# A Solution of the Strong CP Problem Transforming the theta-angle to the KM CP-violating Phase

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**Abstract**

It is shown that in the scheme with a rotating fermion mass matrix (i.e. one with a scale-dependent orientation in generation space) suggested earlier for explaining fermion mixing and mass hierarchy, the theta-angle term in the QCD action of topological origin can be eliminated by chiral transformations, while giving still nonzero masses to all quarks. Instead, the effects of such transformations get transmitted by the rotation to the CKM matrix as the KM phase giving, for $\theta$ of order unity, a Jarlskog invariant typically of order $10^{-5}$ as experimentally observed. Strong and weak CP violations appear then as just two facets of the same phenomenon.

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A long-standing puzzle in particle theory is the so-called strong CP problem. Lorentz and gauge invariance allow in principle in the QCD action a term of the form:

$$\mathcal{L}_\theta = -\frac{\theta}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{1}$$

of topological origin, where $\theta$ can take any arbitrary value. Experiment, on the other hand, does not seem to admit such a term which violates CP invariance and can lead to a nonvanishing electric dipole moment for the neutron of the order [1]:

$$d_n \sim |\theta| e m_\pi^2/m_N^3 \sim 10^{-16}|\theta| e\ cm. \tag{2}$$

The experimental limit for this has by now been pushed to below $2.7 \times 10^{-26} e\ cm$ [2], which means that $|\theta|$ has to have a value below $3 \times 10^{-10}$. It would thus seem that nature has a reason unknown to us either for suppressing this term to such a small value, or else for eliminating it altogether.

The favourite candidate among theoreticians for explaining away the theta-angle term is the axion theory [3, 4, 5], which axions, however, have been diligently searched for in experiment since first suggested, i.e. for over forty years, yet not been found.

Now, it has long been known that the theta-angle term can be eliminated if there are quarks of zero mass. Effecting a chiral transformaton on a quark field, thus:

$$\psi \longrightarrow \exp(i\alpha\gamma_5)\psi \tag{3}$$

will yield a term in the Feynman integral of the same form as the theta-angle term. Hence, if we make a chiral transformation (3) on each quark flavour, we shall end up with a theta-angle term modified to:

$$\theta \longrightarrow \theta + 2\sum_F \alpha_F, \tag{4}$$

which can be made to vanish by a judicious choice of $\alpha_F$. However, a chiral transformation on a massive fermion field will in general make its mass parameter complex:

$$\begin{aligned} m\bar\psi\psi &= m\bar\psi\tfrac{1}{2}(1+\gamma_5)\psi + m\bar\psi\tfrac{1}{2}(1-\gamma_5)\psi \\ &\rightarrow m\exp(2i\alpha)\bar\psi\tfrac{1}{2}(1+\gamma_5)\psi + m\exp(-2i\alpha)\bar\psi\tfrac{1}{2}(1-\gamma_5)\psi, \end{aligned} \tag{5}$$

and lead again to CP-violations. Only when the quark has a zero mass can such a conclusion be avoided. Unfortunately, none of the quarks known can be assigned a zero mass in experiment, and so the problem remains.

A parallel problem actually also exists in the weak sector although it is not usually considered as such. General invariance principles there imply that the weak current is of the form:

$$J^\mu = \left( \begin{array}{c} \bar{u} \\ c \\ t \end{array} \right) \gamma^\mu (1 - \gamma_5) V \left( \begin{array}{c} d \\ s \\ b \end{array} \right), \qquad (6)$$

where $V$, the CKM mixing matrix relating the $U$-type to $D$-type quarks, depends on 4 parameters, one of which is the Kobayashi-Maskawa phase [6] which violates CP. In contrast to the theta-angle term in the strong sector, the effects of the KM phase here have been detected in experiment, but again are suppressed by nature for reasons unknown, though not to the same drastic extent. The Jarlskog invariant [7] $J$ which is a convenient measure of the CP-violating effects here is found to have a value of only [8]:

$$J \sim 3 \times 10^{-5}. \qquad (7)$$

One could well call this in parallel the "weak CP problem", only here, the problem being bound up with the mixing angles, i.e. the 3 other "real" parameters in the matrix $V$, the small values of some of which are equally unexplained in the usual formulation of the Standard Model, one is not so inclined to label it as such. Nevertheless, this "weak CP problem" is quite as puzzling as the strong, and has remained with us almost as long.

