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GAUGE-INVARIANT PROPER SELF-ENERGIES AND VERTICES IN GAUGE THEORIES WITH BROKEN SYMMETRY

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ABSTRACT. Using the pinch technique, we show how to recover, from the S-matrix of a spontaneously-broken non-Abelian gauge theory, proper self-energies and vertices which are fully gauge-invariant when one or more momenta are off-shell. Explicit calculations are carried out at the one-loop level for gauge-boson self-energies and fermion-gauge-boson vertices in a simple SU(2) gauge theory with a Higgs boson. The same technique allows us to calculate, at one-loop order, a neutrino electromagnetic form factor which is gauge-invariant at all photon momenta, thus resolving a long-standing problem. We show how massless Goldstone bosons, not present in the S-matrix, must be introduced into Green's functions in order to reconcile gauge invariance and renormalizability.

1. Introduction.

The standard techniques for quantizing non-Abelian gauge theories (NAGT) in the continuum require gauge fixing. Consequently, the corresponding Green's functions are in general gauge dependent and--with the exception of ghost-free gauges--are related by complicated Ward identities. These facts would not cause any concern if all one cared about was the perturbative computation of S-matrix elements. Indeed, the gauge dependent Green's functions combine to give gauge independent observables order by order in perturbation theory. So, one has the freedom of choosing any convenient gauge for the perturbative calculations knowing that at the end a unique gauge invariant answer will emerge. Many interesting phenomena, however, are believed to have non-perturbative origins. To the extent that one uses the lattice to accommodate such phenomena, one is guaranteed the gauge independence of the results, since no gauge fixing is needed to begin with. Unfortunately, this is hardly the case in the continuum, where the Schwinger-Dyson (SD) equations are put together from guage dependent propagators and vertices and are themselves contaminated with gauge dependences. Of course, if the SD series could be solved exactly, the resulting non-perturbative Green's functions, when put together to form observables, would conspire again to give gauge independent results. For all practical purposes, however, one is compelled to truncate the series at some point or another, and, after doing so, is often led to gauge dependent approximations for ostensibly gauge independent quantities.

This unhappy state of affairs, which signifies the lack of a consistent non-perturbative approximation scheme, can be avoided if one adopts a different approach, as proposed in a number of papers[1-5] over the past few years. The main idea of this method can be summarized as follows: The

Feynman diagrams contributing to a gauge invariant process (like an S-matrix element) must be resummed via a well-defined algorithm--known as the "pinch technique"--in such a way as to form new gauge independent proper vertices and new propagators with gauge independent self-energies and only a trivial gauge dependence--that of their free parts. These new entities satisfy naive (QED-like) Ward identities and their dynamics are governed by modified SD equations, which are themselves, even in their truncated version, manifestly gauge independent.

The program described above has been partially carried out in the context of QCD and many interesting results have been reported so far. In particular, a gauge invariant gluon self-energy was defined and its SD equation was constructed.[1] This SD equation admits "massive" solutions which signify the dynamical generation of gluon mass. Similarly, a gauge invariant three-gluon vertex was calculated at one-loop level and was shown to satisfy a very simple Ward identity.[6] Although the respective SD equation has not been rigorously constructed in this case, several non-perturbative features of the theory have been explored--at least qualitatively--with the aid of a nonlinear toy model.

In the present paper we extend this program at the level of one-loop perturbation theory, to the case of non-Abelian gauge theories with tree-level symmetry breaking, i.e. elementary Higgs particles. Our main motivation is to begin an inquiry into the structure of **gauge-invariant** SD equations for an NAGT either with elementary Higgs particles or with dynamical symmetry breaking (DSB), i.e., **composite** Higgs particles. At first it may seem that the study of elementary Higgs theories in perturbation theory, which is all we study here, must be far removed from DSB, which is fully non- perturbative. But in some respects the structure of these two kinds of symmetry-breaking must be rather similar; indeed the standard way of guessing the dynamical structure of Green's

functions in DSB is to extract it from Higgs theories by replacing those masses generated by $\langle\phi\rangle$ by running masses, then using the SD equations without Higgs fields to determine the momentum dependence of the running masses. In DSB it is necessary to introduce longitudinally-coupled massless poles, which are the Goldstone particles of the theory, into these equations, in order to satisfy the Ward identities for the broken-symmetry mass spectrum. For Abelian gauge theories the DSB program has been carried out quite far,[16] because no particular problems of gauge-dependence arise for the Green's functions of interest (typically gauge-boson proper self-energies).

Much less has been done with DSB in NAGT's for several reasons: The necessary Green's functions have complicated gauge-transformation properties, and are coupled to ghost Green's functions. The role of dynamical Goldstone bosons is obscure, because these massless poles (which do not appear in the S-matrix to the extent that they are "eaten" by gauge bosons) can be changed to poles of arbitrary mass by a gauge transformation, such as going to an R_ξ gauge.[14] Moreover, any approximation made in the course of deriving or solving the coupled Green's function equations can lead to unphysical gauge dependences.

In this paper we construct, for the first time, off-shell gauge-invariant proper two- and three-point functions for an NAGT with symmetry breaking. This involves a generalization of the pinch technique, which makes essential use of Ward identities relating the new vertices and proper self-energies, because these Green's functions have extra pieces coming from symmetry breaking (whether dynamical or with Higgs particles). The Ward identities themselves are the same as for the unbroken theory, but they can only be realized with the help of massless Goldstone bosons. This, as we will see, is quite different from the Ward-identity structure of broken symmetry in the R_ξ gauges, which have massive Goldstone particles in general. Our construction of gauge-invariant gauge-boson proper self-energies and fermion vertices in an NAGT with Higgs breaking automatically leads to a possible Green's function structure for the same theory with DSB and no elementary Higgs particles, simply by replacement of Higgs-generated masses with running masses. It would take us too far afield to explore the (somewhat complicated) SD equations which would determine these running masses, but these will be free of the difficulties discussed in the last paragraph.

The theory we consider here is a slightly modified version of the Georgi-Glashow model.[7] This particular model has the advantage of being simple to deal with, displaying at the same time all the general features we want to study. The gauge group is SU(2) with one fermion doublet in the fundamental and one scalar triplet in the adjoint. The model is vector-like and in addition to the standard gauge couplings we allow Yukawa couplings between fermions and scalars. The gauge symmetry is spontaneously broken via the usual Higgs mechanism giving rise to two massive and one massless gauge bosons and lifting the original mass degeneracy of the fermions. The application of the pinch technique is more complicated than in QCD, mainly because of residual terms proportional to the fermion mass differences, which must be shown to cancel against analogous contributions coming from graphs involving Goldstone bosons. Furthermore, the resulting gauge independent self-energies must be appropriately modified to become transverse--proportional to the usual projection operator $\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right)$--and a similar task must be accomplished for the vertices--they will satisfy QED-like Ward identities. The reasons for such a reformulation will become clear in what follows. Here it should suffice to say that it assures the renormalizability of the respective SD equations. Throughout this paper we consider one-loop contributions only.

Our treatment is therefore purely perturbative but it is very plausible that the particular formulation of the theory in terms of gauge independent and conserved quantities holds true to all orders and can serve as a guidance towards the discovery of the correct set of SD equations.

Although the main motivation of this work is to set up the stage for the non-perturbative treatment of an as-yet undiscovered theory, some of our findings have immediate applications. Indeed, even within the context of the perturbative Standard Model, several problems occur due to the annoying dependence of certain quantities on the gauge choice. It has been known for example, ever since spontaneously broken gauge theories were introduced, that fermion form factors are gauge independent only when the momentum transfer q^2 equals zero.[8,9] Of course, as long as one is only interested in quantities like fermion charges or anomalous magnetic moments, the situation is quite satisfactory. The complications begin when one attempts to relate the off-shell form factor to physical quantities. Such has been the case with the electromagnetic form factor of the neutrino. Bardeen, Gastmans and Lautrup[10] suggested the possibility of relating the neutrino mean square charge radius to the first derivative of the form factor with respect to q^2, only to realize how difficult it was to isolate a physically sensible, e.g. gauge-independent and finite, answer. Various attempts have been made towards this end and although significant progress has been made--especially in a recent work by Degrassi, Sirlin and Marciano[11]--the issue is still not settled. As we will show, the application of the pinch technique in the Standard Model leads us naturally and unambiguously to a gauge independent and finite answer.

The paper is organized as follows: In Section 2 we define the pinch technique and briefly review its application to a gauge theory without symmetry breaking. Section 3 is devoted to the application of the pinch technique in the presence of symmetry breaking. We discuss the technical complications involved and define gauge independent self-energies and vertices. In Section 4 we derive the Ward identities these quantities satisfy and explain how they can be modified to assume a form suitable for non-perturbative treatment, and also give explicit expressions for one-loop boson propagators and fermion vertices. Finally, in Section 5 we show how the use of the pinch technique in the Standard Model gives rise to an off-shell gauge independent neutrino electromagnetic form factor and a finite and gauge independent mean square charge radius.

2. The Pinch Technique.

In this section we will briefly review the application of the pinch technique (PT) in the context of a non-Abelian theory like QCD (no symmetry breaking). The idea has been first introduced in Refs. 1 and 2 and discussed extensively there.

The PT is an algorithm which enables one to **uniquely** decompose an S-matrix element into a set of gauge independent pieces, each with its own characteristic kinematic structure. To be concrete, let us consider the S-matrix element T corresponding to the elastic scattering of two test quarks with masses M_1 and M_2. In general, T will be a function of the two Mandelstam variables s and t and the two quark masses. The application of the PT to the diagrams contributing to T to first order in perturbation theory (Fig. 1) amounts to the following decomposition:

$$T(s,t,M_1,M_2) = T_1(t) + T_2(t,M_1,M_2) + T_3(s,t,M_1,M_2) \tag{2.1}$$

with T_1, T_2 and T_3 individually gauge-independent.

The key observation, which makes the previous decomposition possible, is that graphs which at first glance contribute to a certain set contain pieces which really belong to other sets. For example, Graph (b) of Fig. 1 appears to contribute to T_2 but as we will show, a part of it has the kinematic structure of T_1. Similarly, Graph 1c contains pieces that belong to T_1 and not to T_3. Such parts we call pinch parts (or pinch graphs) and they come about every time a gluon propagator or a three-gluon vertex contribute a longitudinal term k_μ to the graphs' numerator. The action of such term is to trigger an elementary Ward identity of the form:

$$k^{\mu}\gamma_{\mu} \equiv \not k = (\not p+\not k-m) - (\not p-m) = S^{-1}(p+k) - S^{-1}(p) \qquad (2.2)$$

once it gets contracted with a γ matrix. The first term on the RHS of (2.2) will remove the internal fermion propagator of the respective graph--that is a pinch--whereas $S^{-1}(p)$ vanishes on-shell. This last property is characteristic of the S-matrix PT we will be using here, as opposed to the intrinsic PT introduced in Ref. 6.

