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Gravitational Scattering on a Global Monopole*

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ABSTRACT

The scattering amplitude and the total scattering cross-section of massless particles propagating in the gravitational field of a global monopole are derived. We find that the physical signature of such defects is a ring-like angular region where the scattering amplitude is very large. The size of this ring-like region is determined by the ratio of the global monopole mass to the Planck mass and its appearance stems from the fact that the metric of the global monopole is not asymptotically flat but rather displays the characteristic spherical angle defect. The situation is therefore very much reminiscent of scattering in the gravitational field of the (spinning) string.

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Global monopoles are formed in the phase transitions that occur when a global symmetry is broken. The situation is similar to what happens in the condensed matter physics where in the different phases of the liquid helium ^3He one encounters a plethora of different defects classified by the homotopy groups [1]. The simplest model giving rise to global monopoles as well as global textures, is the model with the $O(3)$ order parameter consisting of a triplet of scalar fields Φ^a, $a = 1, 2, 3$. In the case of the liquid helium ^3He we may classify different kinds of defects according to the homotopy groups of the order parameter space ($O(3)$) in the increasing order: domain walls, strings, monopoles, and textures [1]. Those defects were first studied in condensed matter physics [1]. There one writes down the Landau-Ginzburg free energy for the $O(3)$ order parameter. In the field theory case, which is more relevant to the Early Universe, one replaces the L-G free energy density by the Lagrangian

$$L = \frac{1}{2}\partial_\mu \Phi^a \partial^\mu \Phi^a - \frac{1}{4}\lambda(\Phi^a\Phi^a - v^2)^2. \tag{1}$$

The generic $O(3)$ invariant configuration is given by the Skyrme hedgehog ansatz

$$\Phi^a = vf(r)n^a, n^a = \frac{x^a}{r}, \tag{2}$$

The gravitational fields produced by the axisymmetric and spherically symmetric Goldstone field configurations were studied by a number of authors [4,6,7,8] .

In this Letter we are interested in the long distance properties of the gravitational field produced by a global monopole and its effect on the scattering of massless particles, like photons of the cosmic background radiation (CBR). Unlike the magnetic monopole case, where the gauge symmetry is broken down to $U(1)_{local}$ and the total energy is finite and concentrated in the monopole core, a global monopole has a long range Goldstone field Φ^a with energy density falling off like r^{-2}, e. g. $T^0{}_0 = cv^2r^{-2}$, where c is an irrelevant constant. The static and spherically symmetric metric for a global monopole was studied in [4]. The static metric is completely determined by the Hamiltonian constraint equation $G_{00} = 8\pi G T_{00}$, or

$$R = 16\pi G\rho,$$

$$\rho = T^0{}_0. \tag{3}$$

The asymptotic behavior of the spatial metric is determined by the asymptotic behavior of the energy density ρ of the Goldstone field. For $\rho = cv^2r^{-2}$ we have $R = 16\pi cGv^2r^{-2}$. It is easy to convince oneself that the $O(3)$-invariant spatial metric has a form [4]

$$ds^2 = dr^2 + b^2r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{4}$$

where $b^2 = 1 - 8\pi cGv^2$. We notice in passing that in the two-dimensional case it is the Dirac-delta density of energy which produces the spatial metric of this type. On the equatorial plane the metric (4) is exactly the same as that of a local gauge string, in which case the azimuthal angle ϕ has the defect $\Delta = 2\pi(1-b)$ [8]. Exactly for this reason one would expect that the classical and quantum scattering on a global monopole will be reminiscent of the Sommerfeld scattering on wedges [9,10], or what is the same, the scattering on the (spinning) cosmic string [2]. The scattering of massless particles on the spinning cosmic string, first studied in [2], is equivalent to the gravitational Aharonov-Bohm effect (with some modifications caused by the wedge, or conical singularity). Another place where the spatial metric (4) appears is the spatial metric of a global texture at some time $t = t_0$ (see Spergel and Turok [5]). We expect that the scattering on a global texture, at least in the adiabatic approximation, will also bear some similarity to the characteristic scattering on a global monopole. It would be quite interesting to recognize the characteristic physical signature of a global monopole (or global texture) in the scattering of massless particles (like photons of the CBR).

