BROOKHAVEN NATIONAL LABORATORY

September, 1992 BNL – 48001

The Gauge Invariant Four-Gluon Vertex and its Ward Identity.

Joannis Papavassiliou

Department of Physics
Brookhaven National Laboratory
Upton, New York
N. Y. 11973

ABSTRACT

We use the S-matrix pinch technique to construct at one loop order a four-gluon vertex for QCD, which is independent of the choice of the gauge parameter, when one or more of the incoming momenta are off-shell. We discuss some of the technical subtleties in the application of the pinch technique and show that this vertex satisfies a very simple Ward identity, relating it to a previously costructed gauge independent three-gluon vertex, also found with the same technique. This analysis serves as a prelude to the costruction of an effective potential for QCD, which is gauge independent order by order in the dressed-loop expansion.

This manuscript has been authored under contract number DE-AC02-76CH00016 with the U.S. Department of Energy. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes.

1. Introduction.

The pinch technique was first introduced by Cornwall over a decade ago [1] and has received considerable interest ever since. The original motivation was to device a consistent truncation scheme for the Schwinger-Dyson (S.D.) equations that govern the dynamics of gauge theories. These equations are inherrently non-perturbative and could in principle provide important information about a plethora of phenomena in non-Abelian gauge theories not captured by perturbation theory. In practice however, one is severely limited in exploiting them, mainly because they constitute an infinite set of coupled non-linear integral equations. Even though the need for a truncation scheme is evident, particular care is needed for respecting the crucial property of gauge invariance. Indeed, the S.D. equations are conventionally built out of gauge dependent Green's functions. Since the mechanism of gauge cancellation is very subtle and involves a delicate consiracy of terms coming from all orders, a casual trancation of the series often gives rise to gauge dependent approximations for ostencibly gauge independent quantities. The pinch technique attempts to address this problem in its root, namely the building blocks of the S.D. equations. According to this approach, the Feynman graphs contributing to a given gauge invariant process are resummed into new propagators and vertices where the gauge dependence has been reduced to an absolute minimum - that of the free gluon propagator. The proper self-energy of the new propagator and the new vertices are themselves gauge independent and as it turns out so are the S.D. equations governing these new Green's functions. These new S.D. equations are in general more complicated than the usual ones because of the presence of extra terms which enforce gauge invariance. Nonetheless, it is possible to truncate them, usually by keeping only a few terms of a dressed loop expansion, and maintain exact gauge invariance, while at the same time accommodating non-perturbative effects. One very important aspect of gauge invariance in the context of S.D. equations is that the Green's functions defined via the pinch technique satisfy simple QED like Ward identities.This feature is very important since it enables the cancellation of the final gauge dependences stemming from the free parts of the gluon propagators entering in the expressions for the S.D. equations.

Although the program described above is not yet complete, several interesting results have been obtained by the application of P.T. in a variety of physical problems. In the context of QCD a gauge invariant gluon self energy was derived, and its S.D. equation constructed and solved for $T = 0$ [2], as well as finite T [3]. The plasmon decay rate was also calculated at finite T using the same method [4]. Later the QCD gauge invariant three gluon vertex was calculated at one loop level and was shown to satisfy a very simple Ward dentity [5]. Finally, the subleading corrections to the self-energy were calculated by Lavelle [6]. The P.T. was also extended to the case of non-Abelian gauge theories with spontaneously broken gauge symmetry (with elementary Higgs) in the context of a toy field theory based on $SU(2)$ and a gauge independent electromagnetic form factor for the Standard Model neutrino was constructed [7]. The most recent contribution known to the author is that by Degrassi and Sirlin [8], who derived one loop gauge invariant self energies and vertices for the Standard Model and proposed to identify the pinch parts with the contributions of equal time commutators in the relevant Ward identities.

As we already mentioned the upshot of this program is the construction of a S.D. series which is manifestly gauge independent even in its one dressed loop truncated version. The systematic derivation of such a series for QCD has been the focal point of extensive research. In a ghost free gauge, the usual S.D. equations for quarkless QCD are build out of three basic quantities; the gluon propagator Δ , the three gluon vertex Γ_3, and the four gluon vertex Γ_4. One may construct the effective potential Ω [9-10], a functional of the three Greens functions, and then extremise independently the variations of $\Omega(\Delta, \Gamma_3, \Gamma4)$ with respect to Δ, Γ_3 and Γ_4, e.g. $\frac{\delta\Omega}{\delta\Delta} = 0$, $\frac{\delta\Omega}{\delta\Gamma_3} = 0$ and $\frac{\delta\Omega}{\delta\Gamma_4} = 0$ The resulting expresions will be the corresponding S.D. equations for Δ, Γ_3 and Γ_4. In such a picture the solutions to the S.D. equations will in general be gauge dependent in a non-trivial way. If one could solve the entire renormalised S.D. series and then substitute the resulting gauge dependent solutions $\bar{\Delta}$, $\bar{\Gamma}_3$ and $\bar{\Gamma}_4$ back into $\Omega(\Delta, \Gamma_3, \Gamma_4)$ and calculate its value, $\Omega(\bar{\Delta}, \bar{\Gamma}_3, \bar{\Gamma}_4)$, the final answer would be gauge independent, since Ω is a physical quantity (vacuum energy). The way this gauge cancellations would manifest themselves is complicated and involves non-trivial mixing of all orders. However, since solving the entirte S.D. is practically impossible, some form of truncation is necessary. The minimum requirement for any such truncation scheme must be that the solutions of the truncated S.D. equations, when substituted into Ω, should still preserve its gauge invariance. Unfortunately this is not the case if one truncates the series without a particular guiding principle. The alternative approach that has been proposed [11] is to demand from the beginning that the effective potential $\hat{\Omega}(\hat{\Delta}, \hat{\Gamma}_3, \hat{\Gamma}_4)$ as well as the individual expresions for the self energy $\hat{d}$, for $\hat{\Gamma}_3$ and for $\hat{\Gamma}_4$ be gauge independent order by order in the dressed loop expansion. (We use hats to indicate that these expresions are in general different from their conventionally derived unhatted counterparts). Assuming that $\hat{d}$, $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$ are individually gauge independent is not sufficient however to guarantee the order by order gauge independence of $\hat{\Omega}$, because there is a residual dependence on the gauge fixing parameter coming from the free part of the propagators $\hat{\Delta}$ entering in the expression for $\hat{\Omega}$. The necessary and sufficient condition for the order by order cancellation of the residual gauge dependence is that the renormalized self energy $\hat{\Pi}_{\mu\nu}$ is transverse, e.g.

$$q^{\mu}\hat{\Pi}_{\mu\nu} = 0 \tag{1.1}$$

order by order in the dressed expansion. The one loop expresion for $\hat{\Pi}_{\mu\nu}$ in a ghost free gauge is given by the two graphs in Fig. 1 (There is an additional ghost graph in covariant gauges). We see that already at the one loop level the fully dressed expressions for $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$ make their appearance. The second graph would be zero in dimensionally regularized perturbation theory, but that need not be the case beyond perturbation theory (see ref [2]). It turns out that Eq. (1.1) can be satisfied as long as $\hat{d}$, $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$ satisfy the following Ward identities:

$$q_1^{\mu}\hat{\Gamma}_{\mu\nu\alpha}(q_1,q_2,q_3) = P_{\nu\alpha}(q_2)\,\hat{d}^{-1}(q_2) - P_{\nu\alpha}(q_3)\,\hat{d}^{-1}(q_3) \tag{1.2}$$

and

$$q_1^{\mu}\hat{\Gamma}^{abcd}_{\mu\nu\alpha\beta} = f_{abp}\hat{\Gamma}^{cdp}_{\nu\alpha\beta}(q_1+q_2,q_3,q_4) + c.p. \tag{1.3}$$

with $P_{\mu\nu}$ the projection operator defined in Eq. (2.12) and f^{abc} the structure constants of the gauge group. If Eq. (1.2) and Eq. (1.3) are satisfied, than $\hat{\Omega}$ is manifestly gauge independent order by order in the dressed loop expansion and so are the S.D. equations generated after its variation. Once solved they will give rise to gauge independent $\hat{d}$, $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$. Finally, the self-consistency of the whole program requires that, in addition to beeing gauge independent, $\hat{d}$, $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$ satisfy the correct Ward identities, namely Eq. (1.2) and Eq. (1.3), whose validity was assumed.

Although this program has been layed out conceptually, its practical implementation is as yet incomplete. If Green's functions with the properties described above can arise out of a self-consistent treatment of QCD, one should be able to construct Green's functions with the same properties at the level of ordinary perturbation theory, after appropriate resumation. The pinch technique accomplishes this task by providing the systematic algorithm needed to recover the desired Green's functions order by order in perturbation theory.