An explicit calculation, which we will not reproduce (see Ref. 2) shows that the gauge-dependent sum of the usual graphs contributing to the propagator self-energy (Graph 1d) is not all that goes into T_1 and that $T_1(t)$ can be made gauge-independent only if one borrows the propagator-like pieces coming from graphs like 1b and 1c. It is important to notice that **all** gauge dependent parts become propagator-like after pinching. As we will see later, this is not the case any more when the symmetry is broken.

Once it becomes obvious that this technique identifies gauge-independent propagator--and vertex-like quantities (like T_1 and T_2), it is clear that a convenient gauge may be chosen to facilitate the actual calculation. The simplest of all covariant gauges is certainly the Feynman gauge (λ=1) which gets rid of the longitudinal part of the gluon propagtor. Therefore, the only possibility for pinching arises from the three gluon vertices and the only propagator-like pieces come from the graph 1b.

To explicitly demonstrate how the pinching process works for a graph like 1b, it is convenient to decompose the vertex in the following way, first introduced by 't Hooft.[12]

Omitting the group-theoretical factor $i\varepsilon_{abc}$

$$\Gamma_{\mu\nu\alpha} = \Gamma^{P}_{\mu\nu\alpha} + \Gamma^{F}_{\mu\nu\alpha} \qquad (2.3)$$

with

$$\Gamma^{P}_{\mu\nu\alpha} \equiv (q+k)_{\nu} g_{\mu\alpha} + k_{\mu} g_{\nu\alpha} \qquad (2.4)$$

$$\Gamma^{F}_{\mu\nu\alpha} \equiv 2q_{\mu} g_{\nu\alpha} - 2q_{\nu} g_{\mu\alpha} - (2k+q)_{\alpha} g_{\mu\nu} \, . \qquad (2.5)$$

$\Gamma^{P}_{\mu\nu\alpha}$ gives rise to pinch parts when contracted with γ matrices

$$g_{\mu\alpha}(\not q+\not k) = g_{\mu\alpha}[(\not p+\not q-m) - (\not p-\not k-m)] = g_{\mu\alpha}[S^{-1}(p+q) - S^{-1}(p-k)] \qquad (2.6)$$

and

$$g_{\nu\alpha}\not k = g_{\nu\alpha}[(\not p-m) - (\not p-\not k-m)] = g_{\nu\alpha}[S^{-1}(p) - S^{-1}(p-k)] \, . \qquad (2.7)$$

Now $S^{-1}(p+q)$ and $S^{-1}(p)$ vanish on-shell and so we are left with **two** "pinch parts" and a "regular part"

$$\textbf{Pinch parts} = \frac{ig^2}{(2\pi)^4}\left(-\frac{1}{2}C_A\right) 2\int \frac{d^4k}{k^2(k+q)^2}\,(g^{\alpha\rho}\gamma_{\rho}) \qquad (2.8)$$

$$\textbf{Regular part} = \frac{ig^2}{(2\pi)^4}\left(+\frac{1}{2}C_A\right)\int \frac{d^4k\,\gamma^{\mu}S(p-k)\,\gamma_{\nu}\Gamma^{F}_{\mu\nu\alpha}}{k^2(k+q)^2} \qquad (2.9)$$

with C_A the quadratic Casimir operator for the adjoint representation [for SU(N), C_A = N].

When we finally add the pinch contributions of (2.8) to the usual propagator graphs we find the new gauge invariant self-energy:

$$\hat{\Pi}_{\mu\nu} = \left(g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{?} \right) \hat{\Pi}(q) \qquad (2.10)$$

with

$$\hat{\Pi}(q) = -bg^2 \ln\left(\frac{q^2}{\mu^2}\right) \tag{2.11}$$

and $b = 11N/48\pi^2$, the usual β function coefficient.[13]

One additional comment is now needed. The pinch parts of (2.8) are not transverse since they are multiplied by $g^{\mu\nu}$ only. However, making them transverse changes nothing since any term proportional to $q_\mu q_\nu/q^2$ contributes zero on shell. This is true only because at any given vertex both the incoming and outgoing fermions have the same mass. As we will see later on, this is not true any more when the symmetry is broken.

Let us now return to the part (2.9) left over. Since all pinch parts have been removed from Graph (2a), the expression (2.9) is genuinely vertex-like and must be added to the usual QED type graph [Fig. 2]. The final expression is:

$$\hat{\Gamma}_\alpha^{\ a} = \frac{\tau^a i g^2}{(2\pi)^4}\left\{\left[\frac{1}{2}C_A + \frac{N}{d_f}C_f\right]\int \frac{d^4k\ \gamma^\rho S(p+q-k)\ \gamma_\alpha S(p-k)\gamma_\rho}{k^2} + \left(\frac{1}{2}C_A\right)\int d^4k\ \frac{\gamma^\mu S(p-k)\ \gamma^\nu \Gamma^F_{\mu\nu\alpha}}{k^2(k+q)^2}\right\}, \tag{2.12}$$

where τ^a is the fermion representation matrix, d_f its dimension and C_f its Dynkin index.

Using now the fact that

$$q^\alpha \Gamma^F_{\mu\nu\alpha} = [k^2 - (k+q)^2]\ g_{\mu\nu}, \tag{2.13}$$

it immediately follows from (2.12) that $\hat{\Gamma}_\alpha^{\ a}$ satisfies the following QED like

Ward-identity:

$$q^\alpha \hat{\Gamma}_\alpha^{\ a} = \tau^a\left\{\ \Sigma(p) - \Sigma(\hat{p})\ \right\} \tag{2.14}$$

with Σ the usual quark self-energy and $\hat{p} = p+q$.

To summarize: The application of the S-matrix PT gives rise to the three-gauge independent quantities.

(a) A conserved gluon self-energy $\hat{\Pi}_{\mu\nu}(q^2=-t)$ to be identified with $T_1(t)$ when sandwiched between external spinors.

(b) A gluon-quark-quark vertex $\hat{\Gamma}_\alpha^{\ a}\ (q^2 = -t,\ M_i)$, satisfying a naive Ward identity. This vertex, together with its symmetric are to be identified with $T_2(t,M_i)$.

(c) The remainder, given by purely box-like pieces, to be identified with $T_3(t,s,M_i)$.

3. Pinch Technique with Symmetry Breaking.

After this warm-up we proceed to the main subject of this paper and apply the PT to a theory with symmetry breaking. When the gauge symmetry breaks, the tree level Ward identity of (2.2) we use every time we apply the P.T. is modified by a term proportional to the non- vanishing vacuum expectation value $\langle\phi\rangle$. The presence of these new terms amounts to two different things. First, additional gauge-dependent terms emerge when the P.T. is applied at a gauge dependent (proportional to λ) part of a graph, thus complicating the construction of gauge-independent self-energies and proper vertices. Second, the gauge-independent pieces proportional to $\langle\phi\rangle$--which, of course, come about when we apply the P.T. at a gauge-independent part of a graph--make the Ward identities that the self-energy and vertices satisfy different from their QCD counterparts.

This section is motivated by the first problem. We will show explicitly that the P.T. gives rise to gauge-independent self-energies and proper vertices, even when the gauge symmetry is broken.

The model we consider is a simplified version of the Georgi-Glashow model with SU(2) as its gauge group.

There is a doublet of fermions in the fundamental representation and a triplet of scalars in the adjoint. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu} + \bar{\psi}\not{D}\psi - m_o\bar{\psi}\psi + \frac{1}{2}(D_\mu\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \lambda\phi^4$$

$$+ G\bar{\psi}\tau^a\psi\phi_a + \text{gauge fixing term}\,, \qquad (3.1)$$

where

$$\not{D}\psi = i\gamma^\mu(\partial_\mu - igA^a_\mu\tau_a)\,\psi \qquad (3.2)$$

$$D^\mu\phi = (\partial^\mu - ig\,A^{a\mu}R_a)\,\phi \qquad (3.3)$$

with

$$\tau^a_{ij} = \frac{1}{2}\sigma^a_{ij} \qquad \text{(fundamental)} \qquad (3.4)$$

and

$$R^a_{ij} = -i\varepsilon^a_{ij} \qquad \text{(adjoint)}\ . \qquad (3.5)$$

Since this model--with some minor variations--has been thoroughly discussed in the literature,[7,8,9] we will only give a brief account of its main characteristics:

a. The symmetry breaking takes place spontaneously via the usual Higgs mechanism. There are two massive vector mesons with mass $M = g\langle v\rangle$ and one massless vector corresponding to the remaining U(1) symmetry.

b. The two fermions of the theory originally have equal masses. After the symmetry breaking the Yukawa term $\frac{G}{2}\bar{\psi}\sigma^a\psi\phi_a$ lifts this degeneracy:

$$\frac{G}{2}\bar{\psi}\sigma^a\psi\phi_a = \frac{G}{2}\bar{\psi}\sigma^a\psi(\chi_a + \langle v\rangle\delta_{3a}) = \frac{G}{2}\bar{\psi}\sigma^a\psi\chi_a + \frac{G}{2}\bar{\psi}\sigma_3\psi\langle v\rangle\ . \qquad (3.6)$$

Setting $G\langle v\rangle \equiv m$ we have:

$$\frac{G}{2}\bar{\psi}\sigma^a\psi\phi_a = \left(\frac{m}{M}\right) g\bar{\psi}\frac{\sigma^a}{2}\psi\chi_a + m\bar{\psi}\frac{\sigma_3}{2}\psi\ . \qquad (3.7)$$

The first term in (3.7) is the familiar interaction term between fermions and Goldstone bosons. The second term may be added to the fermion mass matrix:

$$\bar{\psi}\left[m_o\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}\right]\psi = \bar{\psi}\begin{bmatrix} m_o + \frac{m}{2} & 0 \\ 0 & m_o - \frac{m}{2} \end{bmatrix}\psi$$

$$= \bar{\psi}\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\psi$$

where we set $m_1 = m_o + \frac{m}{2}$, $m_2 = m_o - \frac{m}{2}$ and $m = m_1 - m_2$. So the two fermions have different masses after the symmetry breaking.

c. The model is asymptotically free. The one-loop β function has the usual form $\beta = -bg^3$ with $b = \frac{1}{16\pi^2}\left[\frac{11}{3}C_A - \frac{4}{3}T_f - \frac{1}{6}T_s\right]$. For the particular particle arrangement we have chosen, the Dynkin indices are: $C_A = T_s = 2$, $T_f = \frac{1}{2}$ and so $b = \frac{19}{3}\left(\frac{1}{16\pi^2}\right)$.

d. We define the usual charge eigenstates:

$$W_\mu^{\pm} = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2), \qquad A_\mu = A_\mu^3 \tag{3.8}$$

$$s^{\pm} = \frac{1}{\sqrt{2}}(\chi^1 \pm i\chi^2) \tag{3.9}$$

This corresponds to using the Cartan instead of the defining representation. Since the only group theoretical quantities involved in our calculations are the group invariants, we will freely switch from one representation to the other.

e. We choose the R_ξ gauge[8,9] to quantize the theory. Such a gauge choice removes the tree-level mixing between W and s by introducing gauge dependent poles in both the W and s propagators. So:

$$W^{\pm} \text{ propagator:} \quad -i\,\Delta_{\mu\nu} = -i\left[g_{\mu\nu} - \frac{q_\mu q_\nu(1-\lambda)}{q^2-\lambda M^2}\right]\frac{1}{q^2-M^2} \tag{3.10}$$

$$s^{\pm} \text{ propagator:} \quad i\,\Delta_s = \frac{i}{q^2-\lambda M^2} \tag{3.11}$$

with $\lambda = 1/\xi$. In addition to that, we have the usual gauge-independent propagator for the massive Higgs field H.

$$\text{Higgs propagator:} \quad i\Delta_H = \frac{i}{q^2-M_H^2}, \tag{3.12}$$

where the $i\epsilon$ prescription in the propagators is understood.