In view of these problems, we were interested first to note recently [9] that in the framework of a rotating mass matrix (i.e. one which has an orientation in generation space which is scale-dependent) we have constructed earlier [10] to explain fermion mixing and mass hierarchy, there is a possible solution to the strong CP problem by means of chiral transformations which, nevertheless, because of the rotation, preserve the hermiticity of the mass matrix and give nonzero masses to all quarks. Then subsequently, following through these earlier considerations down to the mixing matrix level, we have since found, as we shall show below, that the elimination of the theta-angle term in the strong sector automatically leads to a mixing matrix with a Kobayashi-Maskawa phase, the CP-violating effects of which, being themselves contingent on the low speed of the rotation, will naturally yield, for $\theta$ of order unity, small values for the Jarlskog invariant, typically of the order $10^{-5}$ as observed in experiment. In other words, the rotating mass matrix framework seems to offer a simultaneous solution to both the strong and "weak" CP problems, while linking the two intriguingly together - and these in addition to its original offer of an explanation for fermion mixing and mass hierarchy.

That a rotating fermion mass matrix can give rise to both mixing and mass hierarchy is in itself a very simple idea which can easily be seen as follows. One starts with a fermion mass matrix of the usual form:

$$m\tfrac{1}{2}(1+\gamma_5) + m^\dagger \tfrac{1}{2}(1-\gamma_5), \tag{8}$$

which, following Weinberg [11], one can always rewrite by a relabelling of the singlet right-handed fields, with no change in physics, in a hermitian form independent of $\gamma_5$, a form we shall henceforth adopt. Suppose now this matrix is factorizable, meaning that it is of the form:

$$m = m_T \boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger, \tag{9}$$

where $\boldsymbol{\alpha}$ is a normalized global (i.e. $x$-independent) vector in generation space. We can even suppose that $\boldsymbol{\alpha}$ is universal and that only the numerical coefficient $m_T$ depends on the fermion type. Obviously, such a mass matrix has only one massive state represented by the vector $\boldsymbol{\alpha}$, and zero mixing, i.e. only the identity matrix as the mixing matrix. This is not unattractive as a starting point for quarks, as has occurred already a long time ago to several authors [12, 13], but is clearly insufficiently realistic in detail.

However, if one now says that the vector $\boldsymbol{\alpha}$ rotates with changing scale $\mu$, as proposed, then the situation becomes very interesting. All quantities now depend on the scale $\mu$ and one has to specify at which scale the mass or state vector of each particle is to be measured. Suppose one follows the usual convention and defines the mass of each particle as that measured at the scale equal to its mass, we find then that mixing and mass hierarchy would immediately result.

To see how this comes about, it is sufficient for illustration to consider a situation where one takes account only of the two heaviest states in each fermion type. By (9) then, taking for the moment $\boldsymbol{\alpha}$ to be real and $m_T$ to be $\mu$-independent for simplicity, we would have $m_t = m_U$ as the mass of $t$ and the eigenvector $\boldsymbol{\alpha}(\mu = m_t)$ as its state vector $\mathbf{v}_t$. Similarly, we have $m_b = m_D$ as the mass and $\boldsymbol{\alpha}(\mu = m_b)$ as the state vector $\mathbf{v}_b$ of $b$. Next, the state vector $\mathbf{v}_c$ of $c$ must be orthogonal to $\mathbf{v}_t$, $c$ being by definition an independent quantum state to $t$. Similarly, the state vector $\mathbf{v}_s$ of $s$ is orthogonal to $\mathbf{v}_b$. So we have the situation as illustrated in Figure 1, where the vectors $\mathbf{v}_t$ and $\mathbf{v}_b$ are not aligned, being the vector $\boldsymbol{\alpha}(\mu)$ taken at two different values of its argument $\mu$, and $\boldsymbol{\alpha}$ by assumption rotates. This gives then the following CKM mixing

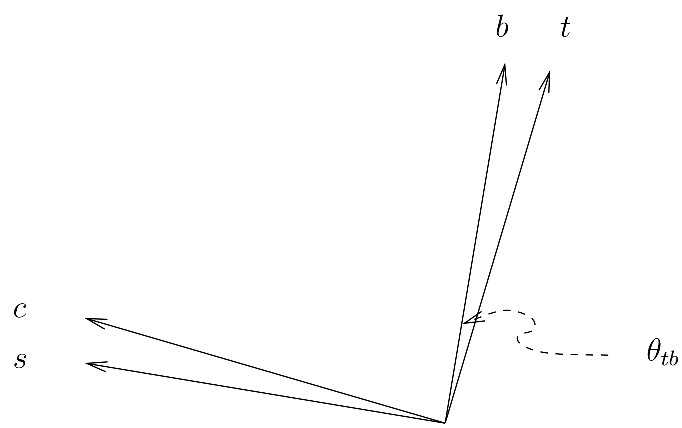


Figure 1: Mixing between up and down fermions from a rotating mass matrix.