Both the gluon self-energy and the three-gluon vertex have already been studied in detail in references [2], [3], and [5] in the context of the pinch technique at one loop order; explicit gauge independent expressions were derived and the validity of the Ward identity Eq. (1.2) was verified. On the contrary, very little has been said thus far about the new four gluon vertex, even at the level of perturbation theory.

In this paper we apply the pinch technique to the case of the of the four gluon vertex. This is a non-trivial task, not only because of the large number of graphs, but also because certain subtleties of the pinch technique, not encounterted before, need be discused. We show that

a) At one loop level the pinch technique gives rise to a gauge independent four gluon vertex.

b) The new four gluon vertex satisfies the Ward identity given in Eq. (1.3).

The paper is organised as follows. In section 2 we review the S-matrix pinch technique and discuss some of the most relevant results for our purposes. In section 3 we present the analysis for the construction of the gauge independent four gluon vertex and discuss the subtleties involved in the application of the pinch technique in this case. In section 4 we prove the Ward identity relating the gauge independent four gluon vertex constructed in the previous section to the gauge independent three gluon vertex of reference [5]. Finally, we present our conclusions in section 5.

2. The Pinch Technique.

In this section we briefly review the S-matrix P.T. and summarise results obtained in the past few years, partially in an attempt to establish notation and partially because we will need the results presented in this section in the analysis that follows. In particular we outline the method of derivation of a gauge independent proper self-energy and a gauge independent proper three gluon vertex, and comment on the simple QED-like Ward identity relating them.

The S-matrix pinch technique is an algorithm that allows the construction of modified gauge independent n-point functions, through the order by order resummation of Feynman graphs contributing to a certain physical and therefore ostencibly gauge independent process (an S-matrix in our case). The simplest example that demonstrates how the P.T. works is the two point function (gluon propagator).

A. Gluon Self-Energy

Consider the S-matrix element T for the elastic scattering of two test quarks of masses M_1 and M_2. To any order in perturbation theory T is independent of the gauge fixing parameter λ, defined by the free gluon propagator

$$\Delta_{\mu\nu}(q) = \frac{-g_{\mu\nu} + (1-\lambda)\frac{q_\mu q_\nu}{q^2}}{q^2} \tag{2.1}$$

On the other hand, as an explicit calculation shows, the conventionally defined proper self-energy (collectively depicted in graph 2a) depends on λ. At the one loop level this dependence is canceled by contributions from other graphs, like 2b and 2c, which do not seem to be of propagator type at first glance. That this must be so is evident from the form of T:

$$T(s,t,M_1,M_2) = T_1(t) + T_2(t,M_1,M_2) + T_3(s,t,M_1,M_2) \tag{2.2}$$

where the function $T_1(t)$ depends only on the Mandelstam variable $t = -(\hat{p}_1 - p_1)^2 = -q^2$, and not on $s = (p_1+p_2)^2$ or on the external masses. The propagator-like parts of graphs like 1e and 1f, which enforce the gauge independence of $T_1(t)$, are called "pinch parts". The pinch parts emerge every time a gluon propagator or an elementary three gluon vertex contribute a longitudinal term k_μ to the original graph's numerator. The action of such a term is to trigger an elementary Ward identity of the form

$$\begin{aligned} k^\mu\gamma_\mu \equiv \not{k} &= (\not{p} + \not{k} - m) - (\not{p} - m) \\ &= S^{-1}(p+k) - S^{-1}(p) \end{aligned} \tag{2.3}$$

once it gets contracted with a γ matrix. The first term on the right-hand side of Eq. (2.3) will remove the internal fermion propagator - that is a "pinch" - whereas $S^{-1}(p)$ vanishes on shell. This last property characterises the S-matrix P.T. we will use troughout this paper. Returning to the decomposition of Eq. (2.2), the function $T_1(t)$ is gauge invariant and unique and represents the contribution of the new propagator. We can construct the new propagator, or equivalently $T_1(t)$, directly from the Feynman rules. In doing so it is evident that any value for the gauge parameter λ may be chosen, since T_1, T_2, and T_3 are all independent of λ. The simplest of all covariant gauges is certainly the Feynman gauge ($\lambda = 1$), which gets rid of the longitudinal part of the gluon propagator. Therefore, the only possibility for pinching arises from the four-momentum of the three-gluon vertices, and the only propagator-like contributions come from graph 2b.

To explicitly calculate the pinching contribution of a graph such as 2b it is convenient to decompose the vertex in the following way, first proposed by 't Hooft. Group theory factors aside,

$$\Gamma_{\mu\nu\alpha} = \Gamma^P_{\mu\nu\alpha} + \Gamma^F_{\mu\nu\alpha} \tag{2.4}$$

with

$$\Gamma^P_{\mu\nu\alpha} \equiv (q+k)_\nu \, g_{\mu\alpha} + k_\mu g_{\nu\alpha} \tag{2.5}$$

and

$$\Gamma^F_{\mu\nu\alpha} \equiv 2q_\mu g_{\nu\alpha} - 2q_\nu g_{\nu\alpha} - (2k+q)_\alpha \, g_{\mu\nu} \tag{2.6}$$

$\Gamma^F_{\mu\nu\alpha}$ satisfies a Feynman-gauge Ward identity:

$$q^\alpha \Gamma^F_{\mu\nu\alpha} = \left[k^2 - (k+q)^2\right] g_{\mu\nu} \tag{2.7}$$

where the RHS is the difference of two inverse propagators. As for $\Gamma^P_{\mu\nu\alpha}$ (P for "pinch") it gives rise to pinch parts when contracted with γ matrices

$$\begin{aligned} g_{\mu\alpha}\left(\not{q} + \not{k}\right) =& g_{\mu\alpha}\left[(\not{p} + \not{q} - m) - (\not{p} - \not{k} - m)\right] \\ =& g_{\mu\alpha}\left[S^{-1}(p+q) - S^{-1}(p-k)\right] \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} g_{\nu\alpha}\not{k} =& g_{\nu\alpha}\left[(\not{p} - m) - (\not{p} - \not{k} - m)\right] \\ =& g_{\nu\alpha}\left[S^{-1}(p) - S^{-1}(p-k)\right] \end{aligned} \tag{2.9}$$

Now both $S^{-1}(p+q)$ and $S^{-1}(p)$ vanish on shell, whereas the two terms proportional to $S^{-1}(p-k)$ pinch out the internal quark propagator in graph 2b. The total pinch contribution $\Pi^P(q)$ from graph 2b and its counterpart mirror image graph with the bubble attached to the left line is given by:

$$\begin{aligned}\Pi^{P}(q) &= \left(\frac{1}{2}N\right)\times 2\times 2\times\left[\frac{ig^{2}}{(2\pi)^{4}}\right]\int\frac{d^{4}k}{k^{2}(k+q)^{2}} \\ &= -\frac{2Ng^{2}}{16\pi^{2}}\ln\left(\frac{-q^{2}}{\mu^{2}}\right)\end{aligned} \tag{2.10}$$

where in the second equality we give the renormalized version of the integral. The factors in front of the integral are a group-theoretic factor $\frac{1}{2}N$ [N = number of colors in $SU(N)$]; one factor of 2 for the two pinching terms from Eq. (2.8) and Eq. (2.9); another factor of 2 from the contribution of the mirror graph. In adding the pinch parts to the usual gluon self-energy one ambiguity needs resolution. Because we are working with the on-shell S-matrix, any terms $\sim q_{\mu}q_{\nu}$ in the pinch parts do not show up in $T_1(t)$. We define uniquely the proper self energy associated with the pinch parts by demanding that it be conserved [12]. So we define $\Pi^{P}_{\mu\nu}(q)$ as

$$\Pi^{P}_{\mu\nu}(q) = P_{\mu\nu}(q)\,\Pi^{P}(q) \tag{2.11}$$

where

$$P_{\mu\nu}(q) \equiv -q^{2}g_{\mu\nu} + q_{\mu}q_{\nu} \tag{2.12}$$

Adding this to the usual Feynman-gauge proper self energy

$$\Pi^{(\lambda=1)}_{\mu\nu}(q) = P_{\mu\nu}(q)\,\Pi^{(\lambda=1)}(q) \tag{2.13}$$

with

$$\Pi^{(\lambda=1)}(q) = -\frac{5}{3}N\frac{g^{2}}{16\pi^{2}}\ln\left(\frac{-q^{2}}{\mu^{2}}\right) \tag{2.14}$$

we find for $\hat{\Pi}_{\mu\nu}(q)$ the gauge invariant combination:

$$\hat{\Pi}_{\mu\nu}(q) = P_{\mu\nu}(q)\,\hat{\Pi}(q) \tag{2.15}$$

with

$$\hat{\Pi}(q) = -bg^{2}\ln\left(\frac{-q^{2}}{\mu^{2}}\right) \tag{2.16}$$

and

$$b = \frac{11N}{48\pi^2} \tag{2.17}$$

the coefficient of $-g^3$ in the usual one loop β function. Finally, the full modified propagator $\hat{\Delta}_{\mu\nu}(q)$ at one-loop order reads

$$\hat{\Delta}_{\mu\nu}(q) = P_{\mu\nu}(q)\,\hat{d}(q) - \lambda\frac{q_\mu q_\nu}{q^4} \tag{2.18}$$

with

$$\begin{aligned} \hat{d}^{-1}(q) =& 1 - \hat{\Pi}(q) \\ =& 1 + bg^2 \ln\left(\frac{-q^2}{\mu^2}\right) \end{aligned} \tag{2.19}$$

We see that the modified propagator has a gauge independent self energy and only a trivial gauge dependence originating from the tree part given by Eq. (2.1).

