at the low energy fixed point $\mathbf{r}_0 = \frac{1}{\sqrt{3}}(1,1,1)$, namely the state vectors $\mathbf{v}_{\nu_2}$ and $\mathbf{v}_{\nu_1}$, and hence all the other elements of the MNS mixing matrix, in particular the solar neutrino angle $U_{e2}$. However, as we shall see in the next section, due to the limitations of the 1-loop calculation so far performed, such results would be unreliable.

## 3 Limitations of 1-loop

The 1-loop approximation with the parameters fitted as above and as in [5] is valid only near the fixed point $(1,0,0)$ where the rotation is slow. Away from the fixed point, the RGE will receive large logaritmic terms from higher-loop contributions. As seen in Figure 1, however, down to the top mass scale at 175 GeV, the vector $\mathbf{r}$ has rotated from the fixed point at infinite scale by an angle of only about 0.012 radians, while even from the top mass scale to that at the muon mass at 105 MeV, the vector has rotated by only about another 0.3 radians. For scales above the muon mass, therefore, one can hope a 1-loop approximation to have some validity. But at scales much below the $\mu$ mass, the 2-loop correction, which is expected to be of the order of the square of the 1-loop result, will become increasingly significant, thus making the 1-loop result unreliable.

Given then the limited validity range for the 1-loop calculation so far performed, one has to re-examine the results obtained to ascertain which could be regarded as reliable. Let us accept for present purposes the 1-loop approximation as sufficiently accurate for scales down to about the muon mass, in which range would then be included the 2 heavier generations of the 3 fermion species $U, D, L$, i.e. all species except the neutrinos. We notice from (4), however, that for $U, D, L$, the triad of state vectors for all 3 generations are in fact already determined in this range, the vector $\mathbf{v}_3$ for the lightest generation being given as the normal once the vectors for the 2 heavier generations are specified. Hence, given the triads for both the $U$ and $D$ quarks, one can evaluate the whole CKM matrix with confidence. Indeed, this is born out by the comparison given in [5] of the calculation result:

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.9745-0.9762 & 0.217-0.224 & 0.0043-0.0046 \\ 0.217-0.224 & 0.9733-0.9756 & 0.0354-0.0508 \\ 0.0120-0.0157 & 0.0336-0.0486 & 0.9988-0.9994 \end{pmatrix}, \qquad (11)$$

with the then available experimental numbers quoted from the databook [6]:

$$\begin{pmatrix} 0.9745-0.9760 & 0.217-0.224 & 0.0018-0.0045 \\ 0.217-0.224 & 0.9737-0.9753 & 0.036-0.042 \\ 0.004-0.013 & 0.035-0.042 & 0.9991-0.9994 \end{pmatrix}, \qquad (12)$$

which are seen also to be within the most recent limits [19]. On the other hand, the masses for the lightest generation, according to (6), requires running the vector $\mathbf{r}$ down to the mass scales of the lightest generation, and hence cannot be achieved reliably by the 1-loop approximation. And indeed, the value obtained in [5] for the electron mass was 6 MeV, to be compared with the correct empirical value of 0.5 MeV. The mass values for the light quarks $u, d$ were also not well reproduced, but that may be due to intrinsic ambiguities in defining light quark masses.

Next, turning to the MNS mixing matrix for leptons, we note first that the state vectors for all 3 charged leptons $\tau, \mu, e$ as quoted in (8) ought, according to the above criterion, be reliable. The same, however, cannot be said for the state vectors of neutrinos, the mass scales of which are all way outside the range of validity of the 1-loop calculation. The only exception is the vector for the heaviest neutrino $\nu_3$ which, as noted in the previous section, can be identified to a good approximation with the low energy fixed point $\mathbf{r}_0 = \frac{1}{\sqrt{3}}(1,1,1)$ known to be valid beyond 1-loop [20]. In consequence, one obtains the result (9) in good agreement with experiment. The other 2 vectors in the neutrino triad both depend on the tangent vector to the trajectory and hence cannot be reproduced by the 1-loop calculation. Indeed, the value obtained in [5] for the solar neutrino angle $U_{e2}$, which depends on the state vector for the second heaviest neutrino $\nu_2$, is about 0.23 and lies outside the experimental limits of about 0.4 to 0.7.