In the following we will consider, for simplicity, the spin zero massless particles propagating in the gravitational field of a global monopole. We will be primarily interested in the small angle scattering amplitude for massless particles like photons of the CBR. For this reason, obviously, only the long distance behavior of the metric is important. The details of the core of a global monopole are completely irrelevant in this case. Also, the mass of a global monopole is assumed to be much smaller than the Planck mass, which implies the small defect angle Δ. We consider the covariant d'Alambert equation

$$\nabla_\mu \nabla^\mu \Phi = 0, \tag{5}$$

in the (asymptotic) gravitational field of a global monopole [4]

$$ds^2 = -dt^2 + dr^2 + b^2 r^2 (d\theta^2 + sin^2\theta d\phi^2), \tag{6}$$

where the free parameter b^2 is related to the vacuum expectation value v of the Goldstone field Φ^a, $b^2 = 1 - 8\pi c G v^2$, and, therefore, to the mass of a global monopole. Because the metric (6) is static we consider the wave of a given frequency ω

$$\Phi = e^{-i\omega t}\psi(r, \theta, \phi), \tag{7}$$

which leads to the Helmholtz type equation [2,10]

$$-\nabla^i \nabla_i \psi = \Delta\psi = \omega^2 \psi. \tag{8}$$

The covariant spatial Laplacian Δ has a form

$$\Delta = -r^{-2}\partial_r(r^2\partial_r) + (b^2 r^2)^{-1} L^2, \tag{9}$$

where L^2 is the square of the flat space angular momentum operator, or the Laplace-Beltrami operator on the two-sphere S^2. As usual in the problem of scattering in the spherically symmetric field we employ the method of separation of variables. It is obvious that only $m = 0$ spherical harmonics will enter the sum of the partial waves (10)

$$\psi(r,\theta) = \sum_{l=0}^{\infty} a_l R_l(r) P_l(cos\theta). \tag{10}$$

We reduced the problem to the radial scattering problem with the radial equation

$$R_l^{''} + 2r^{-1}R_l^{'} + (\omega^2 - l(l+1)b^{-2}r^{-2})R_l = 0. \tag{11}$$

As usual, changing variables $R_l = r^{-1/2}G_l$ leads to the radial equation whose solutions can be described in terms of the Bessel functions

$$r^2 G_l^{''} + rG_l^{'} + \left(\omega^2 r^2 - (l(l+1)b^{-2} + \frac{1}{4})\right) G_l = 0. \tag{12}$$

When $b = 1$ we have the solution corresponding to the spherical Bessel function $j_l(\omega r)$ or $r^{-1/2}J_{l+\frac{1}{2}}(\omega r)$. Actually, our scattering problem should reduce to the no-interaction situation when $b = 1$. This condition puts a strong constraint on the form of the coefficients a_l in the generic scattering solution when $b \neq 1$. When $b \neq 1$ the radial solution which is regular at $r = 0$ is

$$G_l = J_{\nu(l)}(\omega r), \tag{13}$$

where

$$\nu(l) = b^{-1}\sqrt{(l + \frac{1}{2})^2 - \frac{1-b^2}{4}}. \tag{14}$$

The general scattering solution is

$$\psi = \psi_{in} + \psi_{sc}, \tag{15}$$

where $\psi_{in} = e^{ikrcos\theta} = e^{ikz}$, and $\psi_{sc} = r^{-1}f(\theta)e^{ikr}$, as $r \to \infty$. Also, we demand that $\psi_{sc} = 0$, when $b = 1$. This condition uniquely singles out the total scattering amplitude. The total dependence on b of the scattering phase shifts $\delta_l(b)$ in the scattering amplitude is due to the dependence of the radial functions G_l on b (in the order of the Bessel function $\nu(l)$). Standard arguments described above lead to the following scattering amplitude $f(\theta), (\omega = k)$

$$f(\theta) = \frac{1}{2i\omega} \sum_l^{\infty} (2l+1)(e^{2i\delta_l} - 1) P_l(cos\theta), \tag{16}$$

where the phase shift δ_l depends on the "defect angle parameter" b in the following way

$$\delta_l(b) = \frac{\pi}{2}\left(l + \frac{1}{2} - \nu(l)\right) = \frac{\pi}{2}\left(l + \frac{1}{2} - b^{-1}\sqrt{(l+\frac{1}{2})^2 - \frac{1-b^2}{4}}\right). \tag{17}$$

The standard expression for the differential scattering cross-section in terms of $f(\theta)$ is:

$$\frac{d\sigma}{sin\theta d\theta d\phi} = |f(\theta)|^2. \tag{18}$$

We are really interested in the small angle behavior of the differential scattering cross-section, or the scattering amplitude $f(\theta)$. This is because we know only the asymptotic form of the metric of a global monopole. From the small angle behavior of $f(\theta)$ we are able to infer the total scattering cross-section σ as a function of the "defect parameter" b via the optical theorem

$$\sigma = \frac{4\pi}{k} Imf(0). \tag{19}$$

The phase shift δ_l is not a simple function of l (or $l + \frac{1}{2}$), and because of that the exact form of the scattering amplitude $f(\theta)$ is not available at hand. However, at small angles $\theta = O(\Delta)$, where Δ is the "defect angle" which is much less than one, we have a systematic asymptotic expansion for the scattering amplitude $f(\theta)$. In fact, $f(\theta)$ in this regime of angles θ can be easily calculated. In order to evaluate the scattering amplitude $f(\theta)$ it is useful first to change variables