(sub)matrix, in the situation considered with only the two heaviest states:

$$\begin{pmatrix} V_{cs} & V_{cb} \\ V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{v}_c|\mathbf{v}_s\rangle & \langle \mathbf{v}_c|\mathbf{v}_b\rangle \\ \langle \mathbf{v}_t|\mathbf{v}_s\rangle & \langle \mathbf{v}_t|\mathbf{v}_b\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_{tb} & \sin\theta_{tb} \\ -\sin\theta_{tb} & \cos\theta_{tb} \end{pmatrix}, \tag{10}$$

which is no longer the identity, hence mixing.

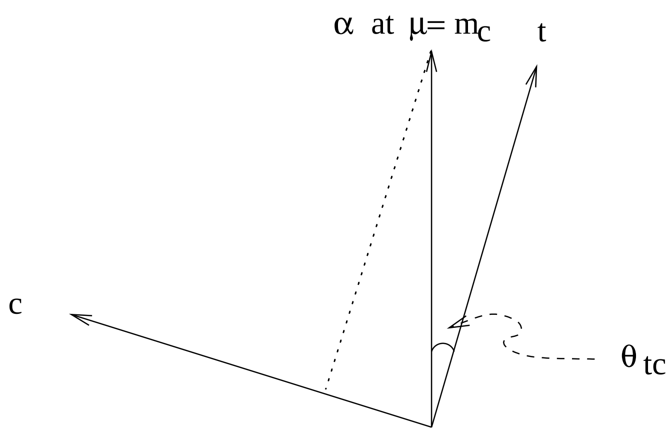


Figure 2: Masses for lower generation fermions from a rotating mass matrix.

Next, what about hierarchical masses? From (9), it follows that $\mathbf{v}_c$ must have zero eigenvalue at $\mu = m_t$. But this value is not to be taken as the mass of $c$ which we agreed has to be measured at $\mu = m_c$. In other words, $m_c$ is instead to be taken as the solution to the equation:

$$\mu = \langle \mathbf{v}_c|m(\mu)|\mathbf{v}_c\rangle = m_U|\langle \mathbf{v}_c|\boldsymbol{\alpha}(\mu)\rangle|^2. \tag{11}$$

A nonzero solution exists since $\boldsymbol{\alpha}$ by assumption rotates so that at $\mu \neq m_t$, it would have rotated to some direction different from $\mathbf{v}_t$, as illustrated in Figure 2, and acquired a component $\sin\theta_{tc}$ in the direction of $\mathbf{v}_c$ giving thus:

$$m_c = m_t \sin^2 \theta_{tc}, \tag{12}$$

which is nonzero, but will be small if the rotation is not too fast, hence mass hierarchy.

Clearly, the same arguments can be applied to the full 3-generation case, to down quarks, and to leptons as well, although with the last we shall not here be directly concerned. The question, however, is whether the mixing matrices and hierarchical masses so obtained are anything like those experimentally observed, and whether viable theoretical models can be constructed to reproduce the required rotation. This question we have studied for some time. The first point is adressed in [14, 10], for example, and the second in [15, 16], to which the interested reader is referred for a summary, and to references therein for details, of the results so far obtained.

As regards the CP problem of present interest, the above scenario has one quite striking feature in that, despite the conclusion that all fermions have nonzero, though hierarchical, masses the mass matrix that appears in the action remains factorizable (rank 1) for all scales. This comes about basically from unitarity which means that the physically measured masses of the two lower generation fermion states are not the eigenvalues of the mass matrix appearing in the action but those of some truncations of that mass matrix [9]. Thus, as far as the mass matrix appearing in the action is concerned, there are still directions in generation space, namely on the plane orthogonal to $\boldsymbol{\alpha}$, in which the fermion field has zero eigenvalues. And in those directions, chiral transformation can still be performed on the fermion field without affecting the hermiticity of the mass matrix, as emphasized after (5) above. In other words, for a mass matrix of rank 1, the relabelling of right-handed fields required to make it appear hermitian as in (9) is nonunique, and leaves sufficient freedom to allow some chiral transformations on the fermion fields. Hence, by a judicious choice of these, any theta-angle term in the action can be eliminated. This point was noted in [9] already, but that was as far as we had got there.

Our intention now is to follow through the argument with rotation and see what other information this will give us. In doing so, we shall retain the simplifying assumption of a real $\boldsymbol{\alpha}$ made above for ease of illustration. We keep the assumption now, however, deliberately, on purpose to demonstrate that even for $\boldsymbol{\alpha}$ real, a CP-violating Kobayashi-Maskawa phase will authomatically appear in the CKM matrix when the theta-angle is eliminated. Only at the end shall we return to discuss what happens when $\boldsymbol{\alpha}$ becomes complex.