In other words, the results of the 1-loop calculation on fermion mass and mixing parameters have now all been checked to agree with present experiment in the range they are expected to be valid, and are seen to deviate from data only outside the expected validity range.

It should be emphasized, however, that this apparent success achieved by calculating only the diagram with a single (dual colour) Higgs loop makes sense only if one regards both the mass hierarchy and mixing as consequences of the mass matrix rotation. As far as the orientation of the rotating vector $\mathbf{r}(\mu)$ rotation is concerned, it was shown in [4] that radiative corrections due to standard model particles are zero, and those due to other dual colour particles are small, leaving thus the calculated (dual colour) Higgs loop with no competition. The same approximation, however, would not be applicable, for example, to the normalization of the rotating vector.

# 4 Analytic Approach

Previous calculations with the DSM scheme, e.g. [5], were done numerically, on which calculations the above remarks also rely. It is found, however, that with some simple yet quite reasonable approximations, an analytic approach is possible which provides first a check on the numerical results, and second an easier means for the reader to scrutinize and verify them. It also gives algebraic relations between fermion mixing elements and mass ratios which may be useful for certain purposes.

First, let us parametrize the vector $\mathbf{r}(\mu)$ in terms of the usual polar coordinates, thus:

$$\mathbf{r}(\mu) = (\cos\theta(\mu), \sin\theta(\mu)\cos\phi(\mu), \sin\theta(\mu)\sin\phi(\mu)), \tag{13}$$

and denote by $\theta_i, \phi_i$ the corresponding angles taken at the scale $\mu = m_i$ with $m_i$ being the mass of the state $i$. Next, we recall from the last section that the

1-loop approximation is valid only in the region near the high energy fixed point $(1,0,0)$ so that there is no loss of generality at 1-loop to make a small angle approximation, i.e. assuming that all $\theta$'s are small and that, although the $\phi$'s are in principle arbitrary, their differences are small. In the same spirit, we assume also that $\theta_1$ is small compared to $\theta_2$. In this approximation then, one has:

$$\begin{aligned}
\mathbf{v}_1 &= (1-\frac{\theta_1^2}{2}, \theta_1\cos\phi_1, \theta_1\sin\phi_1), \\
\mathbf{v}_2 &= (0, \cos\phi_2, \sin\phi_2) \\
&\quad +\theta_1(-\cos(\phi_2-\phi_1), -\frac{1}{\theta_2}\sin(\phi_2-\phi_1)\sin\phi_2, \frac{1}{\theta_2}\sin(\phi_2-\phi_1)\cos\phi_2), \\
\mathbf{v}_3 &= (0, -\sin\phi_2, \cos\phi_2) \\
&\quad +\theta_1\sin(\phi_2-\phi_1)(1, -\frac{1}{\theta_2}\cos\phi_2, -\frac{1}{\theta_2}\sin\phi_2).
\end{aligned} \tag{14}$$

Using (14) and (5), one obtains the approximate forms of the CKM matrix elements:

$$\begin{aligned}
V_{tb} &= 1-\frac{1}{2}(\theta_b-\theta_t)^2, \\
V_{cs} &= \cos(\phi_s-\phi_c)+\sin(\phi_s-\phi_c)\left\{\frac{\theta_t}{\theta_c}\sin(\phi_c-\phi_t)-\frac{\theta_b}{\theta_s}\sin(\phi_s-\phi_b)\right\}, \\
V_{ud} &= -V_{cs}, \\
V_{cd} &= \sin(\phi_s-\phi_c)+\cos(\phi_s-\phi_c)\left\{\frac{\theta_b}{\theta_s}\sin(\phi_s-\phi_b)-\frac{\theta_t}{\theta_c}\sin(\phi_c-\phi_t)\right\}, \\
V_{us} &= -V_{cd}, \\
V_{ts} &= -\theta_b\cos(\phi_s-\phi_b)+\theta_t\cos(\phi_s-\phi_t), \\
V_{cb} &= \theta_b\cos(\phi_c-\phi_b)-\theta_t\cos(\phi_c-\phi_t), \\
V_{td} &= \theta_b\sin(\phi_s-\phi_b)-\theta_t\sin(\phi_s-\phi_t), \\
V_{ub} &= -\theta_b\sin(\phi_c-\phi_b)+\theta_t\sin(\phi_c-\phi_t).
\end{aligned} \tag{15}$$