B. The 3-gluon vertex

The gauge independent three gluon vertex is constructed by considering the connected S-matrix element for scattering of three test quarks of arbitrary masses. As with the propagator $\hat{\Delta}_{\mu\nu}$ the Feynman gauge $\lambda = 1$ is the most convenient. The relevant graphs in this gauge are given in Ref [5]. We can extract a gauge independent improper vertex by identifying all the parts of Feynman graphs which are independent of the external momenta p_i and $\hat{p}_i$, and the test quark masses M_i, and depend only on the momentum transfers $q_i = \hat{p}_i - p_i$. The sum $\hat{T}(q_1, q_2, q_3)$ of all these contributions has the form

$$\hat{T}(1,2,3) \sim \bar{u}_1\gamma_\rho u_1 \bar{u}_2\gamma_\sigma u_2 \bar{u}_3\gamma_\tau u_3 \hat{\Delta}^{\rho\mu}(1)\,\hat{\Delta}^{\sigma\nu}(2)\,\hat{\Delta}^{\tau\alpha}(3)\,\hat{\Gamma}_{\mu\nu\alpha} \tag{2.20}$$

where the propagators $\hat{\Delta}$ are now the new ones defined in Eq. (2.18). $\hat{T}$ is gauge invariant, and the trivial gauge dependence of $\hat{\Delta}$ in Eq. (2.18) does not appear in $\hat{T}$, since the external quarks are on their mass shell ($\hat{\not{p}}_i = \not{p}_i = M_i$). So we can recover the gauge independent $\hat{\Gamma}_{\mu\nu\alpha}$ from $\hat{T}$ by stripping off the $\hat{\Delta}_{\mu\nu}$ as if they had no $q_\mu q_\nu$ terms at all. The final vertex has full Bose symmetry and has the form [13]

$$\begin{aligned} \hat{\Gamma}_{\mu\nu\alpha} =& \Gamma^{(\lambda=1)}_{\mu\nu\alpha} - \frac{1}{2}\Gamma_{\mu\nu\alpha}\left[\Pi^P(q_1) + \Pi^P(q_2) + \Pi^P(q_3)\right] \\ &+ \left[V^\rho_{\nu\alpha}\Delta_0{}^{-1}_{\rho\mu}(q_1) + c.p.\right] \end{aligned} \tag{2.21}$$

In the formula above $\Gamma_{\mu\nu\alpha}$ is the bare three vertex, $\Gamma^{(\lambda=1)}_{\mu\nu\alpha}$ is the contribution of the usual graphs in the Feynman gauge. The third term, as the explicit presence of the free inverse propagator Δ_0^{-1} indicates, contains the pinch contributions from graphs like 3a and 3c, graphically shown in 3b and 3d respectively. It is important to notice the relative minus sign of the second term and that the third term is ultraviolet finite. The exact closed form of $\hat{\Gamma}_{\mu\nu\alpha}$ is lengthy and has been reported in Ref [5]. We will not reproduce it here.

$\hat{\Gamma}_{\mu\nu\alpha}$ was shown to obey a simple Ward identity:

$$q_1^{\mu}\hat{\Gamma}_{\mu\nu\alpha}(q_1,q_2,q_3) = P_{\nu\alpha}(q_2)\,\hat{d}^{-1}(q_2) - P_{\nu\alpha}(q_3)\,\hat{d}^{-1}(q_3) \tag{2.22}$$

with similar Ward identities upon multiplication of $\hat{\Gamma}_{\mu\nu\alpha}$ by q_2^{ν} or q_3^{α}. It is very important to notice that the Ward identity of Eq. (2.22) makes no reference to ghost Green's functions as the usual covariant-gauge Ward identities do. Finally we note that with the exception of ghost-free gauges the RHS of Eq. (2.22) is not the difference of two inverse propagators, because the projection operators $P_{\mu\nu}$ have no inverse.

3. The four-gluon vertex.

In the previous section we established the general rules of the S-matrix pinch technique and showed how a gauge independent self energy and three gluon vertex can be constructed. In this section we use these rules to construct the gauge independent four gluon vertex at one loop order. This is a more formidable task mainly because of the larger number of Feynman graphs involved. In addition new complications arise from the fact that one-particle irreducible and one-particle reducible graphs exchange contributions in a non-trivial way that we will explain shortly. Our analysis will be essentially diagramatic and we will mainly focus on the subtleties of the method. The main points are the following:

a) The bare four-gluon vertex $\Gamma^{abcd}_{\mu\nu\alpha\beta}$ is given by

$$\begin{aligned}\Gamma^{abcd}_{\mu\nu\alpha\beta} &= f^{abe}f^{cde}\left(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}\right)\\ &+ f^{ace}f^{dbe}\left(g_{\mu\rho}g_{\lambda\nu} - g_{\mu\nu}g_{\lambda\rho}\right)\\ &+ f^{ade}f^{bce}\left(g_{\mu\nu}g_{\rho\lambda} - g_{\mu\lambda}g_{\rho\nu}\right)\end{aligned} \tag{3.1}$$

and is related to the three gluon bare vertex $\Gamma^{abc}_{\mu\nu\alpha}$ by the following elementary Ward identity:

$$\begin{aligned}q_1^{\mu}\Gamma^{abcd}_{\mu\nu\alpha\beta} &= f_{abp}\Gamma^{cdp}_{\nu\alpha\beta}\left(q_1+q_2, q_3, q_4\right)\\ &+ f_{acp}\Gamma^{bdp}_{\nu\alpha\beta}\left(q_2, q_1+q_3, q_4\right)\\ &+ f_{adp}\Gamma^{bcp}_{\nu\alpha\beta}\left(q_2, q_3, q_1+q_4\right)\end{aligned} \tag{3.2}$$

and similarly for any other of the q_2^{ν}, q_3^{α} and q_4^{β} (Fig. 4a). Both $\Gamma_{\mu\nu\alpha}$ and $\Gamma_{\mu\nu\alpha\beta}$ are manifestly gauge independent at tree level. If we consider the usual one loop corrections to these vertices they become gauge dependent. Moreover Eq. (3.2) is not satisfied any more. As we will show in the next section Eq. (3.2) can be generalised to one loop for the gauge independent three- and four-gluon vertices constructed via the pinch technique.

b) We consider the S-matrix element of four test "quarks" of arbitrary masses. The external fermions are considered to be on shell, e.g. $\not{p}_i = \not{p}_i - \not{q}_i = M_i$. The proper 4-gluon vertex will be extracted from the part of the S-matrix $\hat{T}(q_1, q_2, q_3, q_4)$ that only depends on the momentum tranfers q_i. The general form of $\hat{T}(q_1, q_2, q_3, q_4)$ is shown in Fig 4b. As before we use the Feynman gauge $\lambda = 1$ since it involves the smallest number of graphs. This is no loss of generality since $\hat{T}(q_1, q_2, q_3, q_4)$ is ostencibly gauge independent. Moreover, we only keep the graphs of quark-less QCD (no quark loops), since the quark contributions are individually gauge-independent and do not affect the Ward identity.

c) The Feynman graphs contributing to $\hat{T}(q_1, q_2, q_3, q_4)$ in the Feynman gauge are shown in Fig 5. Graph 5a stands collectively for the one particle irreducible (1PI) graphs, explicitly shown in Fig 6. (see also Ref [14]). Graph 5b stands for all the one particle reducible (1PR) graphs, like those explicitly shown in 7a, 7b and 7e. One particle reducible graphs certainly contribute to $\hat{T}(q_1, q_2, q_3, q_4)$ and are essential for its gauge independence.