At each scale $\mu$, there are then two independent directions with zero eigenvalues to the mass matrix $m$, namely the two directions on the plane orthogonal to $\boldsymbol{\alpha}(\mu)$. It would thus seem that chiral transformations can be effected on any two of these directions without affecting the hermiticity of the mass matrix so as to eliminate the theta-angle term from the action at $\mu$. On closer examination, however, this does not appear to be the case when rotation is involved. The action at $\mu$ prescribes not only the vector $\boldsymbol{\alpha}$ at $\mu$, but also, via renormalization, the vector $\boldsymbol{\alpha}$ at $\mu + d\mu$, and it is this bit of information that on iteration gives us the rotation trajectory for $\boldsymbol{\alpha}$. At both $\mu$ and $\mu + d\mu$, however, the mass matrix is supposed to be of the form (9), and therefore hermitian. Hence, in performing a chiral transformation to eliminate the theta-angle term at $\mu$, we would want not only to keep hermitian the mass matrix at $\mu$, but also the mass matrix at $\mu + d\mu$. We notice, however, that although the mass matrix at $\mu$, namely $m(\mu) = m_T \boldsymbol{\alpha}(\mu)\boldsymbol{\alpha}(\mu)^\dagger$ has zero eigenvalue in the direction $\dot{\boldsymbol{\alpha}}(\mu)$, this being orthogonal to $\boldsymbol{\alpha}(\mu)$, the mass matrix at the neighbouring point $\mu + d\mu$, namely:

$$m(\mu + d\mu) = m_T[\boldsymbol{\alpha}(\mu) + \dot{\boldsymbol{\alpha}}(\mu)d\mu][\boldsymbol{\alpha}(\mu) + \dot{\boldsymbol{\alpha}}(\mu)d\mu]^\dagger \tag{13}$$

has a nonzero value in the direction $\dot{\boldsymbol{\alpha}}(\mu)$. Thus, a chiral transformation in the direction $\dot{\boldsymbol{\alpha}}(\mu)$ will not leave the mass matrix $m(\mu + d\mu)$ hermitian. Only a chiral transformation in the direction orthogonal to both $\boldsymbol{\alpha}(\mu)$ and $\dot{\boldsymbol{\alpha}}(\mu)$ will leave both the matrices $m(\mu)$ and $m(\mu + d\mu)$ hermitian.

Let us set up then at each point $\mu$ of the rotation trajectory a (Darboux) triad, namely the radial vector $\boldsymbol{\alpha}(\mu)$, the tangent vector $\boldsymbol{\tau}(\mu) = \dot{\boldsymbol{\alpha}}(\mu)$ and the normal vector $\boldsymbol{\nu}(\mu)$ orthogonal to both. The conclusion of the preceding paragraph is that the chiral tansformation to eliminate the theta-angle term should be effected in the direction $\boldsymbol{\nu}(\mu)$ at every $\mu$. To be explicit, let us choose a reference frame in 3-D generation space such that at $\mu = \infty$, we have:

$$\boldsymbol{\alpha}(\infty) = \boldsymbol{\alpha}_0 = (1, 0, 0)^\dagger; \quad \boldsymbol{\tau}(\infty) = \boldsymbol{\tau}_0 = (0, 1, 0)^\dagger; \quad \boldsymbol{\nu}(\infty) = \boldsymbol{\nu}_0 = (0, 0, 1)^\dagger, \tag{14}$$

and define a rotation $A(\mu)$ such that:

$$\boldsymbol{\alpha}(\mu) = A(\mu)\boldsymbol{\alpha}_0; \quad \boldsymbol{\tau}(\mu) = A(\mu)\boldsymbol{\tau}_0; \quad \boldsymbol{\nu}(\mu) = A(\mu)\boldsymbol{\nu}_0. \tag{15}$$

The chiral transformation needed at scale $\mu$ can then be represented as:

$$P(\mu) = A(\mu) P_0 A^{-1}(\mu), \quad P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\theta\gamma_5/2} \end{pmatrix}. \tag{16}$$

Effecting such a chiral transformation on the 3-component quark field $\boldsymbol{\psi}$ (for 3 generations) at every $\mu$ removes the theta-angle term entirely and leave the action CP-invariant and the mass matrix hermitian at every $\mu$.