The application of the pinch technique to this theory is slightly more involved than in the QCD case. The main reason is that the massive vector mesons couple to fermions with different masses and, as a result, the elementary Ward identity of (2.2) will be modified. The pinch will be initiated exactly as before by a longitudinal momentum k^μ hitting a γ_μ. But now:

$$\not{k} = (\not{p}+\not{k}-m_1) - (\not{p}-m_2) + (m_1-m_2) = S_1^{-1}(p+k) - S_2^{-1}(\not{p}) + m\,. \tag{3.13}$$

The first two terms of (3.13) will pinch and vanish on-shell respectively, as they did before. But in addition, a term proportional to m is left over. Such terms represent a potential threat to our program because they are in general gauge-dependent. However, as we will show in a moment, all these additional gauge dependent contributions coming from (3.13) cancel exactly against contributions coming from graphs involving Goldstone bosons, whose coupling to fermions is also proportional to m.

At this point we would like to emphasize that the purpose of this exercise is not to demonstrate the gauge independence of the S-matrix of a theory with symmetry breaking. This issue has been resolved long ago[14] and the role of the Goldstone bosons is very well understood. What we rather want to prove is that the breaking of the symmetry does not really interfere with our ability to identify gauge-independent propagator and vertex-like quantities when we properly apply the pinch technique.

The graphs that contain pinch parts are shown in Figs. 3 and 4.[15] The propagator-like gauge dependent pinch parts will cancel against the gauge dependent pieces from the WW, Ws and sW self energies. But unlike the unbroken case, we now have additional vertex and box-like gauge dependent contributions proportional to m which must add up to zero. To find these, we first decompose the W propagator in the standard way[8]

$$\Delta^{\mu\nu}(q) = \Delta_1^{\mu\nu}(q) + \Delta_2^{\mu\nu}(q) \tag{3.14}$$

with

$$\Delta_1^{\mu\nu}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{M^2} \right) \frac{1}{q^2-M^2} \tag{3.15}$$

$$\Delta_2^{\mu\nu}(q) = \frac{q_\mu q_\nu}{M^2} \frac{1}{q^2-\lambda M^2} = \frac{q_\mu q_\nu}{M^2} \Delta_s(q) \tag{3.16}$$

and then we apply the PT, which amounts to saving only λ-dependent terms generated by $\Delta_2^{\mu\nu}$. Setting

$$D_1 \equiv \frac{1}{k^2-\lambda M^2} \tag{3.17}$$

and

$$D_2 \equiv \frac{1}{(k^2-\lambda M^2)(k+q)^2} \tag{3.18}$$

and omitting external spinors and the loop momentum integration symbol $\frac{1}{2\sqrt{2}} \int \frac{d^4k}{(2\pi)^4}$ we find the following vertex-like gauge dependent leftover pieces:

3a: $\left(\frac{m}{M^2} \right) \left[\gamma_\alpha S_2(p-k)\, D_1 + (-q^2 g_{\alpha\mu} + q_\alpha q_\mu)\, \gamma^\mu S_2(p-k)\, D_2 \right] \Delta^{\alpha\nu}(q)$

3b: $m\gamma_\alpha\, S_2(p-k)\, D_2\, \Delta^{\alpha\nu}(q)$

3c: $-\left[\frac{m}{M^2} \gamma_\alpha\, S_2(p-k) + \left(\frac{m}{M} \right)^2 S_1(\hat{p}-k)\, \gamma_\alpha\, S_2(p-k) \right] D_1\, \Delta^{\alpha\nu}(q)$

3d: $\left(\frac{m}{M} \right)^2 S_1(\hat{p}-k)\, \gamma_\alpha\, S_2(p-k)\, D_1\, \Delta^{\alpha\nu}(q)$

4a: $\left(\frac{m}{M} \right)^2 \not{q}\, S_2(p-k)\, D_1\, \Delta_s(q)$

4b: $\left(\frac{m}{M} \right)^3 S_1(\hat{p}-k)\, S_2(p-k)\, D_1\, \Delta_s(q)$

4c: $\left(\frac{m}{M} \right)^3 S_1(\hat{p}-k)\, S_2(p-k)\, D_1\, \Delta_s(q)$

whereas Graphs 3e, 4d and 4e give no vertex-like leftover contributions, since the pinching occurs at their photon-fermion vertices. Summing up all the above terms and calling their sum L_1^{ν}:

$$L_1^{\nu} = \left(\frac{m}{M^2} \right) \left[(M^2-q^2)\, g_{\alpha\mu} + q_\alpha q_\mu \right] \gamma^\mu S_2(p-k)\, D_2\, \Delta^{\alpha\nu}(q) + \left(\frac{m}{M} \right)^2 \not{q}\, S_2(p-k)\, D_1\, \Delta_s(q) \,. \tag{3.19}$$

To L_1^{ν} we now must add the respective leftover contributions coming from the box diagrams 3f and 4f. They can be separated into two classes: vertex-like ones (one internal fermion propagator already pinched away) and box-like ones (both internal propagators survive). The box-like contributions from 3f cancel against the entire 4f diagram. For the vertex-like contribution of 3f we have:

$$L_2^{\nu} = \left(\frac{m}{M^2} \right) \gamma^{\alpha} S_2(p-k)\, g_{\alpha}^{\nu} D_2 \ . \tag{3.20}$$

If we now write:

$$g^{\nu}{}_{\alpha} = \Delta^{\nu\mu}(q)\, \Delta^{-1}_{\mu\alpha}(q) \tag{3.21}$$

with

$$\Delta^{-1}_{\mu\alpha}(q) = \left[(q^2-M^2)\, g_{\mu\alpha} - q_{\mu}q_{\alpha} + \frac{1}{\lambda} q_{\mu}q_{\alpha} \right] \tag{3.22}$$

or alternatively

$$g^{\nu}{}_{\alpha} = \Delta^{\nu\mu}(q)\, [(q^2-M^2)\, g_{\alpha\mu} - q_{\alpha}q_{\mu}] + \Delta_s(q)\, q^{\nu}q_{\alpha} \tag{3.23}$$

we have for L_2^{ν}:

$$L_2^{\nu} = \left(\frac{m}{M^2} \right) [(q^2-M^2)\, g_{\alpha\mu} - q_{\alpha}q_{\mu}]\, \gamma^{\mu} S_2(p-k)\, D_2\, \Delta^{\alpha\nu}(q)$$
$$- \left(\frac{m}{M} \right)^2 \not{q}\, S_2(p-k)\, D_2\, \Delta_s(q) \tag{3.24}$$

where we used the on-shell condition

$$\bar{u}_1(\hat{\ell})\, \not{q}\, u_2(\ell) = -m\bar{u}_1(\hat{\ell})\, u_2(\ell) \tag{3.25}$$

since both L_1^{ν} and L_2^{ν} must be contracted with $\bar{u}_1\gamma_{\nu}u_2$. As we can see immediately from (3.19) and (3.24)

$$L_1^{\nu} + L_2^{\nu} = 0 \tag{3.26}$$

that is, the gauge dependent vertex-like terms proportional to m cancel exactly against each other as needed. The proof of an analogous cancellation between the additional gauge-dependent contributions coming from graphs containing Higgs instead of photon propagators proceeds in exactly the same way.

Having proved that PT works, the actual calculation of the gauge-independent quantities can be performed in any gauge as long as we assign the pinch parts appropriately. For example, let us consider the gauge-independent W self-energy $\hat{\Pi}_{\mu\nu}$ and the gauge-independent Wf_1f_2 vertex. In the Feynman gauge (λ=1) the only propagator-like pinch part comes from Graph 3a. It is important to emphasize again that the box diagrams do not give any pinch parts in the Feynman gauge. Of course, they do pinch in any other R_{ξ} gauge ($\xi \neq 1$). The contribution from Graph 3a must be subtracted from the vertex and added to the self-energy. So we have:

$$\hat{\Pi}_{\mu\nu}(q) = \Pi^{(\lambda=1)}_{\mu\nu}(q) + 2(q^2-M^2)\, \Pi_{pinch}(q)\, g_{\mu\nu} \tag{3.27}$$

$$\hat{\Gamma}_{\mu} = \Gamma_{\mu}{}^{(\lambda=1)} - \Pi_{pinch}(q)\, \gamma_{\mu} \ , \tag{3.28}$$

where $\Pi^{(\lambda=1)}_{...}$ and $\Gamma^{(\lambda=1)}$ are the conventional self-energy and vertex respectively, calculated in the Feynman gauge, and

$$\Pi_{pinch}(q) = \frac{2ig^2}{(2\pi)^4}\int \frac{d^4k}{(k^2-M^2)(k+q)^2} . \tag{3.29}$$

The factor of 2 in (3.27) accounts for the pinch contribution of the Wf_2f_1 graph, which is the mirror image of Graph 3a.

It is interesting to notice that the addition of the pinch to the self-energy does not shift the position of the pole. So, any normalization condition imposed on $\Pi_{\mu\nu}^{(\lambda=1)}$ at $q^2 = M^2$ will also hold for $\hat{\Pi}_{\mu\nu}(q)$. This was, of course, expected since at $q^2 = M^2$ the W self-energy is an observable and therefore we must have $\Pi^{(\lambda=1)}(q^2=M^2) = \Pi^{(\lambda\neq 1)}(q^2=M^2) = \hat{\Pi}(q^2=M^2)$.

We will postpone recording the explicit closed forms of the gauge independent $\hat{\Gamma}_\mu$ and $\hat{\Pi}_{\mu\nu}$ till the end of the next section and will instead focus our interest on the Ward identities they satisfy.