Finally, graphs like 5c and 5d will contribute to $\hat{T}(q_1, q_2, q_3, q_4)$ only trough their pinched four vertex-like pieces (shown in 8a and 8b) and will ensure both the gauge independence of the vertex under construction and the generalisation of the Ward identity in Eq. (3.2) to this order. We note that, as in the case of the pinched parts of the three vertex, the pinched graphs of the four vertex are ultraviolet finite, as one can easily verify by simple power counting. Finally, for calculational purposes we mention that there is a Bose statistics factor of $\frac{1}{2}$ multiplying graph 6b, a Fermi statistics factor of (-1) for the ghost loop of graph 6a, and the number of permutations for each topology is given in Ref [14].

d) We make extensive use of the following group-theoretical identities for the structure constants f^{abc}:

$$f^{ape} f^{bpe} = N \delta^{ab} \tag{3.3}$$

$$f^{ape} f^{bpm} f^{cem} = \frac{1}{2} N f^{abc} \tag{3.4}$$

and the usual Jaccobi identity:

$$f^{abe} f^{cde} + f^{ace} f^{bde} + f^{ade} f^{bce} = 0 \tag{3.5}$$

In addition, we use that

$$[T_a, T_b] = i f_{abc} T^c \tag{3.6}$$

where T_a is the representation of the external "test" fermions.

e) For the construction of the gauge independent 1PI vertex out of $\hat{T}(q_1, q_2, q_3, q_4)$ particular care is needed. Specificaly, $\hat{T}(q_1, q_2, q_3, q_4)$ contains 1PR graphs, like those in 7a, 7b, and 7e, which, as we already emphasised, are instrumental for its gauge independence. On the other hand 1PR graphs should not be included in a definition of a 1PI four vertex. The resolution of this dilemma lies on the following observation: Up to residual pinch parts which are effectively 1PI, all 1PR graphs organise themselves into gauge independent substructures built out of the gauge independent two-point function $\hat{\Delta}_{\mu\nu}$ and the gauge independent three point function $\hat{\Gamma}_{\mu\nu\alpha}$, defined in the previous section. To demonstrate with an example how this arrangement takes place, we concentrate on the 1PR graphs 7a, 7b, and 7e. Graph 7e contains all the usual one loop self-energy Feynman graphs of $\Delta_{\mu\nu}^{(\lambda=1)}$, whereas graphs contain some of the usual three gluon vertex graphs of $\Gamma_{\mu\nu\alpha}^{(\lambda=1)}$ (ghost graphs omitted). It is important to notice that there is another set of graphs like 7a and 7b, which we obtain by rotating the figures by 180^0. So in effect we have *two* sets of three-gluon vertex graphs appearing along the diagonal of our figure, one in the low left corner (shown) and one in the upper right corner (not shown). In order to construct a gauge independent self energy $\hat{\Delta}_{\mu\nu}$, out of $\Delta_{\mu\nu}^{(\lambda=1)}$, and a gauge independent three gluon vertices $\hat{\Gamma}_{\mu\nu\alpha}$ out of $\Gamma_{\mu\nu\alpha}^{(\lambda=1)}$, the appropriate pinch parts must be supplemented. We can convert $\Delta_{\mu\nu}^{(\lambda=1)}$ to $\hat{\Delta}_{\mu\nu}$ by adding the pinch Π^P and then subtracting $\frac{1}{2}\Pi^P$ from each one of the two $\Gamma_{\mu\nu\alpha}^{(\lambda=1)}$ (one in the low left and one in the upper right corner). Two more terms like this will be suplemented to $\Gamma_{\mu\nu\alpha}^{(\lambda=1)}$ from the two legs hooked on the two external fermions. Thus we have accounted for the second term in Eq. (2.21). The pinch parts of 7c and 7d will contribute two out of the three pieces of the third term in Eq. (2.21), namely $V_{\rho\nu\alpha}{q_1}^2 g^{\rho\mu} + V_{\mu\rho\alpha}{q_2}^2 g^{\rho\nu}$ We now add and subtract the missing part $V_{\mu\nu\rho}{q_3}^2 g^{\rho\alpha}$ and so we end up with exactly $\hat{\Gamma}_{\mu\nu\alpha}$ of Eq. (2.21) and the leftover $-V_{\mu\nu\rho}{q_3}^2 g^{\rho\alpha}$, which is *effectively* 1PI, due to the presence of ${q_3}^2$, which cancells the propagator $\frac{1}{{q_3}^2}$in the original 1PR diagram, as shown in Fig 8c. All such terms are instrumental for the gauge invariance of the vertex under construction. We will collectively call $\Gamma_{\mu\nu\alpha\beta}^{3P}$ all 1PI terms left over from the process of converting 1PR conventional three gluon vertices to gauge independent ones. Similarly, we call $\Gamma_{\mu\nu\alpha\beta}^{2P}$ all 1PI contributions left over after converting conventional self-energies to gauge independent ones.

f) Finally we must isolate all the pinch parts that originate from 1PI graphs like 5c and 5d. Such pinch contributions are schematically shown in 8a and 8b. It is important to include the contributions of the crossed graphs corresponding to 8a and 8c (not shown). Such contributions are essential because, when added to the regular ones, they give rise to group-theoretical factors of the form $[T_a, T_b]$, which, after use of Eq. (3.6), give rise to the correct group theoretical structures. We collectively call all the pinch parts $\Gamma^{4P}_{\mu\nu\alpha\beta}$. Clearly they are all multiplied by a free inverse propagator Δ_0^{-1}.

The final form of the gauge invariant and fully Bose symmetric four gluon vertex $\hat{\Gamma}_{\mu\nu\alpha\beta}$ is given by the following sum:

$$\begin{aligned}\hat{\Gamma}_{\mu\nu\alpha\beta} &= \Gamma^{(\lambda=1)}_{\mu\nu\alpha\beta} + \Gamma^{2P}_{\mu\nu\alpha\beta} \\ &+ \Gamma^{3P}_{\mu\nu\alpha\beta} + \Gamma^{4P}_{\mu\nu\alpha\beta}\end{aligned} \tag{3.7}$$

The exact closed form is very lengthy and of limited usefulness for our general program, and we do not report it here. Far more interesting is the Ward identity that this vertex satisfies. We will undertake the proof of the Ward identity in the next section. Here it should suffice to mention that knowledge of the exact closed form of $\hat{\Gamma}_{\mu\nu\alpha\beta}$ is *not* neccessary for proving the Ward identity.

4. The Ward Identity

In the previous section we described the construction of the gauge independent four gluon vertex $\hat{\Gamma}_{\mu\nu\alpha\beta}$ through the S-matrix pinch technique. In this section we deal with the second central topic of this paper, namely the Ward identity that $\hat{\Gamma}_{\mu\nu\alpha\beta}$ satisfies. It turns out that $\hat{\Gamma}_{\mu\nu\alpha\beta}$ and $\hat{\Gamma}_{\mu\nu\alpha}$ are related by the following Ward identity:

$$\begin{aligned}q_1^{\mu}\hat{\Gamma}^{abcd}_{\mu\nu\alpha\beta} &= f_{abp}\hat{\Gamma}^{cdp}_{\nu\alpha\beta}(q_1+q_2,q_3,q_4) \\ &+ f_{acp}\hat{\Gamma}^{bdp}_{\nu\alpha\beta}(q_2,q_1+q_3,q_4) \\ &+ f_{adp}\hat{\Gamma}^{bcp}_{\nu\alpha\beta}(q_2,q_3,q_1+q_4)\end{aligned} \tag{4.1}$$

Eq. (4.1) is the generalisation of the tree level Ward identity of Eq. (3.2) and the major result of this paper. It must be emphasised that as in the case of the Ward identity in Eq. (2.22) this new Ward identity makes no reference to ghost Green's functions; Eq. (4.1) is completely gauge invariant. Before diving into the complexities of the full proof, it is instructive and relatively straightforward to first establish the validity of Eq. (4.1) only for the ultraviolet divergent parts of $\hat{\Gamma}^{\mu\nu\alpha}$ and $\hat{\Gamma}^{\mu\nu\alpha\beta}$, ${Z_3}^{\mu\nu\alpha}$ and ${Z_4}^{\mu\nu\alpha\beta}$ respectively. Saving only terms $\sim \frac{1}{\epsilon}$ we have that their tensorial structure is simply

$$
{Z_3}^{\mu\nu\alpha} = Z_3 \Gamma^{\mu\nu\alpha} \tag{4.2}
$$

and

$$
{Z_4}^{\mu\nu\alpha\beta} = Z_4 \Gamma^{\mu\nu\alpha\beta} \tag{4.3}
$$

where Z_3 and Z_4 are scalar quantities. From Eq. (2.21) we see that only the first two terms are ultraviolet divergent and contribute to Z_3, whereas the third is ultraviolet finite. Similarly, the first two terms of Eq. (3.7) contribute to Z_4, whereas the last two are finite. So, using the fact [14] that

$$
Z_3^{(\lambda=1)} = 1 + \frac{2g^2 N}{48\pi^2}\left(\frac{1}{\epsilon}\right) \tag{4.4}
$$

and

$$
Z_4^{(\lambda=1)} = 1 - \frac{g^2 N}{48\pi^2}\left(\frac{1}{\epsilon}\right) \tag{4.5}
$$

and the value of Π^P from Eq. (2.10) we have

$$
\begin{aligned}
Z_3 &= \left[1 + \left(\frac{2g^2 N}{48\pi^2}\right)\frac{1}{\epsilon}\right] - \left[3 \times \left(-\frac{3g^2 N}{48\pi^2}\right)\frac{1}{\epsilon}\right] \\
&= 1 + bg^2\left(\frac{1}{\epsilon}\right)
\end{aligned} \tag{4.6}
$$

and

$$
\begin{aligned}
Z_4 &= \left[1 - \left(\frac{g^2 N}{48\pi^2}\right)\frac{1}{\epsilon}\right] - \left[4 \times \left(-\frac{3g^2 N}{48\pi^2}\right)\frac{1}{\epsilon}\right] \\
&= 1 + bg^2\left(\frac{1}{\epsilon}\right)
\end{aligned} \tag{4.7}
$$

Clearly

$$
Z_3 = Z_4 \tag{4.8}
$$

and so from Eq. (3.2), Eq. (4.2), Eq. (4.3) and Eq. (4.8) immediately follows that ${Z_3}^{abc}_{\mu\nu\alpha}$ and ${Z_4}^{abcd}_{\mu\nu\alpha\beta}$ satisfy Eq. (4.1) as advertised.