Notice, however, that although by choosing to perform the chiral transformation in the normal direction $\boldsymbol{\nu}(\mu)$ at scale $\mu$, we have ensured that the mass matrix is hermitian not only at $\mu$ but also at a neighbouring $\mu + d\mu$, it will generally make $m$ non-hermitian at $\mu'$ for $\mu - \mu'$ finite, since $\boldsymbol{\nu}$ would have rotated at $\mu'$ to a different direction giving then the chiral transformation $P(\mu)$ there a massive component and hence a non-hermitian mass matrix. Explicitly, under a chiral transformation $P(\mu)$, the mass term at scale $\mu'$, in close parallel to (5), would transform as:

$$\begin{aligned} & m_T \bar{\boldsymbol{\psi}} \boldsymbol{\alpha}(\mu') \boldsymbol{\alpha}(\mu')^\dagger \boldsymbol{\psi} \\ & \quad \to m_T \bar{\boldsymbol{\psi}} A(\mu) P_0 A^{-1}(\mu) \boldsymbol{\alpha}(\mu') \boldsymbol{\alpha}(\mu')^\dagger \tfrac{1}{2}(1+\gamma_5) A(\mu) P_0 A^{-1}(\mu) \boldsymbol{\psi} \\ & \quad + m_T \bar{\boldsymbol{\psi}} A(\mu) P_0 A^{-1}(\mu) \boldsymbol{\alpha}(\mu') \boldsymbol{\alpha}(\mu')^\dagger \tfrac{1}{2}(1-\gamma_5) A(\mu) P_0 A^{-1}(\mu) \boldsymbol{\psi}. \end{aligned} \tag{17}$$

If $\mu' = \mu$, then $A^{-1}\boldsymbol{\alpha} = (1,0,0)^\dagger$, in which case $P_0$ operating on it will leave it invariant. But for $\mu' \neq \mu$, this no longer applies, and $P_0$ will give in general a phase which is different in the 2 terms after the transformation and make the mass matrix non-hermitian. Notice that although this mass matrix is still, of course, of the general form (8) we started with, we are not allowed now to relabel the right-handed fields to recast it in a hermitian form as we did before in deriving (9), for this would change the relative phases between the right- and left-handed fields, which is equivalent to a chiral transformation and can therefore resuscitate the theta-angle term that we have taken such care earlier to eliminate. To recover a hermitian mass matrix and yet eliminate the theta-angle term, we shall need to first undo the chiral transformation performed above at $\mu$ and then perform the chiral transformation again at $\mu'$ instead. In other words, we need to apply to the quark field the operator $P(\mu')P^{-1}(\mu)$ to obtain the desired result.

Indeed, the chiral transformation $P(\mu)$ guarantees only that the mass matrix is hermitian at the two neighbouring points $\mu$ and $\mu + d\mu$. If we wish to iterate the procedure so as to eliminate the theta-angle term at $\mu + d\mu$

while ensuring the hermiticity of the mass matrix at the next neighbouring point too, then we have first to undo the chiral transformation performed at $\mu$ before, and then effect again the chiral transformation at $\mu + d\mu$, namely to apply the operator

$$P(\mu + d\mu)P^{-1}(\mu) \tag{18}$$

to the quark fields, in order to obtain the desired result. This operator (18) acts thus as a sort of parallel transport, detailing effectively what is meant by the same or parallel (chiral) phases at two neighbouring points along the trajectory, and hence, by iteration, at any two points a finite distance apart as detailed in the paragraph above.

To see how these chiral transformations will affect the conclusions above on quark masses and mixing, let us start with just one species of quarks, say, the U-type quarks, i.e. $t, c, u$. The state vector of $t$, i.e. $\mathbf{v}_t$, or just $\mathbf{t}$ for short, is defined as $\boldsymbol{\alpha}(\mu = m_t)$ and the state vectors $\mathbf{c}, \mathbf{u}$ are to be orthogonal to it and are themselves mutually orthogonal. It follows therefore that the dyad $\mathbf{c}, \mathbf{u}$ is related to the dyad $\boldsymbol{\tau}(\mu = m_t) = \boldsymbol{\tau}_U, \boldsymbol{\nu}(\mu = m_t) = \boldsymbol{\nu}_U$ just by an orthogonal transformation, thus:

$$\boldsymbol{\tau}_U = \Omega_U \mathbf{c}; \quad \boldsymbol{\nu}_U = \Omega_U \mathbf{u}, \tag{19}$$

with

$$\Omega_U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_U & -\sin\omega_U \\ 0 & \sin\omega_U & \cos\omega_U \end{pmatrix}, \tag{20}$$

$\omega_U$ being just the angle between $\mathbf{c}$ and $\boldsymbol{\tau}_U$. This angle is small but nonzero, since $\mathbf{c} = \mathbf{v}_c$ is the vector which is orthogonal to $\boldsymbol{\alpha}(\mu = m_t)$ and lies on the plane containing both the vectors $\boldsymbol{\alpha}(\mu = m_t)$ and $\boldsymbol{\alpha}(\mu = m_c)$ [10], while $\boldsymbol{\tau}_U$ is the tangent to the trajectory at $\mu = m_t$; it is thus a measure of how much $\boldsymbol{\alpha}(\mu)$ has rotated from $\mu = m_t$ to $\mu = m_c$.