4. Ward Identities, S-Matrix Reformulation, and Explicit One-Loop Results.

As we demonstrated in the last section, the application of the PT enables us to identify gauge-independent propagator and vertex-like quantities even when the gauge symmetry of the theory is broken. We now turn to investigate the Ward identities these quantities satisfy. We also give explicit formulas for the one-loop gauge-invariant boson propagators and fermion vertices. In dealing with the Ward identities, knowledge of the specific closed form of the quantities involved is not required, since all one needs to use is the fact that the PT works in our case and that the S-matrix is gauge-independent. To see this exactly, let us consider the Wf_1f_2 and sf_1f_2 vertices we examined in the previous section. We call $\hat{\Gamma}_\mu$ the Wf_1f_2 vertex-like contribution and $\hat{\Lambda}$ the sf_1f_2 contribution (see Fig. 5a). Both $\hat{\Gamma}$ and $\hat{\Lambda}$ are individually gauge-independent after the application of PT but there is a remaining gauge-dependence coming from the external free W and s propagators. The value of $T_2(q,m_i)$ [15] in this case is:

$$T_2 = \bar{u}_1\gamma_\nu u_2\left[\Delta_1^{\nu\mu}(q) + \frac{q^\nu q^\mu}{M^2}\Delta_s \right] \bar{u}_2\hat{\Gamma}_\mu u_1 - \frac{m}{M}\bar{u}_1 u_2 \Delta_s \hat{\Lambda}\, \bar{u}_2 u_1 . \tag{4.1}$$

Gauge invariance of the S-matrix piece T_2 requires:

$$\frac{m}{M^2}\Delta_s[q^\mu\hat{\Gamma}_\mu + M\hat{\Lambda}] = 0 \tag{4.2}$$

from which immediately follows that $\hat{\Gamma}_\mu$ and $\hat{\Lambda}$ must be related by the following Ward identity:

$$q\cdot\hat{\Gamma} + M\hat{\Lambda} = 0 \tag{4.3}$$

when the fermions f_1 and f_2 are on-shell. Obviously, (4.3) is the one-loop generalization of the tree level Ward identity (3.13).

The gauge-independent value of T_2 is:

$$T_2 = \bar{u}_1\gamma_\nu u_2\left[g_{\mu\nu} - \frac{q^\nu q^\mu}{M^2} \right] \frac{\bar{u}_2\hat{\Gamma}_\mu u_1}{q^2-M^2} \tag{4.4}$$

or after using (4.3):

$$T_2 = \bar{u}_1\gamma_\nu u_2\left(\frac{1}{q^2-M^2} \right) \bar{u}_2\left[\hat{\Gamma}_\nu + \frac{q_\nu\hat{\Lambda}}{M} \right] u_1 \ . \tag{4.5}$$

It is relatively straightforward to check the validity of (4.3). As explained in Section 3, since both $\hat{\Gamma}_\mu$ and $\hat{\Lambda}$ are gauge-independent, the Feynman gauge (λ=1) may be used. $\hat{\Gamma}_\mu$ is then given by Graphs 3a to 3e and the only propagator-like contribution that must be removed comes from Graph 3a. We can now follow the steps of Section 2 almost exactly. When acting with q^μ on $\hat{\Gamma}_\mu$ we get the difference of the two fermion self-energies, $\Sigma_1(p) - \Sigma_2(p+q)$, as in (2.14), but in addition we have leftovers proportional to the fermion mass difference m and the W mass M. The two fermion self-energies vanish on-shell. The additional contributions are:

3a: $-\left[m\not{q}\ S_2(p-k) + M^2\gamma^\mu S_2(p-k)\ \gamma_\mu \right] \hat{D}_2 + m\not{q} S_2(p-k)\ \hat{D}_4$

3b: $m\not{q}\ S_2(p-k)\ \hat{D}_2 + \left(\frac{m}{M} \right)^2 [M^2-M_H^2]\ S_2(p-k)\ \hat{D}_4$

3c: $m\gamma_\mu\ S_1(\hat{p}-k)\ S_2(p-k)\ \gamma^\mu\ \hat{D}_1$

3d: $\left(\frac{m^3}{M^2} \right) S_1(\hat{p}-k)\ S_2(p-k)\ \hat{D}_1$

3e: $\frac{1}{2}\left\{ -m\gamma^\mu S_2(\hat{p}-k)\ S_1(p-k)\ \gamma^\mu\left(\frac{1}{k^2} \right) + m\left(\frac{m}{M} \right)^2 S_2(\hat{p}-k)\ S_1(p-k)\ \hat{D}_3 \right\}$

with

$$\hat{D}_1 = D_1(\lambda=1) = \frac{1}{k^2-M^2}\ ;\quad \hat{D}_2 = D_2(\lambda=1) = \frac{1}{(k+q)^2\ (k^2-M^2)}\ ;$$

$$\hat{D}_3 = \frac{1}{k^2-M_H^2}\ ;\qquad \hat{D}_4 = \frac{1}{(k^2-M^2)[(k+q)^2-M_H^2]}\ . \tag{4.6}$$

As for $\hat{\Lambda}$, it is given by Graphs 4a to 4e. Multiplying by M we get:

4a: $M^2\gamma^\mu S_2(p-k)\ \gamma_\mu\hat{D}_2 - m(m+\not{q})\ S_2(p-k)\ \hat{D}_4$

4b: $-m\gamma^\mu S_1(\hat{p}-k)\ S_2(p-k)\ \gamma_\mu\hat{D}_1$

4c: $-\left(\frac{m^3}{M^2} \right) S_1(\hat{p}-k)\ S_2(p-k)\ \hat{D}_1$

4d: $\left(\frac{m}{M} \right)^2 M_H^2\ S_2(p-k)\ \hat{D}_4$

4e: $\frac{1}{2}\left\{ m\gamma_\mu\ S_2(\hat{p}-k)\ S_1(p-k)\ \gamma^\mu\left(\frac{1}{k^2} \right) - m\left(\frac{m}{M} \right)^2 S_2(\hat{p}-k)\ S_1(p-k)\ \hat{D}_3 \right\}$.

Clearly the sum 4a+4b+4c+4d+4e is exactly equal and opposite to 3a+3b+3c+3d+3e, which concludes the proof of (4.3). It is interesting to notice by the way, that Graph 4d is **entirely propagator-like**, since the $s^+s^-A_\alpha$ vertex gives a factor $(q-k)_a$ to the numerator, which, when contracted with γ^α, exactly cancels the internal fermion propagator, leaving no residual pieces. Therefore, the contribution marked as 4d above comes entirely from the graph containing a Higgs instead of a photon propagator.

We can now repeat a similar argument for the propagator-like pieces. We start from the fact that with the application of PT we have isolated the gauge independent quantities $\hat{\Pi}_{\mu\nu}$, $\hat{V}_{\mu}$, $\hat{V}_{\nu}$ and $\hat{B}$, shown in Fig. 5b. As before, the gauge-dependence introduced by the external W and s propagators must be canceled for T_1 to be gauge-independent. For T_1 we have:

$$T_1 = \bar{u}_1\gamma_\nu u_2\left[\left(\Delta_1^{\nu\alpha} + \frac{q^\nu q^\alpha}{M^2}\Delta_s\right)\hat{\Pi}_{\alpha\beta}\left(\Delta_1^{\beta\mu} + \frac{q^\beta q^\mu}{M^2}\Delta_s\right)\right]\bar{u}_2\gamma_\mu u_1$$

$$- \bar{u}_1\gamma_\nu u_2\left[\Delta_1^{\nu\alpha} + \frac{q^\nu q^\alpha}{M^2}\Delta_s\right]\hat{V}^\alpha \Delta_s\left(\frac{m}{M}\right)\bar{u}_2 u_1$$

$$- \bar{u}_1 u_2\left(\frac{m}{M}\right)\Delta_s \hat{V}_\beta\left[\Delta_1^{\beta\mu} + \frac{q^\beta q^\mu}{M^2}\Delta_s\right]\bar{u}_2\gamma_\mu u_1$$

$$+ \bar{u}_1\left(\frac{m}{M}\right) u_2\Delta_s \hat{B}\Delta_s\bar{u}_2\left(\frac{m}{M}\right) u_2 . \tag{4.7}$$

Gauge independence for T_1 implies:

$$q^\alpha\hat{\Pi}_{\alpha\beta} + M\hat{V}_\beta = 0 \tag{4.8}$$

and

$$q^\alpha q^\beta\hat{\Pi}_{\alpha\beta} - M^2\hat{B} = 0 . \tag{4.9}$$

From (4.8) and (4.9) immediately follows:

$$q^\beta\hat{V}_\beta + M\hat{B} = 0 . \tag{4.10}$$

The gauge-independent value for T_1 is then:

$$T_1 = \bar{u}_1\gamma_\nu u_2\left[\Delta_1^{\nu\alpha}\hat{\Pi}_{\alpha\beta}\Delta_1^{\beta\mu}\right]\bar{u}_2\gamma_\mu u_1 . \tag{4.11}$$

A straightforward calculation we will not reproduce here, will show the reader that Eqs. (4.8), (4.9), and (4.10) are indeed correct.

It is now clear from the previous analysis that the gauge-independent quantities $\hat{\Pi}_{\mu\nu}$ and $\hat{\Gamma}_\mu$ we have defined do not quite satisfy the same Ward identities as their respective counterparts of the unbroken case. So, in contrast to (4.8) or (4.9), the self-energy defined in (2.10) is transverse and unlike (4.3), $\hat{\Gamma}_\mu$ of (2.14) satisfies a QED-type Ward identity. This situation is not satisfactory for two reasons. First, the quantities identified as gauge-invariant self-energies and vertices should satisfy the same Ward identities both in the unbroken and broken cases, as happens with their conventionally defined gauge-dependent counterparts. Furthermore, as Eq. (4.11) indicates, the self-energy $\hat{\Pi}(q)$ is multiplied by the factor $\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M^2}\right)$ and it would therefore give rise to non-renormalizable Schwinger-Dyson equations, exactly as happens in the unitary gauges. It is fairly straightforward, however, to reformulate the S-matrix in such a way as to define a new gauge-independent and transverse self-energy $\tilde{\Pi}_{\mu\nu}$ and a new gauge-independent vertex $\tilde{\Gamma}$ satisfying (2.14). The cost of such a reformulation is the appearance of massless Goldstone poles in our expressions. However, since both the old and the new quantities originate from the same unique S-matrix, all poles introduced by this reformulation cancel against each other, because the S-matrix contains no massless poles to begin with.