We now turn to the full proof of Eq. (4.1). To prove Eq. (4.1). we act with q_1^μ on $\hat{\Gamma}_{\mu\nu\alpha\beta}$ given by the RHS of Eq. (3.7) and try to recover the appropriate combinations of $\hat{\Gamma}$, as they appear in the RHS of Eq. (4.1). In that vein it is far more convenient to act on the individual graphs that define $\hat{\Gamma}_{\mu\nu\alpha\beta}$ instead of first evaluating them and then acting with q_1^μ on the final answer. In this way it is much easier to immediately identify entire structures that contribute to the RHS of Eq. (4.1) as well as major cancellations among different graphs. This in retrospect explains the distinction between the topologically equivalent graphs 6a and 6c, since when acting with q_1^μ on them we hit on different elementary vertices (four-gluon vertex for 6a, three-gluon vertex for 6c). Troughout the algebraic manipulations we make extensive use of the tree level Ward identity of Eq. (3.2) to convert $\Gamma_{\mu\nu\alpha\beta}$ to $\Gamma_{\mu\nu\alpha}$, the tree level version of Eq. (2.22) namely

$$q_1^\mu \Gamma_{\mu\nu\alpha}\left[q, k, -(k+q)\right] = P_{\nu\alpha}(k) - P_{\nu\alpha}(k+q) \tag{4.9}$$

to convert $\Gamma_{\mu\nu\alpha}$ to inverse gluon propagators, as well as the identities of Eq. (3.3), Eq. (3.4), Eq. (3.5), and Eq. (3.6). The actual algebra is straightforward but tedious. In what follows we will provide for the reader a roadmap of the proof and highlight some of the most important steps.

a) Acting with q_1^μ on $\Gamma^{(\lambda=1)}_{\mu\nu\alpha\beta}$, the first term of Eq. (3.7), we find three different types of terms, namely

$$q_1^\mu \Gamma^{(\lambda=1)}_{\mu\nu\alpha\beta} = f \otimes \Gamma^{(\lambda=1)}_{\nu\alpha\beta}(q_1+q_2, q_3, q_4) + K^\rho_{\nu\alpha} \Delta_0{}^{-1}_{\rho\beta}(q_1) + S_{\nu\alpha\beta} \\ + c.p. \tag{4.10}$$

where $f \otimes \Gamma$ is shorthand for the usual color index contraction. The second term in Eq. (4.10) stands collectively for all those terms arising every time an inverse external propagator $\Delta_0^{-1}(q_i)$ will be generated from the elementary Ward identity of Eq. (4.9); it is finite in the ultraviolet and will be cancelled by the appropriate pinch contributions, coming from $\Gamma^{4P}_{\mu\nu\alpha\beta}$, introduced in the previous section. The third term $S_{\nu\alpha\beta}$ is also ultraviolet finite, but does not have the typical pinch structure, e.g. it is not multiplied by free inverse propagators.

b) The second term of Eq. (3.7) upon contraction yields

$$q_1^{\mu}\Gamma^{2P}_{\mu\nu\alpha\beta} = \frac{1}{2}\left[\Pi^P(q_2) + \Pi^P(q_3) + \Pi^P(q_4)\right] f \otimes \Gamma_{\nu\alpha\beta}(q_1 + q_2, q_3, q_4) + c.p. \tag{4.11}$$

This term has the same structure as the second term in Eq. (2.21). It is important to notice however that in the equation above the first argument of $\Gamma_{\nu\alpha\beta}$ is $(q_1 + q_2)$ as it should, but the argument of the first one of the Π^P is q_1 instead of $(q_1 + q_2)$. In order to obtain the complete second term of Eq. (2.21) [defined at $(q_1 + q_2, q_3, q_4)$] we add and subtract $\frac{1}{2}\Pi^P(q_1 + q_2)$. The difference

$$R = \frac{1}{2}\left[\Pi^P(q_1 + q_2) - \Pi^P(q_2)\right] \tag{4.12}$$

is ultraviolet finite, and we are leftover with a term $R\Gamma_{\nu\alpha\beta}$.

c) Next we act on the fourth term in equation Eq. (3.7) and get two types of terms

$$q_1^{\mu}\Gamma^{4P}_{\mu\nu\alpha\beta} = f \otimes V^{\rho}_{\alpha\beta}\Delta_0^{-1}{}_{\rho\nu}(q_1) - K_{\nu\alpha\beta}\Delta_0^{-1}{}_{\rho\nu}(q_1) + c.p. \tag{4.13}$$

The first term of Eq. (4.13) provides the last term of $\hat{\Gamma}_{\mu\nu\alpha}$ (third term in RHS of Eq. (2.21)), whereas the second term in Eq. (4.13) exactly cancels against the second term of Eq. (4.10).

d) Finally, contracting the third term of Eq. (3.7) and adding it to the leftover finite parts given in Eq. (4.10) and Eq. (4.12), we get exactly zero, e.g.

$$q_1^{\mu}\Gamma^{3P}_{\mu\nu\alpha\beta} + S_{\nu\alpha\beta} + R\Gamma_{\nu\alpha\beta} = 0 \tag{4.14}$$

Adding Eq. (4.10),Eq. (4.11), Eq. (4.13) and Eq. (4.14) we arrive at the desired result of Eq. (4.1). It is obvious from the above presentation that the inclusion of the pinch contribution is instrumental for the validity of Eq. (4.1).

5. Conclussions

In this paper we showed how to use the S-matrix pinch technique to construct at one loop order a gauge independent four gluon vertex. As it turned out, the new vertex satisfies a very simple Ward identity, relating it to a previously constructed gauge independent three gluon vertex. The exact closed expression is very lengthy and of limited usefulness; we did not report it here. Far more interesting for practical purposes would be a gauge technique inspired Ansatz for the new vertex. The gauge technique [15-16] expresses a vertex in terms of other Green's functions in such a way as to satisfy by construction a given Ward identity exactly, to all orders in perturbation theory. The vertex so constructed is certainly not unique, since any transverse (divergenceless) part may be added, without affecting the Ward identity. Even though the correct transverse part must be eventually supplemented for the exact cancellation of overlapping ultraviolet divergences [17-18], its omission does not affect the infrared domain significantly. In the context of quarkless QCD the gauge technique has been successfully implemented by Cornwall and Hou [3] for the gauge invariant three gluon vertex $\hat{\Gamma}_{\mu\nu\alpha}$, which was expressed in terms of the gauge independent self energy $\hat{\Pi}$, in such a way as to exactly satisfy the Ward identity in Eq. (2.22). The Cornwall-Hou vertex is given by:

$$\begin{aligned} \hat{\Gamma}^{\mu\nu\alpha}_{CH} &= \Gamma^{\mu\nu\alpha} - \frac{1}{2}\frac{q_1^{\mu}q_2^{\nu}}{q_1^2 q_2^2}(q_1-q_2)^{\rho}\hat{\Pi}^{\alpha}_{\rho}(3) \\ &+ \left[Q^{\mu}_{\rho}(1)\,\hat{\Pi}^{\rho\nu}(2) - \hat{\Pi}^{\rho\nu}(1)\,Q^{\mu}_{\rho}(2)\right]\frac{q_3^{\alpha}}{q_3^2} \\ &+ c.p. \end{aligned} \tag{5.1}$$

with $Q_{\mu\nu}(q) = \frac{1}{q^2}P_{\mu\nu}(q)$. It would be interesting to construct a similar Ansatz expressing the gauge invariant four gluon vertex in terms of the three gluon vertex, so that the Ward identity in Eq. (4.1) is automatically satisfied. Calculations in this direction are already in progress.

6. Acknowledgments.

The author is indebted to J. M. Cornwall for numerous very useful discussions.