Suppose we wish again to evaluate the mass of the $c$ quark in the rotation scenario as we did before but incorporating now the above procedure for eliminating the theta-angle term. We agreed that this has to be done at the scale $\mu = m_c$ so that all scale-dependent quantities involved should be evaluated at this scale also. Thus, with the $c$ state vector $\mathbf{c}$ defined originally at $\mu = m_t$, the $c$ quark field is given there as $\psi_c(\mu = m_t) = \mathbf{c}^\dagger \cdot P(m_t)\boldsymbol{\psi}$, but it will now have to be parallelly transported by (18) to $\mu = m_c$, giving instead $\psi_c(\mu = m_c) = \mathbf{c}^\dagger \cdot P(m_c)\boldsymbol{\psi}$. The mass term also, according to (17) above, will now appear as:

$$m_T \bar{\boldsymbol{\psi}} P(m_c) \boldsymbol{\alpha}(m_c) \boldsymbol{\alpha}^\dagger(m_c) P(m_c) \boldsymbol{\psi}, \tag{21}$$

where the operators $P(m_c)$ can in fact be omitted since the vector $\boldsymbol{\alpha}(m_c)$ is invariant under $P(m_c)$. What interests us here as far as the $c$ mass is concerned, according to the analysis in, for example, [9], is the diagonal contribution from the $c$ quark, namely:

$$\begin{aligned} & m_T \bar{\boldsymbol{\psi}} P(m_c) \mathbf{c}\mathbf{c}^\dagger \boldsymbol{\alpha}(m_c) \boldsymbol{\alpha}^\dagger(m_c) \mathbf{c}\mathbf{c}^\dagger P(m_c) \boldsymbol{\psi} \\ & \quad = m_T |\mathbf{c}^\dagger \cdot \boldsymbol{\alpha}(m_c)|^2 \bar{\psi}_c(\mu = m_c) \psi_c(\mu = m_c), \end{aligned} \tag{22}$$

giving then the $c$ mass as:

$$m_c = m_T |\mathbf{c}^\dagger \cdot \boldsymbol{\alpha}(m_c)|^2, \tag{23}$$

i.e., exactly the same as before when no consideration was given to the elimination of the theta-angle term, as in (12) above for the simplified 2-generation version.

The same considerations will apply also to $m_u$ as for $m_c$. One sees therefore that so long as there is only one type of quarks, one can always manage, with a rotating factorizable mass matrix, to eliminate any theta-angle term so as to maintain CP-conservation, while keeping the mass matrix hermitian, and having at the same time hierarchical but nonzero masses for all quarks.

What happens, however, when there are both up-type and down-type quarks? In that case, the two types can be coupled by the weak current via the CKM mixing matrix, and one has again to follow through the preceding arguments and trace out the consequence of eliminating the theta-angle term. To do so, let us denote the state vectors of the U-type quarks defined above at $\mu = m_t$ together as:

$$V_U = (\mathbf{t}, \mathbf{c}, \mathbf{u}) = \begin{pmatrix} t_1 & c_1 & u_1 \\ t_2 & c_2 & u_2 \\ t_3 & c_3 & u_3 \end{pmatrix}, \tag{24}$$

and similarly the state vectors of the D-type quarks defined at $\mu = m_b$ as:

$$V_D = (\mathbf{b}, \mathbf{s}, \mathbf{d}) = \begin{pmatrix} b_1 & s_1 & d_1 \\ b_2 & s_2 & d_2 \\ b_3 & s_3 & d_3 \end{pmatrix}, \tag{25}$$

where in the notation introduced above, we have from rotation:

$$\begin{aligned} V_U &= A_U \Omega_U; \quad A_U = A(\mu = m_t); \\ V_D &= A_D \Omega_D; \quad A_D = A(\mu = m_b). \end{aligned} \tag{26}$$

What interests us is the relative orientation of $V_U$ and $V_D$, the matrix of inner products between the state vectors of the U-type and D-type quarks being the CKM mixing matrix we seek. In order to compare the orientation of the state vectors of one type to those of the other, the two types being defined as they are at two different scales, we need first to "parallelly transport" the (chiral) phase of each to the same scale. Chiral transformations should, of course, be performed in principle on the quark fields, which can be done directly to the weak current in (6), but since in (6) only left-handed fields are involved, the factor $\exp(-i\theta\gamma_5)$ in all chiral transformations can be replaced just by the phase factor $\exp(i\theta)$ and can thus be conveniently transferred from the quark fields to their state vectors for ease of presentation. Parallelly transporting then to a common scale, say $X$, for comparison, we have:

$$\begin{aligned} P_X P_U^{-1} V_U &= P_X (A_U P_0^{-1} A_U^{-1})(A_U \Omega_U) = \tilde{V}_U; \\ P_X P_D^{-1} V_D &= P_X (A_D P_0^{-1} A_D^{-1})(A_D \Omega_D) = \tilde{V}_D. \end{aligned} \tag{27}$$