Let us start with $T_2(t)$. Using the following identity

$$\frac{1}{M^2} = \frac{1}{q^2} + \frac{(q^2-M^2)}{q^2M^2} . \tag{4.12}$$

T_2 from (4.4) becomes:

$$T_2 = \bar{u}_1\gamma_\nu u_2\Big[\, g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} - \frac{(q^2-M^2)\; q_\mu q_\nu}{q^2 M^2}\,\Big]\, \frac{\bar{u}_2\hat{\Gamma}_\mu u_1}{(q^2-M^2)}$$

$$= \bar{u}_1\gamma_\nu u_2\; \bar{u}_2\Big[\, g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\,\Big]\, \frac{\hat{\Gamma}_\mu}{q^2-M^2}\, u_1 - \bar{u}_1\Big(\frac{m}{M}\Big)\, u_2\, \frac{\hat{\Lambda}}{q^2}\, \bar{u}_2 u_1 \;, \qquad (4.13)$$

where we used (3.25) and (4.3). If we now define:

$$\tilde{\Gamma}_\nu = \Big(\, g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\,\Big)\, \hat{\Gamma}_\mu = \hat{\Gamma}_\nu + \frac{q_\nu}{q^2}\, M\hat{\Lambda} \qquad (4.14)$$

it is obvious that

$$q^\nu\tilde{\Gamma}_\nu = \Sigma_1(p) - \Sigma_2(\hat{p}) = 0 \;. \qquad (4.15)$$

In terms of the new vertex, T_2 becomes:[15]

$$T_2 = \bar{u}_1\gamma_\nu u_2\Big[\, \frac{g_{\mu\nu}}{q^2-M^2}\,\Big]\, \bar{u}_2\tilde{\Gamma}_\mu u_1 - \bar{u}_1\Big(\frac{m}{M}\Big)\, u_2\Big(\frac{1}{q^2}\Big)\, \bar{u}_2\, \hat{\Lambda}\, u_1 \;. \qquad (4.16)$$

It is obvious from the form of (4.16) that T_2 can be thought of as consisting of two different pieces, one due to a massive vector boson and one due to a **massless** Goldstone boson.

We now turn to T_1. Using (4.12) in (4.11) and setting

$$\hat{V}_\beta = \hat{V}\cdot q_\beta \qquad (4.17)$$

we get:

$$T_1 = \bar{u}_1\gamma_\nu u_2\Big(\frac{1}{q^2-M^2}\Big)\Big[\, \hat{\Pi}^{\mu\nu} + M\hat{V}\,\frac{q^\mu q^\nu}{q^2}\,\Big]\, \frac{1}{(q^2-M^2)}\, \bar{u}_2\gamma_\mu u_1$$

$$+ \bar{u}_1\Big(\frac{m}{M}\Big)\frac{1}{q^2}\,\hat{B}\,\frac{1}{q^2}\,\bar{u}_2\Big(\frac{m}{M}\Big)\, u_1 \;. \qquad (4.18)$$

Defining

$$\tilde{\Pi}_{\mu\nu} = \hat{\Pi}_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\, M\hat{V} \qquad (4.19)$$

we immediately have:

$$q^\mu\tilde{\Pi}_{\mu\nu} = 0 \qquad (4.20)$$

and so:

$$T_1 = \bar{u}_1\gamma_\nu u_2\Big(\frac{1}{q^2-M^2}\Big)\, \tilde{\Pi}_{\mu\nu}\Big(\frac{1}{q^2-M^2}\Big)\, \bar{u}_2\gamma_\mu u_1 + u_1\Big(\frac{m}{M}\Big)\frac{1}{q^2}\,\hat{B}\,\frac{1}{q^2}\,\bar{u}_2\Big(\frac{m}{M}\Big)\, u_1 \;. \qquad (4.21)$$

As we can see in Fig. 5b, T_1 consists of the sum of two self-energies, one corresponding to a regular massive vector field and one to a massless Goldstone boson. It is interesting to notice that the above rearrangements have removed the one-loop mixing between s and W.

Since $\tilde{\Pi}_{\mu\nu}$ is transverse, it can be written in the form

$$\tilde{\Pi}_{\mu\nu} = \Big(\, g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\,\Big)\, \tilde{\Pi}(q^2) \;. \qquad (4.22)$$

Writing $\hat{\Pi}_{\mu\nu}$ as

$$\hat{\Pi}_{\mu\nu}(q) = \hat{\Pi}_1\, q^2 g_{\mu\nu} + \hat{\Pi}_2\, q_\mu q_\nu + \hat{\Pi}_3^{\,i}\, \hat{M}^2\, g \qquad (4.23)$$

with $\hat{M}_i \in \{M^2, M_H^2, m_1^2, m_2^2\}$, it is straightforward to show that

$$\tilde{\Pi}(q^2) = \hat{\Pi}_1 q^2 + \hat{\Pi}_3^{\,i} M_i^2 \ . \tag{4.24}$$

Therefore, $\tilde{\Pi}_{\mu\nu}$ will obey the same normalization condition as $\hat{\Pi}_{\mu\nu}$ or $\Pi_{\mu\nu}^{(\lambda=1)}$ at $q^2 = M^2$.

At this point we would like to remind the reader of the fact that the massless poles would not have appeared had we not insisted on the transversality of the W self-energy and vertex. They are therefore not related to any particular gauge choice, such as the R gauge for example.[17] In fact, as clearly stated in Section 3, the R_ξ gauge we used to quantize the theory introduces **massive** gauge dependent poles.

Our next step is to express the S-matrix in terms of the newly defined quantities. Combining Eqs. (4.16) and (4.21) and keeping terms up to order g^4, we have:

$$T(s,t,m_1,m_2) = \bar{u}_1 G_\mu u_2 \frac{P^{\mu\nu}}{(q^2-M^2)^2} \bar{u}_2 G_\nu u_1 + \bar{u}_1 L u_2 \frac{H}{q^4} \bar{u}_2 L u_1$$
$$+ T_3(t,s,m_1,m_2) \tag{4.25}$$

with

$$G_\mu = \left(\gamma_\mu + \frac{mq_\mu}{q^2} \right) + \tilde{\Gamma}_\mu$$

$$L = \frac{m}{M} + \hat{\Lambda}$$

$$P_{\mu\nu} = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \left[(q^2-M^2) + \tilde{\Pi}(q^2) \right]$$

$$H = q^2 [1 + \hat{B}] \ . \tag{4.26}$$

Eq. (4.25) includes the zeroth and first-order contribution to the S matrix. Clearly

$$q^\mu G_\mu = q^\mu P_{\mu\nu} = 0 \ . \tag{4.27}$$

It would be straightforward to show that (4.25) holds true to all orders in perturbation theory, assuming that PT actually works to all orders. Although no such proof will be given in this paper, this assumption is very plausible. In such a case, the quantities defined in (4.26) will be perturbative series with an infinite number of gauge independent terms satisfying (4.27).

Finally, we will now elaborate on the closed forms of the vertex $\tilde{\Gamma}_\mu$ and self-energy $\tilde{\Pi}_{\mu\nu}$ defined in (4.14) and (4.19) respectively. Let us start with $\tilde{\Gamma}_\mu$:

Define:

$$S_{[i,j]} \equiv S_i(\hat{p}+k) - S_j(p-k)$$

$$S^\nu_{ij} \equiv S_i(\hat{p}-k)\ \gamma^\nu S_j(p-k)$$

$$S_{ij} \equiv S_i(\hat{p}-k)\ S_j(p-k) \ .$$

Then we have:

$$\tilde{\Gamma}_\alpha = \frac{ig}{2\sqrt{2}} \int \frac{d^4k}{(2\pi)^4} \left\{ \Gamma^F_{\mu\nu\alpha}\, \gamma^\mu S_{[1,2]}\, \gamma^\nu \hat{D}_2 - \left[\left(\frac{m}{M} \right)^2 (2k+q)_\alpha + m\gamma_\alpha \right] S_{[1,2]} \hat{D}_4 + \right.$$

$$+\left[1+\left(\frac{m}{M}\right)^2\right] S_{12}^{\ \nu} D_1 g_{\nu\alpha} - \frac{1}{2}\left[\frac{1}{k^2}+\left(\frac{m}{M}\right)^2 \hat{D}_3\right] S_{21}^{\ \nu}\, g_{\nu\alpha}\Bigg\}$$

$$+\frac{ig}{2\sqrt{2}}\frac{q_\alpha}{q^2}\int\frac{d^4k}{(2\pi)^4}\left\{\left[\left(\frac{m}{M}\right)^2 (M^2-M_H^2) + m\not{q}\right] S_{[1,2]}\hat{D}_4 - M^2\gamma_\mu S_{[1,2]}\,\gamma^\mu \hat{D}_2\right.$$

$$\left. - \frac{1}{2} m S_{21}\left[\frac{1}{k^2}+\left(\frac{m}{M}\right)^2 \hat{D}_3\right] - m\left[1+\left(\frac{m}{M}\right)^2\right] S_{12}\hat{D}_1\right\} . \qquad (4.28)$$

It is now quite elementary to explicitly verify that the action of q^μ on $\tilde{\Gamma}_\mu$ of (4.28) gives rise to the difference of two fermion propagators, $\Sigma_1(p) - \Sigma_2(p+q)$, which of course vanish on-shell.

The W self-energy $\tilde{\Pi}(q^2)$ can also be calculated in a closed form if one follows the instructions of Eqs. (4.24) and (3.27). The one-loop result is lengthy and complicated; we do not record it in full. To $O(g^2)$ it can be factored into a pole term and a term displaying the β-function of the theory; we give here only the form of this latter term at $q^2 \gg M^2$:

$$q^2-\tilde{\Pi}(q^2) = (q^2-M^2)(1 + bg^2\,\ell n(-q^2/\mu^2)) , \qquad (4.29)$$

where μ is an arbitrary mass scale. The important properties of this simple self-energy are:

(a) It is **gauge-independant** and has the **correct** renormalization group coefficient b in front.

(b) It **explicitly** exhibits a pole at $q^2 = M^2$ whereas for $q^2 \gg M^2$ it reduces to the usual ultraviolet result.

5. Neutrino Electromagnetic Form Factor.

It has been known since the early days of gauge theories with spontaneous symmetry breaking[8,9] that both the electric and magnetic form factors of fermions--$F_1(q^2)$ and $F_2(q^2)$ respectively--defined by[18]

$$\Gamma_\mu = \bar{u}(\hat{p})\left[\gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m} F_2(q^2)\right] u(p) \qquad (5.1)$$

are gauge-dependent for general values of the momentum transfer q^2. It is only at $q^2 = 0$ when the gauge dependence drops so that $F_1(0)$ can be identified with the fermion charge and $F_2(0)$ with the anomalous magnetic moment.

In the context of the Standard Model the effective electromagnetic form factor $F(q^2)$ of the neutrino has been a longstanding puzzle. It has been argued[10] that the neutrino mean square radius $\langle r^2\rangle$ and $F(q^2)$ are related by

$$\langle r^2\rangle = 6\left.\frac{\partial F(q^2)}{\partial q^2}\right|_{q^2=0} , \qquad (5.2)$$

but it was soon realized that the conventional definition of F would give rise to gauge-dependent and divergent expressions for $\langle r^2\rangle$. This of course comes as no surprise. There is no *a priori* reason, for example, why even if $F(q^2)$ is gauge-independent at $q^2 = 0$, its derivative will be also.