7. References.

1. J. M. Cornwall, in Deeper Pathways in High Energy Physics, edited by B. Kursunoglu, A. Perlmutter, and L. Scott (Plenum, NewYork, 1977), p.683.
2. J. M. Cornwall, Phys. Rev. D 26, 1453 (1982).
3. J. M. Cornwall, W. S. Hou, and J. E. King, Phys. Lett. B 153, 173 (1988).
4. S. Nadkarni, Phys. Rev. Lett 61, 396 (1988).
5. J. M. Cornwall and J. Papavassiliou, Phys. Rev. D 40, 3474 (1989).
6. M. Lavelle, Phys. Rev. D 44, 26 (1991).
7. J. Papavassiliou, Phys. Rev. D 41, 3179 (1990).
8. G. Degrassi and A. Sirlin, NYU-TR 92-05-02.
9. J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D 10, 2428 (1974).
10. J. M. Cornwall and R. E. Norton, Ann. Phys. (N.Y.) 91, 106 (1975).
11. J. M. Cornwall, Phys. Rev. D 38, 656 (1988).
12. This conserved form is, in fact, automatic in other forms of the pinch technique, e.g., the off-shell approach of Refs [1] and [2], or the intrinsic pinch discussed in Ref [5].
13. To avoid notational clutter, in what follows we omit the color indices whenever possible..
14. P. Pascual and R. Tarrach, Nucl. Phys. B 174, 123 (1980).
15. A. Salam, Phys. Rev. 130, 1287 (1963).
16. R. Delbourgo and A. Salam, Phys. Rev. D 135, B1398 (1964).

17. J. E. King, Phys. Rev. D 27, 1821 (1983).

18. B. Haeri, Phys. Rev. D 38, 3799 (1988).

8. Figure Captions.

1) One-dressed-loop Feynman graphs for the renormalized $\hat{\Pi}_{\mu\nu}$ (in a ghost free gauge) necessary to implement the gauge invariance of the effective potential

2) Graphs (a)-(c) are some of the contributions to the S-matrix T. Graphs (e) and (f) are pinch parts, which, when added to the original propagator parts (d), give a gauge-invariant effective propagator.

3) Graphs (a) and (c) contribute to the gauge invariant three-gluon vertex trough their pinch parts, graphs (b) and (d) respectively.

4) Graph (a) is the bare four-gluon vertex, with incoming momenta, Lorenz- and color indices displayed. Graph (b) represents the general form of the part $\hat{T}(q_1, q_2, q_3, q_4)$ of the S-matrix, that only depends on the momentum transfers q_i.

5) The three different kinds of graphs that contribute to $\hat{T}$. Graphs (c) and (d) contribute trough their pinch parts, shown in Fig 8.

6) Some of the one particle irreducible (1PI) graphs. As explained in the text, graphs (a) and (c), even though topologically equivallent, contribute differently to the Ward identity.

7) Some of the one particle reducible (1PR) graphs.

8) Graphs (a) and (b) are the pinch contributions from graphs (5c) and (5d) respectively. Graph (c) originates from the conversion of gauge dependent three-gluon vertices to gauge independent ones, as explained in section 3.

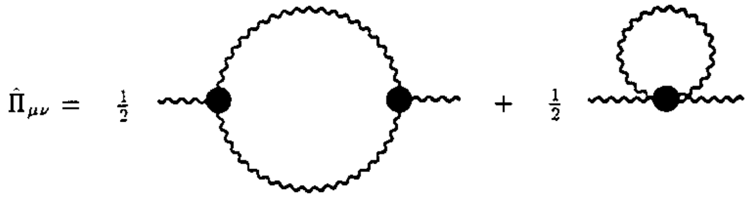


Figure 1

Joannis Papavassiliou

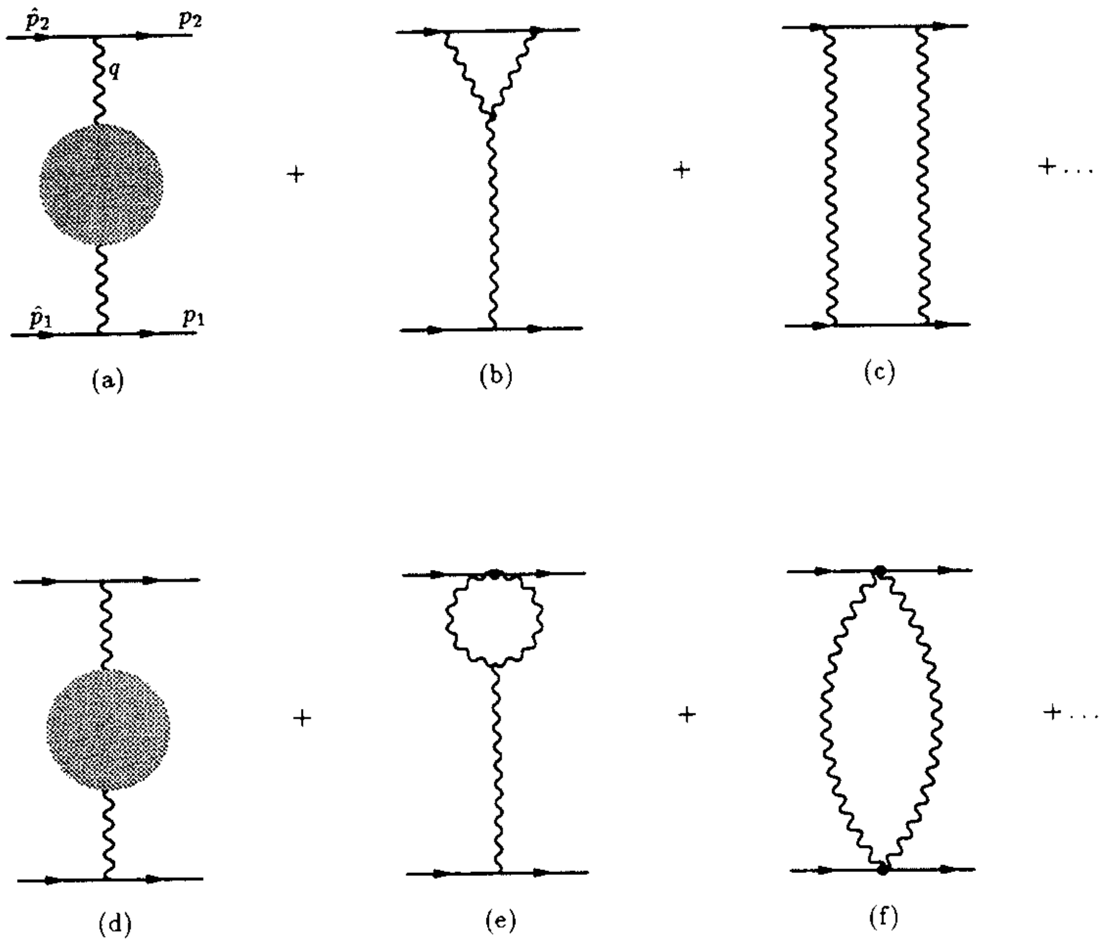
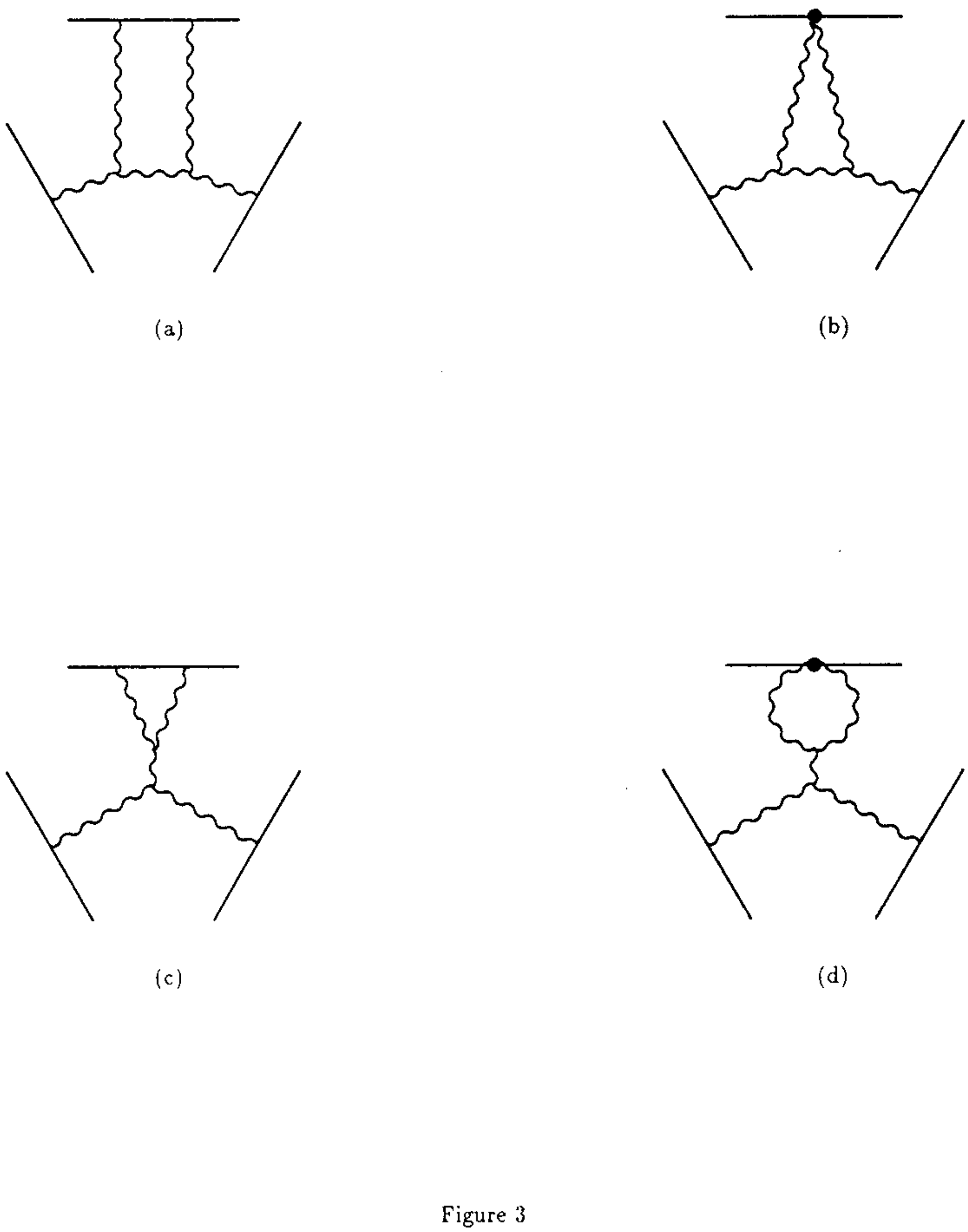