From these formulae, one deduces immediately that:

$$\begin{aligned}
|V_{tb}| &\sim \cos\Delta\theta \sim 1, \\
|V_{cs}| &= |V_{ud}| \sim \cos(\Delta\phi) \sim 1, \\
|V_{cd}| &= |V_{us}| \sim -\sin(\Delta\phi) \sim \text{small}, \\
|V_{ts}| &= |V_{cb}| \sim \Delta\theta \sim \text{small}, \\
|V_{td}| &= |V_{ub}| \sim \theta\sin(\Delta\phi) \sim \text{very small}.
\end{aligned} \tag{16}$$

In other words, simply on the premises of a rotating mass matrix as given in (1) without using even the evolution equation (2) for the rotating vector $\mathbf{r}$ apart from the condition that $\mathbf{r}$ should remain near the high energy fixed point $(1,0,0)$, one has already derived the well-known, and long-wondered at, hierarchy of CKM elements as observed in experiment. This result was anticipated already using some elementary differential geometry in [21] but is here now made completely explicit.

Next, let us rewrite the evolution equation (2) for the unnormalized vector $(x,y,z)$ as:

$$\frac{1}{x}\frac{dx}{dt} = f_1, \quad \frac{1}{y}\frac{dy}{dt} = f_2, \quad \frac{1}{z}\frac{dz}{dt} = f_3, \tag{17}$$

with $t = \ln\mu,\ k = 3\rho^2/(32\pi^2)$, where

$$f_1 = k\left(\frac{x^2-y^2}{x^2+y^2} + \frac{x^2-z^2}{x^2+z^2}\right),\ \text{cyclic.} \tag{18}$$

To get rid of the normalization, we take differences of these equations, so that we can still use the parametrization (13), which, with arbitrary $\phi$ and small $\theta$, becomes:

$$\mathbf{r} \simeq (1, \theta\cos\phi, \theta\sin\phi), \tag{19}$$

and

$$f_1 \simeq 2k,\ f_2 \simeq k(\cos 2\phi - 1),\ f_3 \simeq -k(1+\cos 2\phi). \tag{20}$$

Hence we get for $f_1 - f_3$ and $f_2 - f_3$ respectively:

$$-\frac{1}{\theta\sin\phi}\frac{d(\theta\sin\phi)}{dt} = k(3+\cos 2\phi), \tag{21}$$

$$\frac{1}{\theta\cos\phi}\frac{d(\theta\cos\phi)}{dt} - \frac{1}{\theta\sin\phi}\frac{d(\theta\sin\phi)}{dt} = 2k\cos 2\phi, \tag{22}$$

which simplify to

$$\frac{1}{\sin 2\phi\cos 2\phi}\frac{d\phi}{dt} = -k, \tag{23}$$

$$\frac{1}{\theta}\frac{d\theta}{dt} = -k(2+\sin^2 2\phi), \tag{24}$$

giving

$$\frac{\tan 2\phi}{\tan 2\phi_0} = \left(\frac{\mu_0}{\mu}\right)^{2k}, \tag{25}$$

$$\frac{\theta}{\theta_0} = \left(\frac{\mu_0}{\mu}\right)^{2k}\left(\frac{\cos 2\phi_0}{\cos 2\phi}\right)^{1/2}. \tag{26}$$

Equation (26) reduces to

$$\frac{\theta}{\theta_0} = \left(\frac{\mu_0}{\mu}\right)^{2k} \tag{27}$$

if we further assume that $\phi^2$ is small.