Several authors have tried to remedy this problem in a series of papers.[19-21] The underlying idea most of these papers have in common is to increase the number of graphs involved in the definition of $F(q^2)$, in an attempt to find the right combination that would yield gauge-independent and finite results. Due to the lack of a guiding principle, these attempts proceeded in a more or less "trial and error" fashion and their results have therefore not been entirely successful.

The root of the problem lies in the fact that, although everyone agrees that the Feynman diagrams are just convenient visualizations of a complex underlying formalism, the prevailing attitude is to treat them as individually inseparable entities. According to this logic, a Feynman diagram either contributes to $F(q^2)$ in its entirety or it does not contribute at all. This sort of logic is not part of the PT. Certain diagrams, not relevant to the definition of $F(q^2)$ at first glance, contain pieces which cannot be distinguished from the contributions of the regular graphs and must therefore be included. It is precisely the inclusion of these contributions which renders the answer gauge-independent and finite.

As will become clear in a moment, several aspects of our analysis have already appeared in the literature, especially in a recent paper by Degrassi, Sirlin, and Marciano[12] [DSM]. However, both our philosophy and--as it turns out--our results are different. Therefore, while explaining our own method, we will at the same time point out the differences between the two approaches.

We start by considering the S-matrix element between two charged target fermions and a neutrino antineutrino pair. The diagrams contributing to the process are given in Fig. 6 and we employ again the renormalizable (R_ξ) gauges. The most obvious definition for $F(q^2)$ is certainly just the Graphs 6e but the answer is gauge-dependent. The next possibility is to include Graphs 6d but this combination is gauge-dependent as well. This fact comes of course as no surprise to the reader familiar with the PT. Indeed, one immediately recognizes that the propagator-like pinch parts contained in Graphs 6c and 6f are instrumental for the gauge-independence of the γZ and ZZ self-energies of 6d and 6a. It was Degrassi, Sirlin, and Marciano who first realized the importance of the above-mentioned diagrams for the cancellation of gauge-dependences. However, since their analysis was not guided by PT,

they were eventually faced with the following dilemma: On the one hand Graphs 6c and 6f seem to be essential for gauge-independence (and they are, indeed); on the other hand they depend on the external momentum ℓ of the target and cannot be thought of as arising from a $A\bar{\nu}\nu$ effective vertex. The PT resolves this dilemma **automatically**. The important point is that it is **only** the pinch parts of 6c and 6f one really needs to consider. All gauge-dependent contributions coming from 6c and 6f are propagator-like pinch parts and are therefore independent of the external momentum ℓ. So, after all pinch parts have been removed, the rest of 6c and 6f can be discarded from the definition of F.

As for the ZZ box diagram of Figure 6g, the fact that it is individually gauge-independent, as DSM explain, can be easily understood from the point of view of PT: There is no other self-energy graph containing two Z's, the gauge-dependent pinch parts of the ZZ box could cancel against.

It is important to emphasize at this point that the advantage of our approach over the one by DSM is that after the PT has been used, no additional assumptions are necessary. In particular, we do not need to resort to an effective low-energy theory, as they did, to justify the inclusion of unwanted contributions from 6c and 6f, since all such contribution have already been naturally discarded. Therefore, our results are valid for all energies.

It is now very easy to establish a straightforward recipe for calculating electromagnetic form factors:[22]

1. Fix the gauge at $\lambda = 1$ (Feynman gauge). Of course, any other gauge will give the same answer, but this choice certainly facilitates the calculations enormously.
2. Identify all the pinch parts. Now that $\lambda = 1$ (no longitudinal pieces in the propagators), pinching occurs only at graphs containing three-

vector meson vertices.

3. Isolate those pinch parts that:

   a. are attached to the target electromagnetically (e.g. $-\gamma_\mu$);

   b. do not depend on the external momentum of the target.

4. Add the pinch parts of (3) to the regular graphs which also satisfy conditions (3a) and (3b) [see Fig. 7].

The emerging result after these four steps is both gauge-independent and finite. The generalization of this recipe to more complicated theories than the Standard Model is obvious.

One additional comment related to Step 3a of the previous procedure is now warranted. The pinch contribution from Graph 6c is:

$$P = \frac{1}{2} g^2 \bar{f}\gamma^\mu \left[\frac{1-\gamma_5}{2} \right] f \, \frac{A(q^2)}{q^2-M_Z^2} \, \bar{v}\gamma_\mu \left[\frac{1-\gamma_5}{2} \right] v \tag{5.3}$$

with

$$A(q^2) = \frac{g^2}{(2\pi)^4} \int \frac{d^4k}{(k^2-M_W^2)[(k+q)^2-M_W^2]} \tag{5.4}$$

and is obviously proportional to $\gamma^\mu\left[\frac{1-\gamma_5}{2}\right]$. In identifying the electromagnetic piece of (5.3) we must write $\gamma^\mu\left[\frac{1-\gamma_5}{2}\right]$ as a linear combination of the $A\bar{f}f$ and $Z\bar{f}f$ vertices, e.g.

$$P = \frac{1}{2} g^2 \bar{f}\gamma^\mu \left[\left(\frac{1-\gamma_5}{2} \right) - 2s_w^2 \right] f \, \frac{A(q^2)}{q^2-M_Z^2} \, \bar{v}\gamma_\mu \left(\frac{1-\gamma_5}{2} \right) v$$

$$+ g^2 s_w^2 \, \bar{f}\gamma^\mu f \, \frac{A(q^2)}{q^2-M_Z^2} \, \bar{v}\gamma_\mu \left[\frac{1-\gamma_5}{2} \right] v \tag{5.5}$$

with

$$s_w^2 = \sin^2\theta_W = 1 - \left(\frac{M_W}{M_Z} \right)^2 = 1-c_w^2 \,.$$

The first part on the RHS of (5.5) together with the contribution of the symmetric graph must be alotted to the ZZ self-energy. The second part belongs to $F(q^2)$. This particular decomposition is, of course, identical to the one appearing in the classic paper by Marciano and Sirlin,[23] now motivated by the straightforward application of the PT. The reader should also notice that (5.3) is precisely Eq. (10) of Ref. 23 if we neglect powers of q^2/M_W^2 in evaluating (5.4) as Marciano and Sirlin did.

The actual calculation of the neutrino form factor resulting from Steps 1-4 is almost immediate. All we have to do is subtract the contribution of the W^+W^- box diagram from the result by DSM.

For $F(q^2)$ we have:

$$F_{PT}(q^2) = \frac{q^2}{2c_w^2(q^2-M_Z^2)} \, \hat{\Delta}^{(v)}(q^2) \tag{5.6}$$

where we pulled out a factor $1/q^2$ to account for the virtual photon propagator. Obviously $F(0) = 0$.

The difference between $\hat{\Delta}^{(v)}(q^2)$ from (5.6) and $\Delta^{(v)}(q^2)$ of DSM is the box contribution:

$$\Delta^{(v)}(q^2) - \hat{\Delta}^{(v)}(q^2) = \frac{\alpha}{2\pi s_w^2} \left(\frac{5}{4} \right) . \tag{5.7}$$

For $\langle r^2 \rangle$ we have:

$$\langle r^2 \rangle_{PT} = 6 \left. \frac{\partial F_{PT}}{\partial q^2} \right|_{q^2=0} = - \frac{3}{M_W^2} \hat{\Delta}^{(v)}(0) = - \frac{3}{M_W^2} \left[\Delta^{(v)}(0) - \frac{\alpha}{2\pi s_w^2} \left(\frac{5}{4} \right) \right] =$$

$$= \langle r^2 \rangle_{DSM} + \left(\frac{3}{M_W^2} \right) \frac{5\alpha}{8\pi s_w^2} . \tag{5.8}$$

The difference between the two results is rather significant. Using the same values for the constants as DSM $\left[s_w^2 = 0.23,\ M_W = 81 \text{ GeV},\ \alpha(m_W^2) \approx \frac{1}{128} \right]$ we find:

$$\left(\frac{3}{M_W^2} \right) \frac{5\alpha}{8\pi s_w^2} \approx 12 \times 10^{-34} \text{ cm}^2 .$$

So instead of the DSM values:

$$\langle r^2 \rangle_{DSM} = \begin{cases} (-37.7 \times 10^{-34}) \text{ cm}^2 & \text{for } \nu_e \\ (\ -5.9 \times 10^{-34}) \text{ cm}^2 & \text{for } \nu_\mu \\ (+11.0 \times 10^{-34}) \text{ cm}^2 & \text{for } \nu_\tau \end{cases}$$

we have:

$$\langle r^2 \rangle_{PT} = \begin{cases} (-25.7 \times 10^{-34}) \text{ cm}^2 & \text{for } \nu_e \\ (+\ 6.1 \times 10^{-34}) \text{ cm}^2 & \text{for } \nu_\mu \\ (+23.0 \times 10^{-34}) \text{ cm}^2 & \text{for } \nu_\tau \end{cases} \tag{5.9}$$

We would like to point out once again that these values of $\langle r^2 \rangle$ arise naturally after the application of the PT in the Standard Model with no additional assumptions. It should prove interesting to confront the values of (5.9) with experiment some day.

5. Conclusions

In this paper we took the first step towards a consistent non-perturbative treatment of NAGT with symmetry breaking in the continuum. Since our analysis has not been extended beyond one-loop perturbation theory, the main problem of constructing a set of gauge-independent SD equations remains unresolved. Nonetheless, our results, and in particular the discovery of the new gauge independent proper self-energies and vertices satisfying naive Ward identities, provide a natural starting point for accomplishing such a task and deserve further investigation. Finally, the straightforward application of our techniques in the Standard Model gives rise to a gauge-independent and finite value for the neutrino mean-square charge radius, suitable for comparison with experimental observations.

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Figure Captions.

Fig. 1. Graphs a,b,c are some of the contributions to the S-matrix T. Graphs e and f are pinch parts, stemming from b and c respectively, which when added to the usual propagator parts (d) give a gauge-invariant effective propagator.

Fig. 2. The two graphs defining the gauge-invariant vertex $\hat{\Gamma}_\alpha^a$. The circle in Graph (a) indicates the Γ_a^F part of the three-gluon vertex.

Fig. 3. Some of the graphs giving gauge-dependent non-propagatorlike leftovers.

Fig. 4. The rest of the graphs giving gauge-dependent non-propagatorlike contributions.

Fig. 5. a: Graphical representation of the gauge-independent vertex-like pieces $\hat{\Gamma}$ and $\hat{\Lambda}$.

b: Graphical representation of the gauge-independent self-energies $\hat{\Pi}_{\mu\nu}$, $\hat{V}_\mu$, $\hat{V}_\nu$ and B.