Hence we obtain the CKM matrix in this scenario as:

$$V_{CKM} = \tilde{V}_U^{-1} \tilde{V}_D = (\Omega_U^{-1} P_0 \Omega_U) V_U^{-1} V_D (\Omega_D^{-1} P_0^{-1} \Omega_D), \tag{28}$$

where the factors $P_X$ cancel, meaning that it does not matter at which common scale we choose to make the comparison, as expected.

Notice that had there been no theta-angle term to contend with, we would have obtained for the CKM matrix just the factor $V_U^{-1} V_D$, which will be a real matrix with no CP-violating phase in it, having started with a real $\boldsymbol{\alpha}$ as we have done. By insisting on the chiral transformations to eliminate the theta-angle term throughout, we have then injected some new phases into the mixing matrix elements, and hence the possibility of CP-violation, which will be the case if the phases introduced by the said chiral transformations cannot be removed by any changes in phase of the quark states. Indeed, if we were to put $\omega_U = \omega_D = 0$, with $\omega_D$ similarly defined as $\omega_U$ in (20), we would have obtained a vanishing value for the Jarlskog invariant and no CP-violation. The reason is clear, since in that case the vector $\mathbf{u}$ would coincide with the normal vector $\boldsymbol{\nu}$ at $\mu = m_t$ , and $\mathbf{d}$ with $\boldsymbol{\nu}$ at $\mu = m_d$, on which the chiral transformations are performed, and the effect on the CKM matrix would be the same as that of changing the phases of the $u$ and $d$ fields, which are in any case arbitrary. If we were to calculate the Jarlskog invariant from (28) in this case, the phases would cancel and one obtains a zero value. Since, however, $\omega_U$ and $\omega_D$ are nonzero by virtue of the rotation as explained above, this cancellation has now no reason to occur and one has in general nonvanishing Jarlskog invariants and CP-violations as the result.

But will this yield Jarlskog invariants and CP-violating effects of the order observed in experiment? We recall that the strong CP angle $\theta$ from which this effect is supposed to originate can take in principle any arbitrary value and so should be taken, without prejudice, as of order unity, whereas the measured value of the Jarlskog invariant is of order $3 \times 10^{-5}$ [8], so that a suppression by about 4 orders of magnitude is required in the process of transmitting the CP-violating effects from the strong sector to the weak sector via rotation. This is possible, so long as the rotation is relatively slow as is envisaged. To see whether it is indeed the case, one can evaluate the Jarlskog invariant for (28) with, for example, its $2 \times 2$ submatrix labelled by the 2 heaviest states $t, c$ and $b, s$. One obtains then an explicit expression for $J$ in terms of $\theta$, $\omega_U$, $\omega_D$ and elements of the matrix $V_U^{-1} V_D$. The angles $\omega_U$ and $\omega_D$, one has already noted to be of order $\epsilon$, the angle rotated by the vector $\boldsymbol{\alpha}$ from the scale of the heaviest to that of the second generation. Further, from an earlier analysis of the rotation picture [17], one has learned that the CKM matrix elements $V_{ts}, V_{cb}, V_{cd}, V_{us}$ proportinal to the curvatures of the rotation trajectory are all of order $\epsilon$, while the corner elements $V_{td}, V_{ub}$ proportional to its torsion are of order $\epsilon^2$. This is not to say, of course, that all four elements of order $\epsilon$ need be of the the same size, for the two curvatures, normal and geodesic, can have quite different values, as seems to be the case for a trajectory fitted to experiment. But, for the order-of-magnitude estimate generic to the rotation scheme that we aim for at the moment, we shall deliberately ignore such details specific to a particular trajectory. Substituting then the above estimates into the formula for $J$, one finds that $J$ is of order $\epsilon^4$ and proportional to $\sin(\theta/2)$. An estimate for the value of $\epsilon$ can be obtained from the rotation formula (12) for the mass ratio of the second generation to the heaviest, leading to $\epsilon \sim 0.08$ from $m_c/m_t$, and $\epsilon \sim 0.14$ from $m_s/m_b$, [8]. This then gives an order-of-magnitude estimate for the Jarlskog invariant as:

$$J \sim \sin(\theta/2) \times 10^{-4}, \tag{29}$$

which is quite consistent with the experimentally measured value (7) for a strong CP angle $\theta$ of order unity