With this solution, one easily checks that, for consistency with (6), $\rho$ has to be the same for all fermion species $T$ to a good approximation, a result first found in [4] numerically, as follows. From (6), one has:

$$m_2 = m_1 \sin^2(\theta_2 - \theta_1), \tag{28}$$

or else, recalling $\theta_2 \gg \theta_1$, which is an approximation numerically similar to that for obtaining the linearized solution (27) above:

$$m_2 \sim m_1 \theta_2^2. \tag{29}$$

Hence, using (27), one has:

$$\sqrt{\frac{m_2}{m_1}} \sim \theta_I \left(\frac{\mu_I}{m_2}\right)^{2k}, \tag{30}$$

with $\theta_I$ being the value of $\theta$ at some chosen (large) initial value $\mu_I$. Applying the formula (30) to successively the fermion species $U, D, L$ and eliminating $\theta_I$, one obtains:

$$\begin{aligned} \left(\frac{m_c m_b}{m_t m_s}\right)^{1/2} &= \left(\frac{\mu_I}{m_c}\right)^{2k_U} \left(\frac{m_s}{\mu_I}\right)^{2k_D}, \\ \left(\frac{m_c m_\tau}{m_t m_\mu}\right)^{1/2} &= \left(\frac{\mu_I}{m_c}\right)^{2k_U} \left(\frac{m_\mu}{\mu_I}\right)^{2k_L}, \end{aligned} \tag{31}$$

which is consistent for arbitrary $\mu_I$ only when $k_U = k_D = k_L$, as required. We note that this result is independent of the choice of the 3 model parameters.

Turning next to parameter-dependent results, we shall first determine the parameter values by fitting them to the 3 best known empirical quantities as was done in [5], namely the 2 mass ratios $m_c/m_t$ and $m_\mu/m_\tau$, and the Cabbibo angle. From (30) using the $t$ scale as $I$, one deduces the approximate relations:

$$\begin{aligned} \frac{m_c}{m_t} &= \theta_t^2 \left(\frac{m_t}{m_c}\right)^{4k}, \\ \frac{m_\mu}{m_\tau} &= \theta_t^2 \left(\frac{m_t}{m_\mu}\right)^{4k}. \end{aligned} \tag{32}$$

Inputting then the empirical values in GeV of $m_t \sim 175, m_c \sim 1.25, m_\tau \sim 1.777, m_\mu \sim 0.105$, one easily obtains;

$$k \sim 0.21, \quad \theta_t \sim 0.011. \tag{33}$$

These values compare very well with the values $k \sim 0.20, \theta_t \sim 0.012$ obtained before from the numerical fit of [5], affording thus a check on both the present approximation and the previous numerical result. The good agreement is actually a little fortuitous, being beyond what can be expected from the crudeness of the approximations made in deriving the formulae used above.

With now $k$ and $\theta_t$ as 2 of our parameters, we can next proceed to evaluate other mass ratios and mixing matrix elements which depend only on $\theta$. For mass ratios within the range of validity, we have only one relation (from (31) using $m_b = 4.2$ GeV):

$$m_s \sim m_c \left(\frac{m_b}{m_t}\right)^{1/(4k+1)}, \tag{34}$$

giving $m_s \sim 160$ MeV (cf. 75 to 170 MeV [6]). For the CKM matrix elements, we have:

$$V_{tb} \sim 1 - \theta_b^2/2 \sim 1 - \frac{1}{2}\frac{m_c}{m_t}\left(\frac{m_c}{m_b}\right)^{4k}, \tag{35}$$

giving $V_{tb} \sim 1 - 0.001$ (cf. 0.9990 to 0.9993 [6]), and

$$V_{ts} = -V_{cb} \sim -\theta_b + \theta_t \sim -\sqrt{\frac{m_c}{m_t}}\left(\frac{m_c}{m_b}\right)^{2k}, \tag{36}$$

giving $V_{ts} = -V_{cb} \sim 0.05$ (cf. 0.035 to 0.043 [6]). These number also agree well with those obtained from the numerical result of [5] quoted in (11).