Figure 2

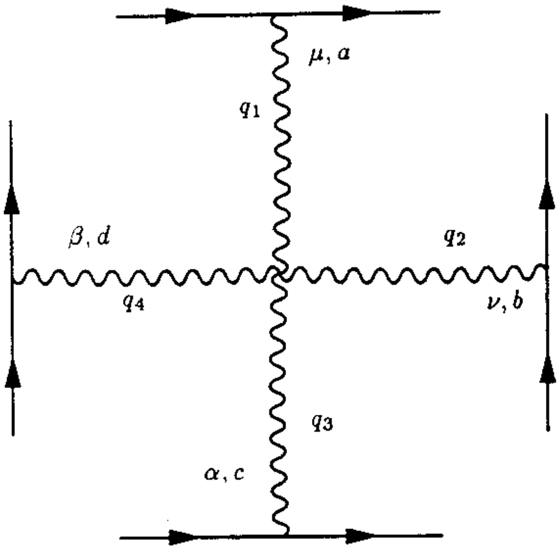
Joannis Papavassiliou



(a) (b) (c) (d)

Figure 3

Joannis Papavassiliou



(a)

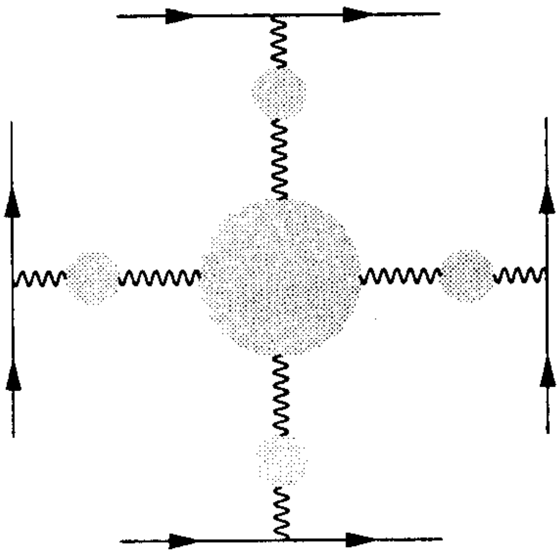

(b)