Fig. 6. Graphs contributing to the $f\nu \to f\nu$ scattering.

Fig. 7. Graphs involved in the definition of $F(q^2)$ in the Feynman gauge. Notice the absence of box diagrams. The counterterms e,f and d have been defined in Ref. 23.

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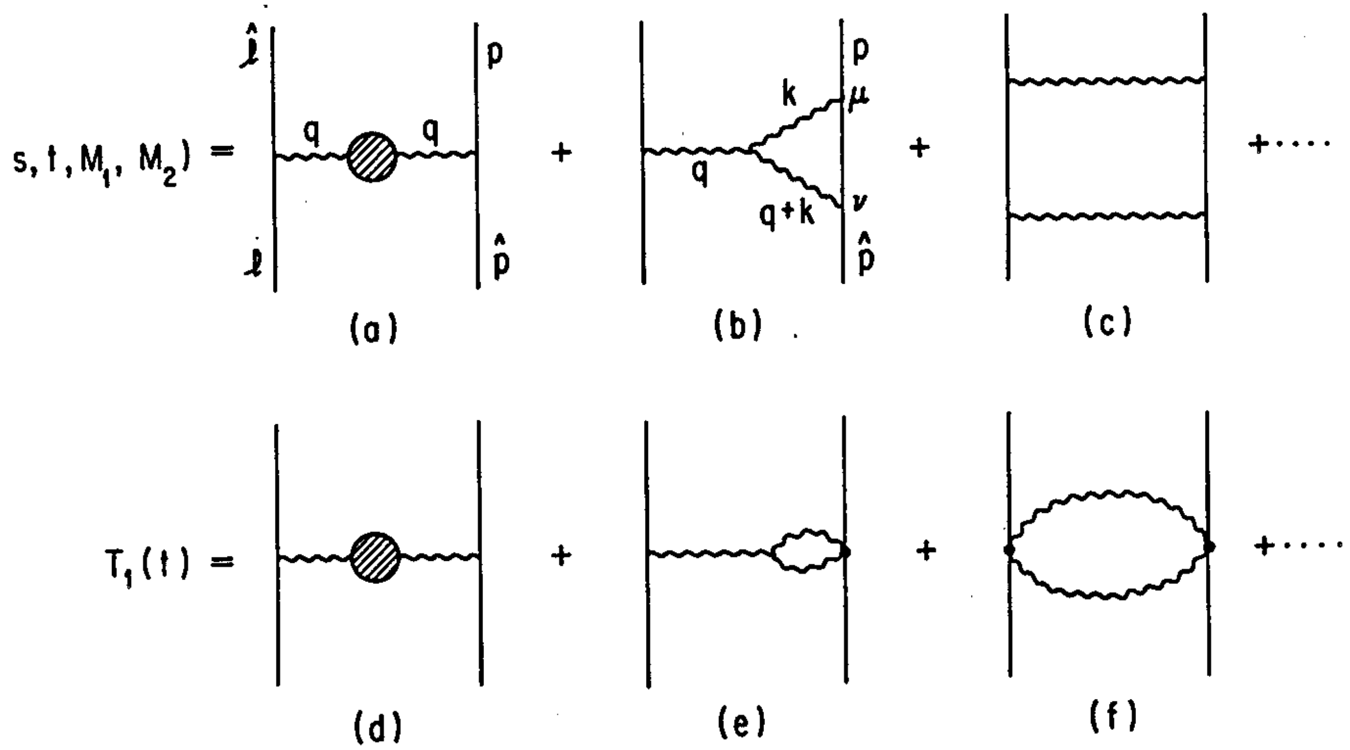


Fig. 1

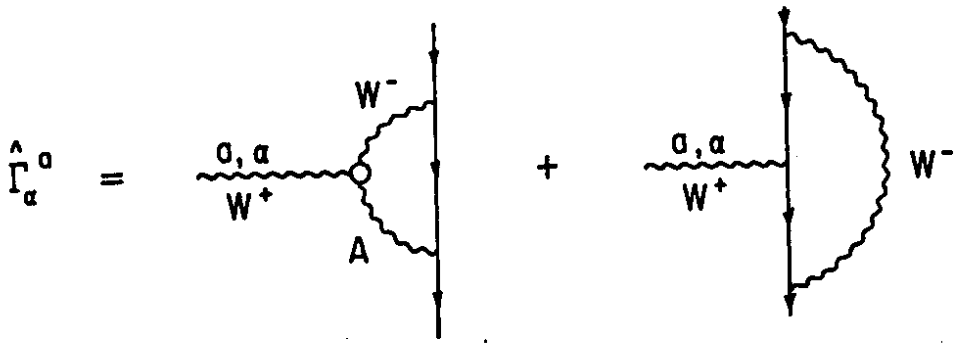


Fig. 2

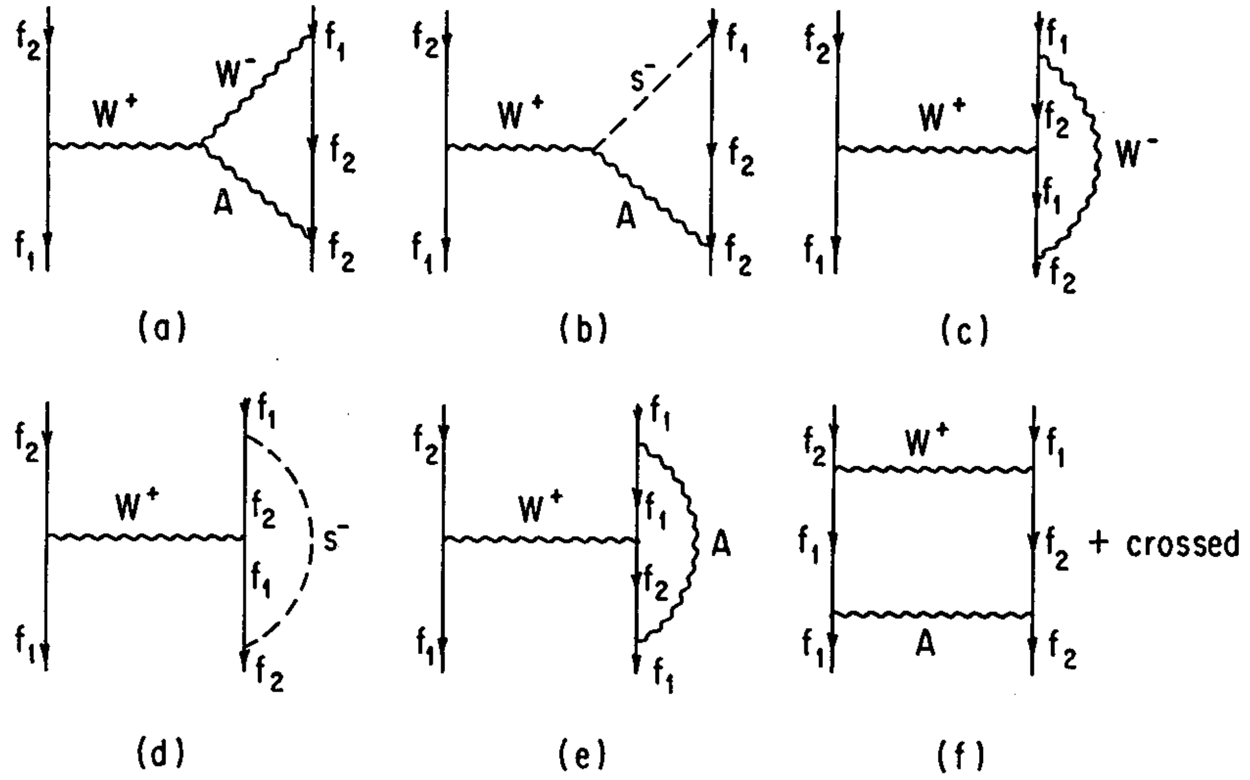
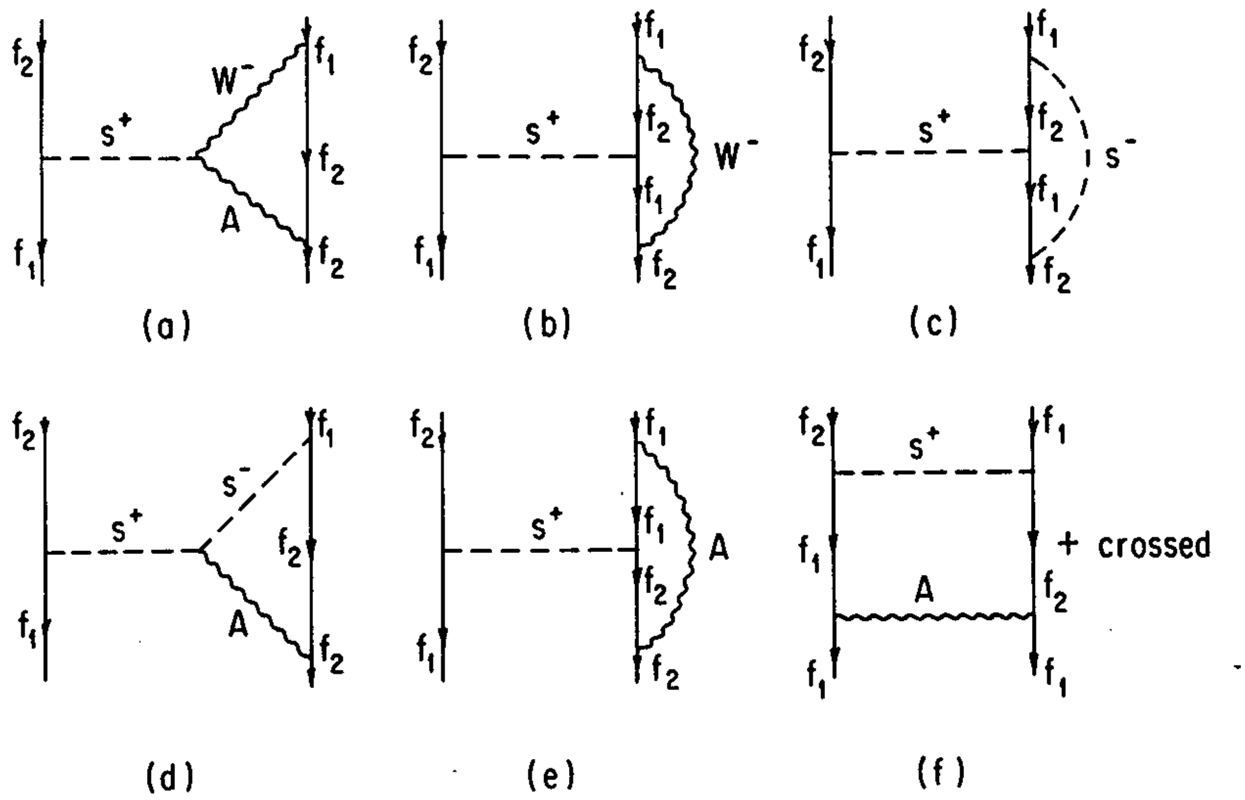