$l \to z = l + \frac{1}{2}$. The reason for that change of variables is that for large z the phase shift $\delta(z)$ has a simple expansion

$$\delta(z) = \frac{\pi}{2}\left(z - b^{-1}\sqrt{z^2 - a^2}\right) = \frac{\pi}{2}z(1 - b^{-1}) + O(z^{-1}), \tag{20}$$

where $a^2 = \frac{1-b^2}{4}$. In the case of interest a^2 is very small and may serve as a convenient expansion parameter. In general, we can write $\delta(z) = \delta^{(0)}(z) + \delta^{(1)}(z) + ... = \sum_{n=0}^{\infty}\delta^{(n)}(z)$, where $\delta^{(0)}(z) = \frac{\pi}{2}(1-b^{-1})z$, and $\delta^{(n)}(z) = O(z^{-2n+1})$. The next to the leading term $\delta^{(0)}(z)$ in this expansion is proportional to a^2 and so on. The exponent of the phase shift is suitable for the power expansion

$$e^{2i\delta(z)} = e^{2i\delta^{(0)}(z)} \times \exp(2i\sum_{n=1}^{\infty}\delta^{(n)}(z)). \tag{21}$$

The second term in the product (21) can be expanded systematically in the powers of a^2. Below we calculate only the leading and the next to the leading contributions to the scattering amplitude $f(\theta)$. Those are the only "singular" contributions which are significant for the small angles θ. This means that only the following terms in the expansion will be important

$$\delta(z) = \frac{\pi}{2}\left((1 - b^{-1})z + \frac{1}{2}a^2 b^{-1} z^{-1} + O(z^{-3})\right). \tag{22}$$

We find that the scattering amplitude can be expressed in terms of the function $h(\theta, \alpha)$ defined as follows

$$h(\theta, \alpha) = \sum_{l=0}^{\infty} e^{\pi i \alpha z(l)} P_l(cos\theta) = \frac{1}{\sqrt{2(cos\pi\alpha - cos\theta)}}, \tag{23}$$

where we have used the generating function for the Legendre polynomials

$$(1 - 2t cos\theta + t^2)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} t^l P_l(cos\theta). \tag{24}$$

The leading contribution to the scattering amplitude $f^{(0)}(\theta)$ is proportional to the derivative of $h(\theta,\alpha)$ with respect to α, where $\alpha = 1 - b^{-1}$

$$f^{(0)}(\theta) = \frac{1}{2\sqrt{2}k} sin\pi\alpha(cos\pi\alpha - cos\theta)^{-3/2}. \tag{25}$$

From the optical theorem (19) we obtain easily the leading contribution $\sigma^{(0)}$ to the total scattering cross-section $\sigma_{tot} = \sigma^{(0)} + \sigma^{(1)} + ...$,

$$\sigma^{(0)} = \frac{\pi}{k^2} \frac{cos\frac{\pi\alpha}{2}}{sin^2\frac{\pi\alpha}{2}}. \tag{26}$$

The next to the leading order contribution to the scattering amplitude $f^{(1)}(\theta)$ comes from the expansion of (21) up to the $O(a^2)$ term

$$f^{(1)}(\theta) = \frac{\pi a^2}{2bk} h(\theta,\alpha). \tag{27}$$

The higher order corrections are given in terms of integrals of $h(\theta,\alpha)$ with respect to α and they are obviously nonsingular at $\theta = \theta_0 = \pi\alpha$. This means that their contribution to the total scattering cross-section will be negligible term by term for the small defect parameter α. From (21) we find that the contribution to the cross-section is negative and equal

$$\sigma^{(1)} = -\frac{\pi a^2}{bk^2} \frac{1}{sin\frac{\pi\alpha}{2}}. \tag{28}$$

The higher order in a^2 contributions to σ_{tot} are in principle calculable but are very small for all practical purposes. This completes our calculation of the scattering amplitude. We observe that the amplitude $f(\theta)$ has a universal singular behavior for angles θ close to $\theta_0 = \pi\alpha$. This is the basic physical signature of a global monopole in the gravitational scattering. The amplitude is very large in the ring-like region with the angular size of order of θ_0. In principle, by measuring the size θ_0 of this ring-like region we can determine the parameter b in the asymptotic metric of the global monopole, and, therefore, the mass of the monopole.

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