Inputting next the Cabbibo angle $V_{us} \sim 0.22$ to determine the last parameter which we can take as $\sin(\phi_s - \phi_c)$ and which for small $\phi$ differences, according to (15), can be take approximately as:

$$\sin(\phi_s - \phi_c) = \frac{V_{us}}{1 + (m_s/m_b)^{2k}}, \tag{37}$$

we obtain $\sin(\phi_s - \phi_c) \sim 0.18$. With this value fixed, we can now evaluate the remaining CKM matrix elements, thus:

$$V_{ud} \sim -V_{cs} \sim 1 - \frac{1}{2}V_{us}^2, \tag{38}$$

giving $V_{ud} \sim -V_{cs} \sim 0.976$ (cf. 0.9734 to 0.9749 [6]),

$$V_{td} \sim \theta_b \sin(\phi_s - \phi_c) \sim \sqrt{\frac{m_c}{m_t}} \left(\frac{m_c}{m_b}\right)^{2k} \frac{V_{us}}{1 + (m_s/m_b)^{2k}}, \tag{39}$$

giving $V_{td} \sim .009$ (cf. 0.004 to 0.014 [6]), and

$$V_{ub} \sim \theta_b \frac{\theta_c - \theta_b}{\theta_s} \sin(\phi_s - \phi_c) \sim \sqrt{\frac{m_c}{m_t}} \left(\frac{m_s}{m_b}\right)^{2k} \frac{V_{us}}{1 + (m_s/m_b)^{2k}}, \tag{40}$$

giving $V_{ub} \sim 0.004$ (cf. 0.002 to 0.005 [6]). Again, these estimates agree well with the values (11) obtained before numerically in [5]. Contrary to usual belief the CKM matrix elements appear related to the ratios of the two heaviest generations, which is expected if both come from a rotating mass matrix.

Finally, using the values of the parameters determined, one can calculate also from (14) the 3 vectors $\mathbf{v}_\tau, \mathbf{v}_\mu$ and $\mathbf{v}_e$ for the charged leptons, as explained in the section above. We notice in (14), however, that these vectors depend on the angles $\phi_i$ which may not be small, not just on their differences. These can be determined by solving equation (25) and using the result $\sin(\phi_s - \phi_c) \sim 0.18$. This gives 2 solutions, $\phi_s \sim 0.61, 0.35$, of which the fit in [5] correspond to the former. With the values $\theta_t \sim 0.011$ obtained before and $\phi_s \sim 0.61$, one can then evaluate the corresponding angles for $\tau$ and $\mu$, and hence the vectors $\mathbf{v}_\tau, \mathbf{v}_\mu, \mathbf{v}_e$. One obtains:

$$\begin{aligned} |\tau\rangle &= (0.994, 0.070, 0.029), \\ |\mu\rangle &= (-0.074, 0.757, 0.657), \\ |e\rangle &= (0.0188, -0.657, 0.757), \end{aligned} \tag{41}$$

not far from the values quoted in (8) from [5], and corresponding to the MNS lepton mixing matrix elements:

$$\begin{aligned} U_{\mu 3} = \langle \mu | \nu_3 \rangle &= 0.77, \\ U_{e3} = \langle e | \nu_3 \rangle &= 0.07, \end{aligned} \tag{42}$$

which is seen to compare well with (9) and (10) above.

Hence, one sees that all the previous results obtained numerically before together with their agreement with experiment have now been confirmed by analytic considerations although in a rather crude approximation.

In summary, we conclude that a closer examination of the tenets of the DSM scheme has removed a previous restriction on its predictions on neutrino oscillations, making them now consistent with the favoured LMA solution for solar neutrinos. It has also explained away the couple of discrepancies noted before as limitations of the 1-loop calculation so far performed, which is seen otherwise to be in full agreement with experiment within its perceived range of validity. Furthermore, all previous numerical results have now been confirmed by analytic considerations.

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