One knows of course, whether in the rotation framework or otherwise, that once given the small values observed in experiment for mixing angles involving the two heaviest states $t$ and $b$, it will follow already that the CP-violating effects of the KM phase will be small, since it is known that for two generations there is no CP-violation, which can thus arise only through mixing with $t$ and $b$. A priori, however, one can give no actual estimate for the size of the effect, not knowing how or where the KM phase originates. The difference with the scheme here is that, first, having traced the origin of the KM phase via rotation back to the strong sector, one can give now an actual estimate for $J$, and second, since the rotation relates also the mixing angles of fermions to their hierarchical masses, as explained in (10) and (12) above, the estimate can be derived with only mass ratios as inputs, with no empirical knowledge of the mixing angles being required at all.

If one is willing further to supply the experimentally measured values of the mixing angles as inputs, then one would obtain in the rotation scheme a more accurate estimate of the Jarlskog invariant. Thus, if we take for $V_U^{-1} V_D$ in (28) the matrix of absolute values of CKM elements given in [8], one is left only with $\omega_U$ and $\omega_D$ as unknowns. Both these angles, however, are seen to be related to the geodesic curvature of the rotation trajectory [17], which is itself related to the Cabibbo angle. Hence, reasoning along these lines, and inputting the measured value $\sim 0.22$ of the Cabibbo angle, one obtains the following rough estimates: $\omega_U \sim 0.16$ and $\omega_D \sim 0.28$, which when substituted into (28) give:

$$J \sim \sin(\theta/2) \times 7.7 \times 10^{-5}. \tag{30}$$

This will coincide with the experimentally measured value of $J$ in (7) for the strong CP angle $\theta \sim 0.8$.

We have now shown that even starting with a real $\boldsymbol{\alpha}$, a CP-violating phase in the CKM matrix of roughly the right order of magnitude will be automatically generated. What happens when $\boldsymbol{\alpha}$ is allowed to be complex? If $\boldsymbol{\alpha}$ has complex elements but the phases of which do not change with scale, then these phases will just cancel in taking inner products of vectors, leading to exactly the same conclusions as for real $\boldsymbol{\alpha}$. Indeed, this fact has been a major obstacle to our attempts at model building where we have never yet succeeded in generating an $\boldsymbol{\alpha}$ with a scale-dependent phase. The reason for this failure appears to be as follows. Our models are based on the, to us, attractive assumption that fermion generation is dual to colour, which assumption guarantees that there are exactly 3 generations of fermions as observed in experiment. In that case, it seems natural that the rotation we want in generation space should have its origin in colour dynamics. But if the QCD action is itself CP-conserving, then it cannot reasonably be expected to give rotations in generation space producing in turn a mixing matrix which violates CP. Although we have not yet proved that this is indeed the reason for our erstwhile failure, we have not succeeded either to obtain the contrary, and this is not without trying. Given now, however, that the QCD action is actually CP-violating by virtue of the theta-angle term, it is natural to expect that CP-violating effects will result. And this is what actually happens, although luckily not in the strong sector explicitly, so long as we take care to eliminate the theta-angle term from the strong action by the appropriate chiral transformations. CP-violating effects will appear only in the weak sector where it is wanted. Thus, the above result would seem also to have removed a major obstacle in our attempts at model building which, if substantiated, would be to us a great relief.

The conclusion at present is thus that with a rotating mass matrix one seems on the one hand to be able to eliminate any theta-angle term by chiral transformations and yet retain a hermitian mass matrix and nonzero masses for all quarks, leaving the strong sector still CP-invariant. In the weak sector, on the other hand, the CKM mixing matrix appearing in the weak current acquires now an extra phase via the chiral transformations and leads to CP-violation there of roughly the order experimentally observed. Thus, it seems to offer a solution simultaneously to both the strong and "weak" CP problems while, for the first time to our knowledge, linking the two together in a quite appealing manner, so that strong and weak CP-violations appear now as just two different facets of one and the same phenomenon. Hence the theta-angle term in the QCD action in the strong sector need not be at all suppressed by nature, as is usually thought, but can instead simply be re-expressed, via chiral transformations and mass matrix rotations, to reveal itself in the weak sector as the KM phase in the CKM mixing matrix, and to give rise to the familiar CP-violating effects observed there. That this is the case adds much, we think, to the attractiveness of the rotation framework for explaining fermion mixing. And as a bonus to us as model builders on the practical side, the new result has for the first time yielded a CP-violating phase in the CKM matrix which has so far eluded us in all the models that we have tried; this we find encouraging.

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