Fig. 3



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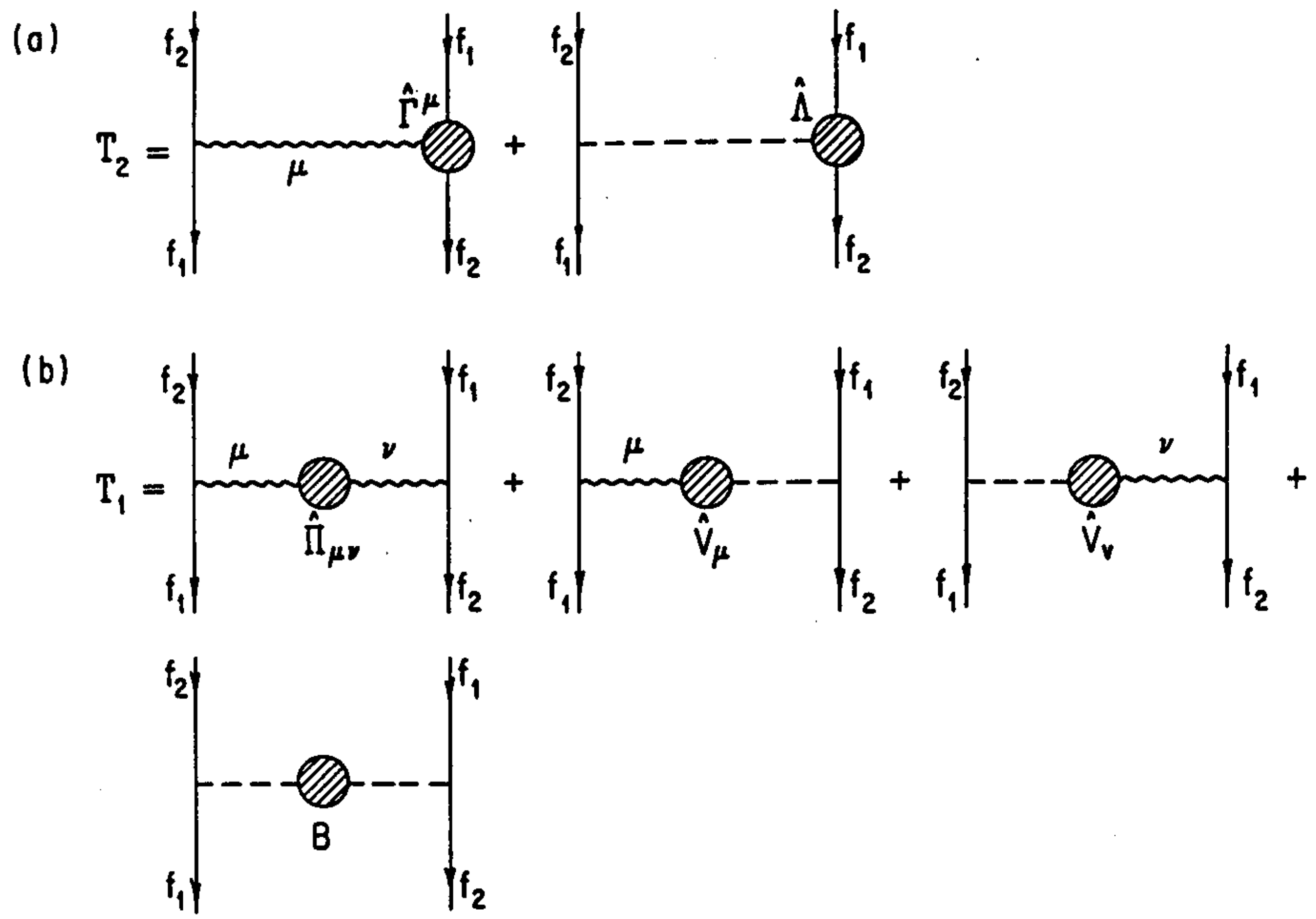
(a)
T_2 =
f_2
f_1
$\hat{\Gamma}^\mu$
μ
+
$\hat{\Lambda}$
(b)
T_1 =
μ
ν
$\hat{\Pi}_{\mu\nu}$
$\hat{V}_\mu$
$\hat{V}_\nu$
B

Fig. 5

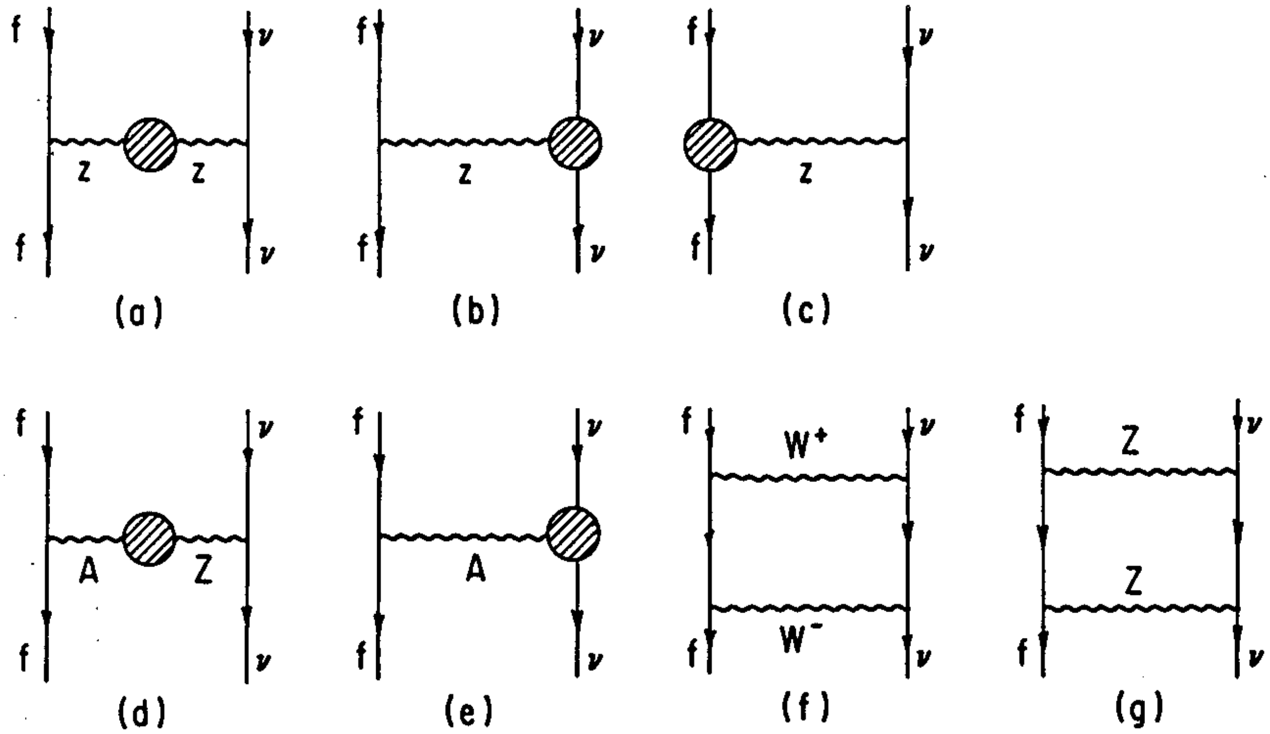
f
ν
Z
A
W^+
W^-
(a)
(b)
(c)
(d)
(e)
(f)
(g)

Fig. 6

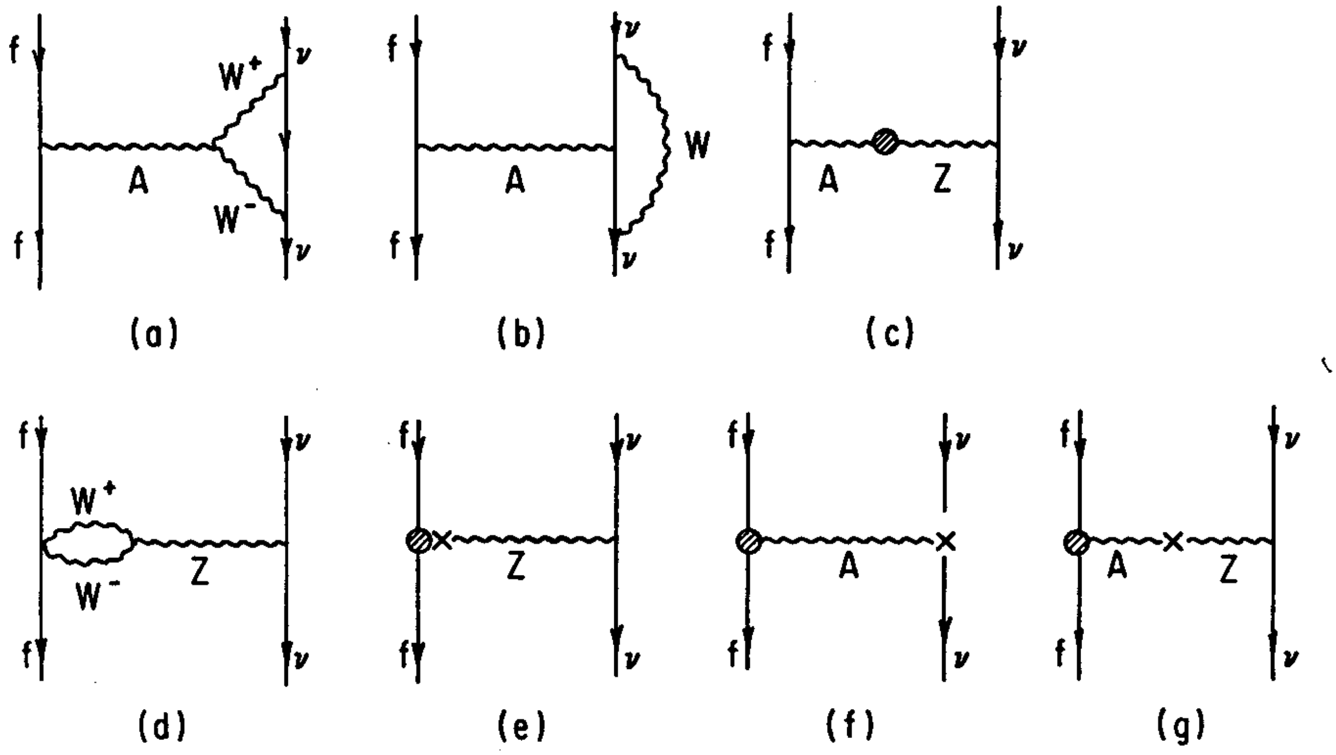

Fig. 7