Figure 4

Joannis Papavassiliou

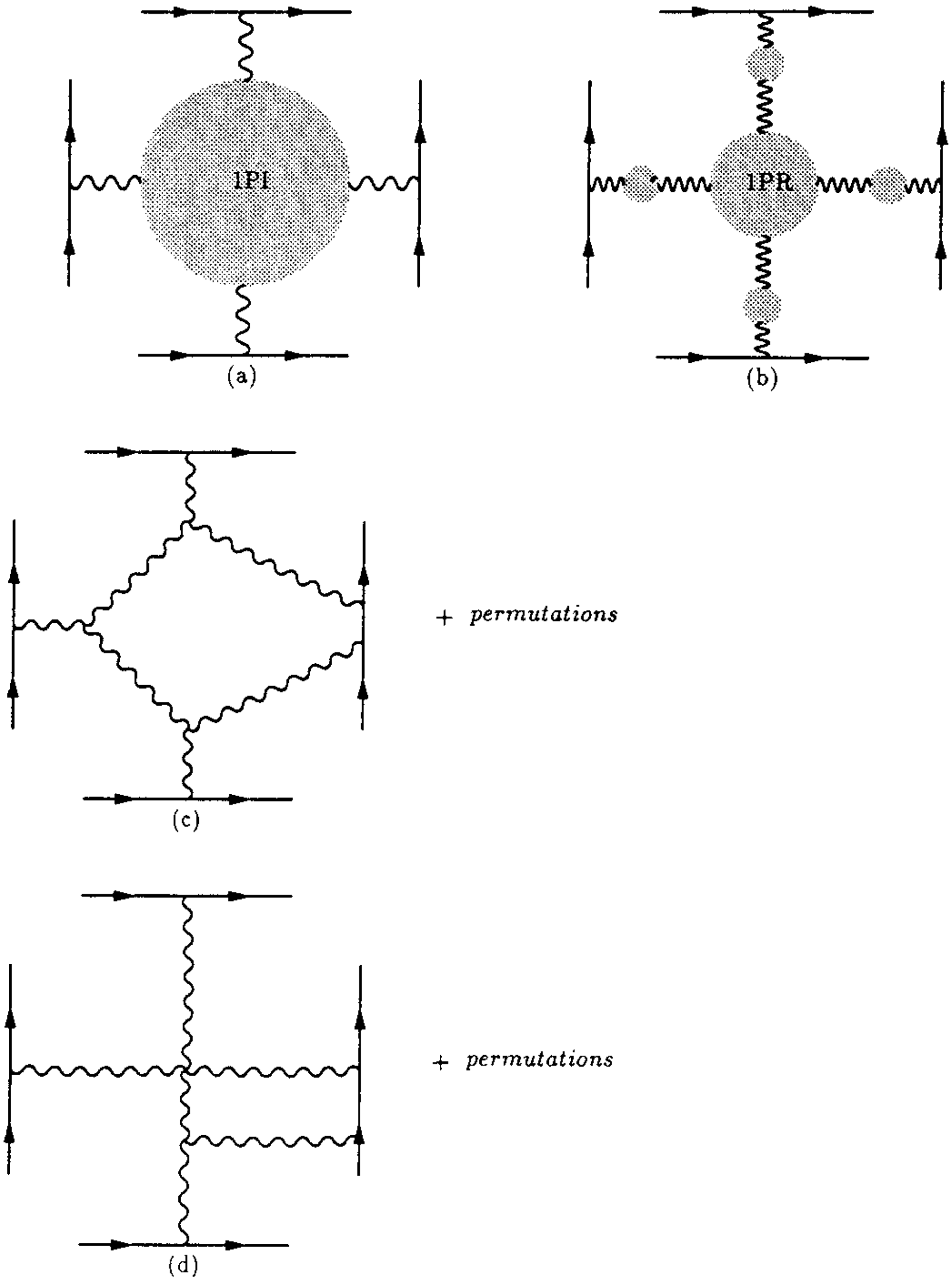


Figure 5

Joannis Papavassiliou

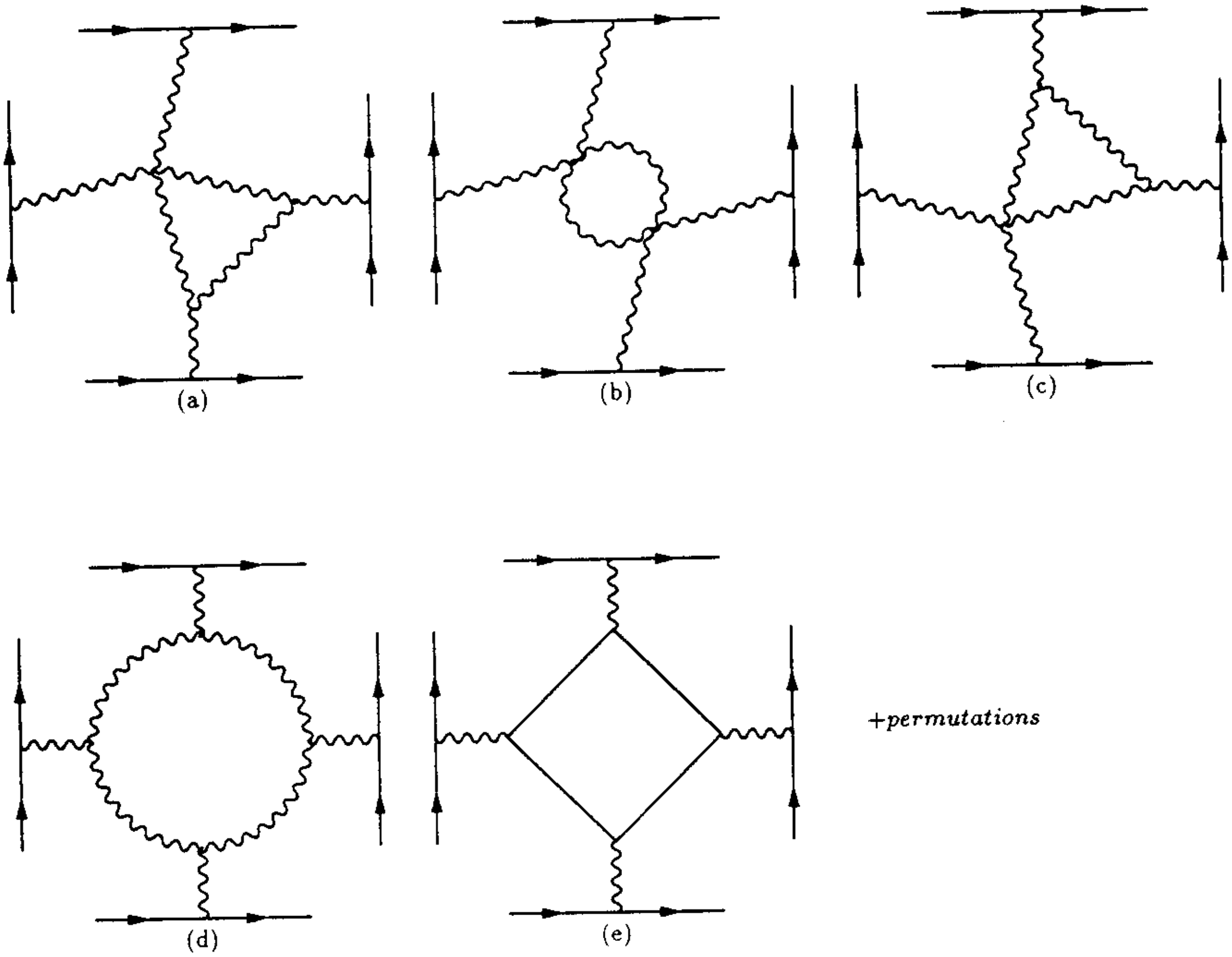


Figure 6

Joannis Papavassiliou

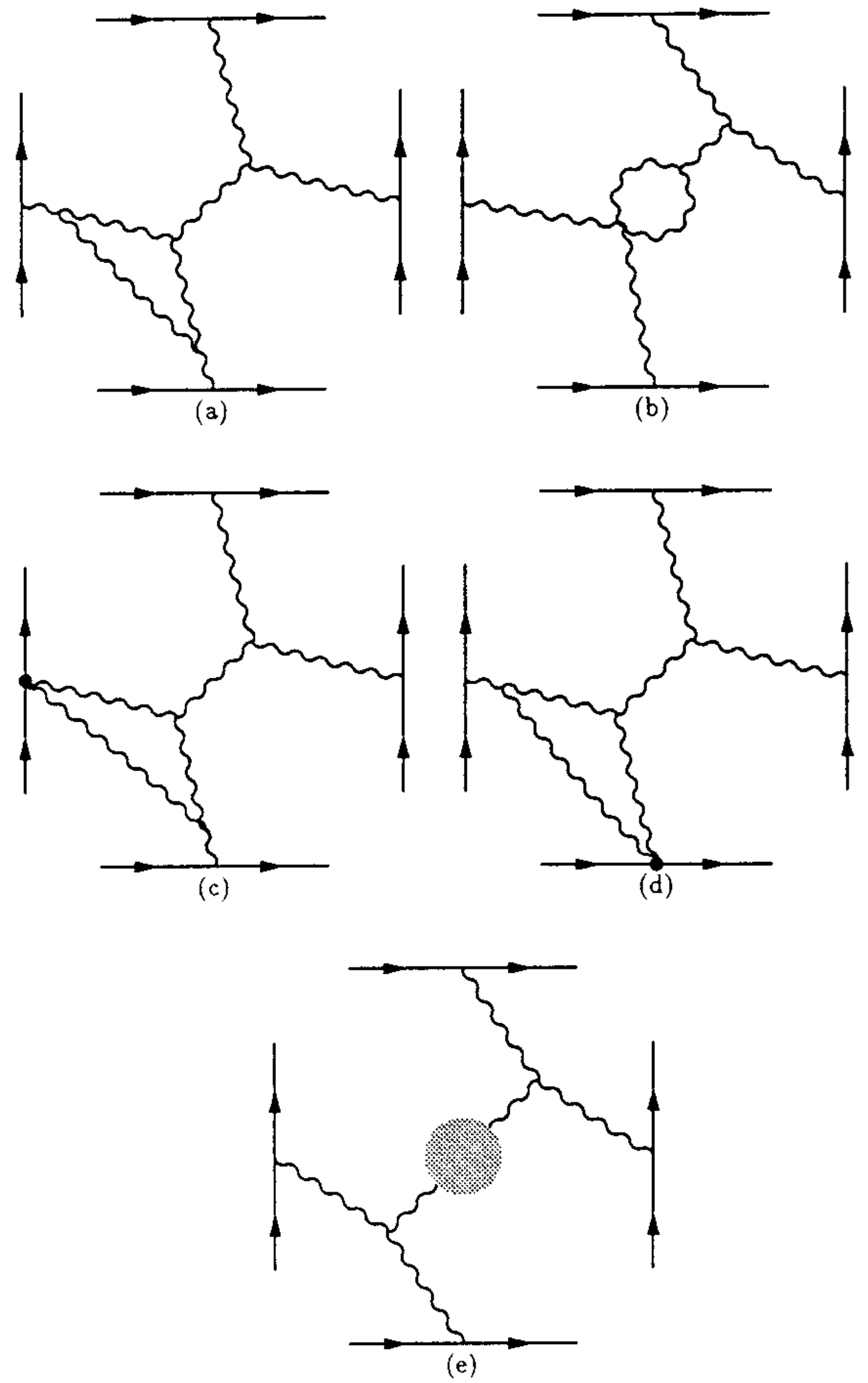
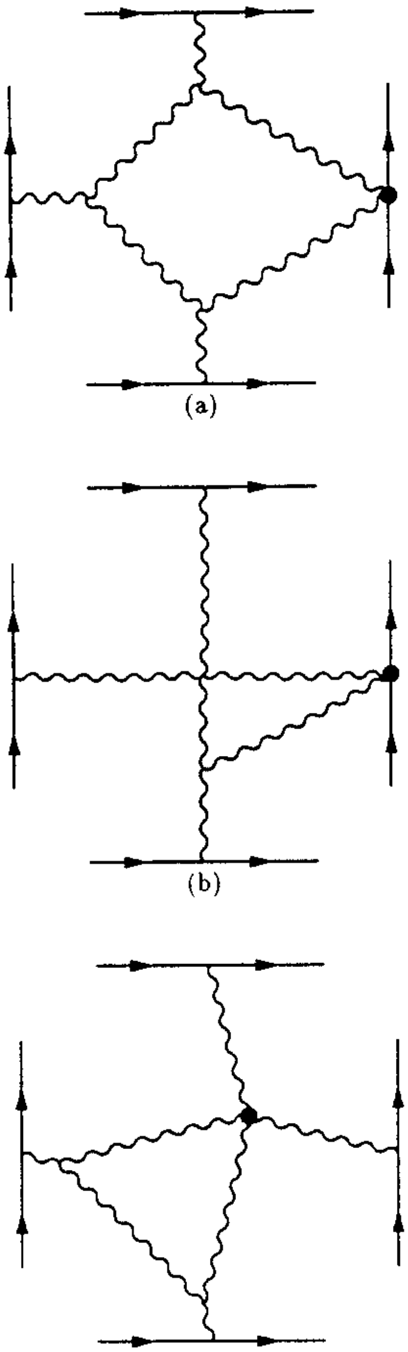


Figure 7

Joannis Papavassiliou



(a)

(b)

(c)

Figure 8

Joannis Papavassiliou